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Cells in affine Weyl groups and tilting modules

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Abstract.

Let G be a reductive algebraic group over a field of positive characteristic. In this paper we explore the relations between the behaviour of tilting modules for G and certain Kazhdan-Lusztig cells for the affine Weyl group associated with G. In the corresponding quantum case at a complex root of unity V. Ostrik has shown that the weight cells defined in terms of tilting modules coincide with right Kazhdan-Lusztig cells. Our method consists in comparing our modules for G with quantized modules for which we can appeal to Ostrik's results. We show that the minimal Kazhdan-Lusztig cell breaks up into infinitely many "modular cells" which in turn are determined by bigger cells. At the opposite end we call attention to recent results by T. Rasmussen on tilting modules corresponding to the cell next to the maximal one. Our techniques also allow us to make comparisons with the mixed quantum case where the quantum parameter is a root of unity in a field of positive characteristic.

§1. Introduction

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and denote by U_q the associated quantum group at a primitive l'th root of unity $q \in \mathbb{C}$. If $A = \mathbb{Z}[v, v^{-1}]$ then U_q is obtained by specializing Lusztig's A-form of the quantized enveloping algebra of $\mathfrak{g}$ at q.

Following Ostrik [17] the set of dominant weights for $\mathfrak{g}$ is divided into weight cells. Two weights are in the same cell if the two corresponding indecomposable tilting modules have the property that each of them occurs as a summand of the tensor product of the other by some third tilting module. Ostrik proved [16] that these weight cells coincide with the Kazhdan-Lusztig cells in the affine Weyl group W_l associated with U_q.

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Denote by k an algebraically closed field of characteristic $p > 0$ and let G be a connected reductive algebraic group over k with the same root system as $\mathfrak{g}$. In an effort to gain some better understanding of the tilting modules for G we lift them to U_{A_p} and then study their specializations at q. Here $A_p = A_{(v-1,p)}$ is the localization of A at the maximal ideal $(v-1, p)$ and $\mathbb{C}$ is made into an A_p-algebra by specializing v at q (q now being a p'th root of 1). The (wide open) problem of finding the characters of the indecomposable tilting modules for G is then equivalent to the problem of decomposing these specializations into indecomposable U_q-modules.

When λ is a dominant weight we denote by $T(\lambda)$, respectively $T_q(\lambda)$ the indecomposable tilting module for G, respectively for U_q with highest weight λ. The U_q-module obtained from $T(\lambda)$ by the "quantization" described above is denoted $T(\lambda)_q$. With this notation the conjecture stated in [3, 5.1] says that $T(\lambda)_q = T_q(\lambda)$ as long as λ belongs to the lowest p^2-alcove. We shall demonstrate that this conjecture combined with the work of Ostrik on weight cells allow us to carry over some of Ostrik's results on tensor ideals in the category of tilting modules. This relies on some of the known properties of Kazhdan-Lusztig cells.

In a different direction we also report on some recent work by T. E. Rasmussen [19] concerning second cell tilting modules. By the second cell we understand the one next to the cell consisting of weights in the first alcove. Rasmussen's main result (so far only valid in type ADE)[1] can be phrased as saying that $T(\lambda)_q \equiv T_q(\lambda)$ modulo tilting modules with highest weights belonging to smaller cells. Recall that Soergel [21] has determined the characters of all $T_q(\lambda)$'s. Hence this result of Rasmussen gives for a given tilting module with known character a way of finding its indecomposable summands with highest weights in the second cell.

Let now $r \in \mathbb{N}$ and denote by $H_r(q)$ the Hecke algebra over $\mathbb{C}$ of the symmetric group Σ_r with parameter q. Denote for a partition λ the simple $k\Sigma_r$-module, respectively $H_r(q)$-module by D^λ, respectively D_q^λ. Then taking $G = GL(V)$ in Rasmussen's theorem he obtains via Schur-Weyl duality that $\dim_k D^\lambda = \dim_{\mathbb{C}} D_q^\lambda$ for all λ in the second cell. Again the right hand side is known (e.g. via Soergel's results mentioned above). This proves parts of a conjecture by Mathieu [15, Conjecture 15.4].

Finally, we describe how the above technique for comparing tilting modules for G with those for U_q can similarly be used to compare also

[1] In the meantime Rasumssen has generalized his results to arbitrary types.

with tilting modules for the quantum group U_ζ at an l'th root of unity $\zeta \in k$. In [4, Section 4] we conjectured that if λ belongs to the lowest lp-alcove then the character of the indecomposable tilting module $T_\zeta(\lambda)$ for U_ζ coincides with the character for $T_q(\lambda)$. As above this implies that for p large we can locate some of the tensor ideals in the category of tilting modules for U_ζ.

Rasmussen's results about second cell tilting modules also generalize to this case. They lead to an equality of dimensions for certain simple modules of $H_r(\zeta)$ and $H_r(q)$. Here $H_r(\zeta)$ is the Hecke algebra over k with parameter ζ.

§2. Notation and recollection

In addition to the notation already introduced above we shall need the following.

The root system for $\mathfrak{g}$ (with respect to a Cartan subalgebra $\mathfrak{h}$) and for G (with respect to a maximal torus T) will be denoted R. We shall fix a set of positive roots R^+ in R. The set of characters $X = X(T)$ of T (which may be identified with the integral weights of $\mathfrak{h}$) then contains a cone X^+ consisting of the dominant weights, namely

$$X^+ = \left\{\lambda \in X | \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\right\}.$$

Here $\alpha^\vee$ is the coroot of $\alpha \in R$. This set is a fundamental domain for the action of the Weyl group W on X. We shall often use the 'dot' action of W on X given by $w \cdot \lambda = w(\lambda + \rho) - \rho, w \in W, \lambda \in X$. As usual ρ is half the sum of the positive roots.

If $l \in \mathbb{N}$ then the affine Weyl group W_l is the group generated by the affine reflections $s_{\alpha,n}, \alpha \in R^+, n \in \mathbb{Z}$ given by

$$s_{\alpha,n} \cdot \lambda = s_\alpha \cdot \lambda + nl\alpha, \quad \lambda \in X.$$

Here s_α is the reflection in W attached to α. As fundamental domain for this action we choose the closure $\bar{C}_l$ (obtained by replacing all $<$ by $\leq$ below) of the bottom alcove C_l in X^+ determined by

$$C_l = \left\{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l, \quad \alpha \in R^+\right\}.$$

Consider now $\lambda \in X^+$. We have four G-modules associated with λ (all having λ as their unique highest weight)

- the simple module $L(\lambda)$,
- the Weyl module $\Delta(\lambda)$,
- the dual Weyl module $\nabla(\lambda)$,

and

- the indecomposable tilting module $T(\lambda)$.

These modules may be constructed as follows. Choose $B \subset T$ to be the Borel subgroup corresponding to the negative roots $-R^+$. Let then $\nabla(\lambda)$ be the G-module induced from the 1-dimensional B-module determined by λ and set $\Delta(\lambda) = \nabla(\lambda)^*$ (for a G-module M we let G act on the dual module $M^* = \mathrm{Hom}_k(M, k)$ via the Chevalley antiautomorphism on G so that the set of weights of M^* coincides with the set of weights on M). It is an easy observation to see that $L(\lambda)$ is the image of the natural homomorphism $\Delta(\lambda) \to \nabla(\lambda)$. Finally, following Ringel [20] we may construct $T(\lambda)$ by first setting $E_1 = \Delta(\lambda)$, then choose $\lambda_1 \in X^+$ maximal with $d_1 = \dim_k \mathrm{Ext}^1_G(\Delta(\lambda_1), \Delta(\lambda)) > 0$ and set E_2 equal to the corresponding extension $0 \to E_1 \to E_2 \to \Delta(\lambda_1)^{\oplus d_1} \to 0$. Now we repeat this process after having replaced E_1 by E_2. After a finite number of such steps we arrive at a G-module E_r which has $\mathrm{Ext}^1_G(\Delta(\mu), E_r) = 0$ for all $\mu \in X^+$. Then $T(\lambda) = E_r$ (see [3] for details).

The above constructions work equally well in the quantum case (using the concept of induction from a "Borel subalgebra" of U_q introduced in [7]). We shall denote the corresponding modules for U_q by $L_q(\lambda), \Delta_q(\lambda), \nabla_q(\lambda)$, and $T_q(\lambda)$, respectively.

Moreover, we have A-forms $\Delta_A(\lambda)$ and $\nabla_A(\lambda)$ of the Weyl module and its dual. These are U_A-modules which are free over A and satisfy

$$\Delta_A(\lambda) \otimes_A \mathbb{C} \simeq \Delta_q(\lambda), \quad \nabla_A(\lambda) \otimes_A \mathbb{C} \simeq \nabla_q(\lambda),$$

and

$$\Delta_A(\lambda) \otimes_A k \simeq \Delta(\lambda), \quad \nabla_A(\lambda) \otimes_A k \simeq \nabla(\lambda).$$

Here $\mathbb{C}$, respectively k is considered an A-algebra by specializing v to $q \in \mathbb{C}$, respectively to $1 \in k$. In the last case we have also used the identification between G-modules and modules for the hyperalgebra $U_k = U_A \otimes_A k$, see [7].

Recall that $A_p = A_{(v-1,p)}$. If in the above construction of $T(\lambda)$ we replace the appropriate G-modules by the corresponding U_A-modules and use 'minimal number of generators' instead of '$\dim_k$' then we obtain a tilting module $T_{A_p}(\lambda)$ for U_{A_p}, which satisfies $T_{A_p}(\lambda) \otimes_{A_p} k \simeq T(\lambda)$. The "quantization" $T(\lambda)_q$ referred to in the introduction is then the specialization of this module at q, i.e.

$$T(\lambda)_q = T_{A_p}(\lambda) \otimes_{A_p} \mathbb{C}.$$

(Now q is a p'th root of unity in $\mathbb{C}$).

The indecomposable tilting modules for U_q have the form $T_q(\mu), \mu \in X^+$. Hence there are unique (fusion numbers) $a_{\mu\lambda} \in \mathbb{N}$ such that

$$T(\lambda)_q = \bigoplus_{\mu \in X^+} a_{\mu\lambda} T_q(\mu).$$

Clearly, $a_{\lambda\lambda} = 1$ for all λ and $a_{\mu,\lambda} = 0$ unless $\mu \leq \lambda$.

Problem 1. Determine $a_{\mu\lambda}$ for all $\mu, \lambda \in X^+$.

Remark 2. i) This problem is wide open. Since by [21] the characters of $T_q(\mu), \mu \in X^+$ are known (with the exception of certain small values of p for some types) this problem is equivalent to the problem of finding the characters of $T(\lambda), \lambda \in X^+$.

ii) Set $X_p = \{\lambda \in X^+ \mid \langle \lambda, \alpha^\vee \rangle < p$ for all simple roots $\alpha\}$. If $p \geq 2h - 2$ then knowledge of the finite set of fusion numbers $\{a_{\mu\lambda} \mid \lambda \in (p-1)\rho + X_p\}$ is equivalent to knowledge of the characters of all $L(\lambda), \lambda \in X^+$ (see [3]).

iv) The only group for which the $a_{\mu\lambda}$'s have been found for all $\mu, \lambda \in X^+$ is $G = SL_2(k)$, see e.g. [10]. For $G = SL_3(k)$ some partial results can be found in [12]. Even in that case a complete solution seems to be far away.

Conjecture 3. ([3, Conjecture 5.4]). If $\lambda \in C_{p^2}$ then $T(\lambda)_q = T_q(\lambda)$.

If $p \geq 2h-2$ then $(p-1)\rho + X_p \subseteq C_{p^2}$. The strongest evidence for this conjecture is that it holds in this subset if and only if Lusztig's conjecture on simple G-modules (known for $p \gg 0$ by [5]) holds. Conjecture 3 is also known to hold (for all $p > 2$) for types A_1 and A_2.

It should be noted that the conjecture is also of interest for $2 < p < 2h - 2$. In this case its verification will not give the characters of all irreducible G-modules. However, for small primes we still have no general conjecture which covers all irreducible characters for G.

§3. Cells and tensor ideals for quantum groups

In this section $q \in \mathbb{C}$ will be a primitive l'th root of 1 with l odd (and prime to 3 if R contains a component of type G_2).

Let $\mathcal{T}_q$ denote the category consisting of all tilting modules for U_q. Recall that the tensor product of two tilting modules is again tilting [18], i.e. $\mathcal{T}_q$ is a tensor category. Clearly, if Q_1, Q_2 are two U_q-modules then $Q_1 \oplus Q_2 \in \mathcal{T}_q$ if and only if $Q_1, Q_2 \in \mathcal{T}_q$.

Definition 4. Let $\lambda, \mu \in X^+$. We write $\lambda \underset{\mathcal{T}_q}{\leq} \mu$ if there exists $Q \in \mathcal{T}_q$ such that $T_q(\lambda)$ is a summand of $T_q(\mu) \otimes_{\mathbb{C}} Q$. If both $\lambda \underset{\mathcal{T}_q}{\leq} \mu$ and $\mu \underset{\mathcal{T}_q}{\leq} \lambda$ then we write $\lambda \underset{\mathcal{T}_q}{\sim} \mu$. The equivalence classes for $\underset{\mathcal{T}_q}{\sim}$ are called weight cells (after Ostrik [17]).

Example 5. Let $\lambda \in X^+$. Then $\lambda + \nu \underset{\mathcal{T}_q}{\leq} \lambda$ for all $\nu \in X^+$. In fact, $T_q(\lambda + \nu)$ is clearly a summand of $T_q(\lambda) \otimes_{\mathbb{C}} T_q(\nu)$.

Consider the special weight $(l-1)\rho$. In this case we have

$$L_q((l-1)\rho) = \Delta_q((l-1)\rho) = \nabla_q((l-1)\rho) = T_q((l-1)\rho)$$

and this module is both simple and injective in the category of finite dimensional U_q-modules. It is denoted St_q and called the Steinberg module for U_q.

Proposition 6. The set $(l-1)\rho + X^+$ is a weight cell.

Proof. By Example 5 we have $(l-1)\rho+\nu \underset{\mathcal{T}_q}{\leq} (l-1)\rho$ for all $\nu \in X^+$. To see that also $(l-1)\rho \underset{\mathcal{T}_q}{\leq} (l-1)\rho+\nu$ it is (by the above properties of St_q) enough to check that there is a non-zero homomorphism $St_q \to T_q((l-1)\rho+\nu) \otimes_{\mathbb{C}} T_q(\nu)$. But since $T_q(\nu) \simeq T_q(\nu)^*$ we have $\mathrm{Hom}_{U_q}(St_q, T_q((l-1)\rho+\nu) \otimes_{\mathbb{C}} T_q(\nu)) \simeq \mathrm{Hom}_{U_q}(St_q \otimes_{\mathbb{C}} T_q(\nu), T_q((l-1)\rho+\nu))$, which is clearly non-zero.

To finish the proof we check that if $\nu \underset{\mathcal{T}_q}{\leq} (l-1)\rho$ then $\nu \in (l-1)\rho+X^+$. But if $\nu \underset{\mathcal{T}_q}{\leq} (l-1)\rho$ then $T_q(\nu)$ is injective and the only indecomposable tilting modules, which are injective, are those with highest weight in $(l-1)\rho + X^+$, see [5].

Remark 7.

i) The partial order $\underset{\mathcal{T}_q}{\leq}$ on X^+ induces a partial order (denoted in the same way) on the set of weight cells. The observation in Example 5 shows that the weight cell $(l-1)\rho+X^+$ from Proposition 6 is the unique smallest cell in this ordering.

ii) The proof of Proposition 6 shows that $(l-1)\rho + X^+$ parametrizes the injective indecomposable tilting modules (the PIM's in the category of finite dimensional U_q-modules).

The linkage principle for U_q [7] implies that $\mathcal{T}_q$ divides into blocks corresponding to the orbits of W_l. Moreover, for each $\lambda, \mu \in \bar{C}_l$ we have a translation functor T_λ^μ from the λ-block in $\mathcal{T}_q$ to the μ-block.

Assume for the rest of this section that $l \geq h$. This is equivalent to C_l being non-empty. If $\lambda, \mu \in C_l$ then T_λ^μ is an equivalence of categories. This means then that if for $w \in W_l$ the alcove $C = w.\ C_l \subset X^+$ contains one weight in a weight cell $\underline{c}$ then all weights in C belong to $\underline{c}$. A little more elaborate argument (see [4]) shows that also the intersection of X^+ with the lower closure of C is in fact contained in $\underline{c}$. Hence

Proposition 8. Each weight cell is the union of lower closures of alcoves (intersected with X^+).

Note that the intersection of X^+ with the lower closure of C_l equals C_l. The result in [2, Theorem 3.4] says

Proposition 9. C_l is a weight cell.

Clearly C_l is the unique maximal cell in the ordering $\underset{\mathcal{T}_q}{\leq}$.

When $\underline{c}$ is a weight cell in X^+ we denote by $\mathcal{T}_q(\leq \underline{c})$ the subcategory in $\mathcal{T}_q$ whose objects are direct sums of $T_q(\lambda)$ with λ in a cell $\underline{c}'$ which satisfies $\underline{c}' \underset{\mathcal{T}_q}{\leq} \underline{c}$. Clearly, $\mathcal{T}_q(\leq c)$ is a tensor ideal in $\mathcal{T}_q$. The following result allows us to determine completely all such ideals.

We identify W_l with the set of alcoves in X by matching $w \in W_l$ with $w \cdot C_l$. Then the above division of X^+ into weight cells gives a corresponding division of $W_l^+ = \{w \in W_l \mid w \cdot C_l \subset X^+\}$. For this we have

Theorem 10 (Ostrik [16]). *The weight cells in X^+ correspond to the right Kazhdan-Lusztig cells in W_l^+.*

For later use we record the following consequence of this theorem.

Corollary 11. Let $\underline{c}$ be a weight cell in X^+. Then there exist finitely many alcoves $A_1, \cdots, A_r$ in $\underline{c}$ such that any alcove $C \subset \underline{c}$ can be reached from some A_i via a sequence $A_i = C_1 < C_2 < \cdots < C_m = C$ of alcoves C_j in $\underline{c}$ where C_j and C_{j+1} share a common wall, $j = 1, \cdots, m-1$.

Proof. Use Theorem 10 and the arguments in [23], Section 3.

§4. Modular tensor ideals

Let $\mathcal{T}$ denote the category of tilting modules for G. We can then define $\underset{\mathcal{T}}{\leq}$ and $\underset{\mathcal{T}}{\sim}$ in analogy with the corresponding quantum case studied in Section 3. The equivalence classes of $\underset{\mathcal{T}}{\sim}$ are called modular weight cells.

The same arguments as in Section 3 give that (for $p \geq h$) each modular weight cell is a union of lower closures of alcoves intersected with X^+. Also we have in analogy with Proposition 9 (see [6], [11]).

Proposition 12. C_p is a modular weight cell.

On the other hand the modular analogue of Proposition 6 fails: For each $r \in \mathbb{N}$ we have a Steinberg module St_r in $\mathcal{T}$ given by

$$\mathrm{St}_r = L((p^r - 1)\rho) = \Delta((p^r - 1)\rho) = \nabla((p^r - 1)\rho) = T((p^r - 1)\rho).$$

However, St_r is not injective (in fact, there are no injective finite dimensional G-modules at all). Moreover, it is clear (e.g. because St_{r+1} is injective for the $(r+1)$'th Frobenius kernel in G whereas St_r is not) that St_r is not a summand of $\mathrm{St}_{r+1} \otimes_k Q$ for any $Q \in \mathcal{T}$. Hence $(p^r - 1)\rho \underset{\mathcal{T}}{\not\leq} (p^{r+1} - 1)\rho$.

This observation shows (in contrast to the case considered in Section 3) that there are infinitely many modular weight cells (as pointed out by Ostrik in [17]).

Set now $Y_r = (p^r - 1)\rho + X^+$. If $\lambda \in Y_r$ then we can write uniquely $\lambda = \lambda_0^r + p^r \lambda_1^r$ with $\lambda_0^r \in (p^r - 1)\rho + X_{p^r}$ and $\lambda_1^r \in X^+$. At least for $p \geq 2h - 2$ (see [9]) we have

$$T(\lambda) \cong T(\lambda_0^r) \otimes_k T(\lambda_1^r)^{(r)}. \tag{1}$$

Here ${}^{(r)}$ denotes twist by the r'th Frobenius homomorphism on G.

Lemma 13. Suppose $p \geq 2h - 2$ and let $\lambda, \mu \in Y_r$. Then $\lambda \underset{\mathcal{T}}{\leq} \mu$ if and only if $\lambda_1^r \underset{\mathcal{T}}{\leq} \mu_1^r$.

Proof. Arguing as in the proof of Proposition 6 we see that $\lambda_0^r \underset{\mathcal{T}}{\sim} (p^r - 1)\rho$ (the assumption $p \geq 2h-2$ gives that St_r is injective among G-modules whose dominant weights are in $C_{2p^r(h-1)}$). When we combine this with (1) we see that $\lambda \underset{\mathcal{T}}{\sim} (p^r - 1)\rho + p^r \lambda_1^r$. Hence we may assume $\lambda_0^r = (p^r - 1)\rho = \mu_0^r$.

Suppose $T(\lambda_1^r)$ is a summand of $T(\mu_1^r) \otimes_k Q$ for some $Q \in \mathcal{T}$. Then $\mathrm{St}_r \otimes_k T(\lambda_1^r)^{(r)}$ is a summand of $\mathrm{St}_r \otimes_k T(\mu_1^r)^{(r)} \otimes_k Q^{(r)}$. Now St_r is a summand of $\mathrm{St}_r \otimes_k \mathrm{St}_r \otimes_k \mathrm{St}_r$ and hence the latter module is a summand of $\mathrm{St}_r \otimes_k T(\mu_1^r)^{(r)} \otimes_k (\mathrm{St}_r \otimes_k \mathrm{St}_r \otimes Q^{(r)})$. Noting that $\mathrm{St}_r \otimes_k Q^{(r)}$ and therefore also $Q^1 = \mathrm{St}_r \otimes_k \mathrm{St}_r \otimes Q^{(r)}$ are in $\mathcal{T}$ we conclude that $T(\lambda)$ is a summand of $T(\mu) \otimes_k Q^1$, i.e. $\lambda \underset{\mathcal{T}}{\leq} \mu$.

Conversely, suppose $T(\lambda)$ is a summand of $T(\mu) \otimes_k Q$ for some $Q \in \mathcal{T}$. Recalling that $T(\mu) = \mathrm{St}_r \otimes_k T(\mu_1^r)^{(r)}$ (by our assumption on μ_0^r and (1)) we consider first $\mathrm{St}_r \otimes_k Q \in \mathcal{T}$. The summands of this module all have the form $T(\nu)$ with $\nu \in Y_r$. Hence

$$T(\mu) \otimes_k Q \simeq \bigoplus_{\nu \in Y_r} c_\nu T(\nu_0^r) \otimes_k (T(\mu_1^r) \otimes_k T(\nu_1^r))^{(r)}$$

for some $c_\nu \in \mathbb{N}$. We see that since $T(\lambda) = \mathrm{St}_r \otimes_k T(\lambda_0^r)^{(r)}$ occurs in this sum there must exist $\nu \in Y_r$ such that $\nu_0^r = (p^r - 1)\rho$ and $T(\lambda_1^r)$ is a summand of $T(\mu_1^r) \otimes_k T(\nu_1^r)$. In particular $\lambda_1^r \underset{\mathcal{T}}{\leq} \mu_1^r$ as desired.

Combining Proposition 12 with this lemma we get

Proposition 14. Assume $p \geq 2h - 2$. Then the set

$$\underline{c}_1^r = (p^r - 1)\rho + X_{p^r} + p^r C_p$$

is a modular weight cell for each $r \in \mathbb{N}$.

Note that for $r = 0$ we have $\underline{c}_1^0 = C_p$. For $r > 0$ the cell $\underline{c}_1^r$ contains the Steinberg weight $(p^r - 1)\rho$. All $\underline{c}_1^r$ are finite.

Example 15. Consider the case where R has type A_2 and $p > 3$. Then it follows from [13] (see also Theorem 19 below) that the set $\underline{c}_2 = X^+ \backslash (C_p \cup ((p-1)\rho + X^+))$ is a modular weight cell. Using the notation from Proposition 14 we set

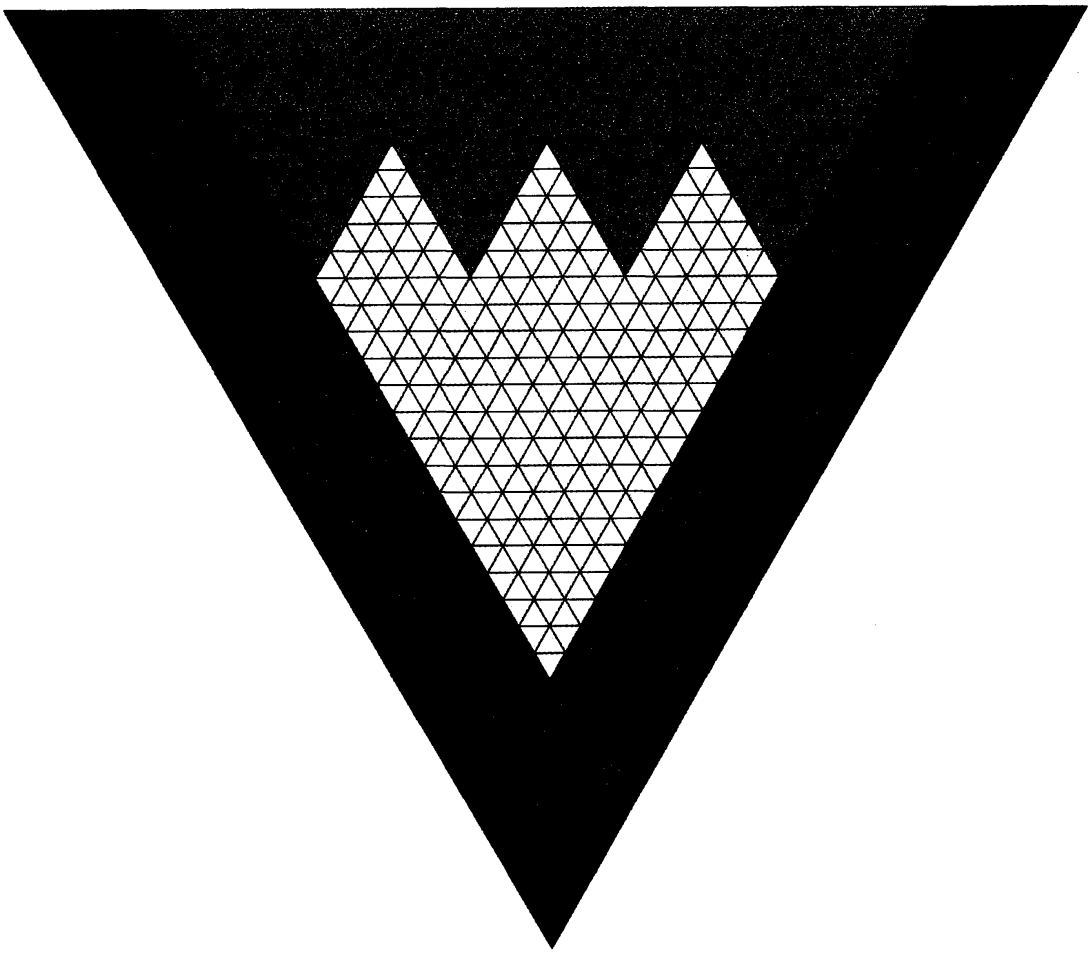
$$\underline{c}_2^r = Y_r \backslash (Y_{r+1} \cup \underline{c}_1^r).$$

Then the modular weight cells in X^+ are $\{\underline{c}_i^r \mid i = 1, 2; r \in \mathbb{N}\}$. We illustrate this for the case $p = 5$ in Figure 1. As a consequence the modular tensor ideals (i.e. the tensor ideals in $\mathcal{T}$) can be described as $\mathcal{T}(\leq \underline{c}_i^r)$ for some $i \in \{1, 2\}, r \in \mathbb{N}$.

In this example, we see that each weight cell breaks up into a union of modular weight cells. We expect this to be true in general. The following result presents some evidence for this.

Proposition 16. Assume Conjecture 3 and let $p \gg 0$. If $a_{\mu\lambda} \neq 0$ for some $\mu, \lambda \in X^+$ then $\mu \underset{\mathcal{T}_q}{\leq} \lambda$.

Proof. We assume (as we may, see Proposition 8) that λ is p-regular. Let C be the alcove which contains λ and let $\underline{c}$ denote the weight cell containing C. Then by Corollary 11 there exists a sequence $A_i = C_1 < C_2 < \cdots < C_m = C$ of alcoves in $\underline{c}$ such that C_j and C_{j+1} have a common wall. Denote by Θ_j the wallcrossing functor associated to this wall. Then $T(\lambda)$ is a summand of $\Theta_{m-1} \cdots \Theta_2 \Theta_1 T(\lambda^1)$ where $\lambda^1 \in A_i$ is in $W_p.\lambda$. Since $p \gg 0$ we have $A_i \subset C_{p^2}$ and hence by Conjecture 3 we get $T(\lambda^1)_q = T_q(\lambda^1)$. If therefore $T_q(\mu)$ is a summand of $T(\lambda)_q$ then it is also a summand of $\Theta_{m-1} \cdots \Theta_1 T_q(\lambda^1)$, i.e. $\mu \underset{\mathcal{T}_q}{\leq} \lambda^1 \underset{\mathcal{T}_q}{\sim} \lambda$.



c_1^0 black, c_2^0 blue, c_1^1 red, c_2^1 green, c_1^2 yellow, c_2^2 grey.
Figure 1: Type $A_2, p = 5$

Corollary 17. Let $\lambda, \mu \in X^+$. With the same assumptions as in Proposition 16 we have that if $\lambda \underset{\mathcal{T}}{\leq} \mu$ then also $\lambda \underset{\mathcal{T}_q}{\leq} \mu$. In particular, each weight cell is a union of modular weight cells.

Proof. Suppose $T(\lambda)$ is a summand of $T(\mu) \otimes_k Q$ for some $Q \in \mathcal{T}$. Then $T(\lambda)_q$ and in particular $T_q(\lambda)$ is a summand of $T(\mu)_q \otimes_{\mathbb{C}} Q_q$. It follows that there exists $\nu \in X^+$ with $a_{\nu\mu} \neq 0$ such that $T_q(\lambda)$ is a summand of $T_q(\nu) \otimes_{\mathbb{C}} Q_q$, i.e. $\lambda \underset{\mathcal{T}_q}{\leq} \nu$. By Proposition 16 we also have $\nu \underset{\mathcal{T}_q}{\leq} \mu$ and the Corollary follows.

As an immediate consequence of this corollary we get

Corollary 18. With assumptions as in Proposition 16 we have that $\mathcal{T}(\underset{\mathcal{T}_q}{\leq} \underline{c})$ is a tensor ideal of $\mathcal{T}$ for any weight cell $\underline{c}$ in X^+.

§5. The second cell

In this section we report on some recent work by T. E. Rasmussen [19].

Assume throughout this section that R is irreducible of rank > 1. We shall denote by $\underline{c}_2$ the weight cell which contains the upper closure of C_p. We refer to $\underline{c}_2$ as the second cell ($\underline{c}_1 = C_p$ being the first cell).

Recall that if $Q \in \mathcal{T}$ then we denote by $Q_q \in \mathcal{T}_q$ the "quantization" of Q.

Theorem 19 ([19]). *Let $Q \in \mathcal{T}$. Then we have for all $\lambda \in \underline{c}_2$*

$$[Q : T(\lambda)] = [Q_q : T_q(\lambda)]$$

This may be thought of as a generalization of [11] and [6] which show that the equality in Theorem 19 holds for $\lambda \in \underline{c}_1$. More explicitly, these results give the following explicit formula

Theorem 20 ([6], [11]). *If $Q \in \mathcal{T}$ (respectively $\mathcal{T}_q$) then we have for all $\lambda \in \underline{c}_1$*

$$[Q : T(\lambda)] = \sum_{\substack{w \in W_p \\ w \cdot \lambda \in X^+}} (-1)^{\ell(w)} [Q : \Delta(w \cdot \lambda)]$$

(respectively

$$[Q : T_q(\lambda)] = \sum_{\substack{w \in W_p \\ w \cdot \lambda \in X^+}} (-1)^{\ell(w)} [Q : \Delta_q(w \cdot \lambda)]).$$

Consider now $r \in \mathbb{N}$ and let λ be a partition of r. Associated to λ we have a simple $k[\Sigma_r]$-module D^λ, respectively, a simple $H_r(q)$-module D_q^λ. Here $H_r(q)$ denotes the Hecke algebra over $\mathbb{C}$ for Σ_r with parameter q. These modules are 0 unless λ is p-regular (i.e. no p lines in λ are equal). Then

Corollary 21 ([19]). Suppose the partition $\lambda = (\lambda_1, \cdots, \lambda_n)$ satisfies either $\lambda_1 - \lambda_{n-1} < p - n + 2$ or $\lambda_2 - \lambda_n < p - n + 2$. Then

$$\dim_k D^\lambda = \dim_{\mathbb{C}} D_q^\lambda.$$

Proof. This is a consequence of Theorem 19 for the case $G = GL_n(k)$ via Schur-Weyl duality. In fact, this gives us (for any partition/dominant weight λ)

$$[V^{\otimes r} : T(\lambda)] = \dim_k D^\lambda.$$

Here V is the natural n-dimensional module for $GL_n(k)$. This is a tilting module and hence so is $V^{\otimes r}$. There is a completely analogous equality in the quantum case. Hence the corollary follows once it is checked that the weight λ belongs to $\underline{c}_1 \cup \underline{c}_2$ if and only if the corresponding partition λ satisfies one of the equalities stated. See [19] for details.

Remark 22. i) For $n = 3$ the corollary was proved by Jensen and Mathieu [13]

ii) If λ satisfies $\lambda_1 - \lambda_n < p - n$ then λ belongs to the first weight cell $\underline{c}_1$ (the bottom alcove). In this case Mathieu has given an explicit algorithm for $\dim_k D^\lambda$ (and for $\dim_{\mathbb{C}} D_q^\lambda$), see [14].

iii) The corollary verifies in part a conjecture by Mathieu [15, Conj. 15.4] (In my lecture at the conference I hinted that there seemed to be some evidence of inconsistency between this conjecture and our Conjecture 3. This turned out to rely on a misunderstanding).

§6. The mixed case

In this section we fix $l \in \mathbb{N}$ with $(p, l) = 1$. As in Section 3 we shall also assume l odd (and prime to 3 if R has a component of type G_2).

We choose a primitive l'th root of 1 in k which we denote ζ. In this section we shall consider tilting modules for $U_\zeta = U_A \otimes_A k$ where the A-algebra structure of k is given by specializing v to $\zeta \in k$. Let m be the kernel of the structure homomorphism $A \to k$ and set $A_{l,p} = A_m$. Then k is also an $A_{l,p}$-algebra and so is $\mathbb{C}$ via $v \mapsto q$ (where $q \in \mathbb{C}$ now again is a primitive l'th root of 1).

The theory in Section 3 may again be carried over to this situation. We denote the simple module, Weyl module, dual Weyl module and indecomposable tilting module for U_ζ with highest weight $\lambda \in X^+$ by $L_\zeta(\lambda), \Delta_\zeta(\lambda), \nabla_\zeta(\lambda)$ and $T_\zeta(\lambda)$, respectively. We have

$$\Delta_\zeta(\lambda) \simeq \Delta_A(\lambda) \otimes_A k \text{ and } \nabla_\zeta(\lambda) \simeq \nabla_A(\lambda) \otimes_A k.$$

Again, $L_\zeta(\lambda)$ is the image of the natural homomorphism $\Delta_\zeta(\lambda) \to \nabla_\zeta(\lambda)$. Moreover, there exists a tilting module $T_{p,l}(\lambda)$ for $U_{A_{p,l}}$ which satisfies $T_{p,l}(\lambda) \otimes_{A_{p,l}} k = T_\zeta(\lambda)$.

In analogy with Section 2 we then define $T_\zeta(\lambda)_q$ by

$$T_\zeta(\lambda)_q = T_{p,l}(\lambda) \otimes_{A_{p,l}} \mathbb{C}.$$

This U_q-module decomposes into a direct sum of $T_q(\mu)$'s and we define the corresponding (fusion) multiplicities $a^\zeta_{\mu\lambda} \in \mathbb{N}$ by

$$T_\zeta(\lambda)_q = \bigoplus_{\mu \in X^+} a^\zeta_{\mu\lambda} T_q(\mu)$$

These numbers again satisfy

$$a^\zeta_{\lambda\lambda} = 1 \text{ and } a^\zeta_{\mu\lambda} = 0 \text{ unless } \mu \leq \lambda.$$

Problem 23. Determine $a^\zeta_{\mu\lambda}$ for all $\mu, \lambda \in X^+$.

Remark 24. Just as in the case $\zeta = 1$ treated in Section 2 (see Problem 1) this problem is wide open. Only the case where R is of type A_1 is known. The problem is equivalent to the problem of finding the characters of all $T_\zeta(\lambda)$'s.

We have also a conjecture analogous to Conjecture 3. It was discussed in [4, Section 4] (see in particular Proposition 4.6 and Remark 4.7) as a consequence of a stronger conjecture.

Conjecture 25. If $\lambda \in C_{lp}$ then $T_\zeta(\lambda)_q = T_q(\lambda)$.

Remark 26. This conjecture is known to hold only for types A_1 and A_2 (see [4], 4.5). If $p \geq 2h - 2$ it implies (as pointed out in loc. cit Remark 4.7 (v)) that $\operatorname{ch} L_\zeta(\lambda) = \operatorname{ch} L_q(\lambda)$ for all $\lambda \in X_l$. This last identity is conjectured in [1, 4c.3 (iv)]. It is known to hold for p large. For arbitrary p it was verified for rank 2 and for type A_3 by Thams [22].

Let now $\mathcal{T}_\zeta$ denote the category of tilting modules for U_ζ. We then have relations $\underset{\mathcal{T}_\zeta}{\leq}$ and $\underset{\mathcal{T}_\zeta}{\sim}$ on X^+ analogous to those studied in Section 3-4. The equivalence classes for $\underset{\mathcal{T}_\zeta}{\sim}$ are called mixed weight cells.

As before we have

(2) A mixed weight cell is the union of lower closures of alcoves (for W_l) intersected with X^+.

(3) C_l is a mixed cell.

Consider now the weight cell $(l-1)\rho + X^+$, see Proposition 6. If $\nu \in (l-1)\rho + X^+$ we write $\nu = \nu_0 + l\nu_1$ with $\nu_0 \in (l-1)\rho + X_l$ and

$\nu_1 \in X^+$. The same argument as in the modular case shows that for $p \geq 2h-2$ the module $T_\zeta(\nu_0)$ restricts to a PIM for the small quantum group $u_\zeta \subset U_\zeta$. Then we can also reuse Donkin's argument from [9] to obtain

$$T_\zeta(\nu) \simeq T_\zeta(\nu_0) \otimes_k T(\nu_1)^{[l]}. \tag{4}$$

Here $^{[l]}$ denotes twist by the quantum Frobenius homomorphism from U_ζ to the hyperalgebra of G_1, see [8].

If ch $L_\zeta(\lambda) =$ ch $L_q(\lambda)$ for all $\lambda \in X_l$ (which we know is true for $p \gg 0$, see Remark 26) then we have $T_\zeta(\nu_0)_q = T_q(\nu_0)$ for all $\nu_0 \in (l-1)\rho + X^+$. Hence (4) shows that the determination of $T_\zeta(\nu)$ for $\nu \in (l-1)\rho + X^+$ is equivalent to the determination of all modular indecomposable tilting modules. On the other hand a solution of the modular problem (Problem 1) is not enough to give the mixed tilting modules with highest weights outside this smallest weight cell.

As in Section 4 we can combine Conjecture 25 and Corollary 11 to see that weight cells divide into mixed weight cells when p is large. Here we only state the analogue of Corollary 18 giving some of the tensor ideals in $\mathcal{T}_\zeta$.

Corollary 27. Assume Conjecture 25 holds and let $p \gg 0$. Then for each weight cell $\underline{c}$ in X^+ the subcategory $\mathcal{T}_\zeta(\underset{\mathcal{T}_q}{\leq} \underline{c})$ is a tensor ideal in $\mathcal{T}_\zeta$.

Finally, assume R is irreducible and simply laced of rank > 1. Consider the second cell $\underline{c}_2$ as in Section 5. The same arguments as used by Rasmussen in [19] give

Proposition 28. Let $Q = \mathcal{T}_\zeta$. Then we have for all $\lambda \in \underline{c}_1 \cup \underline{c}_2$

$$[Q : T_\zeta(\lambda)] = [Q_q : T_q(\lambda)].$$

It follows that $\underline{c}_2$ is also a mixed weight cell.

Consider type A and let $r \in \mathbb{N}$. Set $H_r(\zeta)$ equal to the Hecke algebra over k corresponding to Σ_r and with parameter ζ. Then Schur-Weyl duality gives the following application of Proposition 28, cf. Corollary 21.

Corollary 29. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ is a partition of r with either $\lambda_1 - \lambda_{n-1} < p - n + 2$ or $\lambda_2 - \lambda_n < p - n + 2$ then $\dim_k D_\zeta^\lambda = \dim_{\mathbb{C}} D_q^\lambda$.

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Hecke algebras with a finite number of indecomposable modules

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Abstract.

Recently, there has been progress in determining the representation type of the Hecke algebras of finite Weyl groups. We report on these results.

§1. Introduction

Recall that an Artin algebra A has *finite representation type* if A has finitely many isomorphism classes of indecomposable modules; otherwise, A has *infinite representation type.* In this short article, we report on a criterion for when the Hecke algebra of a finite Weyl group has finite representation type.

Let W be a finite Weyl group, K be an algebraically closed field and let q be a non-zero element of K. The K-algebra $\mathcal{H}_W(q)$ is the Hecke algebra associated with W.

First assume that $q = 1$. Then $\mathcal{H}_W(q)$ is the group algebra KW. Let l be the characteristic of K. It is well-known that if G is a finite group then the group algebra KG has finite representation type if and only if Sylow l–subgroups of G are cyclic; see [13] and [7]. In the case where W is a Weyl group, this implies the following.

Theorem 1. [4, Theorem 2] *Let W be a finite Weyl group. Then KW has finite representation type if and only if l^2 does not divide the order of W.*

Thus, we may assume that $q \neq 1$ in the rest of the paper. A criterion for $\mathcal{H}_W(q)$ to have finite representation type was conjectured by Uno [16]. To explain this, we recall the Poincaré polynomial of W.

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Definition 2. Let W be as above and let x be an indeterminate over K. Then the Poincaré polynomial $P_W(x)$ of W is the polynomial

$$P_W(x) = \sum_{w \in W} x^{l(w)} \in K[x],$$

where $l(w)$ is the length of $w \in W$.

The following is the conjecture of Uno's.

Conjecture 3. *(Conjecture–Theorem) Let $q \neq 1$ and $\mathcal{H}_W(q)$ be as above. Then $\mathcal{H}_W(q)$ has finite representation type if and only if $(x-q)^2$ does not divide $P_W(x)$.*

Uno's conjecture is now a theorem when W does not have a component of exceptional type. If W does have a component of exceptional type then the conjecture is known to be true under a mild assumption on the field K.

Let us explain the strategy used to prove the conjecture. Using the notion of complexity, we can reduce to the case where W is an irreducible Weyl group; see [4, Proposition 8]. We now proceed with a case–by–case analysis. When W is of type A the conjecture was already confirmed by Uno [16]. Uno also proved his conjecture for $\mathcal{H}_W(q)$ whenever W is a finite Coxeter group of rank two. For exceptional types, the conjecture has been proved by Miyachi [15] under the assumption that the characteristic of K is not too small; this uses computational results which had been obtained by Geck, Lux et al.

We now consider the cases where W is of type B or type D. Then, as is explained in [4], the conjecture is a corollary of [6, Theorem 1.4] (Theorem 4 below); see [4] and [6] for the details. Note that we excluded the case $q = -1$ in [6]. However, as we show below, a similar argument works in this case also and the main theorem [6, Theorem 1.4] is true when $q = -1$. In the next section, we explain the proof of this main theorem taking the case $q = -1$ as an example.

§2. Theorem 1.4 of [6] and the case $q = -1$

Recall that we are assuming that $q \neq 1$. Let W_n be the Weyl group of type B_n. Fix a non–negative integer f and let $\mathcal{H}_n = \mathcal{H}_{W_n}(q, -q^f)$ be the K–algebra with generators $T_0, T_1, \ldots, T_{n-1}$ and relations

$$\begin{aligned} &(T_0 - 1)(T_0 - q^f) = 0, \qquad (T_i + 1)(T_i - q) = 0, \qquad \text{for } 1 \leq i \leq n-1,\\ &T_0T_1T_0T_1 = T_1T_0T_1T_0, \qquad T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i \qquad \text{for } 1 \leq i \leq n-2,\\ &\qquad\qquad T_iT_j = T_jT_i, \ \text{ for } 0 \leq i < j-1 \leq n-2. \end{aligned}$$

We are really considering the two parameter Hecke algebra of type B here; by a Morita equivalence argument the general two parameter case for type B reduces to considering the algebras above.

By renormalizing T_0 if necessary (see [6]) we may assume that q is a primitive e^{th} root of unity, where $e \geq 2$, and that $0 \leq f \leq \frac{e}{2}$. The main result of [6] asserts that the following is true.

Theorem 4 ([6, Theorem 1.4]). *Suppose that K is an algebraically closed field, $e \geq 2$ and that $0 \leq f \leq \frac{e}{2}$. Then $\mathcal{H}_n$ is of finite representation type if and only if $n < \min(e, 2f+4)$.*

In fact, in [6] Theorem 4 is proved only for the cases with $e \geq 3$; or, equivalently, when $q \neq \pm 1$. We first discuss the main ideas behind the proof of [6, Theorem 1.4]. We then illustrate how we use them in the argument by giving a proof of Theorem 4 in the case $q = -1$.

To prove that $\mathcal{H}_n$ has finite representation type if $n < \min(e, 2f+4)$ we used the combinatorics of path sequences together with the Jantzen-Schaper sum formula [14] for $\mathcal{H}_n$. Note that the case $q = -1$ (which was not considered in [6]), corresponds to $e = 2$; therefore, if $q = -1$ then $n < \min(e, 2f+4)$ only if $n = 1$. Thus, when $e = 2$ it is automatic that $\mathcal{H}_n$ has finite representation type if $n < \min(e, 2f+4)$.

We now consider the converse. To prove that $\mathcal{H}_n$ has infinite representation type when $n \geq \min(e, 2f+4)$ we rely on two theories. One is the Specht module theory developed by Dipper, James and Murphy [9]. The other is the description of the decomposition numbers of $\mathcal{H}_n$ as the coefficients of the canonical basis elements of a certain level 2 Fock space [1, 5]; we call this Fock space theory.

The Specht module theory provides us with a set of $\mathcal{H}_n$–modules, called Specht modules, indexed by bipartitions. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ be a bipartition of n and let S^λ be the corresponding Specht module. Then each S^λ is equipped with an invariant bilinear form. Let $\mathrm{rad}(S^\lambda)$ be the radical of the bilinear form and set $D^\lambda = S^\lambda/\mathrm{rad}(S^\lambda)$. Then the non–zero D^λ form a complete set of pairwise non–isomorphic $\mathcal{H}_n$–modules. Define P^λ to be the projective cover of $D^\lambda \neq 0$.

Let $\rhd$ be the dominance ordering on the set of bipartitions of n.

Proposition 5. [6, 3.12,3.13]

1. *If $D^\lambda \neq 0$ then S^λ is an indecomposable $\mathcal{H}_n$–module and D^λ is the unique head of S^λ.*
2. *Each projective $\mathcal{H}_n$–module P has a Specht filtration; thus, there exist bipartitions $\nu_1, \ldots, \nu_k$ and a filtration*

$$P = P^k > P^{k-1} > \cdots > P^1 > P^0 = 0$$

such that $P^i/P^{i-1} \cong S^{\nu_i}$, *for* $1 < i \leq k$, *and* $i < j$ *whenever* $\nu_i \rhd \nu_j$.

3. *Suppose that* $P = P^\mu$ *for some bipartition* μ *with* $D^\mu \neq 0$. *Then the Specht filtration of* (2) *can be chosen so that*

$$d_{\lambda\mu} = \#\{\, 1 \leq i \leq k \mid \nu_i = \lambda \,\}.$$

In particular, if λ *is maximal in the dominance ordering such that* $d_{\lambda\mu} \neq 0$ *then* P^μ *has a submodule isomorphic to* S^λ.

The non–zero D^λ were classified by the first author in [2].

Now we turn to the Fock space theory. We begin by recalling the following theorem; see [3, Theorem 12.5] or [1], [5]. For the statement, let $\Lambda_0, \dots, \Lambda_{e-1}$ be the fundamental weights for the Kac–Moody Lie algebra $U(\widehat{sl}_e)$ and, for a dominant weight Λ, let $L(\Lambda)$ be the corresponding integrable highest weight module.

Theorem 6. *For* $i = 0, 1, \dots, e-1$ *there exist exact functors*

$$e_i, f_i : \mathcal{H}_n\text{-}\mathbf{mod} \longrightarrow \mathcal{H}_{n\pm1}\text{-}\mathbf{mod}$$

such that the operators induced by these, and suitably defined operators d *and* h_i, *for* $i = 0, 1, \dots, e-1$, *give* $\mathcal{K}_0 = \bigoplus_{n\geq 0} \mathcal{K}_0(\mathcal{H}_n\text{-}\mathbf{proj}) \otimes_{\mathbb{Z}} \mathbb{Q}$ *the structure of a* $U(\widehat{sl}_e)$*-module. Moreover,* $\mathcal{K}_0 \cong L(\Lambda_0 + \Lambda_f)$ *as a* $U(\widehat{sl}_e)$*-module and if* K *is a field of characteristic zero then the principal indecomposable* $\mathcal{H}_n$*-modules correspond to elements of the Lusztig–Kashiwara canonical basis of* $L(\Lambda_0 + \Lambda_f)$ *under this isomorphism.*

As a consequence of this result, when K is a field of characteristic zero the decomposition numbers of $\mathcal{H}_n$ can be computed using the canonical basis of a certain v–deformed Fock space $\mathcal{F}_v = \mathcal{F}_v(\Lambda_0 + \Lambda_f)$; see [3] for details. In our case, the set of bipartitions form a basis of $\mathcal{F}_v$. Let $U_v(\widehat{sl}_e)$ be the quantum group of $U(\widehat{sl}_e)$; then $\mathcal{F}_v$ is a $U_v(\widehat{sl}_e)$–module. Let $L_v(\Lambda_0 + \Lambda_f)$ be the integrable highest weight module for $U_v(\widehat{sl}_e)$ of highest weight $\Lambda_0 + \Lambda_f$. Then, by definition, the canonical basis of $L(\Lambda_0 + \Lambda_f)$ is the canonical basis of $L_v(\Lambda_0 + \Lambda_f)$ specialized at $v = 1$.

The action of $U(\widehat{sl}_e)$ on the Fock space is the specialization at $v = 1$ of the action of $U_v(\widehat{sl}_e)$ on $\mathcal{F}_v$. In order to describe this let x and y be nodes of a bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)})$. We say that x is *above* y if either (i) $x \in \lambda^{(1)}$ and $y \in \lambda^{(2)}$, or (ii) x and y are both in the same component of λ (i.e. in $\lambda^{(1)}$ or in $\lambda^{(2)}$), and x is above y. (We follow the English convention for describing partitions as Young diagrams.) For each $i \in \mathbb{Z}/e\mathbb{Z}$, write $\lambda \xrightarrow{i} \mu$ if μ can be obtained by adding a single

i–node to λ; see [6]. Then the action of the Chevalley generator f_i of $U_v(\widehat{sl}_e)$ on $\mathcal{F}_v$ is given by

$$f_i\lambda = \sum_{\mu:\lambda \xrightarrow{i} \mu} v^{N_i^b(\mu/\lambda)}\mu,$$

where $N_i^b(\mu/\lambda)$ is the number of addable i–nodes below the node μ/λ minus the number of removable i–nodes below the node μ/λ. (The action of $f_i \in U(\widehat{sl}_e)$ on the Fock space is given by setting $v = 1$.)

The submodule of $\mathcal{F}_v$ generated by the empty bipartition is isomorphic to $L_v(\Lambda_0 + \Lambda_f)$ – the corresponding integrable highest weight module of $U_v(\widehat{sl}_e)$; this module becomes $L(\Lambda_0 + \Lambda_f)$ when we specialize v to 1. Denote the empty bipartition in $\mathcal{F}_v$ by $v_{\Lambda_0+\Lambda_f}$; then $L_v(\Lambda_0 + \Lambda_f) \cong U_v(\widehat{sl}_e)v_{\Lambda_0+\Lambda_f}$.

Corollary 7. [6, Corollary 3.16] *Suppose that $D^\mu \neq 0$ and that, in characteristic zero, $[P^\mu]$ corresponds to an element of the canonical basis which has the form $f_{i_1}^{(m_1)} \dots f_{i_l}^{(m_l)} v_{\Lambda_0+\Lambda_f}$ under the isomorphism of Theorem 6. Then P^μ has the same Specht filtration in positive characteristic as in characteristic zero.*

This corollary, together with the characterization of the canonical basis, implies that if

$$f_{i_1}^{(m_1)} \dots f_{i_l}^{(m_l)} v_{\Lambda_0+\Lambda_f} \in \lambda + \sum_\mu v\mathbb{Z}[v]\mu$$

in the Fock space $\mathcal{F}_v$ then the column of the decomposition matrix of $\mathcal{H}_n$ corresponding to λ does not depend on the characteristic of the base field K. Thus, the corollary gives us a way of applying Theorem 6 to compute decomposition numbers of $\mathcal{H}_n$ when K is a field of positive characteristic.

Using this, and the properties of the Specht modules listed above, we can prove that if $n \geq \min(e, 2f+4)$ then $\mathcal{H}_n$ has infinite representation type. The reader can experience the flavour of the arguments of [6] from the following two lemmas which extend Theorem 4 to the case $q = -1$. Note that we only have to consider the cases $f = 0, 1$ since $0 \leq f \leq \frac{e}{2}$.

Lemma 8. *Assume that $q = -1$, $f = 1$ and $n \geq 2$. Then $\mathcal{H}_n$ has infinite representation type.*

Proof. By [6, Lemma 2.5] we may assume that $n = 2$. The defining relations of $\mathcal{H}_2$ are

$$T_0^2 - 1 = 0, \quad (T_1+1)^2 = 0, \quad (T_0T_1)^2 = (T_1T_0)^2.$$

Let $\lambda_1 = ((0), (1^2))$ and $\lambda_2 = ((1), (1))$. The Fock space has highest weight $\Lambda_0 + \Lambda_1$ and the decomposition matrix is as follows.

	λ_1	λ_2
$((0), (1^2))$	1	0
$((0), (2))$	1	0
$((1), (1))$	1	1
$((1^2), (0))$	0	1
$((2), (0))$	0	1

If M is a finite dimensional $\mathcal{H}_n$–module let $[M]$ denote the corresponding equivalence class in the Grothendieck group of $\mathcal{H}_n$ and let $\operatorname{Rad}(M)$ denote the radical of M. By the decomposition matrix above, we have $[P^{\lambda_1}] = 3[D^{\lambda_1}] + [D^{\lambda_2}]$ and $[P^{\lambda_2}] = [D^{\lambda_1}] + 3[D^{\lambda_2}]$. Observe that S^{λ_2} is indecomposable with head D^{λ_2} and socle D^{λ_1}. Since its dual module is indecomposable with head D^{λ_1} and socle D^{λ_2}, so that D^{λ_2} must appear in $\operatorname{Rad}(P^{\lambda_1})/\operatorname{Rad}^2(P^{\lambda_1})$. On the other hand, $\operatorname{Rad}(P^{\lambda_1})$ has a Specht filtration whose successive quotients are $S^{((0),(2))} = D^{\lambda_1}$ and S^{λ_2}. Thus D^{λ_1} must appear in $\operatorname{Rad}(P^{\lambda_1})/\operatorname{Rad}^2(P^{\lambda_1})$.

Using a similar argument we can prove that D^{λ_1} and D^{λ_2} must appear in $\operatorname{Rad}(P^{\lambda_2})/\operatorname{Rad}^2(P^{\lambda_2})$.

Considering the separation diagram, we conclude that the $\mathcal{H}_2$ has infinite representation type; see [6, Theorem 2.7]. □

Lemma 9. *Assume that $q = -1$, $f = 0$ and $n \geq 2$. Then $\mathcal{H}_n$ has infinite representation type.*

Proof. As before we may assume that $n = 2$. This time the defining relations of $\mathcal{H}_2$ are

$$(T_0 - 1)^2 = 0, \quad (T_1 + 1)^2 = 0, \quad (T_0T_1)^2 = (T_1T_0)^2.$$

Let $\lambda = ((0), (1^2))$. The element of the canonical basis corresponding to λ is given by

$$((0), (1^2)) + v((0), (2)) + v((1^2), (0)) + v^2((2), (0)).$$

The other element of the canonical basis corresponding to $((1), (1))$ is $((1), (1)) = f_0^{(2)}((0), (0))$. Thus, $[P^\lambda] = 4[D^\lambda]$. Looking at the defining relations, we can define a representation of $\mathcal{H}_2$ by

$$T_0 = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} -1 & 0 & c \\ 0 & -1 & d \\ 0 & 0 & -1 \end{pmatrix}.$$

We choose $a, b, c, d \in K$ so that $ad - bc \neq 0$. Then this representation gives an indecomposable module with head D^λ and socle $D^\lambda \oplus D^\lambda$. Therefore, $\mathrm{End}_{\mathcal{H}_2}(P^\lambda) \not\simeq K[x]/(x^m)$ for any $m \geq 0$ (it has two independent generators); so we conclude that the $\mathcal{H}_2$ has infinite representation type by [6, Lemma 2.6]. □

§3. A result of Erdmann and Nakano

In this section, we assume that W has type A_{n-1}. Let e be the multiplicative order of q as before. Recall that an e–core is a partition which does not contain a removable e–hook. Then the blocks of $\mathcal{H}_W(q)$ are labelled by e–cores such that $n - |\kappa|$ is divisible by e. We denote by $\mathcal{B}_\kappa$ the block labelled by an e–core κ.

Artin algebras fall into three categories; finite, tame and wild. Erdmann and Nakano [10] have determined the representation type of the block algebras $\mathcal{B}_\kappa$.

Recall that if κ is an e–core then the e–weight of κ is

$$w(\kappa) := \frac{n - |\kappa|}{e}.$$

Theorem 10. [10, Theorem 1.2] *Maintain the notation above.*

(1) $\mathcal{B}_\kappa$ *is semisimple if and only if* $w(\kappa) = 0$.
(2) $\mathcal{B}_\kappa$ *has finite representation type (and is not semisimple) if and only if* $w(\kappa) = 1$.
(3) $\mathcal{B}_\kappa$ *has tame representation type if and only if* $e = 2$ *and* $w(\kappa) = 2$.
(4) $\mathcal{B}_\kappa$ *has wild representation type if and only if either* $e \geq 3$ *and* $w(\kappa) \geq 2$, *or* $e = 2$ *and* $w(\kappa) \geq 3$.

Generalization of this theorem to other types remains open.

§4. Appendix

The aim of the paper [6] was to determine when the two parameter Hecke algebra $\mathcal{H}_n(q, Q)$ of type B, which is defined by

$$\begin{gathered}(T_0 - 1)(T_0 + Q) = 0, \qquad (T_i + 1)(T_i - q) = 0, \qquad \text{for } 1 \leq i \leq n-1,\\ T_0T_1T_0T_1 = T_1T_0T_1T_0, \qquad T_{i+1}T_iT_{i+1} = T_iT_{i+1}T_i, \qquad \text{for } 1 \leq i \leq n-2,\\ T_iT_j = T_jT_i \text{ for } 0 \leq i < j-1 \leq n-2,\end{gathered}$$

has finite representation type. The Morita equivalence theorem of Dipper and James [8] implies that it is enough to consider the algebras $\mathcal{H}_n = \mathcal{H}_n(q, -q^f)$ of section 2, where $f \in \mathbb{Z}$. Recall that we assumed

$q \neq 1$ in section 2; however, as we now show, it is easy to determine when $\mathcal{H}_n(1,Q)$ has finite representation type.

Assume that $q = 1$. Then, as an algebra, $\mathcal{H}_n(1,Q)$ is isomorphic to the semidirect product of the commutative algebra $\mathcal{L}_n$ and the group algebra of the symmetric group KS_n, where

$$\mathcal{L}_n = \big(K[L]/(L^2 - (Q-1)L - Q)\big)^{\otimes n}$$

and S_n acts on $\mathcal{L}_n$ by conjugation in the natural way.

If $Q = -1$ and $n = 2$ then $\mathcal{L}_2 = \big(K[L]/(L+1)^2\big)^{\otimes 2}$ is the Kronecker algebra and $\mathcal{H}_2(1,Q) = \mathcal{L}_2 \oplus \mathcal{L}_2 T_1 \mathcal{L}_2$. Thus, $\mathcal{H}_n(1,-1)$ has infinite representation type when $n \geq 2$. Hence, we have proved the following.

Proposition 11. *Suppose that K is a field. Then $\mathcal{H}_n(1,-1)$ has finite representation type if and only if $n = 1$.*

If $Q \neq -1$ then the Dipper–James Morita equivalence theorem combined with Uno's proof of Conjecture 3 for type A gives the following.

Proposition 12. *Suppose that K is a field. Then $\mathcal{H}_n(1,Q)$ with $Q \neq -1$ has finite representation type if and only if $n < 2l$ where l is the characteristic of the base field.*

Remark 13. We can prove this statement without appealing to the Dipper–James Morita equivalence theorem. If $l \neq 2$ then

$$K[L]/(L^2 - (Q-1)L - Q) \simeq K \oplus K \simeq KC_2$$

and thus $\mathcal{H}_n(1,Q) \simeq KW_n$ where W_n is the Weyl group of type B_n. Therefore, by Theorem 1, $\mathcal{H}_n(1,Q)$ has finite representation type if and only if $n < 2l$.

Next assume that $l = 2$. Since KS_n is a factor algebra of $\mathcal{H}_n(1,Q)$, Theorem 1 again implies that $\mathcal{H}_n(1,Q)$ has infinite representation type when $n \geq 4$. Let $G_n = C_3 \wr \mathfrak{S}_n$. To prove that $\mathcal{H}_n(1,Q)$ has finite representation type when $n < 4$ it is enough to observe that there is a surjective homomorphism

$$KG_n = (K \oplus K \oplus K)^{\otimes n} KS_n \to (K \oplus K)^{\otimes n} KS_n = \mathcal{H}_n(1,Q).$$

By the remarks before Theorem 1, KG_n has finite representation type if $n < 4$; hence, $\mathcal{H}_n(1,Q)$ has finite representation type when $n < 4$.

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Representation Theory of Algebraic Groups and Quantum Groups
pp. 27–68

Algebraic construction of contragradient quasi-Verma modules in positive characteristic

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Representation Theory of Algebraic Groups and Quantum Groups
pp. 69–90

On tensor categories attached to cells in affine Weyl groups

Roman Bezrukavnikov

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Advanced Studies in Pure Mathematics 40, 2004
Representation Theory of Algebraic Groups and Quantum Groups
pp. 91–100

Appendix: Braiding compatibilities

Dennis Gaitsgory

§1. Introduction

Let us recall the following basic constructions from [3]: [1] for $\mathcal{S} \in \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$, by taking nearby cycles we obtain $Z(\mathcal{S}) \in \mathrm{Perv}_I(\mathrm{Fl})$. Moreover, for $\mathcal{S}$ as above and $\mathcal{T} \in \mathrm{Perv}_I(\mathrm{Fl})$ we have a perverse sheaf $\mathcal{C}(\mathcal{S},\mathcal{T}) \in \mathrm{Perv}_I(\mathrm{Fl})$ with isomorphisms

$$Z(\mathcal{S}) \star \mathcal{T} \to \mathcal{C}(\mathcal{S},\mathcal{T}) \leftarrow \mathcal{T} \star Z(\mathcal{S}).$$

We will denote the resulting isomorphism $Z(\mathcal{S}) \star \mathcal{T} \to \mathcal{T} \star Z(\mathcal{S})$ by $u_{\mathcal{S},\mathcal{T}}$.

In addition, we will denote by $v_{\mathcal{S}_1,\mathcal{S}_2}$ the morphism $Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \to Z(\mathcal{S}_1 \star \mathcal{S}_2)$ for $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$.

There are 3 properties to check:

1) Let $\mathcal{T}_1, \mathcal{T}_2$ be two I-equivariant perverse sheaves on Fl, and $\mathcal{S}$ be a $G(\widehat{\mathcal{O}})$-equivariant perverse sheaf on Gr. We must have a commutative diagram:

$$\begin{tikzcd}
Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2 \arrow[r, "u_{\mathcal{S},\mathcal{T}_1} \star \mathrm{id}_{\mathcal{T}_2}"] \arrow[d, "u_{\mathcal{S},\mathcal{T}_1 \star \mathcal{T}_2}"'] & \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2 \arrow[d, "\mathrm{id}_{\mathcal{T}_1} \star u_{\mathcal{S},\mathcal{T}_2}"'] \\
\mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S}) \arrow[r, "\mathrm{id}"] & \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S}).
\end{tikzcd}$$

2) Let $\mathcal{S}_1, \mathcal{S}_2$ be two $G(\widehat{\mathcal{O}})$-equivariant perverse sheaves on Gr and $\mathcal{T}$–an I-equivariant perverse sheaf on Fl. We must have a commutative diagram:

$$\begin{tikzcd}
Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \star \mathcal{T} \arrow[r, "\mathrm{id}_{Z(\mathcal{S}_1)} \star u_{\mathcal{S}_2,\mathcal{T}}"] \arrow[d, "v_{\mathcal{S}_1,\mathcal{S}_2} \star \mathrm{id}_{\mathcal{T}}"'] & Z(\mathcal{S}_1) \star \mathcal{T} \star Z(\mathcal{S}_2) \arrow[r, "u_{\mathcal{S}_1,\mathcal{T}} \star \mathrm{id}_{Z(\mathcal{S}_2)}"] & \mathcal{T} \star Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \arrow[d, "\mathrm{id}_{\mathcal{T}} \star v_{\mathcal{S}_1,\mathcal{S}_2}"'] \\
Z(\mathcal{S}_1 \star \mathcal{S}_2) \star \mathcal{T} \arrow[r, "u_{\mathcal{S}_1 \star \mathcal{S}_2,\mathcal{T}}"] & \mathcal{T} \star Z(\mathcal{S}_1 \star \mathcal{S}_2) \arrow[r, "\mathrm{id}"] & \mathcal{T} \star Z(\mathcal{S}_1 \star \mathcal{S}_2).
\end{tikzcd}$$

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[1]Our notations follow those of [3].

3) For $\mathcal{S}_1$ and $\mathcal{S}_2$ as above, we must have:

$$\begin{array}{ccc} Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) & \xrightarrow{u_{\mathcal{S}_1, Z(\mathcal{S}_2)}} & Z(\mathcal{S}_2) \star Z(\mathcal{S}_1) \\ {\scriptstyle v_{\mathcal{S}_1,\mathcal{S}_2}} \downarrow & & {\scriptstyle v_{\mathcal{S}_2,\mathcal{S}_1}} \downarrow \\ Z(\mathcal{S}_1 \star \mathcal{S}_2) & \longrightarrow & Z(\mathcal{S}_2 \star \mathcal{S}_1), \end{array}$$

where the bottom arrow comes from the commutativity constraint on the category of spherical perverse sheaves on Gr.

As we will see, 1) and 2) amount to chasing along the diagrams defining u and v, whereas for 3) we will have to consider nearby cycles along a 2-dimensional base.

§2. Verification of Property 1

We will connect the three objects appearing in the commuatative diagram through an intermediate one.

Consider the scheme Fl', classifying the data of

$$(y, \mathcal{F}_G, \mathcal{F}_G \simeq \mathcal{F}^0_G|_{X-\{x,y\}}, \epsilon),$$

and recall that $\mathrm{Fl}'_{X-x} \simeq \mathrm{Gr}_{X-x} \times \mathrm{Fl}$. We cosider the sheaf $\mathcal{A} := \mathcal{S}_{X-x} \boxtimes (\mathcal{T}_1 \star \mathcal{T}_2)$ on it. The intermediate object is $\Psi(\mathcal{A})$. We will show that the three isomorphisms appearing in our commutative diagram come as compositions from isomorphisms

$$\mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2 \simeq \Psi(\mathcal{A}),\ Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2 \simeq \Psi(\mathcal{A}),\ \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S}) \simeq \Psi(\mathcal{A}).$$

We introduce the schemes Fl^i, $i = 1, 2, 3$ classifying, repsectively, the data of:

$$\begin{gathered}(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-x}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon', \epsilon'') \\ (y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-y}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon', \epsilon'') \\ (y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-x}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon', \epsilon'').\end{gathered}$$

Let us denote by π^i, $i = 1, 2, 3$ the natural projection of each of these schemes to Fl' (π^i remembers only $\mathcal{F}_G$ and ϵ). In addition, as in [3], on Fl^i_{X-x}, $i = 1, ..., 3$, there exist the perverse sheaves

$$\mathcal{A}^1 := \mathcal{T}_1 \boxtimes \mathcal{S}_{X-x} \boxtimes \mathcal{T}_2,\ \mathcal{A}^2 := \mathcal{S}_{X-x} \boxtimes \mathcal{T}_1 \boxtimes \mathcal{T}_2,\ \mathcal{A}^3 := \mathcal{T}_1 \boxtimes \mathcal{T}_2 \boxtimes \mathcal{S}_{X-x}$$

all having the property that $(\pi^i_{X-x})_!(\mathcal{A}^i) \simeq \mathcal{A}$ on Fl'_{X-x}. Hence,

$$(\pi^i_x)_!(\Psi(\mathcal{A}^i)) \simeq \Psi(\mathcal{A}).$$

Moreover, by construction,

$$\Psi(\mathcal{A}^1) \simeq \mathcal{T}_1 \widetilde{\boxtimes} Z(\mathcal{S}) \widetilde{\boxtimes} \mathcal{T}_2,\ \Psi(\mathcal{A}^2) \simeq Z(\mathcal{S}) \widetilde{\boxtimes} \mathcal{T}_1 \widetilde{\boxtimes} \mathcal{T}_2,\ \Psi(\mathcal{A}^3) \simeq \mathcal{T}_1 \widetilde{\boxtimes} \mathcal{T}_2 \widetilde{\boxtimes} Z(\mathcal{S}),$$

and hence

$$\begin{aligned}
(\pi^1_x)_!(\Psi(\mathcal{A}^1)) &\simeq \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2, \\
(\pi^2_x)_!(\Psi(\mathcal{A}^2)) &\simeq Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2, \\
(\pi^3_x)_!(\Psi(\mathcal{A}^2)) &\simeq \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S}).
\end{aligned}$$

Therefore, it remains to establish the required relation between these isomorphisms and $\mathrm{id}_{\mathcal{T}_1} \star u_{\mathcal{S},\mathcal{T}_2}$, $u_{\mathcal{S},\mathcal{T}_1} \star \mathrm{id}_{\mathcal{T}_2}$ and $u_{\mathcal{S},\mathcal{T}_1 \star \mathcal{T}_2}$. For that we introduce several auxiliary schemes Fl^{12}, Fl^{13}, $\mathrm{Fl}^{23'}$, $\mathrm{Fl}^{23''}$, which classify, respectively, the data of:

$$\begin{aligned}
&(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-\{x,y\}}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon') \\
&(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-\{x,y\}}, \epsilon, \epsilon') \\
&(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon') \\
&(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon').
\end{aligned}$$

We have the projections $\pi^{ij} : \mathrm{Fl}^{ij} \to \mathrm{Fl}'$ and $\pi^{i,ij} : \mathrm{Fl}^i \to \mathrm{Fl}^{ij}$ with $\pi^{ij} \circ \pi^{i,ij} = \pi^i$. Let $\mathcal{A}^{ij}$ be the following sheaves defined on Fl^{ij}_{X-x} as in [3]:

$$\begin{aligned}
&\mathcal{A}^{12} := \mathcal{S}_{X-x} \boxtimes \mathcal{T}_1 \widetilde{\boxtimes} \mathcal{T}_2, \mathcal{A}^{13} := \mathcal{T}_1 \widetilde{\boxtimes} \mathcal{T}_2 \boxtimes \mathcal{S}_{X-x} \\
&\mathcal{A}^{23'} := \mathcal{S}_{X-x} \boxtimes (\mathcal{T}_1 \star \mathcal{T}_2), \mathcal{A}^{23''} := (\mathcal{T}_1 \star \mathcal{T}_2) \boxtimes \mathcal{S}_{X-x}.
\end{aligned}$$

We have $(\pi^{ij}_{X-x})_!(\mathcal{A}^{ij}) \simeq \mathcal{A}$, and $(\pi^{i,ij}_{X-x})_!(\mathcal{A}^i) \simeq \mathcal{A}^{ij}$.

Note that

$$\begin{aligned}
&\Psi(\mathcal{A}^{13}) \simeq \mathcal{C}(\mathcal{S}, \mathcal{T}_1) \widetilde{\boxtimes} \mathcal{T}_2,\ \Psi(\mathcal{A}^{12}) \simeq \mathcal{T}_1 \widetilde{\boxtimes} \mathcal{C}(\mathcal{S}, \mathcal{T}_2), \\
&\Psi(\mathcal{A}^{23'}) \simeq Z(\mathcal{S}) \widetilde{\boxtimes} (\mathcal{T}_1 \star \mathcal{T}_2),\ \Psi(\mathcal{A}^{23''}) \simeq (\mathcal{T}_1 \star \mathcal{T}_2) \widetilde{\boxtimes} Z(\mathcal{S}).
\end{aligned}$$

Therefore, our isomorphism

$$Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2 \simeq (\pi^2_x)_!(\Psi(\mathcal{A}^2)) \to \Psi(\mathcal{A}) \to (\pi^1_x)_!(\Psi(\mathcal{A}^1)) \simeq \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2$$

can be factored as

$$(\pi_x^2)_!(\Psi(\mathcal{A}^2)) \simeq (\pi_x^{12} \circ \pi_x^{2,12})_!(\Psi(\mathcal{A}^2)) \simeq (\pi_x^{12})_!(\Psi(\mathcal{A}^{12}))$$
$$\simeq (\pi_x^1 \circ \pi_x^{1,12})_!(\Psi(\mathcal{A}^1)) \simeq (\pi_x^1)_!(\Psi(\mathcal{A}^1)),$$

and, therefore, coincides with $u_{\mathcal{S},\mathcal{T}_1} \star \mathrm{id}_{\mathcal{T}_2} : Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2 \to \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2$, because $\mathcal{T}_2$ enters through the twisted external product construction.

Similarly, the isomorphism

$$\mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2 \simeq (\pi_x^1)_!(\Psi(\mathcal{A}^1)) \to \Psi(\mathcal{A}) \to (\pi_x^3)_!(\Psi(\mathcal{A}^3)) \simeq \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S})$$

coincides with $\mathrm{id}_{\mathcal{T}_1} \star u_{\mathcal{S},\mathcal{T}_2} : \mathcal{T}_1 \star Z(\mathcal{S}) \star \mathcal{T}_2 \to \mathcal{T}_1 \star \mathcal{T}_2 \star Z(\mathcal{S})$.

Finally, the isomorphism

$$(\mathcal{T}_1 \star \mathcal{T}_2) \star Z(S) \simeq (\pi_x^{23''})_!(\Psi(\mathcal{A}^{23''})) \simeq \Psi(\mathcal{A})$$
$$\simeq (\pi_x^{23'})_!(\Psi(\mathcal{A}^{23'})) \simeq Z(\mathcal{S}) \star (\mathcal{T}_1 \star \mathcal{T}_2)$$

coincides with $u_{\mathcal{S},\mathcal{T}_1 \star \mathcal{T}_2}$. Therefore, the isomorphism

$$\mathcal{T}_1 \star \mathcal{T}_2 \star Z(S) \simeq (\pi_x^3)_!(\Psi(\mathcal{A}^3)) \to \Psi(\mathcal{A}) \to (\pi_x^2)_!(\Psi(\mathcal{A}^2)) \simeq Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2,$$

which equals

$$(\pi_x^3)_!(\Psi(\mathcal{A}^3)) \simeq (\pi_x^{23''})_!(\Psi(\mathcal{A}^{23''})) \simeq \Psi(\mathcal{A})$$
$$\simeq (\pi_x^{23'})_!(\Psi(\mathcal{A}^{23'})) \simeq (\pi_x^2)_!(\Psi(\mathcal{A}^2))$$

induces $u_{\mathcal{S},\mathcal{T}_1 \star \mathcal{T}_2} : \mathcal{T}_1 \star \mathcal{T}_2 \star Z(S) \to Z(\mathcal{S}) \star \mathcal{T}_1 \star \mathcal{T}_2$.

This establishes the commutativity of the first diagram.

§3. Verification of Property 2

This case is very similar to the previous one. On the scheme Fl'_{X-x} we consider the sheaf $\mathcal{A} := (\mathcal{S}_1 \star \mathcal{S}_2)_{X-x} \boxtimes \mathcal{T}$. We introduce the schemes Fl^i, $i = 1, 2, 3$, $\mathrm{Fl}^{12}, \mathrm{Fl}^{13}, \mathrm{Fl}^{23'}, \mathrm{Fl}^{23''}$, which classify, respectively, the data of

$$(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-y}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon', \epsilon'')$$
$$(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-y}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-x}, \epsilon, \epsilon', \epsilon'')$$
$$(y, \mathcal{F}_G, \mathcal{F}''_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}''_G|_{X-x}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon', \epsilon'')$$
$$(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-\{x,y\}}, \epsilon, \epsilon')$$
$$(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-\{x,y\}}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y}, \epsilon, \epsilon')$$
$$(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-x}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-y}, \epsilon, \epsilon')$$
$$(y, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y}, \mathcal{F}''_G \simeq \mathcal{F}'_G|_{X-x}, \epsilon, \epsilon')$$

We have natural proper maps $\pi^i : \mathrm{Fl}^i \to \mathrm{Fl}'$, $\pi^{ij} : \mathrm{Fl}^{ij} \to \mathrm{Fl}'$, and $\pi^{i,ij} : \mathrm{Fl}^i \to \mathrm{Fl}^{ij}$. The corresponding perverse sheaves are

$$\mathcal{A}^1 := (\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}\mathcal{T}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x}, \quad \mathcal{A}^2 := (\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x} \boxtimes \mathcal{T},$$
$$\mathcal{A}^3 := \mathcal{T}\widetilde{\boxtimes}(\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x},$$
$$\mathcal{A}^{12} := (\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}((\mathcal{S}_2)_{X-x} \boxtimes \mathcal{T}), \quad \mathcal{A}^{13} := ((\mathcal{S}_1)_{X-x} \boxtimes \mathcal{T})\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x},$$
$$\mathcal{A}^{23'} := \mathcal{T}\widetilde{\boxtimes}(\mathcal{S}_1 \star \mathcal{S}_2)_{X-x}, \quad \mathcal{A}^{23''} := (\mathcal{S}_1 \star \mathcal{S}_2)_{X-x}\widetilde{\boxtimes}\mathcal{T},$$

$$(\pi^i_{X-x})_!(\mathcal{A}^i) \simeq \mathcal{A}, \quad (\pi^{ij}_{X-x})_!(\mathcal{A}^{ij}) \simeq \mathcal{A}, \quad (\pi^{i,ij}_{X-x})_!(\mathcal{A}^i) \simeq \mathcal{A}^{ij}.$$

As in [3], we obtain:

$$\Psi(\mathcal{A}^1) \simeq Z(\mathcal{S}_1)\widetilde{\boxtimes}\mathcal{T}\widetilde{\boxtimes}Z(\mathcal{S}_2), \quad \Psi(\mathcal{A}^2) \simeq Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2)\widetilde{\boxtimes}\mathcal{T}$$
$$\Psi(\mathcal{A}^3) \simeq \mathcal{T}\widetilde{\boxtimes}Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2),$$
$$\Psi(\mathcal{A}^{12}) \simeq Z(\mathcal{S}_1)\widetilde{\boxtimes}\mathcal{C}(\mathcal{S}_2, \mathcal{T}), \quad \Psi(\mathcal{A}^{13}) \simeq \mathcal{C}(\mathcal{S}_1, \mathcal{T})\widetilde{\boxtimes}\mathcal{S}_2,$$
$$\Psi(\mathcal{A}^{23'}) \simeq \mathcal{T}\widetilde{\boxtimes}Z(\mathcal{S}_1 \star \mathcal{S}_2), \quad \Psi(\mathcal{A}^{23''}) \simeq Z(\mathcal{S}_1 \star \mathcal{S}_2)\widetilde{\boxtimes}\mathcal{T}.$$

As in the previous section, we obtain that the isomorphism

$$\begin{aligned} Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \star \mathcal{T} &\simeq (\pi^2_x)_!(\Psi(\mathcal{A}^2)) \simeq \Psi(\mathcal{A}) \\ &\simeq (\pi^1_x)_!(\Psi(\mathcal{A}^1)) \simeq Z(\mathcal{S}_1) \star \mathcal{T} \star Z(\mathcal{S}_2) \end{aligned}$$

coincides with $\mathrm{id}_{Z(\mathcal{S}_1)} \star u_{\mathcal{S}_2,\mathcal{T}}$, and

$$\begin{aligned} Z(\mathcal{S}_1) \star \mathcal{T} \star Z(\mathcal{S}_2) &\simeq (\pi^1_x)_!(\Psi(\mathcal{A}^1)) \simeq \Psi(\mathcal{A}) \\ &\simeq (\pi^3_x)_!(\Psi(\mathcal{A}^3)) \simeq \mathcal{T} \star Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \end{aligned}$$

coincides with $u_{\mathcal{S}_1,\mathcal{T}} \star \mathrm{id}_{Z(\mathcal{S}_2)}$.

The isomorphisms

$$\mathcal{T} \star Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \simeq (\pi_x^3)_!(\Psi(\mathcal{A}^3)) \simeq \Psi(\mathcal{A})$$
$$\simeq (\pi_x^{23'})_!(\Psi(\mathcal{A}^{23'})) \simeq \mathcal{T} \star Z(\mathcal{S}_1 \star \mathcal{S}_2)$$

and

$$Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) \star \mathcal{T} \simeq (\pi_x^3)_!(\Psi(\mathcal{A}^3)) \simeq \Psi(\mathcal{A})$$
$$\simeq (\pi_x^{23''})_!(\Psi(\mathcal{A}^{23''})) \simeq Z(\mathcal{S}_1 \star \mathcal{S}_2) \star \mathcal{T}$$

coincide with $\mathrm{id}_{\mathcal{T}} \star v_{\mathcal{S}_1,\mathcal{S}_2}$ and $v_{\mathcal{S}_1,\mathcal{S}_2} \star \mathrm{id}_{\mathcal{T}}$, respectively, and the isomorphism

$$\mathcal{T} \star Z(\mathcal{S}_1 \star \mathcal{S}_2) \simeq (\pi_x^{23'})_!(\Psi(\mathcal{A}^{23'})) \simeq \Psi(\mathcal{A})$$
$$\simeq (\pi_x^{23''})_!(\Psi(\mathcal{A}^{23''})) \simeq Z(\mathcal{S}_1 \star \mathcal{S}_2) \star \mathcal{T}$$

coincides with $u_{\mathcal{S}_1 \star \mathcal{S}_2,\mathcal{T}}$.

This establishes Property 2.

§4. Nearby cycles along a two-dimensional base

To verify the remaining property 3 we need to make a brief digression on the notion of nearby cycles along a 2-dimensional base.

We do not have a good definition. We will, however, introduce some functor Υ, which will suffice for our purposes. But we do not claim neither that it is exact nor that it commutes with the Verdier duality.

Let $\mathcal{Y}$ be a scheme over $X \times X$ and $\mathcal{A}$ be a sheaf on $\mathcal{Y}_{(X-x)\times(X-x)}$. We would like to compare two sheaves on $\mathcal{Y}_x$: one, which we denote by $\Psi_\Delta(\mathcal{A})$ and the other by $\Psi \circ \Psi(\mathcal{A})$:

The functor $\mathcal{A} \mapsto \Psi_\Delta(\mathcal{A})$ is defined as $\Psi_X(\mathcal{A}|_{\mathcal{Y}_{\Delta(X-x)}})[-1]$, where $\mathcal{Y}_{\Delta(X-x)}$ is the preimage of the diagonal $X - x$ in $\mathcal{Y}$.

The functor $\mathcal{A} \mapsto \Psi \circ \Psi(\mathcal{A})$ is iterated nearby cycles: we first take nearby cycles of $\mathcal{A}$ with respect to the projection $\mathcal{Y} \to X \times X \xrightarrow{p_1} X$ and obtain a perverse sheaf on $\mathcal{Y}_{x\times(X-x)}$, and then take the nearby cycles of the latter with respect to $\mathcal{Y}_{x\times X} \to X$.

We claim that $\Psi_{\mathcal{Y}_\Delta}(\mathcal{A})$ and $(\Psi \circ \Psi)_{\mathcal{Y}}(\mathcal{A})$ are connected by a *correspondence*. I.e., there exists a third functor $\mathcal{A} \mapsto \Upsilon(\mathcal{A}) \in \mathrm{D}^b(\mathcal{Y}_x)$ and functorial maps

$$\Psi_\Delta(\mathcal{A}) \leftarrow \Upsilon(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A}).$$

In case when both maps are isomorphisms we will denote by $w_{\mathcal{A}}$ the resulting map

$$\Psi_\Delta(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A}).$$

The construction of $\Upsilon(\mathcal{A})$ is sketched below. Here are its two basic properties:

1. If $\mathcal{Y}$ is $\mathcal{Y}_1 \times \mathcal{Y}_2$ and $\mathcal{A} = \mathcal{A}^1 \boxtimes \mathcal{A}^2$, then both maps $\Upsilon(\mathcal{A}) \to \Psi_\Delta(\mathcal{A})$ and $\Upsilon(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A})$ are isomorphisms and the resulting isomorphism $w_{\mathcal{A}} : \Psi_\Delta(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A})$ coincides with

$$\Psi_\Delta(\mathcal{A}) \simeq \Psi(\mathcal{A}_1 \boxtimes \mathcal{A}_2|_{\mathcal{Y}_1 \underset{X}{\times} \mathcal{Y}_2}) \simeq \Psi(\mathcal{A}_1) \otimes \Psi(\mathcal{A}_2) \simeq \Psi \circ \Psi(\mathcal{A}),$$

where the second arrow is the "Künneth" formula for nearby cycles, cf. [1].

2. For a map $\pi : \mathcal{Y} \to \mathcal{Y}'$ and $\mathcal{A} \in \mathrm{Perv}(\mathcal{Y}_{(X-x)\times(X-x)})$ there is a canonical isomorphism

$$\pi_x(\Upsilon(\mathcal{A})) \simeq \Upsilon((\pi_{(X-x)\times(X-x)})_!(\mathcal{A}))$$

and the following diagram is commutative:

$$\begin{tikzcd}
\Psi_\Delta((\pi_{(X-x)\times(X-x)})_!(\mathcal{A})) \arrow[r, "\sim"] & (\pi_x)_!(\Psi_\Delta(\mathcal{A})) \\
\Upsilon((\pi_{(X-x)\times(X-x)})_!(\mathcal{A})) \arrow[r, "\sim"] \arrow[u] \arrow[d] & (\pi_x)_!(\Upsilon(\mathcal{A})) \arrow[u] \arrow[d] \\
\Psi \circ \Psi((\pi_{(X-x)\times(X-x)})_!(\mathcal{A})) \arrow[r, "\sim"] & (\pi_x)_!(\Psi \circ \Psi(\mathcal{A})).
\end{tikzcd}$$

Here is the construction of $\Upsilon(\mathcal{A})$. (As in [2], we will treat only the unipotent part of nearby cycles.)

Let f denote the structure map $Y \to X \times X$. Let j denote the embedding $\mathcal{Y}_{(X-x)\times(X-x)} \hookrightarrow \mathcal{Y}$, and let i_x denote the embedding of $\mathcal{Y}_x$ into $\mathcal{Y}$.

Following [2], let $\mathcal{E}_n$ be an n-dimensional local system on $X - x$, whose monodromy around x is an n-dimensional nilpotent Jordan block (we can assume that X is affine, hence such a local system exists). The $\mathcal{E}_n$'s form a directed system as $n \in \mathbb{N}$.

Consider the sheaf $\mathcal{F}_{n,m} := j_*(\mathcal{A} \otimes f^*(\mathcal{E}_n \boxtimes \mathcal{E}_m))$. We set $\Upsilon(\mathcal{A})$ to be the direct homotopy limit $\underrightarrow{lim}\ i_x^*(\mathcal{F}_{n,m}[-2])$.

Let us now construct the maps $\Upsilon(\mathcal{A}) \to \Psi_\Delta(\mathcal{A})$ and $\Upsilon(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A})$.

Let $j_{\Delta(X-x)}$ be the embedding of $\mathcal{Y}_{\Delta(X-x)}$. We have a canonical map

$$\mathcal{F}_{n,m}|_{\mathcal{Y}_{\Delta(X)}} \to (j_{\Delta(X-x)})_*(\mathcal{A}|_{\mathcal{Y}_{\Delta(X-x)}} \otimes f^*(\mathcal{E}_n \otimes \mathcal{E}_m)).$$

In addition, we have the maps $\mathcal{E}_n \otimes \mathcal{E}_m \to \mathcal{E}_k$, where $k = \max\{m, n\}$, and by composing we obtain a map

$$\mathcal{F}_{n,m}|_{\mathcal{Y}_{\Delta(X)}}[-1] \to (j_{\Delta(X-x)})_*(\mathcal{A}|_{\mathcal{Y}_{\Delta(X-x)}}[-1] \otimes f^*(\mathcal{E}_k)).$$

By applying the functor i_x^* to both sides and passing to the direct limit, we obtain the required map.

Now, let j_1 be the embedding of $\mathcal{Y}_{x\times(X-x)}$ into $\mathcal{Y}$. Again, we have a natural map

$$\mathcal{F}_{n,m}|_{\mathcal{Y}_{x\times X}} \to (j_1)_*(j_*(\mathcal{A} \otimes (p_1 \circ f)^*(\mathcal{E}_n))|_{\mathcal{Y}_{x\times X}} \otimes (p_2 \circ f)^*(\mathcal{E}_m)).$$

Note that the direct limit of $j_*(\mathcal{A} \otimes (p_1 \circ f)^*(\mathcal{E}_n))|_{\mathcal{Y}_{x\times X}}[-1]$ with respect to n is the 1-step nearby cycles $\Psi(\mathcal{A}) \in \mathrm{Perv}(\mathcal{Y}_{x\times(X-x)})$. Threfeore, by applying i_x^* and taking the direct limit with respect to m as well, we obtain the required map

$$\Upsilon(\mathcal{A}) \to \Psi \circ \Psi(\mathcal{A}).$$

§5. Verification of Property 3

Recall the Beilinson-Drinfeld scheme Gr'' over $X \times X$. By definition, it classifies the data of $(y_1, y_2, \mathcal{F}_G, \mathcal{F}_G \simeq \mathcal{F}_G^0|_{X-\{y_1,y_2\}})$. We have:

$$\mathrm{Gr}''_{X\times X-\Delta} \simeq (\mathrm{Gr}_X \times \mathrm{Gr}_X)_{X\times X-\Delta},$$
$$\mathrm{Gr}''_\Delta \simeq \mathrm{Gr}_X,$$

where $\Delta \subset X \times X$ denotes the diagonal.

Starting with $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$ we consider the corresponding sheaf $(\mathcal{S}_1)_{X-x} \boxtimes (\mathcal{S}_2)_{X-x}$ on $\mathrm{Gr}''_{X\times X-\Delta}$ and denote by $\mathcal{B}$ its Goresky-MacPherson extension to the diagonal. It is known that the extension is acyclic and $\mathcal{B}|_{\mathrm{Gr}''_\Delta}$ identifies with $(\mathcal{S}_1 \star \mathcal{S}_2)_X \simeq (\mathcal{S}_2 \star \mathcal{S}_1)_X$, which is in fact the definition of the commutativity constraint on $\mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$.

Consider now the scheme Fl'' fibered over $X \times X$: A point of Fl'' is a data of

$$(y_1, y_2, \mathcal{F}_G, \mathcal{F}_G \simeq \mathcal{F}_G^0|_{X-\{y_1,y_2\}}, \epsilon),$$

where ϵ is as usual a reduction of $\mathcal{F}_G|_x$ to B. We have:

$$\mathrm{Fl}''_{(X-x)\times(X-x)} \simeq (\mathrm{Gr} \times \mathrm{Gr})_{(X-x)\times(X-x)} \times G/B,$$

and we define the sheaf $\mathcal{A}$ on it equal to $\mathcal{B} \boxtimes \delta_1$.

We introduce two auxiliary schemes Fl^1 and Fl^2 over $X \times X$, which classify the data of $(y_1, y_2, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y_2}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y_1}, \epsilon, \epsilon')$, $(y_1, y_2, \mathcal{F}_G, \mathcal{F}'_G, \mathcal{F}_G \simeq \mathcal{F}'_G|_{X-y_1}, \mathcal{F}'_G \simeq \mathcal{F}^0_G|_{X-y_2}, \epsilon, \epsilon')$, respectively.

We have the natural proper projections $\pi^i : \mathrm{Fl}^i \to \mathrm{Fl}''$ for $i = 1, 2$, which induce isomorphisms

$$\mathrm{Fl}^i_{X\times X-\{x\times X, X\times x, \Delta\}} \simeq \mathrm{Gr}''_{X\times X-\{x\times X, X\times x, \Delta\}} \times G/B \times G/B.$$

Let $\mathcal{A}^i$ be the corresponding perverse sheaves on $\mathrm{Fl}^i_{(X-x)\times(X-x)}$, i.e. we take $\mathcal{B} \boxtimes \delta_1 \boxtimes \delta_1$ on $\mathrm{Fl}^i_{X\times X-\{x\times X, X\times x, \Delta\}}$ and extend it minimally to the preimage of $\Delta(X - x)$.

We obtain a commutative diagram:

$$\begin{array}{ccc}
(\pi^1_x)_!(\Psi_\Delta(\mathcal{A}^1)) & \xrightarrow{w_{\mathcal{A}^1}} & (\pi^1_x)_!(\Psi \circ \Psi(\mathcal{A}^1)) \\
\sim\downarrow & & \sim\downarrow \\
\Psi_\Delta(\mathcal{A}) & \xrightarrow{w_{\mathcal{A}}} & \Psi \circ \Psi(\mathcal{A}) \\
\sim\downarrow & & \sim\downarrow \\
(\pi^2_x)_!(\Psi_\Delta(\mathcal{A}^2)) & \xrightarrow{w_{\mathcal{A}^2}} & (\pi^2_x)_!(\Psi \circ \Psi(\mathcal{A}^2)).
\end{array}$$

As in [3], the terms of this diagram identify, respectively, with

$$\begin{array}{ccc}
(\pi^1_x)_!(\Psi_{\mathrm{Fl}^1_\Delta}((\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x})) & \xrightarrow{w_{\mathcal{A}^1}} & (\pi^1_x)_!(Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2)) \\
\sim\downarrow & & \sim\downarrow \\
\Psi(\mathcal{A}|_\Delta[-1]) & \xrightarrow{w_{\mathcal{A}}} & \mathcal{C}(\mathcal{S}_1, Z(\mathcal{S}_2)) \\
\sim\downarrow & & \sim\downarrow \\
(\pi^2_x)_!(\Psi_{\mathrm{Fl}^2_\Delta}((\mathcal{S}_2)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_1)_{X-x})) & \xrightarrow{w_{\mathcal{A}^2}} & (\pi^2_x)_!(Z(\mathcal{S}_2)\widetilde{\boxtimes}Z(\mathcal{S}_1)).
\end{array}$$

Note first, that we have isomorphisms

$$Z(\mathcal{S}_1 \star \mathcal{S}_2) \simeq \Psi(\mathcal{A}|_\Delta[-1]) \simeq Z(\mathcal{S}_2 \star \mathcal{S}_1),$$

whose composition, by definition, comes from the commutativity constraint on $\mathrm{Perv}_{G(\widehat{\mathcal{O}})}(\mathrm{Gr})$.

The morphism

$$Z(\mathcal{S}_1 \star \mathcal{S}_2) \simeq \Psi(\mathcal{A}|_\Delta[-1]) \simeq$$
$$(\pi^1_x)_!(\Psi_{\mathrm{Fl}^1_\Delta}((\mathcal{S}_1)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_2)_{X-x})) \to (\pi^1_x)_!(Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2)) \simeq Z(\mathcal{S}_1) \star Z(\mathcal{S}_2)$$

coincides with the isomorphism $v_{\mathcal{S}_1,\mathcal{S}_2} : Z(\mathcal{S}_1 \star \mathcal{S}_2) \to Z(\mathcal{S}_1) \star Z(\mathcal{S}_2)$ of [3], by the construction of the latter, and **2** above.

Similarly, the morphism

$$\begin{aligned} Z(\mathcal{S}_2 \star \mathcal{S}_1) &\simeq \Psi(\mathcal{A}|_\Delta[-1]) \\ &\simeq (\pi_x^2)_!(\Psi_{\mathrm{Fl}^2_\Delta}((\mathcal{S}_2)_{X-x}\widetilde{\boxtimes}(\mathcal{S}_1)_{X-x})) \to (\pi_x^2)_!(Z(\mathcal{S}_2)\widetilde{\boxtimes}Z(\mathcal{S}_1)) \\ &\simeq Z(\mathcal{S}_2) \star Z(\mathcal{S}_1) \end{aligned}$$

coincides with $v_{\mathcal{S}_2,\mathcal{S}_1}$.

Finally, the composed isomorphism

$$\begin{aligned} Z(\mathcal{S}_1) \star Z(\mathcal{S}_2) &\simeq (\pi_x^1)_!(Z(\mathcal{S}_1)\widetilde{\boxtimes}Z(\mathcal{S}_2)) \simeq \mathcal{C}(\mathcal{S}_1, Z(\mathcal{S}_2)) \\ &\simeq (\pi_x^2)_!(Z(\mathcal{S}_2)\widetilde{\boxtimes}Z(\mathcal{S}_1)) \simeq Z(\mathcal{S}_2) \star Z(\mathcal{S}_1) \end{aligned}$$

coincides with $u_{\mathcal{S}_1,Z(\mathcal{S}_2)}$, which is what we had to prove.

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On tensor categories attached to cells in affine Weyl groups II

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Abstract.

We prove a weak version of Lusztig's Conjecture on explicit description of the asymptotic affine Hecke algebras in terms of convolution algebras.

§1. Introduction

Let R be a root system. Let W be the corresponding affine Weyl group, and let $\hat{W}$ be an extended affine Weyl group. Let $\mathcal{H}$ (respectively $\hat{\mathcal{H}}$) be the corresponding Hecke algebras. George Lusztig defined an asymptotic version of the Hecke algebra, the ring J, see [10]. By definition the ring J is a direct sum $J = \bigoplus_{\mathbf{c}} J_{\mathbf{c}}$ where summation is over the set of *two-sided cells* in the affine Weyl group. Further, G. Lusztig proved that the set of two-sided cells in W is bijective to the set of unipotent conjugacy classes in an algebraic group over $\mathbb{C}$ with root system R, see [10] IV. Moreover, he proposed a Conjecture describing rings $J_{\mathbf{c}}$ in terms of convolution algebras, see [10] IV, 10.5 (a), (b). This Conjecture was verified in many cases by Nanhua Xi, see [16, 17, 18]. In this note we give a more conceptual proof of all previously known results. Our proof also works in some new cases. In general, we prove a statement (see Theorem 4 below) which is weaker than Lusztig's Conjecture.

The proof relies on many results of G.Lusztig in [10]. Our new essential tool is the theory of *central sheaves* on affine flag manifold due to A. Beilinson, D. Gaitsgory, R. Kottwitz, see [6]. One of us used this theory to prove a part of Lusztig's Conjecture, see [4].

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§2. Recollections

2.1. Notations

Let G be an algebraic reductive connected group over the field of l-adic numbers $\bar{\mathbb{Q}}_l$. Let X denote the weight lattice of G and let $R \subset X$ denote the root system of G. Let W_f denote the Weyl group of G and let $\hat{W}$ be the extended Weyl group, that is the semidirect product of W_f and X. Let $l : \hat{W} \to \mathbb{Z}$ be the length function. Let $W \subset \hat{W}$ be the affine Weyl group, that is subgroup generated by W_f and $R \subset X$. Let $S = \{s \in W | l(s) = 1\}$ be the set of simple reflections. It is well known that pair (W, S) is a Coxeter system.

It is well known that any right W_f-coset in $\hat{W}$ contains unique shortest element. Let $\hat{W}^f \subset \hat{W}$ denote the subset of such representatives, so the set $\hat{W}^f$ is in natural bijection with $\hat{W}/W_f$.

Warning. The notations of this paper are different from the notations of [4], for example the group G is denoted by ${}^L G$ in [4].

2.2. Affine Hecke algebra

Let $\mathcal{A} = \mathbb{C}[v, v^{-1}]$. The affine Hecke algebra $\hat{\mathcal{H}}$ is a free $\mathcal{A}$-module with basis $H_w (w \in \hat{W})$ with an associative $\mathcal{A}$-algebra structure defined by $H_w H_{w'} = H_{ww'}$ if $l(ww') = l(w) + l(w')$ and $(H_s + v^{-1})(H_s - v) = 0$ if $s \in S$. The algebra $\hat{\mathcal{H}}$ is endowed with the *Kazhdan-Lusztig basis* $C_w, w \in \hat{W}$, see e.g. [10] IV 1.1. Let $h_{x,y,z} \in \mathcal{A}$ be the structure constants of $\hat{\mathcal{H}}$ with respect to this basis, that is

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z.$$

We say that (left, right or two-sided) ideal $I \subset \hat{\mathcal{H}}$ is KL-ideal if it admits an $\mathcal{A}$-basis consisting of some elements C_x. For $x, y \in \hat{W}$ we write $x \leq_L y$ (resp. $x \leq_R y, x \leq_{LR} y$) if left (resp. right, two-sided) KL-ideal generated by x is contained in left (resp. right, two-sided) KL-ideal generated by y, cf. [9]. The relations $\leq_L, \leq_R, \leq_{LR}$ are preorders. Let $\sim_L, \sim_R, \sim_{LR}$ be the associated equivalence relations. The corresponding equivalence classes are called left, right and two-sided cells, see *loc. cit.* Each two-sided cell is a union of left (resp. right) cells. The map $w \mapsto w^{-1}$ induces a bijection of the set of left cells to the set of right cells. This map induces identity on the set of two-sided cells.

A deep Theorem due to G. Lusztig (see [10] IV 4.8) states that the set of two-sided cells is bijective to the set of unipotent orbits in G.

2.3. Asymptotic Hecke algebra J

There are well defined functions $a : \hat{W} \to \mathbb{N}, \gamma : \hat{W} \times \hat{W} \times \hat{W} \to \mathbb{N}$ such that

$$v^{a(z)} h_{x,y,z} - \gamma_{x,y,z^{-1}} \in v\mathbb{Z}[v] \text{ for all } x, y, z \in \hat{W}$$

and such that for any $z \in \hat{W}$ there exist $x, y \in \hat{W}$ with $\gamma_{x,y,z} \neq 0$. The function a is constant on two-sided cells, see [10] I 5.4.

Let J be a free $\mathbb{Z}-$module with basis $t_x, x \in \hat{W}$. It has a unique structure of an associative $\mathbb{Z}-$algebra such that $t_x t_y = \sum_{z \in \hat{W}} \gamma_{x,y,z} t_{z^{-1}}$, see [10] II. It has a unit element $\sum_{t \in \mathcal{D}} t_d$ where the summation is over the set $\mathcal{D} \subset W$ of *distinguished involutions*, see *loc. cit.* Each left (resp. right) cell contains exactly one element of $\mathcal{D}$. For any two-sided cell $\mathbf{c}$ let $J_{\mathbf{c}} \subset J$ be the $\mathbb{Z}-$submodule generated by $t_x, x \in \mathbf{c}$. The submodule $J_{\mathbf{c}}$ is in fact a subalgebra; moreover $J_{\mathbf{c}} \cdot J_{\mathbf{c}'} = 0$ for $\mathbf{c} \neq \mathbf{c}'$, see [10] II, hence $J = \bigoplus_{\mathbf{c}} J_{\mathbf{c}}$.

We will use many times the following characterization of cells due to G. Lusztig: $w \sim_L w'$ if and only if $t_w t_{w'^{-1}} \neq 0$, see [11] 3.1 (k).

Algebras $J_{\mathbf{c}}$ are examples of *based algebras*, that is algebras over $\mathbb{Z}$ endowed with a basis over $\mathbb{Z}$ such that the structure constants in this basis are nonnegative integers. Another example of a based algebra can be constructed as follows: let F be a reductive algebraic group acting on the finite set X; then the Grothendieck group $K_F(X \times X)$ of the category of $F-$equivariant coherent sheaves on $X \times X$ is a based algebra with the basis given by classes of irreducible $F-$bundles and multiplication given by convolution, see [10] IV 10.2.

Assume for a moment that group G is simply connected. For any two-sided cell $\mathbf{c}$ let $u_{\mathbf{c}}$ be the unipotent element in G corresponding to $\mathbf{c}$ under Lusztig's bijection [10] IV 4.8. Let $F_{\mathbf{c}}$ be the Levi factor of the centralizer $Z_G(u_{\mathbf{c}})$ of $u_{\mathbf{c}}$ in G. In [10] IV 10.5 G. Lusztig conjectured that there exists a finite set $\mathbf{X}$ endowed with an action of $F_{\mathbf{c}}$ such that the based algebras $J_{\mathbf{c}}$ and $K_{F_{\mathbf{c}}}(\mathbf{X} \times \mathbf{X})$ are isomorphic as based algebras, that is the isomorphism respects bases. The aim of this note is to prove a weak form of this Conjecture; more precisely, we replace finite $F_{\mathbf{c}}-$set by a somewhat more general object — finite $F_{\mathbf{c}}-$set of *centrally extended points*, see below.

2.4.

We will need the following well known

Lemma. *Let Γ_1 and Γ_2 be two left cells lying in the same two-sided cell. Then the intersection $\Gamma_1 \cap (\Gamma_2)^{-1}$ is non empty.*

Proof. Let $w \in \Gamma_1$ and $w' \in \Gamma_2^{-1}$. By [11] 3.1 (l) $w \sim_{LR} w'$ if and only if there exists $y \in \hat{W}$ such that $t_w t_y t_{w'} \neq 0$. We see from the characterization of left cells above that $w \sim_L y^{-1}$ and $y \sim_L {w'}^{-1}$. Thus $y^{-1} \in \Gamma_1 \cap \Gamma_2^{-1}$. □

§3. Affine flags

3.1. Notations

Let ${}^L G$ be a split reductive algebraic group over $\mathbb{Z}$ which is Langlands dual to G. To ${}^L G$ one associates the following "loop objects" defined over $\mathbb{Z}$: the (inifinite type) group schemes $\mathbf{K}_{\mathbb{Z}}$ of maps from a formal disc to ${}^L G$, and the Iwahori group $\mathbf{I}_{\mathbb{Z}}$ of maps whose value at the origin lies in a fixed Borel; and ind-schemes $\mathfrak{Fl}_{\mathbb{Z}}$ (the affine flag variety), and $\mathfrak{Gr}_{\mathbb{Z}}$ (the affine Grassmanian). For a field $\mathfrak{k}$ we have $\mathbf{K}(\mathfrak{k}) = {}^L G(O)$, $\mathbf{I}(\mathfrak{k}) = I$, $\mathfrak{Gr}(\mathfrak{k}) = {}^L G(F)/{}^L G(O)$ and $\mathfrak{Fl}(\mathfrak{k}) = {}^L G(F)/I$ where $F = \mathfrak{k}((t))$, $O = \mathfrak{k}[[t]]$, and $I \subset {}^L G(O)$ is an Iwahori subgroup.

We fix a field $\mathfrak{k}$ which is either $\overline{\mathbb{F}}_p$ or complex numbers; we change scalars from $\mathbb{Z}$ to $\mathfrak{k}$ (and drop the subscript $\mathbb{Z}$). By the (derived) category of sheaves we will mean either the (derived) category of l-adic sheaves, $l \neq char(\mathfrak{k})$, or the (derived) category of constructible sheaves on the complex variety for $\mathfrak{k} = \mathbb{C}$. We will denote $\overline{\mathbb{Q}}_l$ by $\mathbb{C}$ in the first case.

The orbits of $\mathbf{I}$ on $\mathfrak{Fl}$, $\mathfrak{Gr}$ are finite dimensional and isomorphic to affine spaces; it is well known that orbits (called *Schubert cells*) are labelled by elements of $\hat{W}$ for $\mathfrak{Fl}$ and $\hat{W}/W_f$ for $\mathfrak{Gr}$. For $w \in \hat{W}$ (respectively $w \in \hat{W}/W_f$) let $\mathfrak{Fl}_w$, $\mathfrak{Gr}_w$ be the corresponding Schubert cells.

Let D^I be the $\mathbf{I}$−equivariant derived category of sheaves on $\mathfrak{Fl}$, and let $\mathcal{P}^I \subset D^I$ be the full subcategory of perverse sheaves. The convolution product defines a functor $* : D^I \times D^I \to D^I$; moreover, $*$ is equipped with a natural associativity constraint (cf. e.g. [7], §1.1.2-1.1.3, or [3], §7.6.1, p. 260).

Let $j_w : \mathfrak{Fl}_w \to \mathfrak{Fl}$ be the natural inclusion and let

$$L_w = j_{w!*}(\underline{\mathbb{Q}_l}[\dim \mathfrak{Fl}_w]),$$

where $\mathbb{Q}_l$ is the constant sheaf. Simple objects in $\mathcal{P}^I$ are exhausted by $L_w, w \in \hat{W}$.

Remark. Following the standard yoga one can consider the "graded" versions of D^I_{mix}, $\mathcal{P}^I_{mix}$ of D^I, $\mathcal{P}^I$; here D^I_{mix}, $\mathcal{P}^I_{mix}$ are subcategories in the derived category of mixed l-adic sheaves if $\mathfrak{k}$ is of finite characteristic, and they are objects of the (derived) category of mixed Hodge

D-modules if $\mathfrak{k} = \mathbb{C}$. The convolution on D^I_{mix} is defined. It provides D^I_{mix} with the structure of a monoidal category, and thus equips the Grothendieck group $K(D^I_{mix})$ with an algebra structure; this algebra is isomorphic to $\mathcal{H}$.

We will not use this theory below; however, it underlies the relation between the categories considered in this note and affine Hecke algebras. Also, since the set of representations of an affine Hecke algebra injects into the set of representations of the corresponding p-adic group ${}^LG\,(\mathbb{F}_q((t)))$, appearance of the Langlands dual group in the statements below is a manifestation of the geometric Langlands duality.

Notice also that mixed sheaves are used in [4] (in the proof of Theorem 2); the results of this note are based on those of [4].

3.2. Central sheaves

Recall the following definition, see e.g. [4]

Definition. *Let $\mathcal{A}$ be a monoidal category, and $\mathcal{B}$ be a tensor (symmetric monoidal) category. A central functor from $\mathcal{B}$ to $\mathcal{A}$ is a monoidal functor $F : \mathcal{B} \to \mathcal{A}$ together with a functorial isomorphism*

$$\sigma_{X,Y} : F(X) \otimes Y \cong Y \otimes F(X)$$

fixed for all $X \in \mathcal{B}$, $Y \in \mathcal{A}$ subject to the following compatibilities:

(i) *For $X, X' \in \mathcal{B}$ the isomorphism $\sigma_{X,F(X')}$ coincides with the isomorphism induced by commutativity constraint in $\mathcal{B}$.*

(ii) *For $Y_1, Y_2 \in \mathcal{A}$ and $X \in \mathcal{B}$ the composition*

$$F(X) \otimes Y_1 \otimes Y_2 \xrightarrow{\sigma_{X,Y_1} \otimes id} Y_1 \otimes F(X) \otimes Y_2 \xrightarrow{id \otimes \sigma_{X,Y_2}} Y_1 \otimes Y_2 \otimes F(X)$$

coincides with $\sigma_{X,Y_1 \otimes Y_2}$.

(iii) *For $Y \in \mathcal{A}$ and $X_1, X_2 \in \mathcal{B}$ the composition*

$$F(X_1 \otimes X_2) \otimes Y \cong F(X_1) \otimes F(X_2) \otimes Y \xrightarrow{id \otimes \sigma_{X_2,Y}} F(X_1) \otimes Y \otimes F(X_2) \xrightarrow{\sigma_{X_1,Y} \otimes id}$$

$$Y \otimes F(X_1) \otimes F(X_2) \cong Y \otimes F(X_1 \otimes X_2)$$

coincides with $\sigma_{X_1 \otimes X_2, Y}$.

Let $\mathcal{P}_{\mathfrak{Gr}}$ be the category of $\mathbf{K}-$ equivariant perverse sheaves on $\mathfrak{Gr}$. The convolution endows $\mathcal{P}_{\mathfrak{Gr}}$ with monoidal structure and this structure naturally extends to a structure of a commutative rigid tensor category with a fiber functor, and this category is equivalent to $Rep(G)$, see [6, 14]; [3], §5.3, pp 199–215. We will identify $Rep(G)$ with $\mathcal{P}_{\mathfrak{Gr}}$.

In [6] a functor $Z : Rep(G) = \mathcal{P}_{\mathfrak{Gr}} \to \mathcal{P}^I(\mathfrak{Fl})$ was constructed. It enjoys the following properties:

(i) We have a natural isomorphism of functors $\pi_* \circ Z \cong id$, where $\pi : \mathfrak{Fl} \to \mathfrak{Gr}$ is the projection.

(ii) For $\mathcal{F} \in \mathcal{P}_{\mathfrak{Gr}}$, $\mathcal{G} \in \mathcal{P}^I$ we have $\mathcal{G} * Z(\mathcal{F}) \in \mathcal{P}^I$.

(iii) Z is endowed with a natural structure of a central functor from the tensor category $\mathcal{P}_{\mathfrak{Gr}}$ to the monoidal category D^I.

(iv) A unipotent automorphism (monodromy) $\mathfrak{M}$ of the tensor functor Z is given; centrality isomorphism from (iii) commutes with $\mathfrak{M}$.

3.3. Monoidal category $\mathcal{A}_{\mathbf{c}}$

For a subset $S \subset W$ let $\mathcal{P}^I_S$ denote the Serre subcategory of $\mathcal{P}^I$ with simple objects L_w, $w \in S$. Let $\hat{W}_{\leq \mathbf{c}} = \bigcup_{\mathbf{c}' \leq_{LR} \mathbf{c}} \mathbf{c}'$ and $\hat{W}_{<\mathbf{c}} = \bigcup_{\mathbf{c}' <_{LR} \mathbf{c}} \mathbf{c}'$. We abbreviate $\mathcal{P}^I_{\leq \mathbf{c}} = \mathcal{P}^I_{\hat{W}_{\leq \mathbf{c}}}$ and $\mathcal{P}_{<\mathbf{c}} = \mathcal{P}_{\hat{W}_{<\mathbf{c}}}$. Let $\mathcal{P}^I_{\mathbf{c}}$ denote the Serre quotient category $\mathcal{P}^I_{\leq \mathbf{c}}/\mathcal{P}^I_{<\mathbf{c}}$.

For any object $X \in D^I$ and integer i let $H^i(X) \in \mathcal{P}^I$ denote i-th perverse cohomology. For any $X, Y \in \mathcal{P}^I_{\mathbf{c}}$ let us define truncated convolution $X \bullet Y \in \mathcal{P}^I_{\mathbf{c}}$ by $X \bullet Y = H^{a(\mathbf{c})}(X * Y) \mod \mathcal{P}^I_{<\mathbf{c}}$. Let $\mathcal{M}_{\mathbf{c}}$ be the full subcategory of $\mathcal{P}^I_{\mathbf{c}}$ consisting of semisimple objects. It follows from the Decomposition Theorem [2] that the functor $\bullet$ preserves category $\mathcal{M}_{\mathbf{c}}$. The fact the convolution of pure perverse sheaves is pure implies (see [12], 2.6) that the Grothendieck group $K(\mathcal{M}_{\mathbf{c}})$ with the multiplication induced by $\bullet$ is isomorphic to the algebra $J_{\mathbf{c}}$. In [12] a natural associativity constraint was constructed for $\bullet$. Let $\mathbb{I}_{\mathbf{c}} = \bigoplus_{d \in \mathbf{c} \cap \mathcal{D}} L_d \in \mathcal{M}_{\mathbf{c}}$ (recall that $\mathcal{D}$ is the set of distinguished involutions). It is clear that $\mathbb{I}_{\mathbf{c}} \bullet X \simeq X \bullet \mathbb{I}_{\mathbf{c}} \simeq X$ for any $X \in \mathcal{M}_{\mathbf{c}}$. Thus a choice of an isomorphism $\mathbb{I}_{\mathbf{c}} \bullet \mathbb{I}_{\mathbf{c}} \to \mathbb{I}_{\mathbf{c}}$ defines a structure of a monoidal category on $\mathcal{M}_{\mathbf{c}}$, see [12]. We will fix such a choice for the rest of this paper.

Let $\mathcal{A}_{\mathbf{c}}$ be the full subcategory of $\mathcal{P}^I_{\mathbf{c}}$ consisting of all subquotients of $L_w * Z(\mathcal{F}) \mod \mathcal{P}^I_{<\mathbf{c}}$ where $w \in \mathbf{c}$ and $\mathcal{F} \in \mathcal{P}_{\mathfrak{Gr}}$. The following Proposition is proved in [4], Proposition 2.

Proposition. *Restriction of $\bullet$ to $\mathcal{A}_{\mathbf{c}}$ takes values in $\mathcal{A}_{\mathbf{c}}$, is exact in each variable, and it equips $\mathcal{A}_{\mathbf{c}}$ with a structure of a monoidal category with unit object $\mathbb{I}_{\mathbf{c}}$.*

It is clear from the definitions that Lusztig's category $\mathcal{M}_{\mathbf{c}}$ is a monoidal subcategory of $\mathcal{A}_{\mathbf{c}}$ consisting of semisimple objects in $\mathcal{A}_{\mathbf{c}}$.

3.4. Some results from [4]

Let $d \in \mathbf{c}$ be a Duflo involution. Let $\mathcal{A}_d \subset \mathcal{A}_{\mathbf{c}}$ be the full subcategory consisting of all subquotients of $L_d * Z(\mathcal{F})$, $\mathcal{F} \in \mathcal{P}_{\mathfrak{Gr}}$. This category is endowed with a functor $\mathrm{Res}_d : Rep(G) \to \mathcal{A}_d$ defined by $\mathrm{Res}_d(\mathcal{F}) = L_d * Z(\mathcal{F}) \mod \mathcal{P}^I_{<\mathbf{c}}$. The functor Res_d has natural automorphism $\mathfrak{M}_d$

induced by the automorphism $\mathfrak{M}$ of monodromy. The following Theorem is proved in [4] Theorems 1 and 2:

Theorem. (a) *The category $\mathcal{A}_d$ has a natural structure of a tensor category with unit object L_d, functor Res_d has a natural structure of a tensor functor and $\mathfrak{M}_d$ is an automorphism of the tensor functor Res_d.*

(b) *Moreover, there exists a subgroup $H_d \subset G$, a unipotent element $N_d \in G$ commuting with H_d, an equivalence of tensor categories $\Phi_d : Rep(H_d) \to \mathcal{A}_d$, and a natural transformation of functors $Res^G_{H_d} \simeq \Phi_d \circ Res_d$ which interwines the tensor automorphism $\mathfrak{M}_d$ with the action of N_d. The pair (H_d, N_d) is unique up to a simultaneous conjugacy. The element N_d is conjugate to $u_{\mathbf{c}}$.*

It is proved in [13] that the intersection $\mathbf{c} \cap \hat{W}^f$ consists of a unique *canonical* left cell which we will denote $\Gamma_{\mathbf{c}}$ (recall that $\hat{W}^f$ is a set of shortest representatives of right W_f−cosets in $\hat{W}$). In particular, there exists a unique distinguished involution $d = d^f \in \mathbf{c} \cap \hat{W}^f$. We call d^f a *canonical* distinguished involution.

Theorem. (see [4] Theorem 3) (a) *The set of irreducible objects of $\mathcal{A}_{d^f}$ is $\{L_w | w \in \Gamma_{\mathbf{c}} \cap (\Gamma_{\mathbf{c}})^{-1}\}$.*

(b) *The subgroup H_{d^f} contains a maximal reductive subgroup of the centralizer $Z_G(u_{\mathbf{c}})$.*

3.5. Central action of $Rep(F_{\mathbf{c}})$

Consider the functor $\tilde{F} : Rep(G) = \mathcal{P}_{\mathfrak{G}r} \to \mathcal{A}_{\mathbf{c}}$ defined by $\tilde{F}(\mathcal{F}) = Z(\mathcal{F}) * \mathbb{I}_{\mathbf{c}} \mod \mathcal{P}^I_{<\mathbf{c}}$. It is easy to see from 3.2 that the functor $\tilde{F}$ has a natural structure of a central functor. Moreover, this functor has a canonical tensor unipotent automorphism $\mathfrak{M}$ (monodromy) commuting with the centrality isomorphism.

Theorem 1. *There exists a central functor $F : Rep(Z_G(u_{\mathbf{c}})) \to \mathcal{A}_{\mathbf{c}}$ such that $\tilde{F} = F \circ Res^G_{Z_G(u_{\mathbf{c}})}$. Moreover, automorphism $\mathfrak{M}$ is induced by the action of $u_{\mathbf{c}}$ on $Res^G_{Z_G(u_{\mathbf{c}})}$.*

Proof. Let $\mathcal{D}(\mathcal{A}_{\mathbf{c}})$ denote the Drinfeld double of the monoidal category $\mathcal{A}_{\mathbf{c}}$, see e. g. [8]. By the universal property of double the functor $\tilde{F}$ can be factorized as

$$\begin{tikzcd} Rep(G) \arrow[rr, "\tilde{F}"] \arrow[dr, "\tilde{F}_0"'] & & \mathcal{A}_{\mathbf{c}} \\ & \mathcal{D}(\mathcal{A}_{\mathbf{c}}) \arrow[ur] & \end{tikzcd}$$

where $\tilde{F}_0$ is a braided functor. Recall that the unit object of $\mathcal{D}(\mathcal{A}_{\mathbf{c}})$ is $\mathbb{I}_{\mathbf{c}}$ endowed with the centrality isomorphism induced by the unity isomorphisms:

$$\mathbb{I}_{\mathbf{c}} \bullet X \cong X \cong X \bullet \mathbb{I}_{\mathbf{c}}.$$

We remark that $\mathbb{I}_{\mathbf{c}}$ considered as an object of $\mathcal{D}(\mathcal{A}_{\mathbf{c}})$ is irreducible: it is easy to see from Lemma 2.4 that any subobject of $\mathbb{I}_{\mathbf{c}}$ in $\mathcal{A}_{\mathbf{c}}$ does not lie in the center of $\mathcal{A}_{\mathbf{c}}$ even on the level of $K-$theory. Now consider the full subcategory $\tilde{\mathcal{D}} \subset \mathcal{D}(\mathcal{A}_{\mathbf{c}})$ consisting of all subquotients of objects $\tilde{F}_0(X)$, $X \in Rep(G)$. Then the conditions of Proposition 1 [4] are satisfied for the pair $(\tilde{\mathcal{D}}, \tilde{F}_0)$. Consequently, the functor $\tilde{F}_0$ factors through the restriction functor Res^G_H

$$\begin{tikzcd} Rep(G) \arrow[rr, "\tilde{F}_0"] \arrow[dr, "\mathrm{Res}^G_H"'] & & \tilde{\mathcal{D}} \\ & Rep(H) \arrow[ur, "F_0"'] & \end{tikzcd}$$

for some subgroup $H \subset G$ and the action of $\mathfrak{M}$ is given by some unipotent element $u \in Z_G(H)$. Theorem 2 of [4] identifies u with $u_{\mathbf{c}}$. Hence the subgroup H is contained in $Z_G(u_{\mathbf{c}})$. Without loss of generality we can assume that $H = Z_G(u_{\mathbf{c}})$. We set the functor F to be equal to the composition

$$Rep(Z_G(u_{\mathbf{c}})) \xrightarrow{F_0} \tilde{\mathcal{D}} \to \mathcal{D}(\mathcal{A}_{\mathbf{c}}) \to \mathcal{A}_{\mathbf{c}}.$$

The Theorem is proved. $\square$

Let us restrict F to the semisimple part of the category $Rep(Z_G(u_{\mathbf{c}}))$, that is to the category $Rep(F_{\mathbf{c}})$ where $F_{\mathbf{c}}$ is the maximal reductive factor of $Z_G(u_{\mathbf{c}})$.

Proposition. *For any $X \in Rep(F_{\mathbf{c}})$ the object $F(X) \in \mathcal{A}_{\mathbf{c}}$ is semisimple.*

Proof. We can assume that X is simple. Let $Y \in Rep(G)$ be an object such that X is a subquotient of $\mathrm{Res}^G_{F_{\mathbf{c}}}(Y)$. The object $\tilde{F}(Y)$ carries the monodromy filtration; by Gabber's Theorem (see [1], Theorem 5.1.2) it coincides with the weight filtration, so the associated graded object $gr\tilde{F}(Y)$ is semisimple by [2], 5.4.6. By the Theorem 1 we get the same filtration from the action of $u_{\mathbf{c}}$ on $F(\mathrm{Res}^G_{Z_G(u_{\mathbf{c}})})$. But the object X is a direct summand of $gr\ \mathrm{Res}^G_{Z_G(u_{\mathbf{c}})}(Y)$ with respect to this filtration. $\square$

As a corollary we get

Theorem 2. *The functor F restricts to a central functor $F : Rep(F_{\mathbf{c}}) \to \mathcal{M}_{\mathbf{c}}$.*

§4. Canonical cell

4.1. Module categories

In this subsection we review basic theory of module categories. A more detailed exposition will appear in [15]. We will work over a fixed field k.

Let $\mathcal{C}$ be an abelian monoidal category with biexact tensor product and with unit object $\mathbf{1}$.

Definition. *A module category $\mathcal{M}$ over $\mathcal{C}$ is an abelian category $\mathcal{M}$ endowed with*

1) *An exact bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$,*

2) *Functorial* associativity isomorphisms $V \otimes (V' \otimes M) \simeq (V \otimes V') \otimes M$ *for any $V, V' \in \mathcal{C}, M \in \mathcal{M}$,*

3) *Functorial* unit isomorphisms $\mathbf{1} \otimes M \to M$ *for any $M \in \mathcal{M}$ subject to the usual pentagon and triangle axioms: the following diagrams where all arrows are associativity and unit isomorphisms commute:*

$$\begin{tikzcd}
 & ((V_1 \otimes V_2) \otimes V_3) \otimes M \arrow[dl] \arrow[dr] & \\
(V_1 \otimes (V_2 \otimes V_3)) \otimes M \arrow[d] & & (V_1 \otimes V_2) \otimes (V_3 \otimes M) \arrow[d] \\
V_1 \otimes ((V_2 \otimes V_3) \otimes M) \arrow[rr] & & V_1 \otimes (V_2 \otimes (V_3 \otimes M))
\end{tikzcd}$$

$$\begin{tikzcd}
(V \otimes \mathbf{1}) \otimes M \arrow[rr] \arrow[dr] & & V \otimes (\mathbf{1} \otimes M) \arrow[dl] \\
 & V \otimes M &
\end{tikzcd}$$

The notions of module functors, and, in particular, equivalences of module categories are defined in the obvious way.

Remark. Module categories over general monoidal categories were considered by L. Crane and I. Frenkel, see [5]. The name comes from considering the notion of a monoidal category as categorification of the notion of a ring. Module categories seem to be of importance in Conformal Field Theory where they are implicitly considered in the context of Boundary Conformal Field Theory.

Of course the category $\mathcal{C}$ is a module category over itself with associativity and unit isomorphisms induced by ones in tensor category $\mathcal{C}$. Another example can be obtained as follows. Let $A \in \mathcal{C}$ be an associative algebra with unit, that is associative multiplication $A \otimes A \to A$ is defined and there is an inclusion $\mathbf{1} \to A$ satisfying unit axioms. Then category $Mod_{\mathcal{C}}(A)$ of *right* A–modules in the category $\mathcal{C}$ has an obvious structure of a module category.

We will say that a module category $\mathcal{M}$ is generated by objects $M_1, M_2, \ldots \in \mathcal{M}$ over $\mathcal{C}$ if any object of $\mathcal{M}$ is a subquotient of $V \otimes M_i$ for some $V \in \mathcal{C}$. We will say that $\mathcal{M}$ is finitely generated over $\mathcal{C}$ if there exists finitely many (equivalently one) objects $M_1, \ldots \in \mathcal{M}$ such that $\mathcal{M}$ is generated by them over $\mathcal{C}$.

Assume from now on that the category $\mathcal{C}$ is rigid. Then there exists a canonical isomorphism $Hom(V \otimes M, N) \cong Hom(M, V^* \otimes N)$ for any $V \in \mathcal{C}, M, N \in \mathcal{M}$.

Now assume that both categories $\mathcal{C}$ and $\mathcal{M}$ are semisimple. For any two objects $M, N \in \mathcal{M}$ the functor $\mathcal{C} \to Vec_k$, $V \mapsto Hom(V \otimes M, N)$ is representable by an ind-object $\underline{\mathrm{Hom}}(M, N)$ of $\mathcal{C}$. By Yoneda's Lemma $\underline{\mathrm{Hom}}$ is a bifunctor $\mathcal{M}^{op} \times \mathcal{M} \to$ ind-objects of $\mathcal{C}$.

Lemma. *Assume that $\omega : \mathcal{M} \to \mathcal{C}$ is an exact faithful tensor functor. Then for any $M, N \in \mathcal{M}$ $\underline{\mathrm{Hom}}(M, N) \in \mathcal{C}$.*

Proof. It is clear that the map $\underline{\mathrm{Hom}}(M, N) \to \underline{\mathrm{Hom}}(\omega(M), \omega(N))$ is an imbedding. $\square$

Assume that for any $M, N \in \mathcal{M}$ the ind-object $\underline{\mathrm{Hom}}(M, N)$ is an object of $\mathcal{C}$. For any three objects $M, N, K \in \mathcal{M}$ a functorial and associative multiplication $\underline{\mathrm{Hom}}(N, K) \otimes \underline{\mathrm{Hom}}(M, N) \to \underline{\mathrm{Hom}}(M, K)$ is defined (note that the order of factors is opposite to the intuitive one). In particular, for any object $M \in \mathcal{C}$ the object $\underline{\mathrm{Hom}}(M, M)$ has a natural structure of an associative algebra in $\mathcal{C}$. Assume that $\underline{\mathrm{Hom}}(M, X) \neq 0$ for any $X \in \mathcal{M}$, that is the category $\mathcal{M}$ is generated by M over $\mathcal{C}$. It is easy to see that the functor $F_M : \mathcal{M} \to Mod_{\mathcal{C}}(A)$, $F_M(X) = \underline{\mathrm{Hom}}(M, X)$ is a tensor functor. Moreover, we claim that this functor is an equivalence of categories. The proof is straightforward: first one shows that the functor F_M induces an isomorphism on Hom's for objects of the form $V \otimes M$, $V \in \mathcal{C}$, and then one uses the fact that any object of $\mathcal{M}$ admits a resolution by objects of the form $V \otimes M$. Summarizing we get the following

Proposition. *Let $\mathcal{C}$ be a semisimple rigid monoidal category and let $\mathcal{M}$ be a semisimple module category over $\mathcal{C}$. Assume that there exists an exact faithful module functor $\omega : \mathcal{M} \to \mathcal{C}$. Then the category*

$\mathcal{M}$ is equivalent to the category $Mod_{\mathcal{C}}(A)$ for some associative algebra A. Moreover one can choose $A = \underline{\mathrm{Hom}}(M, M)$ for any object $M \in \mathcal{C}$ generating $\mathcal{M}$ over $\mathcal{C}$.

Let $\mathcal{M} = Mod_{\mathcal{C}}(A)$ be a module category. Consider the category $Fun(\mathcal{M}, \mathcal{M})$ consisting of module functors $\mathcal{M} \to \mathcal{M}$. It is clear that the category $Fun(\mathcal{M}, \mathcal{M})$ is a monoidal category with tensor product induced by the composition of functors and identity functor as unit object. One shows easily that the monoidal category $Fun(\mathcal{M}, \mathcal{M})$ is equivalent to the category of $A - A$ bimodules in $\mathcal{C}$ with the obvious monoidal structure.

4.2. Module categories over *Rep*(H)

In this subsection we specialize ourselves to the case when $\mathcal{C} = Rep(H)$ for some reductive group H over an algebraically closed field k of characteristic zero.

Examples. (i) $Rep(H)$ with the associativity and unit isomorphisms induced from those in the monoidal category $Rep(H)$ is of course a module category over $Rep(H)$.

(ii) More generally, let X be a variety endowed with an $H-$action. The category $Coh_H(X)$ of coherent $H-$equivariant sheaves on X is a module category. We get example (i) by letting X =point.

(iii) Let $1 \to \mathbb{G}_m \to \tilde{H} \to H \to 1$ be a central extension of H whose kernel is identified with the multiplicative group $\mathbb{G}_m$ (we will call such a data just "a central extension"); of course, such an extension is necessarily the pushforward if a central extension $1 \to C \to \tilde{H}' \to H \to 1$ under a homomorphism $C \to \mathbb{G}_m$ for a finite cyclic group C. Then the category $Rep^1(\tilde{H})$ of representations V of $\tilde{H}$ such that $\mathbb{G}_m$ acts on V via identity character is a module category over $Rep(H)$. We will also consider the category $Rep^{-1}(\tilde{H})$ of representations of $\tilde{H}$ on which $\mathbb{G}_m$ acts via character $x \mapsto x^{-1}$.

We will say that a module category $\mathcal{C}$ has a *quasifiber functor* if there exists a faithful exact module functor $\omega : \mathcal{C} \to Rep(H)$. The quasifiber functor if it exists is not unique: for any $V \in Rep(H)$ and quasifiber functor ω the functor $M \mapsto \omega(M) \otimes V$ is again a quasifiber functor.

Example. (iv) Let $H' \subset H$ be a subgroup of finite index. Let $1 \to \mathbb{G}_m \to \tilde{H}' \to H' \to 1$ be a central extension. The category $Rep^1(\tilde{H}')$ is a module category over $Rep(H)$ with the $Rep(H)-$action which factors through the restriction functor $Rep(H) \to Rep(H')$. Let $V_0 \in Rep^{-1}(\tilde{H}')$ be a fixed object. It is easy to see that the functor $V \mapsto Ind_{H'}^{H}(V \otimes V_0), V \in Rep^1(\tilde{H}')$ is a quasifiber functor ($\mathbb{G}_m$ acts trivially on $V \otimes V_0$ so $V \otimes V_0$ can be considered as a representation of H').

Example (iv) reduces to the example (i) with $X = H/H'$ if the central extension splits.

Example. (iv′) Finite sums of categories considered in Example (iv) admit the following invariant description.

A finite *H-set of centrally extended points* is the following collection of data:

(a) A finite set X with an H action;

(b) For any $x \in X$ a central extension $\mathbb{G}_m \to \tilde{H}(x) \to H(x)$ of the stabilizer $H(x) = Stab_H(x)$. These should be *equivariant* under the action of H, i.e. for every $g \in G$ an isomorphism of $i_x^g : \tilde{H}(x) \xrightarrow{\sim} \tilde{H}(gx)$ identical on $\mathbb{G}_m$ and covering the map $C_g : H(x) \to H(g(x))$ (conjugation by x) should be given. i_g^x should coincide with the conjugation by g when $g \in H(x)$ and should satisfy $i_{g_1 g_2}^x = i_{g_1}^{g_2(x)} \circ i_{g_2}^x$.

Let $\mathbf{X}$ be a finite set of centrally extended points. An *equivariant sheaf* on $\mathbf{X}$ is a sheaf $\mathcal{F}$ of finite dimensional $\mathbb{C}$-vector spaces on the underlying set X together with

(a) a projective H-equivariant structure on $\mathcal{F}$.

(b) For every $x \in X$ an action of $\tilde{H}(x)$ on the stalk $\mathcal{F}_x$, comprising an object of $Rep^1(\tilde{H}(x))$.

The data (a) and (b) should be compatible, i.e. (b) should be H-equivariant, and the projective action of $H(x)$ arising from (b) must coincide with the one arising from (a).

Equivariant sheaves on $\mathbf{X}$ obviously form a category, which we denote by $Coh_H(\mathbf{X})$.

Choosing a set of representatives x_i for H-orbits on X we see that the data of a centrally extended set with underlying equivariant set X is equivalent to a collection $\tilde{H}_i$ of central extensions $\mathbb{G}_m \to \tilde{H}_i \to H_i = H(x_i)$. The category $Coh_H(\mathbf{X})$ is then canonically equivalent to the direct sum $\oplus_i Rep^1(\tilde{H}_i)$.

Theorem 3. *Let $\mathcal{M}$ be a semisimple module category over $Rep(H)$ finitely generated over $Rep(H)$. Assume that $\mathcal{M}$ admits a quasifiber functor. Then $\mathcal{M}$ is equivalent to $Coh_H(\mathbf{X})$ for some centrally extended finite H-set $\mathbf{X}$ (i.e. to a finite direct sum of some categories of the type described in Example* (iv) *above).*

Proof. By Proposition 4.1 the module category $\mathcal{M}$ is equivalent to the module category $Mod_{Rep(H)}(A)$ for some finite dimensional H-algebra A.

Lemma. *Semisimplicity of $\mathcal{M}$ implies semisimplicity of A as an algebra in the category of vector spaces.*

Proof. Consider the regular representation A_{reg} of A as an object of $Mod_{Rep(H)}(A)$. Let $r(A)$ be the Jacobson radical of A. It is clear that $r(A)$ is H–invariant, hence $r(A)$ is subobject of A_{reg} in $Mod_{Rep(H)}(A)$. Suppose $A_{reg} = r(A) \oplus A_1$ for some $A_1 \in Mod_{Rep(H)}(A)$. Applying the forgetful functor $Mod_{Rep(H)}(A) \to Mod(A)$ to A_{reg} we would get a complement to $r(A)$, which is impossible unless $r(A) = 0$. $\square$

Now let $A \ni 1 = \sum e_i$ is the decomposition of 1 in the sum of minimal central orthogonal idempotents. The group H acts on the set $\{e_i\}$. We may assume that this action is transitive. Let $H_1 \subset H$ be the stabilizer of e_1, the subgroup of finite index in H. The algebra e_1Ae_1 is isomorphic to the matrix algebra and the group H_1 acts on e_1Ae_1. We can choose a projective representation V of H_1 and an isomorphism $e_1Ae_1 \cong End(V)$. It is clear that $A \cong Ind_{H_1}^{H}(e_1Ae_1) = Ind_{H_1}^{H} End(V)$.

The projective action of H_1 on V comes from an action of a central extension $\tilde{H}_1$ of H_1. Let us consider the corresponding category from example (iv) $Rep^1(\tilde{H}_1)$. The representation V can be viewed as an object of this category and one easily calculates $\underline{\mathrm{Hom}}(V, V) = Ind_{H_1}^{H} End(V)$. The Theorem is proved. $\square$

4.3. Module category corresponding to the canonical cell

For any subset $S \subset \mathbf{c}$ let $\mathcal{M}_S \subset \mathcal{M}_{\mathbf{c}}$ denote the full Serre subcategory with simple objects L_w, $w \in S$. Let $\Gamma \subset \mathbf{c}$ be the canonical right cell, see [13]. Let $\mathcal{M}_\Gamma \subset \mathcal{M}_{\mathbf{c}}$ be the corresponding subcategory. By the definition of a right cell we have $\mathcal{M}_\Gamma \bullet \mathcal{M}_{\mathbf{c}} \subset \mathcal{M}_\Gamma$. Define on $\mathcal{M}_\Gamma$ a structure of a module category over $Rep(F_{\mathbf{c}})$ by the formula $V \otimes M = F(V) \bullet M$ where F is a functor from Theorem 2. Note that due to the centrality of functor F we have $F(Rep(F_{\mathbf{c}})) \bullet \mathcal{M}_\Gamma = \mathcal{M}_\Gamma \bullet F(Rep(F_{\mathbf{c}})) \subset \mathcal{M}_\Gamma$ so this is well defined. We claim that this category admits a quasifiber functor. Indeed, let $\{w_1, w_2, \dots\} \subset \Gamma^{-1}$ be a set of representatives of all *right* cells contained in $\mathbf{c}$ (such a set exists by the Lemma 2.4 and is finite since Lusztig proved (see [10] II 2.2) that the number of cells in an affine Weyl group is finite). Consider the functor $\mathcal{M}_\Gamma \to \mathcal{M}_{\Gamma\cap\Gamma^{-1}}$, $X \mapsto X \bullet (\oplus L_{w_i})$. Recall that in [4] the monoidal category $\mathcal{M}_{\Gamma\cap\Gamma^{-1}}$ was identified with $Rep(F_{\mathbf{c}})$, see 3.4. It is a simple exercise to check that this functor is module functor with the module structure induced by the associativity isomorphism in $\mathcal{M}_{\mathbf{c}}$, and it is clear that it is exact and faithful. So this is quasifiber functor, and we can apply Theorem 3. We get

Proposition. *The category $\mathcal{M}_\Gamma$ as a module category over $Rep(F_{\mathbf{c}})$ is equivalent to the category $Coh_{F_{\mathbf{c}}}(\mathbf{X})$ of coherent sheaves on a finite $F_{\mathbf{c}}$−set $\mathbf{X}$ of possibly centrally extended points.*

Note that the inclusion $\mathcal{M}_{\Gamma\cap\Gamma^{-1}} \subset \mathcal{M}_\Gamma$ gives us a distinguished point $\mathbf{0} \in \mathbf{X}$ which is just a usual (not centrally extended) point fixed by the $F_{\mathbf{c}}$−action.

§5. Square of a finite set

5.1. Monoidal category $Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$

Consider the category $Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$ consisting of all module functors $Coh_{F_{\mathbf{c}}}(\mathbf{X}) \to Coh_{F_{\mathbf{c}}}(\mathbf{X})$. It is a monoidal category with the tensor product induced by the composition of functors and unit object equal to the identity functor. Since the category $Coh_{F_{\mathbf{c}}}(\mathbf{X})$ is semisimple, any functor $F \in Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$ has left and right adjoint functors F^* and ${}^*F$. Observe that adjoint of tensor functor has a natural structure of a module functor and hence $F^*, {}^*F \in Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$. Standard properties of adjoint functors show that F^* and ${}^*F$ are right and left duals of F in the monoidal category $Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$. So the category $Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$ is rigid.

Lemma. *The category $Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$ is semisimple.*

Proof. Let us choose an $F_{\mathbf{c}}$−algebra A and an equivalence $Coh_{F_{\mathbf{c}}}(\mathbf{X}) \to Mod_{Rep(F_{\mathbf{c}})}(A)$. Then the category $Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$ is equivalent to the category of $A-A$ bimodules in $Rep(F_{\mathbf{c}})$, or to the category of $A \otimes A^{op}$−modules in $Rep(F_{\mathbf{c}})$ where A^{op} is A with the opposite multiplication. The latter category is clearly semisimple since $A \otimes A^{op}$ is a semisimple algebra. □

Note that in semisimple monoidal category left and right duals coincide, so in the future we will not distinguish left and right duals.

Remark. For an H-set X it is easy to construct an equivalence $Fun_H(X,X) \cong Coh_H(X \times X)$. Let us spell out a generalization of this statement to centrally extended H-sets.

Recall that for two central extensions $1 \to \mathbb{G}_m \to \tilde{H}_i \to H \to 1$, $i = 1,2$ their product is defined by $\tilde{H}_{12} = \tilde{H}_1 \times_H \tilde{H}_2/\mathbb{G}_m$, where $\mathbb{G}_m$ is embedded antidiagonally; also for a central extension $\tilde{H}$ the opposite central extension $\tilde{H}'$ is the same group with the same homomorphism to H but with the identification of its kernel with $\mathbb{G}_m$ replaced by the opposite one (composition of the original one with the map $x \mapsto x^{-1}$).

Now for two centrally extended H-sets $\mathbf{X}$, $\mathbf{Y}$ one can define their product in the obvious manner: the underlying equivariant set is $X \times$

Y, where X, Y are equivariant sets underlying $\mathbf{X}$ and $\mathbf{Y}$; the central extension $\tilde{H}(x,y)$ is the product of restrictions of $\tilde{H}(x)$ and of $\tilde{H}(y)$ to $H(x,y)$. For a centrally extended H-set $\mathbf{X}$ we obtain the *opposite* centrally extended set $\mathbf{X}'$ replacing each of the central extensions $\mathbb{G}_m \to \tilde{H}(x) \to H(x)$ by the opposite one.

If $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are centrally extended H-sets with underlying H-sets X, Y, Z, then for $\mathcal{F} \in Coh_H(X \times Y')$, $\mathcal{G} \in Coh_H(Y \times Z)$ the sheaf $\mathcal{F} \boxtimes \underline{\mathbb{C}} \otimes \underline{\mathbb{C}} \otimes \mathcal{G}$ on $X \times Y \times Z$ carries a natural structure of an equivariant sheaf on $\mathbf{X} \times \mathbf{Y}_0 \times \mathbf{Z}$ (here $\mathbf{Y}_0$ is Y equipped with the trivial (split) centrally extended structure). Thus the *convolution* $\mathcal{F} * \mathcal{G} = \mathrm{pr}_{13*}(\mathcal{F} \boxtimes \underline{\mathbb{C}} \bigotimes \underline{\mathbb{C}} \boxtimes \mathcal{G})$ (where $\mathrm{pr}_{13} : X \times Y \times Z \to X \times Z$ is the projection) carries the structure of an equivariant sheaf on $\mathbf{X} \times \mathbf{Z}$. In particular, for $\mathbf{X} = \mathbf{Y} = \mathbf{Z}$ we get a monoidal structure on $Coh_H(\mathbf{X} \times \mathbf{X}')$; and for $\mathbf{X} = \mathbf{Y}$, and $\mathbf{Z}$ being the point with the split central extension we get a monoidal functor of $Coh_H(\mathbf{X} \times \mathbf{X}') \to Fun_H(\mathbf{X})$. It is easy to see that this functor is an equivalence.

5.2. Monoidal functor G

We have a monoidal functor $G : \mathcal{M}_{\mathbf{c}}^{op} \to Fun_{F_{\mathbf{c}}}(\mathbf{X}, \mathbf{X})$, $G(X) = ? \bullet X$ where $\mathcal{M}_{\mathbf{c}}^{op}$ is $\mathcal{M}_{\mathbf{c}}$ with the *opposite* tensor product. It is clear that G is exact and faithful.

The main result of this section is the following

Theorem 4. *The functor G is a tensor equivalence*

$$\mathcal{M}_{\mathbf{c}}^{op} \to Fun_{F_{\mathbf{c}}}(\mathbf{X}, \mathbf{X}).$$

Corollary. *Suppose that any subgroup of finite index in $F_{\mathbf{c}}$ has no nontrivial projective representations. Then Lusztig's Conjecture holds for the cell* $\mathbf{c}$.

5.3. A result of G. Lusztig

The following result cited from [10] II Proposition 1.4 is cruicial for the proof of Theorem 4.

Proposition. (a) *Assume that $L_x \bullet L_y$, $x, y \in \mathbf{c}$ contains as a direct summand L_d, $d \in \mathcal{D}$. Then $x = y^{-1}$ and the multiplicity of L_d in $L_x \bullet L_y$ is one.*

(b) *For any $x \in \mathbf{c}$ the truncated convolution $L_x \bullet L_{x^{-1}}$ contains L_d for a uniquely defined $d \in \mathcal{D} \cap \mathbf{c}$.*

5.4. Proof of the Theorem 4.

Since the category $Fun_{F_{\mathbf{c}}}(\mathbf{X}, \mathbf{X})$ is semisimple it is enough to prove the following statements:

(i) Any functor from $Fun_{F_{\mathbf{c}}}(\mathbf{X},\mathbf{X})$ appears as a direct summand of $G(L)$, $L\in\mathcal{M}_{\mathbf{c}}^{op}$.

(ii) For any $w\in\mathbf{c}$ the functor $G(L_w)$ is irreducible.

(iii) For $w,w'\in\mathbf{c}$ an isomorphism $G(L_w)=G(L_{w'})$ implies $w=w'$.

5.4.1. We begin with the following

Lemma. (a) *For any $w\in\Gamma$ the functor $G(L_w)$ is irreducible. Moreover G induces an equivalence $\mathcal{M}_\Gamma\to\{$module functors $Rep(F_{\mathbf{c}})\to\mathcal{M}_\Gamma\}$.*

(b) *For any $w\in\Gamma^{-1}$ the functor $G(L_w)$ is irreducible. Moreover, G induces an equivalence $\mathcal{M}_{\Gamma^{-1}}\to\{$module functors $\mathcal{M}_\Gamma\to Rep(F_{\mathbf{c}})\}$.*

Proof. (a) It is clear that $L_v\bullet L_w=0$ for any $w\in\Gamma$, $v\in\Gamma-\Gamma\cap\Gamma^{-1}$. So the functor $G(L_w)$ can be considered as a functor $Rep(F_{\mathbf{c}})=\mathcal{M}_{\Gamma\cap\Gamma^{-1}}\to\mathcal{M}_\Gamma$. For any module category $\mathcal{M}$ over $Rep(F_{\mathbf{c}})$ the map $f\mapsto f(\mathbf{1})$ defines an equivalence of categories $Fun_{Rep(F_{\mathbf{c}})}(Rep(F_{\mathbf{c}}),\mathcal{M})\to\mathcal{M}$. In our case $G(L_w)(\mathbf{1})=L_{d^f}\bullet L_w=L_w$ and (a) is proved.

(b) Let us first check that for $w\in\Gamma^{-1}$ we have

$$G(L_w)^*\cong G(L_{w^{-1}}), \qquad (*)$$

where $G(L_w)^*$ is the functor adjoint to $G(L_w)$.

For $w\in\Gamma^{-1}$ the functor $G(L_w)$ maps $\mathcal{M}_\Gamma$ to $\mathcal{M}_{\Gamma\cap\Gamma^{-1}}=Rep(F_{\mathbf{c}})\subset\mathcal{M}_\Gamma$. Thus $G(L_w)^*$ sends L_v to zero unless $v\in\Gamma\cap\Gamma^{-1}$; hence part (a) of the Lemma implies that $G(L_w)^*$ is isomorphic to $G(L)$ for some $L\in\mathcal{M}_\Gamma$. To see that $L\cong L_{w^{-1}}$ it is enough to check that for $v\in\Gamma$ the space $Hom((G(L_v),G(L_w)^*))$ is one dimensional if $v=w^{-1}$, and is zero otherwise. We have $Hom((G(L_v),G(L_w)^*))=Hom(G(L_w)\circ G(L_v),Id_{\mathcal{M}_\Gamma})$. Notice that $G(L_w)\circ G(L_v)$ preserves the direct summand $\mathcal{M}_{\Gamma\cap\Gamma^{-1}}\subset\mathcal{M}_\Gamma$ and is zero on its complement. It follows that

$$Hom(G(L_w)\circ G(L_v),Id_{\mathcal{M}_\Gamma})=Hom(G(L_w)\circ G(L_v)|_{\mathcal{M}_{\Gamma\cap\Gamma^{-1}}},Id_{\mathcal{M}_{\Gamma\cap\Gamma^{-1}}}).$$

Since $\mathcal{M}_{\Gamma\cap\Gamma^{-1}}\cong Rep(F_{\mathbf{c}})$ by 3.4, and the category of module functors from $Rep(H)$ to $Rep(H)$ (considered as the free module over itself) is equivalent to $Rep(H)$, we see that

$$Hom(G(L_w)\circ G(L_v),Id_{\mathcal{M}_{\Gamma\cap\Gamma^{-1}}})=Hom_{\mathcal{M}_{\Gamma\cap\Gamma^{-1}}}(L_w\bullet L_v,L_{d^f}),$$

thus $(*)$ follows from Proposition 5.3.

Irreducibility of $G(L_w)$ follows from $(*)$ and part (a), because the dual object of an irreducible object (in the category of functors) is irreducible. It remains to check that any module functor $\phi:\mathcal{M}_\Gamma\to Rep(F_{\mathbf{c}})$ is isomorphic to the one coming from some $L\in\mathcal{M}_{\Gamma^{-1}}$. Consider ϕ as an endofunctor $\mathcal{M}_\Gamma$ (i.e. take its composition with the imbedding

$Rep(F_{\mathbf{c}}) = \mathcal{M}_{\Gamma\cap\Gamma^{-1}} \hookrightarrow \mathcal{M}_\Gamma$); then by (a) the adjoint functor ϕ^* is isomorphic to $G(L)$ for $L \in \mathcal{M}_\Gamma$, so this statement also follows from $(*)$. $\square$

We can now prove (i).

Corollary. *Any irreducible functor from $Fun_{Rep(F_{\mathbf{c}})}(\mathbf{X}, \mathbf{X})$ appears as a direct summand of $G(L_{w'} \bullet L_w), w \in \Gamma, w' \in \Gamma^{-1}$.*

Proof. We need to prove that any irreducible functor

$$f \in Fun_{Rep(F_{\mathbf{c}})}(\mathbf{X}, \mathbf{X})$$

is a direct summand of a composite functor $Coh_{F_{\mathbf{c}}}(\mathbf{X}) \to Rep(F_{\mathbf{c}}) \to Coh_{F_{\mathbf{c}}}(\mathbf{X})$. For this we choose any functor $g : Coh_{F_{\mathbf{c}}}(\mathbf{X}) \to Rep(F_{\mathbf{c}})$ such that the composition $g \circ f$ is nonzero. Let g^* be the adjoint functor to g. Then f is evidently a direct summand of $(g^* \circ g) \circ f$ which admits the required factorisation $g^* \circ (g \circ f)$. $\square$

5.4.2. **Lemma.** *For any $w \in \mathbf{c} - \mathcal{D}$ we have $Hom(G(L_w), id) = 0$.*

Proof. We note that L_w as an object of $\mathcal{M}_{\mathbf{c}}$ is a direct summand of $L_u \bullet L_v$ where $u \in \Gamma^{-1}$ and $v \in \Gamma$ (this follows easily for the example from Theorem 1.8 in [10] II). Assume that $Hom(G(L_w), id) \neq 0$ and hence $Hom(G(L_u \bullet L_v), id) \neq 0$. Then we get a nonzero transformation $G(L_u) \to G(L_v)^*$; since both functors are irreducible by Lemma 5.4.1 they are actually isomorphic. On the other hand, Proposition 5.3 provides a non-zero transformation $G(L_u) \circ G(L_u^{-1}) \to Id$, which also yields an isomorphism $G(L_u)^* \cong G(L_{u^{-1}})$. Thus $G(L_{u^{-1}}) \cong G(L_v)$, and by Lemma 5.4.1 this yields $L_{u^{-1}} \cong L_v$, so $v = u^{-1}$. Furthermore, $dimHom(G(L_u \bullet L_v), id) = dimHom(G(L_v), G(L_u)^*) = 1$; and by Proposition 5.3 the object $L_u \bullet L_v$ contains L_d for a uniquely defined $d \in \mathcal{D}$. Since $Hom(G(L_d), id) \neq 0$ we have that $Hom(L_w, id) = 0$ if $w \neq d$. The Lemma is proved. $\square$

Now we can prove (ii)

Corollary. *For any $w \in \mathbf{c}$ the functor $G(L_w)$ is irreducible.*

Proof. Consider the adjoint functor $G(L_w)^*$. By Lemma 5.4.1 any summand of $G(L_w)^*$ appears as a direct summand of $G(L_{w'})$. For such w' we have a non-zero transformation $G(L_{w'}) \circ G(L_w) = G(L_w \bullet L_{w'}) \to Id$, and by Lemma 5.4.2 and Proposition 5.3 this is possible only when $w' = w^{-1}$. If $G(L_w)$ is reducible this implies that $dimHom(G(L_w) \circ G(L_{w^{-1}}), id) > 1$. On the other hand $dimHom(G(L_w) \circ G(L_{w^{-1}}), id) = dimHom(G(L_{w^{-1}} \bullet L_w), id) = 1$ by Proposition 5.3 and Lemma 5.4.2, and we get a contradiction. $\square$

5.4.3. We can now prove (iii). Assume that $G(L_w) = G(L_{w'})$. Then $G(L_w) \circ G(L_{w^{-1}}) = G(L_{w'}) \circ G(L_{w^{-1}})$. Since $Hom(G(L_w) \circ G(L_{w^{-1}}), id) \neq 0$ we have that $Hom(G(L_{w'}) \circ G(L_{w^{-1}}), id) \neq 0$ and by the Proposition 5.3 $w' = (w^{-1})^{-1} = w$.

Since we proved (i) (ii) and (iii) the Theorem 4 is proved. □

5.5. Examples

The Corollary 5.2 can be applied in the following cases:

(a) Let G be a simply connected group and let $\mathbf{c}$ be the lowest cell. In this case $u_{\mathbf{c}} = e \in G$ and $F_{\mathbf{c}} = G$. In this case Corollary 5.2 is a result of [16].

(b) Let $G = GL_n$. In this case all groups $F_{\mathbf{c}}$ are connected and have no nontrivial projective representations (these groups are products of various GL_m). In this case we get a result of [18].

(c) Let G be a simply connected group of rank 2. In this case one easily verifies that the condition of Corollary 5.2 is satisfied and we get a result of [17].

(d) Let G be a simple simply connected group. Let $\mathbf{c}$ be the subregular cell, that is the cell corresponding to the subregular nilpotent orbit. Again one easily verifies that the condition of Corollary 5.2 is satisfied except if G is of type C_n. In the latter case $F_{\mathbf{c}} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ where one of the factors comes from the center of G. One can exlude centrally extended points in this case by considering a reductive group $G_1 = G \times T/(z, -1)$ where T is the one dimensional torus, $z \in G$ is the unique nontrivial central element and $-1 \in T$ is the unique nontrivial involutive element. So we get another result of [17].

Finally note that centrally extended points naturally appear in the description of truncated convolution categories for simple non simply-connected groups, see in [18] 8.3 example with $G = PSL_2$.

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Zeta Functions and Functional Equations Associated with the Components of the Gelfand-Graev Representations of a Finite Reductive Group

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§0. Introduction

Zeta functions and functional equations associated with them for representations of finite groups were first discussed by Springer [18] and Macdonald [14] for certain representations over the complex field $\mathbb{C}$ of $GL_n(k)$ for a finite field $k = \mathbb{F}_q$. Their results, with one additional assumption, hold for irreducible representations over $\mathbb{C}$ of an arbitrary finite group G embedded in $GL(V)$, for an n-dimensional vector space V over k. In §1, a related functional equation is obtained for irreducible representations of Hecke algebras (or endomorphism algebras) $\mathcal{H}$ of multiplicity free induced representations of finite groups.

The functional equation 1.2.1 for an irreducible representation π of G involves an ε-factor $\varepsilon(\pi, \chi)$ which is given by

$$\varepsilon(\pi, \chi) = q^{-n^2/2}(\deg \pi)^{-1} \sum_{g \in G} \zeta_{\pi^*}(g)\chi(\mathrm{Tr}\,(g)),$$

where ζ_{π^*} is the character of the contragredient representation π^* of π, χ is a nontrivial additive character of k, and $\mathrm{Tr}\,(g)$ is the trace of g in $GL(V)$. The functional equations satisfied by irreducible representations f_π of $\mathcal{H}$, with π an irreducible component of the induced representation, have the form (see Proposition 1.5, §1)

$$f_\pi(\widetilde{h}) = \varepsilon(\pi, \chi) f_\pi(h),$$

with $h \in \mathcal{H}$, and $\widetilde{h}$ a twisted Fourier transform of h (to be defined in §1). The ε-factor $\varepsilon(\pi, \chi)$ is also given by the formula

$$\varepsilon(\pi, \chi) = f_\pi(\widetilde{e}),$$

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where $\widetilde{e}$ is the twisted Fourier transform of the identity element e of $\mathcal{H}$.

In §2, the results are applied to the representations of the Hecke algebra $\mathcal{H}$ of an arbitrary Gelfand-Graev representation Γ of a finite reductive group $G = \mathbf{G}^F$, for a connected reductive algebraic group $\mathbf{G}$ defined over k, with Frobenius endomorphism F, as in [3]. The Gelfand-Graev representations Γ of G are multiplicity free induced representations parametrized and decomposed into irreducible components by Digne, Lehrer, and Michel [9].

In [3] the irreducible representations of $\mathcal{H}$ were parametrized by pairs $(\mathbf{T}, \theta)$ with $\mathbf{T}$ an F-stable maximal torus in $\mathbf{G}$, and θ an irreducible representation of the finite torus $T = \mathbf{T}^F$. In §2 we review the main theorem of [3], which states that each representation $f_{\mathbf{T},\theta}$ of $\mathcal{H}$ has a factorization $f_{\mathbf{T},\theta} = \hat{\theta} \circ f_{\mathbf{T}}$, with $f_{\mathbf{T}}$ a homomorphism of algebras from $\mathcal{H}$ to the group algebra of $T = \mathbf{T}^F$, and $\hat{\theta}$ an extension of θ to an irreducible representation of the group algebra of the torus T.

For a general finite reductive group, a formula is obtained in §2 for an ε-factor $\varepsilon(\pi, \chi)$ of an irreducible component π of Γ of the form $\pi = (-1)^{\sigma(\mathbf{G})+\sigma(\mathbf{T})} R_{\mathbf{T},\theta}$, where $\sigma(\mathbf{G}), \sigma(\mathbf{T})$ are the k-ranks of the reductive groups $\mathbf{G}$ and $\mathbf{T}$ respectively, and $R_{\mathbf{T},\theta}$ is the virtual representation of G constructed by Deligne and Lusztig [8], with θ a character of T in general position. In this situation, the ε-factor $\varepsilon(\pi, \chi)$ is a Gauss sum of the representation π, and is expressed as a character sum over the finite torus $T = \mathbf{T}^F$ by a result in ([16], Theorem 1.2). Using the known structure of the finite tori, the ε-factors $\varepsilon(\pi, \chi)$ have been computed in [16] and [17] for some classical groups, and for the exceptional groups of type G_2. The formulas obtained in [16] and [17] involve Gauss sums, Kloosterman sums, and unitary Kloosterman sums (cf. [5]) associated with finite extensions of k.

In §3 more complete results concerning ε-factors are obtained for $GL_n(k)$. These are based on a formula for $f_{\mathbf{T},\theta}(c_{\dot{w}})$ as a character sum over the finite torus $T = \mathbf{T}^F$, for certain standard basis elements $c_{\dot{w}}$ of $\mathcal{H}$. Applications of this result include a formula for $f_{\mathbf{T},\theta}(\tilde{e})$ for all pairs $(\mathbf{T}, \theta)$. In the case of $GL_n(k)$, the ε-factors $\varepsilon(\pi, \chi)$ were computed for all irreducible representations by Kondo [11] and Macdonald [15] and expressed as products of Gauss sums of finite fields, using Green's results on the irreducible characters of $GL_n(k)$. Our results give formulas for the ε-factors as character sums over the finite tori $T = \mathbf{T}^F$. The last result in §3 is a formula expressing the twisted Fourier transform of the identity element of $\mathcal{H}$ in terms of the standard basis elements. In §4 another application of the formula for $f_{\mathbf{T},\theta}(c_{\dot{w}})$, in case $G = SL_n(k)$, gives a formula for the Gauss sums of unipotent representations.

In §5 the formula for $f_{\mathbf{T},\theta}(c_{\dot{w}})$ is applied to the computation of the norm map $\Delta : \mathcal{H}' \to \mathcal{H}$ ([6]), where $\mathcal{H}'$ is the Hecke algebra of the Gelfand-Graev representation of $GL_n(k')$, and k' is the extension of k of degree m. The result is that

$$\Delta(\widetilde{e'}) = (-1)^{n(m-1)}\tilde{e}^m.$$

As a corollary, we obtain an extension of the Davenport-Hasse theorem for Gauss sums of field extensions to Gauss sums associated with certain irreducible components of the Gelfand-Graev representation of $GL_n(k')$.

§1. The zeta function of a representation of a finite group

1.1. Let G be a finite group. We consider a faithful representation ρ of G, $\rho : G \to GL(V)$, where V is an n-dimensional vector space over a finite field $k = \mathbb{F}_q$, so that G can be identified with a subgroup of $GL(V)$. We shall identify an element $g \in G$ with the corresponding linear transformation $\rho(g)$. Let $X = \mathrm{End}_k(V)$ and let $\mathbb{C}(X)$ be the space of complex valued functions on X. Following Springer, [18], or Macdonald, [14], we introduce the notion of the Fourier transform and zeta function of complex representations of G as follows. Let χ be a nontrivial additive character of k, which is fixed throughout this paper. Then for $\Phi \in \mathbb{C}(X)$, the Fourier transform $\widehat{\Phi}$ of Φ is defined by

$$\widehat{\Phi}(x) = q^{-n^2/2} \sum_{y \in X} \Phi(y)\chi(\mathrm{Tr}\,(xy)).$$

Then we have $\widehat{\widehat{\Phi}}(x) = \Phi(-x)$ for all $x \in X$. For a finite dimensional complex representation π of G, and for $\Phi \in \mathbb{C}(X)$, define the zeta function $Z(\Phi, \pi)$ by

$$Z(\Phi, \pi) = \sum_{g \in G} \Phi(g)\pi(g);$$

then $Z(\Phi, \pi) = \pi(a_\Phi)$ where $a_\Phi = \sum_{g \in G} \Phi(g)g$ is the element of the group algebra $\mathbb{C}G$ of G over $\mathbb{C}$ with coefficients $\Phi(g)$.

For $x \in X$, define

$$W(\pi, \chi; x) = q^{-n^2/2} \sum_{g \in G} \chi(\mathrm{Tr}\,(gx))\pi(g).$$

Then

$$Z(\Phi,\pi)=\sum_{x\in X}\widehat{\Phi}(-x)W(\pi,\chi;x).$$

For $g\in G$, one has

$$\begin{aligned} W(\pi,\chi;xg) &= \pi(g)^{-1}W(\pi,\chi;x),\\ W(\pi,\chi;gx) &= W(\pi,\chi;x)\pi(g)^{-1}. \end{aligned}$$

Putting $x=1$, these imply that $\pi(g)$ commutes with $W(\pi,\chi;1)$, so if π is irreducible,

$$W(\pi,\chi;1)=w(\pi,\chi)\pi(1),$$

where $w(\pi,\chi)\in\mathbb{C}$. Define the ε-factor $\varepsilon(\pi,\chi)$ by

$$\varepsilon(\pi,\chi)=w(\pi^*,\chi),$$

where π^* is the contragredient representation of π.

Proposition 1.2. *Let π be an irreducible representation of G and let $\Phi\in\mathbb{C}(X)$ vanish outside G. Then*

$$(1.2.1)\qquad {}^tZ(\widehat{\Phi},\pi^*)=\varepsilon(\pi,\chi)Z(\Phi,\pi).$$

Proof.

$$\begin{aligned} {}^tZ(\widehat{\Phi},\pi^*) &= \sum_{x\in X}\widehat{\widehat{\Phi}}(-x)\,{}^tW(\pi^*,\chi;x)\\ &= \sum_{x\in X}\Phi(x)\,{}^tW(\pi^*,\chi;x)\\ &= \sum_{g\in G}\Phi(g)\,{}^tW(\pi^*,\chi;g)\\ &= \sum_{g\in G}\Phi(g)\,{}^t\pi^*(g^{-1})\,{}^tW(\pi^*,\chi;1)\\ &= \sum_{g\in G}\Phi(g)\pi(g)w(\pi^*,\chi). \end{aligned}$$

□

For all irreducible representations π of $GL_n(k)$ having no one component, Macdonald proved that $W(\pi^*,\chi;x)$ has support contained in $GL_n(k)$, so that the functional equation 1.2.1 holds for all functions Φ (see [14], and [18] for the case of an irreducible cuspidal representation of G). With the assumption that Φ has support in G the formula given in

Proposition 1.2 for an arbitrary finite group embedded in $GL(V)$ follows from Macdonald's argument, as given above. In case π_ϕ is an irreducible cuspidal representation of $GL_n(k)$ associated with a regular character ϕ of the multiplicative group $k_n^\times$ of the extension k_n of k of degree n, Springer proved that the ε-factor is a Gauss sum

$$(-1)^n q^{-n/2} \sum_{x \in k_n^\times} \chi(\mathrm{Tr}_{\,k_n/k} x)\phi(x).$$

Springer also gave an example to show that no functional equation of the above form holds for all irreducible representations π of $GL_n(k)$ and all functions Φ. The zeta function is an analogue for finite fields of a concept introduced by Godement and Jacquet (SLN 260).

1.3. Let U be a subgroup of G and ψ a complex linear character of U. We use the notation concerning the Hecke algebra of the induced representation ψ^G introduced in [3, §2B]. In particular, ψ^G is afforded by the left ideal $\mathbb{C}Ge_\psi$ in the group algebra of G generated by the idempotent

$$e_\psi = \mid U \mid^{-1} \sum_{u \in U} \psi(u^{-1})u.$$

The Hecke algebra $\mathcal{H}$ associated with the induced representation ψ^G is defined by

$$\mathcal{H} = e_\psi \mathbb{C} G e_\psi.$$

We assume $\mathcal{H}$ is commutative (so that (G, H, ψ) is a twisted Gelfand pair according to [15, p.397]).

Lemma 1.4. *Let $\Phi \in \mathbb{C}(X)$ and assume that Φ vanishes outside G. Then $\sum_{g \in G} \Phi(g)g \in \mathcal{H}$ implies $\sum_{g \in G} \widehat{\Phi}(g)g^{-1} \in \mathcal{H}$.*

Proof. First we notice that $\sum_{g \in G} \Phi(g)g \in \mathcal{H}$ if and only if $\Phi(ug) = \Phi(gu) = \psi(u^{-1})\Phi(g)$ for $u \in U$, $g \in G$. So we have to prove that Ψ satisfies these conditions where $\Psi(g) = \widehat{\Phi}(g^{-1})$. We have, using the assumption that Φ is supported on G,

$$\Psi(ug) = \widehat{\Phi}(g^{-1}u^{-1}) = q^{-n^2/2} \sum_{y \in G} \Phi(y)\chi(\mathrm{Tr}\,(g^{-1}u^{-1}y)).$$

Putting $z = g^{-1}u^{-1}y$, the right hand side becomes

$$\begin{aligned} q^{-n^2/2}\sum_{z\in G}\Phi(ugz)\chi(\mathrm{Tr}\,(z)) &= \psi(u^{-1})q^{-n^2/2}\sum_{z\in G}\Phi(gz)\chi(\mathrm{Tr}\,(z)) \\ &= \psi(u^{-1})q^{-n^2/2}\sum_{y\in X}\Phi(y)\chi(\mathrm{Tr}\,(g^{-1}y)) \\ &= \psi(u^{-1})\widehat{\Phi}(g^{-1}) = \psi(u^{-1})\Psi(g) \end{aligned}$$

as required. The formula $\Psi(gu) = \lambda(u^{-1})\Psi(g)$ follows similarly. □

We remark that the converse holds if $-1 \in G$, since $\widehat{\widehat{\Phi}}(x) = \Phi(-x)$. For $h = \sum_{g\in G}\Phi(g)g \in \mathcal{H}$ with Φ supported on G, the element $\widetilde{h} = \sum_{g\in G}\widehat{\Phi}(g)g^{-1} \in \mathcal{H}$ will sometimes be called the twisted Fourier transform of h.

Proposition 1.5. *Let π be an irreducible constituent in ψ^G, and let f_π be the corresponding representation of $\mathcal{H}$. Then*

$$f_\pi(\widetilde{h}) = \varepsilon(\pi,\chi)f_\pi(h)$$

where $h = \sum_{g\in G}\Phi(g)g \in \mathcal{H}, \widetilde{h} = \sum_{g\in G}\widehat{\Phi}(g)g^{-1}$, and Φ vanishes outside G, so $\widetilde{h} \in \mathcal{H}$.

Proof. Taking traces of (1.2.1), one has

$$\sum_{g\in G}\widehat{\Phi}(g)\,\mathrm{Tr}\,(\pi(g^{-1})) = \varepsilon(\pi,\chi)\sum_{g\in G}\Phi(g)\,\mathrm{Tr}\,(\pi(g)).$$

Then the Proposition follows from the previous Lemma. □

We note that $\widetilde{\widetilde{h}}$ is not related to $\widehat{\widehat{\Phi}}$, since $\widehat{\Phi}$ is not supported by G in general, even if Φ is supported by G.

Corollary 1.6. $f_\pi(\widetilde{e}_\psi) = \varepsilon(\pi,\chi)$ *and* $\widetilde{h} = \widetilde{e}_\psi h$.

Proof. Putting $h = e_\psi$ in the above Proposition, we have the first assertion. Then we have

$$f_\pi(\widetilde{h}) = f_\pi(\widetilde{e}_\psi h),$$

for every irreducible representation f_π of the semisimple algebra $\mathcal{H}$, which proves the second. □

§2. Zeta functions and Gelfand-Graev representation of a finite reductive group

2.1. Let $\mathbf{G}$ be a connected reductive algebraic group defined over a finite field $k = \mathbb{F}_q$ with Frobenius map F, and let $G = \mathbf{G}^F$ be the finite group consisting of elements in $\mathbf{G}$ fixed by F. We choose an F-stable Borel subgroup $\mathbf{B}_0$ and an F-stable maximal torus $\mathbf{T}_0$ contained in $\mathbf{B}_0$; and denote by $\mathbf{U}_0$ the unipotent radical of $\mathbf{B}_0$. We put $B_0 = \mathbf{B}_0^F$, $T_0 = \mathbf{T}_0^F$, and $U_0 = \mathbf{U}_0^F$.

Let ρ be a faithful representation of $\mathbf{G}$,

$$\rho : \mathbf{G} \to GL_n(\overline{k}),$$

with $\overline{k}$ the algebraic closure of k. We assume that ρ commutes with Frobenius maps as follows: $\rho \circ F = F' \circ \rho$, where $F'(x) = x^{(q)} = (x_{ij}^q)$ for $x = (x_{ij}) \in GL_n(\overline{k})$. Thus G can be identified with a subgroup of $GL_n(k)$.

2.2. Before discussing representations, it is necessary to change the field from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$, the algebraic closure of the field of ℓ-adic numbers with ℓ a prime different from the characteristic of k, as in the Deligne-Lustzig paper [8].

As for Gelfand-Graev representations of G, we shall follow the notation and preliminary discussion from [3]. We also carry over the notation from the preceding section. In particular, $\Gamma = \psi^G$ denotes a fixed Gelfand-Graev representation of G, parametrized by an element $z \in H^1(F, Z(\mathbf{G}))$ as in [3]; while $\mathcal{H}$ denotes the Hecke algebra of Γ, $e = e_\psi$ the identity element of $\mathcal{H}$, etc. As in [3], $f_{\mathbf{T},\theta}$ denotes the irreducible representation of the Hecke algebra $\mathcal{H}$ associated with the pair consisting of an F-stable maximal torus $\mathbf{T}$ and a character θ of $T = \mathbf{T}^F$. We recall the following factorization theorem ([3, Theorem (4.2)]).

Theorem 2.3. *For each pair $(\mathbf{T}, \theta)$ as above, the corresponding representation $f_{\mathbf{T},\theta} : \mathcal{H} \to \overline{\mathbb{Q}}_l$ can be factored,*

$$f_{\mathbf{T},\theta} = \widehat{\theta} \circ f_{\mathbf{T}},$$

with $f_{\mathbf{T}}$ a homomorphism of algebras from $\mathcal{H}$ to $\overline{\mathbb{Q}}_\ell T$, independent of θ. Let $f_{\mathbf{T}}(c) = \sum f_{\mathbf{T}}(c)(t)t \in \overline{\mathbb{Q}}_l T$, for $c \in \mathcal{H}$. Then the value of the coefficient function $f_{\mathbf{T}}(c_n)(t)$, for a standard basis element c_n of $\mathcal{H}$ and

$t \in T$, is given by the following formula:

$$f_{\mathbf{T}}(c_n)(t) = \text{ind } n < Q_{\mathbf{T}}^{\mathbf{G}}, \Gamma >^{-1} | U_0 |^{-1} | C_{\mathbf{G}}(t)^{\circ F} |^{-1}$$

$$\times \sum_{\substack{g \in G, u \in U_0 \\ (gung^{-1})_{ss} = t}} \psi(u^{-1}) Q_{\mathbf{T}}^{C_{\mathbf{G}}(t)^\circ}((gung^{-1})_{uni}). \tag{2.3.1}$$

2.3.2. *Remark* In what follows, we shall denote $(-1)^{\sigma(\mathbf{G})-\sigma(\mathbf{T})}$ by $\varepsilon(\mathbf{T})$. In case the center of $\mathbf{G}$ is connected, we have $< Q_{\mathbf{T}}^{\mathbf{G}}, \Gamma >= \varepsilon(\mathbf{T})$ from §10 of [8]. In the case of $GL_n(k)$ and if $\mathbf{T}$ corresponds to $w \in S_n$, we have $\varepsilon(\mathbf{T}) = \text{sgn}(w)$.

Theorem 2.4. *Let π be an irreducible representation of G.*

(i) *The ε-factor corresponding to π is given by*

$$\begin{aligned} \varepsilon(\pi, \chi) &= \frac{1}{\deg \pi} \text{Tr } W(\pi^*, \chi; 1) \\ &= \frac{q^{-n^2/2}}{\deg \pi} \sum_{g \in G} \zeta_{\pi^*}(g) \chi(\text{Tr } (g)), \end{aligned}$$

where ζ_{π^*} is the character of the contragredient representation π^*.

(ii) *In case π is a component of Γ corresponding to the representation $f_{\mathbf{T},\theta}$ of $\mathcal{H}$, we have*

$$f_{\mathbf{T},\theta}(\widetilde{h}) = \varepsilon(\pi, \chi) f_{\mathbf{T},\theta}(h),$$

for all $h \in \mathcal{H}$, $h = \sum \Phi(g) g$, with Φ vanishing outside G.

(iii) *In case the irreducible representation π has the form $\varepsilon(\mathbf{T}) R_{\mathbf{T},\theta}$ with θ in general position, one has*

$$\varepsilon(\pi, \chi) = \varepsilon(\mathbf{T}) q^{-n^2/2} | G |_p \sum_{t \in T} \theta^{-1}(t) \chi(\text{Tr } (t)).$$

Proof. The first statement follows from the definition of $\varepsilon(\pi, \chi)$ in §1.1. Part (ii) follows from (1.5), while (iii) follows from ([16], Theorem 1.2) and the fact that $R_{\mathbf{T},\theta}^* = R_{\mathbf{T},\theta^{-1}}$. □

Corollary 2.5. *With π corresponding to $f_{\mathbf{T},\theta}$ as in part* (ii) *of the Theorem, we have by* (1.6)

$$f_{\mathbf{T},\theta}(\widetilde{e}) = \varepsilon(\pi, \chi).$$

Remarks 2.6. (i) For any irreducible representation π of G, the sum

$$\tau(\pi) = \sum_{g \in G} \mathrm{Tr}\,(\pi(g))\chi(\mathrm{Tr}\,(g))$$

is called a Gauss sum of G associated with (π, χ). These have been computed in the case of $G = GL_n(k)$ for all irreducible representations ([11], [15]). In the situation of part (iii) of the Theorem, and also for unipotent representations, the Gauss sums have been computed for several other classical groups and for G_2 ([16], [17]).

(ii) Let $\phi(g) = \chi(\mathrm{Tr}\,(g))$ for $g \in G$ and let $<,>_G$ be the inner product of class functions on G. Then we have

$$\begin{aligned}\tau(\pi) &= |G| < \zeta_{\pi^*}, \phi >_G \\ \varepsilon(\pi, \chi) &= (\deg \pi)^{-1} q^{-n^2/2} |G| < \zeta_\pi, \phi >_G .\end{aligned}$$

We also notice that since the value of ϕ depends only on the semisimple part of the element $g \in G$, ϕ is expressed as a linear combination of the virtual characters of Deligne-Lusztig by [8, (7.12.1)] (see also [1, Proposition 7.6.4]).

§3. ε-Factors for $GL_n(k)$

In this section, let $G = GL_n(k)$ and let U be the upper triangular unipotent subgroup of G. Then $G = \mathbf{G}^F$ for $\mathbf{G} = GL_n(\bar{k})$ with the usual Frobenius endomorphism F. In this case there is, up to equivalence, just one Gelfand-Graev representation $\Gamma = \psi^G$, for the linear character ψ of U given by $\psi(u) = \chi(u_{12} + \cdots + u_{n-1n})$ with $u = (u_{ij}) \in U$.

We begin with some computations of the homomorphisms $f_{\mathbf{T}}$ on standard basis elements of $\mathcal{H}$.

Lemma 3.1. *For $a \in k^*$, let*

$$\dot{w}(a) = \begin{pmatrix} & & & a \\ -1 & & & \\ & \ddots & & \\ & & -1 & \end{pmatrix} \in G. \tag{3.1.1}$$

Then for all $u \in U$, $u\dot{w}(a)$ is a regular element, i.e. $(u\dot{w}(a))_{uni}$ is a regular unipotent element in $C_G((u\dot{w}(a))_{ss})$.

Proof. It is enough to show that the minimal polynomial of $u\dot{w}(a)$ is the characteristic polynomial of $u\dot{w}(a)$ and for that it is enough to show that

(3.1.2)
$$(x_1 I - u\dot{w}(a)) \cdots (x_{n-1} I - u\dot{w}(a)) \neq 0, \quad \text{for all } x_1, \dots, x_{n-1} \in \bar{k},$$

where I is the identity matrix in G. Let $u = (u_{ij})$ and $A = u\dot{w}(a)$. Thus

$$A = \begin{pmatrix} -u_{12} & -u_{13} & \cdots & -u_{1n} & a \\ -1 & -u_{23} & \cdots & -u_{2n} & \\ & -1 & \cdots & -u_{3n} & \\ & & \ddots & \vdots & \\ & & & -1 & \end{pmatrix}.$$

Let $A_i = x_i I - A, (i = 1, \dots, n-1)$, then it is easy to see that the $(n,1)$-entry of $A_1 \cdots A_{n-1}$ is nonzero, which proves (3.1.2). □

Lemma 3.2. *We have*
(i) ${}^{\dot{w}(a)}\psi = \psi$ *on* $U \cap {}^{\dot{w}(a)} U$, *and*
(ii) $[U : U \cap {}^{\dot{w}(a)} U] = q^{n-1}$.
for all nonzero elements $a \in k$.

Proof. For $u = (u_{ij}) \in U$, we have

$${}^{\dot{w}(a)}u = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -a^{-1}u_{1n} & 1 & u_{12} & \cdots & u_{1n-1} \\ -a^{-1}u_{2n} & 0 & 1 & \cdots & u_{2n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a^{-1}u_{n-1n} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Thus the condition for ${}^{\dot{w}(a)}u \in U$ is $u_{1n} = u_{2n} = \cdots = u_{n-1n} = 0$, which proves (ii).

Take any $u_0 \in U \cap {}^{\dot{w}(a)} U$, then there exists $u = (u_{ij}) \in U$ such that $u_0 = {}^{\dot{w}(a)} u$. Therefore, using the first part of the proof,

$$\begin{aligned} {}^{\dot{w}(a)}\psi(u_0) &= \psi({}^{\dot{w}(a)^{-1}}u_0) = \psi(u) \\ &= \chi(u_{12} + \cdots + u_{n-2,n-1}) = \psi(u_0), \end{aligned}$$

which proves the first assertion. □

Theorem 3.3. *Let $G = GL_n(k)$ and let $\dot{w}(a)$ be defined as in (3.1). Then $c_{\dot{w}(a)}$ is a standard basis element of $\mathcal{H}$. For each F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, we have, for all $t \in T$,*

$$f_{\mathbf{T}}(c_{\dot{w}(a)})(t) = \delta_{\det t, a}\varepsilon(\mathbf{T})\chi(\mathrm{Tr}\ t), \tag{3.3.1}$$

where $\delta_{\det t,a} = 1$, if $\det t = a$, and $= 0$, otherwise. Therefore

$$f_{\mathbf{T},\theta}(c_{\dot{w}(a)}) = \varepsilon(\mathbf{T}) \sum_{t\in T, \det t = a} \chi(\mathrm{Tr}\ t)\theta(t). \tag{3.3.2}$$

Proof. By Theorem 2.3, Lemma 3.1, and Lemma 3.2 (2), together with the fact that $Q_{\mathbf{T}}^{\mathbf{G}}(u) = 1$ if u is regular unipotent by [8, Theorem 9.16], we have

$$f_{\mathbf{T}}(c_{\dot{w}(a)})(t) = q^{n-1}\varepsilon(\mathbf{T}) \mid U \mid^{-1}\mid C_G(t) \mid^{-1} \sum_{\substack{g\in G, u\in U \\ (gu\dot{w}(a)g^{-1})_{ss}=t}} \psi(u^{-1}).$$

Two semisimple elements, $(u\dot{w}(a))_{ss}$ and t are conjugate if and only if their characteristic polynomials are the same. Let t be conjugate to $\mathrm{diag}(\alpha_1, \alpha_2, ..., \alpha_n)$ in $\mathbf{G} = GL_n(\bar{k})$, and let $u = (u_{ij})$, where $u_{ij} = 0$, if $i > j$ and $u_{ii} = 1$. Regarding u_{ij} $(i < j)$ as variables and defining polynomials $p_m(u) = p_m(u_{12}, u_{13}, ...)$ over k by $\det(xI - u\dot{w}(a)) = \sum_{m=0}^{n} p_m(u)x^{n-m}$ we can show easily that

$$p_m(u) = (-1)^{m+1}u_{1,m+1} + q_m(u), \quad \text{for } m = 1, ..., n-1,$$

where $q_m(u)$ is a polynomial in the variables u_{1j} $(1 < j < m+1)$ and u_{ij} $(1 < i < j)$. In particular $p_1(u) = \sum_{i=1}^{n-1} u_{ii+1}$.

Thus $(u\dot{w}(a))_{ss}$ and t are conjugate if and only if

$$(-1)^m p_m(u) = \sum_{1\le i_1 < i_2 \cdots < i_m \le n} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_m}, \text{ for } m = 1, ..., n. \tag{3.3.3}$$

These simultaneous equations have solutions if $\det t = a$ and in this case the number of solutions is $q^{(n-1)(n-2)/2}$ since for any values of u_{ij} $(2 \le i < j \le n)$, u_{1j} $(2 \le j \le n)$ are uniquely determined by the equations (3.3.3). Notice that $\mathrm{Tr}\ t = -\sum_{i=1}^{n-1} u_{ii+1}$. Moreover if $(u\dot{w}(a))_{ss}$ and t are conjugate, then the set $\{g \in G \mid g(u\dot{w}(a))_{ss}g^{-1} = t\}$ is a coset of $C_G(t)$. Putting these facts together we have the equations in the theorem. □

Corollary 3.4. *If* $(\mathbf{T},\theta)$ *and* $(\mathbf{T}',\theta')$ *are geometrically conjugate, we have*

$$\varepsilon(\mathbf{T})\sum_{t\in T,\det t=a}\chi(\mathrm{Tr}\ t)\theta(t)=\varepsilon(\mathbf{T}')\sum_{t\in T',\det t=a}\chi(\mathrm{Tr}\ t)\theta'(t).$$

Proof. If $(\mathbf{T},\theta)$ and $(\mathbf{T}',\theta')$ are geometrically conjugate, we have $f_{\mathbf{T},\theta}=f_{\mathbf{T}',\theta'}$ (cf. [3]). By evaluating them on $c_{\dot{w}(a)}$, the assertion follows. □

We remark that the corollary is a generalization of [2, Lemma (5.1)]. In particular if we apply (3.4) to $GL_2(q)$, $(\mathbf{T}_1,1)$ and $(\mathbf{T}_w,1)$ (cf. the notation in [5]), we have

$$\sum_{x\in k^\times}\chi(x+ax^{-1})=-\sum_{y\in k_2^\times,N_{2,1}y=a}\chi(y+y^q),$$

which is (1.3) of [2].

To obtain the value of $f_{\mathbf{T}}$ on $c_{{}^t\dot{w}(a)}$, we consider the following automorphism α on G. Let $w_0=(w_{0,ij})$ be the matrix in G, with $w_{0,ij}=\delta_{i+j,n+1}(-1)^{i-1}$ and put $\alpha(g)=({}^tg^{-1})^{w_0}$ for $g\in\mathbf{G}$. Then α is an involutive automorphism of $\mathbf{G}$, G, and U. It can be checked easily that $\psi\circ\alpha=\psi$. The extension of α to an automorphism of $\mathbb{C}G$ induces an automorphism of $\mathcal{H}$.

Noting that for an F-stable maximal torus $\mathbf{T}$, $\mathbf{T}$ and $\alpha(\mathbf{T})$ are G-conjugate, and using Theorem 2.3, we obtain without difficulty that

(3.4.1) $\quad f_{\mathbf{T}}(c_{\alpha(n)})(t) = f_{\alpha(\mathbf{T})}(c_n)(\alpha(t))$, and

(3.4.2) $\quad f_{\mathbf{T},\theta}(c_{\alpha(n)}) = f_{\alpha(\mathbf{T}),\theta\circ\alpha}(c_n)$.

Lemma 3.5. *We have*

$$f_{\mathbf{T},\theta}(c_{\alpha(\dot{w}(a))})=f_{\mathbf{T},\bar{\theta}}(c_{\dot{w}(a)}),$$

where $\bar{\theta}=\theta^{-1}$. *Therefore*

(3.5.1) $$f_{\mathbf{T},\theta}(c_{-{}^t\dot{w}(a)})=\varepsilon(\mathbf{T})\sum_{t\in T,\det t=(-1)^na^{-1}}\chi(\mathrm{Tr}\ t)\theta(t^{-1}).$$

Proof. From the preceding discussion, we have

$$\begin{aligned} f_{\mathbf{T},\theta}(c_{\alpha(\dot{w}(a))}) &= f_{\alpha(\mathbf{T}),\theta\circ\alpha}(c_{\dot{w}(a)}) \quad \text{(by the equation (3.4.2))} \\ &= \varepsilon(\alpha(\mathbf{T})) \sum_{t'\in\alpha(T),\det t'=a} \chi(\mathrm{Tr}\ t')\theta(\alpha(t')) \\ &= \varepsilon(\mathbf{T}) \sum_{t\in T,\det t=a^{-1}} \chi(\mathrm{Tr}\ t^{-1})\theta(t) \\ &= \varepsilon(\mathbf{T}) \sum_{t\in T,\det t=a} \chi(\mathrm{Tr}\ t)\theta(t^{-1}) \\ &= f_{\mathbf{T},\bar{\theta}}(c_{\dot{w}(a)}), \end{aligned}$$

by Theorem 3.3. The second assertion follows from this and $\alpha(\dot{w}(a)) = -{}^{t}(\dot{w}((-1)^n a^{-1}))$.

□

We remark that the equations (3.3.2) and (3.5.1), together with Theorem 4.2 in [3], generalize Theorem 4.1 in [2] to $GL_n(q)$.

The following theorem was proved by Kondo [11] for all irreducible characters of $G = GL_n(k)$, using the results of J. A. Green on the irreducible characters of G. Kondo stated the theorem in terms of Gauss sums of field extensions of k. Our theorem is stated in terms of character sums over a torus, and is proved using the Deligne-Lusztig theory [8].

Theorem 3.6. *Let ζ be an irreducible character of $G = GL_n(k)$ and let ζ be a component of $R_{\mathbf{T},\theta}$. Then the Gauss sum of the character ζ is given by*

$$\tau(\zeta) = \sum_{g\in G} \zeta(g)\chi(\mathrm{Tr}\,(g)) = \deg\zeta \mid G \mid_p \varepsilon(\mathbf{T}) \sum_{t\in T} \chi(\mathrm{Tr}\,(t))\theta(t).$$

Proof. We shall denote by $\rho_{\mathbf{T},\theta}$ the character of the virtual representation $R_{\mathbf{T},\theta}$. From ([13], §3) and ([8], Prop. 5.11) we have

$$\zeta = \sum_{[(\mathbf{T}',\theta')]} c_{(\mathbf{T}',\theta')}\rho_{\mathbf{T}',\theta'},$$

for some $c_{(\mathbf{T}',\theta')} \in \mathbb{Q}$, where $(\mathbf{T}',\theta')$ runs over members of the geometric conjugacy class of $(\mathbf{T},\theta)$. Since τ is additive (cf. [16]), we have

$$\tau(\zeta) = \sum_{[(\mathbf{T}',\theta')]} c_{(\mathbf{T}',\theta')}\tau(\rho_{\mathbf{T}',\theta'}).$$

By [loc.cit.,(1.2)], the Gauss sums of the virtual characters $\rho_{\mathbf{T}',\theta'}$ are given by

$$\tau(\rho_{\mathbf{T}',\theta'}) = \frac{|G|}{|T'|}\sum_{t'\in T'}\theta'(t')\chi(\mathrm{Tr}\,(t')).$$

Then by (3.4) we have

$$\varepsilon(\mathbf{T})\sum_{t\in T}\chi(\mathrm{Tr}\ t)\theta(t) = \varepsilon(\mathbf{T}')\sum_{t\in T'}\chi(\mathrm{Tr}\ t)\theta'(t).$$

for pairs $(\mathbf{T},\theta)$ and $(\mathbf{T}',\theta')$ in the same geometric conjugacy class. Therefore

$$\tau(\zeta) = \{\varepsilon(\mathbf{T})\sum_{t\in T}\theta(t)\chi(\mathrm{Tr}\ t)\}\{\sum_{[(\mathbf{T}',\theta')]} c_{(\mathbf{T}',\theta')}\varepsilon(\mathbf{T}')\frac{|G|}{|T'|}\}.$$

Since

$$\deg\zeta = \sum_{[(\mathbf{T}',\theta')]} c_{(\mathbf{T}',\theta')}\varepsilon(\mathbf{T}')\frac{|G|_{p'}}{|T'|},$$

the result follows. □

Corollary 3.7. *Let $\pi_{\mathbf{T},\theta}$ be an irreducible component of the Gelfand-Graev representation, associated with the representation $f_{\mathbf{T},\theta}$ of $\mathcal{H}$, for an arbitrary pair $(\mathbf{T},\theta)$ as in ([8], §10). Then we have*

$$f_{\mathbf{T},\theta}(\widetilde{e}) = \varepsilon(\pi_{\mathbf{T},\theta},\chi) = q^{-n/2}\varepsilon(\mathbf{T})\sum_{t\in T}\theta^{-1}(t)\chi(\mathrm{Tr}\ t).$$

Proof. We have

$$f_{\mathbf{T},\theta}(\widetilde{e}) = \varepsilon(\pi_{\mathbf{T},\theta},\chi) = \frac{q^{-n^2/2}}{\deg\pi}\sum_{g\in G}\chi^*_{\mathbf{T},\theta}(g)\chi(\mathrm{Tr}\,(g)),$$

by (2.4), where $\chi^*_{\mathbf{T},\theta}$ is the character of the contragredient representation $\pi^*_{\mathbf{T},\theta}$. By ([3], Theorem (2.1)), $\pi_{\mathbf{T},\theta}$ is a component of $R_{\mathbf{T},\theta}$, and is associated with the geometric conjugacy class $[(\mathbf{T},\theta)]$. Then $\chi_{\mathbf{T},\theta}$ is a linear combination of Deligne-Lusztig characters, so $\chi^*_{\mathbf{T},\theta} = \chi_{\mathbf{T},\theta^{-1}}$ as this is true for the Deligne-Lusztig characters. The Corollary now follows from the preceding Theorem. □

As an application of Lemma 3.5 and Corollary 3.7, we give a formula for the twisted Fourier transform of the identity element e of $\mathcal{H}$ in terms of the standard basis elements of $\mathcal{H}$. It would be interesting to know a version of this formula for other types of finite reductive groups.

We recall the notation for the twisted Fourier transform

$$\widetilde{h} = \sum_{g \in G} \widehat{\Phi}(g) g^{-1} \in \mathcal{H} \text{ for } h = \sum \Phi(g) g \in \mathcal{H},$$

with Φ vanishing outside G.

Theorem 3.8. *We have*

$$\widetilde{e} = q^{-n/2} \sum_{a \in k^\times} c_{-^t\dot{w}(a)},$$

and

$$\widetilde{h} = q^{-n/2} \left(\sum_{a \in k^\times} c_{-^t\dot{w}(a)} \right) h,$$

for all $h \in \mathcal{H}$.

Proof. By the above Corollary together with equation (3.5.1), it follows that

$$f_{\mathbf{T},\theta}(\widetilde{e}) = q^{-n/2} f_{\mathbf{T},\theta}\big(\sum_{a \in k^\times} c_{-^t\dot{w}(a)} \big),$$

for all pairs $(\mathbf{T}, \theta)$, and the first equation follows. The second equation follows from (1.6). $\square$

§4. Gauss sums of unipotent characters of $SL_n(k)$

For the definitions and notation we refer to [16]. We first notice that by Theorem 3.3 above and Theorem 1.2 of [16] we have

$$\tau(R_{\mathbf{T},\theta}) = [G_0 : T] \varepsilon(\mathbf{T}) f_{\mathbf{T},\theta}(c_{\dot{w}}),$$

where $G_0 = SL_n(k)$ and $\dot{w} = \dot{w}(1)$. Let

$$S = \sum_{\substack{x_1, x_2, \ldots, x_n \in k \\ x_1 \cdots x_n = 1}} \chi(x_1 + \cdots + x_n).$$

Then we have

Theorem 4.1. *Let ρ be any irreducible character of $W = S_n$. For the unipotent character R_ρ of $SL_n(k)$ defined by*

$$R_\rho = \frac{1}{|W|}\sum_{w\in W} \mathrm{Tr}\rho(w) R_{\mathbf{T}_w,1},$$

we have

$$w(R_\rho) = q^{n(n-1)/2}S.$$

Proof. If $\mathbf{T}_0$ is a maximal split torus and $\mathbf{T}$ is an arbitrary F-stable maximal torus in $\mathbf{G}_0$, then the pairs $(\mathbf{T}_0, 1)$ and $(\mathbf{T}, 1)$ are geometrically conjugate. Corollary 3.4 holds for G_0, and we have $S = f_{\mathbf{T},1}(c_{\dot{w}})$, since $S = f_{\mathbf{T}_0,1}(c_{\dot{w}})$. Therefore, by the additivity of τ, we have

$$\begin{aligned}
\tau(R_\rho) &= \frac{1}{|W|}\sum_{w\in W} \mathrm{Tr}\,\rho(w)\tau(R_{\mathbf{T}_w,1}) \\
&= \frac{1}{|W|}\sum_{w\in W} \mathrm{Tr}\,\rho(w)[G_0 : T_w]\varepsilon(\mathbf{T}_w)S \\
&= \frac{q^{n(n-1)/2}S}{|W|}\sum_{w\in W} \mathrm{Tr}\,\rho(w)R_{\mathbf{T}_w,1}(1) \\
&= q^{n(n-1)/2}SR_\rho(1).
\end{aligned}$$

Since $w(R_\rho) = R_\rho(1)^{-1}\tau(R_\rho)$, we have proved the assertion in the theorem. □

We remark that if ρ is the trivial representation, the above result is proved in [12].

§5. On the norm map $\Delta : \mathcal{H}' \to \mathcal{H}$

We mention here another application of the preceding results to a computation of the norm map $\Delta : \mathcal{H}' \to \mathcal{H}$ on $\widetilde{e}' \in \mathcal{H}'$, in the case of $\mathbf{G} = GL_n(\bar{k})$. In this case the norm map is a homomorphism of algebras from the Hecke algebra $\mathcal{H}'$ of a Gelfand-Graev representation of $G' = GL_n(k')$, $k' = k_m = \mathbb{F}_{q^m}$, to the Hecke algebra $\mathcal{H}$ of a Gelfand-Graev representation of $G = GL_n(k)$ (cf. [6]) and it is known to be surjective. Moreover it gives a correspondence of representations of Hecke algebras (or spherical functions) $f_{\mathbf{T},\theta} \to f_{\mathbf{T},\theta}\circ\Delta$. Let $\mathbf{T}$ be an F-stable maximal torus, $T = \mathbf{T}^F, T' = \mathbf{T}^{F^m}$, $N_{\mathbf{T}} : T' \to T$ be the (usual) norm map, and let $\widetilde{N}_{\mathbf{T}}$ be the extension of $N_{\mathbf{T}}$ to a homomorphism of group algebras of T' and T. Then the norm map Δ is characterized as the unique linear map $\Delta : \mathcal{H}' \to \mathcal{H}$ with the property that for each F-stable maximal torus $\mathbf{T}$, one has

$$f_{\mathbf{T}} \circ \Delta = \widetilde{N}_{\mathbf{T}} \circ f'_{\mathbf{T}}.$$

Theorem 5.1. *Let e' be the identity element of $\mathcal{H}'$. Then*

$$\Delta(\widetilde{e}') = (-1)^{n(m-1)}\widetilde{e}^m.$$

Proof. In the discussion to follow, we shall use the notation k_m for the extension of k of degree m, along with $\mathrm{Tr}\,_{a,b} = \mathrm{Tr}\,_{k_a/k_b}$ and $N_{a,b} = N_{k_a/k_b}$ for trace and norm maps of field extensions, as in [5], where b is a divisor of a.

By the definition of the norm map, it is enough to show that

$$\widetilde{N}_{\mathbf{T}}(f'_{\mathbf{T}}(\widetilde{e}')) = f_{\mathbf{T}}((-1)^{n(m-1)}\widetilde{e}^m),$$

for each F-stable maximal torus $\mathbf{T}$. From the known structure of the F-stable maximal tori, it is not difficult to verify that it is enough to prove the above formula in case $\mathbf{T}$ is isomorphic to $\{\mathrm{diag}(a_1, ..., a_n) \mid a_i \in \overline{k}^{\times}\}$ where the Frobenius map F acts as $F(\mathrm{diag}(a_1, ..., a_n)) = \mathrm{diag}(a_2^q, ..., a_n^q, a_1^q)$. Hence T is isomorphic to $k_n^{\times}$ and T' is isomorphic to $(k_{nm/d}^{\times})^d$, with $d = g.c.d.(m, n)$. Under this identification of T and T', we have

$$\mathrm{Tr}\,(t') = \mathrm{Tr}\,_{nm/d,m}(a'_1 + \cdots + a'_d)$$

and

$$N_{\mathbf{T}}(t') = N_{nm/d,n}(a'_1 {a'_2}^q \cdots {a'_d}^{q^{d-1}})$$

with $t' = (a'_1, ..., a'_d) \in (k_{nm/d}^{\times})^d$. Let $\chi' = \chi \circ \mathrm{Tr}\,_{m,1}$ and $\chi_n = \chi \circ \mathrm{Tr}\,_{n,1}$. Finally, we note that $\varepsilon'(\mathbf{T}) = (-1)^{\sigma'(\mathbf{G}) - \sigma'(\mathbf{T})} = (-1)^{n-d}$, where $\sigma'(\mathbf{G})$, $\sigma'(\mathbf{T})$ are the k'-ranks of $\mathbf{G}$ and $\mathbf{T}$, and $\varepsilon(\mathbf{T}) = (-1)^{n-1}$. Then for each irreducible representation θ of T we have by Corollary 3.7,

$$\begin{aligned}
&\widetilde{\theta}(\widetilde{N}_{\mathbf{T}}(f'_{\mathbf{T}}(\widetilde{e}')))\\
= \; & q^{-nm/2}\varepsilon'(\mathbf{T}) \sum_{t' \in T'} \theta^{-1}(N_{\mathbf{T}}(t'))\chi'(\mathrm{Tr}\,(t'))\\
= \; & q^{-nm/2}(-1)^{n-d} \sum_{a'_1, \ldots, a'_d} \theta^{-1}(N_{nm/d,n}(a'_1 {a'_2}^q \cdots))\\
&\qquad \times \chi_n(\mathrm{Tr}\,_{nm/d,n}(a'_1 + \cdots + a'_d))\\
= \; & q^{-nm/2}(-1)^{n-d} \prod_{i=0}^{d-1} G(\chi_n \circ \mathrm{Tr}\,_{nm/d,n}, \theta^{-1} \circ N_{nm/d,n} \circ F_q^i)\\
= \; & q^{-nm/2}(-1)^{n-d} G(\chi_n \circ \mathrm{Tr}\,_{nm/d,n}, \theta^{-1} \circ N_{nm/d,n})^d,
\end{aligned}$$

where $F_q(a) = a^q$ for $a \in k_{nm/d}^{\times}$ and $G(\chi_n \circ \mathrm{Tr}_{\,nm/d,n}, \theta \circ N_{nm/d,n})$ is the Gauss sum over $k_{nm/d}$ with $\chi_n \circ \mathrm{Tr}_{\,nm/d,n}$ (resp. $\theta \circ N_{nm/d,n}$) as its additive (resp. multiplicative) character. Now the Davenport-Hasse theorem implies

$$-G(\chi_n \circ \mathrm{Tr}_{\,nm/d,n}, \theta^{-1} \circ N_{nm/d,n}) = (-G(\chi_n, \theta^{-1}))^{m/d}.$$

Thus we have

$$\widetilde{\theta}(\widetilde{N}_{\mathbf{T}}(f'_{\mathbf{T}}(\widetilde{e}'))) = q^{-nm/2}(-1)^{m+n}G(\chi_n, \theta^{-1})^m.$$

On the other hand we have $f_{\mathbf{T},\theta}(\widetilde{e}) = q^{-n/2}(-1)^{n-1}G(\chi_n, \theta^{-1})$, and the result follows. □

As a corollary we obtain what may be viewed as an extension of the Davenport-Hasse relation for Gauss sums of field extensions to Gauss sums of irreducible components of the Gelfand-Graev representation of $GL_n(k')$ and $GL_n(k)$.

Corollary 5.2. *Keep the notation of the previous theorem and Corollary 3.7. For each irreducible representation θ of T, we have*

$$\varepsilon(\pi'_{\mathbf{T},\theta\circ\widetilde{N}_{\mathbf{T}}}, \chi') = (-1)^{n(m-1)}\varepsilon(\pi_{\mathbf{T},\theta}, \chi)^m,$$

for components of the Gelfand-Graev representations of $GL_n(k')$ and $GL_n(k)$ respectively which correspond by the norm map Δ.

The proof is immediate by the previous Theorem and Corollary 3.7.

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Cellular algebras and diagram algebras in representation theory

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Abstract.

We discuss a circle of ideas for addressing problems in representation theory using the philosophy of cellular algebras, applied to algebras described in terms of diagrams. Cellular algebras are often generically semisimple, and have non-semisimple specialisations whose representation theory may be discussed by solving problems in linear algebra, which are formulated in the semisimple context, and are therefore tractable in some significant cases. This applies in particular to certain "Temperley-Lieb" quotients of Hecke algebras, both finite dimensional and affine, which may be described in terms of bases consisting of diagrams. This leads to the application of cellular algebra theory to an analysis of their representation theory, with corresponding consequences for the relevant Hecke algebras. A particular case is the determination of the decomposition numbers of some standard modules for the affine Hecke algebra of GL_n. These decomposition numbers are known (by Kazhdan-Lusztig) to be expressible in terms of the dimensions of the stalks of certain intersection cohomology sheaves, and we discuss how our results imply the rational smoothness of some varieties associated with quiver representations.

§1. Introduction

We discuss a circle of ideas for addressing problems in representation theory using the philosophy of cellular algebras, applied to algebras described in terms of diagrams. Cellular algebras are often generically semisimple, and have non-semisimple specialisations whose representation theory may be discussed by solving problems in linear algebra, which are formulated in the semisimple context, and are therefore tractable in

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some significant cases. This applies in particular to certain "Temperley-Lieb" quotients of Hecke algebras, both finite dimensional and affine, which may be described in terms of bases consisting of diagrams. This leads to the application of cellular algebra theory to an analysis of their representation theory, with corresponding consequences for the relevant Hecke algebras. A particular case is the determination of the decomposition numbers of some standard modules for the affine Hecke algebra of GL_n. These decomposition numbers are known (by Kazhdan-Lusztig) to be expressible in terms of the dimensions of the stalks of certain intersection cohomology sheaves, and we discuss how our results imply the rational smoothness of some varieties associated with quiver representations.

This paper is organised as follows. In §2 we recall the basic notion of a cellular algebra, and how the cellular structure may be used to analyse the representations of its non-semisimple specialisations. The finite dimensional Temperley-Lieb algebras of type A_n and B_n are introduced with their cellular structure and cell modules. In §3 we discuss the infinite dimensional affine Temperley-Lieb algebra $TL_n^a(q)$ from this point of view, recalling the results of [17]. In §4 we describe the interrelationships among these algebras. This is done by relating them all to the affine Hecke algebra of GL_n. An understanding of the interconnections among the underlying generalised Artin braid groups is useful here, to make the connection. In §5 we discuss the lifting of cell modules from $TL_n^a(q)$ to modules for the affine Hecke algebra, and explain the results of [18] which identify these liftings in the Grothendieck ring as "standard modules". In §6 implications for the decomposition numbers of certain standard modules are treated, and finally, in §7, we explain their interpretation in terms of the intersection cohomology of certain quiver varieties.

The approach outlined here could also be used to study the "modular representation theory" of $\widehat{H_n^a}(q)$, but we do not do this here. The results may also be interpreted in terms of a generalisation to the non-generic case of the "multisegments" of Zelevinsky and Bernstein. Since the "annular algebras" of V. Jones ([20]) are quotients of the algebras $T_n^a(q)$, their representation theory may be thought of as a subset of the story below. Hence our work throws light on the connection between the work of Jones on link invariants (cf. [21]) and affine Hecke algebras. There are also close connections between this work and subfactors of C^*-algebras (see [22, 23]).

§2. Cellular algebras and Temperley-Lieb algebras

Cellular algebras were first defined in [15] in terms of the existence of a basis with specified multiplicative properties. Subsequently, there has been a significant literature (cf. [25], [26], [27], [28] and [13]) in which structural definitions have been given, in terms of the existence of certain filtrations, and in which cellular algebras have been compared with quasi-hereditary algebras; in particular much is now known about their characterisation in terms of global dimension, Cartan matrix and representation type. The main point of a cellular structure is that it is preserved under specialisation, so that cellularity is a particularly effective way of studying algebras which are not semisimple, but are deformations of semisimple algebras.

In this work, we shall use the original definition. Let R be a commutative ring with 1, and let A be an R-algebra which is free over R, and which we assume for the moment to be of finite rank (although many of the concepts will be applied when A has infinite rank). To specify a cellular structure for A, we require a "cell datum", which consists of: (i) a poset $\mathcal{T}$, (ii) for each $t \in \mathcal{T}$, a set $M(t)$, and (iii) an injection $\amalg_{t\in\mathcal{T}} M(t) \times M(t) \xrightarrow{C} A$, whose image is an R-basis $\{C^t_{S,T}\}$ of A which satisfies

$$(2.1)\qquad \begin{gathered} aC^t_{S,T} = \sum_{S'\in M(t)} r_a(S',S)C^t_{S',T} + \text{ lower terms} \\ (a \in A, r_a(S',S) \in R), \end{gathered}$$

where "lower terms" indicates a linear combination of basis elements $C^{t'}_{S'',T''}$ with $t' < t$. In addition, one requires that the map $C^t_{S,T} \mapsto C^t_{T,S}$, extended R-linearly, be an anti-automorphism $a \mapsto a^*$ of A.

We shall review briefly the representation theory of cellular algebras. To start with, for each $t \in \mathcal{T}$, one has the *cell module* $W(t)$; this has R-basis $\{C_S \mid S \in M(t)\}$, and A acts on the left by the rule

$$(2.2)\qquad a.C_S = \sum r_a(S',S)C_{S'} \text{ for } a \in A,\ S \in M(t).$$

It follows from 2.1 that 2.2 defines an action.

The cell module $W(t)$ has a natural R-bilinear form $\phi_t : W(t) \times W(t) \longrightarrow R$, defined by

$$(2.3)\qquad (C^t_{S,T})^2 = \phi_t(C_S, C_T)C^t_{S,T} + \text{ lower terms.}$$

It is easily shown ([15]) that the form ϕ_t is symmetric and invariant under the action of A, i.e. that for any two elements $x, y \in W(t)$ and

$a \in A$, we have $\phi_t(x,y) = \phi_t(y,x)$ and $\phi_t(a.x,y) = \phi_t(x,a^*.y)$. An immediate consequence is that the radical $\mathrm{Rad}(t) = \mathrm{Rad}\,\phi_t = \{x \in W(t) \mid \phi_t(x,y) = 0 \text{ for all } y \in W(t)\}$ of ϕ_t is an A-submodule of $W(t)$. Write $L(t) := W(t)/\mathrm{Rad}(t)$; clearly $L(t)$ is zero if and only if the form $\phi_t = 0$.

The next statement summarises the representation theory of A when R is a field.

Proposition 2.4 ([15]). *Maintain the above notation, and in addition assume that R is a field. Let $\mathcal{T}^0$ be the set of $t \in \mathcal{T}$ such that $L(t) \neq 0$ (or equivalently $\phi_t \neq 0$). Then* (i) *The modules $L(t)$ $(t \in \mathcal{T}^0)$ are absolutely irreducible and form a complete set of representatives of the distinct isomorphism classes of irreducible A-modules.*

(ii) *For $s \in \mathcal{T}, t \in \mathcal{T}^0$ let d_{st} be the multiplicity of $L(t)$ in $W(s)$. Then the decomposition matrix $D = (d_{st})$ is upper unitriangular. In particular* $\mathrm{Hom}(W(t_1), W(t_2)) = 0$ *unless $t_1 \geq t_2$.*

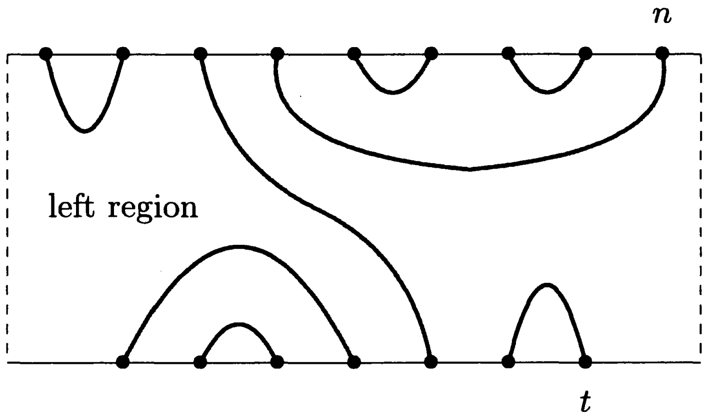
(iii) *For $s_1, s_2 \in \mathcal{T}^0$, let c_{s_1,s_2} be the multiplicity if the irreducible module $L(s_2)$ in the projective cover $P(s_1)$ of $W(s_1)$. If C is the (Cartan) matrix (c_{s_1,s_2}) then $C = D^tD$, where D^t denotes the transpose of D.*

(iv) *The cell module $W(t)$ is irreducible if and only if, for all $t' \in \mathcal{T}, t' \neq t$,* $\mathrm{Hom}(W(t'), W(t)) = 0$. *Equivalently, the form ϕ_t is non-degenerate on $W(t)$.*

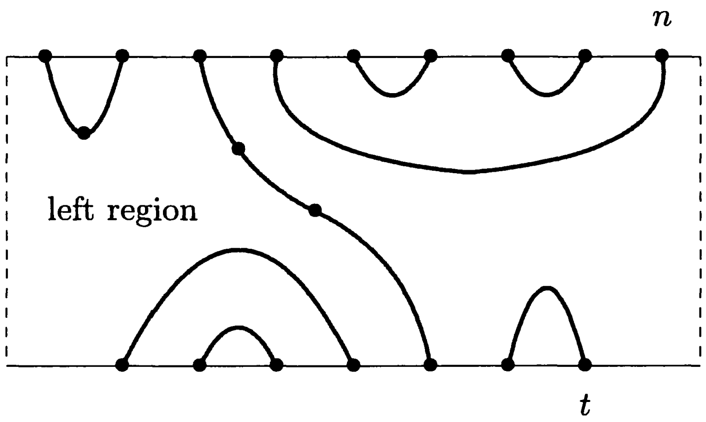
(v) *The following assertions are equivalent. The algebra A is semisimple. Each cell module is absolutely irreducible. The forms ϕ_t are all non-degenerate. There are no non-trivial homomorphisms between distinct cell modules.*

We shall now see how these concepts apply to the Temperley-Lieb algebras of type A and B. Let R be a commutative ring with 1; write $R^\times$ for the group of invertible elements of R and let $q, Q \in R^\times$. For any invertible element $x \in R$, write $\delta_x = -(x + x^{-1})$. Our notation for parameters, which may appear arbitrary, is chosen with a view to the links with Hecke algebras described below. The usual Temperley-Lieb algebra $TL_n(q)$ may be described in terms of planar diagrams (see [20], [10] or [15]) in a way which is well known. The Temperley-Lieb algebra $TLB_n(q,Q)$ of type B_n has a corresponding description in terms of "marked diagrams" (cf. [42], [36], [37]). When the base ring is an algebraically closed field, $TLB_n(q,Q)$ is sometimes known as the "blob algebra" (see [6]). The algebra $TL_n(q)$ is realised as the subalgebra of $TLB_n(q,Q)$ which is spanned by unmarked diagrams. We describe now $TLB_n(q,Q)$ as a cellular algebra, using the language of [16] for diagrams.

If t, n are positive integers of the same parity, a finite (planar) diagram $\mu : t \longrightarrow n$ is represented by a set of non-intersecting arcs which are contained in the "fundamental rectangle" (see below). These arcs divide the fundamental rectangle into regions, among which there is a unique "left region" as shown below.



A *marked diagram* is a (finite planar) diagram, where the interior of the boundary arcs of the leftmost region may be marked with one or more • symbols ("marks") (see below).



The R-linear combinations of unmarked diagrams from t to n constitute the morphisms in the Temperley-Lieb category $\mathbf{T}$, where the objects are the non negative integers $\mathbb{Z}_{\geq 0}$. If t, n respectively denote the number of bottom and top nodes in a diagram D, then D is a morphism from t to n in $\mathbf{T}$. Composition of morphisms corresponds to concatenation of diagrams, with closed loops being deleted and replaced by the scalar δ_q. In the composition $D_1 D_2$ of diagrams D_1 and D_2, D_1 is placed above

D_2. This is consistent with morphisms being composed from right to left. The endomorphisms of $n \in \mathbf{T}$ form an algebra $\mathrm{Hom}_{\mathbb{T}}(n,n) \cong TL_n(q)$. Marked diagrams may be similarly concatenated according to rules we shall now state; this produces a new category, $\mathbb{TB}$, the Temperley-Lieb category of type B. The composition rules are as follows.

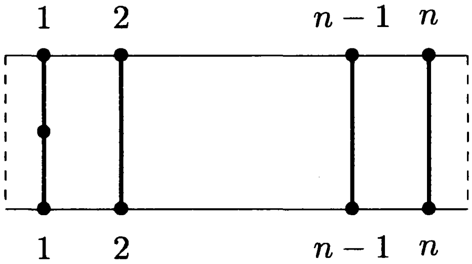
A marked diagram is *proper* if it has no loops and each arc has at most one mark. The following rules reduce the concatenation of any two diagrams to an R linear combination of proper diagrams:

2.5. (i) *If μ is a diagram and L is a loop with no marks, $\mu \amalg L = \delta_q \mu$.*

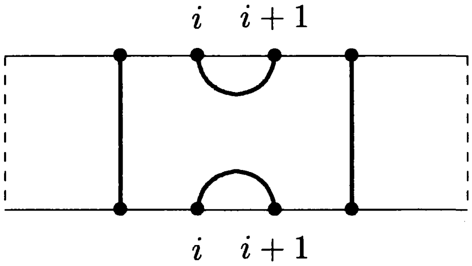
(ii) *If, in (i), L has one mark, $\mu \amalg L = \kappa\mu$, where $\kappa = \frac{q}{Q} + \frac{Q}{q}$.*

(iii) *If some arc of μ has more than one mark and μ' is the diagram obtained by removing a mark from the arc concerned, then $\mu = \delta_Q \mu'$.*

Now consider the following marked diagrams from n to n.



$c_0 : n \to n$



$c_i : n \to n,\ 1 \leq i \leq n-1$

The next result describes $TLB_n(q,Q)$ algebraically.

Proposition 2.6 (cf. [18, 5.8]). (i) *The Temperley-Lieb algebra $TLB_n(q,Q)$ is generated as R-algebra by $\{c_0, c_1, \ldots, c_{n-1}\}$ subject to the relations*

$$\begin{aligned} c_0^2 &= \delta_Q c_0 \\ c_i^2 &= \delta_q c_i \text{ for } 1 \leq i \leq n-1 \\ c_i c_{i+1} c_i &= c_i \text{ for } 1 \leq i \leq n-2 \\ c_i c_{i-1} c_i &= c_i \text{ for } 2 \leq i \leq n-1 \\ c_i c_j &= c_j c_i \text{ if } |i-j| > 1,\ 0 \leq i,j \leq n-1 \\ c_1 c_0 c_1 &= \kappa c_1, \end{aligned} \tag{2.7}$$

where $\kappa = \frac{q}{Q} + \frac{Q}{q}$ and $\delta_x = -(x + x^{-1})$ for $x \in R^\times$.

(ii) *The elements $c_1, \ldots, c_{n-1}$ generate a subalgebra of $TLB_n(q,Q)$ isomorphic to the usual Temperley-Lieb algebra $TL_n(q)$ of type A_{n-1}.*

The stated relations are easily seen to hold by 2.5. The fact that all relations are consequences of these depends on the connection between the Hecke algebra description of $TLB_n(q,Q)$ (§4 below) and the diagrammatic description here. The proof may be found in [42, Satz (4.5) p. 77].

We now specify the various elements of a cellular structure (see above) for $TLB_n(q,Q)$. Take $\mathcal{T} = \{t \in \mathbb{Z} \mid |t| \leq n,\ t \equiv n \pmod 2\}$, partially ordered as follows: $t \leq s$ if $|t| < |s|$ or $|t| = |s|$ and $t \leq s$.

To define the sets $M(t)$, first take $t \in \mathcal{T}, t \geq 0$. Then $M(t)$ is the set of monic diagrams $D : t \to n$ with no marked through strings, where "monic" means that there are t through strings, as in [GL2], where it is shown that this is equivalent to D being a monic morphism in the category-theoretic sense. In general, let $M(t) = M(|t|)$. Then $C : M(t) \times M(t) \to TLB_n(q,Q)$ is defined as follows. Let $S, T \in M(t)$. For $t \geq 0$, define $C^t_{S,T} = S \circ T^*$, where * denotes reflection in a horizontal axis. For $t < 0$, define $C^t_{S,T} = S \circ c_o \circ T^*$, where $c_0 = c_0(t) : t \to t$ is the generator shown below 2.5. This is the diagram $S \circ T^*$, with the leftmost through string marked. The cellular axioms above, in particular 2.1 are easily checked.

The cell modules $W_t(n)$ are now defined in complete analogy with the $W_{t,z}(n)$ of [GL2]. For any $t \in \mathcal{T}$, $W_t(n)$ has basis $M(t)$. If $t \geq 0$, $TLB_n(q,Q)$ acts via composition in the category $\mathbb{TB}$; explicitly, if $D \in M(t)$ and $\omega \in TLB_n(q,Q)$, then $\omega.D = \omega D$ (composition in $\mathbb{TB}$) if $\omega \circ D \in M(t)$, and $\omega.D = 0$ otherwise.

For $t < 0$, one may think of $W_t(n)$ as having basis the set $\{D \circ c_0(t) \mid D \in M(|t|)\}$ of monic diagrams $: t \to n$ in $\mathbb{TB}$ with the leftmost through string marked. Then the action of $TLB_n(q,Q)$ is essentially multiplication in $\mathbb{TB}$, as in the case $t \geq 0$. Thus if $t < 0$, then $\omega.(D \circ c_0(t)) = 0$ if $\omega \circ D$ is not monic, while $\omega.(D \circ c_0) = \omega D c_0$ (composition in $\mathbb{TB}$) if $\omega \circ D$ is monic.

It is easily seen (cf. [42]) that the dimension (i.e. rank over R) of $W_t(n)$ is $\binom{n}{\frac{n-|t|}{2}}$.

For each integer i with $1 \leq i \leq n-1$, the diagrams $\{c_0, c_1, \ldots, c_{i-1}\}$ generate an algebra isomorphic to $TLB_i(q,Q)$, and we shall require information concerning the restriction of the cell modules $W_t(n)$ to the subalgebras $TLB_i(q,Q)$. Let us first consider the restriction of $W_t(n)$ from $TLB_n(q,Q)$ to $TLB_{n-1}(q,Q)$ when $t \in \mathcal{T}, t \geq 0$. Now $TLB_{n-1}(q,Q)$ is spanned by those diagrams in $TLB_n(q,Q)$ with a string joining the rightmost dots in the top and bottom rows. The basis elements of $W_t(n)$ divide naturally into those which have a through string to the rightmost top dot, and those which do not. The former evidently span a

$TLB_{n-1}(q,Q)$- submodule isomorphic to $W_{t-1}(n-1)$. A diagram in the latter set may be identified with a monic diagram $t+1 \to n-1$ by "pulling down" the rightmost dot from the top to the bottom row. In this way, the quotient of the $TLB_{n-1}(q,Q)$-module $W_t(n)$ by $W_{t-1}(n-1)$ is easily identified as $W_{t+1}(n-1)$. Using a similar argument for the case $t \leq 0$ we obtain

Proposition 2.8. *Let $t \in \mathbb{Z}$, $0 \leq |t| \leq n$, $n+t \in 2\mathbb{Z}$. Assume $\delta_q \neq 0$.*

(i) *If $t \geq 0$, we have a short exact sequence*

$$(2.9)\quad 0 \to W_{t-1}(n-1) \to Res^{TLB_n(q,Q)}_{TLB_{n-1}(q,Q)} W_t(n) \to W_{t+1}(n-1) \to 0.$$

(ii) *If $t < 0$, we have a short exact sequence*

$$(2.10)\quad 0 \to W_{t+1}(n-1) \to Res^{TLB_n(q,Q)}_{TLB_{n-1}(q,Q)} W_t(n) \to W_{t-1}(n-1) \to 0.$$

Here we adopt the convention that $W_j(k) = 0$ if $|j| > k$.

Repeated application of 2.8 yields

Corollary 2.11. *Let $t \in \mathbb{Z}$, $0 \leq |t| \leq n$, $n+t \in 2\mathbb{Z}$ and suppose $t \geq 0$. There is a filtration of $W_t(n)$ by R-submodules $W^{(i)}, i = n, n-1, n-2, \ldots, 1$ as in 2.12 below.*

(2.12)

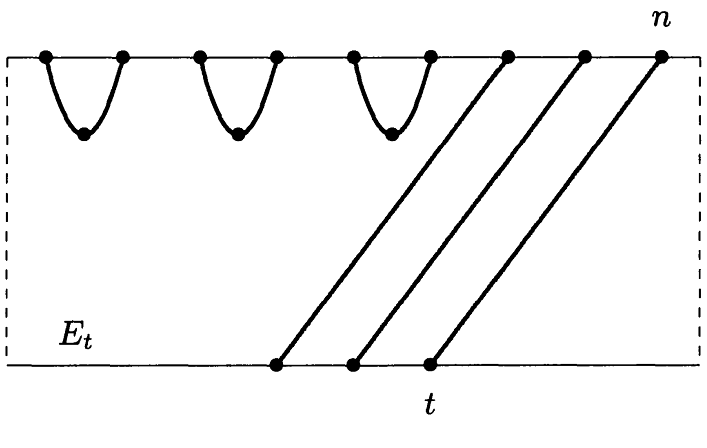
$$\begin{gathered} W_t(n) \supset W_{t-1}(n-1) \supset \cdots \supset W_0(n-t) \supset W_{-1}(n-t-1) \supset \\ W_0(n-t-2) \supset W_{-1}(n-t-3) \supset \cdots \supset W_0(2) \supset W_{-1}(1). \end{gathered}$$

Thus

$$(2.13)\qquad W^{(i)} \cong \begin{cases} W_{t-n+i}(i) \text{ if } n-t \leq i \leq n \\ W_0(i) \text{ if } i \text{ is even and } 0 \leq i \leq n-t \\ W_{-1}(i) \text{ if } i \text{ is odd and } 0 \leq i \leq n-t. \end{cases}$$

For each $i = 1, 2, 3, \ldots, n$, $W^{(i)}$ is a $TLB_i(q,Q)$-submodule of $W_t(n)$.

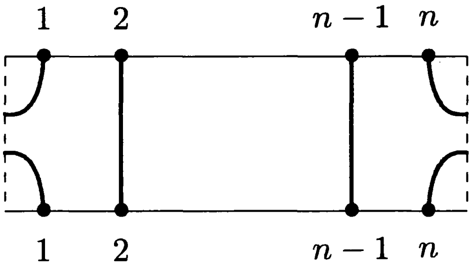
To illustrate the construction in 2.11, we note that the diagram $E_t = E_{t,n}$ below lies in each of the submodules $W^{(i)}$.



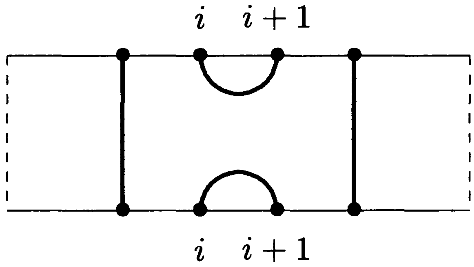
Given the details of the cellular structure of $TLB_n(q, Q)$, all the results of Proposition 2.4 are available. However we turn next to the affine Temperley-Lieb algebra.

§3. The diagram version of the affine Temperley-Lieb algebra

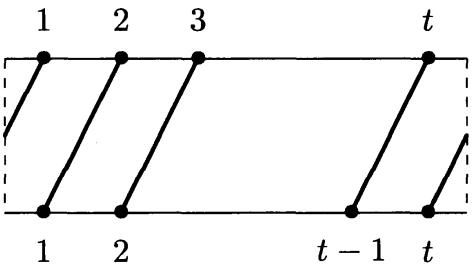
There are several algebras which could be (and have been) called the "affine Temperley-Lieb algebra". In this section we discuss the diagrammatic version $T_n^a(q)$, which was introduced in [16, 2.7] as the algebra of endomorphisms of an object of the affine Temperley-Lieb category $\mathbb{T}^a$ (see [GL2, (2.5)]), and whose representation theory was completely analysed in *loc. cit.* using cellular theory, although the algebras are infinite dimensional. It also occurs independently in the work of Green [12] and Fan–Green [8]. We maintain the notation of the last section for R, q, etc. Details may be found in [GL2], but a good approximation to the picture is obtained if one thinks of affine diagrams as arcs drawn on the surface of a cylinder joining $2n$ marked points, n on each circle component of the boundary, in pairs. The arcs must not intersect, and diagrams are multiplied by concatenation in the usual way. These diagrams are represented by periodic diagrams drawn between two horizontal lines, each diagram being determined by the "fundamental rectangle", from which the cylinder is obtained by identifying vertical edges. In this interpretation, the generators $\{f_1, \ldots, f_n, \tau_n\}$ of $T_n^a(q)$ are represented by the diagrams shown below.



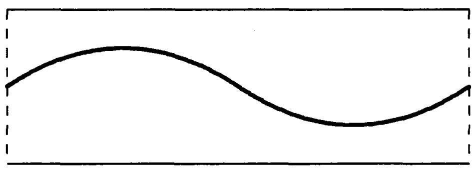
$f_n : n \to n$



$f_i : n \to n,\ 1 \le i \le n-1$



$\tau_t : t \to t$



$\tau_0 : 0 \to 0$

The elements $\{f_1, \dots, f_n\}$ of the algebra $T_n^a(q)$ satisfy the relations 3.1 below.

$$
\begin{aligned}
&f_i^2 = \delta_q f_i \\
(3.1) \qquad &f_i f_{i\pm 1} f_i = f_i \\
&f_i f_j = f_j f_i \text{ if } |i-j| \ge 2 \text{ and } \{i,j\} \ne \{1,n\},
\end{aligned}
$$

Further, $\tau_n f_i \tau_n^{-1} = f_{i+1}$, where the index is taken mod n. We shall sometimes omit the subscript from the "twist" τ_n when there is no ambiguity.

The reduction of a concatenation of affine diagrams to a linear combination of diagrams is simpler that in the case of $TLB_n(q,Q)$: circles which circumnavigate the cylinder (these are called infinite in [16]) remain; they are simply powers of τ_0. If D is an affine diagram and C is a finite or contractible closed loop, then in the category $\mathbb{T}^a$, $D \amalg C = \delta_q D$.

In [16], we defined cell modules $W_{t,z}(n)$ for the algebra $T_n^a(q)$, (where $t \in \mathbb{Z}, 0 \le t \le n, t+n \in 2\mathbb{Z}$, and $z \in R^\times$) and when R is an algebraically closed field of characteristic zero, completely determined their composition factors and multiplicities. We briefly review this material now.

Recall [16] that an affine diagram from t to n is monic if it has t through strings. It follows that $t \le n$ and $t \equiv n (\text{mod } 2)$. For such a positive integer t, let X_t be the $T_n^a(q)$-module with basis *all* monic affine diagrams:$t \longrightarrow n$, with $T_n^a(q)$ action given by composition in the category $\mathbb{T}^a$, modulo diagrams with fewer than t through strings. Thus X_t may be thought of as a quotient of the left $T_n^a(q)$-module $\mathrm{Hom}_{\mathbb{T}^a}(t,n)$ by the submodule spanned by diagrams with fewer than t through strings. Let $z \in R^\times$. The $T_n^a(q)$ module $W_{t,z}(n)$ is defined as the quotient of X_t by the ideal generated by the set

$$I_z := \{\gamma\tau_t - \chi_z(t)\gamma\} \tag{3.2}$$

over all affine diagrams $\gamma \in X_t$, where $\chi_z(t) = z$ if $t \neq 0$ and $\chi_z(0) = z + z^{-1}$.

While the module X_t is evidently of infinite rank over R, $W_{t,z}(n)$ is shown in [16, §2] to be free of rank $\binom{n}{\frac{n-t}{2}}$ over R. A basis of $W_{t,z}(n)$ is provided by the *standard* diagrams (see [16, 1.7, 2.7]) from t to n in $\mathbb{T}^a$.

The R-linear map $w \mapsto w^*$, where d^* denotes the reflection of a diagram $d \in \mathbb{T}^a$ in a horizontal line defines an involution on the category $\mathbb{T}^a$ (i.e. an involutory functor from $\mathbb{T}^a$ to $(\mathbb{T}^a)^{\mathrm{opp}}$), which induces an anti-automorphism of $T_n^a(q)$ for each n. Suppose μ, ν are standard diagrams from t to n. Let $\phi_{t,z} : W_{t,z}(n) \times W_{t,z^{-1}}(n) \longrightarrow R$ be the R-linear extension of the map $(\mu,\nu) \mapsto \phi_{t,z}(\mu,\nu) = \chi_z(\nu^*\mu)$, where $\chi_z : T_t^a(q) \to R$ is the R-linear map which annihilates non-monic diagrams and takes τ_t^i to χ_z^i (see 3.2). Then $\phi_{t,z}$ is bilinear and ([16, (2.7)]) invariant in the sense that $\phi_{t,z}(w\mu,\nu) = \phi_{t,z}(\mu, w^*\nu)$, $(\mu,\nu \in W_{t,z}(n), w \in T_n^a(q))$. We now have all the ingredients of a cellular structure and can state the analogue of 2.4 for $T_n^a(q)$.

Definition 3.3. *Let $\Lambda^a(n)^+$ be the set*

$$\Lambda^a(n)^+ = \{(t,z) \mid t \in \mathbb{Z}_{\geq 0},\ 0 \leq t \leq n,\ n-t \in 2\mathbb{Z};\ z \in R^\times\}. \tag{3.4}$$

Define $\Lambda^a(n)$ by

$$\Lambda^a(n) = \begin{cases} \Lambda^a(n)^+ & \text{if } q^2 \neq -1 \\ \Lambda^a(n)^+ \setminus \{(0,\pm q)\} & \text{if } q^2 = -1. \end{cases} \tag{3.5}$$

Define the equivalence relation $\approx$ on $\Lambda^a(n)^+$ as that which identifies $(0,z)$ and $(0,z^{-1})$ for all $z \in R^\times$, and write

$$\begin{aligned} \Lambda^a(n)^0 &= \Lambda^a(n)/\approx \\ \Lambda^a(n)^{0+} &= \Lambda^a(n)^+/\approx \end{aligned} \tag{3.6}$$

If $(t_1,z_1),(t_2,z_2) \in \Lambda^a(n)^+$, then $W_{t_1,z_1}(n) \cong W_{t_2,z_2}(n)$ if $(t_1,z_1) \approx (t_2,z_2)$. Thus $\Lambda^a(n)^{0+}$ plays the role of $\mathcal{T}$ in §2. We write $\mathrm{Rad}\,\phi_{t,z}$ for the submodule $\{x \in W_{t,z}(n) \mid \phi_{t,z}(x,y) = 0 \text{ for all } y \in W_{t,z^{-1}}(n)\}$ of $W_{t,z}(n)$.

Theorem 3.7 ([16, (2.8)]). *Let R be an algebraically closed field and maintain the above notation.*

(i) *For $(t,z) \in \Lambda^a(n)^+$, $L_{t,z}(n) := W_{t,z}(n)/\operatorname{Rad}\phi_{t,z}$ is either an (absolutely) irreducible $T_n^a(q)$ module or zero, and $L_{t,z}(n) \neq 0$ if and only if $(t,z) \in \Lambda^a(n)$.*

(ii) *All irreducible $T_n^a(q)$ modules are realised thus, and if $(t_1,z_1) \not\sim (t_2,z_2)$, then $L_{t_1,z_1}(n) \not\cong L_{t_2,z_2}(n)$.*

(iii) *The cell module $W_{t,z}(n)$ is irreducible if and only if for any $(t',z') \in \Lambda^a(n)^0$, $(t',z') \neq (t,z)$ we have*

$$\operatorname{Hom}_{T_n^a(q)}(W_{t',z'}(n), W_{t,z}(n)) = 0.$$

Equivalently, the form $\phi_{t,z}$ is a perfect pairing.

It follows that the distinct irreducible $T_n^a(q)$-modules are parametrised by $\Lambda^a(n)^0$, while the distinct cell modules are parametrised by $\Lambda^a(n)^{0+}$. These two sets therefore play the roles of $\mathcal{T}^0$ and $\mathcal{T}$ respectively in §2. Where there is little danger of confusion, we abuse notation by denoting the elements of $\Lambda^a(n)^{0+}$ as pairs (t,z), rather than equivalence classes of pairs. Thus we speak of $W_{t,z}(n)$ and $L_{t,z}(n)$ for $(t,z) \in \Lambda^a(n)^{0+}$. It follows from (3.7) that to understand the composition factors of the $T_n^a(q)$-module $W_{t,z}(n)$, it suffices to understand the spaces $\operatorname{Hom}_{T_n^a(q)}(W_{s,y}(n), W_{t,z}(n))$ for all pairs $(t,z),(s,y)$. This analysis is caried out in [16], and leads to the following results.

Let $\preceq$ be the partial order on $\Lambda^a(n)^+$ which is generated by the preorder $\overset{\circ}{\prec}$ which stipulates that $(t,z) \overset{\circ}{\prec} (s,y)$ if

$$\begin{aligned} &0 \le t \le s \le n, \quad s = t + 2\ell \quad (\ell \in \mathbb{Z},\ \ell > 0) \quad \text{and} \\ &\qquad z^2 = q^{\epsilon(s,z)s} \ \text{ and } \ y = zq^{-\epsilon(s,z)\ell} \ \text{ for } \epsilon(s,z) = \pm 1. \end{aligned} \tag{3.8}$$

Note that 3.8 implies that

$$y^2 = q^{\epsilon(s,z)t} \ \text{ and } \ z^t = y^s \tag{3.9}$$

and

$$(t,z) \preceq (t',z') \implies z^t = (z')^{t'}. \tag{3.10}$$

It suffices to verify 3.10 when $(t,z) \overset{\circ}{\prec} (t',z')$, in which case it follows easily from 3.8.

It is easily verified that [17, 4.1] the partial order $\preceq$ on $\Lambda^a(n)^+$ induces a partial order, also denoted $\preceq$, on the set $\Lambda^a(n)^{0+} = \Lambda^a(n)^+/\approx$.

The following result is proved in [16, Theorem 5.1].

Theorem 3.11. *Let R be a field of characteristic 0 or $p > 0$, where $pe > n$ and e is the multiplicative order of q^2. Then* (i) *We have*

$$\dim \mathrm{Hom}_{T_n^a(q)}(W_{s,y}(n), W_{t,z}(n)) = \begin{cases} 1 & \text{if } (t,z) \preceq (s,y) \\ 0 & \text{otherwise} \end{cases} . \tag{3.12}$$

(ii) *In the Grothendieck group $\Gamma(T_n^a(q))$, we have for any $(t,z) \in \Lambda^a(n)^{0+}$,*

$$W_{t,z}(n) = \sum_{\substack{(s,y)\in\Lambda^a(n)^0 \\ (t,z)\preceq(s,y)}} L_{s,y}(n) \tag{3.13}$$

Thus the matrix expressing the cell modules in terms of the irreducibles in $\Gamma(T_n^a(q))$ is upper unitriangular, and has entries 0 or 1. Now if (t,z) is confined to $\Lambda^a(n)^0$, the relation 3.13 can clearly be inverted.

The result is (cf. [17, Theorem (4.5)])

Theorem 3.14. *In the notation above, if $(t,z) \in \Lambda^a(n)^0$,*

$$L_{t,z}(n) = \sum_{\substack{(s,y)\in\Lambda^a(n)^0 \\ (t,z)\preceq(s,y)}} n_{t,z}^{s,y} W_{s,y}(n) \tag{3.15}$$

where $n_{t,z}^{s,y} = 0$ or ± 1.

§4. Interrelationships among various Hecke and Temperley-Lieb algebras

Our purpose in this section is to give a concise statement of the various connections among several Hecke algebras and Temperley-Lieb algebras. Behind these connections are facts concerning generalised Artin groups and their interpretation in terms of braids and ribbons. The topological background may be found in [7] and [39], and a more leisurely exposition may be found in [18, §2].

4.1. Some generalised Artin braid groups

The generalised Artin group Γ_n of type B_n is well known to be the fundamental group of the space of unordered n-tuples of distinct points in $\mathbb{C}$. It may therefore be described in terms of "cylindrical braids", and the following facts are well known (cf. *loc. cit.*). The group Γ_n has generating set $\{\xi_1, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ with relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ if } |i-j| \neq 1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } i = 1, 2, \ldots, n-1 \\ \xi_1\sigma_1\xi_1\sigma_1 &= \sigma_1\xi_1\sigma_1\xi_1 \\ \xi_1\sigma_i &= \sigma_i\xi_1 \text{ if } i \neq 1. \end{aligned} \tag{4.1}$$

Define elements $\tau = \xi_1\sigma_1\sigma_2\ldots\sigma_{n-1}$ and $\sigma_n = \tau\sigma_{n-1}\tau^{-1}$ in Γ_n. Then τ is represented as a "twist", and the following facts are well known (cf [18, §2]).

Proposition 4.2. (i) *We have $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$ for each i, where the subscripts are taken mod n.*

(ii) *The element τ^n is in the centre of Γ_n (this follows from (i)).*

(iii) *The elements $\{\sigma_1, \ldots, \sigma_{n-1}\}$ are Coxeter generators of the Artin group $\mathcal{B}_n$ of type A_{n-1}, i.e. of the classical braid group on n strings.*

(iv) *The elements $\{\sigma_1, \ldots, \sigma_n\}$ are Coxeter generators of the Artin group Δ_n of affine type $\widetilde{A}_{n-1}$.*

(v) *There is an exact sequence $1 \longrightarrow \Delta_n \longrightarrow \Gamma_n \longrightarrow \mathbb{Z} \longrightarrow 1$, where $\Delta_n \longrightarrow \Gamma_n$ is inclusion and $\Gamma_n \longrightarrow \mathbb{Z}$ is the map taking $\tau^r\sigma_{i_1}^{n_1}\ldots\sigma_{i_l}^{n_l} \in \Gamma_n$ to $r \in \mathbb{Z}$.*

(vi) *If $\xi_{i+1} = \sigma_i\xi_i\sigma_i$ $(i = 1, 2, \ldots, n-1)$, then $\xi_1, \ldots, \xi_n$ generate a free abelian subgroup of rank n of Γ_n and $\tau^n = \xi_1\ldots\xi_n$ is in the centre of Γ_n.*

4.2. Some Hecke algebras

Let R be a commutative ring as in §2, and fix elements $q, Q \in R^\times$. The various Hecke algebras with which we are concerned all emanate from the group ring $R\Gamma_n$, where Γ_n is the Artin braid group of type B_n as above.

Definition 4.3. *Let S_i be the element $S_i = (\sigma_i - q)(\sigma_i + q^{-1})$ of $R\Gamma_n$ $(i = 1, 2, \ldots, n)$. The affine Hecke algebra $\widehat{H_n^a}(q)$ of GL_n over R is defined by*

$$\widehat{H_n^a}(q) = R\Gamma_n/\langle S_1\rangle.$$

Note that since $S_1, \ldots, S_n$ are all conjugate in $R\Gamma_n$, the ideal $\langle S_1\rangle$ is equal to $\langle S_1, \ldots, S_n\rangle$. Let $\eta : R\Gamma_n \longrightarrow \widehat{H_n^a}(q)$ be the natural map. We then write

$$\begin{aligned} \eta(\sigma_i) &= T_i \text{ for } i = 1, \ldots, n \\ \eta(\xi_i) &= X_i \text{ for } i = 1, \ldots, n \\ \eta(\tau) &= V. \end{aligned} \tag{4.4}$$

Note that since $\tau = \xi_1\sigma_1\sigma_2\dots\sigma_{n-1}$, we have

$$V = X_1T_1\dots T_{n-1}. \tag{4.5}$$

We shall need to identify several sub- and quotient algebras of $\widehat{H_n^a}(q)$, and for this purpose we introduce the symmetric group $W \cong \mathrm{Sym}_n$, the corresponding affine Weyl group $W^a \cong \mathrm{Sym}_n \ltimes \mathbb{Z}^{n-1}$, which is a Coxeter group of rank n, and the Weyl group WB_n of type B_n.

The next proposition collects some well known facts concerning $\widehat{H_n^a}(q)$, many of which may be found in §3 of [33].

Proposition 4.6. (i) *The elements $T_1, \dots, T_n$ generate a subalgebra $H_n^a(q)$ of $\widehat{H_n^a}(q)$, which has R-basis $\{T_w \mid w \in W^a \cong Sym_n \ltimes \mathbb{Z}^{n-1}\}$, where, if $w = s_{i_1}\dots s_{i_\ell}$ is a reduced expression for $w \in W^a$, $T_w = T_{i_1}\dots T_{i_\ell}$. We refer to this as the "unextended" Hecke algebra of type $\widetilde{A_{n-1}}$.*

(ii) *The elements $T_1, \dots, T_{n-1}$ generate a subalgebra $H_n(q)$ of $\widehat{H_n^a}(q)$ which has (finite) R-basis $\{T_w \mid w \in W \cong Sym_n\}$.*

(iii) *We have $\widehat{H_n^a}(q) \cong R\mathbb{Z}\otimes H_n^a(q) \cong R\langle V\rangle \otimes H_n^a(q)$, where the tensor product is twisted, using the action of V on $H_n^a(q)$: $VT_iV^{-1} = T_{i+1}$, where the subscript is taken mod n.*

(iv) *(Bernstein) We have $\widehat{H_n^a}(q) \cong H_n(q) \otimes R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ as R-module, and the multiplication is given by the "Bernstein relations": for $i \in \{1, \dots, n-1\}$, write s_i for the corresponding simple reflection in W and ${}^{s_i}f$ for the image of $f \in R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ under the natural action of $W \cong Sym_n$. Then*

$$T_if - ({}^{s_i}f)T_i = (q - q^{-1})\frac{f - ({}^{s_i}f)}{1 - X_iX_{i+1}^{-1}}.$$

The (finite rank) Hecke algebra of type B_n, arises as follows. Let $WB_n := \mathrm{Sym}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ be the hyperoctahedral group. This is the quotient of Γ_n obtained by stipulating that each generator is an involution. Thus if $\{s_0, s_1, \dots, s_{n-1}\}$ are the images under the natural map of the generators $\{\xi_1, \sigma_1, \dots, \sigma_{n-1}\}$, WB_n is generated by the s_i, which are involutions and satisfy the relations analogous to 4.1 above. Let $Q \in R^\times$. The *Hecke algebra* $HB_n(q, Q)$ of type B_n with parameters (q, Q) is defined as

$$\begin{aligned} HB_n(q,Q) =& \widehat{H_n^a}(q)/\langle (X_1 - Q)(X_1 + Q^{-1})\rangle \\ =& R\Gamma_n/\langle (\xi_1 - Q)(\xi_1 + Q^{-1}), (\sigma_1 - q)(\sigma_1 + q^{-1})\rangle. \end{aligned}$$

Proposition 4.7. *Let $\eta_Q : \widehat{H_n^a}(q) \longrightarrow HB_n(q,Q)$ be the natural map. Write $T_i \in HB_n(q,Q)$ for the image of $T_i \in \widehat{H_n^a}(q)$ under η_Q $(i = 1,\dots,n-1)$ (relying on the context to distinguish between them), and write $T_0 = \eta_Q(X_1)$. Then $HB_n(q,Q)$ has R-basis $\{T_w \mid w \in WB_n\}$, where, if $w = s_{i_1}\dots s_{i_\ell}$ is a reduced expression for $w \in WB_n$, $T_w = T_{i_1}\dots T_{i_\ell}$.*

As a special case of 4.7, if w is the Coxeter element $s_0s_1\dots s_{n-1}$ of WB_n, the corresponding basis element T_w of $HB_n(q,Q)$ is written $V(= T_0T_1\dots T_{n-1})$.

Another algebra which is often considered in this context is the affine Hecke algebra of SL_n, which is denoted $\widetilde{H_n^a(q)}$ in [17], and whose representation theory is discussed there. This is defined as the quotient of $\widehat{H_n^a}(q)$ by the ideal generated by V^n-1. It is clear that in the notation of Proposition (4.6), $\widetilde{H_n^a(q)} \cong R(\mathbb{Z}/n\mathbb{Z}) \otimes H_n^a(q)$.

As a summary of the Hecke algebras which play a role in our story, we have the sequence

$$H_n(q) \xrightarrow{\text{incl}} H_n^a(q) \xrightarrow{\text{incl}} \widehat{H_n^a}(q) \xrightarrow{\eta_Q} HB_n(q,Q), \tag{4.8}$$

where the composition of the three maps is the obvious inclusion of $H_n(q)$ in $HB_n(q,Q)$ as the subalgebra generated by $T_1,\dots,T_{n-1}$.

4.3. Temperley-Lieb algebras of various types

We shall next explain how to realise the various algebras which have been introduced above by means of diagrams as quotients of Hecke algebras. Recall that $W = \langle s_1,\dots,s_{n-1}\rangle \cong \mathrm{Sym}_n$ above. Write $W_i = \langle s_i, s_{i+1}\rangle \cong \mathrm{Sym}_3$ for $i = 1,2,\dots,n-2$. Define the element $E_i \in H_n(q) \subset H_n^a(q) \subset \widehat{H_n^a}(q)$ by

$$E_i = \sum_{w\in W_i} q^{\ell(w)} T_w \tag{4.9}$$

where $\ell(w)$ denotes the usual length function. Let I (resp. $\widehat{I}$) denote the ideal of $H_n^a(q)$ (resp. $\widehat{H_n^a}(q)$) generated by E_1. Note that since the E_i are all conjugate (even in $H_n(q)$), this is the same as the ideal generated by all the E_i.

Definition 4.10. *The affine Temperley-Lieb algebras $TL_n^a(q)$ and $\widehat{TL_n^a}(q)$ are defined by*

$$\begin{aligned} TL_n^a(q) &= H_n^a(q)/I \\ \widehat{TL_n^a}(q) &= \widehat{H_n^a}(q)/\widehat{I} \end{aligned} \tag{4.11}$$

It is known (cf. [16], [17]) that if $C_i = -(T_i + q^{-1}) \in H_n^a(q)$ ($i = 1, \dots, n$), then in $H_n^a(q)$, $C_iC_{i+1}C_i - C_i = C_{i+1}C_iC_{i+1} - C_{i+1} = -q^3E_i$, where the indices are taken mod n. If we abuse notation by writing $C_i \in TL_n^a(q)$ for the image of $C_i \in H_n^a(q)$ under the natural map, it follows easily that $TL_n^a(q)$ is generated by $\{C_1, \dots, C_n\}$ subject to the relations

$$\begin{aligned} C_i^2 &= \delta_q C_i \\ C_iC_{i\pm1}C_i &= C_i \\ C_iC_j &= C_jC_i \text{ if } |i-j| \geq 2 \text{ and } \{i,j\} \neq \{1,n\}, \end{aligned} \tag{4.12}$$

Just as in the case of Hecke algebras, we also have the affine Temperley-Lieb algebra $\widetilde{TL_n^a(q)}$ of SL_n (cf. [17]). In analogy with the (4.6), we have

$$\widetilde{TL_n^a(q)} \cong \widehat{TL_n^a}(q)/\langle V^n - 1\rangle \cong R(\mathbb{Z}/n\mathbb{Z}) \otimes TL_n^a(q). \tag{4.13}$$

Proposition 4.14. (i) *We have*

$$\widehat{TL_n^a}(q) \cong R\langle V\rangle \otimes TL_n^a(q), \tag{4.15}$$

where $V \in \widehat{TL_n^a}(q)$ (identified with the image of $V \in \widehat{H_n^a}(q)$) permutes the C_i cyclically.

(ii) *The elements $\{C_1, \dots, C_{n-1}\}$ generate a subalgebra of $\widehat{TL_n^a}(q)$ which is isomorphic to $TL_n(q)$. It may be mapped monomorphically into $T_n^a(q)$ via $C_i \mapsto f_i$ ($i = 1, \dots, n-1$).*

(iii) *There is a family of surjections $\phi_\alpha : \widehat{TL_n^a}(q) \longrightarrow T_n^a(q)$ ($\alpha \in R^\times$), defined by $\phi_\alpha(C_i) = f_i$ and $\phi_\alpha(V) = \alpha\tau_n$. Each ϕ_α restricts to the same monomorphism $TL_n^a(q) \to T_n^a(q)$.*

(iv) *The kernel of ϕ_α is generated by the element $\nu_\alpha = \alpha^{-2}V^2C_{n-1} - C_1C_2 \dots C_{n-1}$ ($= \alpha^{-2}C_1V^2 - C_1C_2 \dots C_{n-1}$) of $\widehat{TL_n^a}(q)$.*

(v) *If R is an algebraically closed field of characteristic prime to n, any irreducible finite dimensional representation of $\widehat{TL_n^a}(q)$ is the pullback via ϕ_α (for some $\alpha \in R^\times$) of an irreducible representation of $T_n^a(q)$.*

Sketch of proof. The proofs of (i) and (ii) are easy. The first part of (iii) follows immediately from the relations above, while the second follows from the fact (cf. [16, (2.9)]) that 4.12 gives a presentation of $TL_n^a(q)$. Next, one verifies easily that $\tau_n^2 f_{n-1} = f_1 f_2 \dots f_{n-1}$ in $T_n^a(q)$ (see [17, 1.11]), which shows that $\nu_\alpha \in \mathrm{Ker}\phi_\alpha$. The fact that ν_α generates the kernel may be found in [12] or [8]. This relation also appears in [20]. We now indicate how to prove (v). Let ρ be an irreducible representation of $\widehat{TL}^a_n(q)$. Since V^n is in the centre of $\widehat{TL}^a_n(q)$ (cf. 4.2(vi)), it follows that V^n acts as a scalar, say $\lambda \in R^\times$ in ρ. Hence ρ factors through $\widehat{TL}^a_n(q)/\langle V^n - \lambda.1\rangle$, which is isomorphic (cf. 4.13) via a map which is the identity on $TL^a_n(q)$, to $\widetilde{TL^a_n(q)}$. But by [17, Theorem (2.8)], every irreducible representation of $\widetilde{TL^a_n(q)}$ is the inflation via a homomorphism like ϕ_α of an irreducible representation of $T^a_n(q)$. □

We shall need an algebraic description of the Temperley-Lieb algebra of type B_n which we have met in §2 as an algebra with a basis of marked planar diagrams. It is constructed as a quotient of $HB_n(q,Q)$ as follows. Recall that since $H_n(q) \subset HB_n(q,Q)$ (4.8) the elements T_i, C_i of 4.14, as well as E_i defined in 4.9 all lie in $HB_n(q,Q)$. In addition, we have (cf. 4.7) $C_0 := -(T_0 + Q^{-1}) \in HB_n(q,Q)$. Clearly $HB_n(q,Q)$ is generated as algebra by $C_0, C_1, \ldots, C_{n-1}$.

Proposition 4.16 ([18], §5). *There is a surjective homomorphism of associative R-algebras $HB_n(q,Q) \xrightarrow{\eta_4} TLB_n(q,Q)$, defined on the generators C_i by $\eta_4(C_i) = c_i$ $(i = 0, 1, \ldots, n-1)$. The kernel of η_4 is generated by E_1 and $C_1C_0C_1 - \kappa C_1 \in HB_n(q,Q)$, where $\kappa = \frac{q}{Q} + \frac{Q}{q}$ as in 2.7.*

Define $t_i \in TLB_n(q,Q)$ by $\eta_4(T_i) = t_i$ for $i = 0, 1, \ldots, n-1$. Then we have $t_0 = -(c_0 + Q^{-1})$ and $t_i = -(c_i + q^{-1})$ for $i = 1, 2, \ldots, n-1$. The t_i are all invertible. Because of its special role, we denote by v the element $\eta_4(V)$ of $TLB_n(q,Q)$ (see immediately following 4.7). Then $TLB_n(q,Q)$ is clearly generated by $\{v, c_1, \ldots, c_{n-1}\}$, and $vc_iv^{-1} = c_{i+1}$ for $1 \le i \le n-2$.

The diagram below summarises the relationships among the algebras we have now introduced.

(4.17)
$$
\begin{tikzcd}
H_n(q) \arrow[r, "\eta_1"] \arrow[d, "\mathrm{incl}"] & TL_n(q) \arrow[d, "\mathrm{incl}"] & \\
H^a_n(q) \arrow[r, "\eta_2"] \arrow[d, "\mathrm{incl}"] & TL^a_n(q) \arrow[d, "\mathrm{incl}"] & \\
\widehat{H}^a_n(q) \arrow[r, "\eta_3"] \arrow[d, "\eta_Q"] & \widehat{TL}^a_n(q) \arrow[r, "\phi_\alpha"] \arrow[d, "\gamma_Q"] & T^a_n(q) \\
HB_n(q,Q) \arrow[r, "\eta_4"] & TLB_n(q,Q) &
\end{tikzcd}
$$

We complete this section with a result which is instrumental in applying our results on $T_n^a(q)$ to the representation theory of $\widehat{H_n^a}(q)$.

Theorem 4.18 ([18], (5.11)). *Let $\beta \in R$ satisfy $\beta^2 = -q^{n-2}$. Then there is a unique surjective homomorphism $g_\beta : T_n^a(q) \longrightarrow TLB_n(q,Q)$ such that $g_\beta(f_i) = c_i$ for $i = 1, 2, \dots, n-1$, and $g_\beta(\tau) = \beta v$. If α, μ satisfy $\alpha^{-1}\mu = \beta$, then the following diagram commutes,*

(4.19)
$$\begin{tikzcd}
\widehat{H_n^a}(q) \arrow[r, "\eta_3"] \arrow[d, "\eta_{Q,\mu}"] & \widehat{TL_n^a}(q) \arrow[r, "\phi_\alpha"] \arrow[d, "\gamma_{Q,\mu}"] & T_n^a(q) \arrow[d, "g_\beta"] \\
HB_n(q,Q) \arrow[r, "\eta_4"] & TLB_n(q,Q) \arrow[r, "id"] & TLB_n(q,Q)
\end{tikzcd}$$

where $\eta_{Q,\mu}(T_i) = \eta_Q(T_i)$ for $1 \le i \le n$, and $\eta_{Q,\mu}(X_1) = \mu\eta_Q(X_1) = \mu T_0$, and $\gamma_{Q,\mu}(T_i) = t_i$ (see immediately following 4.16) and $\gamma_{Q,\mu}(V) = \mu\gamma_Q(V) = \mu v$.

The proof of Theorem (4.18) involves proving the relation $c_1v^2 = -q^{-(n-2)}c_1c_2\dots c_{n-1}$ in $TLB_n(q,Q)$ (cf. the sentence following 4.16), and analysing the diagram 4.17.

§5. Lifting representations from finite to infinite dimensional algebras

5.1. Commutative diagrams and lifting representations

Let A and B be R-algebras and $\psi : A \to B$ be an R-algebra homomorphism. If $\rho : B \to \mathrm{End}_R(M)$ is a representation of B (where M is a free R-module) one may lift ρ to a representation $\psi^*\rho$ of A by composing with ψ. This is called the inflation of ρ via ψ We shall discuss in this section the lifting of representations in the diagram 4.19. Note that by 4.14(v), if R is an algebraically closed field of characteristic not dividing n, then every finite dimensional irreducible representation of $\widehat{TL_n^a}(q)$ is the lift via ϕ_α of an irreducible represetation of $T_n^a(q)$, for some $\alpha \in R^\times$.

It follows that the inflations to $\widehat{H_n^a}(q)$ of irreducible representations of $T_n^a(q)$ are precisely those irreducible representations of $\widehat{H_n^a}(q)$ which factor through $\widehat{TL_n^a}(q)$. Our next result describes the lifting of cell modules from $TLB_n(q,Q)$ to $T_n^a(q)$.

Theorem 5.1 ([18], Corollary (6.15)). *Suppose R is any commutative ring and suppose that q, Q are elements of $R^\times$. Let $\beta \in R^\times$ satisfy $\beta^2 = -q^{n-2}$ and let $g_\beta : T_n^a(q) \to TLB_n(q,Q)$ be the surjection defined in 4.18. For $t \in \mathbb{Z}$ such that $|t| \le n$ and $t \equiv n$ (mod 2), define $\epsilon_t := \frac{t}{|t|}$ for $t \ne 0$, and $\epsilon_t = 1$ if $t = 0$.*

Then the inflation $g_\beta^* W_t(n)$ of the cell module $W_t(n)$ of $TLB_n(q,Q)$ is isomorphic to the cell module $W_{|t|,z_t^{\epsilon_t}}(n)$ of $T_n^a(q)$, where

$$z_t = (-1)^t \beta Q^{-1} q^{-\frac{1}{2}(n+t-2)}. \tag{5.2}$$

It follows from 5.1 that the inflation $g_\beta^* L_t(n)$ of the irreducible quotient $L_t(n)$ (cf. 2.4(i)) of $W_t(n)$ is the irreducible $T_n^a(q)$-module $L_{|t|,z_t^{\epsilon_t}}$. This is because of the implied description of the irreducible quotients of cell modules in terms of homomorphisms between cell modules (see 2.4(ii)). It therefore follows (see 3.7) that

Corollary 5.3. *Fix $q \in R^\times$ and assume R is a field. Then every irreducible $T_n^a(q)$-module is the inflation via g_β of an irreducible $TLB_n(q,Q)$-module for some $Q \in R^\times$.*

To see this, one need only observe that given t, β, q and $z \in R^\times$, the equation $z_t = z$ (see 5.2) has the solution $Q = (-1)^t \beta z^{-1} q^{-\frac{1}{2}(n+t-2)}$.

Our objective is to study the inflations of $T_n^a(q)$-modules to $\widehat{H_n^a}(q)$, and to apply our knowledge of decomposition numbers for $T_n^a(q)$ (see 3.11) to the representations of $\widehat{H_n^a}(q)$. For this purpose we focus on the part of the diagram 4.19 below.

$$\begin{tikzcd} \widehat{H_n^a}(q) \arrow[r, "\psi_\alpha"] \arrow[d, "\xi_{Q,\mu}"] & T_n^a(q) \arrow[d, "g_\beta"] \\ TLB_n(q,Q) \arrow[r, "\mathrm{id}"] & TLB_n(q,Q) \end{tikzcd} \tag{5.4}$$

where $\psi_\alpha = \phi_\alpha \circ \eta_3$ and $\xi_{Q,\mu} = \eta_4 \circ \eta_{Q,\mu}$.

By the commutativity of the diagram 5.4, we have

$$\psi_\alpha^* W_{t,z}(n) \cong \psi_\alpha^* g_\beta^* W_t(n) \cong \xi_{Q,\mu}^* W_t(n), \tag{5.5}$$

where $z = z_t$ as above, and α, β and μ are related by $\beta^2 = -q^{n-2}$ and $\mu\alpha^{-1} = \beta$. We wish to interpret the inflations $\psi_\alpha^* W_{t,z}(n)$ of the cell modules of $T_n^a(q)$ in terms of the "*standard modules*" of $\widehat{H_n^a}(q)$, and we therefore give a brief description of these.

5.2. Standard modules for $\widehat{H_n^a}(q)$

We assume henceforth that the ground ring R is $\mathbb{C}$, although many of our results apply more generally (see [18]). The standard modules for $\widehat{H_n^a}(q)$ may be defined cohomologically ([5, 24, 43]) or as induced modules ([41, 4, 44, 45]). The various definitions agree only up to equivalence in the Grothendieck group $\Gamma(\widehat{H_n^a}(q))$ of finite dimensional $\widehat{H_n^a}(q)$-modules, which suffices for our purpose, since we shall be discussing composition factors. Let $G = \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{G} = \mathrm{Lie}(G)$. The standard modules are parametrised by the set $\widehat{\mathcal{P}}$ of equivalence classes of pairs (s, N) modulo G, where $s \in G$ is semisimple, $N \in \mathfrak{G}$ is nilpotent and $s.N = q^2 N$ (where G acts on $\mathfrak{G}$ through the adjoint representation).

Let us explain a key property of this correspondence. The algebra $\widehat{H_n^a}(q)$ has a subalgebra (4.6(iv)) $U(n) = \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, which may be thought of as the group algebra $\mathbb{C}(\mathbb{Z}^n)$, where $\mathbb{Z}^n = \langle X_1, \ldots, X_n \rangle$. Any one-dimensional representation χ of $U(n)$ therefore corresponds to a character of $\mathbb{Z}^n$ and hence to the sequence $(\chi(X_1), \ldots, \chi(X_n))$. Let $(s, N) \in \widehat{\mathcal{P}}$, so that N corresponds to a partition $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\ell)$, where $\sum \lambda_i = n$ and $s = \mathrm{diag}(s_1, \ldots, s_n)$ where $s_{i+1} = q^{-2} s_i$ whenever $i, i+1$ lie in the same "block" of N. Thus the sequence of eigenvalues of s consists of ℓ subsequences of size $\lambda_1, \lambda_2, \ldots, \lambda_\ell$, which we call blocks, and each block is of the form $a, aq^{-2}, aq^{-4}, \ldots$ for some $a \in \mathbb{C}^\times$. If $M_{s,N}$ is the corresponding standard module, its dimension is $K = \frac{n!}{\lambda_1! \ldots \lambda_\ell!}$ and there is a filtration

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_K = M_{s,N} \tag{5.6}$$

where $\dim M_i = i$, each term M_i is a $U(n)$-submodule of $M_{s,N}$, and the characters of $U(n)$ on the K quotients M_i/M_{i-1} $(i = 1, 2, \ldots, K)$ are given by the K sequences $(\chi_1, \ldots, \chi_n)$ obtained by permuting the sequence $(s_1, \ldots, s_n)$ while keeping each block in its original order.

5.3. Lifting cell modules to standard modules

To state our main result, it is convenient to note that that there is an involution $\iota : \widehat{H_n^a}(q) \to \widehat{H_n^a}(q)$ which takes T_i to $-T_i^{-1}$ $(i = 1, \ldots, n-1)$ and X_j to X_j^{-1} $(j = 1, \ldots, n)$; this follows by noting that the images of the T_i and X_j under ι satisfy the the relations 4.6(iv). The surjection $\theta_\alpha : \widehat{H_n^a}(q) \to T_n^a(q)$ is defined by (cf. 5.4)

$$\theta_\alpha = \psi_\alpha \circ \iota. \tag{5.7}$$

Theorem 5.8 ([18], (9.9)). *Let $\widehat{H_n^a}(q)$ be the affine Hecke algebra of $\mathrm{GL}_n(\mathbb{C})$ and let $\Gamma(\widehat{H_n^a}(q))$ be the Grothendieck group of finite dimensional $\widehat{H_n^a}(q)$-modules. Let $W_{t,z}(n)$ be a cell module for the diagram algebra $T_n^a(q)$. Then, with θ_α as in 5.7, we have*

$$[\theta_\alpha^* W_{t,z}(n)] = [M_{s,N}] \tag{5.9}$$

where $[V]$ denotes the class in $\Gamma(\widehat{H_n^a}(q))$ of an $\widehat{H_n^a}(q)$-module V and $M_{s,N}$ is the Kazhdan-Lusztig standard module of $\widehat{H_n^a}(q)$ corresponding *to the pair* $(s, N) \in \widehat{\mathcal{P}}$ *where* $N = N_k$ *is the nilpotent Jordan matrix corresponding to the 2-step partition* $(n-k, k)$ *and* s *is the diagonal matrix* $\operatorname{diag}(a, aq^{-2}, \ldots, aq^{-(n+t-2)}, b, bq^{-2}, \ldots, bq^{-(n-t-2)})$*, where the parameters* a, b *and* k *are given by*

$$\begin{aligned} k =& \frac{n-t}{2} \\ a =& (-1)^{n+1} \alpha z q^{\frac{1}{2}(n+t-2)} \\ b =& (-1)^{n+1} \alpha z^{-1} q^{\frac{1}{2}(n-t-2)} \end{aligned} \tag{5.10}$$

Some remarks concerning the proof. We begin by noting (see [1, Theorem 3.2, p.798]) that $M_{s,N}$ is equivalent in $\Gamma(\widehat{H_n^a}(q))$ to an induced module, which has an easy characterisation [18, (9.3)] in terms of a generating vector. One then shows that when the pair (q, z) is generic (see [18, (8.1)] for a definition; it suffices that there be no non-trivial polynomial F such that $F(q, z) = 0$) then $\theta_\alpha^* W_{t,z}(n)$ is actually isomorphic to the induced module. Finally a number-theoretic specialisation argument is used to prove the result in general.

The key to the proof is the treatment of the generic case. The heart of the proof in this case is the determination of the characters of the algebra $U(n)$, i.e. the simultaneous eigenvalues of the elements X_i, on $\theta_\alpha^* W_{t,z}(n)$, or equivalently, on $\psi_\alpha^* W_{t,z}(n)$. For this, we use the identification $\psi_\alpha^* W_{t,z}(n) \cong \xi_{Q,\mu}^* W_t(n)$ as in (5.5), where the parameters are as stated there. Now to analyse the action of the X_i on $\xi_{Q,\mu}^* W_t(n)$, we start by observing that $X_1 X_2 \ldots X_n = V^n$ is in the centre of $\widehat{H_n^a}(q)$ (4.2(v)), and acts on $\psi_\alpha^* W_{t,z}(n)$ as a scalar (since τ_n^n acts on $W_{t,z}(n)$ as a scalar), which may be easily computed. We now use the filtration 2.12 of $W_t(n)$ to obtain one for $\psi_\alpha^* W_{t,z}(n) \cong \xi_{Q,\mu}^* W_t(n)$, noting that by the above argument, $X_1 X_2 \ldots X_i$ acts via a (known) scalar on the i^{th} term of the filtration. By this means one finds a simultaneous eigenvector for the X_i, and using the fact that $X_i = (X_1 X_2 \ldots X_i)(X_1 X_2 \ldots X_{i-1})^{-1}$ one knows the corresponding character of $U(n)$. Proceeding by induction, one obtains a filtration of $\psi_\alpha^* W_{t,z}(n)$ like that described in 5.6.

Assuming genericity, the characters of $U(n)$ arising in this filtration are all distinct, so that $\psi_\alpha^* W_{t,z}(n)$ is semisimple as $U(n)$-module. This permits the determination of the action of the T_j on a certain summand, completing the proof for the generic case. The passage from the generic case to the general case involves a number-theoretic argument (cf. [3]). We refer the reader to [18, §§6,7,8 and 9] for all details. □

We denote the pair s, N in the statement 5.8 above by $s(a, b), N_k$.

§6. On the decomposition numbers of standard modules

It is clear from Theorem 5.8 and the remark preceding 5.1 that the standard modules of $\widehat{H}_n^a(q)$ which factor through representations of $\widehat{TL}_n^a(q)$ are precisely the $M_{s,N}$ where $N = N_k$ is a two-step nilpotent element of $\mathfrak{G}$ as above. Since the relations 5.10 may be inverted to give α, z and t in terms of s, k, these modules are all realised, up to Grothendieck equivalence, as inflations of cell modules of $T_n^a(q)$. Hence the results of Theorems 3.11 and 3.14 may be applied to analyse their composition factors.

After working through the details of the correspondence provided by Theorem 5.8 and the properties of the cell modules $W_{t,z}(n)$, the following precise statement results.

Proposition 6.1 ([18], 9.12). *Let $\mathcal{P}$ be the set equivalence classes of pairs $(s, N) \in \widehat{\mathcal{P}}$ where $N \in \mathfrak{G}$ is two-step nilpotent, i.e. $N \sim N_k$ for some k with $0 \leq 2k \leq n$. Let $\widetilde{\Omega}$ be the set of triples (t, α, z) ($t \in \mathbb{Z}, 0 \leq t \leq n, n-t \in 2\mathbb{Z}; \alpha, z \in \mathbb{C}^\times$) and let Ω be the set of equivalence classes of triples in $\widetilde{\Omega}$ under the equivalence generated by the relations $(t, \alpha, z) \sim (t, -\alpha, -z)$, $(n, \alpha, z) \sim (n, y^{-1}z\alpha, y)$ and $(0, \alpha, z) \sim (0, \alpha, z^{-1})$. Then (with the obvious abuse of notation) we have well defined $\widehat{H}_n^a(q)$-modules $M_{s,N}$, $(s, N) \in \mathcal{P}$ and $\theta_\alpha^* W_{t,z}(n)$, $(t, \alpha, z) \in \Omega$, and there is a bijection $f : \mathcal{P} \to \Omega$ such that if $(s, N) \in \mathcal{P}$ corresponds to $(t, \alpha, z) \in \Omega$, $[M_{s,N}] = [\theta_\alpha^* W_{t,z}(n)]$.*

This correspondence permits us to define the irreducible "top quotients" $L_{s,N}$ for $(s, N) \in \mathcal{P}$ as the pullback of the corresponding top quotient of $W_{t,z}(n)$, which is either zero or irreducible. Translating the statement 3.7(i) into the language of pairs, we obtain

Proposition 6.2. *The module $L_{s(a,b),N_k}$ is zero if and only if*

$$(6.3) \quad q^2 = -1,\ n \text{ is even},\ k = \frac{n}{2},\ a = \alpha,\ b = -\alpha,\ \text{for some } \alpha \in \mathbb{C}^\times.$$

We therefore denote by $\mathcal{P}_0$ the set of (equivalence classes of) pairs in $\mathcal{P}$ which do not satisfy the condition 6.3. By Theorem 3.7(i), $\mathcal{P}_0$ parametrises the irreducible $\widehat{H}_n^a(q)$-modules which factor through $\widehat{TL}_n^a(q)$.

The partial order $\preceq$ on $\Lambda^a(n)^{0^+}$ induces one on $\mathcal{P}$ using the correspondence 6.1. Specifically we have

Proposition 6.4. *Suppose that under the correspondence 6.1 above, the triples (t_1, α, z_1), (t_2, α, z_2) correspond to the pairs $(s(a_1, b_1), N_{k_1})$, $(s(a_2, b_2), N_{k_2})$ respectively. Then $(t_1, z_1) \overset{\circ}{\prec} (t_2, z_2)$ if and only if there* *exists* $\ell > 0$ *and* $\epsilon = \pm 1$ *such that if we write* $2k_i = n - t_i$ *for* $i = 1, 2$, *then*

$$(6.5)\qquad \begin{aligned} k_2 &= k_1 - \ell \geq 0 \\ a_1 b_1^{-1} &= q^{t_1 + \epsilon t_2} \\ (a_2, b_2) &= (a_1, b_1) \text{ if } \epsilon = 1 \\ &\quad (b_1, a_1) \text{ if } \epsilon = -1 \end{aligned}$$

Now Theorems 3.11 and 3.14 yield immediately

Theorem 6.6. *Let* $\mathcal{P}$ *and* $\mathcal{P}_0$ *be the sets of semisimple-nilpotent pairs defined above, and let* $\preceq$ *be the partial order on* $\mathcal{P}$ *generated by the relation 6.5. Then in the Grothendieck group of finite-dimensional* $\widehat{H_n^a}(q)$*-modules,*

$$(6.7)\qquad [M_{s,N}] = \sum_{\substack{(s',N') \in \mathcal{P}_0 \\ (s,N) \preceq (s',N')}} [L_{s',N'}] \quad \text{for any pair } (s, N) \in \mathcal{P},$$

and for $(s, N) \in \mathcal{P}_0$,

$$(6.8)\qquad [L_{s,N}] = \sum_{\substack{(s',N') \in \mathcal{P} \\ (s,N) \preceq (s',N')}} n_{s,N}^{s',N'} [M_{s',N'}]$$

where $n_{s,N}^{s',N'} = 0$ *or* ± 1.

We conclude this section by giving some applications to the structure of the standard modules which are easy consequences of Theorem 6.6

Corollary 6.9. (i) *The standard modules* $M_{s,N}$ *(*$(s, N) \in \mathcal{P}$*) are multiplicity free.*

(ii) *If* q *is not a root of unity, the standard modules have at most* 2 *composition factors.*

(iii) *In all cases,* $M_{s,N}$ *has composition length bounded by* $\left[\frac{n}{2}\right]$.

(iv) *The maximum composition length of* $M_{s,N}$ *is* $\left[\frac{n}{2}\right]$, *and therefore is unbounded as* $n \to \infty$.

§7. Connection with representation varieties of quivers

In this section we continue to consider the representations of $\widehat{H_n^a}(q)$, where the underlying ring is $\mathbb{C}$. In this case, we explain an interpretation of 6.7,6.9 in terms of the intersection cohomology of certain varieties.

Specifically, we show how (6.9)(i) is equivalent to the rational smoothness of certain varieties which we define below. This interpretation of decomposition numbers in terms of intersection cohomology is due to Chriss-Ginzburg [5], Kazhdan-Lusztig [24] and Lusztig [33, 34, 35], (see also [11]) and the exposition here owes much to A. Henderson, who has given an alternative direct geometric proof of our (6.9)(i) starting from the geometric point of view (see [19]). Write $G = GL_n(\mathbb{C})$ and $\mathfrak{G} = \mathrm{Lie}(G)$. The multiplicities of the irreducible $\widehat{H_n^a}(q)$-modules in the standard modules $M_{s,N}$ are given in terms of the intersection cohomology of certain complexes on the closures of the orbits of the group $Z_G(s)$ acting on the variety

$$\mathcal{N}_{s,q} = \{N \in \mathfrak{G}_{\mathrm{nil}} \mid s \cdot N = q^2 N\}. \tag{7.1}$$

Specifically, for $N \in \mathcal{N}_{s,q}$, let $\mathcal{O}_N$ be its orbit under $Z_G(s)$. Let $\le$ be the partial order on the (finite) set of orbits of $Z_G(s)$ on $\mathcal{N}_{s,q}$ which is defined by orbit closure. Let $\mathscr{H}^k(IC(\overline{\mathcal{O}_N}))$ denote the k^{th} cohomology sheaf of the perverse extension of the constant sheaf on $\mathcal{O}_N$ to the closure $\overline{\mathcal{O}_N}$. It is known (see below) that $\mathscr{H}^k(IC(\overline{\mathcal{O}_N})) = 0$ if k is odd. For $N' \in \mathcal{N}_{s,q}$ such that $\mathcal{O}_{N'} \le \mathcal{O}_N$, write $\mathscr{H}^k_{N'}(IC(\overline{\mathcal{O}_N})) = 0$ for the stalk at any point of $\mathcal{O}_{N'}$ of this sheaf, and define

$$\tilde{K}_{N,N'}(t) = \sum_{j\ge 0} \dim \mathscr{H}^{2k}_{N'}(IC(\overline{\mathcal{O}_N}))t^k \tag{7.2}$$

Now the modules $M_{s,N}$ generally have a top quotient $L_{s,N}$, which is irreducible or zero, and the non-zero modules among the $L_{s,N}$ form a complete set of irreducible $\widehat{H_n^a}(q)$-modules (see [5, Chapter 8] or [43]).

Theorem 7.3 (Ginzburg [5], 8.6.23) (see also [1] and [11]). *We have the following multiplicity formula for the standard modules $M_{s,N}$. Assume $L_{s,N} \ne 0$. Then*

$$[M_{s',N'} : L_{s,N}] = \begin{cases} 0 \textit{ unless } s' \sim_G s \\ \tilde{K}_{N,N'}(1) \textit{ if } s = s' \textit{ and } \mathcal{O}_{N'} \le \mathcal{O}_N \\ 0 \textit{ if } s = s' \textit{ and } \mathcal{O}_{N'} \not\le \mathcal{O}_N \end{cases} \tag{7.4}$$

Here $\sim_G$ denotes conjugacy in $G = GL_n(\mathbb{C})$.

Let $V = \mathbb{C}^n$, on which both $G = GL_n(\mathbb{C})$ and $\mathfrak{G}$ act via matrix multiplication. For any linear transformation A of V and element $\xi \in \mathbb{C}$, write $V(A,\xi)$ for the ξ-eigenspace of A in V. For any pair (s, N) as above, write $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ for the s-eigenspace decomposition of V, where $V_i = V(s, \xi_i)$ and the ξ_i are distinct. The relation (7.1) shows that

$$N \cdot V(s, \xi_i) \subset V(s, q^2\xi_i). \tag{7.5}$$

Multiplication by q^2 partitions $\mathbb{C}^\times$ into orbits which are infinite if q^2 is not a root of unity and have cardinality e if q^2 has finite multiplicative order e (say). These orbits are linearly (resp. cyclically) ordered if the order of q^2 is infinite (resp. finite). We may assume that the ξ_i are ordered so that each orbit occurs as a sequence of successive elements in $(\xi_1, \cdots, \xi_k)$ in the specified linear or cyclic order, and that the ξ_j in a given orbit occur as a "block", which we denote by s_i $(i = 1, \cdots, r)$, in the matrix s. Let $k_1, \cdots, k_r$ be the sizes of these blocks, so that $k_1 + \cdots + k_r = n$. The nilpotent element N correspondingly decomposes into blocks: $N = N_1 \oplus \cdots \oplus N_r$, where N_i is of size k_i and $s_i \cdot N_i = q^2 N_i$ for $i = 1, \cdots, r$. The following facts may be found in [4, 41, 1].

Proposition 7.6. *(i) The standard module $M_{s,N}$ is Grothendieck equivalent to the induced module*

$$Ind_{\widehat{H_\lambda^a}(q)}^{\widehat{H_n^a}(q)} \widetilde{\otimes_{i=1}^r M_{s_i,N_i}} \tag{7.7}$$

where M_{s_i,N_i} is the standard module of $\widehat{H_{k_i}^a}(q)$, $\lambda = (k_1, \cdots, k_r)$ and $\widetilde{\otimes_{i=1}^r M_{s_i,N_i}}$ is the lift from $\otimes_i \widehat{H_{\lambda_i}^a}(q)$ to $\widehat{H_{\lambda_i}^a}(q)$ of $\otimes_{i=1}^r M_{s_i,N_i}$.

(ii) If ρ_i is a composition factor of M_{s_i,N_i} $(i = 1, \cdots, r)$, then $Ind_{\widehat{H_\lambda^a}(q)}^{\widehat{H_n^a}(q)} \widetilde{\otimes_{i=1}^r \rho_i}$ is irreducible.

(iii) The standard module $M_{s,N}$ is multiplicity free if and only if each component M_{s_i,N_i} is multiplicity free.

Thus the decomposition of $M_{s,N}$ may be discussed in terms of that of the M_{s_i,N_i}; in this way one reduces to the case where all the eigenvalues ξ_j of s are in the same q^2-orbit in $\mathbb{C}$. This may also be seen from the geometric description of the decomposition numbers given above (see (7.3)) as follows.

The centraliser $Z_G(s) \cong \prod_{i=1}^r GL_{n_i}(\mathbb{C})$, where n_i is the size of the block s_i (or N_i). It is clear that the orbit $\mathcal{O}_N \subset \mathcal{N}_{s,q}$ is isomorphic to the product $\prod_{i=1}^r \mathcal{O}_{N_i}$, where $\mathcal{O}_{N_i} := Z_{GL_{n_i}(\mathbb{C})}(s_i) \cdot N_i$. It follows from the Kunneth theorem for intersection cohomology, that in the above notation,

$$\tilde{K}_{N,N'}(t) = \prod_i \tilde{K}_{N_i,N_i'}(t), \tag{7.8}$$

where N'_i is the i-block of N', regarded as an element of $\mathfrak{G}_i$ =Lie $GL_{n_i}(\mathbb{C})$

Thus for the decomposition of $M_{s,N}$, it suffices to consider the case when all eigenvalues of s are in the same q^2-orbit, and hence after multiplication by a scalar, one may assume that they are powers of q^2. The relation (7.5) then shows that the pair (s, N) defines a representation of the cyclic or linear quiver, and this is a convenient context for an explicit discussion of the combinatorial geometric interpretation of our result (6.9)(i).

We shall therefore formulate our result in the language of quiver representations. We start by recalling some elementary facts concerning quiver representations, most of which may be found in [40].

Fix an element $e \in \mathbb{Z}_{\geq 0}$, and let $\mathcal{Q}_e$ be the cyclic quiver of type $\widetilde{A}_{e-1}$. For $e = 0$, this is interpreted as the linear quiver A_∞. If we write $I = \mathbb{Z}/e\mathbb{Z}$, $\mathcal{Q}_e$ has vertex set I and an arrow from i to $i-1$ for each $i \in I$. A *representation* of $\mathcal{Q}_e$ is a pair (V, E), where $V = \oplus_{i\in I} V_i$ is an I-graded $\mathbb{C}$-vector space, and $E = \oplus_{i\in I} E_i \in E_V$, where $E_V = \oplus_{i\in I}\mathrm{Hom}_{i\in I}(V_i, V_{i-1}) \subset \mathrm{End}(V)$. The dimension vector of (V, E) is denoted $\mathbf{d} = (d_1, ..., d_e)$, where d_i =dimV_i, and its total dimension is written $n = \sum_{i=1}^{n} d_i$. We have a notion of equivalence for representations of $\mathcal{Q}_e$, which is defined in the obvious way. For a fixed graded vector space V, the set of all representations (V, E) is identified with the affine space E_V in the obvious way, and the group $G_V := \prod_{i\in I} GL(V_i)$ acts on this space by conjugation. The representation $E \in E_V$ (and its G_V-orbit) is said to be nilpotent if E is nilpotent as an endomorphism of V; the set of nilpotent representations of $\mathcal{Q}_e$ on V is clearly (Zariski)-closed in E_V, and we denote this variety by $\mathcal{N}_V$.

Suppose henceforth that q^2 has multiplicative order e, where $e = 0$ if q is not a root of unity. Let (s, N) be a pair with $N \in \mathcal{N}_{s,q}$, and suppose all eigenvalues of s are powers of q^2, and hence are parametrised by I (we have seen that the general case may be reduced to this one). Then by (7.5), the s-eigenspace decomposition $V = \oplus_{i\in I} V_i$ of $V = \mathbb{C}^n$ defines, together with N, a nilpotent representation of the quiver $\mathcal{Q}_e$. Conversely, given such a representation, $V = \oplus_{i\in I} V_i$, with a nilpotent endomorphism $E : V_j \to V_{j-1}$ $(j \in I)$, define $s \in GL(V)$ as the linear transformation which acts on V_i as multiplication by q^{-2i}. Then $E \in \mathcal{N}_{s,q}$. The following statement is now clear.

Proposition 7.9. *(i) There is a bijection between $GL_n(\mathbb{C})$-conjugacy classes of pairs (s, N) such that the eigenvalues of s are powers of q and $s \cdot N = q^2 N$, and isomorphism classes of nilpotent representations (V, E) of the quiver $\mathcal{Q}_e$, with total dimension n.*

(ii) If (s, N) and (V, E) correspond under this bijection (recall that V comes with its grading), the variety $\mathcal{O}_N$ above is isomorphic to the G_V-orbit of E in $\mathcal{N}_V$.

We next give a parametrisation of the G_V-orbits on $\mathcal{N}_V$. By (7.9)(i), as the pair (V, E) varies, V being an I-grading of $\mathbb{C}^n$ and $E \in \mathcal{N}_V$, these orbits will index the standard modules of $\widehat{H_n^a}(q)$ in (7.9), and by (7.3) and (7.9)(ii), the geometry of any orbit determines the decomposition of the corresponding standard module. Take $E \in \mathcal{N}_V$. Since E is a nilpotent endomorphism of V, it corresponds to a partition $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{p(\lambda)} > 0$ of n (we refer to $p(\lambda)$ as the number of parts of the partition λ and to n as its size). Of course in general λ does not determine the G_V-class of E. To describe the latter, we require the following description of the indecomposable nilpotent representations of $\mathcal{Q}_e$; an arbitrary nilpotent representation is uniquely (up to order of the summands) expessible a sum of nilpotent indecomposables. The nilpotent indecomposables are parametrised by pairs (i, ℓ), $i \in I, \ell \in \mathbb{Z}_{>0}$; denote the corresponding representation of $\mathcal{Q}_e$ by $M(i; \ell)$. If $\ell = ae + b$, $(a, b \in \mathbb{Z}_{\geq 0}, 0 \leq b < e)$, the graded vector space $V = \oplus_{j \in I} V_j$ has

$$(7.10) \qquad d_j = \begin{cases} a + 1 \text{ for } j = i, i+1, \cdots, i+b-1 \\ a \text{ otherwise.} \end{cases}$$

If V_j has basis $\{v_{j,1}, \cdots, v_{j,d_j}\}$, the homomorphisms $E_j : V_{j+1} \to V_j$ are described by

$$(7.11) \qquad v_{j+1,c} \overset{E_j}{\longmapsto} \begin{cases} v_{j,c} \text{ if } j \neq i \\ v_{j,c+1} \text{ if } j = i \\ \text{where } v_{j,c} \text{ is defined to be 0 if } c > d_j. \end{cases}$$

The pair $(i; \ell)$ may be identified with the sequence $(i, i+1, \cdots, i+\ell-1)$ of elements of I, and so is sometimes referred to as a *segment.* Note that the total dimension of $M(i; \ell)$ is ℓ. Any nilpotent representation M of $\mathcal{Q}_e$ of total degree n is equivalent to a sum $M \cong M(i_1; \ell_1) \oplus \cdots \oplus M(i_p; \ell_p)$, where $\sum_j \ell_j = n$. The multiset $\{(i_j, \ell_j) \mid j = 1, \cdots, p\}$ is uniquely determined by the isomorphism class of M and conversely; hence the isomorphism classes are often referred to as *"multisegments"*. If the segments are ordered so that the ℓ_j are in weakly decreasing order, the sequences corresponding to the (i_j, ℓ_j) may be written in an array, which may be thought of as a Young diagram whose boxes are labelled with elements of I, which increase by one across the rows. The dimension vector of the representation M is $\mathbf{d} = (d_0, \cdots, d_{e-1})$, where d_i is the number of entries of the array which are equal to i. If the partition of n which corresponds to the Young diagram is λ, we call the array an I – *labelling* of λ, or simply an I-partition (of n). Clearly λ describes the $GL(V)$-orbit of the nilpotent transformation E.

Remark 7.12. The distinct I-partitions of n parametrise the nilpotent representations of $\mathcal{Q}_e$ of total dimension n, with two I-partitions deemed to be the same if one is obtained from the other by permuting rows (necessarily of the same length). We denote an I-labelling of λ by a symbol (λ, l), where l refers to the labelling. In view of (7.9)(i), corresponding to each I-partition (λ, l), we have a standard module for $\widehat{H_n^a}(q)$, which we denote by $M_{(\lambda,l)}$, which is uniquely defined up to isomorphism; similarly for the top quotient $L_{(\lambda,l)}$ mentioned above. Further, in view of (7.9)(ii), we also have associated to (λ, l) a corresponding variety, which we denote $\mathcal{O}_{(\lambda,l)}$; the closure relation among these varieties defines a partial order among I-partitions, which we denote $\leq$.

Lusztig (see [30, §5 and Appendix]) has given a necessary and sufficient criterion for $L_{(\lambda,l)}$ to be non-zero. Say that (λ, l) is aperiodic if for each $m \in \mathbb{Z}_{>0}$ there exists an element $i \in I$ such that (λ, l) does not have a row of length m starting with i. Then $L_{(\lambda,l)} \neq 0$ if and only if (λ, l) is aperiodic.

We may now express (7.3) in the language of I-partitions as follows. Note that the dimension vector $\mathbf{d}$ of an I-partition determines the "s" in the corresponding pair (s, N).

Corollary 7.13. *Let (λ, l) and (λ', l') be two I-partitions with the same dimension vector such that $(\lambda', l') \leq (\lambda, l)$ in the sense above, and suppose that (λ, l) is aperiodic. Define*

$$\tilde{K}_{(\lambda,l),(\lambda',l')}(t) = \sum_{j\geq 0} dim\, \mathscr{H}^{2k}_{(\lambda',l')}(IC(\overline{\mathcal{O}_{(\lambda,l)}}))t^k. \tag{7.14}$$

Then

$$[M_{(\lambda',l')} : L_{(\lambda,l)}] = \tilde{K}_{(\lambda,l),(\lambda',l')}(1). \tag{7.15}$$

If (λ, l) and (λ', l') do not have the same dimension vector, $[M_{(\lambda,l)} : L_{(\lambda',l')}] = 0$.

Our result (6.9) pertains to I-partitions with at most 2 parts (rows). Note that in this case, the criterion (6.3) given in our result (6.2) for $L_{(\lambda,l)}$ to be non-zero coincides with Lusztig's aperiodicity. Denote the set of I-partitions with dimension vector $\mathbf{d}$ by $\mathcal{P}(\mathbf{d})$, and the set of elements of $\mathcal{P}(\mathbf{d})$ with at most 2 parts by $\mathcal{P}^{\leq 2}(\mathbf{d})$. The subsets corresponding to I-partitions (λ, l) with $L_{(\lambda,l)} \neq 0$ will be denoted respectively by $\mathcal{P}_0(\mathbf{d})$ and $\mathcal{P}_0^{\leq 2}(\mathbf{d})$. Clearly if $(\mu, m) \in \mathcal{P}^{\leq 2}(\mathbf{d})$, $((\lambda, l)) \in \mathcal{P}(\mathbf{d})$ and $(\mu, m) \leq (\lambda, l)$, then $(\lambda, l) \in \mathcal{P}^{\leq 2}(\mathbf{d})$. Our (6.9) may now be stated as follows

Theorem 7.16. *Let* $\mathbf{d}$ *be a fixed dimension vector and suppose that* $(\lambda, l), (\lambda', l') \in \mathcal{P}^{\leq 2}(\mathbf{d})$*, and* $(\lambda', l') \leq (\lambda, l)$*. Then*

$$\tilde{K}_{(\lambda,l),(\lambda',l')}(t) = 1. \tag{7.17}$$

For $(\lambda, l) \in \mathcal{P}_0^{\leq 2}(\mathbf{d})$, (7.16) follows immediately from (6.9) and (7.17), because $K_{(\lambda,l),(\lambda',l')}(1) = 1$ implies that $K_{(\lambda,l),(\lambda',l')}(t) = 1$, since $K_{(\lambda,l),(\lambda',l')}(t) \in \mathbb{Z}_{\geq 0}[t]$. But if (λ, l) is not aperiodic, then as observed above $\lambda = (m, m)$ for some m, and so (λ, l) is minimal in $\mathcal{P}^{\leq 2}(\mathbf{d})$, whence $(\lambda, l) = (\lambda', l')$ and (7.17) holds trivially.

This means that the variety $\overline{\mathcal{O}_{(\lambda,l)}}$ is rationally smooth (but not generally smooth) at the points of $\mathcal{O}_{(\lambda',l')}$. A. Henderson has given ([19]) a geometric proof of Theorem (7.16) using an explicit resolution of $\overline{\mathcal{O}_{(\lambda,l)}}$

Note that in the case $e = 1$ (i.e. $q^2 = 1$), $(\lambda, l) = \lambda$ (i.e. the label is irrelevant) and Lusztig has shown ([34]) that

$$\tilde{K}_{\lambda,\lambda'}(t) = t^{n(\lambda') - n(\lambda)} K_{\lambda,\lambda'}(t^{-1}),$$

where $n(\lambda) = \sum_j (j-1)\lambda_j$, and $K_{\lambda,\lambda'}(t)$ is the Kostka-Foulkes polynomial. In general, Lusztig [35, §11] has shown that for any $(\lambda, l) \in \mathcal{P}(\mathbf{d})$, the variety $\overline{\mathcal{O}_{(\lambda,l)}}$ may be embedded as an open subvariety of an affine Schubert variety of type $\tilde{A}_{n-1}$. It follows from this that $\mathscr{H}^k(IC(\overline{\mathcal{O}_{(\lambda,l)}})) = 0$ for k odd. Furthermore, there is is an order preserving injective map $\mathcal{P}(\mathbf{d}) \hookrightarrow W^a$ (notation as in §2 above) such that if w, w' are the respective images of $(\lambda, l), (\lambda', l')$, then $\tilde{K}_{(\lambda,l),(\lambda',l')}(t) = P_{w,w'}(t)$, the usual Kazhdan-Lusztig polynomial associated with the coxeter system of type $\tilde{A}_{n-1}$. Thus our result also implies the triviality of certain special affine Kazhdan-Lusztig polynomials.

We remark finally is that the results of [18] may be used to discuss aspects of the representation theory of the affine Hecke algebra $\widehat{H_n^a}(q)$ over any algebraically closed field of positive characteristic, i.e. the "modular case". This is carried out for the algebras $TLB_n(q, Q)$ in [6].

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Representations of Lie algebras in positive characteristic

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About 50 years ago it was discovered that finite dimensional Lie algebras in positive characteristic only have finite dimensional irreducible representations. About 15 years ago the irreducible representations for the Lie algebra $\mathfrak{gl}_n$ were classified. About 5 years ago a conjecture was formulated that should lead to a calculation of the dimensions of these simple $\mathfrak{gl}_n$–modules if $p > n$. For Lie algebras of other reductive groups our knowledge is more restricted, but there has been some remarkable progress in this area over the last years. The purpose of this survey is to report on these developments and to update the earlier surveys [H3] and [J3].

Throughout this paper let K be an algebraically closed field of prime characteristic p. All Lie algebras over K will be assumed to be finite dimensional.

A General Theory

A.1. If $\mathfrak{g}$ is a Lie algebra over K, then we denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$.

A *restricted Lie algebra* over K is a Lie algebra $\mathfrak{g}$ over K together with a map $\mathfrak{g} \to \mathfrak{g}$, $X \mapsto X^{[p]}$, often called the *p–th power map*, provided certain conditions are satisfied. The first condition says that for each $X \in \mathfrak{g}$ the element

$$\xi(X) = X^p - X^{[p]} \in U(\mathfrak{g})$$

actually belongs to the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. (Here X^p is the p–th power of X taken in $U(\mathfrak{g})$.) The other condition says that $\xi : \mathfrak{g} \to Z(\mathfrak{g})$ is semi-linear in the following sense: We have

$$\xi(X+Y) = \xi(X) + \xi(Y) \qquad \text{and} \qquad \xi(a\,X) = a^p\,\xi(X)$$

for all $X, Y \in \mathfrak{g}$ and $a \in K$.

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A.2. For example, let A be an associative algebra over K considered as a Lie algebra via $[X,Y] = XY - YX$. Then A becomes a restricted Lie algebra if we set $X^{[p]}$ equal to the p-th power of X taken in A. Indeed, if we write l_X and r_X for left and write multiplication by X in A (so we have, e.g., $l_X(Y) = XY$ for all $Y \in A$), then $\mathrm{ad}(X) = l_X - r_X$. Since l_X and r_X commute and since we are in characteristic p, we get $\mathrm{ad}(X)^p = (l_X)^p - (r_X)^p$. Now $(l_X)^p$ is clearly left multiplication by the p-th power of X that we have decided to denote by $X^{[p]}$; similarly for $(r_X)^p$. We get thus $\mathrm{ad}(X)^p = \mathrm{ad}(X^{[p]})$. Now the same argument used in $U(A)$ instead of A shows that also $\mathrm{ad}(X)^p = \mathrm{ad}(X^p)$, hence that $X^p - X^{[p]}$ commutes with each element of $\mathfrak{g}$. Therefore $\xi(X) = X^p - X^{[p]}$ belongs to $Z(A)$. It remains to check the semi-linearity of ξ: The identity $\xi(a\,X) = a^p\,\xi(X)$ is obvious. The proof of the additivity of ξ requires a formula due to Jacobson expressing $(X+Y)^p - X^p - Y^p$ in terms of commutators, see [Ja], § V.7.

On the other hand, if G is an algebraic group over K, then $\mathrm{Lie}(G)$ has a natural structure as a restricted Lie algebra: One can think of $\mathrm{Lie}(G)$ as the Lie algebra of certain invariant derivations, cf. [H2], 9.1; Then one defines $X^{[p]}$ as the p-th power of X taken as derivation. In case $G = \mathrm{GL}_n(K)$, then one gets thus on $\mathrm{Lie}(G)$ the same structure as from the identification of $\mathrm{Lie}(G)$ with the space $M_n(k)$ of all $(n \times n)$-matrices over K and from the construction in the preceding paragraph. If G is a closed subgroup of $\mathrm{GL}_n(K)$, then we can identify $\mathrm{Lie}(G)$ with a Lie subalgebra of $M_n(k)$ and the p-th power map of $\mathrm{Lie}(G)$ is the restriction of that on $M_n(k)$.

A.3. Let now $\mathfrak{g}$ be an arbitrary restricted Lie algebra over K. Denote by $Z_0(\mathfrak{g})$ the subalgebra of $Z(\mathfrak{g})$ generated by all $\xi(X) = X^p - X^{[p]}$ with $X \in \mathfrak{g}$. This subalgebra is often called the *p-centre* of $U(\mathfrak{g})$.

Let $X_1, X_2, \ldots, X_n$ be a basis for $\mathfrak{g}$. Using the PBW-theorem one checks now easily: The algebra $Z_0(\mathfrak{g})$ is generated by all $X_i^p - X_i^{[p]}$ with $1 \le i \le n$; these elements are algebraically independent over K. Considered as a $Z_0(\mathfrak{g})$-module under left (= right) multiplication $U(\mathfrak{g})$ is free of rank $p^{\dim(\mathfrak{g})}$; all products

$$X_1^{m(1)} X_2^{m(2)} \ldots X_n^{m(n)} \qquad \text{with } 0 \le m(i) < p \text{ for all } i$$

form a basis of $U(\mathfrak{g})$ over $Z_0(\mathfrak{g})$.

The first of these two claims can be restated as follows: The map ξ induces an isomorphism of algebras

$$S(\mathfrak{g}^{(1)}) \xrightarrow{\sim} Z_0(\mathfrak{g}).$$

Here we use the following convention: If V is a vector space over K, we denote by $V^{(1)}$ the vector space over K that is equal to V as an additive group, but where any $a \in K$ acts on $V^{(1)}$ as $a^{1/p}$ does on V. Now the semi-linearity of ξ means that ξ is a linear map $\mathfrak{g}^{(1)} \to Z_0(\mathfrak{g})$, hence induces an algebra homomorphism from the symmetric algebra $S(\mathfrak{g}^{(1)})$ to the commutative algebra $Z_0(\mathfrak{g})$. The claim in the preceding paragraph shows that this map is bijective.

It is now easy to deduce from the results above:

Proposition: *The centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is a finitely generated algebra over K. Considered as a $Z(\mathfrak{g})$–module $U(\mathfrak{g})$ is finitely generated.*

This result actually generalises to all (finite dimensional!) Lie algebras over K.

A.4. Theorem: *Each simple $\mathfrak{g}$–module is finite dimensional. Its dimension is less than or equal to $p^{\dim(\mathfrak{g})}$.*

Proof: Choose $u_1, u_2, \dots, u_r \in U(\mathfrak{g})$ such that $U(\mathfrak{g}) = \sum_{i=1}^r Z(\mathfrak{g})u_i$. This is possible by the proposition; in fact, we may assume that $r \leq p^{\dim(\mathfrak{g})}$ as $U(\mathfrak{g})$ has that rank over the smaller subalgebra $Z_0(\mathfrak{g})$.

Let E be a simple $\mathfrak{g}$–module. Pick $v \in E$, $v \neq 0$. We have then $E = U(\mathfrak{g})v$, hence $E = \sum_{i=1}^r Z(\mathfrak{g})u_i v$. So E is finitely generated as a module over $Z(\mathfrak{g})$. Since $Z(\mathfrak{g})$ is a finitely generated K–algebra, hence a Noetherian ring, there exists a maximal submodule $E' \subset E$. The simple $Z(\mathfrak{g})$–module E/E' is then isomorphic to $Z(\mathfrak{g})/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset Z(\mathfrak{g})$. Now xE is a $\mathfrak{g}$–submodule of E for all $x \in Z(\mathfrak{g})$. If $x \in \mathfrak{m}$, then $x(E/E') = 0$, hence $xE \subset E'$ is a submodule different from E. As E is simple, this implies that $xE = 0$. We get thus that $\mathfrak{m}E = 0$.

A weak version of the Hilbert Nullstellensatz says that $\mathfrak{m}$ has codimension 1 in $Z(\mathfrak{g})$, hence that $Z(\mathfrak{g}) = K1 + \mathfrak{m}$. So $E = \sum_{i=1}^r Z(\mathfrak{g})u_i v$ implies $E = \sum_{i=1}^r K u_i v$, hence $\dim(E) \leq r \leq p^{\dim(\mathfrak{g})}$.

A.5. Thanks to Theorem A.4 we can now associate to $\mathfrak{g}$ the number

$$M(\mathfrak{g}) = \max\{\, \dim E \mid E \text{ a simple } \mathfrak{g}\text{–module} \,\}.$$

Zassenhaus gave in [Za] a ring theoretic interpretation of $M(\mathfrak{g})$. Denote by F_0 a field of fractions for $Z_0(\mathfrak{g})$. Then

$$\mathrm{Frac}(U(\mathfrak{g})) = U(\mathfrak{g}) \otimes_{Z_0(\mathfrak{g})} F_0$$

is a localisation of $U(\mathfrak{g})$. The map $u \mapsto u \otimes 1$ from $U(\mathfrak{g})$ to $\mathrm{Frac}(U(\mathfrak{g}))$ is injective because $U(\mathfrak{g})$ is free over $Z_0(\mathfrak{g})$; we use it to identify $U(\mathfrak{g})$ with a subring of $\mathrm{Frac}(U(\mathfrak{g}))$. Each non-zero element $u \in U(\mathfrak{g})$ is invertible in $\mathrm{Frac}(U(\mathfrak{g}))$ as u is integral over $Z_0(\mathfrak{g})$. Therefore $\mathrm{Frac}(U(\mathfrak{g}))$ is a division ring. It contains a field of fractions F for $Z(\mathfrak{g})$ and F is the centre of $\mathrm{Frac}(U(\mathfrak{g}))$. It now turns out that

$$M(\mathfrak{g})^2 = \dim_F \mathrm{Frac}(U(\mathfrak{g}))$$

cf. [Za], Thms. 1 and 6. On the other hand, we have $\dim_{F_0} \mathrm{Frac}(U(\mathfrak{g})) = p^{\dim(\mathfrak{g})}$ because $U(\mathfrak{g})$ is free of rank $p^{\dim(\mathfrak{g})}$ over $Z_0(\mathfrak{g})$. It follows that $p^{\dim(\mathfrak{g})} = M(\mathfrak{g})^2 \cdot \dim_{F_0} F$, hence that $M(\mathfrak{g})$ is a power of p.

A.6. In [VK], 1.2 Veisfeiler and Kats made a conjecture on the value of $M(\mathfrak{g})$. For each linear form $\chi \in \mathfrak{g}^*$ denote by $\mathfrak{g}_\chi$ its stabiliser in $\mathfrak{g}$ for the coadjoint action:

$$\mathfrak{g}_\chi = \{\, X \in \mathfrak{g} \mid X \cdot \chi = 0 \,\} = \{\, X \in \mathfrak{g} \mid \chi([X, \mathfrak{g}]) = 0 \,\}.$$

This is a restricted Lie subalgebra of $\mathfrak{g}$. For each $\chi \in \mathfrak{g}^*$ the bilinear form $(X, Y) \mapsto \chi([X, Y])$ on $\mathfrak{g}$ induces a non-degenerate alternating form on $\mathfrak{g}/\mathfrak{g}_\chi$; therefore the dimension of this quotient space is even. Set

$$r(\mathfrak{g}) = \min \{\, \dim \mathfrak{g}_\chi \mid \chi \in \mathfrak{g}^* \,\}.$$

Then also $\dim(\mathfrak{g}) - r(\mathfrak{g})$ is even and the conjecture says:

Conjecture ([VK]): $\qquad M(\mathfrak{g}) = p^{(\dim(\mathfrak{g}) - r(\mathfrak{g}))/2}.$

Work by Mil′ner and by Premet and Skryabin (see [Mi], Thm. 3 and [PS], Thm. 4.4) shows:

Theorem: *If there exists a linear form χ on $\mathfrak{g}$ such that $\mathfrak{g}_\chi$ is a toral subalgebra of $\mathfrak{g}$, then this conjecture holds.*

(A subalgebra $\mathfrak{h}$ of a restricted Lie algebra is called toral if it is commutative and if the p–power map $X \mapsto X^{[p]}$ restricts to a bijective map $\mathfrak{h} \to \mathfrak{h}$. This means that $\mathfrak{h}$ is isomorphic as a restricted Lie algebra to the Lie algebra of a torus.)

A.7. Let E be a simple $\mathfrak{g}$–module. Since $\dim(E) < \infty$, Schur's lemma implies that each element in $Z(\mathfrak{g})$ acts as multiplication by a scalar on E. This applies in particular to all $\xi(X) = X^p - X^{[p]}$ with $X \in \mathfrak{g}$. Using the semi-linearity of ξ one shows now that there exists a linear form $\chi_E \in \mathfrak{g}^*$ with

$$(X^p - X^{[p]})_{|E} = \chi_E(X)^p \, \mathrm{id}_E \qquad \text{for all } X \in \mathfrak{g}.$$

One calls χ_E the *p–character* of E.

For each $\chi \in \mathfrak{g}^*$ set

$$U_\chi(\mathfrak{g}) = U(\mathfrak{g})/(X^p - X^{[p]} - \chi(X)^p 1 \mid X \in \mathfrak{g}).$$

This is a finite dimensional algebra over K of dimension $p^{\dim(\mathfrak{g})}$. If X_1, $X_2, \ldots, X_n$ is a basis for $\mathfrak{g}$, then the classes of all $X_1^{m(1)} X_2^{m(2)} \ldots X_n^{m(n)}$ with $0 \le m(i) < p$ for all i are a basis for $U_\chi(\mathfrak{g})$. One calls $U_\chi(\mathfrak{g})$ a *reduced enveloping algebra* of $\mathfrak{g}$. (The special case $U_0(\mathfrak{g})$ is usually called the *restricted enveloping algebra* of $\mathfrak{g}$.)

Each simple $U_\chi(\mathfrak{g})$–module (for any χ) is in a natural way a simple $\mathfrak{g}$–module. The discussion in the first paragraph of this subsection shows: *Each simple $\mathfrak{g}$–module is a simple $U_\chi(\mathfrak{g})$–module for exactly one $\chi \in \mathfrak{g}^*$.*

A.8. If $\gamma : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of $\mathfrak{g}$ as a restricted Lie algebra (i.e., a Lie algebra automorphism with $\gamma(X^{[p]}) = \gamma(X)^{[p]}$ for all $X \in \mathfrak{g}$), then γ induces an isomorphism

$$U_\chi(\mathfrak{g}) \xrightarrow{\sim} U_{\gamma\cdot\chi}(\mathfrak{g})$$

where $(\gamma \cdot \chi)(X) = \chi(\gamma^{-1}(X))$.

In particular, if $\mathfrak{g} = \mathrm{Lie}(G)$ for some algebraic group G over K, then any $g \in G$ acts via the adjoint action $\mathrm{Ad}(g)$ on $\mathfrak{g}$. Each $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$ as a restricted Lie algebra. So we get an isomorphism $U_\chi(\mathfrak{g}) \xrightarrow{\sim} U_{g\cdot\chi}(\mathfrak{g})$ where $g \cdot \chi$ refers to the coadjoint action of g. This implies: If we know all simple $\mathfrak{g}$–modules with a given p–character χ, then we know also all simple $\mathfrak{g}$–modules with a p–character in the coadjoint orbit $G \cdot \chi$.

B Reductive Lie Algebras

B.1. Assume from now on that G is a connected reductive algebraic group over K with a maximal torus T and a Borel subgroup $B^+ \supset T$. Denote the Lie algebras of G, T, B^+ by $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{b}^+$ respectively.

We denote by X the character group of T and set $R \subset X$ equal to the set of roots of G relative to T. We denote by R^+ the set of roots of B^+ relative to T; this is a system of positive roots in R.

For each $\alpha \in R$ let $\mathfrak{g}_\alpha$ denote the corresponding root subspace in $\mathfrak{g}$. Set $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) equal to the sum of all $\mathfrak{g}_\alpha$ with $\alpha \in R^+$ (resp. with $-\alpha \in R^+$). We have then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$.

Fix for each α a basis vector X_α for $\mathfrak{g}_\alpha$. These elements satisfy $X_\alpha^{[p]} = 0$, e.g., since $X_\alpha^{[p]} \in \mathfrak{g}_{p\alpha} = 0$. On the other hand, the Lie algebra $\mathfrak{h}$ of the torus T has a basis $H_1, H_2, \ldots, H_m$ with $H_i^{[p]} = H_i$ for all i.

If $\mathfrak{a}$ is a restricted Lie subalgebra of $\mathfrak{g}$, then we shall usually write $U_\chi(\mathfrak{a}) = U_{\chi|_\mathfrak{a}}(\mathfrak{a})$ for all $\chi \in \mathfrak{g}^*$.

B.2. For each $Y \in \mathfrak{g}$ one can find $g \in G$ with $\mathrm{Ad}(g)(Y) \in \mathfrak{b}^+$, cf. [Bo], Prop. 14.25. One should have analogously:

$$\text{For each } \chi \in \mathfrak{g}^* \text{ there exists } g \in G \text{ with } (g \cdot \chi)(\mathfrak{n}^+) = 0. \qquad (*)$$

This was proved in [KW], Lemma 3.2 for almost simple G except for the case where $G = \mathrm{SO}_{2n+1}$ and $p = 2$. Their argument can be extended to prove $(*)$ whenever the derived group of G is simply connected.

In many cases there exists a non-degenerate G–invariant bilinear form on $\mathfrak{g}$. We can use it to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Let $Y \in \mathfrak{g}$ correspond to $\chi \in \mathfrak{g}^*$. Choose $g \in G$ with $\mathrm{Ad}(g)(Y) \in \mathfrak{b}^+$. Then the image $g \cdot \chi$ of $\mathrm{Ad}(g)(Y)$ vanishes on $(\mathfrak{b}^+)^\perp = \mathfrak{n}^+$. So we get $(*)$ in this case.

Suppose that G satisfies $(*)$. It then follows from A.8 that it suffices to determine the simple $U_\chi(\mathfrak{g})$–modules for all χ with $\chi(\mathfrak{n}^+) = 0$ if we want to describe all simple $\mathfrak{g}$–modules.

B.3. Let $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$. Then each X_α with $\alpha \in R^+$ acts nilpotently on any $U_\chi(\mathfrak{n}^+)$–module since $\chi(X_\alpha) = 0$ and $X_\alpha^{[p]} = 0$. This implies (using an inductive argument) for each $U_\chi(\mathfrak{n}^+)$–module M

$$M \neq 0 \implies M^{\mathfrak{n}^+} \neq 0.$$

(We write generally $M^{\mathfrak{a}}$ for the space of fixed points in a module M over a Lie algebra $\mathfrak{a}$.)

If M is a $U_\chi(\mathfrak{b}^+)$–module, then $\mathfrak{h}$ stabilises $M^{\mathfrak{n}^+}$ as $\mathfrak{n}^+$ is an ideal in $\mathfrak{b}^+$. Since $\mathfrak{h}$ is commutative, it then has a common eigenvector in $M^{\mathfrak{n}^+}$ provided $M \neq 0$. So we get in this case some $v \in M$, $v \neq 0$ and some $\mu \in \mathfrak{h}^*$ with $Hv = \mu(H)v$ for all $H \in \mathfrak{h}$ and with $Yv = 0$ for all $Y \in \mathfrak{n}^+$.

Each $\lambda \in \mathfrak{h}^*$ defines a one dimensional $\mathfrak{b}^+$–module K_λ where $\mathfrak{n}^+$ acts as 0 and where each $H \in \mathfrak{h}$ acts as $\lambda(H)$. Then K_λ is a $U_\chi(\mathfrak{b}^+)$–module if and only if $\lambda \in \Lambda_\chi$ where we set

$$\Lambda_\chi = \{\, \lambda \in \mathfrak{h}^* \mid \lambda(H)^p - \lambda(H^{[p]}) = \chi(H)^p \text{ for all } H \in \mathfrak{h} \,\}.$$

Now the linear function μ in the preceding paragraph has to belong to Λ_χ because M is a $U_\chi(\mathfrak{b}^+)$–module.

Recall that $\mathfrak{h}$ has a basis $H_1, H_2, \dots, H_m$ with $H_i^{[p]} = H_i$ for all i. The semi-linearity of the map $H \mapsto H^p - H^{[p]}$ implies that some $\lambda \in \mathfrak{h}^*$ belongs to Λ_χ if and only if

$$\chi(H_i)^p = \lambda(H_i)^p - \lambda(H_i^{[p]}) = \lambda(H_i)^p - \lambda(H_i)$$

for all i. Given χ, this shows that each $\lambda(H_i)$ can take exactly p distinct values. This implies that

$$|\Lambda_\chi| = p^{\dim(\mathfrak{h})}.$$

B.4. Let again $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0$. For each $\lambda \in \Lambda_\chi$ we can now consider the $U_\chi(\mathfrak{b}^+)$–module K_λ and the induced $U_\chi(\mathfrak{g})$–module

$$Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{b}^+)} K_\lambda$$

which is often called a *baby Verma module.*

We have

$$\dim Z_\chi(\lambda) = p^N \qquad \text{where } N = |R^+|.$$

If we set $v_\lambda = 1 \otimes 1$ and choose a numbering $\alpha_1,\ \alpha_2, \dots, \alpha_N$ of the positive roots, then all

$$X_{-\alpha_1}^{m(1)} X_{-\alpha_2}^{m(2)} \dots X_{-\alpha_N}^{m(N)} v_\lambda$$

with $0 \leq m(i) < p$ for all i form a basis for $Z_\chi(\lambda)$.

If M is a non-zero $U_\chi(\mathfrak{g})$–module, then the discussion in B.3 shows that there exists some $\lambda \in \Lambda_\chi$ with $\mathrm{Hom}_{\mathfrak{b}^+}(K_\lambda, M) \neq 0$. The universal property of the tensor product implies then $\mathrm{Hom}_{\mathfrak{g}}(Z_\chi(\lambda), M) \neq 0$. We get therefore (as observed in [Ru]):

Lemma: *If E is a simple $U_\chi(\mathfrak{g})$–module, then E is a homomorphic image of some $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$.*

B.5. Let us assume that the derived group of G is simply connected. If the p–character χ of a simple $\mathfrak{g}$–module E satisfies $\chi(\mathfrak{n}^+) = 0$, then the results in B.4 imply that $\dim E \leq \dim Z_\chi(\lambda)$ for a suitable λ, hence $\dim E \leq p^N$. By our assumption G satisfies B.2($*$), so this inequality holds for all simple $\mathfrak{g}$–modules E. On the other hand, one knows that G has a Steinberg module that is irreducible of dimension p^N and remains irreducible under restriction to $\mathfrak{g}$. This shows (in the notation from A.5) that

$$M(\mathfrak{g}) = p^N.$$

Note that this result is compatible with the conjecture mentioned in A.6. Since $\dim(\mathfrak{g}) = 2N + \dim(\mathfrak{h})$, we just have to check that $\dim(\mathfrak{h})$ is the minimal dimension of all $\mathfrak{g}_\chi$ with $\chi \in \mathfrak{g}^*$. Well, our assumption on the derived group of G implies that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$ for all $\alpha \in R$. Therefore we can find $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{n}^+) = 0 = \chi(\mathfrak{n}^-)$ and $\chi([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \neq 0$ for all $\alpha \in R$. Then χ satisfies $\mathfrak{g}_\chi = \mathfrak{h}$. (This shows, by the way, that the assumption in Theorem A.6 is satisfied.) One can now use semi-continuity arguments to show that $\dim \mathfrak{g}_\chi \geq \dim(\mathfrak{h})$ for all χ. (The union of the orbits $G \cdot \chi$ with $\mathfrak{g}_\chi = \mathfrak{h}$ is dense in $\mathfrak{g}^*$.)

B.6. Let us make from now on the following simplifying assumptions:

(H1) The derived group of G is simply connected.

(H2) The prime p is good for G.

(H3) There exists a non-degenerate G–invariant bilinear form on $\mathfrak{g}$.

The assumption (H2) excludes $p = 2$ if R has a component not of type A, it excludes $p = 3$ if R has a component of exceptional type, and it excludes $p = 5$ if R has a component of type E_8. If G is almost simple and and if (H2) holds, then (H3) holds unless R has type A_n with $p \mid n+1$. Note that $G = \mathrm{GL}_n$ satisfies all three conditions for all n and p: In (H3) one can take the bilinear form $(Y, Z) = \mathrm{trace}(YZ)$ on $\mathrm{Lie}(G) = M_n(K)$.

One nice aspect of (H1)–(H3) is the following: With G also each Levi subgroup of G satisfies these hypotheses.

B.7. Premet has shown in [Pr], Thm. 3.10 (see also [PS], Thm. 5.6) under our assumptions or slightly weaker ones (proving a conjecture from [VK], 3.5):

Theorem: *Let* $\chi \in \mathfrak{g}^*$. *Then* $p^{\dim(G \cdot \chi)/2}$ *divides* $\dim(M)$ *for each finite dimensional* $U_\chi(\mathfrak{g})$*–module* M.

(It turns out that under our assumption each orbit $G \cdot \chi$ has an even dimension; so the claim makes sense.) For an introduction to Premet's original proof one may also compare [J3], sections 7 and 8.

B.8. Let $\psi : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ be an isomorphism of G–modules induced by a bilinear form as in (H3). We can use ψ to transport notions like *nilpotent* or *semi-simple* from $\mathfrak{g}$ to $\mathfrak{g}^*$ and call $\chi \in \mathfrak{g}^*$ nilpotent (or semi-simple) if $\psi^{-1}(\chi)$ is so. One can also define these notions intrinsically saying that χ is semi-simple if and only if the orbit $G \cdot \chi$ is closed if and only if there exists $g \in G$ with $(g \cdot \chi)(\mathfrak{n}^+ \oplus \mathfrak{n}^-) = 0$. And χ is nilpotent if and only if $0 \in \overline{G \cdot \chi}$ if and only if there exists $g \in G$ with $(g \cdot \chi)(\mathfrak{b}^+) = 0$.

A general $\chi \in \mathfrak{g}^*$ has then a Jordan decomposition $\chi = \chi_s + \chi_n$ with χ_s semi-simple and χ_n nilpotent such that $\psi^{-1}(\chi) = \psi^{-1}(\chi_s) + \psi^{-1}(\chi_n)$ is the Jordan decomposition in $\mathfrak{g}$. (Again, this can be defined directly in $\mathfrak{g}^*$, see [KW], section 3.)

Consider a Jordan decomposition $\chi = \chi_s + \chi_n$ as above and set $\mathfrak{l} = \mathfrak{g}_{\chi_s}$. Our assumption (H2) implies that there exists a Levi subgroup L of G with $\mathfrak{l} = \mathrm{Lie}(L)$. (Replacing χ by an element in $G \cdot \chi$, one may assume that $\chi_s(\mathfrak{n}^+) = 0 = \chi_s(\mathfrak{n}^-)$. Then $\mathfrak{l}$ is equal to the sum of $\mathfrak{h}$ and all $\mathfrak{g}_\alpha$ with $\chi_s([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) = 0$. The set of all α with this property is conjugate under the Weyl group to a set of the form $R \cap \mathbf{Z}I$ where I is a subset of the set of simple roots; here one uses that p is good.) There exists then a parabolic subgroup P in G such that P is the semi-direct product of its unipotent radical U_P and of L. Then $\mathfrak{p} = \mathrm{Lie}(P)$ satisfies $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{u} = \mathrm{Lie}(U_P)$.

We have now $\chi(\mathfrak{u}) = 0$. (This follows from the fact that $\psi^{-1}(\chi_s)$ and $\psi^{-1}(\chi_n)$ commute.) If we extend a $U_\chi(\mathfrak{l})$–module to a $\mathfrak{p}$–module letting the ideal $\mathfrak{u}$ act via 0, then we get therefore a $U_\chi(\mathfrak{p})$–module. Now one can show:

Theorem: *The functors $V \mapsto U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} V$ and $M \mapsto M^{\mathfrak{u}}$ are inverse equivalences of categories between $\{\, U_\chi(\mathfrak{l})$–modules$\,\}$ and $\{\, U_\chi(\mathfrak{g})$–modules$\,\}$. They induce bijections of isomorphism classes of simple modules.*

This goes back to Veisfeiler and Kats who showed in [VK], Thm. 2 that the simple $U_\chi(\mathfrak{g})$–modules are the $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} E$ with E a simple $U_\chi(\mathfrak{l})$–module. The more precise statement here is due to Friedlander and Parshall, see [FP1], Thm. 3.2 and Thm. 8.5. (See also [J3], 7.4 for a proof of this result based on a theorem of Premet.)

The unipotent group U_P satisfies $U_\chi(\mathfrak{u})^{\mathfrak{u}} = K1$. Using the PBW theorem one can now check that the equivalence of categories $M \mapsto M^{\mathfrak{u}}$ take $U_\chi(\mathfrak{g})$ to a direct sum of p^d copies of $U_\chi(\mathfrak{l})$ where $d = \dim(\mathfrak{g}/\mathfrak{p})$. This implies that $U_\chi(\mathfrak{g})$ is isomorphic to the matrix ring $M_{p^d}(U_\chi(\mathfrak{l}))$.

B.9. Theorem B.8 reduces the problem of finding the simple $\mathfrak{g}$–modules to the investigation of the simple $U_\chi(\mathfrak{g})$–modules for $\chi \in \mathfrak{g}^*$ with $\mathfrak{g}_{\chi_s} = \mathfrak{g}$ and to the analogous problem for Lie algebras of smaller reductive groups that again satisfy (H1)–(H3).

By definition $\mathfrak{g}_{\chi_s} = \mathfrak{g}$ means that $\chi_s([\mathfrak{g}, \mathfrak{g}]) = 0$. Under our assumptions $[\mathfrak{g}, \mathfrak{g}]$ is the Lie algebra of the derived group of G. So $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a restricted Lie algebra. Let E be a simple $U_\gamma(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$–module where $\gamma(Y + [\mathfrak{g}, \mathfrak{g}]) = \chi_s(Y)$ for all $Y \in \mathfrak{g}$. Then E has dimension 1 because $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is commutative, and E is a $U_{\chi_s}(\mathfrak{g})$–module when considered as a $\mathfrak{g}$–module. Now $V \mapsto V \otimes E$ and $V' \mapsto V' \otimes E^*$ are inverse equivalences of categories between $\{\, U_{\chi_n}(\mathfrak{g})$–modules$\,\}$ and $\{\, U_\chi(\mathfrak{g})$–modules$\,\}$.

This shows that it suffices to study $U_{\chi_n}(\mathfrak{g})$–modules. So we have a reduction to the case where χ is nilpotent.

B.10. Before we investigate the nilpotent case, let us look at a special case of Theorem B.8. Suppose that χ is *regular semi-simple*, i.e., that $\chi = \chi_s$ and that $\dim(\mathfrak{g}_\chi) = \dim(\mathfrak{h})$. Replacing χ by a conjugate under G, we may assume that $\mathfrak{g}_\chi = \mathfrak{h}$. This means that $\chi(\mathfrak{n}^+) = 0 = \chi(\mathfrak{n}^-)$ and that $\chi([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \neq 0$ for all $\alpha \in R$. We have $\mathfrak{l} = \mathfrak{h}$ in the notation of B.8 and may choose $P = B^+$. So Theorem B.8 says in this case that all $U_\chi(\mathfrak{g})$–modules are semi-simple (since $U_\chi(\mathfrak{h})$–modules are so) and that the simple $U_\chi(\mathfrak{g})$–modules are the $Z_\chi(\lambda)$ with $\lambda \in \Lambda_\chi$. These $Z_\chi(\lambda)$ are pairwise non-isomorphic.

C Nilpotent Forms

We keep throughout the assumptions and notations from B.1 and B.6.

C.1. By B.8 and B.9 we may restrict to the case where χ is nilpotent. Replacing χ by some $g \cdot \chi$ with $g \in G$, we may assume that $\chi(\mathfrak{b}^+) = 0$. We have then in particular $\Lambda_\chi = \Lambda_0$.

One can identify the multiplicative group over K with GL_1, hence its Lie algebra with $M_1(K)$. This Lie algebra has basis H equal to the (1×1)–matrix (1). This shows that $H^{[p]} = H$ and that each character $\varphi_n : t \mapsto t^n$ with $n \in \mathbf{Z}$ of the multiplicative group has tangent map $d\varphi_n$ mapping H to n. A linear form λ on $M_1(K)$ satisfies $\lambda(H)^p - \lambda(H^{[p]}) = 0$ if and only if $\lambda(H) \in \mathbf{F}_p$ if and only if $\lambda = d\varphi_n$ for some n. Furthermore, we have $d\varphi_n = d\varphi_m$ if and only if $n \equiv m \pmod p$.

This implies now for T, a direct product of multiplicative groups, that

$$\Lambda_0 = \{\, d\lambda \mid \lambda \in X \,\}$$

and that $d\lambda = d\mu$ if and only if $\lambda \equiv \mu \pmod{pX}$.

We shall often write K_μ instead of $K_{d\mu}$ and $Z_\chi(\mu)$ instead of $Z_\chi(d\mu)$ for $\mu \in X$ and $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$. We have to keep in mind that then $Z_\chi(\mu) = Z_\chi(\mu + p\nu)$ for all $\mu, \nu \in X$.

Let $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$. We know by B.4 that each simple $U_\chi(\mathfrak{g})$–module is the homomorphic image of some $Z_\chi(\mu)$ with $\mu \in X$. The problem now is that μ is not necessarily uniquely determined; we shall see an example of this in a moment. Furthermore, some $Z_\chi(\mu)$ may have more than one simple homomorphic image. In fact, there exist even $Z_\chi(\mu)$ that are decomposable, see [J3], 6.9.

C.2. Fix $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$. For each finite dimensional $U_\chi(\mathfrak{g})$–module M denote by $[M]$ the class of M in the Grothendieck group of all finite dimensional $U_\chi(\mathfrak{g})$–modules. If $E_1, E_2, \dots, E_r$ is a system of representatives for the isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules, then $[E_1], [E_2], \dots, [E_r]$ is a basis over $\mathbf{Z}$ for this Grothendieck group. An arbitrary M then satisfies $[M] = \sum_{i=1}^r [M : E_i][E_i]$ where $[M : E_i]$ is the multiplicity of E_i as a composition factor of M.

Let W denote the Weyl group of G with respect to T. For each $\alpha \in R$ denote by $s_\alpha \in W$ the corresponding reflection given by $s_\alpha(\mu) = \mu - \langle \mu, \alpha^\vee \rangle \alpha$ where $\alpha^\vee$ is the coroot to α. We shall often use the "dot action" of W on X, given by $w \bullet \mu = w(\mu + \rho) - \rho$ where $\rho \in X \otimes_{\mathbf{Z}} \mathbf{Q}$ is half the sum of all positive roots.

Proposition: *We have $[Z_\chi(w\bullet\lambda)] = [Z_\chi(\lambda)]$ for all $w \in W$ and $\lambda \in X$.*

This was first shown in [H1], Thm. 2.2 in case $\chi = 0$; the proof in that case generalises. It suffices to take $w = s_\alpha$ with α a simple root. Let d denote the integer with $\langle \lambda + \rho, \alpha^\vee \rangle \equiv d \pmod p$ and $0 \leq d < p$. Then $s_\alpha \bullet \lambda \equiv \lambda - d\alpha \pmod{pX}$; so we have to show that $[Z_\chi(\lambda - d\alpha)] = [Z_\chi(\lambda)]$.

This is trivial if $d = 0$. If $d > 0$, then rank 1 calculations show that there exists a homomorphism of $\mathfrak{g}$–modules $\varphi : Z_\chi(\lambda - d\alpha) \to Z_\chi(\lambda)$ given by $\varphi(v_{\lambda - d\alpha}) = X_{-\alpha}^d v_\lambda$. (We use here notations like v_λ as in B.4.)

If $\chi(X_{-\alpha}) \neq 0$, then $X_{-\alpha}^p - X_{-\alpha}^{[p]} = X_{-\alpha}^p$ acts as the non-zero scalar $\chi(X_{-\alpha})^p$ on $Z_\chi(\lambda)$. It then follows that

$$v_\lambda = \chi(X_{-\alpha})^{-p} X_{-\alpha}^p v_\lambda = \chi(X_{-\alpha})^{-p} X_{-\alpha}^{p-d} \varphi(v_{\lambda - d\alpha})$$

belongs to the image of φ. Therefore φ is surjective, hence bijective by dimension comparison. So in this case φ is an isomorphism $Z_\chi(\lambda - d\alpha) \xrightarrow{\sim} Z_\chi(\lambda)$.

If $\chi(X_{-\alpha}) = 0$, then one checks — working with bases as in B.4 such that $\alpha_N = \alpha$ — that the kernel of φ is generated by $X_{-\alpha}^{p-d} v_{\lambda - d\alpha}$. Furthermore there is a homomorphism ψ from $Z_\chi(\lambda - p\alpha) = Z_\chi(\lambda)$ to $Z_\chi(\lambda - d\alpha)$ with $\psi(v_\lambda) = X_{-\alpha}^{p-d} v_{\lambda - d\alpha}$. One gets then $\ker(\varphi) = \mathrm{im}(\psi)$ and $\ker(\psi) = \mathrm{im}(\varphi)$, hence

$$[Z_\chi(\lambda)] = [\ker(\psi)] + [\mathrm{im}(\psi)] = [\mathrm{im}(\varphi)] + [\ker(\varphi)] = [Z_\chi(\lambda - d\alpha)]$$

as claimed.

C.3. If χ (as in C.2) satisfies $\chi(X_{-\alpha}) \neq 0$ for *all* simple roots α, then the (sketched) proof of Proposition C.2 shows that $Z_\chi(w \bullet \lambda) \simeq Z_\chi(\lambda)$ for all $w \in W$ and $\lambda \in X$.

In this case χ is "regular nilpotent", i.e., satisfies $\dim(G \cdot \chi) = 2\dim(\mathfrak{g}/\mathfrak{b}) = 2N$. (If χ corresponds to $Y \in \mathfrak{g}$ under an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ as in B.8, then $Y = \sum_{\alpha \in R^+} a_\alpha X_\alpha$ with suitable $a_\alpha \in K$ and with $a_\alpha \neq 0$ for all simple roots α. Such elements in $\mathfrak{g}$ are regular nilpotent, see [J6], 6.7(1).)

Now Theorem B.7 says in this case that p^N divides the dimension of each $U_\chi(\mathfrak{g})$–module. Since all $Z_\chi(\lambda)$ have this dimension, they have to be simple. We have thus shown most of the following result (proved in [FP1], 4.2/3 for certain types, in [FP2], 2.2–4 in general under slightly more restrictive conditions on p; for $G = \mathrm{SL}_n$ see also [P1], Thm. 5):

Proposition: *Let $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$ and $\chi(X_{-\alpha}) \neq 0$ for all simple roots α. Then each $Z_\chi(\lambda)$ with $\lambda \in X$ is simple and each simple*

$U_\chi(\mathfrak{g})$–module is isomorphic to some $Z_\chi(\lambda)$ with $\lambda \in X$. Given $\lambda, \mu \in X$ we have $Z_\chi(\lambda) \simeq Z_\chi(\mu)$ if and only if $\lambda \in W \bullet \mu + pX$.

C.4. The only argument missing above is the proof of the claim: If $Z_\chi(\lambda) \simeq Z_\chi(\mu)$, then $\lambda \in W \bullet \mu + pX$. For this one looks at the subalgebra $U(\mathfrak{g})^G$ of all $\mathrm{Ad}(G)$–invariant elements in $U(\mathfrak{g})$. This subalgebra is contained in the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. If we were working in characteristic 0, then $U(\mathfrak{g})^G$ would be all of $Z(\mathfrak{g})$. In our present set-up, however, there are many elements of the form $Y^p - Y^{[p]}$ with $Y \in \mathfrak{g}$ that belong to $Z(\mathfrak{g})$, but not to $U(\mathfrak{g})^G$. One can show that $Z(\mathfrak{g})$ is generated by $U(\mathfrak{g})^G$ and $Z_0(\mathfrak{g})$, cf. [BG], Thm. 3.5. (However, this does not imply that the canonical map $U(\mathfrak{g}) \to U_\chi(\mathfrak{g})$ maps $U(\mathfrak{g})^G$ onto the centre of $U_\chi(\mathfrak{g})$; see the counter-example by Premet in [BG], 3.17.)

One checks now that there exists for each $u \in U(\mathfrak{g})^G$ and each $\lambda \in X$ a scalar $\mathrm{cen}_\lambda(u) \in K$ such that $u\, v_\lambda = \mathrm{cen}_\lambda(u)\, v_\lambda$ in $Z_\chi(\lambda)$ for all $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^*) = 0$. It then follows u acts as multiplication with $\mathrm{cen}_\lambda(u)$ on all of $Z_\chi(\lambda)$, hence also on all composition factors of $Z_\chi(\lambda)$. Each cen_λ is an algebra homomorphism from $U(\mathfrak{g})^G$ to K. Now one has analogously to the Harish-Chandra theorem in characteristic 0:

Proposition: *If $\lambda, \mu \in X$, then $\mathrm{cen}_\lambda = \mathrm{cen}_\mu$ if and only if $\lambda \in W \bullet \mu + pX$.*

In fact, one has as in characteristic 0 a Harish-Chandra isomorphism $U(\mathfrak{g})^G \xrightarrow{\sim} U(\mathfrak{h})^W$. This was first proved in [H1], Thm. 3.1 for large p (larger than the Coxeter number) and in [KW] for almost simple G. (The arguments there extend to the present set-up. See [J3], 9.6 for a proof that uses reduction modulo p techniques, as in [H1].)

C.5. Proposition C.4 has an obvious corollary for the description of the blocks of $U_\chi(\mathfrak{g})$: If two simple $U_\chi(\mathfrak{g})$–modules E and E' belong to the same block, then $U(\mathfrak{g})^G$ has to act via the same character on both E and E'. So, if E is a composition factor of $Z_\chi(\lambda)$ and if E' is one of $Z_\chi(\lambda')$, and if E, E' belong to the same block, then $\lambda' \in W \bullet \lambda + pX$. But then Proposition C.2 implies that E' is also a composition factor of $Z_\chi(\lambda)$. This proves the implications "(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)" and "(iii) $\Rightarrow$ (ii)" in:

Proposition: *Let $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$. Let E and E' be simple $U_\chi(\mathfrak{g})$–modules. Then the following are equivalent:*

(i) *E and E' belong to the same block of $U_\chi(\mathfrak{g})$.*

(ii) *$U_\chi(\mathfrak{g})^G$ acts via the same character on E and E'.*

(iii) *There exists $\lambda \in X$ such that both E and E' are composition factors of $Z_\chi(\lambda)$.*

Remark: If (i) or (ii) holds for one χ, then it holds for all $g \cdot \chi$ with $g \in G$. Therefore the equivalence of (i) and (ii) holds for all nilpotent $\chi \in \mathfrak{g}^*$, not only for those with $\chi(\mathfrak{b}^+) = 0$.

Note that the implication "(iii) $\Rightarrow$ (i)" [that we did not prove here] is obvious in the cases where all $Z_\chi(\lambda)$ are indecomposable. This holds for $\chi = 0$ by [H1], Prop. 1.5, and more generally for χ in standard Levi form, see D.1 below. (It actually suffices to find for each $\lambda \in X$ one $\lambda' \in W \bullet \lambda + pX$ such that $Z_\chi(\lambda')$ is indecomposable; that is in many additional cases possible, for example always when R has no component of exceptional type, see [J5], C.3 and H.1.)

The first general proof of "(ii) $\Rightarrow$ (i)" was given in [BG], Thm. 3.18 (assuming $p > 2$). A more direct proof (that works also for $p = 2$) is due to Gordon, see [Go], Thm. 3.6. Let m_0 denote the number of orbits of W on X/pX with respect to the dot action. The implication "(i) $\Rightarrow$ (iii)" shows that the number of blocks of $U_\chi(\mathfrak{g})$ is at least equal to m_0 for all $\chi \in (\mathfrak{b}^+)^\perp = \{\chi \in \mathfrak{g}^* \mid \chi(\mathfrak{b}^+) = 0\}$. So the implication "(iii) $\Rightarrow$ (i)" is equivalent to the claim that each $U_\chi(\mathfrak{g})$ has at most m_0 blocks. Now one checks for each m that the set

$$D_m = \{\, \chi \in (\mathfrak{b}^+)^\perp \mid U_\chi(\mathfrak{g}) \text{ has at most } m \text{ blocks} \,\}$$

is closed in $(\mathfrak{b}^+)^\perp$. Proposition C.3 implies that all regular nilpotent elements in $(\mathfrak{b}^+)^\perp$ belong to D_{m_0}. As these elements are dense in $(\mathfrak{b}^+)^\perp$, we get $D_{m_0} = (\mathfrak{b}^+)^\perp$, hence the claim.

C.6. Semi-continuity arguments like the one used in C.5 can be used for many purposes in the present theory. They often rely on the following observation: Fix $\lambda \in X$ and choose a numbering $\alpha_1, \alpha_2, \dots, \alpha_N$ of the positive roots. For all $\chi \in (\mathfrak{b}^+)^\perp$ and $\mathbf{m} = (m(1), m(2), \dots, m(N))$ in $\mathbf{Z}^N$ with $0 \le m(i) < p$ for all i let $z_{\mathbf{m},\chi}$ denote the basis element $X_{-\alpha_1}^{m(1)} X_{-\alpha_2}^{m(2)} \dots X_{-\alpha_N}^{m(N)} v_\lambda$ of $Z_\chi(\lambda)$, cf. B.4. There are then for all $Y \in \mathfrak{g}$ and all $\mathbf{m}$, $\mathbf{n}$ elements $c_{\mathbf{n},\mathbf{m}}(Y, \chi) \in K$ such that

$$Y\, z_{\mathbf{m},\chi} = \sum_{\mathbf{n}} c_{\mathbf{n},\mathbf{m}}(Y, \chi)\, z_{\mathbf{n},\chi}$$

cf. the proof of A.7(2) in [J5].

Each $c_{\mathbf{n},\mathbf{m}} : \mathfrak{g} \times (\mathfrak{b}^+)^\perp \to K$ is a linear function of $Y \in \mathfrak{g}$ and a polynomial function of $\chi \in (\mathfrak{b}^+)^\perp$. Using this one can check that both

$$\{\, \chi \in (\mathfrak{b}^+)^\perp \mid Z_\chi(\lambda) \text{ is simple} \,\} \text{ and } \{\, \chi \in (\mathfrak{b}^+)^\perp \mid Z_\chi(\lambda) \text{ is projective} \,\}$$

are open subsets of $(\mathfrak{b}^+)^\perp$. They are also $\mathrm{Ad}(T)$–stable. It is easy to see that $0 \in \overline{\mathrm{Ad}(T)\cdot\chi}$ for all $\chi \in (\mathfrak{b}^+)^\perp$. So if $Z_\chi(\lambda)$ is not simple (or not projective) for some $\chi \in (\mathfrak{b}^+)^\perp$, then also $Z_0(\lambda)$ is not simple (or not projective). Now classical results on the Steinberg module in the restricted case (cf. [J1], II.3.18 and II.10.2) yield:

Proposition: *Let $\lambda \in X$ with $\langle \lambda+\rho, \alpha^\vee\rangle \equiv 0 \pmod p$ for all roots α. Then $Z_\chi(\lambda)$ is simple and projective for all $\chi \in (\mathfrak{b}^+)^\perp$.*

Kac seems to have had in mind a proof along the lines indicated above when he claimed the simplicity of $Z_\chi(\lambda)$ for these λ in [Ka]. Proofs of these results appeared in [FP2], Thms. 4.1/2. Compare also [BG], Cor. 3.11.

Remark: Let me indicate how one can prove that the two sets above are open in $(\mathfrak{b}^+)^\perp$ or rather that their complements are closed.

All $Z_\chi(\lambda)$ have dimension $r = p^N$; we have bijections $f_\chi : K^r \to Z_\chi(\lambda)$ mapping a family $(a_{\mathbf{m}})_{\mathbf{m}}$ to $\sum_{\mathbf{m}} a_{\mathbf{m}} z_{\mathbf{m},\chi}$. Set N_d equal to the set of all $\chi \in (\mathfrak{b}^+)^\perp$ for which $Z_\chi(\lambda)$ has a submodule of dimension d. We want to show that each N_d with $0 < d < r$ is closed in $(\mathfrak{b}^+)^\perp$. Let $\mathbf{G}_{d,r}$ denote the Grassmannian of all d–dimensional subspaces of K^r. The description of the action of $Y \in \mathfrak{g}$ on $z_{\mathbf{m},\chi}$ implies that the set $M_d(Y)$ of all $(V,\chi) \in \mathbf{G}_{d,r} \times (\mathfrak{b}^+)^\perp$ with $Y f_\chi(V) \subset f_\chi(V)$ is closed. Hence so is the intersection M_d of all $M_d(Y)$ with $Y \in \mathfrak{g}$. Now the second projection maps M_d onto N_d, and this image is closed because $\mathbf{G}_{d,r}$ is a complete variety.

In order to get the claim on projectivity I shall use support varieties. Set $\mathcal{N}_p(\mathfrak{g})$ equal to the set of all $x \in \mathfrak{g}$ with $x^{[p]} = 0$. This is a closed and conic subvariety of $\mathfrak{g}$. So the image $\mathbf{P}\mathcal{N}_p(\mathfrak{g})$ of $\mathcal{N}_p(\mathfrak{g}) \setminus \{0\}$ in the projective space $\mathbf{P}(\mathfrak{g})$ is closed. For any $U_\chi(\mathfrak{g})$–module M set $\Phi_\mathfrak{g}(M)$ equal to the set of all $Kx \in \mathbf{P}\mathcal{N}_p(\mathfrak{g})$ such that the rank of $x-\chi(x)$ acting on M is strictly less than $(p-1)\dim(M)/p$. Then M is a projective $U_\chi(\mathfrak{g})$–module if and only if $\Phi_\mathfrak{g}(M) = \emptyset$, see [FP1], Thm. 6.4.

Now $Kx \in \mathbf{P}\mathcal{N}_p(\mathfrak{g})$ belongs to $\Phi_\mathfrak{g}(Z_\chi(\lambda))$ if and only if all $(m\times m)$–minors with $m = (p-1)p^{N-1}$ of the matrix of $x - \chi(x)$ with respect to the $z_{\mathbf{m},\chi}$ are 0. Therefore the set of all $(Kx,\chi) \in \mathbf{P}\mathcal{N}_p(\mathfrak{g}) \times (\mathfrak{b}^+)^\perp$ with $Kx \in \Phi_\mathfrak{g}(Z_\chi(\lambda))$ is closed. Hence so is its image under the second projection since $\mathbf{P}\mathcal{N}_p(\mathfrak{g})$ is a complete variety. That image is exactly the set of all $\chi \in (\mathfrak{b}^+)^\perp$ such that $Z_\chi(\lambda)$ is not projective.

D Standard Levi Form

We keep throughout the same assumptions and notations as in the preceding section.

D.1. A linear form $\chi \in \mathfrak{g}^*$ is said to have *standard Levi form* if $\chi(\mathfrak{b}^+) = 0$ and if there exists a subset I of the set of simple roots such that $\chi(X_{-\alpha}) \neq 0$ for all $\alpha \in I$ while $\chi(X_{-\beta}) = 0$ for all $\beta \in R^+ \setminus I$. (This definition goes back to [FP2], 3.1.)

If χ has standard Levi form, then $\chi([\mathfrak{n}^-, \mathfrak{n}^-]) = 0$ and $\chi(\mathfrak{n}^{-[p]}) = 0$. This implies that χ defines a one-dimensional $\mathfrak{n}^-$–module that is then a $U_\chi(\mathfrak{n}^-)$–module. Since $\mathfrak{n}^-$ is unipotent, this is the only simple $U_\chi(\mathfrak{n}^-)$–module (up to isomorphism), cf. [J3], 3.3. It then follows that $U_\chi(\mathfrak{n}^-)$ is the projective cover of this simple module, hence has a unique maximal submodule. Each $Z_\chi(\lambda)$ with $\lambda \in X$ is isomorphic to $U_\chi(\mathfrak{n}^-)$ as an $\mathfrak{n}^-$–module. Any proper $\mathfrak{g}$–submodule of $Z_\chi(\lambda)$ is then contained in that unique maximal $\mathfrak{n}^-$–submodule. Taking the sum of all these $\mathfrak{g}$–submodules we see:

Lemma: *If $\chi \in \mathfrak{g}^*$ has standard Levi form, then each $Z_\chi(\lambda)$ with $\lambda \in X$ has a unique maximal submodule.*

We then denote by $L_\chi(\lambda)$ the unique simple quotient of $Z_\chi(\lambda)$. Lemma B.4 tells us now that each simple $U_\chi(\mathfrak{g})$–module is isomorphic to some $L_\chi(\lambda)$ with $\lambda \in X$. However, λ will not be unique: We have at least $L_\chi(\lambda) \simeq L_\chi(\lambda + p\mu)$ for all $\mu \in X$; but there may be additional isomorphisms.

D.2. Before returning to the question when $L_\chi(\lambda) \simeq L_\chi(\lambda')$ in D.1, let us look at the special case $\chi = 0$ that clearly has standard Levi form.

This was the first case to be investigated. If V is a G–module (i.e., a vector space over K with a representation $G \to \mathrm{GL}(V)$ that is a homomorphism of algebraic groups), then V becomes a $\mathfrak{g}$–module taking the tangent map at 1 of the representation. One gets thus $U_0(\mathfrak{g})$–modules.

The simple G–modules are classified by their highest weight. There is for each $\lambda \in X$ with $0 \leq \langle \lambda, \alpha^\vee \rangle$ for all $\alpha \in R^+$ a simple G–module $L(\lambda)$ with highest weight λ; each simple G–module is isomorphic to exactly one of these $L(\lambda)$, cf. [J1], II.2.7. Now Curtis proved in [Cu]:

Theorem: a) *The $L(\lambda)$ with $\lambda \in X$ and $0 \leq \langle \lambda, \alpha^\vee \rangle < p$ for all simple roots α are simple also as $\mathfrak{g}$–modules.*

b) *Each simple $U_0(\mathfrak{g})$–module is isomorphic to one of these $L(\lambda)$. If $\lambda, \lambda' \in X$ both satisfy the condition in* a), *then $L(\lambda)$ and $L(\lambda')$ are isomorphic as $\mathfrak{g}$–modules if and only if $\lambda - \lambda' \in pX$.*

Note: if $\lambda, \lambda' \in X$ both satisfy the condition in a) and if $\lambda - \lambda' \in pX$, then $\langle \lambda, \alpha^\vee \rangle = \langle \lambda', \alpha^\vee \rangle$ for all $\alpha \in R$; in case G is semi-simple this implies that $\lambda = \lambda'$.

If one compares with the set up in D.1, specialised to the case $\chi = 0$, then we get that $L(\lambda)$ with λ as in a) is isomorphic as a $\mathfrak{g}$–module to $L_0(\lambda)$. And we get for all $\mu, \nu \in X$ that $L_0(\mu) \simeq L_0(\nu)$ if and only if $\mu \equiv \nu \pmod{pX}$.

D.3. Return to the more general situation in D.1 and consider $\chi \in \mathfrak{g}^*$ in standard Levi form with a set I of simple roots as in the definition. The (sketched) proof of Proposition C.2 shows that $Z_\chi(\lambda) \simeq Z_\chi(w \bullet \lambda)$ for all $\lambda \in X$ and $w \in W_I$ where W_I is the subgroup of the Weyl group W generated by all reflections s_α with $\alpha \in I$. We get thus one direction of:

Proposition: *Suppose that χ has standard Levi form and that $I = \{\alpha \in R \mid \chi(X_{-\alpha}) \neq 0\}$. Let $\lambda, \mu \in X$. Then $L_\chi(\mu) \simeq L_\chi(\lambda)$ if and only if $\mu \in W_I \bullet \lambda + pX$.*

The "only if" part is proved in [FP2], 3.2/4; for $G = \mathrm{SL}_n$ see also [P1], Thm. 3. Generalisations were found by Shen Guangyu and by Nakano, cf. [J3], 10.7.

D.4. The classification above of the simple $U_\chi(\mathfrak{g})$–modules for χ in standard Levi form leads immediately to a classification of the simple $U_\chi(\mathfrak{g})$–modules for χ in the G–orbit of some χ' in standard Levi form. Consider an isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ arising from a bilinear form as in (H3). If χ has standard Levi form and if $I = \{\alpha \in R \mid \chi(X_{-\alpha}) \neq 0\}$, then χ corresponds under this isomorphism to some $Y \in \mathfrak{g}$ that is regular nilpotent in the Levi factor $\mathfrak{g}_I$ of $\mathfrak{g}$ spanned by $\mathfrak{h}$ and all $\mathfrak{g}_\alpha$ with $\alpha \in R \cap \mathbf{Z}I$.

If $G = \mathrm{GL}_n$ or $G = \mathrm{SL}_n$, then the classification of nilpotent orbits in $\mathfrak{g}$ by the Jordan normal form shows that each nilpotent element in $\mathfrak{g}$ is conjugate to a regular nilpotent one in some Levi factor, hence each nilpotent linear form on $\mathfrak{g}$ conjugate to one in standard Levi form. Therefore Proposition D.3 and the earlier reductions yield a complete classification of all simple $\mathfrak{g}$–modules in case G is isomorphic to some GL_n (or a product of such groups).

Also for R of type B_2 or B_3 each nilpotent linear form on $\mathfrak{g}$ is conjugate to one in standard Levi form. This is no longer true for the other types. In those cases Proposition D.3 does not yield a complete classification of all simple modules.

D.5. Let us stay with χ in standard Levi form. Having achieved a classification of the simple modules one would like to know more about their structure; at least their dimensions ought to be determined. In the case $\chi = 0$ Lusztig's conjecture in [L1] on formal characters of simple G–modules yields via Theorem D.2 a conjecture for simple $\mathfrak{g}$–modules that determines (among other things) their dimensions provided p is not too small. One may hope that p greater than the Coxeter number of R will do. It is known that Lusztig's conjecture is true for p larger than an unknown bound that depends on the root system, see [AJS].

There is a similar conjecture for any χ in standard Levi form. In order to formulate it, we have to replace the category of $U_\chi(\mathfrak{g})$–modules by a certain category of graded $U_\chi(\mathfrak{g})$–modules.

Fix $\chi \in \mathfrak{g}^*$ in standard Levi form and set $I = \{\alpha \in R \mid \chi(X_{-\alpha}) \neq 0\}$. The enveloping algebra $U(\mathfrak{g})$ is in a natural way $\mathbf{Z}R$–graded such that each X_α has degree α and each $H \in \mathfrak{h}$ degree 0. However, it will now be more useful to regard $U(\mathfrak{g})$ as $\mathbf{Z}R/\mathbf{Z}I$–graded such that each X_α has degree $\alpha + \mathbf{Z}I$ and each $H \in \mathfrak{h}$ degree $0 + \mathbf{Z}I$. This has the advantage that the kernel of the canonical map $U(\mathfrak{g}) \to U_\chi(\mathfrak{g})$ is homogeneous: It is generated by all $H^p - H^{[p]}$ with $H \in \mathfrak{h}$ — homogeneous of degree $0+\mathbf{Z}I$ — and by all $X_\alpha^p - \chi(X_\alpha)^p$ with $\alpha \in R$ — homogeneous of degree $p\alpha + \mathbf{Z}I$ since $p\alpha \in \mathbf{Z}I$ whenever $\chi(X_\alpha) \neq 0$.

We get now a $\mathbf{Z}R/\mathbf{Z}I$–grading on $U_\chi(\mathfrak{g})$. It will be convenient to regard this as a grading by the larger group $X/\mathbf{Z}I$ and now to study $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–modules. For example, we can give each $Z_\chi(\lambda)$ with $\lambda \in X$ a grading such that each basis element $X_{-\alpha_1}^{m(1)} X_{-\alpha_2}^{m(2)} \dots X_{-\alpha_N}^{m(N)} v_\lambda$ as in B.4 is homogeneous of degree $\lambda - \sum_{i=1}^N m(i)\alpha_i + \mathbf{Z}I$. Denote $Z_\chi(\lambda)$ with this grading by $\widehat{Z}_\chi(\lambda)$.

General results on graded modules (cf. [J4], 1.4/5) imply that the radical of any $Z_\chi(\lambda)$ is a graded submodule. It follows that each $L_\chi(\lambda)$ has a grading such that the image of v_λ in $L_\chi(\lambda)$ is homogeneous of degree $\lambda + \mathbf{Z}I$. Denote this graded module by $\widehat{L}_\chi(\lambda)$. Note that any $\widehat{Z}_\chi(\lambda+p\nu)$ is just $\widehat{Z}_\chi(\lambda)$ with the grading shifted by $p\nu+\mathbf{Z}I$, similarly for $\widehat{L}_\chi(\lambda+p\nu)$ and $\widehat{L}_\chi(\lambda)$. If $\nu \in \mathbf{Z}I$, then we still have $\widehat{Z}_\chi(\lambda+p\nu) \simeq \widehat{Z}_\chi(\lambda)$; but this is no longer true when $\nu \notin \mathbf{Z}I$. One can now show (see [J3], 11.9):

Proposition: *Let $\lambda, \mu \in X$. Then*

$$\widehat{L}_\chi(\lambda) \simeq \widehat{L}_\chi(\mu) \iff \widehat{Z}_\chi(\lambda) \simeq \widehat{Z}_\chi(\mu) \iff \lambda \in W_I \bullet \mu + p\mathbf{Z}I.$$

D.6. Keep the assumptions on χ and I until the end of Section D. We began in D.5 with a discussion of graded $U_\chi(\mathfrak{g})$–modules. However, we shall not consider all possible modules of this type. If $M = \bigoplus_{\gamma \in X/\mathbf{Z}I} M_\gamma$ is an $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–module, then each M_γ is $\mathfrak{h}$–stable, hence a direct sum of weight spaces for $\mathfrak{h}$. We now make the additional condition:

If $\gamma = \mu + \mathbf{Z}I$ with $\mu \in X$, then all weights of $\mathfrak{h}$ on M_γ have the form $d(\mu + \nu)$ with $\nu \in \mathbf{Z}I$. We denote by $\mathcal{C}$ the category of all $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–modules satisfying this condition.

(In case $\chi = 0$ we have $I = \emptyset$ and deal here with X–graded $U_0(\mathfrak{g})$–modules $M = \bigoplus_{\mu \in X} M_\mu$. In this case the extra condition means that $\mathfrak{h}$ acts on each M_μ via $d\mu$. It follows that we can identify $\mathcal{C}$ for $\chi = 0$ with the category of G_1T–modules as in [J1], II.9.)

Now one easily checks (in the general case) that all $\widehat{Z}_\chi(\lambda)$ belong to $\mathcal{C}$. So do all their composition factors, in particular all $\widehat{L}_\chi(\lambda)$. An argument like that in B.4 shows that each simple module in $\mathcal{C}$ is isomorphic to some $\widehat{L}_\chi(\lambda)$ with $\lambda \in X$, cf. [J4], 2.5.

Set

$$C_I = \{\, \lambda \in X \mid 0 \le \langle \lambda + \rho, \alpha^\vee \rangle \le p \text{ for all } \alpha \in R^+ \cap \mathbf{Z}I \,\}.$$

This is a fundamental domain for the dot action on X of the affine reflection group generated by W_I and by the translations by all $p\alpha$ with $\alpha \in I$. Therefore Proposition D.5 implies:

Corollary: *Each simple module in $\mathcal{C}$ is isomorphic to exactly one $\widehat{L}_\chi(\lambda)$ with $\lambda \in C_I$.*

D.7. Let W_p denote the affine Weyl group of R, generated by W and the translations by elements in $p\mathbf{Z}R$. It is also generated by all affine reflections $s_{\alpha,rp}$ with $\alpha \in R$ and $r \in \mathbf{Z}$ where $s_{\alpha,rp}(\mu) = \mu - (\langle \mu, \alpha^\vee \rangle - rp)\alpha$. We shall usually consider the dot action of W_p on X where $w \bullet \mu = w(\mu + \rho) - \rho$.

Let $\le$ denote the usual order relation on X: We have $\mu \le \nu$ if and only if $\nu - \mu \in \sum_{\alpha \in R^+} \mathbf{N}\alpha$.

The reflections $s_{\alpha,mp} \in W_p$ are used to define another order relation on X that will be denoted by $\uparrow$. If $\alpha \in R^+$, $m \in \mathbf{Z}$, and $\lambda \in X$, then we say that $s_{\alpha,mp} \bullet \lambda \uparrow \lambda$ if and only if $\langle \lambda + \rho, \alpha^\vee \rangle \ge mp$. In general $\uparrow$ is defined as the transitive closure of this relation. It is then clear that $\mu \uparrow \lambda$ implies $\mu \le \lambda$ and $\mu \in W_p \bullet \lambda$. (Cf. [J1], II.6.4.)

One has now a strong linkage principle, see [J3], 11.11 and [J4], 4.5:

Proposition: *Let $\lambda, \mu \in C_I$. If $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)] \neq 0$, then $\mu \uparrow \lambda$.*

This is analogous to classical results for G–modules or G_1T–modules (the case $\chi = 0$ here), cf. [J1], II.6.16 and II.9.12.

We have furthermore (see [J4], 2.8(1))

$$[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\lambda)] = 1 \qquad \text{for all } \lambda \in C_I.$$

It follows that the (infinite) "decomposition matrix" of all $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]$ is lower triangular with ones on the diagonal (with respect to some total ordering on X that refines $\leq$. This will be crucial for us when we shall want to use information on this decomposition matrix to determine characters and dimensions of the simple modules, cf. the discussion in D.12 below. Without introducing the grading this would not work because Proposition C.2 says that all $Z_\chi(\lambda)$ in a block have the same composition factors.

D.8. One possible proof of Proposition D.7 involves a sum formula for a certain filtration: Each $\widehat{Z}_\chi(\lambda)$ has a filtration

$$\widehat{Z}_\chi(\lambda) \supset \widehat{Z}_\chi(\lambda)^1 \supset \widehat{Z}_\chi(\lambda)^2 \supset \cdots$$

with $\widehat{Z}_\chi(\lambda)/\widehat{Z}_\chi(\lambda)^1 = \widehat{L}_\chi(\lambda)$ and $\widehat{Z}_\chi(\lambda)^i = 0$ for $i \gg 0$ such that for all simple modules E in $\mathcal{C}$

$$\sum_{i>0} [\widehat{Z}_\chi(\lambda)^i : E] = \sum_\beta \Big(\sum_{i\geq 0} [\widehat{Z}_\chi(\lambda-(ip+n_\beta)\beta) : E] - \sum_{i>0} [\widehat{Z}_\chi(\lambda-ip\beta) : E]\Big)$$

where n_β is the integer with $0 < n_\beta \leq p$ and $\langle \lambda+\rho, \beta^\vee\rangle \equiv n_\beta \pmod p$ and where we sum over all $\beta \in R^+ \setminus \mathbf{Z}I$ with $n_\beta < p$. This formula is proved in [J4], 3.10 generalising the case $\chi = 0$ treated in [AJS], 6.6. (The infinite sum in the formula makes sense because one can check for each E that there are only finitely many non-zero terms in the sum.)

D.9. The sum formula in D.8 is also one of the main tools in determining all multiplicities $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]$ in "easy" cases. Other important tools used there are Premet's theorem B.7, translation functors, and indecomposable projective modules. Projective modules will be discussed later on, see G.1. The translation functors on $\mathcal{C}$ are defined in a way similar to the one used for G– or G_1T–modules (for these compare [J1], II.7 and II.9.19) and they have similar properties: Translation from a "regular weight" (i.e., one with trivial stabiliser in W_p) to an arbitrary weight takes a baby Verma module to a baby Verma module, and it takes a simple module to a simple module or to 0, and one knows, when one gets 0, see [J4], 4.9 and 4.11.

Using such techniques I have been able to determine all $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]$ in the following cases (sometimes under additional restrictions on p):

G	A_n	B_n	B_2	A_{n+1}	G_2	D_n
I	A_{n-1}	B_{n-1}	A_1	$A_1 \times A_{n-1}$	$\widetilde{A}_1$	D_{n-1}

Here G is assumed to be semi-simple and simply connected of the type mentioned in the first row. The second row then describes the type of the root system $R \cap \mathbf{Z}I$. In the cases of two root lengths I of type A_1 (resp. $\widetilde{A}_1$) means that I consists of a *long* (resp. *short*) simple root. For $n = 2$ the B_{n-1} under B_n has to be interpreted as $\widetilde{A}_1$.

The first two cases in the table were treated in [J2]; there the only restrictions on p are those imposed by (H3) or (H2): One has $p \nmid n+1$ for type A_n, and $p \neq 2$ for type B_n. (If one works in the first case with GL_{n+1} instead of SL_{n+1}, then no restriction on p is needed according to [GP], Rem. 9.3.) The case (B_2, A_1) was first dealt with (for $p \neq 2$) by brute force calculations in an Aarhus preprint (1997:13); in [J4], 5.2–10 this case is treated for $p > 3$ using the general ideas mentioned above. In the remaining cases I assume that p is larger than the Coxeter number of R. Files with the lengthy calculations in the A_{n+1} and D_n cases are available from me upon request.

D.10. The explicit results referred to in D.9 confirm in each case a conjecture by Lusztig. As in the case $\chi = 0$ (see D.5) one will have to expect some restrictions on p for this conjecture to be true; one may also here hope that p greater than the Coxeter number of R will suffice.

The conjecture says that any multiplicity $[\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]$ with $\lambda, \mu \in C_I$ is the value of a certain polynomial at 1 or -1 (depending on some normalisation of the polynomial). These polynomials were first constructed in [L2] where Lusztig then (in 13.17) expresses his hope that they would play a role similar to that of some previously constructed polynomials in the case $\chi = 0$. (See also the explicit formulation in [J3], 11.24.)

As in other situations one may speculate whether also the coefficients of these polynomials have a representation theoretic interpretation. For example, one may ask whether they yield the multiplicities in the factors of subsequent terms in the filtration mentioned in D.8 or in the radical filtration of $\widehat{Z}_\chi(\lambda)$. [These filtrations may well coincide.]

Set

$$A_0 = \{\, \lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+ \,\}.$$

This is the set of integral weights in the interior of the "first dominant alcove" with respect to W_p. This is the intersection of X with the "first dominant alcove over $\mathbf{R}$" as in [J1], II.6.2(6). Let us assume that $A_0 \neq \emptyset$, i.e., that p is at least equal to the Coxeter number, cf. [J1], II.6.2(10).

We call sets of the form $w \bullet A_0$ with $w \in W_p$ alcoves (in X). We denote the set of all alcoves by $\mathfrak{A}$; then $w \mapsto w \bullet A_0$ is a bijection from W_p onto $\mathfrak{A}$. Set $\mathfrak{A}_I$ equal to the set of all alcoves $A \in \mathfrak{A}$ with $A \subset C_I$.

Choose $\lambda_0 \in A_0$. For each $A \in \mathfrak{A}$ let λ_A denote the unique element in $A \cap W_p \bullet \lambda_0$; if $A = w \bullet A_0$ with $w \in W_p$, then $\lambda_A = w \bullet \lambda_0$. We have now $C_I \cap W_p \bullet \lambda_0 = \{\lambda_A \mid A \in \mathfrak{A}_I\}$. Lusztig's conjecture in the version of [L5], 17.3 says:

Conjecture ([L5]): $[\widehat{Z}_\chi(\lambda_A) : \widehat{L}_\chi(\lambda_B)] = \pi_{B,A}(-1)$ for all $A, B \in \mathfrak{A}_I$.

Here the $\pi_{B,A}$ are Lusztig's "periodic polynomials" normalised as in [L5], 9.17.

Thanks to the linkage principle the conjecture would determine all composition factors of all $\widehat{Z}_\chi(\mu)$ with $\mu \in C_I \cap W_p \bullet \lambda_0$. Then translation functors would yield all composition factors of all $\widehat{Z}_\chi(\mu)$ with $\mu \in C_I$.

Since baby Verma modules are finite dimensional, there are for each $A \in \mathfrak{A}_I$ only finitely many $B \in \mathfrak{A}_I$ with $[\widehat{Z}_\chi(\lambda_A) : \widehat{L}_\chi(\lambda_B)] \neq 0$. According to Conj. 9.20(b) in [L5] the $\pi_{B,A}$ are expected to have alternating signs. Therefore $\pi_{B,A} \neq 0$ should imply $\pi_{B,A}(-1) \neq 0$. So given A there should be only finitely many $B \in \mathfrak{A}_I$ with $\pi_{B,A} \neq 0$. That is known to hold in many examples, but is only conjectured by Lusztig in general, see [L5], 12.7/8. Also, it is not clear whether there is a good algorithm for computing the $\pi_{B,A}$, cf. [L2], 13.19.

D.11. The formal character of a finite dimensional $X/\mathbf{Z}I$–graded $U_\chi(\mathfrak{g})$–module $M = \bigoplus_{\tau \in X/\mathbf{Z}I} M_\tau$ is defined as

$$\operatorname{ch} M = \sum_{\tau \in X/\mathbf{Z}I} \dim(M_\tau)\, e(\tau) \in \mathbf{Z}[X/\mathbf{Z}I]$$

where the $e(\tau)$ with $\tau \in X/\mathbf{Z}I$ form the canonical basis of the group ring $\mathbf{Z}[X/\mathbf{Z}I]$.

Note that $\mathbf{Z}R/\mathbf{Z}I$ is a free abelian group of finite rank: The cosets of the simple roots not in I are a basis. Therefore the group ring $\mathbf{Z}[\mathbf{Z}R/\mathbf{Z}I]$ is a localised polynomial ring, hence an integral domain. The larger group ring $\mathbf{Z}[X/\mathbf{Z}I]$ contains $\mathbf{Z}[\mathbf{Z}R/\mathbf{Z}I]$ as a subring and is a free module over this subring.

The standard basis of $\widehat{Z}_\chi(\lambda)$ shows that

$$\operatorname{ch} \widehat{Z}_\chi(\lambda) = p^{N(I)}\, e(\lambda + \mathbf{Z}I) \prod_{\alpha \in R^+ \setminus \mathbf{Z}I} \frac{1 - e(-p\alpha)}{1 - e(-\alpha)} \tag{1}$$

for all $\lambda \in X$, where $N(I) = |R^+ \cap \mathbf{Z}I|$. It is then clear that

$$\operatorname{ch} \widehat{Z}_\chi(\mu) = e(\mu - \lambda + \mathbf{Z}I)\ \operatorname{ch} \widehat{Z}_\chi(\lambda) \tag{2}$$

for all $\mu, \lambda \in X$. On the other hand, we have

$$\widehat{L}_\chi(\lambda + p\nu) = e(p\nu + \mathbf{Z}I)\ \mathrm{ch}\,\widehat{L}_\chi(\lambda) \tag{3}$$

for all $\lambda, \nu \in X$ because adding $p\nu$ only amounts to a shift of the grading.

We get from the results in D.7 that for all $\lambda \in C_I$

$$\mathrm{ch}\,\widehat{Z}_\chi(\lambda) = \mathrm{ch}\,\widehat{L}_\chi(\lambda) + \sum_{\mu<\lambda, \mu\in C_I} [\widehat{Z}_\chi(\lambda) : \widehat{L}_\chi(\mu)]\,\mathrm{ch}\,\widehat{L}_\chi(\mu).$$

It follows that we can write each $\mathrm{ch}\,\widehat{L}_\chi(\lambda)$ as a (usually infinite) linear combination of the form $\sum_{\mu\leq\lambda} a_{\lambda\mu}\,\mathrm{ch}\,\widehat{Z}_\chi(\mu)$ with all $a_{\lambda\mu} \in \mathbf{Z}$ and $a_{\lambda\lambda} = 1$. Such infinite sums make sense because each $e(\tau)$ with $\tau \in X/\mathbf{Z}I$ occurs only in finitely many $\mathrm{ch}\,\widehat{Z}_\chi(\mu)$, $\mu \in C_I$ with a non-zero coefficient.

Here is an easy example that shows also how one may replace an infinite sum by a finite one. Consider $G = \mathrm{SL}_2$ and $\chi = 0$. Denote the only positive root by α. Consider $\lambda \in X$ and an integer d with $\langle\lambda+\rho, \alpha^\vee\rangle \equiv d \pmod p$ and $0 < d < p$. It is easy to show (and a classical result) for all $i \in \mathbf{Z}$ that $[\widehat{Z}_\chi(\lambda - ip\alpha)] = [\widehat{L}_\chi(\lambda - ip\alpha)] + [\widehat{L}_\chi(\lambda - (ip+d)\alpha)]$ and $[\widehat{Z}_\chi(\lambda - (ip+d)\alpha)] = [\widehat{L}_\chi(\lambda - (ip+d)\alpha)] + [\widehat{L}_\chi(\lambda - (i+1)p\alpha)]$ in the Grothendieck group. It follows that

$$\mathrm{ch}\,\widehat{L}_\chi(\lambda) = \sum_{i\geq 0} \Big(\mathrm{ch}\,\widehat{Z}_\chi(\lambda - ip\alpha) - \mathrm{ch}\,\widehat{Z}_\chi(\lambda - (ip+d)\alpha)\Big).$$

Comparing with the analogous formula for $\mathrm{ch}\,\widehat{L}_\chi(\lambda - p\alpha)$ one gets using (2) and (3)

$$\begin{aligned}(1 - e(-p\alpha))\ \mathrm{ch}\,\widehat{L}_\chi(\lambda) &= \mathrm{ch}\,\widehat{L}_\chi(\lambda) - \mathrm{ch}\,\widehat{L}_\chi(\lambda - p\alpha)\\ &= \mathrm{ch}\,\widehat{Z}_\chi(\lambda) - \mathrm{ch}\,\widehat{Z}_\chi(\lambda - d\alpha) = (1 - e(-d\alpha))\ \mathrm{ch}\,\widehat{Z}_\chi(\lambda).\end{aligned}$$

Now plug in (1):

$$(1 - e(-p\alpha))\ \mathrm{ch}\,\widehat{L}_\chi(\lambda) = \frac{(1 - e(-d\alpha))\,(1 - e(-p\alpha))}{1 - e(-\alpha)}\ e(\lambda)$$

and cancel the common factor $1 - e(-p\alpha)$:

$$\mathrm{ch}\,\widehat{L}_\chi(\lambda) = \frac{1 - e(-d\alpha)}{1 - e(-\alpha)}\ e(\lambda) = \sum_{j=0}^{d-1} e(\lambda - j\alpha).$$

Note that such a cancellation is quite generally permitted because we work in a free module over an integral domain, see above.

D.12. I want to conclude this section by showing that it will always be possible to get each $\operatorname{ch} \widehat{L}_\chi(\lambda)$ as a finite sum modulo two conjectures by Lusztig. For this I need some properties of the polynomials $\pi_{B,A}$ and therefore have to look closer at Lusztig's constructions.

Lusztig works with the localised polynomial ring $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ in the indeterminate v and he considers the free $\mathcal{A}$–module $\mathbf{M}_c$ with basis $\mathfrak{A}_I$ as well as two completions, $\mathbf{M}_\leq$ and $\mathbf{M}_\geq$, of $\mathbf{M}_c$. In order to define them, we need an order relation on $\mathfrak{A}$: We set $A \leq B$ if and only if $\lambda_A \uparrow \lambda_B$ in the set-up of D.10. (This is the order relation on the alcoves denoted by $\uparrow$ in [J1], II.6.5.)

The support of a function $f : \mathfrak{A}_I \to \mathcal{A}$ is the set $\{A \in \mathfrak{A}_I \mid f(A) \neq 0\}$. We identify $\mathbf{M}_c$ with the $\mathcal{A}$–module of all f that have finite support. Then $\mathbf{M}_\leq$ (resp. $\mathbf{M}_\geq$) consists of all functions $f : \mathfrak{A}_I \to \mathcal{A}$ whose support is bounded above (resp. below) relative to $\leq$. We write such functions as formal sums $f = \sum_{A \in \mathfrak{A}_I} f(A) A$.

In [L5], 9.17/19 Lusztig introduces (for each $B \in \mathfrak{A}_I$) elements $B_\leq = \sum_{A \leq B} \tilde{\pi}_{A,B} A \in \mathbf{M}_\leq$ and $B_\geq = \sum_{A \geq B} \pi_{B,A} A \in \mathbf{M}_\geq$ and $\check{B}_\leq = \sum_{A \leq B} \check{\pi}_{A,B} A \in \mathbf{M}_\leq$ where the coefficients ($\tilde{\pi}_{A,B}$ etc.) belong to $\mathbf{Z}[v^{-1}]$. More precisely, we have $\tilde{\pi}_{B,B} = 1$ and $\tilde{\pi}_{A,B} \in v^{-1}\mathbf{Z}[v^{-1}]$ if $A \neq B$, similarly for the other polynomials. Lemma 11.7(b) in [L5] says $\partial(\check{C}_\leq \| B_\geq) = \delta_{C,B}$ which means by the definition in [L5], 11.6 that

$$\sum_A \pi_{B,A} \check{\pi}_{A,C} = \delta_{C,B} \tag{1}$$

for all $B, C \in \mathfrak{A}_I$.

The fundamental domain property of C_I shows for all $A \in \mathfrak{A}_I$ and $\mu \in \mathbf{Z}R$ that there exists an alcove $\gamma_\mu(A) \in \mathfrak{A}_I$ such that $\gamma_\mu(A) = w \bullet (A + p\mu) + p\nu$ for suitable $w \in W_I$ and $\nu \in \mathbf{Z}I$. If $\mu \in \mathbf{Z}I$, then obviously $\gamma_\mu(A) = A$ for all A. It follows that we get an action of $\mathbf{Z}R/\mathbf{Z}I$ on $\mathfrak{A}_I$. We denote this action by $*$ and write $\tau * A = \gamma_\mu(A)$ if $\tau = \mu + \mathbf{Z}I$. This action preserves $\leq$, see [L2], 2.12(c).

The element $B_\geq \in \mathbf{M}_\geq$ is determined by the form of the coefficients stated above together with its invariance under a certain involution on $\mathbf{M}_\geq$. This involution commutes with the action of $\mathbf{Z}R/\mathbf{Z}I$ described above. Using this one can check that $(\tau * B)_\geq = \tau * (B_\geq)$ for all B, hence that $\pi_{\tau * B, \tau * A} = \pi_{B,A}$ for all B, A. Similar results hold for the other polynomials. We shall use below the corresponding result for the $\check{B}_\leq$:

$$\check{\pi}_{\tau * A, \tau * B} = \check{\pi}_{A,B}. \tag{2}$$

In [L5], 8.3 Lusztig constructs a homomorphism $\mu_I' : \mathbf{Z}R \to \mathbf{Z}$ such that $\mu_I'(\alpha) = -2$ for all $\alpha \in I$ and such that there are integers c_α with

$\mu'_I(\lambda) = \sum_{\alpha \in I} c_\alpha \langle \lambda, \alpha^\vee \rangle$ for all $\lambda \in \mathbf{Z}R$. Then μ'_I induces a homomorphism $\tau \mapsto \varepsilon_\tau$ from $\mathbf{Z}R/\mathbf{Z}I$ to $\{\pm 1\}$ such that $\varepsilon_{\lambda + \mathbf{Z}I} = (-1)^{\mu'_I(\lambda)}$ for all λ.

Lusztig extends the $\mathcal{A}$–module structure on $\mathbf{M}_c$, $\mathbf{M}_\leq$, and $\mathbf{M}_\geq$ to the group algebra $\mathcal{A}[\mathbf{Z}R/\mathbf{Z}I]$ by setting

$$e(\tau)\, A = \varepsilon_\tau\, (\tau * A)$$

for all $A \in \mathfrak{A}_I$ and $\tau \in \mathbf{Z}R/\mathbf{Z}I$.

There is a certain element $z = \sum_\tau z_\tau e(\tau) \in \mathcal{A}[\mathbf{Z}R/\mathbf{Z}]$ with $B_\leq = z\check{B}_\leq$, see [L5], 9.19. This means that

$$B_\leq = \sum_A \sum_\tau \check{\pi}_{A,B} \varepsilon_\tau z_\tau (\tau * A) = \sum_A \sum_\tau \check{\pi}_{(\tau *)^{-1} A, B} \varepsilon_\tau z_\tau A,$$

hence

$$\tilde{\pi}_{A,B} = \sum_\tau \varepsilon_\tau z_\tau \check{\pi}_{(\tau *)^{-1} A, B} = \sum_\tau \varepsilon_\tau z_\tau \check{\pi}_{A, \tau * B} \tag{3}$$

where we used (2) for the last step. Combining (3) and (1) we get now

$$\sum_A \pi_{C,A} \tilde{\pi}_{A,B} = \sum_\tau \varepsilon_\tau z_\tau \delta_{C, \tau * B}. \tag{4}$$

If now Lusztig's conjecture as in D.10 holds, then we get for all $B \in \mathfrak{A}_I$ (in the Grothendieck group)

$$\sum_{A \leq B} \tilde{\pi}_{A,B}(-1)\, [\widehat{Z}_\chi(\lambda_A)] = \sum_\tau \varepsilon_\tau z_\tau(-1)\, [\widehat{L}_\chi(\lambda_{\tau * B})]. \tag{5}$$

This means on the character level for each B

$$\sum_{A \leq B} \tilde{\pi}_{A,B}(-1)\, \mathrm{ch}(\widehat{Z}_\chi(\lambda_A)) = \mathrm{ch}(\widehat{L}_\chi(\lambda_B)) \sum_\tau \varepsilon_\tau z_\tau(-1) e(p\tau). \tag{6}$$

There is a ring homomorphism $\varphi : \mathcal{A}[\mathbf{Z}R/\mathbf{Z}I] \to \mathbf{Z}[\mathbf{Z}R/\mathbf{Z}I]$ with $\varphi(v) = -1$ and $\varphi(e(\tau)) = \varepsilon_\tau e(p\tau)$ for all τ. In this notation the right hand side in (6) is equal to $\varphi(z)\, \mathrm{ch}(\widehat{L}_\chi(\lambda_B))$.

The element z is introduced in [L5], Lemma 8.15. By that lemma (compare also the proof of Lemma 3.2.a in [L5]) z is a product of factors of the form $1 - v^m e(\tau)$ with $m \in \mathbf{Z}$ and $\tau \in (\mathbf{Z}R/\mathbf{Z}I) \setminus 0$, in particular with $e(\tau) \neq \pm 1$. It follows that $\varphi(z) \neq 0$. The characters in (6) belong to a free module over $\mathbf{Z}[\mathbf{Z}R/\mathbf{Z}I]$. Therefore $\mathrm{ch}(\widehat{L}_\chi(\lambda_B))$ is uniquely determined by $\varphi(z)\, \mathrm{ch}(\widehat{L}_\chi(\lambda_B))$ and we can use (6) to compute it. Finally, Lusztig conjectures that for each B one should have only finitely many A with $\tilde{\pi}_{A,B} \neq 0$, see [L5], 12.7/8. So the left hand side in (6) should be a finite sum.

E Representations and Springer Fibres

We keep throughout the same assumptions and notations as in the preceding section. Let h denote the Coxeter number of the root system R.

E.1. Set $\mathcal{B} = G/B^+$; this is the flag variety of G. It can be identified via $gB^+ \mapsto gB^+g^{-1}$ with the set of all Borel subgroups of G, or via $gB^+ \mapsto \mathrm{Ad}(g)(\mathfrak{b}^+)$ with the set of all Borel subalgebras of $\mathfrak{g}$.

For each $\chi \in \mathfrak{g}^*$ set

$$\mathcal{B}_\chi = \{\, gB^+ \in \mathcal{B} \mid \chi(\mathrm{Ad}(g)(\mathfrak{b}^+)) = 0 \,\}.$$

This is a closed subvariety of $\mathcal{B}$. Using the identification above we can think of $\mathcal{B}_\chi$ as the set of all Borel subalgebras of $\mathfrak{g}$ contained in the kernel of χ. Note that $\mathcal{B}_\chi \neq \emptyset$ if and only if $(g^{-1} \cdot \chi)(\mathfrak{b}^+) = 0$ for some $g \in G$ if and only if χ is nilpotent, cf. B.8.

Suppose that we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ using a bilinear form as in (H3) and that $Y \in \mathfrak{g}$ corresponds to some nilpotent $\chi \in \mathfrak{g}^*$. Then we get $\mathcal{B}_\chi = \mathcal{B}_Y$ where $\mathcal{B}_Y$ is the set of all gB^+ with $Y \in \mathrm{Ad}(g)(\mathfrak{n}^+)$. So $\mathcal{B}_\chi$ is equal to the fibre over Y in Springer's resolution of the nilpotent cone in $\mathfrak{g}$, cf. [J6], 6.6(3). We therefore call $\mathcal{B}_\chi$ the *Springer fibre* of χ.

The Springer resolution and its fibres have played an important role in other parts of representation theory, e.g., in Springer's theory of Weyl group representations. Humphreys has suggested for some time (cf. [H3], 23) that there might be connections between the theory of $U_\chi(\mathfrak{g})$–modules and the geometry of $\mathcal{B}_\chi$. This motivated geometric constructions of representations by Mirković and Rumynin in [MR]. Lusztig made then some explicit conjectures involving $\mathcal{B}_\chi$ (in [L3], [L4], and [L5]) part of which has now been proved by Bezrukavnikov, Mirković, and Rumynin in [BMR]. We shall describe their main results in this section.

E.2. Recall from C.2 the subalgebra $U(\mathfrak{g})^G$ of the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ and the algebra homomorphisms $\mathrm{cen}_\lambda : U(\mathfrak{g})^G \to K$. For any nilpotent χ and any λ the simple $U_\chi(\mathfrak{g})$–modules on which $U(\mathfrak{g})^G$ acts via cen_λ are the simple modules in one block of $U_\chi(\mathfrak{g})$, see C.5.

For any $\chi \in \mathfrak{g}^*$ let U_χ^0 denote the quotient algebra of $U_\chi(\mathfrak{g})$ by the ideal generated by the image of $\ker(\mathrm{cen}_0)$ under the canonical map $U(\mathfrak{g}) \to U_\chi(\mathfrak{g})$. So the simple U_χ^0–modules are for nilpotent χ the simple $U_\chi(\mathfrak{g})$–modules in one specific block of $U_\chi(\mathfrak{g})$.

Now one of the main results in [BMR] (Cor. 5.2.2 and Section 6) is:

Theorem: *Suppose that $p > 2h - 2$. Let $\chi \in \mathfrak{g}^*$ be nilpotent. Then the number of simple U_χ^0–modules is equal to the rank of the Grothendieck* *group of the category of coherent sheaves on the Springer fibre $\mathcal{B}_\chi$. This rank is also equal to the dimension of the l-adic cohomology of $\mathcal{B}_\chi$.*

Here h is the Coxeter number of R; one should expect that the theorem should extend to all $p > h$. Take for example χ in standard Levi form and I as in the definition in D.1. Then Proposition D.3 shows that the number of simple U_χ^0–modules is equal to $|W/W_I|$ in case $p > h$. This is compatible with the present theorem. For $p \leq h$ the number gets smaller indicating that the theorem cannot extend to such p.

The theorem proves (for $p > 2h - 2$) a conjecture of Lusztig. In his formulation the conjecture actually did not involve $\mathcal{B}_\chi$, but an analogue over $\mathbf{C}$: Let $G_\mathbf{C}$ denote the connected reductive group over $\mathbf{C}$ with the same root data as G, set $\mathfrak{g}_\mathbf{C} = \mathrm{Lie}(G_\mathbf{C})$ and let $\mathcal{B}^\mathbf{C}$ denote the flag variety of $G_\mathbf{C}$. Under our assumption on p we have a bijection between the set of nilpotent orbits in $\mathfrak{g}$ and the set of those in $\mathfrak{g}_\mathbf{C}$, hence also a bijection between the set of nilpotent orbits in $\mathfrak{g}^*$ and the set of those in $\mathfrak{g}_\mathbf{C}^*$. Choose a nilpotent $\chi(\mathbf{C}) \in \mathfrak{g}_\mathbf{C}^*$ in the orbit corresponding to that of χ. Then Lusztig predicts that the number of simple U_χ^0–modules should be equal to the dimension of the ordinary cohomology of $\mathcal{B}_{\chi(\mathbf{C})}^\mathbf{C}$ with coefficients in a field of characteristic 0. However, that dimension is equal to the dimension of the l–adic cohomology of $\mathcal{B}_\chi$, thanks to work by Lusztig, cf. [BMR], 6.4.3.

E.3. Let $\chi \in \mathfrak{g}^*$ be nilpotent. Theorem E.2 yields for $p > 2h - 2$ the number of simple $U_\chi(\mathfrak{g})$–modules in a specific block of $U_\chi(\mathfrak{g})$. One may ask about the remaining blocks.

Consider first $\lambda \in X$ such that $\lambda + pX$ has trivial stabiliser for the dot action on X/pX. Then the number of simple $U_\chi(\mathfrak{g})$–modules in the block determined by cen_λ is equal to the number of simple U_χ^0–modules. In fact there are translation functors that are equivalences between the blocks determined by cen_λ and by cen_0 and that induce inverse bijections between the two sets of simple modules, cf. [J5], Prop. B.5.

It is not clear what will happen when $\lambda + pX$ has a non-trivial stabiliser for the dot action on X/pX. Suppose that $p > h$. If χ has standard Levi form (or if $G \cdot \chi$ contains an element in standard Levi form), then the translation functor from the block of 0 to the block of λ takes simple modules to 0 or to simple modules; we get thus each simple module in the block of λ from exactly one simple module (up to isomorphism) in the block of 0. (This follows from the analogous result in the graded case mentioned in D.9.) We get the same behaviour in the subregular nilpotent cases in Section F. One may speculate whether such results generalise. Maybe further work in the spirit of [BMR] will lead to some answers to these questions.

E.4. Set U^0 equal to the quotient of $U(\mathfrak{g})$ by the ideal generated by $\ker(\text{cen}_0)$. So we can describe any U_χ^0 also as the quotient of U^0 by the ideal generated by the images in U^0 of all $Y^p - Y^{[p]} - \chi(Y)^p$ with $Y \in \mathfrak{g}$.

The work in [BMR] leading to Theorem E.2 is inspired by the paper [BB1]. There Beilinson and Bernstein show that the analogue to U^0 over $\mathbf{C}$ is isomorphic to the algebra of global sections of the sheaf $\mathcal{D}_{\mathbf{C}}$ of differential operators on the flag variety, and that the global section functor induces an equivalence of categories from (certain) sheaves of $\mathcal{D}_{\mathbf{C}}$–modules to U^0–modules. (This result is one of the main steps in Beilinson's and Bernstein's proof of the Kazhdan-Lusztig conjecture on characters of simple highest weight modules over $\mathfrak{g}_{\mathbf{C}}$.)

The generalisation of this result to prime characteristic involves two major changes: One has to replace differential operators by "*crystalline*" *differential operators* (using the terminology of [BMR]), and one gets at the end not an equivalence of categories, but an equivalence of derived categories.

E.5. If A is the algebra of regular functions on a smooth affine variety over K, then the algebra of "crystalline" differential operators on A is an algebra generated by A and the Lie algebra $\text{Der}_K(A)$ of all K–linear derivations of A. One imposes some obvious relations, e.g., $D\,a - a\,D = D(a)$ for all $a \in A$ and $D \in \text{Der}_K(A)$.

For example, if A is a polynomial ring $A = K[X_1, X_2, \dots, X_n]$, then the algebra of "crystalline" differential operators on A is a free module over A with basis all monomials in the partial derivatives $\partial_i = \partial/\partial X_i$, $1 \le i \le n$. This algebra acts on A via "true" differential operators, but this action is not faithful because (e.g.) any $(\partial_i)^p$ acts as 0.

This construction has a sheaf version that leads to a sheaf of "crystalline" differential operators $\mathcal{D}_Y$ on any smooth variety Y over K. One replaces A by the sheaf $\mathcal{O}_Y$ of regular functions on Y, and $\text{Der}_K(A)$ by the tangent sheaf $\mathcal{T}_Y$, cf. [BMR], 1.2. This is a special case of a construction from [BB2], 1.2.5.

E.6. Consider now $Y = \mathcal{B}$, the flag variety, and set $\mathcal{D} = \mathcal{D}_{\mathcal{B}}$. The action of G on $\mathcal{B} = G/B^+$ defines a Lie algebra homomorphism from $\mathfrak{g}$ to the global sections of the tangent sheaf $\mathcal{T}_{\mathcal{B}}$. This induces by construction a homomorphism from the enveloping algebra $U(\mathfrak{g})$ to the algebra $H^0(\mathcal{B}, \mathcal{D})$ of global sections of $\mathcal{D}$.

This homomorphism factors over U^0 ([BMR], 3.1.7) and induces for good p an isomorphism

$$U^0 \xrightarrow{\sim} H^0(\mathcal{B}, \mathcal{D}) \tag{1}$$

see [BMR], Prop. 3.3.1. The proof uses a transition to the associated graded algebras; this technique shows also that $H^i(\mathcal{B},\mathcal{D}) = 0$ for all $i > 0$.

Denote by $\mathrm{mod}_c(\mathcal{D})$ the category of coherent $\mathcal{D}$–modules; here coherent means locally finitely generated. Write $\mathrm{mod}_{fg}(U^0)$ for the category of finitely generated U^0–modules. For each coherent $\mathcal{D}$–module $\mathcal{F}$ the space $H^0(\mathcal{B},\mathcal{F})$ of global sections is via (1) a U^0–module that turns out to be finitely generated. One gets thus a functor that induces a functor on the bounded derived categories $D^b(\mathrm{mod}_c(\mathcal{D})) \to D^b(\mathrm{mod}_{fg}(U^0))$ because $H^i(\mathcal{B},?) = 0$ for $i > \dim(\mathcal{B})$. Now the first main result in [BMR] (Thm. 3.2) says:

Theorem: *Suppose that $p > 2h-2$. Then the functor*

$$D^b(\mathrm{mod}_c(\mathcal{D})) \to D^b(\mathrm{mod}_{fg}(U^0)) \tag{2}$$

is an equivalence of derived categories.

As pointed out in [BMR] (Remarks 1 and 2 following Thm. 3.2) this theorem does not hold without going over to the derived categories; it also will not hold for small p.

The equivalence in the opposite direction in (2) is induced by the "localisation functor" that takes a U^0–module M to the $\mathcal{D}$–module $\mathcal{D} \otimes_{U^0} M$. For this to work one first has to show that the derived functor $\mathcal{D} \,{}^L\!\otimes_{U^0}$ takes $D^b(\mathrm{mod}_{fg}(U^0))$ to the *bounded* derived category $D^b(\mathrm{mod}_c(\mathcal{D}))$, see [BMR], Prop. 3.8.1. One then wants to show that the compositions of these functors are isomorphic to the identity. This is checked only on certain special objects; the proof that it suffices to look at these special objects requires the bound on p in the theorem.

E.7. The transition from Theorem E.6 to Theorem E.2 involves two main steps. Denote by $\mathcal{D}_\chi$ the restriction of $\mathcal{D}$ to $\mathcal{B}_\chi \subset \mathcal{B}$. If we were lucky, then the equivalence in E.6(2) would induce an equivalence between $D^b(\mathrm{mod}_c(\mathcal{D}_\chi))$ and $D^b(\mathrm{mod}_{fg}(U^0_\chi))$ for each nilpotent $\chi \in \mathfrak{g}^*$, and the derived category $D^b(\mathrm{mod}_c(\mathcal{D}_\chi))$ would also be equivalent to the derived category of coherent modules over the structure sheaf of $\mathcal{B}_\chi$. This is almost true: It holds when we replace $\mathcal{B}_\chi$ by a formal neighbourhood, see [BMR], Thm. 5.2.1. That however does not matter in the end because this transition does not change the Grothendieck groups of the categories involved, cf. the proof of Cor. 4.1.4 in [BMR]. Finally one has to observe that the Grothendieck groups of these categories and those of their bounded derived categories coincide.

F The Subregular Nilpotent Case

In addition to our earlier conventions, we assume in this section that G is almost simple.

F.1. Since we assume G to be almost simple there is exactly one nilpotent orbit in $\mathfrak{g}$ of dimension $2(N-1)$ where $N = |R^+|$. Thanks to our general assumption (H3) there is also only one nilpotent orbit in $\mathfrak{g}^*$ of this dimension. We call these orbits the *subregular nilpotent* orbits (in $\mathfrak{g}$ or $\mathfrak{g}^*$).

If R is of type A_n or B_n for some n, then we can choose χ in the subregular nilpotent orbit in $\mathfrak{g}^*$ such that χ has standard Levi form. In these cases the simple $U_\chi(\mathfrak{g})$–modules as well as the structure of the baby Verma modules were determined in [J2].

For R not of type A_n or B_n, then the subregular nilpotent orbit in $\mathfrak{g}^*$ does not contain an element in standard Levi form. In these cases many results on $U_\chi(\mathfrak{g})$–modules were proved in [J5]. The results were most detailed in the simply laced cases (i.e., for R of type D_n or E_n) and proved in these special cases Lusztig's conjectures from [L4], 2.4/6 and [L5], 17.2. For other types the results in [J5] are less complete and can now be improved using the results from [BMR].

Thoughout this section we usually say "simple modules" when we mean "isomorphism classes of simple modules".

F.2. Choose some χ in the subregular nilpotent orbit in $\mathfrak{g}^*$ such that $\chi(\mathfrak{b}^+) = 0$. We pick some $\lambda \in X$ such that $0 \le \langle \lambda + \rho, \alpha^\vee \rangle \le p$ for all $\alpha \in R^+$. Each orbit of W on X/pX has a representative of this form. So we describe an arbitrary block of $U_\chi(\mathfrak{g})$ if we describe the block "of λ", i.e., the block where $U(\mathfrak{g})^G$ acts on all simple modules via cen_λ, cf. C.5. We know also that the simple modules in this block are exactly the composition factors of $Z_\chi(\lambda)$.

We call λ *regular* if p does not divide any $\langle \lambda+\rho, \alpha^\vee \rangle$, or, equivalently, if $0 < \langle \lambda + \rho, \alpha^\vee \rangle < p$ for all $\alpha \in R^+$, i.e., if λ belongs to the interior A_0 of the first dominant alcove as in D.10. We shall assume throughout that $p > h$. This implies that regular λ do exist.

Denote the set of all simple roots by Π and denote by α_0 the largest short root. Set $m_{-\alpha_0} = 1$ and define positive integers $(m_\alpha)_{\alpha \in \Pi}$ by $\alpha_0^\vee = \sum_{\alpha \in \Pi} m_\alpha \alpha^\vee$. Let $(\varpi_\alpha)_{\alpha \in \Pi}$ denote the fundamental weights.

If λ is regular, then we associate to each $U_\chi(\mathfrak{g})$–module M in the block of λ an invariant $\kappa(M)$ defined by

$$\kappa(M) = \{\, \alpha \in \Pi \cup \{0\} \mid T_\lambda^{\varpi_\alpha - \rho}(M) \neq 0 \,\}$$

where we set $\varpi_0 = 0$. Here each T_λ^μ is a translation functor, cf. [J5], B.2.

F.3. Assume in this subsection that R is simply laced, i.e., of type A_n, D_n, or E_n. In type E_8 the results stated below are proved only for $p > h+1$; this additional restriction should be unnecessary.

Suppose first that λ is regular. Then the block of λ contains $|\Pi|+1$ simple modules. We can denote these simple modules by L_α^λ with $\alpha \in \Pi \cup \{-\alpha_0\}$ such that

$$\dim(L_\alpha^\lambda) = \begin{cases} \langle \lambda+\rho, \alpha^\vee \rangle \, p^{N-1}, & \text{if } \alpha \in \Pi, \\ (p - \langle \lambda+\rho, \alpha_0^\vee \rangle) \, p^{N-1}, & \text{if } \alpha = -\alpha_0, \end{cases} \tag{1}$$

(recall that $N = |R^+|$) and

$$\kappa(L_\alpha^\lambda) = \begin{cases} \{\alpha\}, & \text{if } \alpha \in \Pi, \\ \Pi \cup \{0\}, & \text{if } \alpha = -\alpha_0. \end{cases} \tag{2}$$

We have

$$[Z_\chi(\lambda) : L_\alpha^\lambda] = m_\alpha \tag{3}$$

for all $\alpha \in \Pi \cup \{-\alpha_0\}$. Denote by Q_α^λ the projective cover of L_α^λ in the category of all $U_\chi(\mathfrak{g})$–modules. Then we have

$$[Q_\alpha^\lambda : L_\beta^\lambda] = |W| \, m_\alpha \, m_\beta \tag{4}$$

for all $\alpha, \beta \in \Pi \cup \{-\alpha_0\}$. Let us write Ext for Ext-groups in the category of all $U_\chi(\mathfrak{g})$–modules. One gets now

$$\mathrm{Ext}^1(L_\alpha^\lambda, L_\beta^\lambda) \simeq \begin{cases} K, & \text{if } \langle \beta, \alpha^\vee \rangle < 0, \\ 0, & \text{if } \langle \beta, \alpha^\vee \rangle = 0, \end{cases} \tag{5}$$

for all $\alpha, \beta \in \Pi \cup \{-\alpha_0\}$ with $\alpha \neq \beta$ — except that we have to replace K by K^2 if R has type A_1. The size of the Ext group in case $\alpha = \beta$ is unknown to me; one can show that it is non-zero in most cases.

The strategy for the proof of these results in [J5] is as follows: Using homomorphisms between baby Verma modules one constructs a chain of submodules in $Z_\chi(\lambda)$ where the dimensions and κ–invariants are known for the quotients of subsequent terms. Then translation functors and Premet's theorem are used to show that these quotients are simple. Next a deformation argument yields that simple modules with the same κ–invariant are isomorphic to each other. Thus one gets the classification of the simple modules and (1)–(3), cf. [J5], Thm. F.5. Quite general arguments imply in our case that $\dim(Q_\alpha^\lambda) = p^N |W| \, [Z_\chi(\lambda) : L_\alpha^\lambda]$, see

G.3(1) below. Inductive arguments using translation functors show in the present case that $[Q] = (\dim(Q)/p^N)[Z_\chi(\lambda)]$ in the Grothendieck group for all projective modules Q. Now (4) follows from (3), see [J5], Thm. G.6. The proof of (5) uses an idea of Vogan, relating Ext groups to the structure of simple modules translated through a wall, see [J5], Prop. H.12.

The statement about the number of simple modules as well as the result (4) on the Cartan matrix of the block prove conjectures by Lusztig from [L4], 2.4/6 and [L5], 17.2.

Drop now the assumption that λ should be regular, but continue to assume that $\langle \lambda+\rho, \alpha^\vee \rangle < p$. Then the number of simple modules in the block of λ is $1+|\{\alpha \in \Pi \mid \langle \lambda+\rho, \alpha^\vee \rangle > 0\}|$. One can denote these simple modules by L_α^λ with $\alpha = -\alpha_0$ or with $\alpha \in \Pi$ with $\langle \lambda+\rho, \alpha^\vee \rangle > 0$. Then (1) and (3) above still hold as long as we only allow α from this new parameter set. So does (4) if we replace $|W|$ by $|W \bullet (\lambda + pX)|$. Also (2) survives if we modify the definition of κ appropriately. If $\mu \in A_0$, then T_μ^λ takes L_α^μ to L_α^λ if this module is defined, to 0 otherwise. I have no information about Ext groups between simple modules when λ is not regular (except for R of type A_n where one can consult [J2], Prop. 2.19). These results are actually proved also for $p < h$ (except in type E_8) as long as (H2) and (H3) hold.

Suppose on the other hand that $\langle \lambda+\rho, \alpha^\vee \rangle = p$ for some $\alpha \in R^+$. Then we have in particular $\langle \lambda+\rho, \alpha_0^\vee \rangle = p$. In this case the simple modules should be parametrised by the simple roots α with $\langle \lambda+\rho, \alpha^\vee \rangle > 0$, and (1), (3), (4) should still hold with the appropriate modifications as in the preceding paragraph. This is true for R of type A_n and D_n, but open in type E_n.

F.4. As mentioned before, the classification of the simple modules in the regular case in F.3 was the first evidence besides the case of standard Levi form that Theorem E.2 should hold. To see this, let us describe $\mathcal{B}_\chi$ explicitly for subregular nilpotent χ.

For R of type A_n, D_n, or E_n the variety $\mathcal{B}_\chi$ has $|\Pi|$ irreducible components $(\ell_\alpha)_{\alpha\in\Pi}$. Each component ℓ_α is isomorphic to the projective line $\mathbf{P}^1$. Two distinct components ℓ_α and ℓ_β do not intersect if $\langle \beta, \alpha^\vee \rangle = 0$; they intersect transversally in one point if $\langle \beta, \alpha^\vee \rangle < 0$.

This description shows that $H^0(\mathcal{B}_\chi, \mathbf{Q}_l) \simeq \mathbf{Q}_l$ because $\mathcal{B}_\chi$ is connected, and that $\dim H^2(\mathcal{B}_\chi, \mathbf{Q}_l) = |\Pi|$ because all irreducible components of $\mathcal{B}_\chi$ have dimension 1 and there are $|\Pi|$ of them. This fact implies also that $H^i(\mathcal{B}_\chi, \mathbf{Q}_l) = 0$ for $i > 2$. Finally, $H^1(\mathcal{B}_\chi, \mathbf{Q}_l) = 0$ follows (e.g.) from the fact that one can pave $\mathcal{B}_\chi$ by affine lines and one point.

For the remaining root systems one can reduce the description of $\mathcal{B}_\chi$ to the cases already considered: If R has type B_n (resp. C_n, F_4, G_2), then $\mathcal{B}_\chi$ is isomorphic as variety to the analogous object for a group of type A_{2n-1} (resp. D_{n+1}, E_6, D_4). So again each irreducible component of $\mathcal{B}_\chi$ isomorphic to the projective line $\mathbf{P}^1$. But their number is now larger than $|\Pi|$. In types B_n, C_n, and F_4 one has now one component for each short simple root, but two components for each long simple root. In type G_2 there are one component for the short simple root and three components for the long simple root.

In any case the centraliser $\overline{G}_\chi$ of χ in the adjoint group $\overline{G}$ of G acts on $\mathcal{B}_\chi$, hence permutes the irreducible components of $\mathcal{B}_\chi$. The connected component $\overline{G}_\chi^0$ of 1 in $\overline{G}_\chi$ has to stabilise each component of $\mathcal{B}_\chi$, so the *component group* $C(\chi) = \overline{G}_\chi/\overline{G}_\chi^0$ acts as permutation group on the irreducible components of $\mathcal{B}_\chi$. For R of type A_n, D_n, or E_n the group $C(\chi)$ is trivial. For R of type B_n, C_n (with $n \geq 2$ in both cases), or F_4 the group $C(\chi)$ is cyclic of order 2; the nontrivial element interchanges for each long simple root α the two components belonging to α and it fixes the components belonging to short simple roots. For R of type G_2 the group $C(\chi)$ is isomorphic to the symmetric group S_3; it fixes the component belonging to the short simple root and acts as full permutation group on the three components belonging to the long simple root.

The action of $\overline{G}_\chi$ on $\mathcal{B}_\chi$ induces an action of $C(\chi)$ on $H^\bullet(\mathcal{B}_\chi, \mathbf{Q}_l)$. This action is trivial on $H^0(\mathcal{B}_\chi, \mathbf{Q}_l)$. On the other hand, $H^2(\mathcal{B}_\chi, \mathbf{Q}_l)$ has a basis indexed by the irreducible components of $\mathcal{B}_\chi$ and $C(\chi)$ permutes these basis elements in the same way as it permutes the irreducible components.

F.5. Assume in this subsection that R is of type B_n, C_n (with $n \geq 2$ in both cases), or F_4. Assume that $p > 2h - 2$ if R has type C_n with $n \geq 3$ or F_4, i.e., $p > 4n - 2$ in type C_n and $p > 22$ in type F_4. We need this bound so that we can apply the results from [BMR]. Let Π_s (resp. Π_l) denote the set of all short (resp. long) simple roots.

We look again first at the case where λ is regular. Then the block of λ contains $|\Pi_s| + 2|\Pi_l| + 1$ simple modules. We can denote them by L_α^λ with $\alpha \in \Pi_s \cup \{-\alpha_0\}$ and $L_{\alpha,1}^\lambda$, $L_{\alpha,2}^\lambda$ with $\alpha \in \Pi_l$ such that F.3(1)–(3) also hold here for all short α, i.e., for $\alpha \in \Pi_s \cup \{-\alpha_0\}$, and such that for all $\alpha \in \Pi_l$ and all $i \in \{1, 2\}$

$$\dim(L_{\alpha,i}^\lambda) = \langle \lambda + \rho, \alpha^\vee \rangle\, p^{N-1} \tag{1}$$

and

$$\kappa(L^\lambda_{\alpha,i}) = \{\alpha\} \tag{2}$$

and

$$[Z_\chi(\lambda) : L^\lambda_{\alpha,i}] = \frac{m_\alpha}{2}\,. \tag{3}$$

In [J5] I had to leave open the question whether in types C_n and F_4 the two simple modules $L^\lambda_{\alpha,1}$ and $L^\lambda_{\alpha,2}$ corresponding to some $\alpha \in \Pi_l$ are isomorphic to each other or not; the results from [BMR] yielding the total number of simple modules give the answer: they are not.

Denote by Q^λ_α the projective cover of L^λ_α for short α and by $Q^\lambda_{\alpha,i}$ the projective cover of $L^\lambda_{\alpha,i}$ for long α. Then F.3(4) holds for short α and β. If α is short and β is long, then one has for all $i \in \{1,2\}$

$$[Q^\lambda_\alpha : L^\lambda_{\beta,i}] = [Q^\lambda_{\beta,i} : L^\lambda_\alpha] = |W|\,\frac{m_\alpha\, m_\beta}{2}\,. \tag{4}$$

For the remaining Cartan invariants I can only state the obvious:

Conjecture : $$[Q^\lambda_{\alpha,i} : L^\lambda_{\beta,j}] = |W|\,\frac{m_\alpha\, m_\beta}{4}$$

for all long α and β and all $i, j \in \{1,2\}$. (This should be a special case of Lusztig's conjecture in [L5], 17.2.) At least in type B_n this formula holds thanks to the results in [J2].

If both α and β are short with $\alpha \neq \beta$, then F.3(5) extends to the present situation. If α is short and β is long, then one gets for all $i \in \{1,2\}$

$$\operatorname{Ext}^1(L^\lambda_\alpha, L^\lambda_{\beta,i}) \simeq \operatorname{Ext}^1(L^\lambda_{\beta,i}, L^\lambda_\alpha) \simeq \begin{cases} K, & \text{if } \langle\beta,\alpha^\vee\rangle < 0, \\ 0, & \text{if } \langle\beta,\alpha^\vee\rangle = 0. \end{cases} \tag{5}$$

Finally, one can choose the numbering of the $L^\lambda_{\alpha,i}$ such that for all long α and β with $\alpha \neq \beta$ and all $i, j \in \{1,2\}$

$$\operatorname{Ext}^1(L^\lambda_{\alpha,i}, L^\lambda_{\beta,j}) \simeq \begin{cases} K, & \text{if } \langle\beta,\alpha^\vee\rangle < 0 \text{ and } i = j, \\ 0, & \text{if } \langle\beta,\alpha^\vee\rangle = 0 \text{ or } i \neq j. \end{cases} \tag{6}$$

I do not know what $\dim \operatorname{Ext}^1(L^\lambda_{\alpha,1}, L^\lambda_{\alpha,2})$ is for long α. Furthermore, usually the simple modules have non-trivial self-extensions, but as in F.3 I do not know how big the Ext group is. (For all of this, see [J5], Prop. H.12.)

If we twist a $U_\chi(\mathfrak{g})$–module with $\mathrm{Ad}(g)$ for some $g \in \overline{G}_\chi$, the centraliser of χ in the adjoint group of G, then we get again a $U_\chi(\mathfrak{g})$–module. If the original module was simple, then so is the twisted one. If $U(\mathfrak{g})^G$

acts via cen_λ on the original module, then also on the new one. We get thus an action of the component group $C(\chi)$ as in F.4 on the set of isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules in the block of λ. In our present situation $C(\chi)$ is cyclic of order 2 and the non-trivial element in $C(\chi)$ interchanges $L^\lambda_{\alpha,1}$ and $L^\lambda_{\alpha,2}$ for all long α, and it fixes all L^λ_β with β short, see [J5], Prop. F.7.

For non-regular λ the situation should be similar to the one described in F.3. If we still have $\langle\lambda+\rho,\beta^\vee\rangle < p$ for all $\beta\in R^+$, then one gets simple modules L^λ_α and $L^\lambda_{\alpha,i}$ more or less as above except that we drop all α with $\langle\lambda+\rho,\alpha^\vee\rangle = 0$. Each simple module is isomorphic to one of these. The only problem is that we no longer can be sure that $L^\lambda_{\alpha,1}$ and $L^\lambda_{\alpha,2}$ for long α are not isomorphic to each other because [BMR] does not tell us the number of simple modules in the non-regular case. However, in the case where there is only one $\beta\in R^+$ such that p divides $\langle\lambda+\rho,\beta^\vee\rangle$, then one can show that the simple modules $L^\lambda_{\alpha,1}$ and $L^\lambda_{\alpha,2}$ for long α are not isomorphic, see [J5], Lemma H.11.b.

F.6. Consider now R of type G_2 and assume that $p > 2h-2 = 10$. Denote the simple roots by α_1 (short) and α_2 (long). Let λ be regular. Then [J5], D.11 shows: The block of λ contains a simple module $L^\lambda_{-\alpha_0}$ with κ–invariant $\Pi\cup\{0\}$ and dimension $(p-\langle\lambda+\rho,\alpha_0^\vee\rangle)\,p^{N-1}$; this is the only simple module in this block with κ–invariant $\Pi\cup\{0\}$. There are one or two simple module in this block with κ–invariant $\{\alpha_1\}$; this module or these modules have dimension $\langle\lambda+\rho,\alpha_1^\vee\rangle\,p^{N-1}$. There is at least one simple module with κ–invariant $\{\alpha_2\}$ and dimension $\langle\lambda+\rho,\alpha_2^\vee\rangle\,p^{N-1}$. All remaining simple modules have κ–invariant $\{\alpha_2\}$ or $\emptyset$.

Now [BMR] tells us that there are five simple modules. We know also that the component group $C(\chi)\simeq S_3$ permutes these modules. This action preserves the κ–invariant by [J5], D.5(3), and is independent of the choice of λ if we use translation functors to identify the simple modules in different blocks for regular weights, see [J5], B.13(2).

The possible orbit sizes of $C(\chi)$ permuting the five simple modules are 1, 2, and 3. There have to be at least three orbits. If one of the orbits has size 3, then there are exactly three orbits; the other two orbits have cardinality 1. If there is no orbit of size 3, then the alternating subgroup in $C(\chi)\simeq S_3$ has to fix all simple modules.

We have by [BMR] an isomorphism between the Grothendieck group of the block of 0 and the Grothendieck group $K_0(\mathcal{B}_\chi)$ of coherent sheaves on $\mathcal{B}_\chi$. By the naturality of the construction in [BMR] one may hope that this isomorphism is compatible with the actions of $\overline{G}_\chi$.

Let us assume that this is true. Then the action of $\overline{G}_\chi$ on $K_0(\mathcal{B}_\chi)$ factors over $C(\chi)$ since this holds for the action on the block side.

Denote by Y the irreducible component of $\mathcal{B}_\chi$ corresponding to the short simple root, and by U the (open) complement of Y. The group $\overline{G}_\chi$ stabilises Y and hence also U. We have then a surjective map $K_0(\mathcal{B}_\chi) \to K_0(U)$ that is compatible with the action of $\overline{G}_\chi$. Now U is isomorphic to a disjoint union of three affine lines. This implies that $K_0(U) \simeq \mathbf{Z}^3$. Furthermore, the action of $\overline{G}_\chi$ on $K_0(U)$ has to factor over $C(\chi)$, and the action of $C(\chi)$ has to permute the three summands isomorphic to $\mathbf{Z}$ in $K_0(U)$ because $C(\chi)$ permutes the three affine lines. Therefore $C(\chi)$ acts faithfully on $K_0(U)$, hence on $K_0(\mathcal{B}_\chi)$.

This should now imply that also the action of $C(\chi)$ on the Grothendieck group of the block is faithful. As pointed out above this means that there are three orbits, and that they have cardinality 1, 1, and 3. This holds at first for $\lambda = 0$ and then for all λ using translation functors. It then follows that there is only one simple module, say $L^\lambda_{\alpha_1}$, with κ–invariant $\{\alpha_1\}$, and that there are three simple modules, say $L^\lambda_{\alpha_2,1}$, $L^\lambda_{\alpha_2,2}$, $L^\lambda_{\alpha_2,3}$, with κ–invariant $\{\alpha_2\}$, all of dimension $\langle \lambda+\rho, \alpha_2^\vee\rangle\, p^{N-1}$. One gets then (using [J5], D.11 once more) that

$$[Z_\chi(\lambda) : L^\lambda_{\alpha_1}] = 2$$

and

$$[Z_\chi(\lambda) : L^\lambda_{-\alpha_0}] = 1 = [Z_\chi(\lambda) : L^\lambda_{\alpha_2,i}]$$

for all i.

G Projective Modules

We keep the previous assumptions as in Sections D and E.

G.1. Each $U_\chi(\mathfrak{g})$ with $\chi \in \mathfrak{g}^*$ is a finite dimensional algebra. So there is a bijection between the set of isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules and the set of isomorphism classes of indecomposable projective $U_\chi(\mathfrak{g})$–modules. Any simple $U_\chi(\mathfrak{g})$–module E corresponds to its projective cover Q_E, the unique (up to isomorphism) indecomposable projective $U_\chi(\mathfrak{g})$–module with $E \simeq Q_E/\mathrm{rad}(Q_E)$. The matrix of all $[Q_E : E']$ with E, E' running over representatives for the isomorphism classes of simple $U_\chi(\mathfrak{g})$–modules is then the *Cartan matrix* of $U_\chi(\mathfrak{g})$.

G.2. Suppose that $\chi \in \mathfrak{g}^*$ has standard Levi form. Denote the projective cover of any $L_\chi(\lambda)$ as in D.1 by $Q_\chi(\lambda)$. Recall now the graded category $\mathcal{C}$ from D.5/6. Each simple module $\widehat{L}_\chi(\lambda)$ in $\mathcal{C}$ has a projective cover $\widehat{Q}_\chi(\lambda)$ in $\mathcal{C}$. If we forget the grading, then $\widehat{Q}_\chi(\lambda)$ is isomorphic to $Q_\chi(\lambda)$, cf. [J4], 1.4.

Set $I = \{\alpha \in R \mid \chi(X_{-\alpha}) \neq 0\}$. Recall from D.6 the definition of C_I and the fact that $\lambda \mapsto \widehat{L}_\chi(\lambda)$ yields a bijection between C_I and the set of isomorphism classes of simple modules in $\mathcal{C}$. One can show (see [J4], Prop. 2.9) that each $\widehat{Q}_\chi(\lambda)$ with $\lambda \in C_I$ has a filtration with factors $\widehat{Z}_\chi(\mu)$ with $\mu \in C_I$, each $\widehat{Z}_\chi(\mu)$ occurring $|W_I \bullet (\mu + pX)|\,[\widehat{Z}_\chi(\mu) : \widehat{L}_\chi(\lambda)]$ times. (Note that there is a misprint in [J4], page 154, line -2: Replace $|W_I(\lambda + pX)|$ by $|W_I \bullet (\lambda + pX)|$.) This yields in the Grothendieck group of $\mathcal{C}$ using Proposition D.7

$$[\widehat{Q}_\chi(\lambda)] = \sum_{\mu \in C_I,\, \lambda \uparrow \mu} |W_I \bullet (\mu + pX)|\,[\widehat{Z}_\chi(\mu) : \widehat{L}_\chi(\lambda)]\,[\widehat{Z}_\chi(\mu)]. \tag{1}$$

We have in the ungraded category $[Z_\chi(\mu)] = [Z_\chi(\lambda)]$ for all $\mu \in W \bullet \lambda + pX$, hence for all μ with $\lambda \uparrow \mu$, see Proposition C.2. So (1) implies that in the Grothendieck group of the ungraded category

$$[Q_\chi(\lambda)] = [Z_\chi(\lambda)] \sum_{\mu \in C_I,\, \lambda \uparrow \mu} |W_I \bullet (\mu + pX)|\,[\widehat{Z}_\chi(\mu) : \widehat{L}_\chi(\lambda)]. \tag{2}$$

It follows since $\dim Z_\chi(\lambda) = p^N$ with $N = |R^+|$

$$[Q_\chi(\lambda)] = \frac{\dim(Q_\chi(\lambda))}{p^N}[Z_\chi(\lambda)]. \tag{3}$$

So all rows in the Cartan matrix belonging to a fixed block are proportional to each other. Therefore the Cartan matrix has determinant 0 unless each block contains only one simple module. The latter is the case if and only if I consists of all simple roots, i.e., if and only if χ is regular nilpotent. In that special case all composition factors of $Q_\chi(\lambda)$ are isomorphic to $L_\chi(\lambda) = Z_\chi(\lambda)$, and there are $|W_I \bullet (\lambda + pX)|$ of them.

G.3. For an arbitrary nilpotent $\chi \in \mathfrak{g}^*$ with $\chi(\mathfrak{b}^+) = 0$ one knows less about the projective indecomposable modules. If E is a simple $U_\chi(\mathfrak{g})$-module in the block of some $\lambda \in X$, then one can show that

$$\dim(Q_E) = p^N\,|W \bullet (\lambda + pX)|\,[Z_\chi(\lambda) : E] \tag{1}$$

see [J5], B.12(2). For χ in standard Levi form this can be deduced from G.2(2). In general (for χ not in standard Levi form) it is not known whether Q_E has a filtration with factors of the form $Z_\chi(\mu)$ that would "explain" (1) and that would allow us to generalise G.2(3). So we have at present only a

Hope: $$[Q_E] = \frac{\dim Q_E}{p^N}\,[Z_\chi(\lambda)]$$

for all simple E in the block of some λ. Results in the subregular nilpotent case (see F.3 and F.5) support this hope.

G.4. Let $\chi \in \mathfrak{g}^*$ be nilpotent and let $\lambda \in X$ be regular. (This means as in F.2 that p does not divide any $\langle \lambda + \rho, \alpha^\vee \rangle$ with $\alpha \in R$. So we assume that $p \geq h$.)

Lusztig has a general conjecture on the Cartan matrix of the block of λ of $U_\chi(\mathfrak{g})$, see [L4], 2.4 and [L5], 17.2. Actually, the conjecture is stated for a graded version of that category. One chooses a maximal torus T_0 in the centraliser of χ in G. Then the adjoint action of T_0 on $U(\mathfrak{g})$ induces one on $U_\chi(\mathfrak{g})$. This leads then to a grading of $U_\chi(\mathfrak{g})$ by the character group $X(T_0)$ of T_0. One now considers $X(T_0)$–graded $U_\chi(\mathfrak{g})$–modules satisfying a compatibility condition analogous to the one in D.6; finally one takes a block of this category where $U(\mathfrak{g})^G$ acts on all simple modules via cen_λ.

Now Lusztig's conjecture predicts that the simple modules in this block should be in bijection with a certain basis of some equivariant K–group of some variety (a Slodowy variety) related to $\mathcal{B}_\chi$, and that the entry in the Cartan matrix corresponding to two simple modules should be given by a certain bilinear form evaluated at the two corresponding basis elements. (See [L5] for more details.)

In the case where χ is subregular nilpotent and G almost simple not of type A_n or B_n, then $T_0 = 1$, so we can forget about the grading. If G is almost simple of type D_n or E_n, then [L4], 2.6 gives an explicit description of the conjectured Cartan matrix that since has turned out to be correct, see F.3(4).

Suppose that χ has standard Levi form (for arbitrary G). In this case one can choose $T_0 \subset T$ with $X(T_0) = \mathbf{Z}R/\mathbf{Z}I$. In this case the category described above is a direct summand of the category $\mathcal{C}$ from D.6 where gradings by $X/\mathbf{Z}I$ are permitted. However, this smaller category contains the block corresponding to $\lambda = 0$; all other blocks for regular λ are equivalent to this one (assuming $p \geq h$). For such λ Lusztig's conjecture on the Cartan matrix will follow using G.2(1) from Lusztig's conjecture, mentioned in D.10, on the composition factors of the $\widehat{Z}_\chi(\mu)$, see [L5], 17.3.

G.5. Consider the case $\chi = 0$. So the graded category $\mathcal{C}$ from D.6 is the same as that of all G_1T–modules as in [J1], II.9. Let us now drop in this case the index χ and write $\widehat{L}(\lambda)$, $\widehat{Z}(\lambda)$, $\widehat{Q}(\lambda)$ instead of $\widehat{L}_\chi(\lambda)$, etc. So these objects are parametrised by $\lambda \in X$. Each $\widehat{Q}(\lambda)$ has a filtration with factors of the form $\widehat{Z}(\mu)$ such that each $\widehat{Z}(\mu)$ occurs $[\widehat{Z}(\mu) : \widehat{L}(\lambda)]$

times. We have in particular

$$[\widehat{Q}(\lambda) : \widehat{L}(\nu)] = \sum_{\mu \in X} [\widehat{Z}(\mu) : \widehat{L}(\lambda)]\,[\widehat{Z}(\mu) : \widehat{L}(\nu)] \tag{1}$$

for all $\lambda, \nu \in X$.

There is now the older conjecture by Lusztig predicting all $[\widehat{Z}(\mu) : \widehat{L}(\lambda)]$ for $p > h$ and proved in [AJS] for $p \gg 0$. And there is the newer one that predicts all $[\widehat{Q}(\lambda) : \widehat{L}(\nu)]$ for $p > h$, and that follows, according to Lusztig, from the older one using (1). Now I want to point out that in this case ($\chi = 0$) conversely the older conjecture follows from the newer one. Since Lusztig has already shown the other direction, it suffices to show that the Cartan matrix determines the decomposition matrix. More precisely:

Claim: *Let $\lambda \in A_0$. If we know all $[\widehat{Q}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ with $x, w \in W_p$, then we know all $[\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ with $x, w \in W_p$.*

Here $A_0 \subset X$ is as in D.10 the interior of the first dominant alcove inside X, and W_p is the affine Weyl group as in D.7.

G.6. Before we can start proving the claim in G.5 we need some facts on the alcove geometry. Let X_+ denote the set of all dominant weights in X. For each $\mu \in X$ there exists $w \in W$ such that $w\mu$ is antidominant (or, equivalently, such that $-w\mu \in X_+$) and $w\mu$ is unique with this property. Moving the origin to another point, we get:

Fix $\nu \in X$. For each $\mu \in X$ there exists $w \in W_p$ such that $w \bullet (p\nu - \rho) = p\nu - \rho$ and $p\nu - \rho - w \bullet \mu \in X_+$. The weight $w \bullet \mu$ is uniquely determined by this condition; we shall denote it by $a_\nu(\mu)$.

Indeed: There exists $x \in W$ with $x(p\nu - \mu - \rho) \in X_+$. Set $\gamma = \nu - x(\nu) \in \mathbf{Z}R$ and set $w \in W_p$ equal to the composition of first x, then translation with $p\gamma$. Then

$$w \bullet (p\nu - \rho) = x(p\nu) + p\gamma - \rho = p\nu - \rho$$

and

$$p\nu - \rho - w \bullet \mu = p\nu - w(\mu + \rho) = p\nu - p\gamma - x(\mu + \rho) = x(p\nu - \mu - \rho) \in X_+.$$

This yields the existence; the uniqueness is left as an exercise.

Keep ν as above. A trivial remark: If $z \in W_p$, then:

$$z \bullet (p\nu - \rho) = p\nu - \rho \implies a_\nu(z \bullet \mu) = a_\nu(\mu) \text{ for all } \mu \in X. \tag{1}$$

(If $a_\nu(\mu) = w \bullet \mu$ with $w \in W_p$ and $w \bullet (p\nu - \rho) = p\nu - \rho$ then $a_\nu(\mu) = (wz^{-1}) \bullet (z \bullet \mu)$ and $(wz^{-1}) \bullet (p\nu - \rho) = p\nu - \rho$.)

We claim that next

$$a_\nu(\mu) \uparrow \mu \qquad \text{for all } \mu \in X. \tag{2}$$

It is well known that $x \bullet \mu' \uparrow \mu'$ for all $x \in W$ if $\mu' + \rho$ is dominant. One has similarly $\mu' \uparrow x \bullet \mu'$ for all $x \in W$ if $\mu' + \rho$ is antidominant. We can apply this to $a_\nu(\mu) - p\nu$ since $a_\nu(\mu) + \rho - p\nu$ is antidominant. There are $x \in W$ and $\gamma \in \mathbf{Z}R$ with $a_\nu(\mu) = p\gamma + x \bullet \mu$ and $\gamma = \nu - x(\nu)$. Now we get

$$a_\nu(\mu) - p\nu \uparrow x^{-1} \bullet (a_\nu(\mu) - p\nu) = x^{-1}(p\gamma + x(\mu+\rho) - p\nu) - \rho = \mu - p\nu$$

hence (2).

Claim: *If ν and μ are antidominant, then*

$$a_\nu(z \bullet \mu) \uparrow a_\nu(\mu) \qquad \textit{for all } z \in W. \tag{3}$$

This follows by induction on the length of z from the following claim, where ν is assumed to be antidominant and μ is arbitrary: Let $\alpha \in R$. Then

$$\mu \uparrow s_\alpha \bullet \mu \implies a_\nu(s_\alpha \bullet \mu) \uparrow a_\nu(\mu). \tag{4}$$

Proof: We may assume that $\alpha > 0$. We have $s_\alpha \bullet \mu = \mu + m\alpha$ with $m = -\langle \mu + \rho, \alpha^\vee \rangle \geq 0$.

There exists $w \in W_p$ with $w \bullet (p\nu - \rho) = p\nu - \rho$ and $a_\nu(\mu) = w \bullet \mu$. Set $\mu' = w \bullet (s_\alpha \bullet \mu)$. Then (1) says that $a_\nu(\mu') = a_\nu(s_\alpha \bullet \mu)$; so we want to show that $a_\nu(\mu') \uparrow a_\nu(\mu)$.

There are $x \in W$ and $\gamma \in \mathbf{Z}R$ such that w is the composition of first x and then translation by $p\gamma$. The assumption $w \bullet (p\nu - \rho) = p\nu - \rho$ implies that $\gamma = \nu - x(\nu)$.

Set $\beta = x(\alpha)$. We have then $ws_\alpha w^{-1} = s_{\beta,rp}$ with $r = \langle \gamma, \beta^\vee \rangle$. Now $\mu' = ws_\alpha \bullet \mu = s_{\beta,rp} \bullet (w \bullet \mu) = s_{\beta,rp} \bullet a_\nu(\mu)$ shows that $a_\nu(\mu) \uparrow \mu'$ or $\mu' \uparrow a_\nu(\mu)$. On the other hand, we have

$$ws_\alpha \bullet \mu = p\gamma + x \bullet (\mu + m\alpha) = p\gamma + x \bullet \mu + m\beta = a_\nu(\mu) + m\beta.$$

If $\beta < 0$, then $m \geq 0$ shows that $\mu' \uparrow a_\nu(\mu)$, hence $a_\nu(s_\alpha \bullet \mu) = a_\nu(\mu') \uparrow \mu' \uparrow a_\nu(\mu)$ using (2).

Suppose now that $\beta > 0$. Set $n = \langle \nu, \beta^\vee \rangle \in \mathbf{Z}$. It is clear that $s_{\beta,np} \bullet (p\nu - \rho) = p\nu - \rho$. So we have $a_\nu(\mu') = a_\nu(s_{\beta,np} \bullet \mu')$. Note that

$$s_{\beta,np} \bullet \mu' = s_{\beta,np}s_{\beta,rp} \bullet a_\nu(\mu) = a_\nu(\mu) + p(n-r)\beta$$

and

$$n - r = \langle \nu - \gamma, \beta^\vee \rangle = \langle x(\nu), \beta^\vee \rangle = \langle \nu, x^{-1}(\beta)^\vee \rangle = \langle \nu, \alpha^\vee \rangle.$$

Since ν is antidominant, the last equation yields $n-r \leq 0$, hence $a_\nu(\mu)+p(n-r)\beta \uparrow a_\nu(\mu)$. Now we get

$$a_\nu(s_\alpha \bullet \mu) = a_\nu(s_{\beta,np} \bullet \mu') \uparrow s_{\beta,np} \bullet \mu' = a_\nu(\mu) + p(n-r)\beta \uparrow a_\nu(\mu)$$

as claimed.

G.7. (*Proof of the claim in G.5*) Our multiplicities are periodic: We have $[\widehat{Z}(\mu + p\nu) : \widehat{L}(\mu' + p\nu)] = [\widehat{Z}(\mu) : \widehat{L}(\mu')]$ for all $\mu, \mu', \nu \in X$ since adding $p\nu$ amounts just to a shift of the grading. Each $\mu \in X$ can be written in the form $\mu = \mu_0 + p\mu_1$ with $\mu_1 \in X$ and $\mu_0 \in X_1$ where

$$X_1 = \{ \mu \in X \mid 0 \leq \langle \mu, \beta^\vee \rangle < p \text{ for all simple roots } \beta \}.$$

Actually, it will be more convenient for us to replace X_1 by $X_1 - p\rho$.

Set $\mathcal{W} = \{ w \in W_p \mid w \bullet \lambda \in X_1 - p\rho \}$. By the observation above we get all $[\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ if we know them for all $x \in W_p$ and $w \in \mathcal{W}$. We get from [J1], II.9.13 that

$$[\widehat{Z}(yx \bullet \lambda) : \widehat{L}(w \bullet \lambda)] = [\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)] \tag{1}$$

for all $w \in \mathcal{W}$, $x \in W_p$, and $y \in W$. It is therefore enough to know all $[\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ with $w \in \mathcal{W}$ and $x \in W_p$ such that $x \bullet \lambda$ is antidominant.

Set $\mathcal{W}'$ equal to the set of all $x \in W_p$ such that $x \bullet \lambda$ is antidominant and such that there exists $w \in \mathcal{W}$ with $w \bullet \lambda \uparrow x \bullet \lambda$. This is a finite set containing $\mathcal{W}$. We have to show that the Cartan matrix determines all $[\widehat{Z}(x \bullet \lambda) : \widehat{L}(w \bullet \lambda)]$ with $w \in \mathcal{W}$ and $x \in \mathcal{W}'$.

Use the notation $d(C)$ for any alcove C as in [J1], II.6.6(1). Choose a numbering $\mathcal{W}' = \{w_1, w_2, \dots, w_r\}$ such that $d(w_i \bullet A_0) < d(w_j \bullet A_0)$ implies $j < i$. It follows that $w_i \bullet \lambda \uparrow w_j \bullet \lambda$ implies $j \leq i$, cf. [J1], II.6.6. In particular, if $[\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] \neq 0$, then $j \leq i$.

We use induction on i. For $i = 1$ observe that the only relevant multiplicity is $[\widehat{Z}(w_1 \bullet \lambda) : \widehat{L}(w_1 \bullet \lambda)] = 1$. Let now $i > 1$ and assume that the Cartan matrix determines all $[\widehat{Z}(w \bullet \lambda) : \widehat{L}(w_s \bullet \lambda)]$ with $s < i$ and $w \in W_p$. There exists a weight ν_i with $w_i \bullet \lambda \in p\nu_i - \rho - X_1$. Since $w_i \bullet \lambda$ is antidominant, so is ν_i. Now $w_i \bullet \lambda - p\nu_i \in -\rho - X_1$ show that there exists $w_m \in \mathcal{W}$ with $w_i \bullet A_0 = p\nu_i + w_m \bullet A_0$. If $\nu_i \neq 0$, then $d(w_m \bullet A_0) > d(w_i \bullet A_0)$, hence $m < i$. There exists $\lambda' \in A_0$ with $w_i \bullet \lambda - p\nu_i = w_m \bullet \lambda'$. We know by induction (and translation) all $[\widehat{Z}(w \bullet \lambda') : \widehat{L}(w_m \bullet \lambda')]$ and get by periodicity all $[\widehat{Z}(w \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$.

So suppose that $\nu_i = 0$, i.e., that $w_i \in \mathcal{W}$. We have in the Grothendieck group

$$[\widehat{Q}(w_i \bullet \lambda)] = \sum_{j \le i} [\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] \sum_{x \in W} [\widehat{Z}(xw_j \bullet \lambda)]. \tag{2}$$

Here we have used (1).

We want to show by induction on l that all $[\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ are determined. As $[\widehat{Z}(w_i \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] = 1$, we may assume $l < i$. Suppose first that $w_l \in \mathcal{W}$. Then $[\widehat{Q}(w_i \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)]$ is by (1) and (2) equal to

$$\sum_{j \le i} |W| \, [\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)][\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)]. \tag{3}$$

By our induction on i we know all $[\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)]$. This multiplicity is 0 unless $j \le l$. If $j < l$, then we know $[\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ by our induction on l. So the only unknown summand in (3) is the one for $j = l$ where we get $|W| \, [\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$. This shows that $[\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ is determined by the Cartan matrix.

Now turn to the case where $w_l \notin \mathcal{W}$. There exists an antidominant weight $\nu \ne 0$ such that $w_l \bullet \lambda \in p\nu - \rho - X_1$. In this case [J1], II.9.13 implies for all μ that

$$[\widehat{Z}(\mu) : \widehat{L}(w_l \bullet \lambda)] = [\widehat{Z}(a_\nu(\mu)) : \widehat{L}(w_l \bullet \lambda)].$$

So we get from (2) that $[\widehat{Q}(w_i \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)]$ is equal to

$$\sum_{j \le i} [\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] \sum_{x \in W} [\widehat{Z}(a_\nu(xw_j \bullet \lambda)) : \widehat{L}(w_l \bullet \lambda)]. \tag{4}$$

Again we know by our induction on i all $[\widehat{Z}(a_\nu(xw_j \bullet \lambda)) : \widehat{L}(w_l \bullet \lambda)]$. If this multiplicity is non-zero, then $w_l \bullet \lambda \uparrow a_\nu(xw_j \bullet \lambda)$. Since ν and $w_j \bullet \lambda$ are antidominant, we get then from G.6(3) that $w_l \bullet \lambda \uparrow a_\nu(w_j \bullet \lambda)$, hence from G.6(2) that $w_l \bullet \lambda \uparrow w_j \bullet \lambda$ and $j \le l$. If $j < l$, then we know $[\widehat{Z}(w_j \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ by our induction on l. So the only unknown summand in (4) is the one for $j = l$ where we get

$$[\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)] \sum_{x \in W} [\widehat{Z}(xw_l \bullet \lambda) : \widehat{L}(w_l \bullet \lambda)].$$

(We do not need a_ν any longer.) Now the sum over x is positive since all terms are non-negative and the one for $x = 1$ is equal to one. We see as above that $[\widehat{Z}(w_l \bullet \lambda) : \widehat{L}(w_i \bullet \lambda)]$ is determined by the Cartan matrix.

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Quantum affine algebras and crystal bases

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§1. Introduction

The notion of a *quantum group* was introduced by Drinfel'd and Jimbo, independently, in their study of the quantum Yang-Baxter equation arising from 2-dimensional solvable lattice models ([3, 9]). Quantum groups are certain families of Hopf algebras that are deformations of universal enveloping algebras of Kac-Moody algebras. For the past 20 years, the quantum groups turned out to be the fundamental algebraic structure behind many branches of mathematics and mathematical physics.

In [18, 19, 25], Kashiwara and Lusztig independently developed the theory of *crystal bases* (or *canonical bases*) for quantum groups which provides a powerful combinatorial and geometric tool to study the representations of quantum groups. A *crystal basis* can be understood as a basis at $q = 0$ and is given a structure of colored oriented graph, called the *crystal graph*, with arrows defined by the *Kashiwara operators*. The crystal graphs have many nice combinatorial features reflecting the internal structure of integrable modules over quantum groups. In particular, they have extremely simple behavior with respect to taking the tensor product.

In this paper, we will discuss some of the recent developments in crystal basis theory for quantum affine algebras in connection with combinatorics of *Young walls* ([12, 16, 17]). In Section 2, 3 and 4, we briefly review the basic properties of Kac-Moody algebras, quantum groups and

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crystal bases ([8, 11, 19]). In Section 5, we fix some notations for quantum affine algebras, and in Section 6, we recall the *path realization* of crystal graphs for quantum affine algebras using the notion of *perfect crystals* ([13, 14]). In Section 7, we discuss the surprising connection between the crystal basis theory for the quantum affine algebra $U_q(\widehat{sl_n})$ and the modular representation theory of Hecke algebras ([1, 6, 26]). In Section 8, we explain the combinatorics of Young walls using the example of the quantum affine algebra $U_q(B_n^{(1)})$ and give a realization of the crystal graph for the basic representations in terms of *reduced proper Young walls.* In Section 9, we give a construction of the *Fock space representation* of quantum affine algebras using combinatorics of Young walls ([16, 17]).

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§2. Kac-Moody algebras

Let I be a finite index set. A square matrix $A = (a_{ij})_{i,j\in I}$ is called a *generalized Cartan matrix* if it satisfies: (i) $a_{ii} = 2$ for all $i \in I$, (ii) $a_{ij} \in \mathbf{Z}_{<0}$ for all $i, j \in I$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. In this paper, we assume that A is *symmetrizable*; i.e., there is a diagonal matrix $D = \operatorname{diag}(s_i \in \mathbf{Z}_{>0} | \, i \in I)$ with positive integral entries such that DA is symmmteric.

Consider a free abelian group

$$P^\vee = \left(\bigoplus_{i\in I} \mathbf{Z}h_i\right) \oplus \left(\bigoplus_{j=1}^{\operatorname{corank} A} \mathbf{Z}d_j\right),$$

and let $\mathfrak{h} = \mathbf{Q} \otimes_{\mathbf{Z}} P^\vee$. The free abelian group $P^\vee$ is called the *dual weight lattice* and the $\mathbf{Q}$-vector space $\mathfrak{h}$ is called the *Cartan subalgebra.* The *weight lattice* and the set of *simple coroots* are defined to be

$$P = \{\lambda \in \mathfrak{h}^* | \, \lambda(P^\vee) \subset \mathbf{Z}\}, \qquad \Pi^\vee = \{h_i | \, i \in I\}.$$

We denote by $\Pi = \{\alpha_i | \, i \in I\}$ the set of *simple roots*, which is a linearly independent subset of $\mathfrak{h}^*$ satisfying

$$\alpha_i(h_j) = a_{ji} \quad \text{for all} \quad i, j \in I.$$

Definition 2.1. The quintuple $(A, P^\vee, P, \Pi^\vee, \Pi)$ defined above is called a *Cartan datum* associated with A.

We denote by $P^+ = \{\lambda \in P|\ \lambda(h_i) \geq 0 \text{ for all } i \in I\}$ the set of *dominant integral weights*. The free abelian group $Q = \bigoplus_{i\in I} \mathbf{Z}\alpha_i$ is called the *root lattice*. We set $Q_+ = \sum_{i\in I} \mathbf{Z}_{\geq 0}\alpha_i$ and $Q_- = -Q_+$. There is a partial ordering on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. Since the generalized Cartan matrix A is symmetrizable, there is a nondegenerate symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ satisfying

$$s_i = \frac{(\alpha_i|\alpha_i)}{2} \quad \text{and} \quad \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} = a_{ij} \quad \text{for all } i, j \in I.$$

Definition 2.2. The *Kac-Moody algebra* $\mathfrak{g}$ associated with a Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the Lie algebra over $\mathbf{Q}$ generated by the elements e_i, f_i $(i \in I)$ and $h \in \mathfrak{h}$ subject to the defining relations:

$$\begin{gathered}[h, h'] = 0, \quad [e_i, f_j] = \delta_{ij}h_i \quad \text{for all } h, h' \in \mathfrak{h},\ i, j \in I,\\ [h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i \quad \text{for all } h \in \mathfrak{h},\ i \in I,\\ (\mathrm{ad}e_i)^{1-a_{ij}}(e_j) = (\mathrm{ad}f_i)^{1-a_{ij}}(f_j) = 0 \quad \text{for } i \neq j.\end{gathered}$$

Let $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) be the subalgebra of $\mathfrak{g}$ generated by e_i (resp. f_i) for $i \in I$. Then we have the *triangular decomposition*:

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+.$$

A $\mathfrak{g}$-module V is called a *weight module* if it admits a *weight space decomposition* $V = \bigoplus_{\mu\in\mathfrak{h}^*} V_\mu$, where $V_\mu = \{v \in V|\ hv = \mu(h)v \text{ for all } h \in \mathfrak{h}\}$. If $\dim_{\mathbf{Q}} V_\mu < \infty$ for all $\mu \in \mathfrak{h}^*$, we define the *character* of V by

$$\mathrm{ch}V = \sum_{\mu\in\mathfrak{h}^*} (\dim_{\mathbf{Q}} V_\mu)e^\mu,$$

where e^μ are basis elements of the group algebra $\mathbf{Q}[\mathfrak{h}^*]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for all $\mu, \nu \in \mathfrak{h}^*$.

A weight module V over $\mathfrak{g}$ is called a *highest weight module* with highest weight λ if there exists a non-zero vector $v_\lambda \in V$ such that (i) $V = U(\mathfrak{g})v_\lambda$, (ii) $hv_\lambda = \lambda(h)v_\lambda$ for all $h \in \mathfrak{h}$, (iii) $e_iv_\lambda = 0$ for all $i \in I$. For example, let $J(\lambda)$ denote the left ideal of $U(\mathfrak{g})$ generated by e_i, $h - \lambda(h)1$ for $i \in I$, $h \in \mathfrak{h}$, and set $M(\lambda) = U(\mathfrak{g})/J(\lambda)$. Then, via left multiplication, $M(\lambda)$ becomes a highest weight $\mathfrak{g}$-module with highest weight λ. The $\mathfrak{g}$-module $M(\lambda)$ is called the *Verma module*, and it satisfies the following properties:

Proposition 2.3 ([11]).

(a) *$M(\lambda)$ is a free $U(\mathfrak{g}^-)$-module of rank 1.*

(b) *Evey highest weight $\mathfrak{g}$-module with highest weight λ is a homomorphic image of $M(\lambda)$.*

(c) *$M(\lambda)$ contais a unique maximal submodule $R(\lambda)$.*

The irreducible quotient $V(\lambda) = M(\lambda)/R(\lambda)$ is called the *irreducible highest weight $\mathfrak{g}$-module with highest weight λ.*

Definition 2.4. The *category $\mathcal{O}_{int}$* consists of $\mathfrak{g}$-modules M satisfying the following properties:

(i) $M = \bigoplus_{\mu \in P} M_\mu$, where $M_\mu = \{v \in M | \ hv = \mu(h)v \ \text{ for all } h \in \mathfrak{h}\}$,

(ii) there exist finitely many $\lambda_1, \cdots, \lambda_s \in P$ such that

$$\mathrm{wt}(M) := \{\mu \in P | \ M_\mu \neq 0\} \subset \bigcup_{j=1}^{s} (\lambda_j - Q_+),$$

(iii) e_i and f_i $(i \in I)$ are locally nilpotent on M.

The basic properties of the category $\mathcal{O}_{int}$ are given in the following proposition.

Proposition 2.5.

(a) *For each $i \in I$, let $\mathfrak{g}_{(i)}$ be the subalgebra of $\mathfrak{g}$ generated by e_i, f_i, h_i, which is isomorphic to the 3-dimensional simple Lie algebra sl_2. Then every $\mathfrak{g}$-module M in the category $\mathcal{O}_{int}$ is a direct sum of finite dimensional irreducible $\mathfrak{g}_{(i)}$-submodules.*

(b) *The category $\mathcal{O}_{int}$ is semisimple. Moreover, every irreducible object in the category $\mathcal{O}_{int}$ has the form $V(\lambda)$ with $\lambda \in P^+$.*

§3. Quantum groups

Let $(A, P^\vee, P, \Pi^\vee, \Pi)$ be a Cartan datum associated with a symmetrizable generalized Cartan matrix A. For an indeterminate q, set $q_i = q^{s_i}$ and define

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}.$$

We will also use the notation $e_i^{(n)} = e_i^n / [n]_i!$, $f_i^{(n)} = f_i^n / [n]_i!$.

Definition 3.1. The *quantum group* $U_q(\mathfrak{g})$ associated with a Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the associative algebra over $\mathbf{Q}(q)$ with 1 generated by the symbols e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ subject to the following defining relations:

$$
\begin{aligned}
& q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad (h, h' \in P^\vee), \\
& q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad (h \in P^\vee,\ i \in I), \\
& e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where} \quad K_i = q^{s_i h_i}, \\
& \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0 \qquad (i \neq j), \\
& \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0 \qquad (i \neq j).
\end{aligned}
$$

The quantum group $U_q(\mathfrak{g})$ has a *Hopf algebra* structure with the comultiplication Δ, counit ε, and antipode S defined by

$$
\begin{gathered}
\Delta(q^h) = q^h \otimes q^h, \\
\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i, \\
\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \\
S(q^h) = q^{-h}, \quad S(e_i) = -e_i K_i, \quad S(f_i) = -K_i^{-1} f_i
\end{gathered}
$$

for $h \in P^\vee$ and $i \in I$.

Let U^+ (resp. U^-) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i (resp. f_i) for $i \in I$, and let U^0 be the subalgebra of $U_q(\mathfrak{g})$ generated by q^h $(h \in P^\vee)$. Then we have the *triangular decomposition*:

$$U_q(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+.$$

A $U_q(\mathfrak{g})$-module V^q is called a *weight module* if it admits a *weight space decomposition* $V^q = \bigoplus_{\mu \in P} V_\mu$, where $V_\mu^q = \{v \in V^q \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}$. If $\dim_{\mathbf{Q}(q)} V_\mu^q < \infty$ for all $\mu \in P$, we define the *character* of V^q by

$$\mathrm{ch} V = \sum_{\mu \in P} \left(\dim_{\mathbf{Q}(q)} V_\mu\right) e^\mu,$$

where e^μ are basis elements of the group algebra $\mathbf{Q}(q)[P]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for all $\mu, \nu \in P$.

A weight module V^q over $U_q(\mathfrak{g})$ is called a *highest weight module with highest weight* λ if there exists a non-zero vector $v_\lambda \in V^q$ (called the *highest weight vector*) such that (i) $V^q = U_q(\mathfrak{g})v_\lambda$, (ii) $q^h v_\lambda = q^{\lambda(h)} v_\lambda$ for all $h \in P^\vee$, (iii) $e_i v_\lambda = 0$ for all $i \in I$.

We construct the *Verma module* over $U_q(\mathfrak{g})$ as in the Kac-Moody algebra case. Let $J^q(\lambda)$ denote the left ideal of $U_q(\mathfrak{g})$ generated by e_i, $q^h - q^{\lambda(h)}1$ for $i \in I$, $h \in P^\vee$, and set $M^q(\lambda) = U_q(\mathfrak{g})/J^q(\lambda)$. Then, via left multiplication, $M^q(\lambda)$ becomes a highest weight $U_q(\mathfrak{g})$-module with highest weight λ called the Verma module.

Also, as in the Kac-Moody algebra case, we have:

Proposition 3.2 (cf. [8]).
(a) *$M^q(\lambda)$ is a free U^--module of rank 1.*
(b) *Evey highest weight $U_q(\mathfrak{g})$-module with highest weight λ is a homomorphic image of $M^q(\lambda)$.*
(c) *$M^q(\lambda)$ contains a unique maximal submodule $R^q(\lambda)$.*

The irreducible quotient $V^q(\lambda) = M^q(\lambda)/R^q(\lambda)$ is called the *irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ.*

Definition 3.3. The *category $\mathcal{O}^q_{int}$* consists of $U_q(\mathfrak{g})$-modules M^q satisfying the following properties:

(i) $M^q = \bigoplus_{\mu \in P} M^q_\mu$, where $M^q_\mu = \{v \in M^q \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}$,

(ii) there exist finitely many $\lambda_1, \cdots, \lambda_s \in P$ such that

$$\mathrm{wt}(M^q) := \{\mu \in P \mid M^q_\mu \neq 0\} \subset \bigcup_{j=1}^{s} (\lambda_j - Q_+),$$

(iii) e_i and f_i ($i \in I$) are locally nilpotent on M^q.

Proposition 3.4.
(a) *For each $i \in I$, let $U_{(i)}$ be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1}$, which is isomorphic to the quantum group $U_{q_i}(sl_2)$. Then every $U_q(\mathfrak{g})$-module M^q in the category $\mathcal{O}^q_{int}$ is a direct sum of finite dimensional irreducible $U_{(i)}$-submodules.*
(b) *The category $\mathcal{O}^q_{int}$ is semisimple. Moreover, every irreducible object in the category $\mathcal{O}^q_{int}$ has the form $V^q(\lambda)$ with $\lambda \in P^+$.*

Let $\mathbf{A}_1 = \{f/g \in \mathbf{Q}(q) \mid f, g \in \mathbf{Q}[q],\ g(1) \neq 0\}$ be the subring $\mathbf{Q}(q)$ consisting of the rational functions in q that are regular at $q = 1$. We define the $\mathbf{A}_1$*-form* of the quantum group $U_q(\mathfrak{g})$ to be the $\mathbf{A}_1$-subalgebra $U_{\mathbf{A}_1}$ generated by the elements e_i, f_i, q^h, $\dfrac{q^h - 1}{q - 1}$ for $h \in P^\vee$, $i \in I$.

Similarly, for the irreducible highest weight module $V^q(\lambda) = U_q(\mathfrak{g})v_\lambda$ and the Verma module $M^q(\lambda) = U_q(\mathfrak{g})u_\lambda$ over $U_q(\mathfrak{g})$, we define their $\mathbf{A}_1$-*forms* by

$$V^q(\lambda)_{\mathbf{A}_1} = U_{\mathbf{A}_1} v_\lambda, \qquad M^q(\lambda)_{\mathbf{A}_1} = U_{\mathbf{A}_1} u_\lambda.$$

Let $\mathbf{J}_1$ be the unique maximal ideal of $\mathbf{A}_1$ generated by $q-1$ and consider the isomorphism of fields

$$\mathbf{A}_1/\mathbf{J}_1 \xrightarrow{\sim} \mathbf{Q} \qquad \text{given by} \qquad f(q) + \mathbf{J}_1 \longmapsto f(1).$$

(In particular, q is mapped onto 1.) Define the $\mathbf{Q}$-linear vector spaces

$$\begin{aligned} U_1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} U_{\mathbf{A}_1}, \\ V^1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} V^q(\lambda)_{\mathbf{A}_1}, \\ M^1 &= (\mathbf{A}_1/\mathbf{J}_1) \otimes_{\mathbf{A}_1} M^q(\lambda)_{\mathbf{A}_1}. \end{aligned}$$

They are called the *classical limits* of $U_q(\mathfrak{g})$, $V^q(\lambda)$ and $M^q(\lambda)$, respectively.

As we can see in the following proposition, the structure of the quantum group $U_q(\mathfrak{g})$ *tends to* that of $U(\mathfrak{g})$ as $q \to 1$. Similarly, as $q \to 1$, the structure of the irreducible highest weight module $V^q(\lambda)$ (resp. the Verma module $M^q(\lambda)$) over $U_q(\mathfrak{g})$ with highest weight $\lambda \in P^+$ *tends to* that of $V(\lambda)$ (resp. $M(\lambda)$) over $U(\mathfrak{g})$. Moreover, the dimensions of the weight spaces are invariant under the deformation.

Proposition 3.5 ([8, 24]).
(a)

$$U_1 \cong U(\mathfrak{g}), \qquad V^1 \cong V(\lambda), \qquad M^1 \cong M(\lambda).$$

(b)

$$\begin{aligned} \dim_{\mathbf{Q}(q)} V^q(\lambda)_\mu &= \dim_{\mathbf{Q}} V(\lambda)_\mu, \\ \dim_{\mathbf{Q}(q)} M^q(\lambda)_\mu &= \dim_{\mathbf{Q}} M(\lambda)_\mu \quad \textit{for all} \quad \mu \in P. \end{aligned}$$

§4. Crystal bases

In this section, We briefly review the *crystal basis theory* for quantum groups developed by Kashiwara ([18, 19]). For simplicity, we will drop the superscript q from $U_q(\mathfrak{g})$-modules. Fix an index $i \in I$ and let $M = \bigoplus_{\lambda \in P} M_\lambda$ be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$. By the representation theory of $U_q(sl_2)$, every element $v \in M_\lambda$ can be written uniquely as

$$v = \sum_{k \geq 0} f_i^{(k)} v_k,$$

where $k \geq -\lambda(h_i)$ and $v_k \in \ker e_i \cap M_{\lambda + k\alpha_i}$. We define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ on M, called the *Kashiwara operators*, by

$$\tilde{e}_i v = \sum_{k \geq 1} f_i^{(k-1)} v_k, \qquad \tilde{f}_i v = \sum_{k \geq 0} f_i^{(k+1)} v_k.$$

Let $\mathbf{A}_0 = \{f/g \in \mathbf{Q}(q) \mid f, g \in \mathbf{Q}[q],\, g(0) \neq 0\}$ be the subring of $\mathbf{Q}(q)$ consisting of the rational functions in q that are regular at $q = 0$.

Definition 4.1. A *crystal basis* of M is a pair (L, B), where
(i) L is a free $\mathbf{A}_0$-submodule M such that $M \cong \mathbf{Q}(q) \otimes_{\mathbf{A}_0} L$,
(ii) B is a basis of the $\mathbf{Q}$-vector space L/qL,
(iii) $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$,
(iv) $B = \bigsqcup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda / qL_\lambda)$,
(v) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$ for all $i \in I$,
(vi) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$ for all $i \in I$,
(vii) for $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

The set B is given a colored oriented graph structure with the arrows defined by

$$b \xrightarrow{i} b' \quad \text{if and only if} \quad \tilde{f}_i b = b'.$$

The graph B is called the *crystal graph* of M and it reflects the combinatorial structure of M. For instance, we have

$$\dim_{\mathbf{Q}(q)} M_\lambda = \# B_\lambda \quad \text{for all} \ \ \lambda \in P.$$

Let B be a crystal graph for a $U_q(\mathfrak{g})$-module M in the category $\mathcal{O}^q_{int}$. For each $b \in B$ and $i \in I$, we define

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in B\}, \qquad \varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in B\}.$$

Then the crystal graph B satisfies the following properties.

Proposition 4.2 ([19, 20, 21]).
(a) *For all $i \in I$ and $b \in B$, we have*

$$\begin{aligned} \varphi_i(b) &= \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle, \\ \mathrm{wt}(\tilde{e}_i b) &= \mathrm{wt}(b) + \alpha_i, \\ \mathrm{wt}(\tilde{f}_i b) &= \mathrm{wt}(b) - \alpha_i. \end{aligned}$$

(b) *If* $\tilde{e}_i b \in B$, *then*

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1.$$

(c) *If* $\tilde{f}_i b \in B$, *then*

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1.$$

Moreover, the crystal bases have extremely simple behavior with respect to taking the tensor product.

Proposition 4.3 ([18, 19]).
Let M_j $(j = 1, 2)$ *be a* $U_q(\mathfrak{g})$*-module in the category* $\mathcal{O}^q_{int}$ *and let* (L_j, B_j) *be its crystal basis. Set*

$$L = L_1 \otimes_{\mathbf{A}_0} L_2, \quad B = B_1 \times B_2.$$

Then (L, B) *is a crystal basis of* $M_1 \otimes_{\mathbf{Q}(q)} M_2$ *with the Kashiwara operators on* B *given by*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

The *tensor product rule* in Proposition 4.3 gives a very convenient combinatorial description of the action of Kashiwara operators on the multi-fold tensor product of crystal graphs. Let M_j be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$ with a crystal basis (L_j, B_j) $(j = 1, \cdots, N)$. Fix an index $i \in I$ and consider a vector $b = b_1 \otimes \cdots \otimes b_N \in B_1 \otimes \cdots \otimes B_N$. To each $b_j \in B_j$ $(j = 1, \cdots, N)$, we assign a sequence of $-$'s and $+$'s with as many $-$'s as $\varepsilon_i(b_j)$ followed by as many $+$'s as $\varphi_i(b_j)$:

$$\begin{aligned} b &= b_1 \otimes b_2 \otimes \cdots \otimes b_N \\ &\longmapsto (\underbrace{-, \cdots, -}_{\varepsilon_i(b_1)}, \underbrace{+, \cdots, +}_{\varphi_i(b_1)}, \cdots\cdots, \underbrace{-, \cdots, -}_{\varepsilon_i(b_N)}, \underbrace{+, \cdots, +}_{\varphi_i(b_N)}). \end{aligned}$$

In this sequence, we cancel out all the $(+, -)$-pairs to obtain a sequence of $-$'s followed by $+$'s:

$$i\text{-sgn}(b) = (-, -, \cdots, -, +, +, \cdots, +).$$

The sequence i-sgn(b) is called the *i-signature* of b.

Now the tensor product rule tells that $\tilde{e}_i$ acts on b_j corresponding to the rightmost $-$ in the i-signature of b and $\tilde{f}_i$ acts on b_k corresponding to the leftmost $+$ in the i-signature of b:

$$\tilde{e}_i b = b_1 \otimes \cdots \otimes \tilde{e}_i b_j \otimes \cdots \otimes b_N,$$
$$\tilde{f}_i b = b_1 \otimes \cdots \otimes \tilde{f}_i b_k \otimes \cdots \otimes b_N.$$

We define $\tilde{e}_i b = 0$ (resp. $\tilde{f}_i b = 0$) if there is no $-$ (resp. $+$) in the i-signature of b.

We close this section with the existence and uniquenes theorem for crystal bases.

Theorem 4.4 ([19]).

(a) *Let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\lambda \in P^+$ and highest weight vector v_λ. Let $L(\lambda)$ be the free $\mathbf{A}_0$-submodule of $V(\lambda)$ spanned by the vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda$ $(i_k \in I, r \in \mathbf{Z}_{\geq 0})$ and set*

$$B(\lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda + qL(\lambda) \in L(\lambda)/qL(\lambda)\} \setminus \{0\}.$$

Then $(L(\lambda), B(\lambda))$ is a crystal basis of $V(\lambda)$ and every crystal basis of $V(\lambda)$ is isomorphic to $(L(\lambda), B(\lambda))$.

(b) *Define a $\mathbf{Q}$-algebra automorphism of $U_q(\mathfrak{g})$ by*

$$\overline{q} = q^{-1}, \quad \overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q^h} = q^{-h} \quad \textit{for} \ \ i \in I, h \in P^\vee.$$

Let $\mathbf{A} = \mathbf{Q}[q, q^{-1}]$ and define $V(\lambda)_\mathbf{A} = U_\mathbf{A}^- v_\lambda$, where $U_\mathbf{A}^-$ is the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)}$ $(i \in I, n \in \mathbf{Z}_{\geq 0})$. Then there exists a unique $\mathbf{A}$-basis $G(\lambda) = \{G(b) | \, b \in B(\lambda)\}$ of $V(\lambda)_\mathbf{A}$ such that

$$\overline{G(b)} = G(b), \quad G(b) \equiv b \ \ \mathrm{mod}\, qL(\lambda) \quad \textit{for all } b \in B(\lambda).$$

The basis $G(\lambda)$ of $V(\lambda)$ given in Theorem 4.4 is called the *global basis* or the *canonical basis* of $V(\lambda)$ associated with the crystal graph $B(\lambda)$ ([19, 25]).

§5. Quantum affine algebras

Let $I = \{0, 1, \cdots, n\}$ be an index set and let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix of affine type. We denote by

$$P^\vee = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \cdots \oplus \mathbf{Z}h_n \oplus \mathbf{Z}d$$

the dual weight lattice and $\Pi^\vee = \{h_i | \ i \in I\}$ the simple coroots. The simple roots α_i and the fundamental weights Λ_i are given by

$$\begin{aligned} &\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{0,i}, \\ &\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \qquad (i, j \in I). \end{aligned}$$

We define the *affine weight lattice* to be

$$P = \{\lambda \in \mathfrak{h}^* | \ \lambda(P^\vee) \subset \mathbf{Z}\}.$$

The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called an *affine Cartan datum.* To each affine Cartan datum, we can associate an infinite dimensional Lie algebra $\mathfrak{g}$ called the *affine Kac-Moody algebra* ([11]). The center of the affine Kac-Moody algebra $\mathfrak{g}$ is 1-dimensional and is generated by the *canonical central element*

$$c = c_0 h_0 + c_1 h_1 + \cdots + c_n h_n.$$

Moreover, the imaginary roots of $\mathfrak{g}$ are nonzero integral multiples of the *null root*

$$\delta = d_0 \alpha_0 + d_1 \alpha_1 + \cdots + d_n \alpha_n.$$

Here, c_i and d_i $(i \in I)$ are the non-negative integers given in [11].

Using the fundamental weights and the null root, the affine weight lattice can be written as

$$P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_n \oplus \mathbf{Z}\delta.$$

Set

$$P^+ = \{\lambda \in P \,|\, \lambda(h_i) \in \mathbf{Z}_{\geq 0} \quad \text{for all} \ \ i \in I\}.$$

The elements of P (resp. P^+) are called the *affine weights* (resp. *affine dominant integral weights*). The *level* of an affine dominant integral weight $\lambda \in P^+$ is defined to be the nonnegative integer $\lambda(c)$.

Definition 5.1. The *quantum affine algebra* $U_q(\mathfrak{g})$ is the quantum group associated with the affine Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$.

The subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1}$ $(i \in I)$ is denoted by $U_q'(\mathfrak{g})$, and is also called the *quantum affine algebra.*

Let

$$\overline{P}^\vee = \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \cdots \oplus \mathbf{Z}h_n \quad \text{and} \quad \overline{\mathfrak{h}} = \mathbf{Q} \otimes_{\mathbf{Z}} \overline{P}^\vee.$$

Consider α_i and Λ_i $(i \in I)$ as linear functionals on $\overline{\mathfrak{h}}$, and set

$$\overline{P} = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_n.$$

The elements of $\overline{P}$ are called the *classical weights.* The algebra $U_q'(\mathfrak{g})$ can be regarded as the quantum affine algebra associated with the *classical Cartan datum* $(A, \Pi, \Pi^\vee, \overline{P}, \overline{P}^\vee)$.

The projection $\mathrm{cl} : P \longrightarrow \overline{P}$ will be denoted by $\lambda \longmapsto \overline{\lambda}$ and we will fix an embedding $\mathrm{aff} : \overline{P} \longrightarrow P$ such that

$$\mathrm{cl} \circ \mathrm{aff} = \mathrm{id}, \qquad \mathrm{aff} \circ \mathrm{cl}(\alpha_i) = \alpha_i \quad \text{for} \quad i \neq 0.$$

We define

$$\overline{P}^+ = \mathrm{cl}(P^+) = \{\lambda \in \overline{P} \mid \lambda(h_i) \geq 0 \ \text{ for all } \ i \in I\}.$$

The elements of $\overline{P}^+$ are called the *classical dominant integral weights.* A classical dominant integral weight $\lambda \in \overline{P}^+$ is said to have *level* $l \in \mathbf{Z}_{\geq 0}$ if $\lambda(c) = l$. Note that it has the same level as its affine counterpart.

§6. Perfect crystals and paths

By extracting properties of the crystal graphs, we define the notion of abstract *crystals* as follows ([20, 21]).

Definition 6.1. An *affine crystal* (resp. *classical crystal*) is a set B together with the maps $\mathrm{wt} : B \to P$ (resp. $\mathrm{wt} : B \to \overline{P}$), $\varepsilon_i : B \to \mathbf{Z} \cup \{-\infty\}$, $\varphi_i : B \to \mathbf{Z} \cup \{-\infty\}$, $\tilde{e}_i : B \to B \cup \{0\}$, and $\tilde{f}_i : B \to B \cup \{0\}$ satisfying the following conditions:

(i) for all $i \in I$, $b \in B$, we have

$$\begin{aligned} \varphi_i(b) &= \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle, \\ \mathrm{wt}(\tilde{e}_i b) &= \mathrm{wt}(b) + \alpha_i, \\ \mathrm{wt}(\tilde{f}_i b) &= \mathrm{wt}(b) - \alpha_i, \end{aligned}$$

(ii) if $\tilde{e}_i b \in B$, then

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1,$$

(iii) if $\tilde{f}_i b \in B$, then

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1,$$

(iv) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for all $i \in I$, $b, b' \in B$,

(v) if $\varepsilon_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

For instance, the crystal graphs for $U_q(\mathfrak{g})$-modules (resp. $U'_q(\mathfrak{g})$-modules) in the category $\mathcal{O}^q_{int}$ are affine crystals (resp. classical crystals).

Definition 6.2. Let B_1 and B_2 be (affine or classical) crystals. A *morphism* $\psi : B_1 \to B_2$ of crystals is a map $\psi : B_1 \cup \{0\} \to B_2 \cup \{0\}$ satisfying the conditions:

(i) $\psi(0) = 0$,

(ii) if $b \in B_1$ and $\psi(b) \in B_2$, then

$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b),$$

(iii) if $b, b' \in B_1$, $\psi(b), \psi(b') \in B_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \psi(b) = \psi(b')$.

A morphism of crystals is said to be *strict* if it commutes with the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$.

Definition 6.3. The *tensor product* $B_1 \otimes B_2$ of the crystals B_1 and B_2 is defined to be the set $B_1 \times B_2$ whose crystal structure is given by

$$\begin{aligned}
\mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \\
\varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \mathrm{wt}(b_1)\rangle), \\
\varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varepsilon_i(b_1) + \langle h_i, \mathrm{wt}(b_2)\rangle), \\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}
\end{aligned}$$

Here, we denote $b_1 \otimes b_2 = (b_1, b_2)$ and use the convention that $b_1 \otimes 0 = 0 \otimes b_2 = 0$.

We now define the notion of *perfect crystals.* Let B be a classical crystal. For $b \in B$, we define

$$\varepsilon(b) = \sum_i \varepsilon_i(b)\Lambda_i \quad \text{and} \quad \varphi(b) = \sum_i \varphi_i(b)\Lambda_i.$$

Note that

$$\mathrm{wt}(b) = \varphi(b) - \varepsilon(b).$$

For a positive integer $l > 0$, set

$$\overline{P}^+_l = \{\lambda \in \overline{P}^+ \mid \langle c, \lambda\rangle = l\}. \tag{6.1}$$

Definition 6.4. For $l \in \mathbf{Z}_{>0}$, we say that a finite classical crystal $\mathbf{B}$ is a *perfect crystal of level l* if

(i) there is a finite dimensional $U_q'(\mathfrak{g})$-module with a crystal basis whose crystal graph is isomorphic to $\mathbf{B}$,

(ii) $\mathbf{B} \otimes \mathbf{B}$ is connected,

(iii) there exists some $\lambda_0 \in \overline{P}$ such that

$$\mathrm{wt}(\mathbf{B}) \subset \lambda_0 + \frac{1}{d_0} \sum_{i \neq 0} \mathbf{Z}_{\leq 0} \alpha_i, \qquad \#(\mathbf{B}_{\lambda_0}) = 1,$$

(iv) for any $b \in \mathbf{B}$, we have $\langle c, \varepsilon(b) \rangle \geq l$,

(v) for each $\lambda \in \overline{P}_l^+$, there exist unique $b^\lambda \in \mathbf{B}$ and $b_\lambda \in \mathbf{B}$ such that

$$\varepsilon(b^\lambda) = \lambda, \qquad \varphi(b_\lambda) = \lambda.$$

A finite dimensional $U_q'(\mathfrak{g})$-module $\mathbf{V}$ is called a *perfect representation* of *level l* if it has a crystal basis (L, B) such that B is isomorphic to a perfect crystal of level l.

Remark. For a perfect crystal $\mathbf{B}$ of level l, define

$$\mathbf{B}^{\min} = \{ b \in \mathbf{B} \mid \langle c, \varepsilon(b) \rangle = l \}.$$

Then the maps $\varepsilon, \varphi : \mathbf{B}^{\min} = \{ b \in B \mid \langle c, \varepsilon(b) \rangle = l \} \longrightarrow \overline{P}_l^+$ are bijective.

In the following, we give an example of perfect crystals of level 1 for the quantum affine algebra $U_q(B_n^{(1)})$ $(n \geq 3)$.

Example 6.5.

$$\begin{tikzcd} \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "n-1"] & \boxed{n} \arrow[r, "n"] & \boxed{0} \arrow[r, "n"] & \boxed{\bar{n}} \arrow[r, "n-1"] & \cdots \arrow[r, "2"] & \boxed{\bar{2}} \arrow[r, "1"] \arrow[lllllll, bend right, "0"'] & \boxed{\bar{1}} \arrow[lllllll, bend left, "0"] \end{tikzcd}$$

Here, we have

$$b^{\Lambda_0} = \boxed{1}, \quad b_{\Lambda_0} = \boxed{\bar{1}}; \quad b^{\Lambda_1} = \boxed{\bar{1}}, \quad b_{\Lambda_1} = \boxed{1};$$
$$b^{\Lambda_n} = \boxed{0}, \quad b_{\Lambda_n} = \boxed{0}.$$

Fix a positive integer $l > 0$ and let $\mathbf{B}$ be a perfect crystal of level l. By definition, for any classical dominant integral weight $\lambda \in \overline{P}_l^+$, there exists a unique element $b_\lambda \in \mathbf{B}$ such that $\varphi(b_\lambda) = \lambda$. Set

$$\mu = \lambda - \mathrm{wt}(b_\lambda) = \varepsilon(b_\lambda),$$

and denote by u_μ the highest weight vector of the crystal graph $B(\mu)$. Then, using the fact that $\mathbf{B}$ is perfect, one can show that the vector $u_\mu \otimes b_\lambda$ is the unique maximal vector in $B(\mu) \otimes \mathbf{B}$. Moreover, we have:

Theorem 6.6 ([13]). *Let $\mathbf{B}$ be a perfect crystal of level $l > 0$. Then for any dominant integral weight $\lambda \in \overline{P}_l^+$, there exists a crystal isomorphism*

$$\Psi : B(\lambda) \xrightarrow{\sim} B(\varepsilon(b_\lambda)) \otimes \mathbf{B} \quad \text{given by} \quad u_\lambda \longmapsto u_{\varepsilon(b_\lambda)} \otimes b_\lambda,$$

where b_λ is the unique element in $\mathbf{B}$ such that $\varphi(b_\lambda) = \lambda$.

Set

$$\lambda_0 = \lambda, \quad \lambda_{k+1} = \varepsilon(b_{\lambda_k}),$$

and

$$b_0 = b_\lambda, \quad b_{k+1} = b_{\lambda_{k+1}}.$$

By taking the composition of crystal isomorphism given in Theorem 6.6 repeatedly, we obtain a sequence of crystal isomorphisms

$$B(\lambda) \xrightarrow{\sim} B(\lambda_1) \otimes \mathbf{B} \xrightarrow{\sim} B(\lambda_2) \otimes \mathbf{B} \otimes \mathbf{B} \xrightarrow{\sim} \cdots\cdots$$

given by

$$u_\lambda \longmapsto u_{\lambda_1} \otimes b_0 \longmapsto u_{\lambda_2} \otimes b_1 \otimes b_0 \longmapsto \cdots\cdots,$$

which yields the infinite sequences

$$\mathbf{w}_\lambda = (\lambda_k)_{k=0}^\infty = (\cdots, \lambda_{k+1}, \lambda_k, \cdots, \lambda_1, \lambda_0) \quad \text{in} \quad (\overline{P}_l^+)^\infty$$

and

$$\mathbf{p}_\lambda = (b_k)_{k=0}^\infty = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0 \quad \text{in} \quad \mathbf{B}^{\otimes\infty}.$$

Thus for each $k \geq 1$, we get a crystal isomorphism

$$\Psi_k : B(\lambda) \xrightarrow{\sim} B(\lambda_k) \otimes \mathbf{B}^{\otimes k}$$

given by

$$u_\lambda \longmapsto u_{\lambda_k} \otimes b_{k-1} \otimes \cdots \otimes b_1 \otimes b_0.$$

Since $\mathbf{B}$ is perfect, we have $\varphi(b_j) = \lambda_j$ and $\varepsilon(b_j) = \lambda_{j+1}$. It follows that the sequences

$$\mathbf{w}_\lambda = (\lambda_k)_{k=0}^{\infty} = (\cdots, \lambda_{k+1}, \lambda_k, \cdots, \lambda_1, \lambda_0)$$

and

$$\mathbf{p}_\lambda = (b_k)_{k=0}^{\infty} = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0$$

are periodic with the same period. That is, there is a positive integer $N > 0$ such that $\lambda_{j+N} = \lambda_j$, $b_{j+N} = b_j$ for all $j = 0, 1, \cdots, N-1$.

Definition 6.7.
(a) The sequence

$$\mathbf{p}_\lambda = (b_k)_{k=0}^{\infty} = \cdots \otimes b_{k+1} \otimes b_k \otimes \cdots \otimes b_1 \otimes b_0$$

is called the *ground-state path* of weight λ.
(b) A *λ-path* in $\mathbf{B}$ is a sequence

$$\mathbf{p} = (\mathbf{p}(k))_{k=0}^{\infty} = \cdots \otimes \mathbf{p}(k) \otimes \cdots \otimes \mathbf{p}(1) \otimes \mathbf{p}(0)$$

such that $\mathbf{p}(k) = b_k$ for all $k \gg 0$.

Let $\mathbf{P}(\lambda) = \mathbf{P}(\lambda, \mathbf{B})$ be the set of all λ-paths in $\mathbf{B}$. We define the $U_q'(\mathfrak{g})$-crystal structure on $\mathbf{P}(\lambda)$ as follows. Let $\mathbf{p} = (\mathbf{p}(k))_{k=0}^{\infty}$ be a λ-path in $\mathbf{P}(\lambda)$ and let $N > 0$ be a positive integer such that $\mathbf{p}(k) = b_k$ for all $k \geq N$. For each $i \in I$, we define

$$\begin{aligned}
\overline{\mathrm{wt}}(\mathbf{p}) &= \lambda_N + \textstyle\sum_{k=0}^{N-1} \overline{\mathrm{wt}}\mathbf{p}(k), \\
\tilde{e}_i \mathbf{p} &= \cdots \otimes \mathbf{p}(N+1) \otimes \tilde{e}_i(\mathbf{p}(N) \otimes \cdots \otimes \mathbf{p}(0)), \\
\tilde{f}_i \mathbf{p} &= \cdots \otimes \mathbf{p}(N+1) \otimes \tilde{f}_i(\mathbf{p}(N) \otimes \cdots \otimes \mathbf{p}(0)), \\
\varepsilon_i(\mathbf{p}) &= \max(\varepsilon_i(\mathbf{p}') - \varphi_i(b_N), 0), \\
\varphi_i(\mathbf{p}) &= \varphi_i(\mathbf{p}') + \max(\varphi_i(b_N) - \varepsilon_i(\mathbf{p}'), 0),
\end{aligned}$$

where $\mathbf{p}' = \mathbf{p}(N-1) \otimes \cdots \otimes \mathbf{p}(1) \otimes \mathbf{p}(0)$.
Then we have the *path realization* of the classical crystal $B(\lambda)$:

Theorem 6.8 ([13]).
(a) *The maps* $\overline{wt} : \mathbf{P}(\lambda) \to \overline{P}$, $\tilde{e}_i, \tilde{f}_i : \mathbf{P}(\lambda) \to \mathbf{P}(\lambda) \cup \{0\}$, $\varepsilon_i, \varphi_i : \mathbf{P}(\lambda) \to \mathbf{Z}$ *define a classical crystal structure on* $\mathbf{P}(\lambda)$.
(b) *There exists an isomorphism of classical crystals*

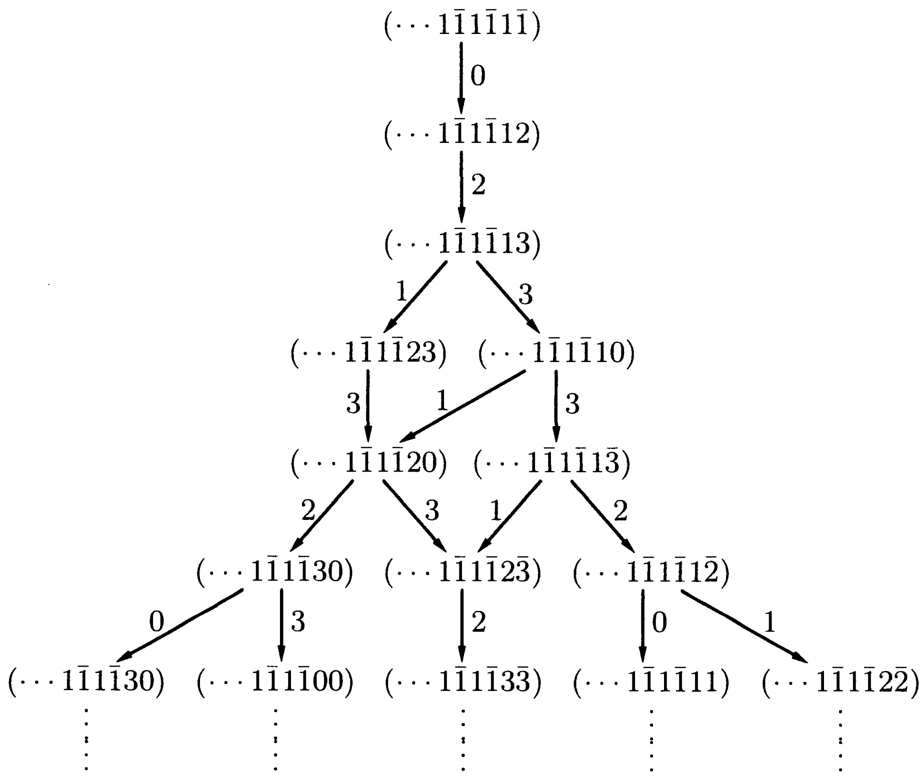
$$\Psi : B(\lambda) \xrightarrow{\sim} \mathbf{P}(\lambda) \quad \textit{given by} \quad u_\lambda \longmapsto \mathbf{p}_\lambda.$$

In the following example, we give the ground-state paths for the basic representations of the quantum affine algebra $U_q(B_3^{(1)})$ and illustrate the top part of their crystal graphs.

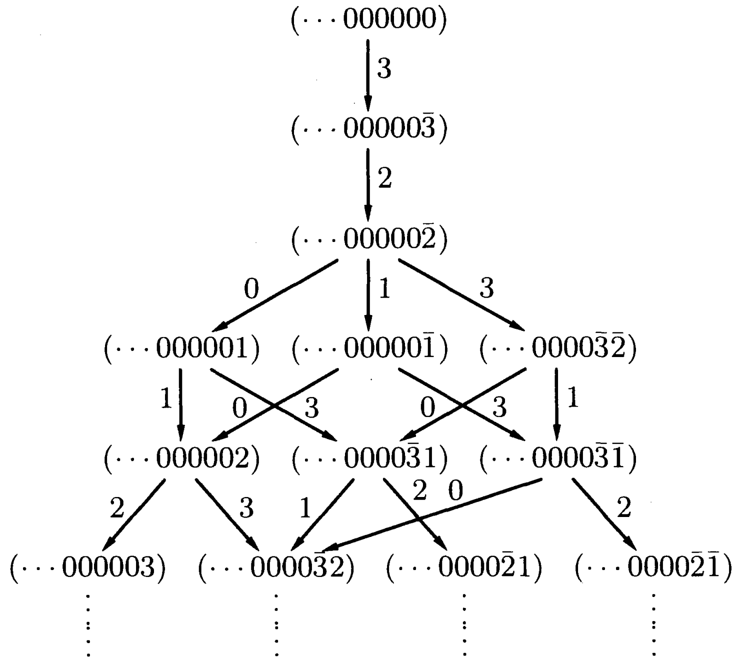
Example 6.9.
(a) Ground-state paths:

$$
\begin{aligned}
\mathbf{p}_{\Lambda_0} &= (\mathbf{p}_{\Lambda_0}(k))_{k=0}^{\infty} = (\cdots, 1, \bar{1}, 1, \bar{1}, 1, \bar{1}),\\
\mathbf{p}_{\Lambda_1} &= (\mathbf{p}_{\Lambda_1}(k))_{k=0}^{\infty} = (\cdots, \bar{1}, 1, \bar{1}, 1, \bar{1}, 1),\\
\mathbf{p}_{\Lambda_n} &= (\mathbf{p}_{\Lambda_n}(k))_{k=0}^{\infty} = (\cdots, 0, 0, 0, 0, 0, 0)
\end{aligned}
$$

(b) Crystal graph $\mathbf{P}(\Lambda_0)$



(c) Crystal graph $\mathbf{P}(\Lambda_3)$



§7. Hecke algebras and crystal bases

Let $U_q(\widehat{sl_n})$ be the quantum affine algebra associated with the Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$, where $A = (a_{ij})_{i,j=1}^{n-1}$ is the generalized Cartan matrix of affine type $A_{n-1}^{(1)}$,

$$\begin{aligned} P^\vee &= \mathbf{Z}h_0 \oplus \mathbf{Z}h_1 \oplus \cdots \oplus \mathbf{Z}h_{n-1} \oplus \mathbf{Z}d, \\ P &= \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_{n-1} \oplus \mathbf{Z}\delta, \\ \Pi^\vee &= \{h_0, h_1, \cdots, h_{n-1}\}, \\ \Pi &= \{\alpha_0, \alpha_1, \cdots, \alpha_{n-1}\}. \end{aligned}$$

Here, Λ_i are the *fundamental weights* defined by

$$\Lambda_i(h_j) = \delta_{ij}, \quad \Lambda_i(d) = 0 \quad \text{for } i = 0, 1, \cdots, n-1,$$

and $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$ is the *null root.*

In [26], Misra and Miwa gave a realization of the crystal graph $B(\Lambda_0)$ for the basic representation $V(\Lambda_0)$ of the quantum affine algebra $U_q(\widehat{sl_n})$

in terms of *n-reduced colored Young diagrams.* More precisely, we will build a colored Young diagram $Y = (y_k)_{k=0}^{\infty}$ with each box colored by $0, 1, \cdots, n-1$ following the pattern given below:

					0	1
					$n-1$	0
					$n-2$	$n-1$
					$\vdots$	$\vdots$
0	1	2	3	$\cdots$	0	1
$n-1$	0	1	2	$\cdots$	$n-1$	0

Here, y_k $(k = 0, 1, 2, \cdots)$ denotes the k-th column of Y reading from right to left. The heights of the columns of Y are weakly decreasing as we proceed from right to left and we have $y_k = 0$ for $k \gg 0$.

A colored Young diagram $Y = (y_k)_{k=0}^{\infty}$ is called *n-reduced* if $y_k - y_{k+1} < n$ for all $k \geq 0$. Let $\mathcal{Y}(\Lambda_0)$ be the set of all n-reduced colored Young diagrams. We will define the Kashiwara operators on $\mathcal{Y}(\Lambda_0)$ as follows. Fix an index $i \in I = \{0, 1, \cdots, n-1\}$. To each column y_k of Y, we assign its *i-signature* by

$$i\text{-signature of } y_k = \begin{cases} + & \text{if the top of } y_k \text{ is } i-1, \\ - & \text{if the top of } y_k \text{ is } i, \\ \cdot & \text{otherwise.} \end{cases}$$

Then we get an infinite sequence of $+$'s and $-$'s. From this infinite sequence, we cancel all the $(+,-)$-pairs to obatin a finite sequence of $-$'s followed by $+$'s:

$$i\text{-signature of } Y = (-, \cdots, -, +, \cdots, +),$$

which is called the *i-signature* of Y. Now we define $\tilde{e}_i Y$ (resp. $\tilde{f}_i Y$) to be the colored Young diagram obtained from Y by removing the i-box from the column (resp. by adding an i-box to the column) corresponding to the rightmost $-$ (resp. leftmost $+$) in the i-signature of Y.

Proposition 7.1 ([26]). *With the Kashiwara operators defined in this way, the set $\mathcal{Y}(\Lambda_0)$ becomes a $U_q(\widehat{sl_n})$-crystal. Moreover, there exists a $U_q(\widehat{sl_n})$-crystal isomorphism*

$$\mathcal{Y}(\Lambda_0) \xrightarrow{\sim} B(\Lambda_0).$$

For a nonzero complex number ζ, let $H_N(\zeta)$ denote the Hecke algebra of type A_{N-1}. Recall that when ζ is a primitive n-th root of unity, every finite dimensional irreducible $H_N(\zeta)$-module appears as the unique irreducible quotient $D(Y)$ of the Specht module $S(Y)$ corresponding to an n-reduced Young diagram Y with N boxes (see, for example, [4, 5]). Since Y can be viewed as a crystal basis element of $B(\Lambda_0)$ for $U_q(\widehat{sl_n})$, using the $U_q(\widehat{sl_n})$-module action on the space of all colored Young diagrams (see, for example, [7, 10, 26]), one should be able to write the global basis element $G(Y)$ as a linear combination of colored Young diagrams:

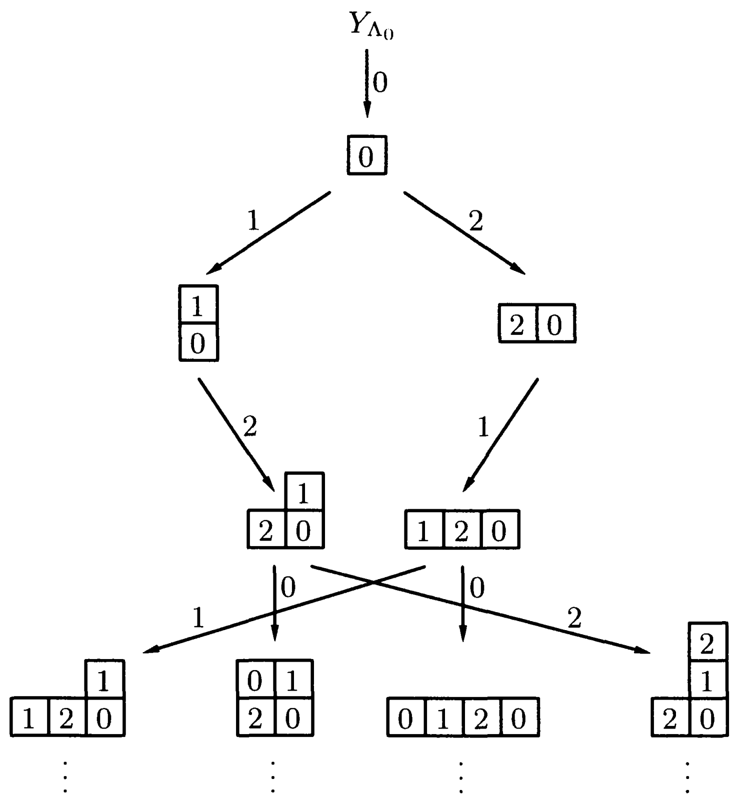
$$G(Y) = \sum_{Y'} d_{Y',Y}(q) Y' \quad \text{for} \quad d_{Y',Y}(q) \in \mathbf{Z}[q].$$

In [23], Lascoux, Leclerc and Thibon gave a recursive algorithm of computing the polynomials $d_{Y',Y}(q)$ and conjectured that

$$d_{Y',Y}(1) = [S(Y') : D(Y)],$$

where $[S(Y') : D(Y)]$ denotes the multiplicity of $D(Y)$ occurring in a composition series of $S(Y')$. This conjecture was proved by Ariki ([1]) and Grojnowski ([6]). In fact, the polynomials $d_{Y',Y}(q)$ coincide with the affine Kazhdan-Lusztig polynomials.

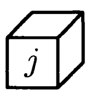
Example 7.2. By Proposition 7.1, the crystal graph $B(\Lambda_0)$ for the quantum affine algebra $U_q(\widehat{sl_3})$ is realized as the set of 3-reduced colored Young diagrams.



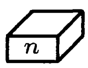
§8. Combinatorics of Young walls

In [12], we generalized the idea of [26] to the other classical quantum affine algebras (except $U_q(C_n^{(1)})$) and gave a realization of the crystal graphs $B(\Lambda)$ for the basic representations in terms of new combinatorial objects called the *Young walls*, which can be viewed as generalizations of colored Young diagrams. We will explain the main idea of [12] with the example of the quantum affine algebra $U_q(B_n^{(1)})$ because this case contains all the characteristics of combinatorics of Young walls.

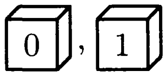
The Young walls are built of colored blocks of three different shapes (called the blocks of type I, type II, and type III, respectively) :



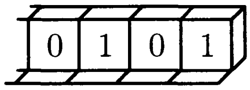
$(2 \leq j \leq n-1)$: unit width, unit height, unit thickness,

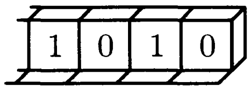


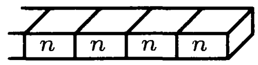
: unit width, half-unit height, unit thickness,

 : unit width, unit height, half-unit thickness.

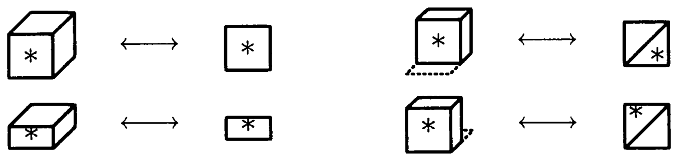
For each fundamental weight Λ of level 1; i.e., for $\Lambda = \Lambda_0$, Λ_1 or Λ_n, we fix a frame Y_Λ called the *ground-state wall of weight* Λ:

$Y_{\Lambda_0} =$  ,

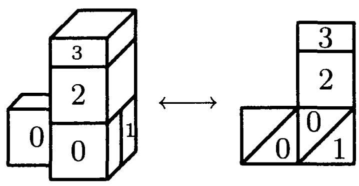
$Y_{\Lambda_1} =$  ,

$Y_{\Lambda_n} =$  .

On these frames, we build the walls of thickness less than or equal to one unit extending to the left. For convenience, we will use the following notations:



For example, when $n = 3$, we have



The rules for building the walls are given as follows:

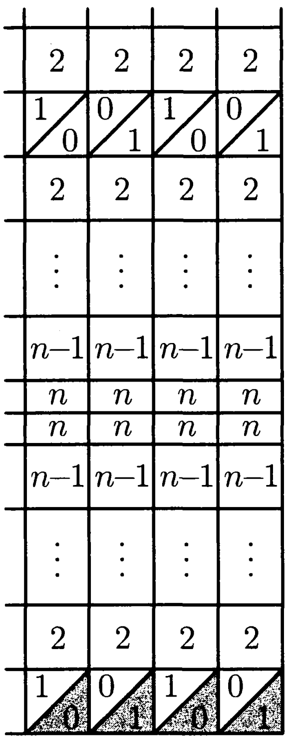
(1) The walls are built on top of the ground-state walls.

(2) The colored blocks should be stacked in columns. No block can be placed on top of a column of half-unit thickness.

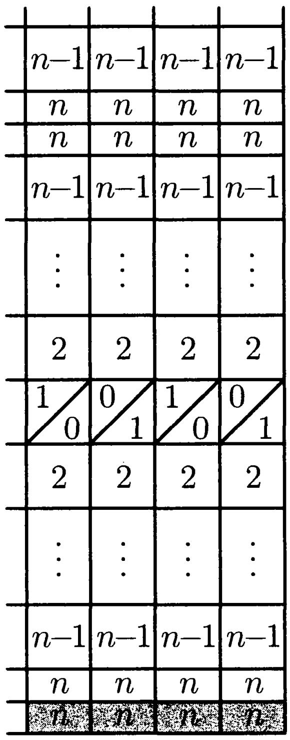
(3) Except for the right-most column, there should be no free space to the right of any block.

(4) The colored blocks should be stacked in the pattern given below:

On Y_{Λ_0} :



On Y_{Λ_n} :



A wall Y built on the ground-state wall Y_Λ following the rules given above is called a *Young wall* on Y_Λ, for the heights of its columns are weakly decreasing as we proceed from right to left. We often write $Y = (y_k)_{k=0}^{\infty} = (\cdots\cdots, y_2, y_1, y_0)$ as an infinite sequence of its colums.

Definition 8.1.
(a) A column of a Young wall is called a *full column* if its height is a multiple of the unit length and its top is of unit thickness.
(b) A Young wall is said to be *proper* if none of the full columns have the same height.

We denote by $\mathcal{P}(\Lambda)$ the set of all proper Young walls on Y_Λ. We will define the action of Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ on $\mathcal{P}(\Lambda)$.

Definition 8.2. Let $Y = (y_k)_{k=0}^{\infty}$ be a proper Young wall on Y_Λ.
(a) A block of color i in Y is called a *removable i-block* if Y remains a proper Young wall after removing the block. A column in Y is said to be *i-removable* if the top of that column is a removable i-block.
(b) A place in Y is called an *i-admissible slot* if one may add an i-block to obtain another proper Young wall. A column in Y is said to be *i-admissible* if the top of that column is an i-admissible slot.

Fix an index $i \in I$ and let $Y = (y_k)_{k=0}^{\infty} \in \mathcal{P}(\Lambda)$ be a proper Young wall. To each column y_k of Y, we assign its *i-signature* as follows:

(1) we assign $--$ if the column y_k is twice i-removable (the i-block will be of half-unit height in this case);
(2) we assign $-$ if the column is once i-removable, but not i-admissible (the i-block may be of unit height or of half-unit height);
(3) we assign $-+$ if the column is once i-removable and once i-admissible (the i-block will be of half-unit height in this case);
(4) we assign $+$ if the column is once i-admissible, but not i-removable (the i-block may be of unit height or of half-unit height);
(5) we assign $++$ if the column is twice i-admissible (the i-block will be of half-unit height in this case).

Then we get an infinite sequence of $+$'s and $-$'s. From this infinite sequence, we cancel out every $(+,-)$-pair to obtain a finite sequence of $-$'s followed by $+$'s, reading from left to right. This sequence is called the *i-signature* of Y. Now, we define the crystal structure on $\mathcal{P}(\Lambda)$ as follows.

(1) We define $\tilde{e}_i Y$ to be the proper Young wall obtained from Y by removing the i-block corresponding to the rightmost $-$ in the i-signature of Y. We define $\tilde{e}_i Y = 0$ if there exists no $-$ in the i-signature of Y.
(2) We define $\tilde{f}_i Y$ to be the proper Young wall obtained from Y by adding an i-block to the column corresponding to the leftmost $+$ in the i-signature of Y. We define $\tilde{f}_i Y = 0$ if there exists no $+$ in the i-signature of Y.

We also define the maps

$$\mathrm{wt} : \mathcal{P}(\Lambda) \longrightarrow P, \quad \varepsilon_i : \mathcal{P}(\Lambda) \longrightarrow \mathbf{Z}, \quad \varphi_i : \mathcal{P}(\Lambda) \longrightarrow \mathbf{Z}$$

by

$$\begin{aligned} \mathrm{wt}(Y) &= \Lambda - \sum_{i \in I} k_i \alpha_i, \\ \varepsilon_i(Y) &= \text{the number of } -\text{'s in the } i\text{-signature of } Y, \\ \varphi_i(Y) &= \text{the number of } +\text{'s in the } i\text{-signature of } Y, \end{aligned}$$

where k_i is the number of i-blocks in Y that have been added to the ground-state wall Y_Λ.

Proposition 8.3 ([12]).
The set $\mathcal{P}(\Lambda)$ together with the maps $\mathrm{wt} : \mathcal{P}(\Lambda) \to P$, $\tilde{e}_i, \tilde{f}_i : \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda) \cup \{0\}$ and $\varepsilon_i, \varphi_i : \mathcal{P}(\Lambda) \to \mathbf{Z}$ becomes a $U_q(B_n^{(1)})$-crystal.

Recall that $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + 2\alpha_n$ is the null root of the quantum affine algebra $U_q(B_n^{(1)})$.

Definition 8.4.

(a) The part of a column in a proper Young wall is called a *δ-column* if it has the same number of colored blocks as the null root δ in some cyclic order.

(b) A δ-column in a proper Young wall is called *removable* if it can be removed to yield another proper Young wall.

(c) A proper Young wall is said to be *reduced* if none of its columns contain a removable δ-column.

Let $\mathcal{Y}(\Lambda) \subset \mathcal{P}(\Lambda)$ be the set of all reduced proper Young walls on Y_Λ.

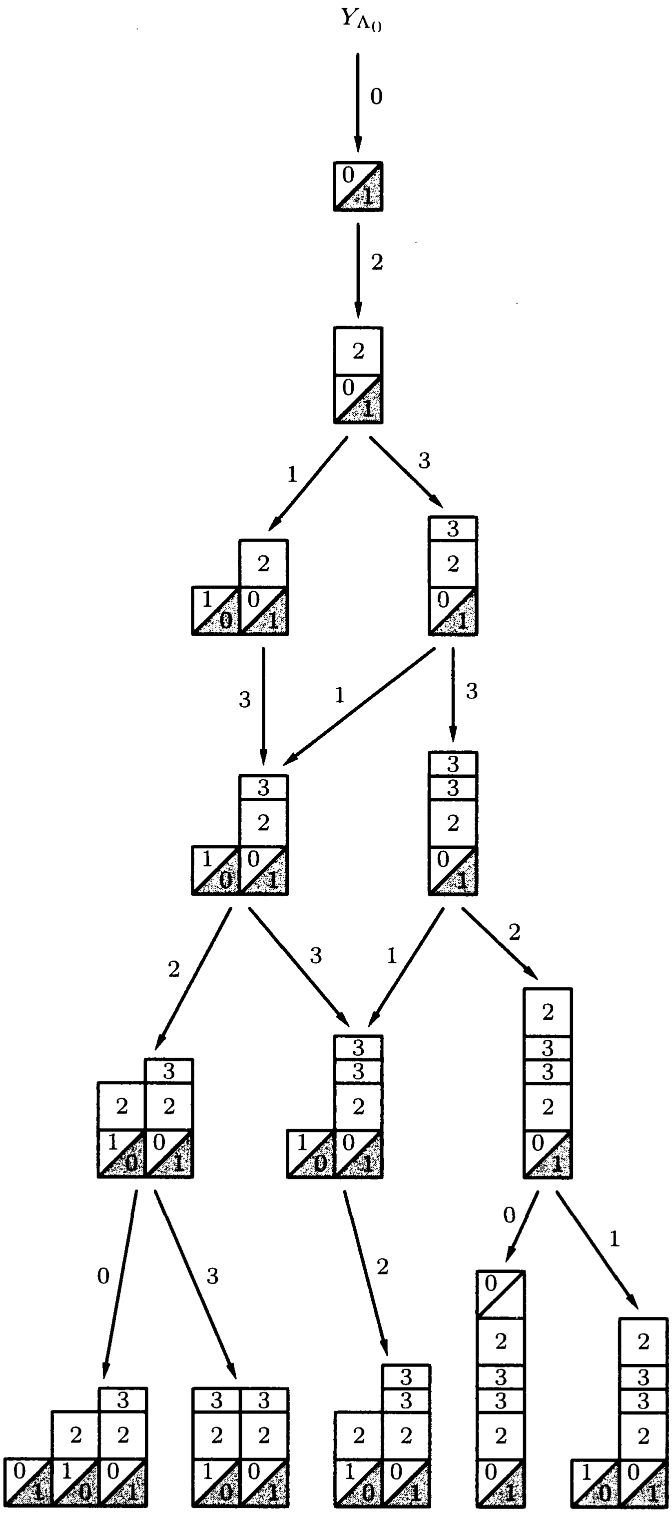
Theorem 8.5 ([12]).

The set $\mathcal{Y}(\Lambda)$ is a connected $U_q(B_n^{(1)})$-crystal. Moreover, there is a $U_q(B_n^{(1)})$-crystal isomorphism

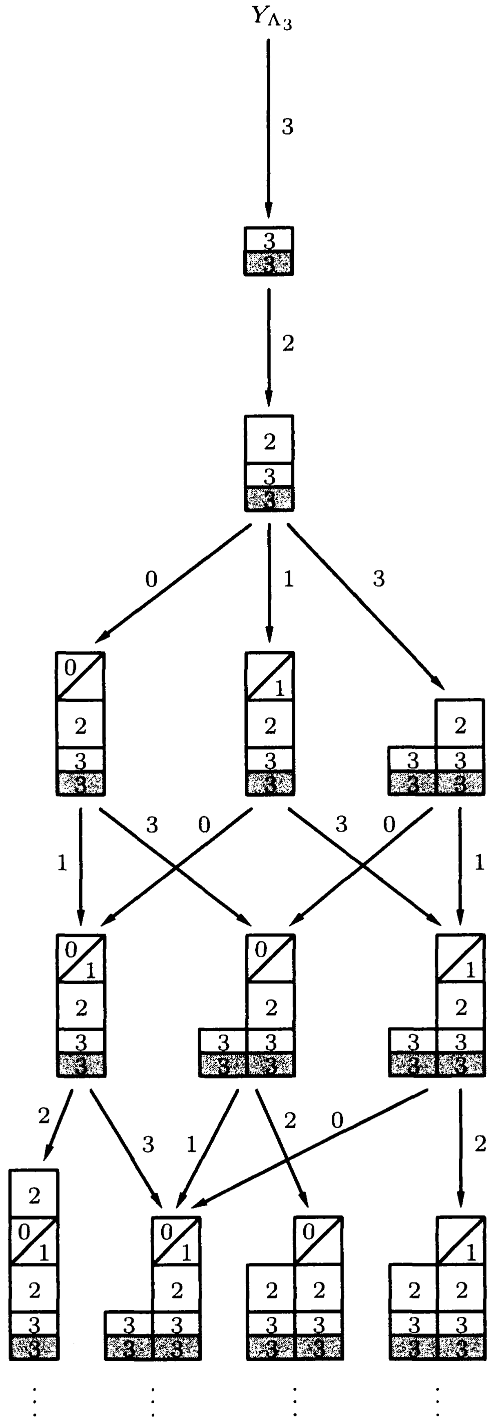
$$\mathcal{Y}(\Lambda) \xrightarrow{\sim} B(\Lambda),$$

where $B(\Lambda)$ is the crystal graph of the basic representation $V(\Lambda)$ of the quantum affine algebra $U_q(B_n^{(1)})$.

Example 8.6. (a) The crystal $\mathbf{Y}(\Lambda_0)$ for $U_q(B_3^{(1)})$



(b) The crystal $\mathbf{Y}(\Lambda_3)$ for $U_q(B_3^{(1)})$



Remark. (a) For the other classical quantum affine algebras (except $U_q(C_n^{(1)})$), one can prove that the crystal graph of a basic representation can be realized as the affine crystal consisting of reduced proper Young walls ([12]).

(b) As an application, we obtain a realization of crystal graphs for finite dimensional irreducible modules over quantum classical algebras (see [15]).

(c) The colored Young diagrams introduced in [26] can be regarded as the Young walls consisting of regular cubes only. In [10], Jimbo, Misra, Miwa and Okado extended the idea of [26] to higher level irreducible representations of $U_q(\widehat{sl_n})$. The crystal graph of a level l irreucible highest weight representation was characterized as the set of l-tuples of n-reduced colored Young diagrams satisfying certain addtional conditions. From our point of view, they can be viewed as reduced proper Young walls with l layers, which provides us with a clue to the construction of crystal bases for the higer level irreducible highest weight representations of other classical quantum affine algebras.

(d) As we have seen in Section 7, when ζ is a primitive n-th root of unity, the finite dimensional irreducible $H_N(\zeta)$-modules can be parametrized by n-reduced colored Young diagrams with N boxes. We expect that there exist some interesting algebraic structures whose irreducible representations (at some specialization) are parametrized by reduced proper Young walls. In [2], Brundan and Kleshchev verified this idea by showing that the irreducible representations of the Hecke-Clifford superalgebra $\mathcal{H}_N(\zeta)$ with ζ a primitive $(2n+1)$-th root of unity are parametrized by reduced proper Young walls of type $A_{2n}^{(2)}$ with N blocks.

§9. Fock space representation

Let $\mathcal{F}(\Lambda) = \bigoplus_{Y \in \mathcal{P}(\Lambda)} \mathbf{Q}(q)Y$ be the $\mathbf{Q}(q)$-vector space with a basis $\mathcal{P}(\Lambda)$. In [16, 17], Kang and Kwon defined a $U_q(\mathfrak{g})$-module structure on $\mathcal{F}(\Lambda)$, the *Fock space representation*, and showed that the crystal $\mathcal{P}(\Lambda)$ is exactly the crystal graph of $\mathcal{F}(\Lambda)$. The Fock space $\mathcal{F}(\Lambda)$ can be regarded as the q-deformed wedge space arising from a level 1 perfect representation ([22]).

We recall how to define the action of e_i, f_i $(i \in I)$ and q^h $(h \in P^\vee)$ on proper Young walls in $\mathcal{P}(\Lambda)$. Let $Y = (y_k)_{k=0}^\infty$ be a proper Young wall on Y_Λ. We denote by $|y_k|$ the number of blocks in y_k added to Y_Λ. Then the *associated partition* is defined to be $|Y| = (\cdots, |y_k|, \cdots, |y_1|, |y_0|)$. For $Y = (y_k)_{k=0}^\infty$, $Z = (z_k)_{k=0}^\infty$ in $\mathcal{P}(\Lambda)$, we define $|Y| \trianglerighteq |Z|$ if and only if $\sum_{k=l}^\infty |y_k| \geq \sum_{k=l}^\infty |z_k|$ for all $l \geq 0$.

Since the action of q^h is easily defined:

$$q^h Y = q^{\langle h, \mathrm{wt}(Y) \rangle} \quad \text{for} \quad h \in P^\vee, \ Y \in \mathcal{P}(\Lambda),$$

we will focus on the action of e_i and f_i on Y $(i \in I)$.

Case 1. Suppose that the i-blocks are of type I.

If b is a removable i-block in y_k of Y, then let $Y_R(b) = (y_{k-1}, \cdots, y_1, y_0)$ be the wall consisting of the columns lying at the right of b, and set $R_i(b; Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b))$. (The wall $Y_R(b)$ should be regarded as a $U_{(i)}$-crystal, where no block can be added on y_l for $l \geq k$.) We denote by $Y \nearrow b$ the Young wall obtained by removing b from Y. Then we define

$$e_i Y = \sum_b q_i^{-R_i(b;Y)}(Y \nearrow b),$$

where b runs over all removable i-blocks in Y.

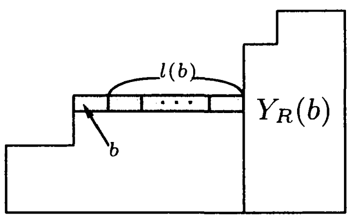
On the other hand, if b is an admissible i-slot in y_k of Y, then let $Y_L(b) = (\cdots, y_{k+2}, y_{k+1})$ be the Young wall consisting of the columns in Y lying at the left of b, and set $L_i(b; Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b))$. (The wall $Y_L(b)$ may be a proper Young wall on another ground-state wall $Y_{\Lambda'}$.) We denote by $Y \swarrow b$ the Young wall obtained by adding an i-block at b. Then we define

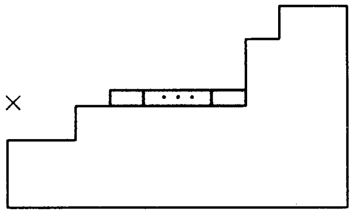
$$f_i Y = \sum_b q_i^{L_i(b;Y)}(Y \swarrow b),$$

where b runs over all admissible i-slots in Y.

Case 2. Suppose that the i-blocks are of type II.

Let b be a removable i-block in y_k of Y. If the i-signature of y_k is $--$, or if the i-signature of y_k is $-$ and there is another i-block below b, define $Y \nearrow b$ to be the Young wall obtained by removing b from Y. If the i-signature of y_k is $-+$, or if the i-signature of y_k is $-$ and there is no i-block below b, define $Y \nearrow b = q^{-1}(1-(-q^2)^{l(b)+1})Z$, where Z is the Young wall obtained by removing b from Y and $l(b)$ is the number of y_l's with $l < k$ such that $|y_l| = |y_k|$.

$Y =$ 

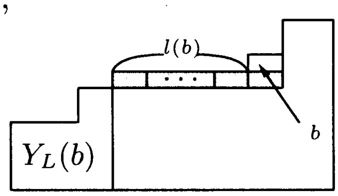
$$Y \nearrow b = \frac{(1-(-q^2)^{l(b)+1})}{q} \times$$ 

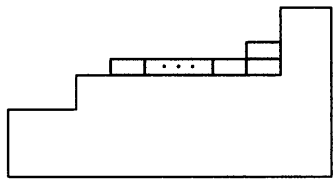
In either case, set $Y_R(b) = (y_{l-1}, \cdots, y_0)$, where l is the integer such that $|y_k| = |y_{k-1}| = \cdots = |y_{l+1}| < |y_l|$, and let $R_i(b;Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b))$. Then we define

$$e_i Y = \sum_b q_i^{-R_i(b;Y)} (Y \nearrow b),$$

where b runs over all removable i-blocks in Y.

On the other hand, suppose that b is an admissible i-slot in y_k of Y. If the i-signature of y_k is $++$, or if the i-signature of y_k is $+$ and there is no i-block below b, then we define $Y \swarrow b$ to be the Young wall obtained by adding an i-block at b. If the i-signature of y_k is $-+$, or if the i-signature of y_k is $+$ and there is another i-block below b, then we define $Y \swarrow b = q^{-1}(1-(-q^2)^{l(b)+1})Z$, where Z is the Young wall obtained by adding an i-block at b and $l(b)$ is the number of y_l's with $l > k$ such that $|y_l| = |y_k|$. That is,

$$Y =$$ 

$$Y \swarrow b = \frac{(1-(-q^2)^{l(b)+1})}{q} \times$$ 

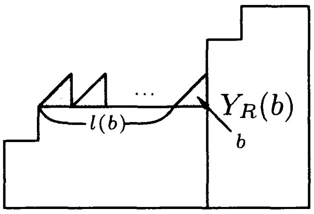
In either case, set $Y_L(b) = (\cdots, y_{l+2}, y_{l+1})$, where l is the integer such that $|y_{l+1}| < |y_l| = |y_{l-1}| = \cdots = |y_k|$, and let $L_i(b;Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b))$. Then we define

$$f_i Y = \sum_b q_i^{L_i(b;Y)} (Y \swarrow b),$$

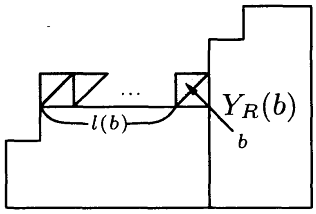
where b runs over all admissible i-slots in Y.

Case 3. Suppose that the i-blocks are of type III.

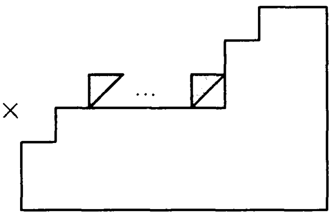
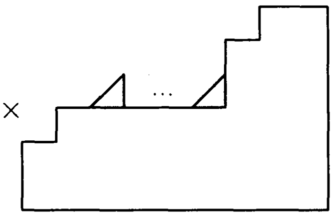
If b is a removable i-block in y_k of Y, then we define $Y \nearrow b$ to be the Young wall obtained by removing b from Y. We also consider the following i-block b in y_k of Y, which we call a *virtually removable i-block*:



or



In this case, we define $Y \nearrow b$ to be

$(-q_i)^{l(b)}\times$  or $(-q_i)^{l(b)}\times$  ,

where $l(b)$ is given in the above figure. In either case, set

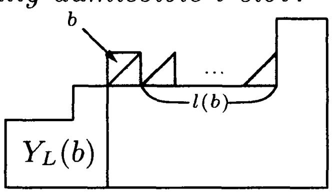
$$Y_R(b) = (y_{k-1}, \cdots, y_0), \quad R_i(b;Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b)),$$

and define

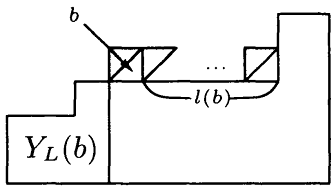
$$e_i Y = \sum_b q_i^{-R_i(b;Y)}(Y \nearrow b),$$

where b runs over all removable and virtually removable i-blocks in Y.

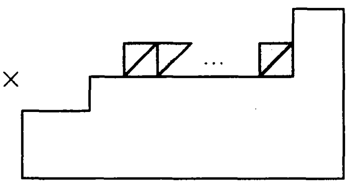
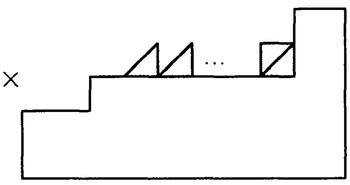
On the other hand, if b is an admissible i-slot in y_k of Y, then we define $Y \swarrow b$ to be the Young wall obtained by adding an i-block at b. We also consider the following i-slot b in y_k of Y, which we call a *virtually admissible i-slot*:



or



In this case, we define $Y \swarrow b$ to be

$(-q_i)^{l(b)}\times$  or $(-q_i)^{l(b)}\times$  ,

where $l(b)$ is given in the above figure. In either case, set

$$Y_L(b) = (\cdots, y_{k+2}, y_{k+1}), \quad L_i(b;Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b)),$$

and define

$$f_i Y = \sum_b q_i^{L_i(b;Y)}(Y \swarrow b),$$

where b runs over all admissible and virtually admissible i-slots in Y.

Theorem 9.1 ([16, 17]).

(a) *The Fock space $\mathcal{F}(\Lambda)$ is an integrable $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{int}$.*

(b) *Let $\mathcal{L}(\Lambda) = \bigoplus_{Y \in \mathcal{P}(\Lambda)} \mathbf{A}_0 Y$. Then the pair $(\mathcal{L}(\Lambda), \mathcal{P}(\Lambda))$ is a crystal basis of the Fock space $\mathcal{F}(\Lambda)$.*

(c)

$$\mathcal{F}(\Lambda) = \begin{cases} \displaystyle\bigoplus_{m=0}^{\infty} V(\Lambda - m\delta)^{\oplus p(m)} & if\ \mathfrak{g} \neq D^2_{n+1}, \\ \displaystyle\bigoplus_{m=0}^{\infty} V(\Lambda - 2m\delta)^{\oplus p(m)} & if\ \mathfrak{g} = D^2_{n+1}. \end{cases}$$

Remark. In [16, 17], Kang and Kwon generalized the Lascoux-Leclerc-Thibon algorithm to obtain an effective algorithm for constructing the global basis $G(\Lambda)$ of the basic representation $V(\Lambda)$ for all classical quantum affine algebras except $U_q(C_n^{(1)})$. More precisely, for each reduced proper Young wall $Y \in \mathcal{Y}(\Lambda)$, the generalized Lascoux-Leclerc-Thibon algorithm yields the global basis element

$$G(Y) = \sum_{Y' \in \mathcal{P}(\Lambda)} K_{Y,Y'}(q) Y',$$

where the coefficients $K_{Y,Y'}(q)$ satisfy the following conditions:

(i) $K_{Y,Y'}(q) \in \mathbf{Z}[q]$,

(ii) $K_{Y,Y'}(q) = 0$ unless $|Y| \trianglerighteq |Y'|$,

(iii) $K_{Y,Y'}(q) = 1$ and $K_{Y,Y'}(0) = 0$ if $Y \neq Y'$.

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Representation Theory of Algebraic Groups and Quantum Groups
pp. 253–259

An induction theorem for Springer's representations

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Representation Theory of Algebraic Groups and Quantum Groups
pp. 261–320

The representation theory of the Ariki-Koike and cyclotomic q-Schur algebras

Andrew Mathas

§1. Introduction

The Ariki–Koike algebras first appeared in the work of Cherednik [**30**] who discovered these algebras in his study of the q–analogue of Drinfeld's duality between the degenerate affine Hecke algebra and the Yangians for $\mathfrak{gl}_N$. Seven years later these algebras were rediscovered by Ariki and Koike [**8**] who were interested in them because they are a natural generalization of the Iwahori-Hecke algebras of types A and B. At almost the same time, Broué and Malle [**21**] also attached to each complex reflection group a *cyclotomic Hecke algebra* which, they conjectured, should play a role in the decomposition of the induced cuspidal representations of the finite groups of Lie type. The Ariki-Koike algebras are a special case of Broué and Malle's construction.

The deepest conjectures of Broué, Malle and Michel concerning the Ariki-Koike algebras have not yet been proved (see §2.5); however, many of the consequences of these conjectures have been established. Further, the representation theory of these algebras is beginning to be well understood. For example, the simple modules of the Ariki-Koike algebras have been classified; the blocks are known; there are analogues of Kleshchev's modular branching rules; and, in principle, the decomposition matrices of the Ariki-Koike algebras are known in characteristic zero. In many respects this theory looks much like that of the symmetric groups; in particular, there is a rich combinatorial mosaic underpinning these results which involves familiar objects like standard tableaux (indexed by multipartitions), Specht modules and so on.

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The cyclotomic Schur algebras were introduced by Dipper, James and the author [**42**]; by definition these algebras are endomorphism algebras of a direct sum of "permutation modules" for an Ariki-Koike algebra. This generalizes the Dipper-James definition [**39**] of the q-Schur algebras as endomorphism algebras of tensor space. We were interested in these algebras both as another tool for studying the Ariki-Koike algebras and because we hoped that there might be a cyclotomic analogue of the famous Dipper-James theory [**27**, **39**].

As with the Ariki-Koike algebras, the representation theory of the cyclotomic Schur algebras is now well developed. They are always cellular algebras; indeed, they are quasi-hereditary. The cellular basis of these algebras is indexed by a generalization of semistandard tableaux and their representation theory looks very much like the representation theory of the q-Schur algebras. In particular, they have a highest weight theory; there is a cyclotomic Schur functor and a double centralizer theorem; the Jantzen filtrations of the cyclotomic Weyl modules satisfy a generalization of the Jantzen sum formula; and the cyclotomic Schur algebras have Borel subalgebras and admit a triangular decomposition.

In the short time since its inception this theory has blossomed producing many interesting results; largely this is because it generalizes the representation theories of the symmetric groups, the Schur algebras and the q-analogues of these. Many of the results in this article have the flavour of results from Lie theory; however, as yet, there are no known connections between the representation theories of the cyclotomic Schur algebras and the finite groups of Lie type except in the case where the underlying complex reflection group is actually a Weyl group.

The aim of this article is to describe the representation theory of these algebras in detail. Throughout we have tried to give an indication of how the results are proved; unfortunately, in distilling one or more papers in to one or more paragraphs some of the finer details have inevitably been lost.

§2. The Ariki-Koike algebras

In this chapter we introduce the Ariki-Koike algebras by giving three different constructions of them. From the point of view of presentations it is clear that all three definitions agree; however, for motivation, and also for proving certain results, it is important to know the different contexts in which the Ariki-Koike algebras arise.

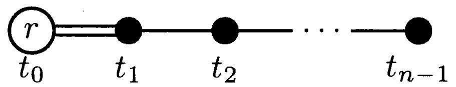
We begin with a brief discussion of the complex reflection groups which underpin the Ariki-Koike algebras. In the final section we give a brief account of the conjectures of Broué and Malle [**21**] which describe the role that the Ariki-Koike algebras should play in the representation theory of the finite groups of Lie type.

2.1. The complex reflection group of type $G(r,1,n)$.

Fix integers $r \geq 1$ and $n \geq 0$ and let $W_{r,n} = \mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n$ be the wreath product of a cyclic group of order r and a symmetric group of degree n. Then $W_{r,n}$ is the complex reflection group of type $G(r,1,n)$ in the Shephard-Todd classification [**116**]; in particular, $W_{r,n}$ has a faithful representation on a complex vector space on which it acts as a group generated by reflections (see section (2.3)).

If $r = 1$ then $W_{1,n} \cong \mathfrak{S}_n$ is just the symmetric group $\mathfrak{S}_n$. If $r = 2$ then $W_{2,n} = \mathbb{Z}/2\mathbb{Z} \rtimes \mathfrak{S}_n$ is the hyperoctohedral group, or the group of signed permutations. In these two cases $W_{r,n}$ is a *Coxeter group* or *real reflection group*; in fact, they are the Weyl groups of type A_{n-1} and B_n respectively.

The group $W_{r,n}$ has the Coxeter like presentation given by the following diagram.



The circle around the r indicates that the corresponding generator t_0 has order r; otherwise, this should be read as a standard Dynkin diagram. Thus, as an abstract group, $W_{r,n}$ is generated by elements $t_0, t_1, \ldots, t_{n-1}$ which are subject to the relations

$$\begin{aligned} t_0^r &= 1, && \\ t_i^2 &= 1, && \text{for } 1 \leq i < n, \\ t_0t_1t_0t_1 &= t_1t_0t_1t_0, && \\ t_it_j &= t_jt_i, && \text{for } 0 \leq j < i-1 < n-1, \\ t_it_{i+1}t_i &= t_{i+1}t_it_{i+1}, && \text{for } 1 \leq i < n-1. \end{aligned}$$

In particular, the subgroup $\langle t_1, \ldots, t_{n-1} \rangle$ of $W_{r,n}$ is isomorphic to the symmetric group $\mathfrak{S}_n$; hereafter, we identify $\mathfrak{S}_n$ and $\langle t_1, \ldots, t_{n-1} \rangle$ via the map $(i, i+1) \longmapsto t_i$, for $1 \leq i < n$.

Let $l_1 = t_0$, $l_2 = t_1t_0t_1, \ldots, l_n = t_{n-1}\ldots t_1t_0t_1\ldots t_{n-1}$. Then $l_1, \ldots, l_n$ generate a subgroup of $W_{r,n}$ isomorphic to $\mathbb{Z}/r\mathbb{Z} \times \cdots \times \mathbb{Z}/r\mathbb{Z}$ (n copies), which is just the base group when we consider $W_{r,n}$ as the semidirect product $(\mathbb{Z}/r\mathbb{Z} \times \cdots \times \mathbb{Z}/r\mathbb{Z}) \rtimes \mathfrak{S}_n$. Thus, as a set, $W_{r,n} = \{\, l_1^{a_1} \ldots l_n^{a_n} w \mid 0 \leq a_i < r \text{ and } w \in \mathfrak{S}_n \,\}$ and these elements are all distinct. In particular, $|W_{r,n}| = r^n n!$.

In general, $W_{r,n}$ is not a Coxeter group so the familiar combinatorics of root systems and length functions cannot be used in understanding $W_{r,n}$ and its representations. (Bremke and Malle [**16**] have defined a root system for $W_{r,n}$.) The theory of complex reflection groups is still very much in its infancy; the major tool being used to understand these groups is the geometry of their reflection representation.

2.2. The Ariki-Koike algebras

The Iwahori-Hecke algebras of Weyl groups play an important role in the representation theory of the groups of Lie type. Two important special cases of these algebras are the Iwahori-Hecke algebras of the Weyl groups of types A_{n-1} and B_n which are the groups $G(1,1,n)$ and $G(2,1,n)$, respectively. Ariki and Koike [**8**] observed that the definition of these algebras could be generalized to give a Hecke algebra, or deformation algebra, for each complex reflection group of type $G(r,1,n)$.

Let R be an integral domain with 1 and let $q, Q_1, \dots, Q_r$ be elements of R with q invertible. Let $\mathbf{Q} = \{Q_1, \dots, Q_r\}$.

Deforming the relations of $W_{r,n}$ we obtain the Ariki-Koike algebra.

Definition 2.1 (Ariki-Koike [**8**]). *The* Ariki-Koike algebra *is the unital associative R-algebra $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$ with generators $T_0, T_1, \dots, T_{n-1}$ and relations*

$$\begin{aligned}
(T_0 - Q_1)\dots(T_0 - Q_r) &= 0, \\
(T_i - q)(T_i + 1) &= 0, && \textit{for } 1 \le i < n, \\
T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\
T_i T_j &= T_j T_i, && \textit{for } 0 \le i < j - 1 < n - 1, \\
T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, && \textit{for } 1 \le i < n - 1.
\end{aligned}$$

The three homogeneous relations are known as *braid relations.*

Typically, we write $\mathscr{H} = \mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$; when we wish to emphasize the ring of definition we will write $\mathscr{H} = \mathscr{H}_{R,q,\mathbf{Q}}(W_{r,n})$.

Notice that if R contains a primitive r^{th} root of unity ζ and we set $q = 1$ and $Q_s = \zeta^s$, for $1 \le s \le r$, then $\mathscr{H} \cong RW_{r,n}$, the group algebra of $W_{r,n}$ (because the relations collapse to give those of $W_{r,n}$ for this choice of parameters).

Let $w \in \mathfrak{S}_n$. Then $w = t_{i_1} \dots t_{i_k}$ for some i_j with $1 \le i_j < n$. If k is minimal we say that $t_{i_1} \dots t_{i_k}$ is a reduced expression for w and define $T_w = T_{i_1} \dots T_{i_k}$. Since the braid relations hold in $\mathscr{H}$ it follows from Matsumoto's monoid lemma (see, for example, [**103**, Theorem 1.8]), that T_w is independent of the choice of reduced expression for w.

Mimicking the definition of the elements l_k in $W_{r,n}$, for $k = 1, \dots, n$ set $L_k = q^{1-k} T_{k-1} \dots T_1 T_0 T_1 \dots T_{k-1}$. (The renormalization by the unit q^{1-k} is there to make the combinatorics more natural later on.) Using the relations it is straightforward to see that $L_1, \dots, L_n$ generate an abelian subalgebra of $\mathscr{H}$ and that the symmetric polynomials in $L_1, \dots, L_n$ belong to the centre of $\mathscr{H}$.

A priori there is no reason to expect that the presentation above will yield an interesting algebra. The first indication that $\mathscr{H}$ is worth studying is the following theorem.

Theorem 2.2 (Ariki-Koike [**8**]). *The Ariki-Koike algebra $\mathscr{H}$ is free as an R-module with basis $\{ L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \le a_i < r$ and $w \in \mathfrak{S}_n \}$.*

In particular, notice that $\mathscr{H}$ is R-free of rank $r^n n! = |W_{r,n}|$ for any choice of R, q and $\mathbf{Q}$. Furthermore, the subalgebra of $\mathscr{H}$ generated by $T_1, \dots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra $\mathscr{H}_q(\mathfrak{S}_n)$ of the symmetric group $\mathfrak{S}_n$. Hereafter, we identify the two algebras $\mathscr{H}_q(\mathfrak{S}_n)$ and $\langle T_1, \dots, T_{n-1} \rangle$.

Using the relations it is not hard to show that $\mathscr{H}$ is spanned by the elements $L_1^{a_1} \dots L_n^{a_n} T_w$; there are $|W_{r,n}|$ such elements. To prove linear independence Ariki and Koike explicitly constructed the simple $\mathscr{H}$-modules using a generalization of Young's seminormal form for the Ariki-Koike algebras when $R = \mathbb{C}(q, u_1, \dots, u_r)$; see Theorem 3.2 below. This shows that $\mathscr{H} / \operatorname{Rad} \mathscr{H}$ has dimension at most $|W_{r,n}|$. Hence, $\mathscr{H}$ is semisimple and Theorem 2.2 is proved when $R = \mathbb{C}(q, Q_1, \dots, Q_r)$. The general case now follows by a specialization argument.

There are now other proofs of Theorem 2.2 available. Broué, Malle and Rouquier [**24**, Theorem 4.24] have given a geometrical argument which results from thinking of $\mathscr{H}$ as a quotient of the group algebra of the braid group of $W_{r,n}$ and studying its monodromy representation; this is the topic of the next section. Sakamoto and Shoji [**113**] also proved Theorem 2.2 as a consequence of an analogue of Schur-Weyl reciprocity for $\mathscr{H}$ and a particular quantum group; we will return to this in §5.4 below. Another proof, using the affine Hecke algebra $\hat{H}_n$ below, can be extracted from related arguments of Brundan and Kleshchev; see the proof of [**28**, Theorem 3.6].

Finally, we remark that Shoji [**117**] has given a different presentation of $\mathscr{H}$ when $R = \mathbb{C}(q, Q_1, \dots, Q_r)$. Shoji's presentation is very interesting and deserves further study.

2.3. The braid group of $W_{r,n}$ and the Hecke algebra

At almost the same time that Ariki and Koike introduced their algebra, Broué and Malle [**21**] associated to each complex reflection group W a *cyclotomic Hecke algebra*; for the group $W_{r,n}$. Broué and Malle's cyclotomic Hecke algebra is precisely the Ariki-Koike algebra. Broué and Malle's motivation was that they expected that the cyclotomic Hecke algebras should play a role in the representation theory of the finite groups of Lie type similar to, but more complicated than, that played by the Iwahori-Hecke algebras (see §2.5).

In this section we briefly describe Broué and Malle's definition in the case of $W_{r,n}$ and some of its consequences.

Let V be the complex vector space with basis $\{\epsilon_1, \dots, \epsilon_n\}$ and let $\zeta \in \mathbb{C}$ be a primitive r^{th} root of unity. The symmetric group $\mathfrak{S}_n = \langle t_1, \dots, t_{n-1}\rangle$ acts on V in the natural way; extend this to an action of $W_{r,n}$ by letting t_0 act via the $n \times n$ matrix $\operatorname{diag}(\zeta, 1, \dots, 1)$. This defines a faithful representation of $W_{r,n}$. Observe that each of the generators of $W_{r,n}$ acts as a *reflection* (that is, fixes a space of codimension 1), so this shows that $W_{r,n}$ is a complex reflection group.

Let $\Omega = \{\, \epsilon_i - \zeta^k\epsilon_j \mid 1 \le j \le i \le n \text{ and } \max(j-i, -1) < k < r\,\}$. Then Ω is in one-to-one correspondence with the set of reflections in $W_{r,n}$, where the correspondence attaches to each reflection its unique eigenvector in Ω with non-trivial eigenvalue; see [**16**, §3]. For each $\omega \in \Omega$ let H_ω be the hyperplane orthogonal to ω, let $\mathscr{M} = V \setminus \bigcup_{\omega\in\Omega} H_\omega$ be the associated hyperplane complement and $\mathscr{M}/W_{r,n}$ its quotient by $W_{r,n}$.

Definition 2.3. *The* braid group *of $W_{r,n}$ is the group*

$$\mathfrak{B}_{r,n} = \pi_1(\mathscr{M}/W_{r,n}, x_0),$$

where $x_0 \in \mathscr{M}/W_{r,n}$.

Here, $\pi_1(\mathscr{M}/W_{r,n}, x_0)$ is the fundamental group of the quotient space $\mathscr{M}/W_{r,n}$ with base point x_0. Because $\mathscr{M}$ is connected $\mathfrak{B}_{r,n}$ is independent of the choice of x_0.

If $r > 1$ then $\mathfrak{B}_{r,n}$ is a braid group of type B_n and as an abstract group it is generated by elements $s_0, \dots, s_{n-1}$ subject to the relations

$$s_0s_1s_0s_1 = s_1s_0s_1s_0, \quad s_is_j = s_js_i, \quad \text{and} \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1},$$

where $1 \le i < n-1$, $0 \le j < n$ and $|i-j| > 1$. In particular, observe that $W_{r,n}$ is a quotient of $\mathfrak{B}_{r,n}$ (via the map which sends s_i to t_i for $0 \le i < n$).

The generators of $\mathfrak{B}_{n,r}$ can be chosen as generators of the monodromy around the hyperplanes. Bessis [**12**] has now given a general argument for the existence of such presentations for the braid groups of complex reflection groups.

Broué and Malle considered the algebra $R\mathfrak{B}_{r,n}/I_{q,\mathbf{Q}}$, where $I_{q,\mathbf{Q}}$ is the ideal of $R\mathfrak{B}_{r,n}$ generated by $(s_0-Q_1)\dots(s_0-Q_r)$ and $(s_i-q)(s_i+1)$, for $1 \le i < n$; evidently, $\mathscr{H} \cong R\mathfrak{B}_{r,n}/I_{q,\mathbf{Q}}$. One consequence of this definition is that we can use the monodromy representation of the braid group $\mathfrak{B}_{r,n}$ to analyze $\mathscr{H}$. This leads to a more conceptual proof of the fact that $\mathscr{H}$ is always free as an R-module of rank $|W_{r,n}|$ (a corollary of Theorem 2.2). Moreover, it yields the following important result.

Theorem 2.4 (Broué-Malle-Rouquier [**24**, Theorem 4.24]). *Let* $\mathbb{K} = \mathbb{C}(q, Q_1, \dots, Q_r)$. *Then the monodromy representation of* $\mathfrak{B}_{r,n}$ *induces an isomorphism of* $\mathbb{K}$*-algebras* $\mathscr{H}_{\mathbb{K},q,\mathbf{Q}} \cong \mathbb{K}W_{r,n}$.

Here, $\mathbb{K}W_{r,n}$ is the group algebra of $W_{r,n}$ over $\mathbb{K}$. That $\mathscr{H}_{\mathbb{K},q,\mathbf{Q}}$ and $\mathbb{K}W_{r,n}$ are isomorphic algebras can be established by a general Tits deformation theory argument (see, for example, [**34**, §66]). The main point of this result is that the isomorphism is canonically determined.

Lusztig [**96**] has proved a similar result for the Iwahori-Hecke algebras of Weyl groups; however, his argument is less elementary relying on a deep property of the cells of Weyl groups. For Weyl groups, Lusztig's isomorphism and that of Theorem 2.4 are different.

2.4. The affine Hecke algebra of type A

The Ariki-Koike algebras should really be considered as affine objects because they are quotients of the (extended) affine Hecke algebra of type A (i.e., the affine Hecke algebra of $\mathrm{GL}_n(\mathbb{C})$). The affine Hecke algebra $\hat{H}_n$ is the R-algebra with generators $T_1, \dots, T_{n-1}$ and $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ and relations

$$\begin{array}{ccc} (T_i - q)(T_i + 1) = 0, & T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & T_iX_iT_i = qX_{i+1} \\ T_iT_k = T_kT_i, & X_iX_k = X_kX_i, & T_iX_k = X_kT_i \end{array}$$

and $X_iX_i^{-1} = 1 = X_i^{-1}X_i$ for all sensible values of i, j, k with $|i-k| > 1$. In particular, abusing notation slightly, notice that there is surjective algebra homomorphism $\hat{H}_n \twoheadrightarrow \mathscr{H}$ given by sending $T_i \longmapsto T_i$ and $X_j \longmapsto L_j$, for $1 \le i < n$ and $1 \le j \le n$ respectively. It is easy to see that $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n}) \cong \hat{H}_n / \langle (X_1 - Q_1) \dots (X_1 - Q_r) \rangle$.

It follows from the relations that $T_1, \dots, T_{n-1}$ generate a subalgebra of $\hat{H}_n$ isomorphic to $\mathscr{H}_q(\mathfrak{S}_n)$ and that $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ generate a Laurent polynomial ring. Therefore, as an R-module, $\hat{H}_n \cong \mathscr{H}_q(\mathfrak{S}_n) \otimes R[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$; so, $\hat{H}_n$ is a twisted tensor product.

Let $\mathcal{P} = \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$ be the free $\mathbb{Z}$-module with basis $\epsilon_1, \dots, \epsilon_n$; so, $\mathcal{P}$ is the weight lattice of $\mathrm{GL}_n(\mathbb{C})$. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{P}$ by permuting the ϵ_i.

If $\lambda \in \mathcal{P}$ set $X^\lambda = X_1^{\lambda_1} \dots X_n^{\lambda_n}$. Then the two commutation relations for the T_i and the X_j can be replaced by the relation

$$T_iX^\lambda = X^{t_i\lambda}T_i + (q-1)\frac{X^\lambda - X^{t_i\lambda}}{1 - X^{\alpha_i}},$$

where $\lambda \in \mathcal{P}$, $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and $1 \leq i < n$. A quick calculation shows that $X^\lambda - X^{t_i\lambda}$ is divisible by $1 - X^{\alpha_i}$ so the right hand side does make sense. Notice that when $q = 1$ this relation becomes $t_i X^\lambda = X^{t_i\lambda} t_i$; this is what we expect because the extended affine Weyl group is the semidirect product $\mathcal{P} \rtimes \mathfrak{S}_n$.

Now suppose that R is an algebraically closed field. Bernstein showed that the centre of $\hat{H}_n$ is the set of symmetric polynomials in $X_1, \dots, X_n$ (Theorem 5.4). Consequently, $\hat{H}_n$ is finite dimensional over its centre; therefore, by Schur's Lemma, every irreducible $\hat{H}_n$-module is finite dimensional (with dimension at most $n!$ since $\dim_R \hat{H}_n/Z(\hat{H}_n) = (n!)^2$ by Theorem 5.4 below).

As remarked above, each Ariki-Koike algebra $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$ is a quotient of $\hat{H}_n$, so every irreducible $\mathscr{H}$-module is also an irreducible $\hat{H}_n$-module. Conversely, suppose that R is algebraically closed and that M is an irreducible $\hat{H}_n$-module. If $c_M(X_1)$ is the characteristic polynomial for the action of X_1 on M then $\mathscr{H}_M := \hat{H}_n/\langle c_M(X_1)\rangle$ is an Ariki-Koike algebra (with parameters the eigenvalues for the action of X_1 on M) and M is an irreducible $\mathscr{H}_M$-module. (More generally, M is an irreducible module for any Ariki-Koike algebra obtained by quotienting out by the ideal generated by any polynomial in X_1 which is divisible by $c_M(X_1)$.) Thus the irreducible $\hat{H}_n$-modules are precisely the irreducible $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$-modules as $\mathbf{Q}$ ranges over the elements of $(R^\times)^r$ for $r \geq 1$.

2.5. The conjectures of Broué, Malle and Michel

The conjectures which we now discuss grew out of the attempts of Broué and others to understand Broué's [**18**] conjectures for blocks with abelian defect groups in the case of the finite reductive groups. We consider only a very special case of these conjectures; for references and further details see the original papers [**21, 22, 25**] and Broué's [**19**] comprehensive survey article.

Let $\mathbf{G}$ be an algebraic group defined over $\overline{\mathbb{F}}_q$, where q is a prime power, and let W be the Weyl group of $\mathbf{G}$. Let $F: \mathbf{G} \longrightarrow \mathbf{G}$ be a Frobenius map and let $G = \mathbf{G}^F$ be the F-fixed points of $\mathbf{G}$. We assume that W is F-split. The simplest example is to take $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $F(a_{ij}) = (a_{ij}^q)$; then $G = \mathrm{GL}_n(q)$ and $W = \mathfrak{S}_n$.

Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$ and set $B = \mathbf{B}^F$. It is well-known that the irreducible constituents of $\mathrm{Ind}_B^G(1)$ are in one-to-one correspondence with the irreducible representations of W; see, for example, [**29**]. The Iwahori-Hecke algebras of Weyl groups play an important role in this theory; indeed, $\mathscr{H}_q(W) \cong \mathrm{End}_G\big(\mathrm{Ind}_B^G(1)\big)$ and this explains why the dimensions of the irreducible representations in the unipotent principal series, the constituents of $\mathrm{Ind}_B^G(1)$, are given by evaluating certain polynomials $D_\chi(x)$ at $x = q$. The conjectures which follow attempt to explain other "generic" features of the representation theory of finite reductive groups.

Let B_W be the Braid group of W and for $w \in W$ let $\underline{w} \in B_W$ be the lift of w (under the canonical embedding of W into the positive braid monoid B_W^+). Brieskorn and Saito [**17**] showed that the centre of B_W is generated by $\pi = \underline{w_0^2}$ (or $\underline{w_0}$ if w_0 is central in W), where w_0 is the unique element of maximal length in W.

Call an element $w \in W$ **good** if $\pi = \underline{w}^d$ for some d. Note that w has order d since $w_0^2 = 1$ in W. Every conjugacy class of regular elements in W contains a good element. Assume that w is good. Then Broué and Michel [**25**] have shown that every good element is regular; so $C_W(w)$ is a complex reflection group by Springer [**118**]. Let $B_w = B(C_W(w))$ be the braid group of $C_W(w)$. It is conjectured that $B_w = C_{B_W}(\underline{w})$; this has now been proved in almost all cases [**13, 14**].

Let X_w be the Deligne-Lusztig variety associated to w; so X_w is the variety of Borel subgroups $\mathbf{B}'$ of $\mathbf{G}$ such that $\mathbf{B}'$ and $F(\mathbf{B}')$ are in relative position w. Fix a prime ℓ not dividing q and consider the étale cohomology groups $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ of X_w. The finite group $G = \mathbf{G}^F$ acts on X_w and hence also on $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$. By [**25, 35**] there is also an action of $C_{B_W}(\underline{w})$ on $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ (this comes from an action of the positive braids in $C_{B_W}(\underline{w})$ on X_w). In many cases the action of $\overline{\mathbb{Q}}_\ell C_{B_W}(\underline{w})$ is known to factor through a cyclotomic Hecke algebra. Conjecturally, the action of $C_{B_W}(\underline{w})$ should generate $\mathrm{End}_{\overline{\mathbb{Q}}_\ell G}\big(H_c^i(X_w, \overline{\mathbb{Q}}_\ell)\big)$; this is one of the key unsolved problems and it appears to be very hard.

Let $\mathcal{H}(\mathbf{G}, F, W, w)$ be the image of $\overline{\mathbb{Q}}_\ell C_{B_W}(\underline{w})$ in the (graded) endomorphism algebra of $\bigoplus_{i \geq 0} H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$. Then $\mathcal{H}(\mathbf{G}, F, W, w)$ is a finite dimensional algebra and the following conjecture is expected to be true.

Conjecture 2.5 (Broué,Malle,Michel [**19, 21, 22, 25**]). *Suppose that w is a good element of order d.*

(i) *If $i \neq j$ then the $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-modules $H_c^i(X_w, \overline{\mathbb{Q}}_\ell)$ and $H_c^j(X_w, \overline{\mathbb{Q}}_\ell)$ have no irreducible constituents in common.*

(ii) *There is a d-cyclotomic Hecke algebra $\mathscr{H}_x\big(C_W(w)\big)$ of the complex reflection group $C_W(w)$ such that*

$$\mathcal{H}(\mathbf{G}, F, W, w) \cong \mathscr{H}_{\overline{\mathbb{Q}}_\ell, q}\big(C_W(w)\big) \cong \mathrm{End}_{\overline{\mathbb{Q}}_\ell \mathbf{G}^F}\Big(\bigoplus_{i \geq 0} H_c^i(X_w, \overline{\mathbb{Q}}_\ell)\Big).$$

(iii) *There is a one-to-one correspondence $\chi \longleftrightarrow \chi_q$ between the irreducible representations of $C_W(w)$ and the irreducible constituents* *of the $\overline{\mathbb{Q}}_\ell \mathbf{G}^F$-module $\bigoplus_{i\ge0} H^i_c(X_w,\overline{\mathbb{Q}}_\ell)$. Moreover, for each irreducible character χ of $C_W(w)$ there is a polynomial $D_\chi(x)$, depending only on χ, such that the degree of χ_q is equal to $D_\chi(q)$.*

We now explain the term *d-cyclotomic Hecke algebra* when $C_W(w) = W_{r,n}$. Let $\xi \in \mathbb{C}$ be a root of unity and x an indeterminate over $\mathbb{Z}[\xi]$ and let ζ_d be a primitive d^{th} root of unity. An Ariki-Koike algebra $\mathscr{H}_x(W_{r,n}) = \mathscr{H}_{R,v,\mathbf{Q}}(W_{r,n})$ is *d-cyclotomic* if $R = \mathbb{Z}[\xi][x,x^{-1}]$ and the parameters of $\mathscr{H}_x(W_{r,n})$ are of the form $v = \zeta^{a_v}x^{b_v}$ and $Q_s = \zeta^{a_s}x^{b_s}$, for some rational numbers a_v, b_v and a_s, b_s, such that:

(a) $\mathscr{H}_{\zeta_d} = \mathscr{H}_x(W_{r,n}) \otimes_R R/(x-\zeta_d) \cong \mathbb{Z}[\xi]W_{r,n}$; and,
(b) $\mathscr{H}_q = \mathscr{H}_x(W_{r,n}) \otimes_R R/(x-q)$ is semisimple over its field of fractions.

For example, take parameters $v = x^d$ and $Q_s = x^{s-1}$ (with $\xi = 1$); then $\mathscr{H}_{\zeta_d} \cong \mathbb{Z}[\zeta_d]W_{r,n}$.

Thus, part (ii) of the conjecture together with (b) implies that the irreducible representations occurring in $\bigoplus_i H^i_c(X_w,\overline{\mathbb{Q}}_\ell)$ are in one-to-one correspondence with the irreducible representations of $\mathscr{H}_q(C_W(w))$; in turn, by (a) these representations are in one-to-one correspondence with the irreducible representations of $C_W(w)$. Importantly, nothing here depends upon the choice of q or ℓ. Conjecturally, these correspondences come from a derived equivalence, so they are really perfect isometries ("bijections with signs"). The polynomials $D_\chi(x)$ in part (iii) are the generic degrees of $\mathscr{H}_x(C_W(w))$; see the remarks after Theorem 3.6.

In fact, part (iii) of the conjecture is already known. The key fact needed to establish this is that the virtual module $\bigoplus_{i\ge0}(-1)^i H^i_c(X_w,\overline{\mathbb{Q}}_\ell)$ is a Deligne-Lusztig representation (specifically, it is $R^G_{T_w}(1)$, where T_w is the maximal torus associated to the conjugacy class of w in W), so its irreducible constituents are known. Parts (i) and (ii) of the conjecture are known in only a small number of cases.

We also mention that everything above is compatible with the decomposition of the unipotent characters of $\mathbf{G}^F$ into d-Harish-Chandra series [**22**]. For these details, and stronger forms of the conjecture, we refer the reader to Broué's article [**19**].

To conclude this section we remark that if $w = 1$ then $X_1 = G/B$ is the flag variety; so, $H^0_c(X_1,\overline{\mathbb{Q}}_\ell) \cong \operatorname{Ind}_B^G(1)$ and all higher cohomology groups are zero. Thus, in this case the conjectures recover the well-known results for the principal unipotent series of $\mathbf{G}^F$. (According to our definitions, $w = 1$ is not a good element of W; however, we have discussed only a special case of the general conjectures.)

§3. The representation theory of the Ariki-Koike algebras

3.1. The semisimple representation theory of $\mathscr{H}$

Because $W_{r,n}$ is the wreath product $\mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n$, its ordinary irreducible representations are indexed by r-tuples of partitions of n. In this section we see that the same is true of the irreducible representations of $\mathscr{H}$ when $\mathscr{H}$ is semisimple.

A partition of n is a sequence $\sigma = (\sigma_1 \geq \sigma_2 \geq \cdots)$ of non-negative integers σ_i such that $|\sigma| = \sum_{i\geq 1} \sigma_i = n$; we write $\sigma = (\sigma_1, \ldots, \sigma_k)$ if $\sigma_i = 0$ for $i > k$. A multipartition of n is an ordered r-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of partitions with $|\lambda^{(1)}| + \cdots + |\lambda^{(r)}| = n$. We write $\lambda \vdash n$ if λ is a multipartition of n.

The multipartitions form a poset under dominance $\trianglerighteq$, where $\lambda \trianglerighteq \mu$ if

$$\sum_{t=1}^{s-1} |\lambda^{(t)}| + \sum_{j=1}^{i} \lambda_j^{(s)} \geq \sum_{t=1}^{s-1} |\mu^{(t)}| + \sum_{j=1}^{i} \mu_j^{(s)}$$

for $s = 1, 2, \ldots, r$ and all $i \geq 1$. If $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$ we write $\lambda \triangleright \mu$.

The diagram of λ is $[\lambda] = \{\, (i,j,s) \mid 1 \leq j \leq \lambda_i^{(s)} \text{ and } 1 \leq s \leq r \,\}$. The elements of $[\lambda]$ are called nodes; more generally, a node is any triple (i,j,s) where $1 \leq s \leq r$ and $i, j \geq 1$.

A λ-tableau is a bijection $\mathfrak{t}\colon [\lambda] \longrightarrow \{1, 2 \ldots, n\}$, which we consider as an r-tuple $\mathfrak{t} = (\mathfrak{t}^{(1)}, \ldots, \mathfrak{t}^{(r)})$ of labeled tableaux where $\mathfrak{t}^{(s)}$ is a $\lambda^{(s)}$-tableau for each s; the tableaux $\mathfrak{t}^{(s)}$ are the components of $\mathfrak{t}$. If $\mathfrak{t}$ is a λ-tableau we write $\operatorname{Shape}(\mathfrak{t}) = \lambda$.

A tableau $\mathfrak{t}$ is standard if, in each component, its entries increase from left to right along each row and from top to bottom down each column. For example,

$$\begin{aligned} \mathfrak{t}^{\lambda} &= \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4 \\\cline{1-1}\end{array}\,,\ \begin{array}{|c|}\hline 5\\\hline 6\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 7&8\\\hline 9\\\cline{1-1}\end{array} \right) \quad \text{and} \\ \mathfrak{t} &= \left(\begin{array}{|c|c|c|}\hline 4&7&9\\\hline 6 \\\cline{1-1}\end{array}\,,\ \begin{array}{|c|}\hline 1\\\hline 8\\\hline\end{array}\,,\ \begin{array}{|c|c|}\hline 2&5\\\hline 3\\\cline{1-1}\end{array} \right) \end{aligned} \tag{3.1}$$

are two standard $\big((3,1),(1^2),(2,1)\big)$-tableaux. Let $\mathcal{T}^{s}(\lambda)$ be the set of standard λ-tableaux.

If $\mathfrak{t}$ is a λ-tableau and $w \in \mathfrak{S}_n$ let $\mathfrak{t}w = \mathfrak{t} \circ w$ be the tableau obtained from $\mathfrak{t}$ by replacing each entry in $\mathfrak{t}$ by its image under w. This defines a right action of $\mathfrak{S}_n$ on the set of all λ-tableaux. For example, $\mathfrak{t} = \mathfrak{t}^{\lambda}(1,4,6,8,5)(2,7)(3,9)$ in (3.1).

If $\mathfrak{t}$ is a tableau and k an integer, with $1 \leq k \leq n$, then the residue of k in $\mathfrak{t}$ is defined to be $\operatorname{res}_{\mathfrak{t}}(k) = q^{j-i} Q_s$, if k appears in row i and column j of $\mathfrak{t}^{(s)}$; that is, $\mathfrak{t}(i,j,s) = k$.

The last ingredient that we need is something like the Poincaré polynomial of a Coxeter group; however, be warned that it is not true that $|W_{r,n}| = P_{\mathscr{H}}(q, \mathbf{Q})$ when $R = \mathbb{C}$, $q = 1$ and $Q_s = \zeta^{s-1}$, where $\zeta = \exp(2\pi i/e)$ (that is, when $\mathscr{H} = RW_{r,n}$). Let

$$P_{\mathscr{H}}(q, \mathbf{Q}) = \prod_{i=1}^{n}(1 + q + \cdots + q^{i-1}) \cdot \prod_{1 \leq i < j \leq r} \prod_{-n<d<n} (q^d Q_i - Q_j).$$

We can now describe the irreducible representations of $\mathscr{H}$ when $P_{\mathscr{H}}(q, \mathbf{Q})$ is invertible. (Note that if R is a field then $P_{\mathscr{H}}(q, \mathbf{Q})$ is invertible if and only if $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$.)

Theorem 3.2 (Hoefsmit [**76**], Cherednik [**30**], Ariki-Koike [**8**]). *Suppose that $P_{\mathscr{H}}(q, \mathbf{Q})$ is invertible in R.*

(i) *For each multipartition λ let V^λ be the R-module with basis*

$$\{ v_{\mathfrak{t}} \mid \mathfrak{t} \text{ a standard } \lambda\text{-tableau} \}.$$

Then V^λ becomes a right $\mathscr{H}$-module via $v_{\mathfrak{t}} T_0 = \operatorname{res}_{\mathfrak{t}}(1) v_{\mathfrak{t}}$ and, for $1 \leq i < n$, if $\mathfrak{s} = \mathfrak{t} t_i$ is not standard then

$$v_{\mathfrak{t}} T_i = \begin{cases} q v_{\mathfrak{t}}, & \text{if } i \text{ and } i+1 \text{ are in the same row of } \mathfrak{t}, \\ -v_{\mathfrak{t}}, & \text{if } i \text{ and } i+1 \text{ are in the same column of } \mathfrak{t}, \end{cases}$$

and if $\mathfrak{s}$ is standard then

$$v_{\mathfrak{t}} T_i = \frac{(q-1)\operatorname{res}_{\mathfrak{t}}(i)}{\operatorname{res}_{\mathfrak{t}}(i) - \operatorname{res}_{\mathfrak{s}}(i)} v_{\mathfrak{t}} + \frac{q \operatorname{res}_{\mathfrak{t}}(i) - \operatorname{res}_{\mathfrak{s}}(i)}{\operatorname{res}_{\mathfrak{t}}(i) - \operatorname{res}_{\mathfrak{s}}(i)} v_{\mathfrak{s}}.$$

(ii) *If R is a field then V^λ is an irreducible $\mathscr{H}$-module for each multipartition λ.*

(iii) *If R is a field then $\{ V^\lambda \mid \lambda \vdash n \}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}$-modules.*

The general case follows from the type A case ($r = 1$); this is due to Hoefsmit who, in turn built upon Young's seminormal form for the symmetric groups. Cherednik does not state the result in this form; it is necessary to do some work to see that his result is equivalent.

Part (i) is proved by a brute force calculation to show that the action of the generators on V^λ respects the relations in $\mathscr{H}$. The remaining parts can be proved by looking at how the commutative subalgebra $\mathscr{L} = \langle L_1, \ldots, L_n \rangle$ of $\mathscr{H}$ acts on V^λ. From Theorem 3.2(i) it follows that $v_{\mathfrak{t}} L_k = \operatorname{res}_{\mathfrak{t}}(k) v_{\mathfrak{t}}$ for all standard tableaux $\mathfrak{t}$, for $1 \leq k \leq n$. Hence $R v_{\mathfrak{t}}$ is an irreducible $\mathscr{L}$-module; in fact, Ariki and Koike [**8**] show that every irreducible $\mathscr{L}$-module is of this form. Moreover, because $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$ if $\mathfrak{s}$ and $\mathfrak{t}$ are standard tableaux then $\mathfrak{s} = \mathfrak{t}$ if and only if $\operatorname{res}_{\mathfrak{t}}(k) = \operatorname{res}_{\mathfrak{s}}(k)$, for $1 \leq k \leq n$; this implies that $Rv_{\mathfrak{s}} \cong Rv_{\mathfrak{t}}$ as $\mathscr{L}$-modules only if $\mathfrak{s} = \mathfrak{t}$. Therefore, V^λ and V^μ have a common composition factor only if $\lambda = \mu$; hence (ii). Part (iii) now follows by counting dimensions because

$$\dim \mathscr{H} \geq \dim(\mathscr{H}/\operatorname{Rad}\mathscr{H}) \geq \sum_{\lambda \vdash n} (\dim V^\lambda)^2 = r^n n! = |W_{r,n}| \geq \dim \mathscr{H}.$$

(The third equality follows from the Robinson-Schensted correspondence which implies that the sum of the squares of the number of standard λ-tableaux, as λ runs over all multipartitions of n, is equal to $|W_{r,n}|$.) As we have equality throughout, this also proves Theorem 2.2 (indeed, this is how Ariki and Koike first proved it).

Corollary 3.3 (Ariki [**2**]). *Suppose that R is a field. Then $\mathscr{H}$ is semisimple if and only if $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$.*

Sketch of proof. By Theorem 3.2 if $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$ then $\mathscr{H}$ is semisimple. For the converse, when $P_{\mathscr{H}}(q, \mathbf{Q}) = 0$ the ideal of $\mathscr{H}$ generated by

$$\Big(\prod_{k=1}^{n}\prod_{s=1}^{r-1}(L_k - Q_s)\Big)\Big(\sum_{w \in \mathfrak{S}_n} T_w\Big),$$

is nilpotent. (This ideal affords the "trivial" representation of $\mathscr{H}$.) □

Halverson and Ram [**75**] have generalized the Murnaghan-Nakayama rule of the symmetric groups to give a method for computing the characters of the irreducible representations V^λ. (In fact, they also compute the characters of the irreducible representations of the cyclotomic Hecke algebras of type $G(r, p, n)$; the irreducible representations of these algebras were constructed by Ariki [**3**].) See also Shoji [**117**].

As remarked earlier the symmetric polynomials in $L_1, \ldots, L_n$ belong to the centre of $\mathscr{H}$. In the semisimple case this is a complete description of the centre.

Theorem 3.4 (Ariki-Koike [**8**]). *Suppose that R is a field and that $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$. Then the centre of $\mathscr{H}$ is equal to the set of symmetric polynomials in $L_1, \ldots, L_n$.*

Graham [**63**] has recently announced that the centre of $\mathscr{H}_{R,q}(\mathfrak{S}_n)$ is always equal to the set of symmetric polynomials in $L_1, \ldots, L_n$ when R is an integral domain (this is the case $r = 1$). Ariki [**4**] has given an example which shows that the centre of $\mathscr{H}$ can be larger than the set of symmetric polynomials when $r > 1$.

When $q \neq 1$ and $P_{\mathscr{H}}(q,\mathbf{Q}) \neq 0$ the author [**104**] has explicitly described the primitive central idempotents as symmetric polynomials in $L_1,\dots,L_n$ (see also Shoji [**117**]); this gives a second proof of Theorem 3.4. In addition, [**104, 117**] construct the primitive idempotents and a Wedderburn basis of $\mathscr{H}$ in the semisimple case.

Define $\tau:\mathscr{H}\longrightarrow R$ to be the R-linear map determined by

$$\tau(L_1^{a_1}\dots L_n^{a_n}T_w)=\begin{cases}1, & \text{if } a_1=\dots=a_n=0 \text{ and } w=1,\\ 0, & \text{otherwise,}\end{cases}$$

for $0\le a_i<r$ and $w\in\mathfrak{S}_n$. Notice that if $q=1$ and $Q_s=\zeta^s$, where $\zeta=\exp(2\pi i/r)\in\mathbb{C}$, then τ is the natural trace function on the group algebra $\mathbb{C}W_{r,n}$. The definition of τ looks quite ad hoc; however, as we explain below, τ is canonically determined.

Proposition 3.5. *Assume that R is an integral domain. Then the following hold.*

(i) (**Bremke-Malle** [**16**]) *τ is a trace form on $\mathscr{H}$.*
(ii) (**Malle-Mathas** [**101**]) *Suppose that $q,Q_1,\dots,Q_r$ are all invertible in R. Then τ is non-degenerate. Consequently, $\mathscr{H}$ is a symmetric algebra.*

Part (i) is straightforward; although we should mention that Bremke and Malle use a different (but, by [**101**], equivalent), definition of the trace form τ. For the Iwahori-Hecke algebras ($r\le 2$), part (ii) is also routine (see, for example, [**103**, Prop. 1.16]); in contrast, whilst not difficult, the proof of (ii) is a laborious calculation when $r>2$. As an indication of the difficulties here, no pair of dual bases for $\mathscr{H}$ is known when $r>2$ (except in the semisimple case; see [**104**, Theorem 3.9]).

As we will describe, Proposition 3.5 provides the strongest known link between the representation theory of $\mathscr{H}$ and that of the finite groups of Lie type (when $r>2$).

If R is a field and $P_{\mathscr{H}}(q,\mathbf{Q})\neq 0$ then $\mathscr{H}$ is semisimple. Let χ^λ be the character of V^λ. Since τ is a trace function we can write

$$\tau=\sum_{\lambda\vdash n}\frac{1}{s_\lambda(q,\mathbf{Q})}\chi^\lambda$$

for some $s_\lambda(q,\mathbf{Q})\in R$. The rational functions $s_\lambda(q,\mathbf{Q})$ are the Schur elements of $\mathscr{H}$; to describe them we need some more notation. In fact, by general arguments (see [**61**, Prop. 7.3.9]), $s_\lambda(q,\mathbf{Q})$ in $\mathbb{Z}[q^\pm,\mathbf{Q}^\pm]$; this is by no means obvious from the explicit formula for $s_\lambda(q,\mathbf{Q})$ given below.

Define the length of a partition σ to be the smallest integer $\ell(\sigma)$ such that $\sigma_i=0$ for all $i>\ell(\sigma)$; the length of a multipartition λ is

$\ell(\lambda) = \max\{\,\ell(\lambda^{(s)}) \mid 1 \le s \le r\,\}$. Suppose that $L \ge \ell(\lambda)$ and set $\beta_i^{(s)} = \lambda_i^{(s)} + L - i$ for $i = 1, \dots, L$ and $1 \le s \le r$; also set $B_s = \{\beta_1^{(s)}, \dots, \beta_L^{(s)}\}$, for $s = 1, \dots, r$. The matrix $B = \big(\beta_i^{(s)}\big)_{s,i}$ is the L-symbol of λ [**20**, **99**].

Theorem 3.6 (Geck-Iancu-Malle [**60**]). *Suppose that λ is a multipartition of n with L-symbol $B = (\beta_i^{(s)})_{s,i}$ such that $L \ge \ell(\lambda)$. Then the Schur element $s_\lambda(q, \mathbf{Q})$ is equal to*

$$(-1)^{a_{rL}} q^{b_{rL}} \frac{\displaystyle\prod_{1\le s<t\le r} (Q_s - Q_t)^L \cdot \prod_{1\le s,t\le r} \prod_{\alpha_s \in B_s} \prod_{1\le k\le \alpha_s} (q^k Q_s - Q_t)}{\displaystyle(q-1)^n (Q_1 \dots Q_r)^n \prod_{1\le s\le t\le r} \prod_{\substack{(\alpha_s,\alpha_t)\in B_s\times B_t\\ \alpha_s>\alpha_t \text{ if } s=t}} (q^{\alpha_s} Q_s - q^{\alpha_t} Q_t)},$$

where $a_{rL} = n(r-1) + \binom{r}{2}\binom{L}{2}$ and $b_{rL} = \frac{rL(L-1)(2rL-r-3)}{12}$.

It is not hard to see that if f_λ is a primitive idempotent in $\mathscr{H}$ which generates the Specht module S^λ then $s_\lambda(q, \mathbf{Q}) = 1/\tau(f_\lambda)$; this observation is used in [**104**] to give a direct proof of Theorem 3.6. (Actually, [**60**] and [**104**] were both written at the same time; however, I obtained a different formula for $s_\lambda(q, \mathbf{Q})$. The final version of my paper shows that these two formulae coincide.)

For $r = 1, 2$ the Schur elements were first computed by Hoefsmit [**76**]. Murphy [**106**] gave a different argument for type A (that is, $r = 1$). For $r > 2$ this result was conjectured by Malle [**99**]. Geck, Iancu and Malle use a clever specialization argument due to Orellana [**108**] to compute the Schur elements using the Markov trace of the Hecke algebras $\mathscr{H}_q(\mathfrak{S}_m)$; in turn, this builds on work of Wenzl [**122**].

Theorem 3.6 is important because when combined with [**99**, 3.16 and 6.11] it implies that Φ_d-blocks [**22**] of the finite reductive groups satisfy a generalized Howlett–Lehrer theory [**77**]. More precisely, Conjecture 2.5(iii) is true and the dimensions of the irreducible representations in a unipotent Φ_d-block are given by specializations of the generic degrees of $\mathscr{H}$; these are the rational functions $D_\lambda(q) = s_\eta(q, \mathbf{Q})/s_\lambda(q, \mathbf{Q})$, where $\eta = ((n), (0), \dots, (0))$. Remarkably, for "spetsial specializations" the generic degrees are actually polynomials; see [**100**]. (The significance of $s_\eta(q, \mathbf{Q})$ is that it is the Poincaré polynomial of the coinvariant algebra of $W_{r,n}$ when $Q_1 = q$ and $Q_s = \zeta^{s-1}$, for $2 \le s \le r$.)

As a second application of Theorem 3.6, it follows from [**60**, Theorem 5.2] and [**23**, Lemma 2.7] that the trace form τ is the unique trace form on $\mathscr{H}$ which, in a precise sense [**23**, Theorem 2.1], is compatible with the usual trace forms on both $W_{r,n}$ and on the braid group

$\mathfrak{B}_{r,n}$. In addition, Malle [**100**] uses Theorem 3.6 to define the notion of "spetsiality" for complex reflection groups; for more details see [**23**].

Finally, Broué and Kim [**20**] use Theorem 3.6, together with the block structure of $\mathscr{H}$, to show that the irreducible representations of $\mathscr{H}$ can be grouped according to a generalization of Lusztig's *families*; a key ingredient in their paper is a block theoretical characterisation of Lusztig's families due to Rouquier [**112**]. Again, the combinatorial description of the spetsial families of $\mathscr{H}$ had previously been conjectured by Malle [**99**].

3.2. The modular representation theory of $\mathscr{H}$

We now turn to the modular representation theory of $\mathscr{H}$; that is, the representation theory when $\mathscr{H}$ is not semisimple. In types A and B the irreducible modular representations were first constructed by Dipper and James [**37**] and Dipper, James and Murphy [**43**], respectively. Graham and Lehrer [**64**] considered the general case using cellular algebra techniques. Even though the papers [**43, 107**] predated Graham and Lehrer, the cellular approach is already implicit in them.

Graham and Lehrer constructed a cellular basis for $\mathscr{H}$ by building upon the Kazhdan-Lusztig basis of $\mathscr{H}_q(\mathfrak{S}_n)$ (which is itself cellular). We will describe a different cellular basis of $\mathscr{H}$ which comes from the work of Dipper, James and the author [**42**]. We prefer this basis because we know how to lift this basis to give a basis for the cyclotomic q-Schur algebras and because this basis has many nice combinatorial and representation theoretic properties.

Let $*$ be the anti-isomorphism of $\mathscr{H}$ determined by $T_i^* = T_i$, for $0 \le i < n$. Then $*$ is an involution and $T_w^* = T_{w^{-1}}$, $L_k^* = L_k$ and $(h_1h_2)^* = h_2^*h_1^*$ for $w \in \mathfrak{S}_n$, $1 \le k \le n$ and $h_1, h_2 \in \mathscr{H}$.

Fix a multipartition λ and let $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda^{(1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(r)}}$ be the associated Young subgroup of $\mathfrak{S}_n$. Equivalently, $\mathfrak{S}_\lambda$ is the row stabilizer of the λ-tableau $\mathfrak{t}^\lambda$ which has the numbers $1, \dots, n$ entered in order from left to right, top to bottom first along the rows of $\mathfrak{t}^{\lambda^{(1)}}$ and then $\mathfrak{t}^{\lambda^{(2)}}$ and so on (for example, see the first tableau in (3.1)).

If $\mathfrak{t}$ is a standard λ-tableau let $d(\mathfrak{t}) \in \mathfrak{S}_n$ be the unique permutation in $\mathfrak{S}_n$ such that $\mathfrak{t} = \mathfrak{t}^\lambda d(\mathfrak{t})$. Define elements x_λ and u_λ^+ in $\mathscr{H}$ by

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w \quad \text{and} \quad u_\lambda^+ = \prod_{s=2}^{r} \prod_{k=1}^{|\lambda^{(1)}|+\dots+|\lambda^{(s-1)}|} (L_k - Q_s).$$

It follows easily from the relations in $\mathscr{H}$ that $x_\lambda u_\lambda^+ = u_\lambda^+ x_\lambda$. Although somewhat ungainly, the function of u_λ^+ is used to control the eigenvalues of the L_k on the modules below. Set $m_\lambda = x_\lambda u_\lambda^+$.

Definition 3.7. *Suppose that* $\mathfrak{s}$ *and* $\mathfrak{t}$ *are standard* λ*-tableaux. Let* $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} m_\lambda T_{d(\mathfrak{t})}$.

Theorem 3.8 (The standard basis theorem [**42**]). *The Ariki-Koike algebra* $\mathscr{H}$ *is free as an* R*-module with cellular basis*

$$\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s} \text{ and } \mathfrak{t} \text{ standard } \lambda\text{-tableaux}, \ \lambda \vdash n \,\}.$$

When $r = 1$ this result is due to Murphy [**107**] and when $r = 2$ it was proved by Dipper, James and Murphy [**43**]. The basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ is called both the Murphy basis and the standard basis of $\mathscr{H}$. As mentioned above, Graham and Lehrer [**64**] were the first to produce a (different) cellular basis of $\mathscr{H}$.

The proof of this theorem starts by observing that $\mathscr{H}$ is spanned by a set of more general elements $m_{\mathfrak{s}\mathfrak{t}}$ where $\mathfrak{s}$ and $\mathfrak{t}$ are *row standard* tableaux of the same shape. (The entries in row standard tableaux increase along rows, but not necessarily down columns.) Next, one shows that if $\mathfrak{s}$ and $\mathfrak{t}$ are not standard tableaux then $m_{\mathfrak{s}\mathfrak{t}}$ can be written as a linear combination of "higher terms" $m_{\mathfrak{u}\mathfrak{v}}$; so, by induction, $\mathscr{H}$ is spanned by standard basis elements (here, "higher" is essentially the Bruhat order on $\mathfrak{S}_n$). The rewriting rules involved in this step are essentially Garnir relations; in fact, they are a little bit easier than the classical Garnir relations because we work modulo a filtration. A counting argument now shows that we have a basis. In order to show that the basis is cellular some accounting details need to be carried through the argument; this adds only minor complications to the proof.

We will not describe the theory of cellular algebras here; instead the reader is referred to the beautiful paper of Graham and Lehrer [**64**] or to Chapter 2 of my book [**103**]. A different approach to cellular algebras can be found in [**93**].

The required indexing of a cellular basis is already implicit in our notation. The two properties that the basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ must satisfy for it to be cellular are: (i) the R-linear map determined by $m_{\mathfrak{s}\mathfrak{t}} \longmapsto m_{\mathfrak{t}\mathfrak{s}}$ must be an algebra anti-isomorphism — this is obvious for us because $m^*_{\mathfrak{s}\mathfrak{t}} = m_{\mathfrak{t}\mathfrak{s}}$; and, (ii) for all λ-tableaux $\mathfrak{t}$ and all $h \in \mathscr{H}$ there exist scalars $r_\mathfrak{v} \in R$ such that for any standard λ-tableau $\mathfrak{s}$

$$m_{\mathfrak{s}\mathfrak{t}} h \equiv \sum_{\mathfrak{v} \in \mathcal{T}^s(\lambda)} r_\mathfrak{v} m_{\mathfrak{s}\mathfrak{v}} \pmod{\mathcal{H}^\lambda}, \tag{3.9}$$

where $\mathcal{H}^\lambda$ is the R-module spanned by the elements $m_{\mathfrak{u}\mathfrak{v}}$ for $\mathrm{Shape}(\mathfrak{u}) = \mathrm{Shape}(\mathfrak{v}) \rhd \lambda$. The point of this equation is that the scalars $r_\mathfrak{v}$ depend only on $\mathfrak{t}, \mathfrak{v}$ and h; importantly, $r_\mathfrak{v}$ does not depend on $\mathfrak{s}$.

Applying the anti-isomorphism $*$ to the last equation gives a left hand analogue of (3.9) for $hm_{\mathfrak{st}}$. It follows that $\mathcal{H}^\lambda$ is a two-sided ideal of $\mathscr{H}$.

Definition 3.10. *Suppose that λ is a multipartition of n. The* Specht module S^λ *is the right $\mathscr{H}$-module generated by $m_\lambda + \mathcal{H}^\lambda$.*

Thus, S^λ is a submodule of the quotient module $\mathscr{H}/\mathcal{H}^\lambda$. Du and Rui [**55**] have shown how to construct the Specht modules as submodules of $\mathscr{H}$ (as distinct from subquotients as we have defined them here).

For each standard λ-tableau $\mathfrak{t}$ let $m_\mathfrak{t} = m_{\mathfrak{t}^\lambda\mathfrak{t}} + \mathcal{H}^\lambda = m_\lambda T_{d(\mathfrak{t})} + \mathcal{H}^\lambda$. It follows from Theorem 3.8 that S^λ is free as an R-module with basis $\{\, m_\mathfrak{t} \mid \mathfrak{t} \text{ a standard } \lambda\text{-tableau} \,\}$; moreover, by (3.9) the action of $\mathscr{H}$ on this basis is given by

$$m_\mathfrak{t} h = \sum_{\substack{\mathfrak{v} \text{ standard} \\ \lambda\text{-tableau}}} r_\mathfrak{v} m_\mathfrak{v},$$

where the scalars $r_\mathfrak{v} \in R$ are the same as those in (3.9). It follows from the left and right handed versions of (3.9) that there is a bilinear form on S^λ which is determined by

$$\langle m_\mathfrak{s}, m_\mathfrak{t} \rangle m_{\mathfrak{uv}} \equiv m_{\mathfrak{us}} m_{\mathfrak{tv}} \pmod{\mathcal{H}^\lambda}$$

for all standard λ-tableaux $\mathfrak{s}$ and $\mathfrak{t}$. This form is $*$–associative; that is, $\langle xh, y \rangle = \langle x, yh^* \rangle$ for all $x, y \in S^\lambda$ and $h \in \mathscr{H}$. Hence, $\operatorname{Rad} S^\lambda = \{\, x \in S^\lambda \mid \langle x, y \rangle = 0 \text{ for all } y \in S^\lambda \,\}$ is a submodule of S^λ and we may make the following definition.

Definition 3.11. *Suppose that λ is a multipartition of n. Then D^λ is the right $\mathscr{H}$-module $D^\lambda = S^\lambda / \operatorname{Rad} S^\lambda$.*

Everything that we have said since Theorem 3.8 is part of the general machinery of cellular algebras. Without too much work, the cellular theory now produces the following result.

Theorem 3.12 (Graham-Lehrer [**64**], Dipper-James-Mathas [**42**]). *Suppose that R is a field.*

(i) *For each multipartition μ, D^μ is either zero or absolutely irreducible.*
(ii) *$\{\, D^\mu \mid \mu \vdash n \text{ and } D^\mu \neq 0 \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{H}$-modules.*
(iii) *If $D^\mu \neq 0$ then the decomposition multiplicity $[S^\lambda{:}D^\mu] \neq 0$ only if $\lambda \trianglerighteq \mu$; further, $[S^\mu{:}D^\mu] = 1$.*

Graham and Lehrer proved this result for a different collection of modules; but this should really be considered their result. Again, for the cases $r = 1, 2$ see [**37**, **43**].

In particular, note that every field is a splitting field for $\mathscr{H}$. The reader might be concerned with the claim that any field R is a splitting field for $\mathscr{H}$ because, for example, $R = \mathbb{Q}$ is not a splitting field for $W_{r,n}$ when $r > 2$; however, this is OK because by definition all of the eigenvalues of T_0 automatically belong to R.

The multiplicities $d_{\lambda\mu} = [S^\lambda{:}D^\mu]$ are the decomposition numbers of $\mathscr{H}$ and the matrix $(d_{\lambda\mu})$ is the decomposition matrix of $\mathscr{H}$. Part (iii) of Theorem 3.12 says that the decomposition matrix of $\mathscr{H}$ is unitriangular when its rows and columns are ordered in a way that is compatible with the dominance order.

Corollary 3.3 and the theory of cellular algebras also gives us the following result.

Theorem 3.13. *Suppose that R is a field. Then the following are equivalent.*

(i) $P_{\mathscr{H}}(q, \mathbf{Q}) \neq 0$;
(ii) *$\mathscr{H}$ is semisimple;*
(iii) *$\mathscr{H}$ is split semisimple; and,*
(iv) *$S^\lambda = D^\lambda$ for all multipartitions λ of n.*

If $1 \leq k \leq n$ let $\mathfrak{t}{\downarrow}k$ be the subtableau of $\mathfrak{t}$ containing the integers $1, 2, \ldots, k$; so, if $\mathfrak{t}$ is standard then $\mathrm{Shape}(\mathfrak{t}{\downarrow}k)$ is a multipartition of k. We extend the dominance ordering to the set of standard tableaux by defining $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if $\mathrm{Shape}(\mathfrak{s}{\downarrow}k) \trianglerighteq \mathrm{Shape}(\mathfrak{t}{\downarrow}k)$ for $k = 1, \ldots, n$. Again we write $\mathfrak{s} \triangleright \mathfrak{t}$ if $\mathfrak{s} \trianglerighteq \mathfrak{t}$ and $\mathfrak{s} \neq \mathfrak{t}$. In fact, this partial order coincides with the Chevalley–Bruhat order $\leq$ on $\mathfrak{S}_n$: $\mathfrak{s} \trianglerighteq \mathfrak{t}$ if and only if $d(\mathfrak{s}) \leq d(\mathfrak{t})$. This result really goes back to Ehresmann and, independently, Dipper and James [**37**]; see also [**103**, Theorem 3.8].

A useful fact about the standard basis of $\mathscr{H}$ is the following.

Proposition 3.14 ([**81**, Prop. 3.7]). *Suppose that $1 \leq k \leq n$ and let $\mathfrak{s}$ and $\mathfrak{t}$ be standard tableaux of the same shape. Then, there exist scalars $r_{\mathfrak{v}} \in R$ such that*

$$m_{\mathfrak{s}\mathfrak{t}} L_k = \mathrm{res}_{\mathfrak{t}}(k) m_{\mathfrak{s}\mathfrak{t}} + \sum_{\mathfrak{v} \triangleright \mathfrak{t}} r_{\mathfrak{v}} m_{\mathfrak{s}\mathfrak{v}} \pmod{\mathcal{H}^\lambda}.$$

As shown in [**81**], the general case can be reduced to the case $r = 1$ where it is a theorem of Dipper and James [**38**]. When $r = 1$ the result can be proved by induction on n and k using the fact that $L_1 + \cdots + L_n$ belongs to the centre of $\mathscr{H}$; see [**103**].

As an application of Proposition 3.14, if R is a field and $P_{\mathscr{H}}(q,\mathbf{Q}) \neq 0$ then we can construct the irreducible $\mathscr{H}$-modules either as the modules V^λ of Theorem 3.2 or as the Specht modules S^λ. By Proposition 3.14 the modules V^λ and S^λ have the same $\mathscr{L}$-module composition factors; this implies that $V^\lambda \cong S^\lambda$ as $\mathscr{H}$-modules.

We close this section with a reduction theorem which shows that, up to Morita equivalence, the only important Ariki-Koike algebras are those with parameters of the form (i) $Q_s = q^{a_s}$ for some integers a_s with $|a_s| < n$, for $1 \le s \le r$, or (ii) $Q_s = 0$ for $1 \le s \le r$. The result actually says that we can reduce to the case where there exists a constant $c \in R$ and integers a_s such that $Q_s = cq^{a_s}$, for all s; However, if $c \neq 0$ then we can renormalize the generator T_0 as $\tilde{T}_0 = c^{-1}T_0$ and then the order relation for $\tilde{T}_0$ becomes $(\tilde{T}_0 - q^{a_1})\dots(\tilde{T}_0 - q^{a_r}) = 0$, so we are back in case (i).

Recall that $\mathbf{Q} = \{Q_1,\dots,Q_r\}$ and fix a partition $\mathbf{Q} = \mathbf{Q}_1 \coprod \cdots \coprod \mathbf{Q}_\kappa$ (disjoint union) of $\mathbf{Q}$ and let

$$P_n(q,\mathbf{Q}_1,\dots,\mathbf{Q}_\kappa) = \prod_{1\le\alpha<\beta\le\kappa} \prod_{\substack{Q_i\in\mathbf{Q}_\alpha\\ Q_j\in\mathbf{Q}_\beta}} \prod_{-n<d<n} (q^dQ_i - Q_j).$$

Observe that $P_n(q,\mathbf{Q}_1,\dots,\mathbf{Q}_\kappa)$ is a factor of the polynomial $P_{\mathscr{H}}(q,\mathbf{Q})$.

Theorem 3.15 (Dipper-Mathas [**44**]). *Suppose that R is an integral domain and that $\mathbf{Q} = \mathbf{Q}_1 \coprod \cdots \coprod \mathbf{Q}_\kappa$ is a partition of $\mathbf{Q}$ such that the polynomial $P_n(q,\mathbf{Q}_1,\dots,\mathbf{Q}_\kappa)$ is invertible in R. For $\alpha = 1,\dots,\kappa$ let $r_\alpha = |\mathbf{Q}_\alpha|$. Then $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$ is Morita equivalent to the R-algebra*

$$\bigoplus_{\substack{n_1,\dots,n_\kappa\ge0\\ n_1+\dots+n_\kappa=n}} \mathscr{H}_{q,\mathbf{Q}_1}(W_{r_1,n_1}) \otimes \cdots \otimes \mathscr{H}_{q,\mathbf{Q}_\kappa}(W_{r_\kappa,n_\kappa}).$$

If $r = 2$ then $|\mathbf{Q}_1| = |\mathbf{Q}_2| = 1$ and this is a result of Dipper and James [**40**]. Du and Rui [**54**] extended the argument of [**40**] to prove the special case of Theorem 3.15 when $|\mathbf{Q}_\alpha| = 1$ for $1 \le \alpha \le \kappa$; notice that in this case $\mathscr{H}$ is Morita equivalent to a direct sum of tensor products of Iwahori-Hecke algebras of type A.

For the proof of Theorem 3.15 observe that by induction it is enough to consider the special case $\kappa = 2$. Without loss of generality we may assume that $\mathbf{Q}_1 = \{Q_1,\dots,Q_s\}$ and $\mathbf{Q}_2 = \{Q_{s+1},\dots,Q_r\}$ for some s. The trick is to consider the right ideals $V^b = v_b\mathscr{H}$, for $0 \le b \le n$, where

$$v_b = \prod_{t=1}^{s}(L_1 - Q_t)\dots(L_{n-b} - Q_t)\cdot T_{w_b}\cdot \prod_{t=s+1}^{r}(L_1 - Q_t)\dots(L_b - Q_t)$$

and $w_b = (n, \dots, 2, 1)^b$. It turns out that the standard basis of $\mathscr{H}$ can be adapted to give a 'standard' basis of V^b. With this basis in hand one sees that V^b is a projective $\mathscr{H}$-module, that

$$\operatorname{End}_{\mathscr{H}}(V^b) \cong \mathscr{H}_{q,\mathbf{Q}_1}(W_{s,b}) \otimes \mathscr{H}_{q,\mathbf{Q}_2}(W_{r-s,n-b})$$

and $\operatorname{Hom}_{\mathscr{H}}(V^b, V^c) = 0$ for $b \neq c$. These results imply that $\bigoplus_{b=0}^{n} V^b$ is a projective generator for $\mathscr{H}$ which gives the result. The Morita equivalence can be described very explicitly; consequently, when R is a field it is easy to compare the dimensions of the simple modules under the equivalence.

3.3. Ariki's theorem

This section discusses a very deep result of Ariki [**4**] which gives a way to compute the decomposition numbers of the Ariki-Koike algebras $\mathscr{H}_{\mathbb{C},q,\mathbf{Q}}(W_{r,n})$ when $q \neq 1$ and $Q_s \neq 0$ for all s. Throughout we assume that R is a field (we won't restrict ourselves to characteristic zero until we have to). For convenience write $\mathscr{H}_n = \mathscr{H}_{q,\mathbf{Q}}(W_{r,n})$ and let $\mathscr{H}_n$-**mod** be the category of finite dimensional right $\mathscr{H}_n$-modules. We begin with some motivation.

If M is an $\mathscr{H}_n$-module let $\operatorname{Res} M$ be the restriction of M to $\mathscr{H}_{n-1}$. Then Res is an exact functor from $\mathscr{H}_n$-**mod** to $\mathscr{H}_{n-1}$-**mod**. Since $\mathscr{H}_n$ is free as an $\mathscr{H}_{n-1}$-module Res has a right adjoint; namely, the induction functor which sends a right $\mathscr{H}_{n-1}$-module N to $\operatorname{Ind} N = N \otimes_{\mathscr{H}_{n-1}} \mathscr{H}_n$.

If λ is a multipartition of $n-1$ and μ is a multipartition of n write $\lambda \longrightarrow \mu$ if the diagrams of λ and μ differ by only one node. From the definition of the Specht modules it is clear that the action of $\mathscr{H}_{n-1}$ on $\operatorname{Res} S^\mu$ is given by ignoring the node in the tableaux with label n. With only a small amount of work this implies the following result.

Proposition 3.16 (Ariki [**4**, Lemma 2.1]). *Suppose that μ is a multipartition of n. Then $\operatorname{Res} S^\mu$ has a filtration with composition factors isomorphic to the Specht modules S^λ, where λ runs over the multipartitions of $n-1$ such that $\lambda \longrightarrow \mu$.*

Let $K_0(\mathscr{H}_n\text{-}\mathbf{mod})$ be the Grothendieck group of $\mathscr{H}_n$-**mod**. Thus, $K_0(\mathscr{H}_n\text{-}\mathbf{mod})$ is the free abelian group generated by all isomorphism classes of finitely generated right $\mathscr{H}_n$-modules where the relations are given by short exact sequences. If M is a right $\mathscr{H}_n$-module let $[M]$ be the corresponding equivalence class in $K_0(\mathscr{H}_n\text{-}\mathbf{mod})$. By Theorem 3.12 $\{\, [S^\mu] \mid D^\mu \neq 0 \,\}$ and $\{\, [D^\mu] \mid D^\mu \neq 0 \,\}$ are both bases of $K_0(\mathscr{H}_n\text{-}\mathbf{mod})$ and the transition matrix between these bases is the decomposition matrix of $\mathscr{H}$.

The functors Res and Ind induce homomorphisms of Grothendieck groups which, by abuse of notation, we also denote by Res and Ind. Thus, $\operatorname{Res}: K_0(\mathscr{H}_n\text{-}\mathbf{mod}) \longrightarrow K_0(\mathscr{H}_{n-1}\text{-}\mathbf{mod})$ and $\operatorname{Ind}: K_0(\mathscr{H}_n\text{-}\mathbf{mod}) \longrightarrow K_0(\mathscr{H}_{n+1}\text{-}\mathbf{mod})$ are the maps given by $\operatorname{Res}[M] = [\operatorname{Res} M]$ and $\operatorname{Ind}[M] = [\operatorname{Ind} M]$. These homomorphisms are completely determined by their actions on the Specht modules and this is given by Proposition 3.16 and Frobenius reciprocity.

Corollary 3.17. *Suppose that λ is a multipartition of n. Then*

$$\operatorname{Res}[S^\lambda] = \sum_{\nu \longrightarrow \lambda} [S^\nu] \quad \textit{and} \quad \operatorname{Ind}[S^\lambda] = \sum_{\lambda \longrightarrow \mu} [S^\mu].$$

Let $c_n = L_1 + \cdots + L_n$; then c_n belongs to the centre of $\mathscr{H}_n$. If M is any $\mathscr{H}_n$-module let $M_\alpha = \{\, m \in M \mid (c_n - \alpha)^k m = 0 \text{ for } k \gg 0 \,\}$ be the corresponding generalized eigenspace for c_n acting on M, for $\alpha \in R$. Then M_α is an $\mathscr{H}_n$-module since $c_n \in Z(\mathscr{H}_n)$; so $M = \oplus_{\alpha \in R} M_\alpha$ as an $\mathscr{H}_n$-module.

Until further notice we assume that $q \neq 1$ and that $Q_s = q^{a_s}$ for some integers a_s, for $1 \leq s \leq r$. In particular, this implies that the eigenvalues of c_n are always linear combinations of powers of q. Let e be the multiplicative order of q; then $e \in \mathbb{N} \cup \{\infty\}$.

Now the Specht module S^λ is irreducible when $R = \mathbb{C}(q)$; therefore, it follows from Proposition 3.14, and a specialization argument, that c_n acts on the Specht module S^λ as multiplication by the scalar $c(\lambda) = \sum_{k=1}^n \operatorname{res}_{\mathfrak{t}^\lambda}(k)$. Therefore, $S^\lambda = (S^\lambda)_{c(\lambda)}$ is a single generalized eigenspace and, by the Corollary, $\operatorname{Res} S^\lambda = \oplus_{i\in\mathbb{Z}} (\operatorname{Res} S^\lambda)_{c(\lambda)-q^i}$ and $\operatorname{Ind} S^\lambda = \oplus_{i\in\mathbb{Z}} (\operatorname{Ind} S^\lambda)_{c(\lambda)+q^i}$. Therefore the eigenvalues of c_n on an arbitrary $\mathscr{H}_n$-module change by $\pm q^i$, for some $i \in \mathbb{Z}/e\mathbb{Z}$, under the functors Res and Ind respectively. Accordingly, we define new functors i-Res and i-Ind on $\mathscr{H}_n$-**mod** by

$$i\text{-}\operatorname{Res} M = \bigoplus_\alpha (\operatorname{Res} M_\alpha)_{\alpha - q^i} \quad \text{and} \quad i\text{-}\operatorname{Ind} M = \bigoplus_\alpha (\operatorname{Ind} M_\alpha)_{\alpha + q^i},$$

for $i = 0, 1, \ldots, e-1$. Then $\operatorname{Res} = \sum_{i=0}^{e-1} i\text{-}\operatorname{Res}$ and $\operatorname{Ind} = \sum_{i=0}^{e-1} i\text{-}\operatorname{Ind}$. These functors also induce group homomorphisms $K_0(\mathscr{H}_n\text{-}\mathbf{mod}) \longrightarrow K_0(\mathscr{H}_{n\pm1}\text{-}\mathbf{mod})$ and these maps are completely determined by their actions on the Specht modules.

Write $\lambda \xrightarrow{i} \mu$ if $\lambda \longrightarrow \mu$ and the node in $[\mu] \setminus [\lambda]$ has residue q^i. Then we have the following refinement of Corollary 3.17.

Corollary 3.18. *Suppose that $0 \le i < e$ and let λ be a multipartition of n. Then*

$$i\text{-}\mathrm{Res}[S^\lambda] = \sum_{\nu \xrightarrow{i} \lambda} [S^\nu] \quad \text{and} \quad i\text{-}\mathrm{Ind}[S^\lambda] = \sum_{\lambda \xrightarrow{i} \mu} [S^\mu].$$

Let $\mathscr{H}_n$-**proj** be the category of finitely generated *projective* $\mathscr{H}_n$-modules and let $K_0(\mathscr{H}_n\text{-}\mathbf{proj})$ be its Grothendieck group. If P is a projective $\mathscr{H}_n$-module let $[\![P]\!]$ denote its image in $K_0(\mathscr{H}_n\text{-}\mathbf{proj})$. Observe that there is a natural non-degenerate paring

$$\langle\ ,\ \rangle : K_0(\mathscr{H}_n\text{-}\mathbf{proj}) \times K_0(\mathscr{H}_n\text{-}\mathbf{mod}) \longrightarrow \mathbb{Z}$$

given by $\langle [\![P]\!], [M] \rangle = \dim_R \mathrm{Hom}_{\mathscr{H}_n}(P, M)$; hence, $K_0(\mathscr{H}_n\text{-}\mathbf{proj}) \cong K_0(\mathscr{H}_n\text{-}\mathbf{mod})^*$. Consequently, if P^μ is the projective cover of D^μ then $\{\, [\![P^\mu]\!] \mid \mu \vdash n \text{ and } D^\mu \ne 0 \,\}$ is a basis of $K_0(\mathscr{H}_n\text{-}\mathbf{proj})$ and we have induced maps $i\text{-}\mathrm{Res}^*, i\text{-}\mathrm{Ind}^* : K_0(\mathscr{H}_n\text{-}\mathbf{proj}) \longrightarrow K_0(\mathscr{H}_{n\pm1}\text{-}\mathbf{proj})$.

We are almost ready to state Ariki's theorem. Let $U(\widehat{\mathfrak{sl}}_e)$ be the Kac-Moody Lie algebra of type $A^{(1)}_{e-1}$. Thus, $U(\widehat{\mathfrak{sl}}_e)$ is the $\mathbb{C}$-algebra generated by d, e_i, f_i and h_i, for $0 \le i < r$, subject to a well-known set of relations; see [**7**, **86**]. Let $\Lambda_0, \dots, \Lambda_{e-1}$ be the fundamental weights of $U(\widehat{\mathfrak{sl}}_e)$ and recall that for each dominant weight $\Lambda \in \sum_{i=0}^{e-1} \mathbb{N}\Lambda_i$ there is a unique integrable highest weight $U(\widehat{\mathfrak{sl}}_e)$-module $L(\Lambda)$ with highest weight Λ.

Theorem 3.19 (Ariki [**4**, **9**]). *Suppose that R is a field and fix $q, Q_1 = q^{a_1}, \dots, Q_r = q^{a_r}$ in R such that $q \ne 1$ is a primitive e^{th} root of unity and integers $a_1, \dots, a_r$ (with $0 \le a_i < e$ if $e < \infty$). Finally, let $\Lambda = \sum_{i=0}^{e-1} a_i\Lambda_i$ and set $V_{q,\mathbf{Q}}(R) = \bigoplus_{n\ge0} K_0(\mathscr{H}_{R,n}\text{-}\mathbf{proj}) \otimes_{\mathbb{Z}} \mathbb{C}$.*

(i) *$V_{q,\mathbf{Q}}(R)$ is an integrable $U(\widehat{\mathfrak{sl}}_e)$-module upon which the Chevalley generators e_i and f_i act as follows:*

$$e_i[\![M]\!] = i\text{-}\mathrm{Res}^*[\![M]\!] \quad \text{and} \quad f_i[\![M]\!] = i\text{-}\mathrm{Ind}^*[\![M]\!],$$

for all $[\![M]\!] \in V_{q,\mathbf{Q}}(R)$. Moreover, $V_{q,\mathbf{Q}}(R) \cong L(\Lambda)$ as a $U(\widehat{\mathfrak{sl}}_e)$-module.

(ii) *If R is a field of characteristic zero then the canonical basis of $V_{q,\mathbf{Q}}(R)$ coincides with the basis*

$$\{\, [\![P^\mu]\!] \mid D^\mu \ne 0 \text{ for some } \mu \vdash n \ge 0 \,\}$$

given by the projective indecomposable $\mathscr{H}_n$-modules.

Some remarks are in order. First, the hard part of this theorem is the case where $R = \mathbb{C}$; this is proved in [**4**]. The result for an arbitrary field follows from the complex case by a modular reduction argument; see [**9**]. Next, by the canonical basis of $L(\Lambda)$ we mean the specialization at $v = 1$ of the Kashiwara-Lusztig canonical basis[1] of $L_v(\Lambda)$, the corresponding integrable highest weight representation of the quantum group $U_v(\widehat{\mathfrak{sl}}_e)$.

Theorem 3.19 is a very deep result which relies upon the topological K-theory of Kazhdan and Lusztig [**88**] and Ginzburg's equivariant K-theory [**31**]; these theories give different constructions of the standard modules of the affine Hecke algebras in characteristic zero. For details of the proof see Ariki's original paper [**4**] and also his forthcoming book [**7**]. Geck [**58**] has also written an excellent survey article on the modular representation theory of Hecke algebras; he includes a detailed account of Ariki's paper.

The special case of Theorem 3.19 with $r = 1$ proves the conjecture of Lascoux, Leclerc and Thibon [**94**] for computing the decomposition matrices of the Iwahori-Hecke algebras $\mathscr{H}_q(\mathfrak{S}_n)$ of the symmetric groups. The main point of [**94**] is that they gave an elementary combinatorial algorithm for computing the canonical basis of the integrable highest weight module $L_v(\Lambda_0)$ for $U_v(\widehat{\mathfrak{sl}}_e)$ — and hence the decomposition matrices of $\mathscr{H}_q(\mathfrak{S}_n)$. This and similar algorithms are described in [**7, 62, 94, 95, 103**]. In contrast to the difficulty of Theorem 3.19, these algorithms involve only basic linear algebra; they amount to computing certain parabolic affine Kazhdan-Lusztig polynomials of type A and evaluating them at 1. This is described explicitly in [**62, 95, 103**].

Uglov [**120**], extending the ideas of Leclerc and Thibon [**95**], has given an algorithm for computing the canonical basis of any integrable highest weight module for $U_v(\widehat{\mathfrak{sl}}_e)$; see also [**119**]. Hence, combining Theorem 3.19(ii) with Uglov's work and Theorem 3.15 we have the following.

Corollary 3.20. *Suppose that R is a field of characteristic zero and that $q \neq 1$ and $Q_s \neq 0$ for $1 \leq s \leq r$. Then the decomposition matrix of $\mathscr{H}_{R,q,\mathbf{Q}}(W_{r,n})$ is known.*

---

[1] Canonical bases of quantum groups were introduced independently by Lusztig [**97**] and Kashiwara [**87**]. Jantzen [**83**] has given an excellent treatment of this theory; unfortunately, he only considers quantum groups of finite type which is insufficient for our purposes. Ariki [**7**] gives a largely self-contained account of the canonical bases of $U_v(\widehat{\mathfrak{sl}}_e)$, which is exactly what we need. See also Lusztig's book [**98**].

In practice there is a bit of work to be done to use this result to compute the decomposition numbers of $\mathscr{H}$. First, Uglov's algorithm computes a canonical basis for a larger space which contains $L_v(\Lambda)$ as a submodule; this is less efficient than the LLT algorithm and its variants. Next, Uglov's indexing of the canonical basis of $L_v(\Lambda)$ is not compatible with Theorem 3.12(ii) and Theorem 3.24 below; a bijection between the different indexing sets for the irreducibles is given by the paths in the associated crystal graphs. Finally, the effect of the Morita equivalence of Theorem 3.15 on the decomposition numbers must be taken into account; this last step is straightforward and is described in [**44**].

3.4. The irreducible $\mathscr{H}$-modules

In principle, the simple $\mathscr{H}_n$-modules are completely determined by Theorem 3.12; that is, the simple $\mathscr{H}_n$-modules are precisely the non-zero modules D^μ for μ a multipartition of n. Unfortunately, it is non-trivial to determine when D^μ is zero and when it is non-zero.

We begin the classification of the simple modules of the Ariki-Koike algebras with the case $r = 1$; that is, when $\mathscr{H} = \mathscr{H}_q(\mathfrak{S}_n)$. Let e be the smallest positive integer such that $1 + q + \cdots + q^{e-1} = 0$. A partition is e-restricted if $\mu_i - \mu_{i+1} < e$ for $i \geq 1$. (This is compatible with our previous definition of e: if $q \neq 1$ then e is the multiplicative order of q in R; otherwise, e is the characteristic of R.)

Theorem 3.21 (Dipper and James [**37**]). *Suppose that R is a field. Then the $\mathscr{H}_q(\mathfrak{S}_n)$-module D^μ is non-zero if and only if μ is e-restricted.*

Dipper and James actually showed that the simple $\mathscr{H}_q(\mathfrak{S}_n)$-modules are indexed by e-regular partitions (that is, a partition with no e non-zero parts being equal). Our statement is different from theirs because our Specht modules are isomorphic to the duals of the Dipper-James Specht modules [**107**].

Using the $\mathscr{L}$-module structure of the Specht modules it is straightforward to see that the $\mathscr{H}_q(\mathfrak{S}_n)$-module D^μ is non-zero whenever μ is e-restricted (recall that $\mathscr{L} = \langle L_1, \ldots, L_n \rangle$). The converse is harder and follows from showing that if μ is not e-restricted then $[e]_q! = \prod_{k=1}^{e}(1 + q + \cdots + q^{k-1})$ divides the Gram determinant of the Specht module defined over $\mathbb{Z}[q, q^{-1}]$. For the proof see [**37, 103, 107**].

Returning to the general case where $r \geq 1$, the next result follows easily from Theorem 3.21. The statement is misleading because two separate, but similar, arguments are needed. For the proof when $q = 1$ see Mathas [**102**]; for the case where $Q_s = 0$, for all s, see Ariki-Mathas [**9**].

Corollary 3.22 (Mathas [**102**], Ariki-Mathas [**9**]). *Suppose that R is a field and that either (i) $q = 1$, or (ii) $Q_1 = \cdots = Q_r = 0$. Let $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ be a multipartition of n. Then $D^\mu \neq 0$ if and only if the following two conditions are satisfied.*

(i) *$\mu^{(s)}$ is e-restricted for $1 \le s \le r$.*
(ii) *$\mu^{(s)} = (0)$ whenever $Q_s = Q_t$ for some $t > s$.*

In the case $Q_1 = \cdots = Q_r = 0$ the last result simplifies to saying that $D^\mu \neq 0$ if and only if $\mu = \big((0), \dots, (0), \mu^{(r)}\big)$ for some e-restricted partition $\mu^{(r)}$.

It remains to treat the cases where $q \neq 1$ and $Q_s \neq 0$ for all s.

Given two nodes $x = (a, b, s)$ and $y = (c, d, t)$ we say that y is below x if either $s < t$, or $s = t$ and $a < c$. Further, $x \in [\lambda]$ is removable if $[\lambda] \setminus \{x\}$ is the diagram of a multipartition; similarly, $y \notin [\lambda]$ is addable if $[\lambda] \cup \{y\}$ is the diagram of a multipartition. If $i = \mathrm{res}(x)$ we call x an i-node.

An i-node x is normal if (i) whenever y is a removable i-node below x then there are more removable i-nodes between x and y than there are addable i-nodes, and (ii) there are at least as many removable i-nodes below x as addable i-nodes below x. In addition, a normal i-node x is good if there are no normal i-nodes above x. If $[\mu] = [\lambda] \cup \{x\}$ for some good node x we write $\lambda \xrightarrow{\text{good}} \mu$.

Definition 3.23. *A multipartition μ is* Kleshchev *if either $\mu = \big((0), , \dots, (0)\big)$ or $\lambda \xrightarrow{\text{good}} \mu$ for some Kleshchev multipartition λ.*

The origin of the definition of the Kleshchev multipartitions is that they are the vertices of the crystal graph of an integrable $U_v(\widehat{\mathfrak{sl}}_e)$-module. (When $Q_s = q^{a_s}$, for all s, then the Kleshchev multipartitions are the vertices of the crystal graph of $L_v(\Lambda)$, where $\Lambda = \sum_{s=1}^{r} \Lambda_{a_s}$. In general, we take a direct sum of tensor products of crystal graphs in accordance with Theorem 3.15.) There is an edge in the crystal graph between two Kleshchev multipartitions if $\lambda \xrightarrow{\text{good}} \mu$; the label of the edge is the residue of the node in $[\mu] \setminus [\lambda]$. For more details see [**9**, **85**].

When $r = 1$ a partition μ is Kleshchev if and only if μ is e-restricted; consequently, as it must, the next result agrees with Theorem 3.21 when $r = 1$.

Theorem 3.24 (Ariki [**6**]). *Suppose that R is a field, $q \neq 1$, $Q_s \neq 0$ for $1 \le s \le r$, and that μ is a multipartition of n. Then $D^\mu \neq 0$ if and only if μ is a Kleshchev multipartition.*

The first step towards Theorem 3.24 is to observe that Theorem 3.15 allows us to reduce to the crucial case where $q \neq 1$ and $Q_s = q^{a_s}$ for some integers a_s (a different argument is given in [**9**]). Using Theorem 3.19(ii), Ariki [**6**] is able to complete the classification of the irreducible $\mathscr{H}$-modules over $\mathbb{C}$. To complete the argument, [**9**] shows that the number of simple modules depends only on the integers a_s and the multiplicative order of q in R.

Finally, we remark that by combining these techniques with results of Ginzburg [**31**], Ariki and the author [**9**] classified the simple modules of the affine Hecke algebras over an algebraically closed field of positive characteristic; again, the hard work is done by Ariki's paper [**4**]. When $R = \mathbb{C}$ and q is not a root of unity the simple $\hat{H}_n$-modules were classified by Zelevinsky [**124**]; see also [**88, 111**]. When $q \in \mathbb{C}^\times$ is a root of unity the simple $\hat{H}_n$-modules were classified by Lusztig and Ginzburg; see [**4, 31**].

3.5. The modular branching rules

One of the most significant results in modular representation theory from the nineties is Kleshchev's modular branching rule for the symmetric groups [**89–92**]. Using a streamlined version of the same techniques Brundan [**26**] extended these results to the Iwahori-Hecke algebra of the symmetric group. Using completely different methods, Grojnowski [**71**] and Grojnowski-Vazirani [**73**] generalized Kleshchev's modular branching rules to the Ariki-Koike algebras and the affine Hecke algebra of type A. (Brundan and Kleshchev [**28**] have also applied Grojnowski's methods to the projective representations of the symmetric groups.)

Grojnowski was mainly interested in representations of the affine Hecke algebra $\hat{H}_n$; however, as remarked in §2.4 every irreducible representation of the affine Hecke algebra is an irreducible representation for a family of Ariki-Koike algebras. He studies the functors given by induction and restriction (from $\hat{H}_n$ to $\hat{H}_{n\pm 1}$), followed by the taking of socles by analyzing the effect of these functors on the central characters of $\hat{H}_n$. Grojnowski shows that these functors can be described in terms of the crystal graphs of integral highest weight modules for the quantum group $U_v(\widehat{\mathfrak{sl}}_e)$; cf. Theorem 3.19(i).

Theorem 3.25 (Grojnowski [**71**], Grojnowski-Vazirani [**73**]). *Suppose that R is a field, $q \neq 1$ and $Q_s \neq 0$, for $1 \leq s \leq r$. Then, for each m, there is an (unknown) permutation π_m of the set Kleshchev multipartitions of m such that if μ is a Kleshchev multipartition of n then*

$$\operatorname{Soc}(\operatorname{Res}(D^\mu)) \cong \bigoplus_{\pi_{n-1}(\lambda) \xrightarrow{\text{good}} \pi_n(\mu)} D^\lambda.$$

In [**73**] Grojnowski-Vazirani prove that $\mathrm{Soc}(\mathrm{Res}(D^\mu))$ is multiplicity free. In [**71**] Grojnowski shows that there exists a set of irreducible $\mathscr{H}$–modules which are indexed by the Kleshchev multipartitions and for which the modular branching rule is given by removing good nodes; Grojnowski does not give an explicit construction of these modules. Conjecturally, π_m is trivial for all m.

Notice that Theorem 3.25 implies that there are at most e direct summands of $\mathrm{Soc}(\mathrm{Res}(D^\mu))$ and that they all belong to different blocks.

As Grojnowski remarks, the assumption that $q \neq 1$ is not essential and can be removed (at the expense of some additional notation). Du and Rui [**55**] also obtained the modular branching rule in the special case where $q^d Q_s \neq Q_t$, for $1 \leq s < t \leq r$ and $|d| < n$. In fact, in their case they obtain the stronger result that $\pi_m = 1$, for all m. By Theorem 3.15 such Ariki-Koike algebras are Morita equivalent to direct sums of tensor products of Iwahori-Hecke algebras $\mathscr{H}_q(\mathfrak{S}_m)$; Du and Rui use this to deduce the result from Brundan's theorem [**26**] for $\mathscr{H}_q(\mathfrak{S}_m)$.

Grojnowski [**71**] shows that the number of irreducible $\mathscr{H}_n$–modules is equal to the number of Kleshchev multipartitions of n; this gives a more elementary proof of part of Theorem 3.24. Grojnowski also counts the number of irreducible modules of the affine Hecke algebra $\hat{H}_n$ over an arbitrary algebraically closed field. In §5.2 below we discuss the application of Theorem 3.25 to classifying the blocks of $\mathscr{H}$.

§4. The cyclotomic q-Schur algebra

This chapter introduces the cyclotomic q-Schur algebras. These algebras are defined as endomorphism algebras

$$\mathscr{S}(\Lambda) = \mathrm{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda} M^\mu\Big),$$

where Λ is a finite set of multicompositions and M^μ is a certain $\mathscr{H}$-module. In the special case where $r = 1$ the cyclotomic q-Schur algebras are the q-Schur algebras of Dipper and James [**39**]; see [**27**, **46**, **65**, **103**]. This was one of the motivations for introducing the cyclotomic q-Schur algebras.

Prior to [**42**] several authors [**41**,**51**,**56**,**59**,**69**,**74**] had studied Schur algebras of type B; these algebras are either subalgebras or special cases of the cyclotomic q–Schur algebras. See [**57**, **70**] for Schur algebras of other types.

4.1. Permutation modules.

We begin by describing the $\mathscr{H}$-modules M^μ.

A composition of n is a sequence $\sigma = (\sigma_1, \sigma_2, \dots)$ of non-negative integers σ_i such that $|\sigma| = \sum_{i\geq 1} \sigma_i = n$; we will sometimes write $\sigma = (\sigma_1, \dots, \sigma_k)$ if $\sigma_i = 0$ for $i > k$. A multicomposition of n is an ordered r-tuple $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ of compositions with $|\mu^{(1)}| + \cdots + |\mu^{(r)}| = n$.

Definition 4.1. *Suppose that μ is a multicomposition of n. Then M^μ is the right ideal $M^\mu = m_\mu \mathscr{H}$ of $\mathscr{H}$ (where $m_\mu = x_\mu u_\mu^+$ as before)*

Given a multicomposition μ let $\vec{\mu} = (\vec{\mu}^{(1)}, \dots, \vec{\mu}^{(r)})$ be the multipartition where $\vec{\mu}^{(s)}$ is the partition obtained by ordering the parts of the composition $\mu^{(s)}$. It is not hard to see that $M^{\vec{\mu}} \cong M^\mu$; indeed, if $d \in \mathfrak{S}_n$ is a right coset representative of $\mathfrak{S}_\mu$ of minimal length such that $\mathfrak{S}_{\vec{\mu}} = d^{-1}\mathfrak{S}_\mu d$ then $T_d x_{\vec{\mu}} = x_\mu T_d$; hence, $T_d m_{\vec{\mu}} = m_\mu T_d$ and an isomorphism $M^{\vec{\mu}} \cong M^\mu$ is given by $h \longmapsto T_d h$, for $h \in M^{\vec{\mu}}$.

When $r = 1$ the module M^μ is the induced trivial representation of the parabolic subalgebra

$$\mathscr{H}_q(\mathfrak{S}_\mu) = \langle T_i \mid t_i \in \mathfrak{S}_\mu \rangle = \sum_{w \in \mathfrak{S}_\mu} RT_w.$$

More precisely, let $\mathbf{1}_\mu$ be the trivial representation of the subalgebra $\mathscr{H}_q(\mathfrak{S}_\mu)$; so $\mathbf{1}_\mu$ is a free R-module of rank 1 on which T_w acts as multiplication by $q^{\ell(w)}$ for all $w \in \mathfrak{S}_\mu$. Then

$$M^\mu \cong \mathbf{1}_\mu \otimes_{\mathscr{H}_q(\mathfrak{S}_\mu)} \mathscr{H}_q(\mathfrak{S}_n).$$

(Note that $\mathscr{H}_q(\mathfrak{S}_n)$ is free as a right $\mathscr{H}_q(\mathfrak{S}_\mu)$-module.)

If $r > 1$ then, in general, the modules M^μ are not obviously induced from subalgebras (except in the case considered by Shoji [**117**]). Even so, the M^μ behave very much like permutation modules, so it is not a bad idea to think of them as such.

In order to describe a basis of M^μ we need to introduce some more notation. Let $\underline{\mathbf{nr}} = \{\, (i,s) \mid i \geq 1 \text{ and } 1 \leq s \leq r \,\}$. If $(i,s), (j,t)$ are elements of $\underline{\mathbf{nr}}$ write $(i,s) \preceq (j,t)$ if either $s < t$, or $s = t$ and $i \leq j$.

Let μ be a multicomposition. Then a λ-tableau of type μ is a map $\mathsf{T} : [\lambda] \longrightarrow \underline{\mathbf{nr}}$ such that $\mu_i^{(s)} = \#\{\, x \in [\lambda] \mid \mathsf{T}(x) = (i,s) \,\}$, for $1 \leq s \leq r$ and all $i \geq 1$; we write $\mathrm{Type}(\mathsf{T}) = \mu$. Again, we will think of a tableau of type μ as being an r-tuple of tableaux. For example, two tableaux of type $\big((3,1), (1^2), (2,1)\big)$ are

$$\left(\begin{array}{|c|c|c|}\hline 1_1 & 1_1 & 1_1 \\ \hline 2_1 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 1_2 \\ \hline 2_2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1_3 & 1_3 \\ \hline 2_3 \\ \cline{1-1}\end{array}\right) \text{ and } \left(\begin{array}{|c|c|c|c|c|}\hline 1_1 & 1_1 & 1_1 & 2_1 & 1_3 \\ \hline 1_2 & 2_3 \\ \cline{1-2}\end{array}, \begin{array}{|c|c|}\hline 2_2 & 1_3 \\ \hline\end{array}, \begin{array}{|c|}\hline 2_3 \\ \hline\end{array}\right),$$

where we write i_s instead of the ordered pair (i,s).

If $\mathfrak{s}$ is a standard λ-tableau let $\mu(\mathfrak{s})$ be the tableau of type μ obtained by replacing each entry k in $\mathfrak{s}$ by (i, s) if k appears in row i of component s of $\mathfrak{t}^\mu$ — as for multipartitions, we define $\mathfrak{t}^\mu$ to be the μ-tableau with the integers $1, \dots, n$ entered from left to right and then top to bottom along the rows of the components of $[\mu]$.

Definition 4.2. *Let λ be a multipartition and μ a multicomposition. A* semistandard *λ-tableau is a λ-tableau $\mathsf{T} = (\mathsf{T}^{(1)}, \dots, \mathsf{T}^{(r)})$ such that*

(i) *the entries in each row of T are non-decreasing in each component (when ordered by $\preceq$); and,*

(ii) *the entries in each column of T are strictly increasing in each component; and,*

(iii) *if $(a, b, c) \in [\lambda]$ and $\mathsf{T}(a, b, c) = (i, s)$ then $s \geq c$.*

Let $\mathcal{T}^{ss}_\mu(\lambda)$ be the set of semistandard λ-tableaux of type μ and let $\mathcal{T}^{ss}_\Lambda(\lambda) = \bigcup_{\mu \in \Lambda} \mathcal{T}^{ss}_\mu(\lambda)$.

When $r = 1$ condition (iii) is redundant and Definition 4.2 becomes the familiar definition of semistandard tableaux from the representation theory of the general linear and symmetric groups.

Write $\mathrm{comp}_\mathfrak{t}(k) = s$ if k appears in component s of $\mathfrak{t}$. For $r > 1$ condition (iii) is unexpected; it has its origin in the fact [**42**, Prop. 3.23] that if $h \in M^\mu$ and $h = \sum_{\mathfrak{s},\mathfrak{t}} r_{\mathfrak{s}\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}$ for some $r_{\mathfrak{s}\mathfrak{t}} \in R$ then $\mathrm{comp}_\mathfrak{s}(k) \leq \mathrm{comp}_{\mathfrak{t}^\mu}(k)$, for $k = 1, \dots, n$. Observe that $\mu(\mathfrak{s})$ satisfies condition (iii) if and only if $\mathrm{comp}_\mathfrak{s}(k) \leq \mathrm{comp}_{\mathfrak{t}^\mu}(k)$ for all k.

For example, if λ is a multipartition then $\mathsf{T}^\lambda = \lambda(\mathfrak{t}^\lambda)$ is the unique semistandard λ-tableau of type λ. The first of the two tableaux in the example above is T^λ for $\lambda = \big((3,1), (1^2), (2,1)\big)$; the second tableau there is also semistandard. Finally, let $\omega = \big((0), \dots, (0), (1^n)\big)$. Then it is easy to see that the map

$$\omega : \mathcal{T}^{s}(\lambda) \xrightarrow{\sim} \mathcal{T}^{ss}_\omega(\lambda); \mathfrak{s} \longmapsto \omega(\mathfrak{s}) \tag{4.3}$$

is a bijection between the set of standard λ-tableaux and the set of semistandard λ-tableaux of type ω. Hereafter, we identity $\mathcal{T}^{s}(\lambda)$ and $\mathcal{T}^{ss}_\omega(\lambda)$ via (4.3).

Definition 4.4. *Suppose that S is a semistandard λ-tableau of type μ and that $\mathfrak{t}$ is a standard λ-tableau. Define*

$$m_{\mathsf{S}\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \mathcal{T}^{s}(\lambda) \\ \mathsf{S} = \mu(\mathfrak{s})}} m_{\mathfrak{s}\mathfrak{t}}.$$

The point of all of this notation is the following useful theorem.

Theorem 4.5. *Suppose that μ is a multicomposition of n. Then M^μ is free as an R-module with basis*

$$\left\{ m_{\mathsf{S}\mathfrak{t}} \;\middle|\; \begin{array}{c} \mathsf{S} \in \mathcal{T}^{ss}_\mu(\lambda) \text{ and } \mathfrak{t} \in \mathcal{T}^s(\lambda) \text{ for} \\ \text{some multipartition } \lambda \text{ of } n \end{array} \right\}.$$

When $r = 1$ this result was first proved by Murphy [**107**]; the general case can be found in Dipper-James-Mathas [**42**].

The proof of this result is straightforward. A small calculation shows that $m_{\mathsf{S}\mathfrak{t}}$ is an element of M^μ. Next, the elements in the statement of Theorem 4.5 are linearly independent by Theorem 3.8. Finally, if $h \in M^\mu$ then h can be written as a linear combination of standard basis elements; in turn, these are a linear combination of the $m_{\mathsf{S}\mathfrak{t}}$.

The importance of Theorem 4.5 stems from the following applications.

Corollary 4.6. *Suppose that μ is a multicomposition of n. Then there exists a filtration $M^\mu = M_1 > M_2 > \cdots > M_{k+1} = 0$ of M^μ such that*

(i) *$M_i/M_{i+1} \cong S^{\lambda_i}$ for some multipartition λ_i for $i = 1, \dots, k$; and,*
(ii) *for each multipartition λ the number of i with $\lambda = \lambda_i$ is equal to the number of semistandard λ-tableaux of type μ.*

Sketch of proof. Fixing S and varying $\mathfrak{t}$ in the basis $\{m_{\mathsf{S}\mathfrak{t}}\}$ of M^μ gives a Specht module modulo higher terms. □

For each semistandard λ-tableau S of type μ and each semistandard λ-tableau T of type ν define

$$m_{\mathsf{ST}} = \sum_{\substack{\mathfrak{t} \in \mathcal{T}^s(\lambda) \\ \mathsf{T} = \nu(\mathfrak{t})}} m_{\mathsf{S}\mathfrak{t}}.$$

By definition, $m_{\mathsf{ST}} = \sum_{\mathfrak{s},\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}}$ where the sum is over the standard λ-tableaux $\mathfrak{s}$ and $\mathfrak{t}$ such that $\mu(\mathfrak{s}) = \mathsf{S}$ and $\nu(\mathfrak{t}) = \mathsf{T}$.

Corollary 4.7. *Suppose that μ and ν are multicompositions of n. Then*

$$\left\{ m_{\mathsf{ST}} \;\middle|\; \begin{array}{c} \mathsf{S} \in \mathcal{T}^{ss}_\mu(\lambda) \text{ and } \mathsf{T} \in \mathcal{T}^{ss}_\nu(\lambda) \text{ for} \\ \text{some multipartition } \lambda \text{ of } n \end{array} \right\}$$

is a basis of $\mathscr{H} m_\nu \cap m_\mu \mathscr{H}$.

We are now ready to tackle the cyclotomic q-Schur algebras.

4.2. The semistandard basis theorem

We give a slightly more general definition for the cyclotomic q-Schur algebras than appeared in [**42**] in that we allow the set Λ to be an arbitrary finite set of multicompositions. We invite the reader to check that the arguments from [**42**] go through without change.

Extend the dominance ordering $\trianglerighteq$ to the set of all multicompositions; by restriction we consider any set of multicompositions as a poset.

Definition 4.8. *Suppose that Λ is a finite set of multicompositions of n. The* cyclotomic q-Schur algebra *is the endomorphism algebra*

$$\mathscr{S}(\Lambda) = \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda} M^\mu\Big).$$

Let $\Lambda^+ = \{\,\lambda \vdash n \mid \lambda \trianglerighteq \mu$ for some $\mu \in \Lambda\,\}$.

We should really write $\mathscr{S}(\Lambda) = \mathscr{S}_{R,q,\mathbf{Q}}(\Lambda)$ since $\mathscr{S}(\Lambda)$ depends on Λ, R, q and $\mathbf{Q}$.

Part of the original definition of the cyclotomic q-Schur algebras in [**42**] was the requirement that $\Lambda^+ \subseteq \Lambda$. Following Donkin [**46**], we say that Λ is saturated if $\Lambda^+ \subseteq \Lambda$. In analogy with representations of Lie groups, Λ^+ should be thought of as the set of dominant weights and Λ the set of weights. Note that Λ^+ is not necessarily a subset of Λ.

Let $\Lambda = \Lambda(d;n)$ be the set of all compositions $\mu = (\mu_1, \dots, \mu_d)$ of n of length at most d (so $\mu_i = 0$ whenever $i > d$). Then $\mathscr{S}_q(d;n) = \mathscr{S}\big(\Lambda(d;n)\big)$ is a q-Schur algebra in the sense of Dipper and James [**39**].

As an R-module we see that

$$\mathscr{S}(\Lambda) = \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\mu\in\Lambda} M^\mu\Big) = \bigoplus_{\mu,\nu\in\Lambda} \operatorname{Hom}_{\mathscr{H}}\Big(M^\nu, M^\mu\Big);$$

so we need to understand the R-modules $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$.

Proposition 4.9. *Suppose that μ and ν are multicompositions of n. Then an R-linear map $\varphi\colon M^\nu \longrightarrow M^\mu$ belongs to $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ if and only if*

$$\varphi(m_\nu) = \sum_{\substack{\mathsf{S}\in\mathcal{T}_\mu^{ss}(\lambda)\\ \mathsf{T}\in\mathcal{T}_\nu^{ss}(\lambda)}} r_{\mathsf{ST}} m_{\mathsf{ST}}$$

for some $r_{\mathsf{ST}} \in R$.

Sketch of proof. If $Q_1, \dots, Q_r$ are invertible elements of R then $\mathscr{H}$ is a symmetric algebra by Proposition 3.5(ii); therefore, $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ and $m_\mu\mathscr{H} \cap \mathscr{H}m_\nu$ are canonically isomorphic R-modules (via the map $\varphi \longmapsto \varphi(m_\nu)$), so the proposition follows by Corollary 4.7.

For the general case, an intricate induction (see [**42**, §5]), which is independent of Proposition 3.5, shows that the double annihilator of m_μ,

$$\{\, x \in \mathscr{H} \mid xh = 0 \text{ whenever } m_\mu h = 0 \text{ for some } h \in \mathscr{H} \,\},$$

is $\mathscr{H} m_\mu$. Hence, it again follows that $\operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu) \cong m_\nu\mathscr{H} \cap \mathscr{H} m_\mu$, so we can complete the proof using the argument of the last paragraph. □

Definition 4.10. *Suppose that $\lambda \in \Lambda^+$ is a multipartition and that $\mu, \nu \in \Lambda$ are multicompositions. For each pair of standard λ-tableaux $\mathsf{S} \in \mathcal{T}_\mu^{ss}(\lambda)$ and $\mathsf{T} \in \mathcal{T}_\nu^{ss}(\lambda)$ let φ_{ST} be the R-linear endomorphism of $\bigoplus_{\mu\in\Lambda} M^\mu$ determined by*

$$\varphi_{\mathsf{ST}}(m_\alpha h) = \delta_{\alpha\nu} m_{\mathsf{ST}} h,$$

for all $\alpha \in \Lambda$ and $h \in \mathscr{H}$ (here $\delta_{\alpha\nu}$ is the Kronecker delta).

By Proposition 4.9 φ_{ST} is an element of $\mathscr{S}(\Lambda)$.

Let $\mathcal{S}^\lambda$ be the R-submodule of $\mathscr{S}(\Lambda)$ spanned by the φ_{UV}, for some $\mathsf{U}, \mathsf{V} \in \mathcal{T}_\Lambda^{ss}(\rho)$ where $\rho \in \Lambda^+$ and $\rho \rhd \lambda$. From the definitions, $\mathcal{S}^\lambda$ consists of those elements of $\mathscr{S}(\Lambda)$ whose image is contained in $\mathcal{H}^\lambda$.

Observe that a map $\varphi \in \operatorname{Hom}_{\mathscr{H}}(M^\nu, M^\mu)$ is completely determined by $\varphi(m_\nu)$ since $\varphi(m_\nu h) = \varphi(m_\nu)h$ for all $h \in \mathscr{H}$. Therefore, we can lift the involution $*$ of $\mathscr{H}$ to give an involutory anti-isomorphism of $\mathscr{S}(\Lambda)$ by defining $\varphi^* \in \operatorname{Hom}_{\mathscr{H}}(M^\mu, M^\nu)$ by $\varphi^*(m_\mu h) = \big(\varphi(m_\nu)\big)^* h$ for all $h \in \mathscr{H}$. In particular, note that $\varphi_{\mathsf{ST}}^* = \varphi_{\mathsf{TS}}$.

We can now state the semistandard basis theorem for the cyclotomic Schur algebras.

Theorem 4.11 (Dipper-James-Mathas [**42**, Theorem 6.6]). *Let Λ be a finite set of multicompositions. Then the cyclotomic q-Schur algebra $\mathscr{S}(\Lambda)$ is free as an R-module with basis*

$$\{\, \varphi_{\mathsf{ST}} \mid \text{ for some } \mathsf{S}, \mathsf{T} \in \mathcal{T}_\Lambda^{ss}(\lambda) \text{ and } \lambda \in \Lambda^+ \,\}.$$

Moreover, this basis is a cellular basis of $\mathscr{S}(\Lambda)$; more precisely, if S *and* T *are semistandard λ-tableaux, for some $\lambda \in \Lambda^+$, then*

(i) $\varphi_{\mathsf{ST}}^* = \varphi_{\mathsf{TS}}$*; and,*
(ii) *for all $\varphi \in \mathscr{S}(\Lambda)$ there exist scalars $r_{\mathsf{V}} = r_{\mathsf{TV}}(\varphi) \in R$, which do not depend on* S*, such that*

$$\varphi_{\mathsf{ST}}\varphi \equiv \sum_{\mathsf{V}\in\mathcal{T}_\Lambda^{ss}(\lambda)} r_{\mathsf{V}} \varphi_{\mathsf{SV}} \pmod{\mathcal{S}^\lambda}.$$

Sketch of proof. Proposition 4.9 implies that these elements give a basis of $\mathscr{S}(\Lambda)$. Using Theorem 3.8 it is not hard to see that the semistandard basis is cellular. □

In particular, notice that $\mathscr{S}(\Lambda)$ is always free as an R-module and that its rank is independent of R, q and $\mathbf{Q}$. The semistandard basis of $\mathscr{S}(\Lambda)$ really comes from Theorem 4.5 and the basis element φ_{ST} really comes from a Specht filtration of M^μ.

It is worthwhile explaining how the multiplication in $\mathscr{S}(\Lambda)$ is determined. Suppose that $\mathsf{S}, \mathsf{T}, \mathsf{U}$ and V are semistandard tableaux and suppose that $\nu = \mathrm{Type}(\mathsf{V})$ and $\mu = \mathrm{Type}(\mathsf{U})$. Then $m_{\mathsf{UV}} = m_\mu h_{\mathsf{UV}}$, for some $h_{\mathsf{UV}} \in \mathscr{H}$, and

$$\varphi_{\mathsf{ST}}\varphi_{\mathsf{UV}} = \sum_{\mathsf{A},\mathsf{B}} r_{\mathsf{AB}}\varphi_{\mathsf{AB}},$$

where the scalars $r_{\mathsf{AB}} \in R$ are determined by $m_{\mathsf{ST}}h_{\mathsf{UV}} = \sum r_{\mathsf{AB}}m_{\mathsf{AB}}$; this makes sense by Proposition 4.7 and is proved by evaluating the functions on both sides at m_ν. Note, in particular, that $r_{\mathsf{AB}} = 0$ unless $\mathrm{Type}(\mathsf{U}) = \mathrm{Type}(\mathsf{T})$, $\mathrm{Type}(\mathsf{A}) = \mathrm{Type}(\mathsf{S})$ and $\mathrm{Type}(\mathsf{B}) = \mathrm{Type}(\mathsf{V})$. In Theorem 4.11(ii), $r_{\mathsf{V}} = r_{\mathsf{SV}}$.

With some work it is possible to show that when $r = 1$ this basis agrees with Richard Green's codeterminant basis of the q-Schur algebra [**68**]; see also [**67, 123**]. When $r = 2$ and $\mathscr{H}$ is symmetric Theorem 4.11 is equivalent to a theorem of Du and Scott [**56**].

4.3. Weyl modules for cyclotomic q-Schur algebras

By the semistandard basis theorem $\mathscr{S}(\Lambda)$ is a cellular algebra. Therefore, exactly as in Definition 3.10 we can write down a collection of *cell modules* for $\mathscr{S}(\Lambda)$ and, up to isomorphism, every irreducible $\mathscr{S}(\Lambda)$-module is a quotient of one of these modules.

Definition 4.12. *Suppose that $\lambda \in \Lambda^+$ is a multipartition. The* Weyl module W^λ *is the free R-module with basis $\{\, \varphi_{\mathsf{T}} \mid \mathsf{T} \in \mathcal{T}^{ss}_\Lambda(\lambda)\,\}$ on which $\varphi \in \mathscr{S}(\Lambda)$ acts via*

$$\varphi_{\mathsf{T}}\varphi = \sum_{\mathsf{V} \in \mathcal{T}^{ss}_\Lambda(\lambda)} r_{\mathsf{V}}\varphi_{\mathsf{V}},$$

where the scalars $r_{\mathsf{V}} \in R$ are as in Theorem 4.11(ii).

It follows from Theorem 4.11 that W^λ is a right $\mathscr{S}(\Lambda)$-module. As with the Specht modules we define a bilinear form on W^λ by requiring that $\langle\varphi_{\mathsf{S}}, \varphi_{\mathsf{T}}\rangle\varphi_{\mathsf{UV}} \equiv \varphi_{\mathsf{US}}\varphi_{\mathsf{TV}} \pmod{\mathcal{S}^\lambda}$ for semistandard tableaux

$\mathsf{S},\mathsf{T},\mathsf{U},\mathsf{V} \in \mathcal{T}^{ss}_{\Lambda}(\lambda)$. Then the radical of this form, $\operatorname{Rad} W^\lambda$, is a submodule of W^λ and we define $L^\lambda = W^\lambda/\operatorname{Rad} W^\lambda$.

Exactly as in Theorem 3.12, the theory of cellular algebras now gives us the following.

Theorem 4.13. *Suppose that R is a field.*

(i) *For each $\lambda \in \Lambda^+$, L^λ is either zero or an absolutely irreducible $\mathscr{S}(\Lambda)$-module.*
(ii) *$\{\, L^\lambda \mid \lambda \in \Lambda^+ \text{ and } L^\lambda \neq 0 \,\}$ is a complete set of pairwise non-isomorphic irreducible $\mathscr{S}(\Lambda)$-modules.*
(iii) *$\mathscr{S}(\Lambda)$ is semisimple if and only if $L^\lambda = W^\lambda$ for all $\lambda \in \Lambda^+$.*
(iv) *Suppose that $\mu, \lambda \in \Lambda^+$ and $L^\lambda \neq 0$. Then $[W^\mu{:}L^\lambda] \neq 0$ only if $\mu \trianglerighteq \lambda$; moreover, $[W^\lambda{:}L^\lambda] = 1$.*

At this level of generality, determining exactly when L^λ is non-zero is a difficult task. To see this notice that if $\Lambda = \{\omega\}$ then $\mathscr{S}(\Lambda) = \operatorname{End}_{\mathscr{H}}(\mathscr{H}) \cong \mathscr{H}$ and Λ^+ is the set of all partitions of n; so Theorem 3.24 is a special case of Theorem 4.13. When the poset Λ is saturated (that is, $\Lambda^+ \subseteq \Lambda$) we can say much more.

Assume now that $\Lambda^+ \subseteq \Lambda$ and let λ be a multipartition of n. Then M^λ is a summand of $\oplus_{\mu\in\Lambda} M^\mu$ and so the identity map $\varphi_\lambda : M^\lambda \longrightarrow M^\lambda$ is an element of $\mathscr{S}(\Lambda)$. Indeed, looking at the definitions, $\varphi_\lambda = \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda}$, where $\mathsf{T}^\lambda = \lambda(\mathsf{t}^\lambda)$ is the unique semistandard λ-tableau of type λ. It follows that the Weyl module W^λ is isomorphic to the submodule of $\mathscr{S}(\Lambda)/\mathcal{S}^\lambda$ generated by $\varphi_\lambda + \mathcal{S}^\lambda$, the isomorphism being given by

$$\varphi_{\mathsf{T}} \longmapsto \varphi_{\mathsf{T}^\lambda\mathsf{T}} + \mathcal{S}^\lambda = (\varphi_\lambda + \mathcal{S}^\lambda)\varphi_{\mathsf{T}^\lambda\mathsf{T}},$$

for all $\mathsf{T} \in \mathcal{T}^{ss}_{\Lambda}(\lambda)$.

Theorem 4.14. *Suppose that R is a field and that $\Lambda^+ \subseteq \Lambda$.*

(i) *L^λ is a non-zero absolutely irreducible $\mathscr{S}(\Lambda)$-module for all $\lambda \in \Lambda^+$.*
(ii) *$\mathscr{S}(\Lambda)$ is a quasi-hereditary algebra.*

Sketch of proof. To prove (i) observe that $\varphi_\lambda = \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda}$ is an element of $\mathscr{S}(\Lambda)$ because $\lambda \in \Lambda$. Therefore, $\varphi_{\mathsf{T}^\lambda} \in W^\lambda$ and so

$$\langle \varphi_{\mathsf{T}^\lambda}, \varphi_{\mathsf{T}^\lambda}\rangle \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda} \equiv \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda}\varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda} = \varphi_{\mathsf{T}^\lambda\mathsf{T}^\lambda} \pmod{\mathcal{S}^\lambda};$$

hence, $\langle \varphi_{\mathsf{T}^\lambda}, \varphi_{\mathsf{T}^\lambda}\rangle = 1$ and $\varphi_{\mathsf{T}^\lambda} \notin \operatorname{Rad} W^\lambda$; so $L^\lambda \neq 0$. Part (ii) follows from (i) and the structure of cellular algebras. □

Parshall and Wang [**109**] were the first to show that the q-Schur algebras are quasi-hereditary. More generally, the argument above shows that the q-Schur algebras and the cyclotomic Schur algebras are integrally quasi-hereditary in the sense of [**50**].

As the example $\Lambda = \{\omega\}$ indicates, when Λ is not saturated the classification of the simple $\mathscr{S}(\Lambda)$-modules is non-trivial. Nor are there obvious necessary and sufficient conditions for when $\mathscr{S}(\Lambda)$ is quasi-hereditary. The answers to these questions will depend on Λ, R and the parameters $q, Q_1, \dots, Q_r$.

The final result of this section is the analogue of Theorem 3.15 for the cyclotomic Schur algebras. In [**44**] a general version of the result below is proved for an arbitrary finite set of (saturated) multicompositions; we state only a special case in order to avoid introducing additional notation.

Let $\Lambda_{r,n}$ be the set of all multicompositions of n of length at most n and let $\Lambda^+_{r,n} \subseteq \Lambda_{r,n}$ be the set of multipartitions of n. We write $\mathscr{S}(\Lambda_{r,n}) = \mathscr{S}_{q,\mathbf{Q}}(\Lambda_{r,n})$ to emphasize the choice of parameters.

Theorem 4.15 (Dipper-Mathas [**44**]). *Suppose that R is an integral domain and let $\mathbf{Q} = \mathbf{Q}_1 \coprod \cdots \coprod \mathbf{Q}_\kappa$ be a partition of $\mathbf{Q}$ and suppose that the polynomial $P_n(q, \mathbf{Q}_1, \dots, \mathbf{Q}_\kappa)$ is invertible in R. Let $r_\alpha = |\mathbf{Q}_\alpha|$, for $1 \le \alpha \le \kappa$. Then $\mathscr{S}(\Lambda_{r,n})$ is Morita equivalent to the R-algebra*

$$\bigoplus_{\substack{n_1,\dots,n_\kappa \ge 0\\ n_1+\dots+n_\kappa = n}} \mathscr{S}_{q,\mathbf{Q}_1}(\Lambda_{r_1,n_1}) \otimes \cdots \otimes \mathscr{S}_{q,\mathbf{Q}_\kappa}(\Lambda_{r_\kappa,n_\kappa}).$$

This result is deduced from Theorem 3.15 using the theory of Young modules for Ariki-Koike algebras [**105**]. In the special case when $|\mathbf{Q}_\alpha| = 1$, for all α, this was first proved by Ariki [**5**]; see also [**54**].

§5. The representation theory of cyclotomic q-Schur algebras

This chapter gives a summary of the main results in the representation theory of the cyclotomic q-Schur algebras. All of these results are generalizations of theorems for the q-Schur algebras.

5.1. A Schur functor and double centralizer property

Throughout this section we assume that $\omega \in \Lambda$; because of this $\varphi_\omega = \varphi_{\mathsf{T}^\omega\mathsf{T}^\omega}$ is an element of $\mathscr{S}(\Lambda)$. Now, φ_ω is the identity map on $\mathscr{H}$; in particular, it is an idempotent. Moreover, it is easy to see that $\mathscr{H} \cong \varphi_\omega \mathscr{S}(\Lambda)\varphi_\omega$. Hence, by general nonsense (see, for example [**27, 65**]), φ_ω gives rise to a functor Φ_ω from the category of $\mathscr{S}(\Lambda)$-modules to the category of $\mathscr{H}$-modules; explicitly, if M is a right $\mathscr{S}(\Lambda)$-module then $\Phi_\omega(M) = M\varphi_\omega$ is a right $\mathscr{H}$-module.

Notice that the condition $\omega \in \Lambda$ is the analogue for the cyclotomic Schur algebras of the familiar requirement that $d \geq n$ for the q-Schur algebra $\mathscr{S}_q(d;n)$.

Theorem 5.1 (The cyclotomic Schur functor [**81**]). *Suppose that R is a field and that $\omega \in \Lambda^+ \subseteq \Lambda$. Let $\lambda \in \Lambda^+$. Then, as right $\mathscr{H}$-modules,*

(i) $\Phi_\omega(W^\lambda) \cong S^\lambda$;
(ii) $\Phi_\omega(L^\lambda) \cong D^\lambda$.

Furthermore, if $D^\mu \neq 0$ then $[W^\lambda{:}L^\mu] = [S^\lambda{:}D^\mu]$.

Sketch of proof. This can be proved either by general arguments as in [**65**]. Alternatively, from the definitions and the semistandard basis theorem it is clear that $\Phi_\omega(W^\lambda) \cong S^\lambda$ (if $\mathsf{T} \in \mathcal{T}_\mu^{ss}(\lambda)$ then $\varphi_\mathsf{T}\varphi_\omega = \delta_{\mu\omega}\varphi_\mathsf{T}$). Next observe that if $\mathfrak{s}$ and $\mathfrak{t}$ are standard tableaux then the definition of the inner product on W^λ is that

$$\langle \varphi_\mathfrak{s}, \varphi_\mathfrak{t} \rangle \varphi_\lambda \equiv \varphi_{\mathfrak{t}^\lambda \mathfrak{s}} \varphi_{\mathfrak{t}\mathfrak{t}^\lambda} \pmod{\mathcal{S}^\lambda}.$$

Evaluating the functions on both sides at m_λ we find that

$$\langle \varphi_\mathfrak{s}, \varphi_\mathfrak{t} \rangle m_\lambda \equiv m_{\mathfrak{t}^\lambda \mathfrak{s}} m_{\mathfrak{t}\mathfrak{t}^\lambda} \equiv \langle m_\mathfrak{s}, m_\mathfrak{t} \rangle m_\lambda \pmod{\mathcal{H}^\lambda}.$$

Hence, $\langle \varphi_\mathfrak{s}, \varphi_\mathfrak{t} \rangle = \langle m_\mathfrak{s}, m_\mathfrak{t} \rangle$ and the remaining claims follow. □

An important consequence of Theorem 5.1 is that the decomposition matrix of $\mathscr{H}$ is a submatrix of the decomposition matrix of $\mathscr{S}(\Lambda)$.

Corollary 5.2. *Suppose that R is a field and that $\omega \in \Lambda^+ \subseteq \Lambda$. Then the decomposition matrix of $\mathscr{H}$ is the submatrix of the decomposition matrix of $\mathscr{S}(\Lambda)$ obtained by deleting those columns indexed by the multipartitions μ such that $D^\mu = 0$.*

Observe that $\bigoplus_{\lambda \in \Lambda} M^\lambda$ is an $(\mathscr{S}(\Lambda), \mathscr{H})$-bimodule. In fact, each algebra is the full centralizer algebra for the other and we have a cyclotomic analogue of Schur-Weyl duality.

Theorem 5.3 (Double centralizer property). *Suppose that $\omega \in \Lambda$ and that $\Lambda^+ \subseteq \Lambda$. Then*

$$\mathscr{S}(\Lambda) \cong \operatorname{End}_{\mathscr{H}}\Big(\bigoplus_{\lambda\in\Lambda} M^\lambda\Big) \quad \textit{and} \quad \mathscr{H} \cong \operatorname{End}_{\mathscr{S}(\Lambda)}\Big(\bigoplus_{\lambda\in\Lambda} M^\lambda\Big).$$

Sketch of proof. The first isomorphism is just the definition of $\mathscr{S}(\Lambda)$ so there is nothing to prove here. For the second isomorphism for each $\lambda \in \Lambda$ let φ_λ be the identity map on M^λ and let $\mathscr{M}^\lambda = \varphi_\lambda\mathscr{S}(\Lambda)$. (So $\mathscr{M}^\lambda$ is an $\mathscr{S}(\Lambda)$–module and M^λ is an $\mathscr{H}$–module.) Then there an isomorphism of $\mathscr{H}$-modules

$$\bigoplus_{\lambda\in\Lambda} M^\lambda \cong \bigoplus_{\lambda\in\Lambda} \Phi_\omega(\mathscr{M}^\lambda) = \bigoplus_{\lambda\in\Lambda} \varphi_\lambda\mathscr{S}(\Lambda)\varphi_\omega.$$

By definition $\sum_\lambda \varphi_\lambda$ is the identity of $\mathscr{S}(\Lambda)$, so $\mathscr{S}(\Lambda) = \bigoplus_\lambda \varphi_\lambda\mathscr{S}(\Lambda)$ and $\bigoplus_{\lambda\in\Lambda} M^\lambda \cong \mathscr{S}(\Lambda)\varphi_\omega$ as a left $\mathscr{S}(\Lambda)$-module. Therefore,

$$\operatorname{End}_{\mathscr{S}(\Lambda)}\big(\bigoplus_{\lambda\in\Lambda} M^\lambda\big) \cong \operatorname{End}_{\mathscr{S}(\Lambda)}\big(\mathscr{S}(\Lambda)\varphi_\omega\big) \cong \varphi_\omega\mathscr{S}(\Lambda)\varphi_\omega.$$

As $\varphi_\omega\mathscr{S}(\Lambda)\varphi_\omega \cong \mathscr{H}$, this completes the proof. □

5.2. The blocks of the cyclotomic Schur algebras

The centre of the affine Hecke algebra $\hat{H}_n$ is given by the following well-known result of Bernstein.

Theorem 5.4 (Bernstein). *Suppose that R is an algebraically closed field. Then the centre of $\hat{H}_n$ is equal to $R[X_1^\pm,\dots,X_n^\pm]^{\mathfrak{S}_n}$, the R-algebra of symmetric Laurent polynomials in $X_1,\dots,X_n$.*

This is quite straightforward to prove given the Bernstein presentation of $\hat{H}_n$.

Now X_k maps to L_k under the natural surjection $\hat{H}_n \longrightarrow \mathscr{H}_n$ so this implies that any symmetric polynomial in $L_1,\dots,L_n$ belongs to the centre of the Ariki-Koike algebra $\mathscr{H}_n$. As we remarked earlier, in the semisimple case the centre of $\mathscr{H}_n$ is always the algebra of symmetric polynomials in $L_1,\dots,L_n$; however, when $\mathscr{H}_n$ is not semisimple the centre of $\mathscr{H}_n$ can be larger than this. Because of this Theorem 5.5 below is a little surprising. First, some notation.

Given a multipartition λ let $\operatorname{res}(\lambda) = \{\operatorname{res}_{\mathfrak{t}^\lambda}(k) \mid 1 \le k \le n\}$, which we consider as a *multiset*. By the remarks above and Proposition 3.14 if two simple $\mathscr{H}_n$-modules D^λ and D^μ are in the same block then $\operatorname{res}(\lambda) = \operatorname{res}(\mu)$ as multisets.

We also note that because $\mathscr{H}_n$ is a cellular algebra all of the composition factors of S^λ belong to the same block; hence, D^λ and D^μ are in the same block if and only if S^λ and S^μ are in the same block. The same remark applies to the simple modules and the Weyl modules of the cyclotomic q-Schur algebras.

Theorem 5.5. *Suppose that R is an algebraically closed field and that λ and μ are multipartitions of n. Then the following are equivalent.*

(i) $\operatorname{res}(\lambda) = \operatorname{res}(\mu)$ *as multisets.*

(ii) S^λ *and* S^μ *are in the same block as* $\mathscr{H}_n$*-modules.*

(iii) S^λ *and* S^μ *are in the same block as* $\hat{H}_n$*-modules.*

(iv) W^λ *and* W^μ *are in the same block as* $\mathscr{S}(\Lambda_{r,n})$*-modules.*

Sketch of proof. As noted above, the implication (ii)⇒(i) follows from Proposition 3.14; this is was first proved by Graham and Lehrer [**64**] who also conjectured that the converse was true. That (i) and (iii) are equivalent follows from Theorem 5.4.

The hard part is proving that (iii) implies (ii); this was done by Grojnowski [**72**] using his modular branching rule. The key point is that if λ and μ are distinct multipartitions with $D^\lambda \neq 0$, $D^\mu \neq 0$ and $\operatorname{res}(\lambda) = \operatorname{res}(\mu)$ then $\operatorname{Hom}_{\hat{H}_{n-1}}(\operatorname{Res} D^\lambda, \operatorname{Res} D^\mu) = 0$ by Theorem 3.25; here Res is the functor $\operatorname{Res} : \hat{H}_n\text{-}\mathbf{mod} \longrightarrow \hat{H}_{n-1}\text{-}\mathbf{mod}$. Grojnowski shows that this implies that whenever $0 \longrightarrow D^\lambda \longrightarrow X \longrightarrow D^\mu \longrightarrow 0$ is an exact sequence of $\hat{H}_n$-modules then it is still exact when considered as a sequence of $\mathscr{H}_n$-modules (for any $\mathscr{H}_n$-module X). This implies (ii).

Finally, by the double centralizer property (Theorem 5.3), $\mathscr{S}(\Lambda_{r,n})$ and $\mathscr{H}_n$ have the same number of blocks (see [**103**, Cor. 5.38]), so it follows that (ii) and (iv) are equivalent. □

Theorem 5.5 does not classify the blocks of an arbitrary cyclotomic Schur algebra; rather it classifies the blocks of $\mathscr{S}(\Lambda)$ for any Λ with $\Lambda^+_{r,n} \subseteq \Lambda$ (by standard arguments, all of these algebras are Morita equivalent). When $r = 1$ the blocks for the q-Schur algebras $\mathscr{S}_q(d;n)$ with $d \geq n$ were classified by Dipper, James and the author [**38, 80**]; the general case was settled by Cox [**33**]. The classification of the blocks of the cyclotomic q–Schur algebras is still open when $r > 1$ and $\omega \notin \Lambda$.

5.3. The Jantzen sum formula

Throughout this section assume that R is a field and that Λ is saturated. Let t be an indeterminate over R and let $\mathfrak{p}$ be the maximal ideal of $R[t,t^{-1}]$ generated by $t - q$. The localization $\mathcal{O} = R[t,t^{-1}]_\mathfrak{p}$ of $R[t,t^{-1}]$ at $\mathfrak{p}$ is a discrete valuation ring and $R \cong \mathcal{O}/\mathfrak{p}$. Let $\nu_\mathfrak{p}$ be the $\mathfrak{p}$-adic valuation on $\mathcal{O}$.

Let $\mathscr{H}_\mathcal{O}$ be the Hecke algebra over $\mathcal{O}$ with parameters qt and $U_s = Q_s t^{ns}$ if $Q_s \neq 0$ and $U_s = (t^{ns} - 1)$ if $Q_s = 0$. Then $\mathscr{H}_{R(t)} = \mathscr{H}_\mathcal{O} \otimes R(t)$ is semisimple by Corollary 3.3 and $\mathscr{H}_R = \mathscr{H}_{q,\mathbf{Q}}(W_{r,n}) \cong \mathscr{H}_\mathcal{O} \otimes_\mathcal{O} R$ is the reduction of $\mathscr{H}_\mathcal{O}$ modulo $\mathfrak{p}$. Let $\mathscr{S}_\mathcal{O}(\Lambda)$, $\mathscr{S}_{R(t)}(\Lambda)$ and $\mathscr{S}_R(\Lambda)$ be the corresponding cyclotomic q-Schur algebras.

Define the $\mathcal{O}$-residue of a node $x = (i, j, s)$ to be $\operatorname{res}_{\mathcal{O}}(x) = (qt)^{j-i}U_s$, an element of $\mathcal{O}$. The connection with our previous definition of residue is that $\operatorname{res}(x) = \operatorname{res}_{\mathcal{O}}(x) \otimes_{\mathcal{O}} 1_R$.

Let λ be a multipartition and for each node $x = (i, j, s) \in [\lambda]$ let $r_x \subseteq [\lambda]$ be the corresponding rim hook (so r_x is a rim hook in $[\lambda^{(s)}]$); then $[\lambda] \setminus r_x$ is the diagram of a multipartition. Let $\ell\ell(r_x)$ be the leg length of r_x and define $\operatorname{res}_{\mathcal{O}}(r_x) = \operatorname{res}_{\mathcal{O}}(f_x)$ where f_x is the foot node of r_x. These definitions can be found in [**79**, **103**].

Suppose that λ and μ are multipartitions of n. If $\lambda \not\rhd \mu$ let $g_{\lambda\mu} = 1$; otherwise set

$$g_{\lambda\mu} = \prod_{x\in[\lambda]} \prod_{\substack{y\in[\mu] \\ [\mu]\setminus r_y=[\lambda]\setminus r_x}} \big(\operatorname{res}_{\mathcal{O}}(r_x) - \operatorname{res}_{\mathcal{O}}(r_y)\big)^{\varepsilon_{xy}},$$

where $\varepsilon_{xy} = (-1)^{\ell\ell(r_x)+\ell\ell(r_y)}$. The scalars $g_{\lambda\mu} \in \mathcal{O}$ have a combinatorial interpretation in terms of moving rim hooks in the diagram of a multipartition; see [**81**, Example 3.39].

Finally, let $W^\lambda_{\mathcal{O}}$ and W^λ_R be the Weyl modules for $\mathscr{S}_{\mathcal{O}}(\Lambda)$ and $\mathscr{S}_R(\Lambda)$ respectively; note that $W^\lambda_R \cong W^\lambda_{\mathcal{O}} \otimes_{\mathcal{O}} R$ as R-modules. For each $i \geq 0$ define

$$W^\lambda_{\mathcal{O}}(i) = \{\, x \in W^\lambda_{\mathcal{O}} \mid \langle x, y\rangle \in \mathfrak{p}^i \text{ for all } y \in W^\lambda_{\mathcal{O}} \,\}$$

and set $W^\lambda_R(i) = \big(W^\lambda_{\mathcal{O}}(i) + \mathfrak{p}W^\lambda_{\mathcal{O}}\big)/\mathfrak{p}W^\lambda_{\mathcal{O}}$. The Jantzen filtration of W^λ_R is

$$W^\lambda_R = W^\lambda_R(0) \geq W^\lambda_R(1) \geq W^\lambda_R(2) \geq \cdots.$$

In particular, note that $\operatorname{Rad} W^\lambda_R = W^\lambda_R(1)$; consequently, $W^\lambda_R(1)$ is a proper submodule of $W^\lambda_R(0)$ and $W^\lambda_R(0)/W^\lambda_R(1) \cong L^\lambda_R$. Note also that $W^\lambda_R(k) = 0$ for $k \gg 0$.

Actually, what we have just given is a special case of the definition of a Jantzen filtration. More generally, the same construction gives a Jantzen filtration for any suitable modular system $(K, \mathcal{O}, \mathfrak{p})$ (with parameters). The point of this remark is that the Jantzen filtration of W^λ_R depends upon a non-canonical choice of modular system.

We can now state the analogue of Jantzen's sum formula for $\mathscr{S}_R(\Lambda)$.

Theorem 5.6 (James-Mathas [**81**, Theorem 4.6]). *Let λ be a multipartition of n. Then*

$$\sum_{i>0}[W^\lambda_R(i)] = \sum_{\mu:\lambda\rhd\mu} \nu_{\mathfrak{p}}(g_{\lambda\mu})[W^\mu_R].$$

*in the Grothendieck group $K_0(\mathscr{S}_R(\Lambda)$-**mod**$)$ of $\mathscr{S}_R(\Lambda)$.*

When $r = 1$ this result describes the Jantzen filtration of the Weyl modules of the q-Schur algebra. The Weyl modules of the q-Schur algebra coincide with the Weyl modules of quantum $\mathfrak{gl}_d$; therefore, when $r = 1$ Theorem 5.6 is a special case of a result of Andersen, Polo and Wen [**1**] who proved the analogue of the Jantzen sum formula for the quantum groups of finite type as a consequence of Kempf's vanishing theorem. For a combinatorial proof which takes place inside the q-Schur algebra see [**80, 103**]. When $r > 1$ there is no geometry to work with. The argument of [**81**] generalizes that of [**80**].

The idea behind the proof of Theorem 5.6 is quite simple. First, for each μ compute the determinant of the Gram matrix $G^\lambda_\mu = (\langle\varphi_\mathsf{S}, \varphi_\mathsf{T}\rangle)$, $\mathsf{S}, \mathsf{T} \in \mathcal{T}^{\mathrm{ss}}_\mu(\lambda)$, of the μ-weight space $W^\lambda_\mathcal{O}\varphi_\mu$ of $W^\lambda_\mathcal{O}$. It turns out that $\det G^\lambda_\mu = g_{\lambda\mu}$. Now, the inner product $\langle\ ,\ \rangle$ on $W^\lambda_\mathcal{O}$ is non-degenerate; so Jantzen's elementary, yet fundamental, lemma says that

$$\sum_{i>0} \dim_R W^\lambda_R(i)\varphi_\mu = \nu_\mathfrak{p}(\det G^\lambda_\mu).$$

This is enough to deduce the result because, by Theorem 4.13(iv), any $\mathscr{S}(\Lambda)$-module is uniquely determined by the dimensions of its weight spaces since $\dim W^\nu\varphi_\nu = 1$ and $W^\mu\varphi_\nu \neq 0$ only if $\mu \trianglerighteq \nu$, for all $\nu \in \Lambda^+$.

Of course, computing $\det G^\lambda_\mu$ is not so easy. This is accomplished using an orthogonal basis of W^λ_R when $P_\mathscr{H}(q, \mathbf{Q}) \neq 0$. With this basis the Gram determinant is easier to calculate because almost all inner products are zero (we are really computing inner products in $W^\lambda_{R(t)}$). The orthogonal basis is constructed using a family of operators which act in a triangular fashion on the semistandard basis of the Weyl modules; intuitively, these operators belong to something like a Cartan subalgebra of $\mathscr{S}(\Lambda)$ — in fact, they are 'lifts' of the elements L_k to $\mathscr{S}(\Lambda)$.

The definition of the Jantzen filtration only requires a finitely generated $\mathcal{O}$-module which possesses a non-degenerate bilinear form. The same construction gives a Jantzen filtration $S^\lambda = S^\lambda_R(0) \geq S^\lambda_R(1) \geq \cdots$ for each Specht module; equivalently, by the proof of Theorem 5.1, we can set $S^\lambda_R(i) = \Phi_\omega(W^\lambda_R(i))$. Applying the Schur functor to Theorem 5.6 yields the following.

Corollary 5.7 (James-Mathas [**81**]). *Let λ be a multipartition of n. Then*

$$\sum_{i>0}[S^\lambda_R(i)] = \sum_{\mu:\lambda\triangleright\mu} \nu_\mathfrak{p}(g_{\lambda\mu})[S^\mu_R].$$

*in the Grothendieck group $K_0(\mathscr{H}_R$-**mod**$)$ of $\mathscr{H}_R$.*

For the symmetric groups (that is, $r = 1$ and $q = 1$) this is a result of long standing known as Schaper's Theorem [**114**]. Schaper's argument is a translation of the Jantzen sum formula for the Weyl modules of the general linear group [**82**] (phrased in terms of the dot action of the symmetric group upon the weight lattice of GL_n), into the combinatorial language of the symmetric group. It is worth remarking that the corresponding result for the Weyl groups of type B (i.e., $r = 2$ and $q = 1$), was obtained only relatively recently [**81**].

The main application of the cyclotomic sum formula has been a classification of the irreducible Weyl modules and the irreducible Specht modules with $S^\lambda = D^\lambda$; see [**81**]. When $r = 1$ the sum formula was used to complete the classification of the blocks of the q-Schur algebras and the Iwahori-Hecke algebras of type A and also to classify the ordinary irreducible $\mathrm{GL}_n(q)$-modules which remain irreducible when reduced mod p when $p \nmid q$; see [**80**]. Ariki and the author [**10**] have also used the Jantzen sum formula to classify the representation type of the Iwahori-Hecke algebras of type B.

5.4. Connections with quantum groups

For this section only we renormalize the basis of the Ariki-Koike algebras so as to be consistent with the notation in [**5, 113**]. We assume that q has a square root in R and let $q = v^2$. As every field is a splitting field for $\mathscr{H}$ and $\mathscr{S}(\Lambda)$ we are free to extend R so that it contains a square root of q if necessary.

Let $\tilde{T}_i = v^{-1}T_i$ for $1 \le i < n$. Then $T_0, \tilde{T}_1, \dots, \tilde{T}_{n-1}$ still generate $\mathscr{H}$ and they are subject to the same relations as before except that the quadratic relation for the $\tilde{T}_i$ becomes $(\tilde{T}_i - v)(\tilde{T}_i + v^{-1}) = 0$, for $1 \le i < n$. Observe that $L_k = \tilde{T}_{k-1} \dots \tilde{T}_1 T_0 \tilde{T}_1 \dots \tilde{T}_{k-1}$ for $k = 1, \dots, n$.

Fix an integer $d \ge 1$ and let $U_v(\mathfrak{gl}_d)$ be the quantized enveloping algebra of $\mathfrak{gl}_d$. Thus, $U_v(\mathfrak{gl}_d)$ is an associative $\mathbb{Q}(v)$-algebra which is generated by elements $E_i, F_i, K_j^{\pm}$, where $1 \le i < n$ and $1 \le j \le n$, which are subject to the quantum Serre relations.

Let V be a d dimensional $\mathbb{Q}(v)$-vector space with basis $\{e_1, \dots, e_d\}$. Then V is naturally a $U_v(\mathfrak{gl}_d)$-module, where the action of $U_v(\mathfrak{gl}_d)$ on V is determined by

$$E_i e_a = \delta_{a,i+1} e_{a-1}, \quad F_i e_a = \delta_{a,i} e_{a+1}, \quad \text{and} \quad K_j e_a = v^{\delta_{j,a}} e_a,$$

for $1 \le i < n$, $1 \le j \le n$ and $1 \le a \le n$. Now, $U_v(\mathfrak{gl}_d)$ is a Hopf algebra with coproduct Δ given by $\Delta(K_j) = K_j \otimes K_j$,

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i \quad \text{and} \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

for $1 \le i < n$ and $1 \le j \le n$. Therefore, $V^{\otimes n}$ is a $U_v(\mathfrak{gl}_d)$-module; let $\rho_n : U_v(\mathfrak{gl}_d) \longrightarrow \operatorname{End}(V^{\otimes n})$ be the corresponding representation.

Let $I(d;n) = \{\, (a_1, \dots, a_n) \mid 1 \le a_1, \dots, a_n \le d \,\}$. If $\mathbf{a} \in I(d;n)$ let $e_{\mathbf{a}} = e_{a_1} \otimes \cdots \otimes e_{a_n}$. Then $\{\, e_{\mathbf{a}} \mid \mathbf{a} \in I(d;n) \,\}$ is a basis of $V^{\otimes n}$.

The symmetric group $\mathfrak{S}_n$ also acts on $V^{\otimes n}$ by place permutations and it acts on $I(d;n)$ by permuting components; indeed, $e_{\mathbf{a}} w = e_{\mathbf{a}w}$ for $\mathbf{a} \in I(d;n)$ and $w \in \mathfrak{S}_n$. Jimbo showed how to deform the action of $\mathfrak{S}_n$ to give an action of $\mathscr{H}_q(\mathfrak{S}_n)$ on $V^{\otimes n}$.

Recall that $\mathfrak{S}_n$ is generated by $t_1, \dots, t_{n-1}$, where $t_i = (i, i+1)$ for $i = 1, \dots, n-1$. Let $\Lambda^+(d;n)$ be the set of partitions in $\Lambda(d;n)$.

Theorem 5.8 (Jimbo [**84**]). *Assume that $\mathscr{H}_q(\mathfrak{S}_n)$ and $U_v(\mathfrak{gl}_d)$ are defined over $\mathbb{Q}(v)$.*

(i) *There is a unique $\mathscr{H}_q(\mathfrak{S}_n)$-module structure on $V^{\otimes n}$ such that*

$$e_{\mathbf{a}} \tilde{T}_j = \begin{cases} v e_{\mathbf{a}}, & \text{if } a_j = a_{j+1}, \\ e_{\mathbf{a}t_j}, & \text{if } a_j > a_{j+1}, \\ e_{\mathbf{a}t_j} + (v - v^{-1}) e_{\mathbf{a}}, & \text{if } a_j < a_{j+1} \end{cases}$$

for $j = 1, \dots, n-1$ and $\mathbf{a} \in I(d;n)$.

(ii) *The algebras $\mathscr{H}_q(\mathfrak{S}_n)$ and $\rho_n\big(U_v(\mathfrak{gl}_d)\big)$ are mutually the full centralizer algebras for each other for their actions on $V^{\otimes n}$. Moreover,*

$$V^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda^+(d;n)} W^\lambda \otimes S^\lambda$$

as an $\big(\mathscr{S}_q(d;n), \mathscr{H}_q(\mathfrak{S}_n)\big)$-bimodule.

It is not hard to see that there is an isomorphism $V^{\otimes n} \cong \oplus_{\lambda \in \Lambda(d;n)} M^\lambda$ of $\mathscr{H}$-modules; consequently, $\rho_n\big(U_v(\mathfrak{gl}_d)\big) \cong \mathscr{S}_q(d;n)$. In part (ii), W^λ is a Weyl module for $U_v(\mathfrak{gl}_d)$; by what we have just said this is the same as a Weyl module for the q-Schur algebra $\mathscr{S}(d;n)$.

Actually, this is a slight modification of Jimbo's original action of $\mathscr{H}_{v^2}(\mathfrak{S}_n)$ on $V^{\otimes n}$; this action comes from Du-Parshall-Wang [**52**].

The proof of Theorem 5.8 is straightforward. Checking the relations it is easy to see that $V^{\otimes n}$ is an $\mathscr{H}$-module and that the actions of $\mathscr{H}$ and $U_v(\mathfrak{gl}_d)$ commute. The double centralizer property can be proved using a highest weight argument to decompose $V^{\otimes n}$ as a $U_v(\mathfrak{gl}_d)$-module.

Notice that Theorem 5.8 is stated over the rational function field $\mathbb{Q}(v)$. Using the work of Beĭlinson, Lusztig and MacPherson [**15**], Du [**48**] showed that when $d \ge n$ Theorem 5.8 holds over the Laurent polynomial ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, where we replace $U_v(\mathfrak{gl}_d)$ with its Lusztig

$\mathcal{A}$-form $U_{\mathcal{A}}(\mathfrak{gl}_d)$; for the case $d < n$ see [**50**]. Further, if $d \ge n$ then $\mathscr{H}_q(\mathfrak{S}_n) \cong \operatorname{End}_{U_{\mathcal{A}}(\mathfrak{gl}_d)}(V^{\otimes n})$.

We remark that Doty and Giaquinto [**47**] have recently used the surjection $U(\mathfrak{gl}_d) \longrightarrow \mathscr{S}_q(d;n)$ to give a presentation of the q-Schur algebras over $\mathbb{Q}(v)$; see also [**49**]. No such presentation is known for the cyclotomic q-Schur algebras.

Now we indicate how Sakamoto and Shoji [**113**] have generalized Theorem 5.8 to the cyclotomic case. We extend the coefficient ring for all our algebras to the rational function field $\mathbb{Q}(v, Q_1, \dots, Q_r)$, where $v, Q_1, \dots, Q_r$ are indeterminates.

Fix positive integers $d_1, \dots, d_r$ with $d = d_1 + \dots + d_r$ and let $\gamma: \{1, \dots, d\} \longrightarrow \{1, \dots, r\}$ be the map such that $\gamma(a) = s$ if s is minimal such that $a \le d_1 + \dots + d_s$ and let V_s be the subspace of V with basis $\{ e_a \mid \gamma(a) = s \}$, for $1 \le s \le r$, and let $\mathfrak{g} = \mathfrak{gl}_{d_1}(V_1) \oplus \dots \oplus \mathfrak{gl}_{d_r}(V_r)$. We consider $U_v(\mathfrak{g})$ as a Levi subalgebra of $U_v(\mathfrak{gl}_d)$ in the natural way. Then $V^{\otimes n}$ is a $U_v(\mathfrak{g})$-module by restriction; let $\rho_{n,r}: U_v(\mathfrak{g}) \longrightarrow \operatorname{End}(V^{\otimes n})$ be the corresponding representation of $U_v(\mathfrak{g})$.

In order to extend the action of $\mathscr{H}_q(\mathfrak{S}_n)$ on $V^{\otimes n}$ to an action of $\mathscr{H}$ define linear operators ϖ and S_j on $V^{\otimes n}$ by

$$e_{\mathbf{a}}\varpi = Q_{\gamma(a_1)} e_{\mathbf{a}} \quad \text{and} \quad e_{\mathbf{a}} S_j = \begin{cases} e_{\mathbf{a}} \tilde{T}_j, & \text{if } \gamma(a_{j-1}) = \gamma(a_j), \\ e_{\mathbf{a}t_j}, & \text{otherwise,} \end{cases}$$

for $\mathbf{a} \in I(d;n)$ and $1 \le j < n$.

Let $\Lambda(d_1, \dots, d_r; n)$ be the set of multicompositions λ of n such that $|d_s| = |\lambda^{(s)}|$ for $1 \le s \le r$ and let $\Lambda^+(d_1, \dots, d_r; n)$ be the set of multipartitions in $\Lambda(d_1, \dots, d_r; n)$. We warn the reader that $\Lambda^+(d_1, \dots, d_r; n) \ne \big(\Lambda(d_1, \dots, d_r; n)\big)^+$, in the sense of Definition 4.8 — unless $d_s \ge n$ for $1 \le s < r$.

The irreducible representations of $U_v(\mathfrak{g})$ can be parametrized by multipartitions in $\Lambda^+(d_1, \dots, d_r; n)$ for $n \ge 0$. If $\lambda \in \Lambda^+(d_1, \dots, d_r; n)$ let $W(\lambda)$ be the corresponding Weyl module for $U_v(\mathfrak{g})$. If $r = 1$ then $W(\lambda) \cong W^\lambda$ as $\mathscr{S}_q(d;n)$-modules; however, in general, W^λ and $W(\lambda)$ are not isomorphic even as vector spaces.

Theorem 5.9 (Sakamoto and Shoji [**113**]). *Assume that $\mathscr{H}$ and $U_v(\mathfrak{g})$ are defined over the field $\mathbb{Q}(v, Q_1, \dots, Q_r)$.*

(i) *The action of $\mathscr{H}_q(\mathfrak{S}_n)$ on $V^{\otimes n}$ extends to give an action of $\mathscr{H}$ on $V^{\otimes n}$ via*

$$e_{\mathbf{a}} T_0 = e_{\mathbf{a}} \varpi S_1 \dots S_{n-1} \tilde{T}_{n-1}^{-1} \dots \tilde{T}_1^{-1}$$

for all $\mathbf{a} \in I(d;n)$.

(ii) *The algebras $\mathscr{H}$ and $\mathscr{S}_n^{(r)} \cong \rho_{n,r}(U_v(\mathfrak{g}))$ are mutually the full centralizer algebras for the others action on $V^{\otimes n}$. Moreover,*

$$V^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda^+(d_1,\ldots,d_r;n)} W(\lambda) \otimes S^\lambda$$

as an $(\mathscr{S}_n^{(r)}, \mathscr{H})$-bimodule.

Sakamoto and Shoji were guided in part by Ariki, Terasoma and Yamada [**11**] who considered the special case when $d_1 = \cdots = d_r = 1$. The proof of part (i) of the theorem is a long calculation building on Theorem 5.8(i). Once again, part (ii) is a highest weight computation.

Sakamoto and Shoji also note that part (i) of Theorem 5.9 is true over an arbitrary integral domain. Using this observation they gave another proof that $\mathscr{H}$ is free of rank $|W_{r,n}|$ (Theorem 2.2).

Ariki [**5**] asked whether Theorem 5.9(ii) is true over an arbitrary field; he was particularly interested in knowing when the dimension of $\rho_{r,n}(U_v(\mathfrak{g})) = \operatorname{End}_{\mathscr{H}}(V^{\otimes n})$ is independent of R, q and $\mathbf{Q}$. Ariki found an example which showed that in general the dimension of $\rho_{r,n}(U_v(\mathfrak{g}))$ does depend upon these choices; nonetheless, he was able to prove the result below.

Let $U_{\mathcal{A}}(\mathfrak{g})$ be the Kostant-Lusztig $\mathcal{A}$-form of $U_v(\mathfrak{g})$ and set $U_{R,v}(\mathfrak{g}) = U_v(\mathfrak{g}) \otimes_{\mathcal{A}} R$, where R is an integral domain. We also consider V to be the free R-module with basis $\{e_1, \ldots, e_d\}$. Finally, define $\mathscr{S}_n^{(r)} = \operatorname{End}_{\mathscr{H}}(V^{\otimes n})$.

Theorem 5.10 (Ariki [**5**]). *Suppose that R is an integral domain and that $q = v^2, Q_1, \ldots, Q_r$ are elements of R such that*

$$P_n(q, \mathbf{Q}) = \prod_{1 \le i < j \le r} \prod_{-n < d < n} (q^d Q_i - Q_j)$$

is invertible in R. Then the following hold.

(i) *Suppose that $d_s \ge n$ for all s. Then there is an isomorphism of R-algebras*

$$\mathscr{S}_n^{(r)} \cong \bigoplus_{\substack{n_1,\ldots,n_r \ge 0 \\ n_1+\cdots+n_r = n}} \mathscr{S}_q(d_1; n_1) \otimes \cdots \otimes \mathscr{S}_q(d_r; n_r).$$

In particular, $\mathscr{S}_n^{(r)}$ is free as an R-module and its rank is independent of the choice of R or the parameters $v, Q_1, \ldots, Q_r$.

(ii) *The algebra $\mathscr{S}_n^{(r)}$ is a quotient of $U_{R,v}(\mathfrak{g})$.*

(iii) *Assume that* $d_s \geq n$ *for* $1 \leq s \leq r$. *Then* $\operatorname{End}_{U_{R,v}(\mathfrak{g})}(V^{\otimes n})$ *is Morita equivalent to the algebra*

$$\bigoplus_{\substack{n_1,\dots,n_r\geq 0\\ n_1+\dots+n_r=n}} \mathscr{H}_q(\mathfrak{S}_{n_1})\otimes\cdots\otimes\mathscr{H}_q(\mathfrak{S}_{n_r}).$$

Observe that $P_n(q,\mathbf{Q})$ is equal to the polynomial $P_n(q,\mathbf{Q}_1,\dots,\mathbf{Q}_r)$ of Theorem 3.15 (with $\mathbf{Q}_\alpha = \{Q_\alpha\}$, for all α). Assume that $d_s \geq n$ for all s. Then, by part (i) and Theorem 4.15, the algebra $\mathscr{S}_n^{(r)}$ is Morita equivalent to the cyclotomic Schur algebra $\mathscr{S}(\Lambda_{r,n})$. Similarly, by part (iii) and Theorem 3.15, if $d_s \geq n$ for all s then $\operatorname{End}_{U_{R,v}}(V^{\otimes n})$ is Morita equivalent to $\mathscr{H}$. Hence, up to Morita equivalence, we have a complete analogue of Schur-Weyl duality linking $U_v(\mathfrak{g})$, $\mathscr{S}(\Lambda_{r,n})$ and $\mathscr{H}$ in this setting; however, note that this is really a type A phenomenon and is not genuinely 'cyclotomic'.

Ariki also uses this result to compute the decomposition matrices of the algebras $\mathscr{S}_n^{(r)}$ when $R=\mathbb{Q}$, $q\neq 1$ and $P_n(q,\mathbf{Q})\neq 0$. To do this he uses part (i) and the LT-conjecture [**95**] which gives an extension of Theorem 3.19(ii) to the q-Schur algebras. The LT-conjecture was proved by Varagnolo and Vasserot [**121**]; see also Schiffmann [**115**].

Combining these results, the decomposition matrices of the cyclotomic Schur algebras are known whenever R is a field of characteristic zero, $q\neq 1$ and $P_n(q,\mathbf{Q})\neq 0$. Actually, we do not need Ariki's work to do this as we already obtain this result from Theorem 4.15 and [**95**, **121**]. (Note that Ariki's paper appeared before [**44**], the source of Theorem 4.15.)

5.5. Borel subalgebras

In this section we show that the cyclotomic Schur algebras admit a "triangular decomposition". For the Schur algebras this is a result of J.A. Green [**66**]; the cyclotomic case is due to Du and Rui [**53**].

For simplicity we consider the case where $\Lambda = \Lambda_{r,n}$ is the set of all multicompositions of n of length at most n. Du and Rui note that the general case can be deduced from this because if Λ is a saturated set of multicompositions then $\mathscr{S}(\Lambda)$ is Morita equivalent to the subalgebra $e\mathscr{S}(\Lambda_{r,n})e$ of $\mathscr{S}(\Lambda_{r,n})$, where e is the idempotent $\sum_{\lambda\in\Lambda^+}\varphi_\lambda$.

Recall that $I(rn;n) = \{\,(a_1,\dots,a_n)\mid 1\leq a_i\leq rn\,\}$. Then $\mathfrak{S}_n$ acts on $I(rn;n)$ by place permutations. Given a multicomposition λ in $\Lambda_{r,n}$ let $\overline{\lambda} = (\overline{\lambda}_1,\dots,\overline{\lambda}_{rn})$ be the composition in $\Lambda(rn;n)$ with $\overline{\lambda}_i = \lambda_j^{(s)}$

if $i = (s-1)n + j$. Define

$$\mathbf{i}_\lambda = (i_{\lambda,1}, \dots, i_{\lambda,n}) = (\underbrace{1,\dots,1}_{\overline{\lambda}_1},\underbrace{2,\dots,2}_{\overline{\lambda}_2}\dots,\underbrace{rn,\dots,rn}_{\overline{\lambda}_{rn}}) \in I(rn;n).$$

Let $\succeq$ be the partial order on $I(rn;n)$ given by $\mathbf{a} \succeq \mathbf{b}$ if $a_k \geq b_k$ for $1 \leq k \leq rn$. Note that for any $d \in \mathfrak{S}_n$ if λ and μ are multicompositions with $\mathbf{i}_\lambda d \succeq \mathbf{i}_\mu$ then $\mu \trianglerighteq \lambda$.

Recall that for each multicomposition $\lambda \in \Lambda_{r,n}$ we have a Young subgroup $\mathfrak{S}_\lambda$ and that $\mathcal{D}_\lambda$ is the set of minimal length coset right representatives for $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$. Moreover, if μ is another multicomposition then $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$ is a set of minimal length $(\mathfrak{S}_\lambda, \mathfrak{S}_\mu)$-double coset representatives. For each $d \in \mathcal{D}_{\lambda\mu}$ define $\varphi^d_{\lambda\mu}$ to be the R-linear endomorphism of $\bigoplus_\alpha M^\alpha$ determined by

$$\varphi^d_{\lambda\mu}(m_\alpha h) = \delta_{\alpha\mu}\Big(\sum_{w \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_w\Big) u^+_\mu h$$

for all $\alpha \in \Lambda_{r,n}$ and all $h \in \mathscr{H}$. If $\mathbf{i}_\lambda d \succeq \mathbf{i}_\mu$ then $\varphi^d_{\lambda\mu} \in \mathscr{S}(\Lambda_{r,n})$ by [**53**, Lemma 5.6]. In particular, if $\nu \in \Lambda_{r,n}$ then $\varphi^1_{\nu\nu} = \varphi_\nu$ restricts to the identity map on M^ν (and is zero on M^α for $\alpha \neq \nu$).

Finally, given multicompositions λ and μ in $\Lambda_{r,n}$ let

$$\Omega_{\lambda\mu} = \{\, d \in \mathcal{D}_{\lambda\mu} \mid \mathbf{i}_\lambda d \succeq \mathbf{i}_\mu \,\}.$$

Define $\mathscr{S}^\pm(\Lambda_{r,n})$ to be the two R-submodules of $\mathscr{S}(\Lambda_{r,n})$ spanned by $\{\, \varphi^d_{\lambda\mu} \mid d^\mp \in \Omega_{\lambda\mu} \,\}$. We can now state the main result.

Theorem 5.11 (Du and Rui [**53**]). *Suppose that R is an integral domain.*

(i) *The two R-modules $\mathscr{S}^\pm(\Lambda_{r,n})$ are subalgebras of $\mathscr{S}(\Lambda_{r,n})$.*

(ii) *$\mathscr{S}^\pm(\Lambda_{r,n})$ is free as an R-module with basis*

$$\{\, \varphi^d_{\lambda\mu} \mid \lambda, \mu \in \Lambda_{r,n} \text{ and } d^\mp \in \Omega_{\lambda\mu} \,\}.$$

(iii) *$\mathscr{S}(\Lambda_{r,n})$ has a triangular decomposition*

$$\mathscr{S}(\Lambda_{r,n}) = \mathscr{S}^-(\Lambda_{r,n}) \cdot \mathscr{S}^+(\Lambda_{r,n}) = \mathscr{S}^-(\Lambda_{r,n}) \cdot \Big(\sum_{\nu \in \Lambda_{r,n}} R\varphi_\nu\Big) \cdot \mathscr{S}^+(\Lambda_{r,n}).$$

Thus, $\{\, \varphi^d_{\lambda\mu}\varphi^e_{\mu\nu} \mid \lambda, \mu, \nu \in \Lambda_{r,n}, d \in \mathcal{D}_{\lambda\mu}$ and $e^{-1} \in \mathcal{D}_{\mu\nu} \,\}$ is a basis of $\mathscr{S}(\Lambda_{r,n})$.

Du and Rui call $\mathscr{S}^-(\Lambda_{r,n})$ and $\mathscr{S}^+(\Lambda_{r,n})$ the Borel subalgebras of $\mathscr{S}(\Lambda_{r,n})$. Surprisingly, the Borel subalgebras of the cyclotomic Schur algebras are isomorphic to the Borel subalgebras of the q-Schur algebras; hence, they are really type A algebras.

The right hand side of part (iii) is written so as to suggest the triangular decomposition of quantum groups; however, this is slightly misleading because $\varphi^d_{\lambda\mu}\big(\sum_\nu r_\nu\varphi_\nu\big)\varphi^e_{\sigma\tau} = \delta_{\mu\sigma}r_\mu\varphi^d_{\lambda\mu}\varphi^e_{\sigma\tau}$, for $r_\nu \in R$.

Du and Rui are able to say quite a lot about the representation theory of these subalgebras. Because $\mathscr{S}^\pm(\Lambda_{r,n})$ are quasi-hereditary, they have standard modules and costandard modules; denote these by $\Delta^\pm(\mu)$ and $\nabla^\pm(\mu)$ respectively, for $\mu \in \Lambda_{r,n}$. Also, if $\mu \in \Lambda^+_{r,n}$ then the Weyl module $W^\mu = \Delta(\mu)$ is a standard module of $\mathscr{S}(\Lambda_{r,n})$ and its contragredient dual $(W^\mu)^* = \nabla(\mu)$ is a costandard module (duality with respect to $*$).

Theorem 5.12 (Du and Rui). *Suppose that R is a field.*

(i) *The Borel subalgebras $\mathscr{S}^-(\Lambda_{r,n})$ and $\mathscr{S}^+(\Lambda_{r,n})$ are quasi-hereditary, with respect to the poset $\Lambda_{r,n}$. Moreover, $\mathscr{S}^-(\Lambda_{r,n})$ and $\mathscr{S}^+(\Lambda_{r,n})$ are Ringel dual to each other.*

(ii) (a) *Each costandard module of $\mathscr{S}^-(\Lambda)$ is one dimensional and, hence, simple; moreover, every simple module appears this way.*

(b) *Dually, each standard module of $\mathscr{S}^-(\Lambda_{r,n})$ is a projective indecomposable $\mathscr{S}^-(\Lambda_{r,n})$-module.*

(c) *Explicitly, if $\mu \in \Lambda^+_{r,n}$ then $\Delta^-(\mu) = \mathscr{S}^-(\Lambda_{r,n})\varphi_\mu$ and $\nabla^-(\mu) = \Delta^-(\mu)/\operatorname{Rad}\Delta^-(\mu)$; moreover, $\{\,\varphi_\mu \mid \mu \in \Lambda_{r,n}\,\}$ is a complete set of primitive idempotents in $\mathscr{S}^-(\Lambda_{r,n})$.*

(iii) *Suppose that $\mu \in \Lambda_{r,n}$. Then*

$$\mathscr{S}(\Lambda_{r,n}) \otimes_{\mathscr{S}^+(\Lambda_{r,n})} \Delta^+(\mu) \cong \begin{cases} \Delta(\mu), & \text{if } \mu \in \Lambda^+_{r,n},\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\operatorname{Hom}_{\mathscr{S}^-(\Lambda_{r,n})}\big(\mathscr{S}(\Lambda_{r,n}), \nabla^-(\mu)\big) \cong \begin{cases} \nabla(\mu), & \text{if } \mu \in \Lambda^+_{r,n},\\ 0, & \text{otherwise,}\end{cases}$$

Ringel duality interchanges the standard and costandard modules of $\mathscr{S}^-(\Lambda_{r,n})$ and $\mathscr{S}^+(\Lambda_{r,n})$, so part (ii) also describes the simple and projective $\mathscr{S}^+(\Lambda_{r,n})$-modules.

Du and Rui also give the dimensions of the standard and costandard modules for the Borel subalgebras and show that the Borel subalgebras are Ringel dual to each other.

5.6. Tilting modules

Let A be a quasi-hereditary algebra (see [**32, 46**]), and let Λ^+ be its poset of weights. Then for each $\lambda \in \Lambda^+$ we have a standard module $\Delta(\lambda)$, a costandard module $\nabla(\Lambda)$ and a simple module $L(\lambda)$. The simple module $L(\lambda)$ is the head of $\Delta(\lambda)$ and the simple socle of $\nabla(\lambda)$; further, $\nabla(\lambda)$ is the contragredient dual of $\Delta(\lambda)$ if A possesses a suitable involution.

Let $\mathcal{F}(\Delta)$ be the full subcategory of A-**mod** consisting of those modules which have a Δ-filtration; thus $X \in \mathcal{F}(\Delta)$ if X has a filtration $X = X_1 \supset X_2 \supset \cdots \supset X_m \supset 0$ with $X_i/X_{i+1} \cong \Delta(\lambda_i)$ for $1 \le i \le m$. If $X \in \mathcal{F}(\Delta)$ and $\lambda \in \Lambda^+$ let $[X{:}\Delta(\lambda)] = \#\{\, 1 \le i \le m \mid X_i/X_{i+1} \cong \Delta(\lambda) \,\}$; this is independent of the choice of filtration because the equivalence classes of standard modules are a basis of the Grothendieck group of A. Similarly, let $\mathcal{F}(\nabla)$ be the full subcategory of A-modules which have a ∇-filtration.

Ringel [**110**] has proved that for each $\lambda \in \Lambda^+$ there is a unique indecomposable A-module $T(\lambda) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ such that $[T(\lambda){:}\Delta(\lambda)] = 1$ and $[T(\lambda){:}\Delta(\mu)] \neq 0$ only if $\mu \geq \lambda$; we call $T(\lambda)$ a (partial) tilting module for A. Moreover, every module in $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$ is isomorphic to a direct sum of tilting modules.

If Λ is saturated then the cyclotomic Schur algebra $\mathscr{S}(\Lambda)$ is quasi-hereditary by Theorem 4.14, so we may ask for a description of the tilting modules of $\mathscr{S}(\Lambda)$. When $r = 1$ Donkin [**45, 46**] determined the tilting modules of the q-Schur algebras. To describe this, recall from the previous section that $\mathscr{S}_q(d;n) = \mathrm{End}_{\mathscr{H}}(V^{\otimes n})$. Donkin showed that the tilting modules of $\mathscr{S}_q(d;n)$ are precisely the indecomposable direct summands of the exterior powers $\wedge^\lambda V = \wedge^{\lambda_1} V \otimes \cdots \otimes \wedge^{\lambda_d} V$. For another approach to the tilting modules of the q-Schur algebras see [**50**].

Even though we do not know how to describe $\oplus_\mu M^\mu$ as a tensor product the exterior powers of $\mathscr{S}(\Lambda)$ still admit a similar description. In introducing M^λ we said that it should be thought of as an induced trivial module; the analogue of an induced sign representation for $\mathscr{H}$ is the module $N^\lambda = n_\lambda \mathscr{H}$, where $n_\lambda = y_\lambda u_\lambda^- = u_\lambda^- y_\lambda$ and

$$y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-q)^{-\ell(w)} T_w \quad \text{and} \quad u_\lambda^- = \prod_{s=1}^{r-1} \prod_{k=1}^{|\lambda^{(s+1)}|+\cdots+|\lambda^{(r)}|} (L_k - Q_s).$$

For each multipartition λ let $E^\lambda = \theta_\lambda \mathscr{S}(\Lambda)$, where $\theta_\lambda \in \mathrm{Hom}_{\mathscr{H}}(\mathscr{H}, N^\lambda)$ is the map $\theta_\lambda(h) = n_\lambda h$ for $h \in \mathscr{H}$. Then E^λ is a right $\mathscr{S}(\Lambda)$-module and we have the following.

Theorem 5.13 (Mathas [**105**]). *Suppose that R is a field, and that Λ is a saturated set of multicompositions containing ω. Then the tilting modules of $\mathscr{S}(\Lambda_{r,n})$ are the indecomposable summands of the modules $\{\, E^\lambda \mid \lambda \in \Lambda^+ \,\}$.*

The key tool in the proof of Theorem 5.13 is the use of Specht filtrations and dual Specht filtrations of $\mathscr{H}$–modules; this is a bit surprising because Specht filtrations are generally not as good as Weyl filtrations (since it can happen that $S^\lambda \cong S^\mu$ when $\lambda \neq \mu$).

The tilting modules of $\mathscr{S}(\Lambda)$ have all of the expected properties. For example, $[T(\lambda){:}\nabla(\mu)] = [\Delta(\mu'){:}L(\lambda')]$ for all $\lambda, \mu \in \Lambda^+$. (Here μ' is the multipartition conjugate to μ.) Furthermore, the Ringel dual of $\mathscr{S}(\Lambda)$ is the algebra $\mathscr{S}'(\Lambda) = \mathrm{End}_{\mathscr{H}}\big(\bigoplus_{\mu\in\Lambda} N^\mu\big)$ and $\mathscr{S}'(\Lambda) \cong \mathscr{S}(\Lambda)$.

The theory of Young modules for $\mathscr{H}$ (cf. [**78**]), is also developed in [**105**]. The Young modules (and twisted Young modules) are the indecomposable direct summands of the modules M^λ and N^λ, for λ a multicomposition of n; they are indexed by the multipartitions of n. The Young modules are the image under the Schur functor of the corresponding indecomposable projective, injective or tilting modules for the algebras $\mathscr{S}(\Lambda)$ or $\mathscr{S}'(\Lambda)$.

§6. Some open problems

In this final chapter we discuss some open problems for the Ariki-Koike algebras and the cyclotomic Schur algebras. We are mostly interested in the connections between the representation theory of the Ariki-Koike algebras and cyclotomic Schur algebras with the representation theory of the finite groups of Lie type.

Problem 6.1. *Prove the conjectures of Broué, Malle and Michel stated in Conjecture 2.5 and* [**19**].

We also pose the more general (and more vague) problem.

Problem 6.2. *Find a link between the representation theory of the cyclotomic Schur algebras and the modular representation theory of the finite groups of Lie type.*

At best, there is only circumstantial evidence for such a connection when $r > 2$. If we believe in the conjectures of the Broué school then there are strong ties between the representation theory of cuspidal representations of $\mathrm{GL}_d(q)$ in characteristic zero, so it is not unreasonable to expect that the modular theory of the cyclotomic Schur algebras and Ariki-Koike algebras also carry information about the modular representations of $\mathrm{GL}_d(q)$.

6.1. Quantum groups and geometry

The results of Ariki and Sakamoto and Shoji from §5.4 show that in some circumstances the module categories of the Ariki-Koike algebras and the cyclotomic Schur algebras are connected with the module categories of Levi subalgebras of $U_v(\mathfrak{gl}_d)$. Unfortunately, these results apply only in cases where the Ariki-Koike algebras are Morita equivalent to direct sums of tensor products of Iwahori-Hecke algebras of type A and when the cyclotomic q-Schur algebras were Morita equivalent to direct sums of tensor products of q-Schur algebras.

Problem 6.3. *Realize the cyclotomic Schur algebras as a quotient of a quantum group $U_{\mathcal{A}}(\mathfrak{g})$ over an arbitrary integral domain.*

We could ask for a generalization of the results of Sakamoto and Shoji (Theorem 5.9) and Ariki (Theorem 5.10); however, as the conjectures of Broué's school only ask for a derived equivalence it seems to me that we cannot expect something so simple here.

That the cyclotomic Schur algebras might be realizable as a quotient of a quantum group is suggested by the cyclotomic Jantzen sum formula (Theorem 5.6) and by the existence of the Borel subalgebras and the triangular decomposition of $\mathscr{S}(\Lambda)$ (Theorem 5.11). Both of these results hint at connections with quantum groups and at some undiscovered geometry.

Note also that the existence of the Borel subalgebras allows us to consider the dual Weyl modules of the cyclotomic Schur algebras as induced modules and so gives us cohomological techniques to play with.

6.2. Tensor products

First consider the case $r = 1$. If λ is a partition of n and μ is a partition of m then $S^\lambda \boxtimes S^\mu$ is a module for the Hecke algebra $\mathscr{H}_q(\mathfrak{S}_n) \otimes \mathscr{H}_q(\mathfrak{S}_m)$. We can identify $\mathscr{H}_q(\mathfrak{S}_n) \otimes \mathscr{H}_q(\mathfrak{S}_m)$ with the subalgebra $\mathscr{H}_q(\mathfrak{S}_{(n,m)})$ of $\mathscr{H}_q(\mathfrak{S}_{n+m})$ where $\mathfrak{S}_{(n,m)} = \mathfrak{S}_n \times \mathfrak{S}_m$. Thus, $\mathscr{H}_q(\mathfrak{S}_{n+m})$ is a free $\mathscr{H}_q(\mathfrak{S}_n) \otimes \mathscr{H}_q(\mathfrak{S}_m)$-module and we can define the $\mathscr{H}_q(\mathfrak{S}_{n+m})$-module

$$S^\lambda \otimes S^\mu = (S^\lambda \boxtimes S^\mu) \otimes_{\mathscr{H}_q(\mathfrak{S}_n)\otimes\mathscr{H}_q(\mathfrak{S}_m)} \mathscr{H}_q(\mathfrak{S}_{n+m}).$$

When $\mathscr{H}_q(\mathfrak{S}_n)$ is semisimple, this decomposes as a direct sum of Specht modules according to the Littlewood-Richardson rule.

In the case of the q-Schur algebras it is even easier. If λ and μ are both partitions of length at most d then the Weyl modules W^λ and W^μ are homogeneous polynomial representations for $U_v(\mathfrak{gl}_d)$ of degree n and m respectively; therefore, $W^\lambda \otimes W^\mu$ is a polynomial representation of

$U_v(\mathfrak{gl}_d)$ of degree $n+m$ — since $U_v(\mathfrak{gl}_d)$ is a Hopf algebra. Hence, $W^\lambda \otimes W^\mu$ is an $\mathscr{S}_q(d; n+m)$-module since $\mathscr{S}(d;N)$-**mod** is the category of polynomial representations of $U_v(\mathfrak{gl}_d)$ of homogeneous degree N. Again, in the semisimple case the decomposition of $W^\lambda \otimes W^\mu$ into irreducibles is given by the Littlewood-Richardson rule.

When we try and extend either of these constructions to the cyclotomic case we run into problems. First, for the Ariki-Koike algebras there is no obvious way to consider $\mathscr{H}_{q,\mathbf{Q}}(W_{r,n}) \otimes \mathscr{H}_{q,\mathbf{Q}'}(W_{s,m})$ as a free submodule of $\mathscr{H}_{q,\mathbf{Q}\cup\mathbf{Q}'}(W_{t,n+m})$ for any t, unless $rs = 0$. Secondly, we do not have an interpretation of the module category of a cyclotomic Schur algebra in terms of homogeneous representations of a quantum group.

Problem 6.4. *Find a good tensor product operation for the categories $\mathscr{H}$-**mod** and $\mathscr{S}(\Lambda)$-**mod**.*

Of course, a strong enough link with quantum groups would give us this for free. The correct approach is probably via the affine Hecke algebra (or possibly the work of Shoji [**117**]).

If we knew how to take tensor products of modules for the cyclotomic Schur algebras then we could try and solve the following problem.

Problem 6.5. *Find an analogue of the Steinberg tensor product theorem for the cyclotomic Schur algebras.*

Evidence for the existence of such a result, as well as an indication of what it might look like, are given by Uglov's [**120**] action of the Heisenberg algebra upon the generalized Fock spaces.

6.3. Decomposition numbers at roots of unity

The decomposition numbers of the Ariki-Koike algebras are known in characteristic zero, thanks to Ariki's theorem and the work of Uglov (assuming that $Q_s \neq 0$ for any s); see Corollary 3.20.

Problem 6.6. *Compute the decomposition numbers of the cyclotomic q-Schur algebras in characteristic zero.*

By Theorem 3.19 the decomposition matrix of $\mathscr{H}_q(\mathfrak{S}_n)$ can be computed from the canonical basis of $L_v(\Lambda_0)$. The easiest way to compute the canonical basis of $L_v(\Lambda_0)$ is to work in the Fock space $\mathcal{F}$, an infinite rank free $\mathbb{C}[v,v^{-1}]$-module with a basis given by the set of all partitions of all integers. Leclerc and Thibon's idea [**95**] was to define a canonical basis on the whole of the Fock space; they did this using the action of a Heisenberg algebra on $\mathcal{F}$. Leclerc and Thibon conjectured that the decomposition matrices of the q-Schur algebra were given by computing the canonical basis of $\mathcal{F}$ and then specializing $v = 1$; this was proved by Varagnolo and Vasserot [**121**].

Hence, this problem has been solved when $r = 1$. Furthermore, as remarked in §5.4, when $P_n(q, \mathbf{Q}) \neq 0$ we also know the answer because by Theorem 4.15 $\mathscr{S}(\Lambda)$ is Morita equivalent to a direct sum of tensor products of q-Schur algebras.

Now, the decomposition matrices of the Ariki-Koike algebras in characteristic zero are obtained by computing the canonical basis of highest weight modules $L_v(\Lambda)$, for the various dominant weights Λ. This time, $L_v(\Lambda)$ embeds in a generalized Fock space $\mathcal{F}_\Lambda$ and Uglov has shown how to compute a canonical basis for the whole of this space; this gives a canonical basis element for each multipartition. For $n \geq 0$ the canonical basis of $\mathcal{F}_\Lambda$ at $v = 1$ gives a square unitriangular matrix, indexed by the multipartitions of n, which contains the decomposition matrix of the Ariki-Koike algebra $\mathscr{H}_n$ as a submatrix (delete those columns corresponding to the multipartitions λ with $D^\lambda = 0$); compare with Corollary 5.2. The indexing of the rows and columns is wrong; however, once this difference in labeling is taken into account, I expect that this will give the decomposition matrix of $\mathscr{S}(\Lambda_{r,n})$.

6.4. Dipper-James theory

Let q be a prime power and let $\mathrm{GL}_n(q)$ be the general linear group over a field with q elements. Dipper and James [**39**] proved that the decomposition matrix of $\mathrm{GL}_n(q)$ in non-defining characteristic is completely determined by the decomposition matrix of the q^d–Schur algebras, for $d \geq 1$. Recently Brundan, Dipper, and Kleshchev [**27**] have rewritten this theory using cuspidal algebras. They also make the Dipper-James result on decomposition matrices much more explicit; see [**27**, Theorem 4.4d].

To date, no one has succeeded in generalizing this theory to the cyclotomic q-Schur algebras. The best results in this direction were obtained by Gruber and Hiss [**74**] who, for linear primes, worked with a Morita equivalent version of the cyclotomic Schur algebras when $r = 2$ (type B), to give similar results for other finite reductive groups $G_n(q)$. See the survey article of Dipper, Geck, Hiss and Malle [**36**] for the current status of this theory.

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Crystal Bases and Diagram Automorphisms

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Abstract.

We prove that the action of an ω-root operator on the set of all paths fixed by a diagram automorphism ω of a Kac–Moody algebra $\mathfrak{g}$ can be identified with the action of a root operator for the orbit Lie algebra $\breve{\mathfrak{g}}$. Moreover, we prove that there exists a canonical bijection between the elements of the crystal base $\mathcal{B}(\infty)$ for $\mathfrak{g}$ fixed by ω and the elements of the crystal base $\breve{\mathcal{B}}(\infty)$ for $\breve{\mathfrak{g}}$. Using this result, we give twining character formulas for the "negative part" of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ and for certain modules of Demazure type.

§0. Introduction.

Let $\mathfrak{g} := \mathfrak{g}(A)$ be the Kac–Moody algebra over $\mathbb{Q}$ associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$ with Cartan subalgebra $\mathfrak{h}$ and Weyl group $W = \langle r_i \mid i \in I \rangle$. A path is, by definition, a piecewise linear, continuous map $\pi : [0,1] \to \mathfrak{h}^*$ such that $\pi(0) = 0$, where $[0,1] := \{t \in \mathbb{Q} \mid 0 \leq t \leq 1\}$. We denote by $\mathbb{P}$ the set of all paths (modulo reparametrization). In [13], Littelmann defined root operators $e_i, f_i : \mathbb{P} \cup \{\theta\} \to \mathbb{P} \cup \{\theta\}$, where θ is an extra element, and introduced the notion of Lakshmibai–Seshadri paths of shape λ, where $\lambda \in \mathfrak{h}^*$ is a dominant integral weight. By using root operators, we can make the set $\mathbb{B}(\lambda)$ of Lakshmibai–Seshadri paths of shape λ into a crystal which is isomorphic to the crystal base $\mathcal{B}(\lambda)$ of an integrable highest weight $U_q(\mathfrak{g})$-module of highest weight λ (see [3] and [10]), where $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra of $\mathfrak{g}$ over $\mathbb{Q}(q)$.

Let $\omega \in \mathrm{Aut}(\mathfrak{g})$ be a diagram automorphism of $\mathfrak{g}$, and $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ the contragredient map of the restriction $\omega|_{\mathfrak{h}}$ of ω to $\mathfrak{h}$. For a path $\pi \in \mathbb{P}$, we define a path $\omega(\pi) \in \mathbb{P}$ by $(\omega(\pi))(t) := \omega^*(\pi(t))$ for $t \in [0,1]$. In [20] and [21], we introduced ω-root operators $\widetilde{e}_i$ and $\widetilde{f}_i$ (see (2.2.2)), and then proved that the Lakshmibai–Seshadri paths fixed by ω can be

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identified with the Lakshmibai–Seshadri paths for the orbit Lie algebra $\breve{\mathfrak{g}}$, which is a certain Kac–Moody algebra corresponding to ω.

In this paper, we first prove that the action of an ω-root operator on the set of all paths fixed by ω can be identified with the action of a root operator for the orbit Lie algebra $\breve{\mathfrak{g}}$, generalizing results in [20] and [22]. Then, using results in [20] and [21], we show that there exists a canonical bijection between the elements of the crystal base $\mathcal{B}(\infty)$ of the negative part $U_q^-(\mathfrak{g})$ of $U_q(\mathfrak{g})$ fixed by ω and the elements of the crystal base $\breve{\mathcal{B}}(\infty)$ of the negative part $U_q^-(\breve{\mathfrak{g}})$ of $U_q(\breve{\mathfrak{g}})$. In addition, we give twining character formulas for $U_q^-(\mathfrak{g})$ and for certain modules $(U_w^-)_q(\mathfrak{g})$ of Demazure type.

Let us explain our results more precisely. We set $(\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}$, $\widetilde{W} := \{w \in W \mid \omega^* w = w\omega^*\}$. Note that there exist a natural $\mathbb{Q}$-linear isomorphism $P_\omega^* : \widehat{\mathfrak{h}} \to (\mathfrak{h}^*)^0$ and a group isomorphism $\Theta : \widehat{W} \to \widetilde{W}$, where $\widehat{\mathfrak{h}}$ is the Cartan subalgebra of the orbit Lie algebra $\breve{\mathfrak{g}}$ and $\widehat{W}$ is the Weyl group of $\breve{\mathfrak{g}}$. Denote by $\widehat{\mathbb{P}}$ the set of all paths (modulo reparametrization) for the orbit Lie algebra $\breve{\mathfrak{g}}$, and by $\widehat{e}_i$, $\widehat{f}_i$: $\widehat{\mathbb{P}} \cup \{\theta\} \to \widehat{\mathbb{P}} \cup \{\theta\}$ root operators for $\breve{\mathfrak{g}}$. For a path $\widehat{\pi} \in \widehat{\mathbb{P}}$, we define a path $P_\omega^*(\widehat{\pi}) \in \mathbb{P}$ by $(P_\omega^*(\widehat{\pi}))(t) := P_\omega^*(\widehat{\pi}(t))$ for $t \in [0,1]$, and set $P_\omega^*(\theta) = \theta$.

In [20] and [22], we proved that the equalities $\widetilde{e}_i \circ P_\omega^* = P_\omega^* \circ \widehat{e}_i$ and $\widetilde{f}_i \circ P_\omega^* = P_\omega^* \circ \widehat{f}_i$ hold on a certain subset of $\widehat{\mathbb{P}}$. In this paper, we extend this result to the whole of $\widehat{\mathbb{P}}$.

Theorem 1. *The set $\mathbb{P}^0 \cup \{\theta\}$ is stable under the ω-root operators, where $\mathbb{P}^0 := \{\pi \in \mathbb{P} \mid \omega(\pi) = \pi\}$. Furthermore, we have $\widetilde{e}_i \circ P_\omega^* = P_\omega^* \circ \widehat{e}_i$ and $\widetilde{f}_i \circ P_\omega^* = P_\omega^* \circ \widehat{f}_i$ on $\widehat{\mathbb{P}}$.*

Denote by $e_i, f_i : \mathcal{B}(\infty) \cup \{0\} \to \mathcal{B}(\infty) \cup \{0\}$ the Kashiwara operators for the crystal base $\mathcal{B}(\infty)$. Let $w \in W$, and $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ its reduced expression. We define a subset $\mathcal{B}_w(\infty)$ of $\mathcal{B}(\infty)$ by

$$\mathcal{B}_w(\infty) := \{f_{i_1}^{m_1} f_{i_2}^{m_2} \ldots f_{i_k}^{m_k} \overline{v}_\infty \mid m_j \in \mathbb{Z}_{\geq 0}\},$$

where $\overline{v}_\infty$ is the (unique) highest weight element of $\mathcal{B}(\infty)$. We know from [8] that $\mathcal{B}_w(\infty)$ is the crystal base of the following module $(U_w^-)_q(\mathfrak{g})$ of Demazure type:

$$(U_w^-)_q(\mathfrak{g}) = \sum_{m_j \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) y_{i_1}^{m_1} y_{i_2}^{m_2} \cdots y_{i_k}^{m_k} \subset U_q^-(\mathfrak{g}),$$

where y_i, $i \in I$, are the Chevalley generators corresponding to negative roots. We also know that $\mathcal{B}_w(\infty)$ (and hence $(U_w^-)_q(\mathfrak{g})$) does not depend on the choice of the reduced expression of w.

There exists a canonical $\mathbb{Q}(q)$-algebra automorphism $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$ of $U_q(\mathfrak{g})$ induced from the diagram automorphism ω. Since the crystal lattice $\mathcal{L}(\infty)$ of $U_q^-(\mathfrak{g})$ is stable under ω, we obtain a $\mathbb{Q}$-linear automorphism $\omega : \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\infty)/q\mathcal{L}(\infty)$ induced from $\omega : \mathcal{L}(\infty) \to \mathcal{L}(\infty)$. Note that the crystal base $\mathcal{B}(\infty)$ and its subset $\mathcal{B}_w(\infty)$ for $w \in \widetilde{W}$ are stable under ω. We set

$$\mathcal{B}^0(\infty) := \{b \in \mathcal{B}(\infty) \mid \omega(b) = b\}, \quad \mathcal{B}^0_w(\infty) := \{b \in \mathcal{B}_w(\infty) \mid \omega(b) = b\}.$$

We denote by $\widehat{e}_i, \widehat{f}_i : \breve{\mathcal{B}}(\infty) \cup \{0\} \to \breve{\mathcal{B}}(\infty) \cup \{0\}$ the Kashiwara operators for the crystal base $\breve{\mathcal{B}}(\infty)$, and by $\breve{\mathcal{B}}_{\widehat{w}}(\infty)$ the crystal base of the module $(U^-_{\widehat{w}})_q(\breve{\mathfrak{g}})$ of Demazure type corresponding to $\widehat{w} \in \widehat{W}$.

By using results in [20] and [21], we prove the following theorem.

Theorem 2. *The set $\mathcal{B}^0(\infty) \cup \{0\}$ is stable under the ω-Kashiwara operators, defined in the same way as (2.2.2). Moreover, there exists a canonical bijection $P_\infty : \mathcal{B}^0(\infty) \xrightarrow{\sim} \breve{\mathcal{B}}(\infty)$ such that*

$$(P^*_\omega)^{-1}(\mathrm{wt}(b)) = \mathrm{wt}(P_\infty(b)) \quad \text{for} \quad b \in \mathcal{B}^0(\infty),$$
$$P_\infty \circ \widetilde{e}_i = \widehat{e}_i \circ P_\infty \quad \text{and} \quad P_\infty \circ \widetilde{f}_i = \widehat{f}_i \circ P_\infty.$$

In addition, we have $P_\infty(\mathcal{B}^0_w(\lambda)) = \breve{\mathcal{B}}_{\widehat{w}}(\infty)$ for each $w \in \widetilde{W}$, where $\widehat{w} := \Theta^{-1}(w)$.

The twining character $\mathrm{ch}^\omega(U_q^-(\mathfrak{g}))$ of $U_q^-(\mathfrak{g})$ is defined to be the following formal sum:

$$\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) = \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\omega|_{(U_q^-(\mathfrak{g}))_\chi}\big) e(\chi).$$

For each $w \in \widetilde{W}$, we define the twining character $\mathrm{ch}^\omega((U^-_w)_q(\mathfrak{g}))$ of $(U^-_w)_q(\mathfrak{g})$ by

$$\mathrm{ch}^\omega((U^-_w)_q(\mathfrak{g})) := \sum_{\chi \in (\mathfrak{h}^*)^0} \mathrm{tr}\big(\omega|_{((U^-_w)_q(\mathfrak{g}))_\chi}\big) e(\chi).$$

As a corollary of Theorem 2, we obtain the following.

Corollary 3. *Let $w \in \widetilde{W}$, and set $\widehat{w} := \Theta^{-1}(w)$. Then we have*

$$\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) = P^*_\omega(\mathrm{ch}\, U_q^-(\breve{\mathfrak{g}})), \qquad \mathrm{ch}^\omega((U^-_w)_q(\mathfrak{g})) = P^*_\omega(\mathrm{ch}\,(U^-_{\widehat{w}})_q(\breve{\mathfrak{g}})).$$

This paper is organized as follows. In §1, we fix our notation for Kac–Moody algebras, and then recall some basic facts about diagram automorphisms and orbit Lie algebras. In §2, we recall the definition of an ω-root operator, and prove Theorem 1. In §3, we study the elements of some crystal bases fixed by a diagram automorphism, and show Theorem 2. In §4, we obtain Corollary 3 as an application of Theorem 2.

§1. Preliminaries.

1.1. Kac–Moody algebras and diagram automorphisms.

Let $\mathfrak{g} := \mathfrak{g}(A)$ be the Kac–Moody algebra over $\mathbb{Q}$ associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,\,j \in I}$, with Cartan subalgebra $\mathfrak{h}$, simple roots $\Pi = \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$, simple coroots $\Pi^\vee = \{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}$, Chevalley generators $\{x_i,\, y_i \mid i \in I\}$, where $\mathfrak{g}_{\alpha_i} = \mathbb{Q}x_i$ and $\mathfrak{g}_{-\alpha_i} = \mathbb{Q}y_i$, and Weyl group $W = \langle r_i \mid i \in I\rangle$.

Let $\omega : I \to I$ be a bijection of order N such that $a_{\omega(i),\,\omega(j)} = a_{ij}$ for all $i,\, j \in I$, which we call a (Dynkin) diagram automorphism. Then ω naturally induces a Lie algebra automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ of order N such that $\omega(\mathfrak{h}) = \mathfrak{h}$, and $\omega(x_i) = x_{\omega(i)}$, $\omega(y_i) = y_{\omega(i)}$, $\omega(\alpha_i^\vee) = \alpha_{\omega(i)}^\vee$ for $i \in I$ (see [23, §1.1]). We define a $\mathbb{Q}$-linear automorphism $\omega^* : \mathfrak{h}^* \to \mathfrak{h}^*$ by $(\omega^*(\lambda))(h) := \lambda(\omega^{-1}(h))$ for $\lambda \in \mathfrak{h}^*$, $h \in \mathfrak{h}$, and set

$$\text{(1.1.1)} \quad \mathfrak{h}^0 := \{h \in \mathfrak{h} \mid \omega(h) = h\}, \qquad (\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\}.$$

We also set

$$\text{(1.1.2)} \qquad \widetilde{W} := \{w \in W \mid \omega^* w = w\omega^*\}.$$

Note that $\omega^* r_i (\omega^*)^{-1} = r_{\omega(i)}$ for every $i \in I$.

1.2. Orbit Lie algebras.

We set $c_{ij} := \sum_{k=0}^{N_j - 1} a_{i,\,\omega^k(j)}$ for $i,\, j \in I$, where $N_i := \#\{\omega^k(i) \mid k \geq 0\}$. We choose a complete set $\widehat{I}$ of representatives of the ω-orbits in I, and set $\check{I} := \{i \in \widehat{I} \mid c_{ii} > 0\}$.

Remark 1.2.1 (cf. [2, §2.2]). Assume that $c_{ii} > 0$. Then $c_{ii} = 1$ or 2. If $c_{ii} = 1$, then $a_{i,\,\omega^{N_i/2}(i)} = -1$ and $a_{i,\,\omega^k(i)} = 0$ for any other $1 \leq k \leq N_i - 1$, $k \neq N_i/2$, with N_i even. Hence the Dynkin diagram corresponding to the ω-orbit of the i is of type $A_2 \times \cdots \times A_2$ ($N_i/2$ times). If $c_{ii} = 2$, then $a_{i,\,\omega^k(i)} = 0$ for all $1 \leq k \leq N_i - 1$. Hence the Dynkin diagram corresponding to the ω-orbit of the i is of type $A_1 \times \cdots \times A_1$ (N_i times).

We set $\widehat{a}_{ij} := 2c_{ij}/c_j$ for $i, j \in \widehat{I}$, where $c_i := c_{ii}$ if $i \in \check{I}$, and $c_i := 2$ otherwise. We know from [1, Lemma 2.1] that a matrix $\widehat{A} = (\widehat{a}_{ij})_{i,j\in\widehat{I}}$ is a symmetrizable Borcherds-Cartan matrix, and its submatrix $\check{A} = (\widehat{a}_{ij})_{i,j\in\check{I}}$ is a symmetrizable generalized Cartan matrix. Let $\widehat{\mathfrak{g}} := \mathfrak{g}(\widehat{A})$ be the generalized Kac–Moody algebra over $\mathbb{Q}$ associated to $\widehat{A}$, with Cartan subalgebra $\widehat{\mathfrak{h}}$, simple roots $\widehat{\Pi} = \{\widehat{\alpha}_i\}_{i\in\widehat{I}}$, simple coroots $\widehat{\Pi}^\vee = \{\widehat{\alpha}_i^\vee\}_{i\in\widehat{I}}$, Chevalley generators $\{\widehat{x}_i, \widehat{y}_i \mid i \in \widehat{I}\}$, and Weyl group $\widehat{W} = \langle \widehat{r}_i \mid i \in \check{I}\rangle$. The orbit Lie algebra $\breve{\mathfrak{g}}$ is defined to be the subalgebra of $\widehat{\mathfrak{g}}$ generated by $\widehat{\mathfrak{h}} \cup \{\widehat{x}_i, \widehat{y}_i \mid i \in \check{I}\}$, which is a Kac–Moody algebra associated to $\check{A}$.

As in [1, §2], we obtain $\mathbb{Q}$-linear isomorphisms $P_\omega : \mathfrak{h}^0 \to \widehat{\mathfrak{h}}$ and $P_\omega^* : \widehat{\mathfrak{h}}^* \to (\mathfrak{h}^0)^* \cong (\mathfrak{h}^*)^0$ such that

$$(1.2.1) \qquad \begin{cases} P_\omega(\widetilde{\alpha}_i^\vee) = \widehat{\alpha}_i^\vee, \ P_\omega^*(\widehat{\alpha}_i) = \widetilde{\alpha}_i & \text{for each } i \in \widehat{I}, \\ \bigl(P_\omega^*(\widehat{\lambda})\bigr)(h) = \widehat{\lambda}(P_\omega(h)) & \text{for } \widehat{\lambda} \in \widehat{\mathfrak{h}}^* \text{ and } h \in \mathfrak{h}^0, \end{cases}$$

where

$$(1.2.2) \qquad \widetilde{\alpha}_i^\vee := \frac{1}{N_i}\sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}^\vee \in \mathfrak{h}^0, \qquad \widetilde{\alpha}_i := \frac{2}{c_i}\sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)} \in (\mathfrak{h}^*)^0.$$

We also know from [1, §3] that there exists a group isomorphism $\Theta : \widehat{W} \to \widetilde{W}$ such that $\Theta(\widehat{w}) = P_\omega^* \circ \widehat{w} \circ (P_\omega^*)^{-1}$ for each $\widehat{w} \in \widehat{W}$.

§2. Properties of ω-root Operators.

2.1. Root operators.

In this subsection, we recall the definition of a root operator from [13]. A path is, by definition, a piecewise linear, continuous map $\pi : [0,1] \to \mathfrak{h}^*$ such that $\pi(0) = 0$, where $[0,1] := \{t \in \mathbb{Q} \mid 0 \le t \le 1\}$. We regard two paths π and π' as equivalent if there exist piecewise linear, nondecreasing, surjective, continuous maps $\psi, \psi' : [0,1] \to [0,1]$ (reparametrization) such that $\pi \circ \psi = \pi' \circ \psi'$. Denote by $\mathbb{P}$ the set of (representatives of) equivalence classes of all paths under this equivalence relation. For $\pi \in \mathbb{P}$ and $i \in I$, we set

$$(2.1.1) \qquad h_i^\pi(t) := \bigl(\pi(t)\bigr)(\alpha_i^\vee), \qquad m_i^\pi := \min\bigl\{h_i^\pi(t) \mid t \in [0,1]\bigr\}.$$

For convenience, we introduce an extra element θ (corresponding to "0" of a crystal).

For each $i \in I$, the raising root operator $e_i : \mathbb{P} \cup \{\theta\} \to \mathbb{P} \cup \{\theta\}$ is defined as follows. We set $e_i\theta := \theta$, and $e_i\pi := \theta$ for $\pi \in \mathbb{P}$ with $m_i^\pi > -1$. If $m_i^\pi \le 1$, then we can take the following points:

$$(2.1.2) \qquad \begin{aligned} t_1 &= \min\{t \in [0,1] \mid h_i^\pi(t) = m_i^\pi\}, \\ t_0 &= \max\{t' \in [0,t_1] \mid h_i^\pi(t) \ge m_i^\pi + 1 \text{ for all } t \in [0,t']\}. \end{aligned}$$

Choose a partition $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ of $[t_0, t_1]$ such that either of the following holds:

$$(2.1.3) \qquad \begin{aligned} &(1) \quad h_i^\pi(s_{k-1}) = h_i^\pi(s_k) \text{ and } h_i^\pi(t) \ge h_i^\pi(s_{k-1}) \\ &\qquad\ \text{for } t \in [s_{k-1}, s_k]. \\ &(2) \quad h_i^\pi(t) \text{ is strictly decreasing on } [s_{k-1}, s_k] \text{ and} \\ &\qquad\ h_i^\pi(t) \ge h_i^\pi(s_{k-1}) \text{ for } t \in [s_0, s_{k-1}]. \end{aligned}$$

Remark 2.1.1. We deduce from the definition of t_0 (resp. t_1) that $h_i^\pi(t)$ is strictly decreasing on $[s_0, s_1]$ (resp. $[s_{r-1}, s_r]$). Namely, $[s_0, s_1]$ and $[s_{r-1}, s_r]$ are of type (2).

We set

(2.1.4)

$$(e_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \pi(t) - (h_i^\pi(s_{k-1}) - m_i^\pi - 1)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h_i^\pi(t) - m_i^\pi - 1)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + \alpha_i & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

The lowering root operator $f_i : \mathbb{P} \cup \{\theta\} \to \mathbb{P} \cup \{\theta\}$ is defined in a similar way: We set $f_i\theta := \theta$, and $f_i\pi := \theta$ for $\pi \in \mathbb{P}$ with $h_i^\pi(1) - m_i^\pi < 1$. If $h_i^\pi(1) - m_i^\pi \ge 1$, then we can take the following points:

$$(2.1.5) \qquad \begin{aligned} t_0 &= \max\{t \in [0,1] \mid h_i^\pi(t) = m_i^\pi\}, \\ t_1 &= \min\{t' \in [t_0,1] \mid h_i^\pi(t) \ge m_i^\pi + 1 \text{ for all } t \in [t',1]\}. \end{aligned}$$

Choose a partition $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ of $[t_0, t_1]$ such that either of the following holds:

$$(2.1.6) \qquad \begin{aligned} &(1) \quad h_i^\pi(s_{k-1}) = h_i^\pi(s_k) \text{ and } h_i^\pi(t) \ge h_i^\pi(s_{k-1}) \\ &\qquad\ \text{for } t \in [s_{k-1}, s_k]. \\ &(2) \quad h_i^\pi(t) \text{ is strictly increasing on } [s_{k-1}, s_k] \text{ and} \\ &\qquad\ h_i^\pi(t) \ge h_i^\pi(s_k) \text{ for } t \in [s_k, s_1]. \end{aligned}$$

We set

(2.1.7)

$$(f_i\pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \pi(t) - (h_i^\pi(s_{k-1}) - m_i^\pi)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h_i^\pi(t) - m_i^\pi)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) - \alpha_i & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

For $\pi \in \mathbb{P}$ and $r \in \mathbb{Q}$, we define a path $r\pi \in \mathbb{P}$ by $(r\pi)(t) := r\pi(t)$ for $t \in [0,1]$. Also, we define the "dual path" $\pi^\vee \in \mathbb{P}$ of a path $\pi \in \mathbb{P}$ by $\pi^\vee(t) := \pi(1-t) - \pi(1)$ for $t \in [0,1]$. Let us recall the following lemma from [13, Lemmas 2.1 and 2.4].

Lemma 2.1.2. (1) *We have* $(f_i\pi)^\vee = e_i\pi^\vee$ *and* $(e_i\pi)^\vee = f_i\pi^\vee$ *for all* $\pi \in \mathbb{P}$.

(2) *We have* $n(e_i\pi) = e_i^n(n\pi)$ *and* $n(f_i\pi) = f_i^n(n\pi)$ *for all* $\pi \in \mathbb{P}$ *and* $n \in \mathbb{Z}_{\geq 0}$.

2.2. ω-root operators.

For a path $\pi \in \mathbb{P}$, we define a path $\omega(\pi) \in \mathbb{P}$ by $(\omega(\pi))(t) := \omega^*(\pi(t))$ for $0 \le t \le 1$, and set $\omega(\theta) := \theta$. We set

$$\mathbb{P}^0 := \{\pi \in \mathbb{P} \mid \omega(\pi) = \pi\}. \tag{2.2.1}$$

Let us recall the following definition of the ω-root operators $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in \check{I}$ from [20, §3.1] (cf. Remark 1.2.1):

$$\widetilde{X}_i = \begin{cases} \prod_{k=1}^{N_i/2} (X_{\omega^k(i)} X^2_{\omega^{k+N_i/2}(i)} X_{\omega^k(i)}) & \text{if } c_{ii} = 1, \\ \prod_{k=1}^{N_i} X_{\omega^k(i)} & \text{if } c_{ii} = 2, \end{cases} \tag{2.2.2}$$

where X is either e or f.

Denote by $\widehat{\mathbb{P}}$ the set of all paths (modulo reparametrization) for the orbit Lie algebra $\breve{\mathfrak{g}}$. For $i \in \check{I}$, we denote by $\widehat{e}_i : \widehat{\mathbb{P}} \cup \{\theta\} \to \widehat{\mathbb{P}} \cup \{\theta\}$ and $\widehat{f}_i : \widehat{\mathbb{P}} \cup \{\theta\} \to \widehat{\mathbb{P}} \cup \{\theta\}$ the raising root operator and the lowering root operator for $\breve{\mathfrak{g}}$, respectively. For a path $\widehat{\pi} \in \widehat{\mathbb{P}}$, we define a path $P_\omega^*(\widehat{\pi}) \in \mathbb{P}^0$ by $(P_\omega^*(\widehat{\pi}))(t) := P_\omega^*(\widehat{\pi}(t))$ for $t \in [0,1]$, and set $P_\omega^*(\theta) = \theta$.

In [20] and [22], we showed that the equalities $\widetilde{e}_i \circ P_\omega^* = P_\omega^* \circ \widehat{e}_i$ and $\widetilde{f}_i \circ P_\omega^* = P_\omega^* \circ \widehat{f}_i$ hold on a certain subset of $\widehat{\mathbb{P}}$. Here we extend this result to the whole of $\widehat{\mathbb{P}}$. The proof below essentially follows the same line as those of [20, Theorem 3.1.2] and [22, Theorem 2.1.2]; however, it is a little simplified by virtue of Lemma 2.1.2.

Theorem 2.2.1. *The set $\mathbb{P}^0 \cup \{\theta\}$ is stable under the ω-root operators. In addition, we have $\widetilde{e}_i \circ P_\omega^* = P_\omega^* \circ \widehat{e}_i$ and $\widetilde{f}_i \circ P_\omega^* = P_\omega^* \circ \widehat{f}_i$ for each $i \in \check{I}$.*

Proof. Let us show the following claim, which generalizes [20, Theorem 3.1.2].

Claim. Let $\pi \in \mathbb{P}^0$. If $m_i^\pi > -1$, then $\widetilde{e}_i \pi = \theta$. If $m_i^\pi \leq -1$, then we have

(2.2.3)
$$(\widetilde{e}_i \pi)(t) := \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_0 = s_0, \\ \pi(t) - (h_i^\pi(s_{k-1}) - m_i^\pi - 1)\widetilde{\alpha}_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h_i^\pi(t) - m_i^\pi - 1)\widetilde{\alpha}_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + \widetilde{\alpha}_i & \text{if } s_r = t_1 \leq t \leq 1, \end{cases}$$

where t_0, t_1 are the points given by (2.1.2) for $\pi \in \mathbb{P}^0$ and $i \in \check{I}$, and $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ is a partition of $[t_0, t_1]$ satisfying Condition (2.1.3) for $\pi \in \mathbb{P}^0$ and $i \in \check{I}$.

(Proof of Claim.) It is obvious that $\widetilde{e}_i \pi = \theta$ if $m_i^\pi > -1$. We will show Equality (2.2.3). If $c_{ii} = 2$, then Equality (2.2.3) immediately follows from the definition of root operators and Remark 1.2.1. Assume that $c_{ii} = 1$. For simplicity, we assume that the Dynkin diagram corresponding to the ω-orbit of the i is of type A_2 (cf. Remark 1.2.1). For $\pi \in \mathbb{P}^0$, we set $h(t) := h_i^\pi(t) = h_j^\pi(t)$ and $m := m_i^\pi = m_j^\pi$ with $j := \omega(i)$. Since $m \leq -1$, it follows from the definition of the raising root operator e_i that

$$\eta_1(t) := (e_i \pi)(t) = \begin{cases} \pi(t) & \text{if } 0 \leq t \leq t_0 = s_0, \\ \pi(t) - (h(s_{k-1}) - m - 1)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h(t) - m - 1)\alpha_i & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + \alpha_i & \text{if } s_r = t_1 \leq t \leq 1. \end{cases}$$

By definition, we have

(2.2.4)

$$h_j^{\eta_1}(t) = \begin{cases} h(t) & \text{if } 0 \le t \le t_0 = s_0, \\ h(t) + h(s_{k-1}) - m - 1 & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ 2h(t) - m - 1 & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ h(t) - 1 & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

Subclaim 1. We have $m_j^{\eta_1} = m - 1$ and $t_1 = \min\{t \in [0,1] \mid h_j^{\eta_1}(t) = m_j^{\eta_1}\}$.

It is obvious that $h_j^{\eta_1}(t) = h(t) - 1 \ge m - 1$ for $t \in [t_1, 1]$, and that $h_j^{\eta_1}(t_1) = m - 1$. So it suffices to show that $h_j^{\eta_1}(t_1) > m - 1$ for all $t \in [0, t_1)$. By the definition of t_0, we have $h_j^{\eta_1}(t) = h(t) \ge m+1 > m-1$ for $t \in [0, t_0]$. Suppose that $h_j^{\eta_1}(t) \le m - 1$ for some $t \in [t_0, t_1)$. If t is in $[s_{k-1}, s_k]$ of type (1), then we have $h(t) + h(s_{k-1}) - m - 1 \le m - 1$, and hence $h(s_{k-1}) \le m$, since $h(t) \ge h(s_{k-1})$ for all $t \in [s_{k-1}, s_k]$ (see (2.1.3)). This contradicts the definition of t_1 (notice that $s_{k-1} < s_r = t_1$). Similarly, if t is in $[s_{k-1}, s_k]$ of type (2), then we have $h(t) \le m$, which is a contradiction. Thus we conclude that $h_j^{\eta_1}(t) > m - 1$ for all $t \in [0, t_1)$.

Subclaim 2. We have $t_0 = \max\{t' \in [0, t_1] \mid h_j^{\eta_1}(t) \ge m_j^{\eta_1}+2 \text{ for all } t \in [0, t']\}$.

It is obvious from the definition of t_0 and Subclaim 1 that $h_j^{\eta_1}(t) = h(t) \ge m + 1 = m_j^{\eta_1} + 2$ for all $t \in [0, t_0]$. We deduce from Remark 2.1.1 and (2.2.4) that $h_j^{\eta_1}(t_0 + \varepsilon) < h_j^{\eta_1}(t_0) = m_j^{\eta_1} + 2$ for sufficiently small $\varepsilon > 0$. Now, Subclaim 2 immediately follows from these facts.

Set $\eta_1' := \frac{1}{2}\eta_1$. It follows from Subclaims 1 and 2 that t_0, t_1 are the points given by (2.1.2) for $\eta_1' \in \mathbb{P}$ and $j \in I$. In addition, we deduce from (2.2.4) that $t_0 = s_0 < s_1 < \cdots < s_r = t_1$ is a partition of $[t_0, t_1]$ satisfying Condition (2.1.3) with $\pi = \eta_1'$ and $i = j$. Therefore, we have

$$(e_j\eta_1')(t) =$$

$$\begin{cases} \eta_1'(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \eta_1'(t) - (h_j^{\eta_1'}(s_{k-1}) - m_j^{\eta_1'} - 1)\alpha_j & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \eta_1'(t) - (h_j^{\eta_1'}(t) - m_j^{\eta_1'} - 1)\alpha_j & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \eta_1'(t) + \alpha_j & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

$$= \begin{cases} \frac{1}{2}\pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \frac{1}{2}\pi(t) - \frac{1}{2}(h(s_{k-1}) - m - 1)(\alpha_i + 2\alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \frac{1}{2}\pi(t) - \frac{1}{2}(h(t) - m - 1)(\alpha_i + 2\alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \frac{1}{2}\pi(t) + \frac{1}{2}\alpha_i + \alpha_j & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

Because $e_j^2\eta_1 = 2(e_j\eta_1')$ by Lemma 2.1.2 (2), we get

$$\eta_2(t) := (e_j^2\eta_1)(t) =$$

$$\begin{cases} \pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \pi(t) - (h(s_{k-1}) - m - 1)(\alpha_i + 2\alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - (h(t) - m - 1)(\alpha_i + 2\alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + \alpha_i + 2\alpha_j & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

Since $h_i^{\eta_2}(t) = h(t)$, we obtain

$$(\widetilde{e}_i\pi)(t) := (e_i\eta_2)(t) =$$

$$\begin{cases} \pi(t) & \text{if } 0 \le t \le t_0 = s_0, \\ \pi(t) - 2(h(s_{k-1}) - m - 1)(\alpha_i + \alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (1)}, \\ \pi(t) - 2(h(t) - m - 1)(\alpha_i + \alpha_j) & \text{if } t \in [s_{k-1}, s_k] \text{ of type (2)}, \\ \pi(t) + 2(\alpha_i + \alpha_j) & \text{if } s_r = t_1 \le t \le 1. \end{cases}$$

This completes the proof of Claim.

It immediately follows from the claim above that $\mathbb{P}^0 \cup \{\theta\}$ is stable under $\widetilde{e}_i$. Also, we deduce from Lemma 2.1.2 (1) that $\mathbb{P}^0 \cup \{\theta\}$ is stable under $\widetilde{f}_i$, since $\widetilde{f}_i\pi = (\widetilde{e}_i\pi^\vee)^\vee$ for all $\pi \in \mathbb{P}^0$ (remark that if $\pi \in \mathbb{P}^0$, then so is $\pi^\vee$). Moreover, because $((P_\omega^*(\widehat{\pi}))(t))(\alpha_i^\vee) = (\widehat{\pi}(t))(\widehat{\alpha}_i^\vee)$ for $\widehat{\pi} \in \widehat{\mathbb{P}}$ and $i \in \check{I}$ (cf. (1.2.1)), we can easily check that $\widetilde{e}_i \circ P_\omega^* = P_\omega^* \circ \widehat{e}_i$ by using (2.2.3). Since $P_\omega^*(\widehat{\pi}^\vee) = (P_\omega^*(\widehat{\pi}))^\vee$ for all $\widehat{\pi} \in \widehat{\mathbb{P}}$, we get by Lemma 2.1.2 (1) that for each $\widehat{\pi} \in \widehat{\mathbb{P}}$,

$$\begin{aligned} \widetilde{f}_i(P_\omega^*(\widehat{\pi})) &= (\widetilde{e}_i(P_\omega^*(\widehat{\pi}))^\vee)^\vee = (\widetilde{e}_i(P_\omega^*(\widehat{\pi}^\vee)))^\vee \\ &= (P_\omega^*(\widehat{e}_i\widehat{\pi}^\vee))^\vee = P_\omega^*((\widehat{e}_i\widehat{\pi}^\vee)^\vee) = P_\omega^*(\widehat{f}_i\widehat{\pi}). \end{aligned}$$

Therefore we get $\widetilde{f}_i \circ P_\omega^* = P_\omega^* \circ \widehat{f}_i$. This completes the proof of the theorem. □

Remark 2.2.2. We can easily check that

(2.2.5) $$\omega \circ e_i = e_{\omega(i)} \circ \omega \quad \text{and} \quad \omega \circ f_i = f_{\omega(i)} \circ \omega \quad \text{on } \mathbb{P}.$$

Therefore we deduce from Theorem 2.2.1 that the ω-root operators on $\mathbb{P}^0$ do not depend on the choice of a representative of the ω-orbit of $i \in I$ with $c_{ii} > 0$.

We define $\widetilde{e}(n)_i$ and $\widetilde{f}(n)_i$ for $i \in \check{I}$ and $n \in \mathbb{Z}_{\geq 0}$ by

(2.2.6) $$\widetilde{X}(n)_i := \begin{cases} \prod_{k=1}^{N_i/2} (X^n_{\omega^k(i)} X^{2n}_{\omega^{k+N_i/2}(i)} X^n_{\omega^k(i)}) & \text{if } c_{ii} = 1, \\ \prod_{k=1}^{N_i} X^n_{\omega^k(i)} & \text{if } c_{ii} = 2, \end{cases}$$

where X is either e or f. As an application of Theorem 2.2.1, we can give a shorter proof of (a generalization of) [22, Proposition 2.1.3].

Corollary 2.2.3. *On $\mathbb{P}^0$, we have $(\widetilde{e}_i)^n = \widetilde{e}(n)_i$ and $(\widetilde{f}_i)^n = \widetilde{f}(n)_i$ for each $n \in \mathbb{Z}_{\geq 0}$ and $i \in I$.*

Proof. Let $\pi \in \mathbb{P}^0$, and set $\pi' = \frac{1}{n}\pi \in \mathbb{P}^0$. We deduce that

$$\begin{aligned} \widetilde{e}(n)_i\pi &= n(\widetilde{e}_i\pi') \quad \text{by Lemma 2.1.2 (2)} \\ &= n(P^*_\omega \circ \widehat{e}_i \circ (P^*_\omega)^{-1}(\pi')) \quad \text{by Theorem 2.2.1} \\ &= P^*_\omega(n\widehat{e}_i((P^*_\omega)^{-1}(\pi'))) \\ &= P^*_\omega((\widehat{e}_i)^n(n(P^*_\omega)^{-1}(\pi'))) \quad \text{by Lemma 2.1.2 (2)} \\ &= P^*_\omega \circ (\widehat{e}_i)^n \circ (P^*_\omega)^{-1}(\pi) \\ &= (P^*_\omega \circ \widehat{e}_i \circ (P^*_\omega)^{-1})^n(\pi) \\ &= (\widetilde{e}_i)^n\pi \quad \text{by Theorem 2.2.1.} \end{aligned}$$

Therefore we get $\widetilde{e}(n)_i = (\widetilde{e}_i)^n$. The equality $\widetilde{f}(n)_i = (\widetilde{f}_i)^n$ can be shown similarly. □

Let $P \subset \mathfrak{h}^*$ be an ω^*-stable integral weight lattice such that $\alpha_i \in P$ for all $i \in I$, and set $P_+ := \{\lambda \in P \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}$. For $\lambda \in P_+$, we denote by $\mathbb{B}(\lambda)$ the set of Lakshmibai–Seshadri paths of shape λ. Recall from [13, §4] that $\mathbb{B}(\lambda) \cup \{\theta\}$ is stable under the root operators, and that every element π of $\mathbb{B}(\lambda)$ is of the form $\pi = f_{i_1} f_{i_2} \cdots f_{i_k} \pi_\lambda$ for some $i_1, i_2, \ldots, i_k \in I$, where $\pi_\lambda(t) := t\lambda$ for $t \in [0,1]$. Let $w \in W$, and $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ its reduced expression. We put

(2.2.7) $$\mathbb{B}_w(\lambda) := \{f^{m_1}_{i_1} f^{m_2}_{i_2} \cdots f^{m_k}_{i_k} \pi_\lambda \mid m_1, m_2, \ldots, m_k \in \mathbb{Z}_{\geq 0}\} \setminus \{\theta\}.$$

We know that $\mathbb{B}_w(\lambda)$ does not depend on the choice of the reduced expression of w (cf. [12, §5] and [11, §6.1]).

If $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$, then $\mathbb{B}(\lambda)$ is stable under ω (cf. (2.2.5)). Furthermore, we deduce from (2.2.5) that $\omega(\mathbb{B}_w(\lambda)) = \mathbb{B}_{\omega^* w (\omega^*)^{-1}}(\lambda)$. Hence, if $w \in \widetilde{W}$, then $\mathbb{B}_w(\lambda)$ is stable under ω. We set

$$(2.2.8) \qquad \begin{aligned} \mathbb{B}^0(\lambda) &:= \{\pi \in \mathbb{B}(\lambda) \mid \omega(\pi) = \pi\}, \\ \mathbb{B}^0_w(\lambda) &:= \{\pi \in \mathbb{B}_w(\lambda) \mid \omega(\pi) = \pi\}. \end{aligned}$$

We have the following theorem (see [20, Theorem 3.2.4] and [21, Theorem 4.2]).

Theorem 2.2.4. *Let $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. Set $\widehat{\lambda} := (P^*_\omega)^{-1}(\lambda)$ and $\widehat{w} := \Theta^{-1}(w)$.*

(1) *The set $\mathbb{B}^0(\lambda) \cup \{\theta\}$ is stable under the ω-root operators.*

(2) *Each element $\pi \in \mathbb{B}^0(\lambda)$ is of the form $\pi = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\pi_\lambda$ for some $i_1, i_2, \ldots, i_k \in \check{I}$.*

(3) *We have $\mathbb{B}^0(\lambda) = P^*_\omega(\breve{\mathbb{B}}(\widehat{\lambda}))$ and $\mathbb{B}^0_w(\lambda) = P^*_\omega(\breve{\mathbb{B}}_{\widehat{w}}(\widehat{\lambda}))$, where $\breve{\mathbb{B}}(\widehat{\lambda})$ is the set of Lakshmibai–Seshadri paths of shape $\widehat{\lambda}$ for the orbit Lie algebra $\breve{\mathfrak{g}}$, and $\breve{\mathbb{B}}_{\widehat{w}}(\widehat{\lambda})$ is the subset of $\breve{\mathbb{B}}(\widehat{\lambda})$ corresponding to $\widehat{w}$ (cf. (2.2.7)).*

§3. Crystal Bases and Diagram Automorphisms.

3.1. Crystal bases $\mathcal{B}(\lambda)$ and $\mathcal{B}_w(\lambda)$.

Set $P^\vee := \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \subset \mathfrak{h}$. Let $U_q(\mathfrak{g}) = \langle x_i, y_i, q^h \mid i \in I, h \in P^\vee\rangle$ be the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in q, and $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the $\mathbb{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ generated by $\{x_i \mid i \in I\}$ (resp. $\{y_i \mid i \in I\}$).

For $\lambda \in P_+$, let $V(\lambda) = \bigoplus_{\chi \in P} V(\lambda)_\chi$ be the integrable highest weight $U_q(\mathfrak{g})$-module of highest weight λ. Denote by e_i and f_i the raising Kashiwara operator and the lowering Kashiwara operator for $V(\lambda)$, respectively, by $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ the crystal base of $V(\lambda)$, and by $\{G_\lambda(b) \mid b \in \mathcal{B}(\lambda)\}$ the global base of $V(\lambda)$ (see [6]).

For $w \in W$, let $V_w(\lambda) = U_q^+(\mathfrak{g})V(\lambda)_{w(\lambda)}$ be the quantum Demazure module of lowest weight $w(\lambda)$. We know from [8, Proposition 3.2.3] that there exists a subset $\mathcal{B}_w(\lambda)$ of $\mathcal{B}(\lambda)$ such that $V_w(\lambda) = \bigoplus_{b \in \mathcal{B}_w(\lambda)} \mathbb{Q}(q)G_\lambda(b)$. We see from [8, Proposition 3.2.3] that if $w = r_{i_1}r_{i_2}\cdots r_{i_k}$ is a reduced expression of w, then

$$(3.1.1) \quad \mathcal{B}_w(\lambda) = \{f_{i_1}^{m_1} f_{i_2}^{m_2} \cdots f_{i_k}^{m_k}\overline{v}_\lambda \mid m_1, m_2, \ldots, m_k \in \mathbb{Z}_{\geq 0}\} \setminus \{0\},$$

where $\overline{v}_\lambda$ is the image of a (nonzero) highest weight vector v_λ of $V(\lambda)$ in $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$.

Let $U_q(\breve{\mathfrak{g}}) = \langle \widehat{x}_i, \widehat{y}_i, q^{\widehat{h}} \mid i \in \breve{I}, \widehat{h} \in \widehat{P}^\vee \rangle$ be the quantized universal enveloping algebra of the orbit Lie algebra $\breve{\mathfrak{g}}$, where $\widehat{P}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\widehat{P}, \mathbb{Z})$ with $\widehat{P} := (P_\omega^*)^{-1}(P \cap (\mathfrak{h}^*)^0)$. Denote by $\breve{\mathcal{B}}(\widehat{\lambda})$ the crystal base of the integrable highest weight $U_q(\breve{\mathfrak{g}})$-module $\breve{V}(\widehat{\lambda})$ of dominant integral highest weight $\widehat{\lambda}$, and by $\widehat{e}_i$ (resp. $\widehat{f}_i$) the raising (resp. lowering) Kashiwara operator for $\breve{\mathcal{B}}(\widehat{\lambda})$. For $\widehat{w} \in \widehat{W}$, we denote by $\breve{\mathcal{B}}_{\widehat{w}}(\widehat{\lambda})$ the crystal base of the quantum Demazure module $\breve{V}_{\widehat{w}}(\widehat{\lambda}) \subset \breve{V}(\widehat{\lambda})$ of lowest weight $\widehat{w}(\widehat{\lambda})$.

3.2. Fixed point subsets of $\mathcal{B}(\lambda)$ and $\mathcal{B}_w(\lambda)$.

Since $P^\vee$ is ω-stable, we obtain a $\mathbb{Q}(q)$-algebra automorphism $\omega \in \mathrm{Aut}(U_q(\mathfrak{g}))$ such that $\omega(x_i) = x_{\omega(i)}$, $\omega(y_i) = y_{\omega(i)}$, and $\omega(q^h) = q^{\omega(h)}$ for $i \in I$ and $h \in P^\vee$ (cf. [23, Lemma 1.2]). Remark that $U_q^-(\mathfrak{g})$ is stable under ω. If $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$, then we have a $\mathbb{Q}(q)$-linear automorphism $\omega : V(\lambda) \to V(\lambda)$ induced from $\omega : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$. Because

$$\omega \circ e_i = e_{\omega(i)} \circ \omega \quad \text{and} \quad \omega \circ f_i = f_{\omega(i)} \circ \omega \tag{3.2.1}$$

on $V(\lambda)$ (see [22, Lemma 2.3.2]), the crystal lattice $\mathcal{L}(\lambda)$ is stable under ω. Therefore, we have a $\mathbb{Q}$-linear automorphism $\omega : \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \to \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ induced from $\omega : \mathcal{L}(\lambda) \to \mathcal{L}(\lambda)$. We deduce from (3.2.1) that the crystal base $\mathcal{B}(\lambda)$ is stable under ω. Moreover, we obtain by (3.1.1) and (3.2.1) that $\omega(\mathcal{B}_w(\lambda)) = \mathcal{B}_{\omega^* w (\omega^*)^{-1}}(\lambda)$. Hence, if $w \in \widetilde{W}$, then $\mathcal{B}_w(\lambda)$ is stable under ω. We set

$$\begin{aligned} \mathcal{B}^0(\lambda) &:= \{ b \in \mathcal{B}(\lambda) \mid \omega(b) = b \}, \\ \mathcal{B}^0_w(\lambda) &:= \{ b \in \mathcal{B}_w(\lambda) \mid \omega(b) = b \}. \end{aligned} \tag{3.2.2}$$

We see from [13] that $\mathbb{B}(\lambda)$ has a natural (normal) crystal structure for each $\lambda \in P_+$. We know from [3, Corollary 6.4.27] or [10, Theorem 4.1] that there exists an isomorphism $\Phi_\lambda : \mathbb{B}(\lambda) \xrightarrow{\sim} \mathcal{B}(\lambda)$ of crystals, and from [11, §5.6] that $\Phi(\mathbb{B}_w(\lambda)) = \mathcal{B}_w(\lambda)$ for every $w \in W$. If $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$, then we obtain the following commutative diagram (cf. (2.2.5) and (3.2.1)):

$$\begin{array}{ccc} \mathbb{B}(\lambda) & \xrightarrow{\omega} & \mathbb{B}(\lambda) \\ {\scriptstyle \Phi_\lambda}\downarrow & & \downarrow{\scriptstyle \Phi_\lambda} \\ \mathcal{B}(\lambda) & \xrightarrow{\omega} & \mathcal{B}(\lambda). \end{array} \tag{3.2.3}$$

Therefore, we obtain $\Phi_\lambda(\mathbb{B}^0(\lambda)) = \mathcal{B}^0(\lambda)$ and $\Phi_\lambda(\mathbb{B}^0_w(\lambda)) = \mathcal{B}^0_w(\lambda)$ for each $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. Combining this fact with Theorems 2.2.1 and 2.2.4, we get the following proposition.

Proposition 3.2.1. *Let $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. Set $\widehat{\lambda} := (P^*_\omega)^{-1}(\lambda)$ and $\widehat{w} := \Theta^{-1}(w)$.*

(1) *The set $\mathcal{B}^0(\lambda) \cup \{0\}$ is stable under the ω-Kashiwara operators $\widetilde{e}_i$ and $\widetilde{f}_i$, defined in the same way as (2.2.2).*

(2) *Each element $b \in \mathcal{B}^0(\lambda)$ is of the form $b = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\overline{v}_\lambda$ for some $i_1, i_2, \ldots, i_k \in \check{I}$.*

(3) *There exists a canonical bijection $P_\lambda : \mathcal{B}^0(\lambda) \xrightarrow{\sim} \breve{\mathcal{B}}(\widehat{\lambda})$ such that*

$$(3.2.4)\qquad \begin{aligned} &(P^*_\omega)^{-1}(\mathrm{wt}(b)) = \mathrm{wt}(P_\lambda(b)) \quad \textit{for each} \ \ b \in \mathcal{B}^0(\lambda), \\ &P_\lambda \circ \widetilde{e}_i = \widehat{e}_i \circ P_\lambda \quad \textit{and} \quad P_\lambda \circ \widetilde{f}_i = \widehat{f}_i \circ P_\lambda \quad \textit{for all} \ \ i \in \check{I}. \end{aligned}$$

In addition, we have $P_\lambda(\mathcal{B}^0_w(\lambda)) = \breve{\mathcal{B}}_{\widehat{w}}(\widehat{\lambda})$.

3.3. Crystal bases $\mathcal{B}(\infty)$ and $\mathcal{B}_w(\infty)$.

We denote by e_i and f_i the raising Kashiwara operator and the lowering Kashiwara operator for $U_q^-(\mathfrak{g})$, respectively, and by $(\mathcal{L}(\infty), \mathcal{B}(\infty))$ the crystal base of $U_q^-(\mathfrak{g})$. Denote by $\{G(b) \mid b \in \mathcal{B}(\infty)\}$ the global base of $U_q^-(\mathfrak{g})$ (see [6]).

Let $Q_+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\, \alpha_i$, and set $Q_+(n) := \{\alpha \in \mathbb{Q}_+ \mid \mathrm{ht}(\alpha) \leq n\}$ for each $n \in \mathbb{Z}_{\geq 0}$, where $\mathrm{ht}(\alpha) := \sum_{i\in I} k_i$ for $\alpha = \sum_{i \in I} k_i\alpha_i \in Q_+$. Let us recall the following theorem from [6, Theorem 5 and Corollary 4.4.5].

Theorem 3.3.1. *Let $\varphi_\lambda : U_q^-(\mathfrak{g}) \to V(\lambda)$ be the canonical $U_q^-(\mathfrak{g})$-module homomorphism sending 1 to v_λ.*

(1) *We have $\varphi_\lambda(\mathcal{L}(\infty)) = \mathcal{L}(\lambda)$. Hence we have a $\mathbb{Q}$-linear homomorphism*

$$(3.3.1)\qquad \overline{\varphi}_\lambda : \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$$

induced from $\varphi_\lambda : \mathcal{L}(\infty) \to \mathcal{L}(\lambda)$. The restriction of $\overline{\varphi}_\lambda$ to $\mathcal{B}(\infty) \setminus \overline{\varphi}_\lambda^{-1}(\{0\})$ is a bijection from $\mathcal{B}(\infty) \setminus \overline{\varphi}_\lambda^{-1}(\{0\})$ to $\mathcal{B}(\lambda)$.

(2) *We have $f_i \circ \overline{\varphi}_\lambda = \overline{\varphi}_\lambda \circ f_i$ for each $i \in I$. In addition, if $b \in \mathcal{B}(\infty)$ satisfies $\overline{\varphi}_\lambda(b) \neq 0$, then $e_i\overline{\varphi}_\lambda(b) = \overline{\varphi}_\lambda(e_ib)$ for each $i \in I$.*

(3) *Fix $n \in \mathbb{Z}_{\geq 0}$. If $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$, then, for every $\xi \in Q_+(n)$, the restriction of $\overline{\varphi}_\lambda$ to $\mathcal{B}(\infty)_{-\xi}$ is a bijection from $\mathcal{B}(\infty)_{-\xi}$ to $\mathcal{B}(\lambda)_{\lambda-\xi}$. Here, for a crystal $\mathcal{B}$, we denote by $\mathcal{B}_\mu$ the set of elements of weight μ in $\mathcal{B}$.*

Let $w \in W$, and $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ its reduced expression. We define a module $(U_w^-)_q(\mathfrak{g})$ of Demazure type by

$$(U_w^-)_q(\mathfrak{g}) := \sum_{m_j \in \mathbb{Z}_{\geq 0}} \mathbb{Q}(q) y_{i_1}^{m_1} y_{i_2}^{m_2} \cdots y_{i_k}^{m_k}. \tag{3.3.2}$$

We know from [8, Proposition 3.2.5] that $(U_w^-)_q(\mathfrak{g}) = \bigoplus_{b \in \mathcal{B}_w(\infty)} \mathbb{Q}(q) G(b)$, where

$$\mathcal{B}_w(\infty) := \{ f_{i_1}^{m_1} f_{i_2}^{m_2} \cdots f_{i_k}^{m_k} \overline{v}_\infty \mid m_1, m_2, \ldots, m_k \in \mathbb{Z}_{\geq 0} \}, \tag{3.3.3}$$

with $\overline{v}_\infty$ the image of $1 \in U_q^-(\mathfrak{g})$ in $\mathcal{L}(\infty)/q\mathcal{L}(\infty)$. Furthermore, we can easily show the following theorem, by using [8, Proposition 3.2.5], Theorem 3.3.1, (3.1.1), and (3.3.3).

Theorem 3.3.2. (1) *The restriction of $\overline{\varphi}_\lambda$ to $\mathcal{B}_w(\infty) \setminus \overline{\varphi}_\lambda^{-1}(\{0\})$ is a bijection from $\mathcal{B}_w(\infty) \setminus \overline{\varphi}_\lambda^{-1}(\{0\})$ to $\mathcal{B}_w(\lambda)$.*
(2) *Fix $n \in \mathbb{Z}_{\geq 0}$. If $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$, then, for every $\xi \in Q_+(n)$, the restriction of $\overline{\varphi}_\lambda$ to $\mathcal{B}_w(\infty)_{-\xi}$ is a bijection from $\mathcal{B}_w(\infty)_{-\xi}$ to $\mathcal{B}_w(\lambda)_{\lambda-\xi}$.*

Remark 3.3.3. It follows from Theorem 3.3.2 that $\mathcal{B}_w(\infty)$ (and hence $(U_w^-)_q(\mathfrak{g})$) does not depend on the choice of the reduced expression of w.

Denote by $\breve{\mathcal{B}}(\infty)$ the crystal base of $U_q^-(\breve{\mathfrak{g}}) := \langle \widehat{y}_i \mid i \in \breve{I} \rangle$, and by $\widehat{e}_i$ (resp. $\widehat{f}_i$) the raising (resp. lowering) Kashiwara operator for $\breve{\mathcal{B}}(\infty)$. For $\widehat{w} \in \widehat{W}$, we denote by $\breve{\mathcal{B}}_{\widehat{w}}(\infty)$ the crystal base of the module $(U_{\widehat{w}}^-)_q(\breve{\mathfrak{g}})$ of Demazure type corresponding to $\widehat{w}$.

3.4. Fixed point subsets of $\mathcal{B}(\infty)$ and $\mathcal{B}_w(\infty)$.

In a way similar to the case of $V(\lambda)$, we can show that $\omega \circ e_i = e_{\omega(i)} \circ \omega$ and $\omega \circ f_i = f_{\omega(i)} \circ \omega$ on $U_q^-(\mathfrak{g})$. Thus, $\mathcal{L}(\infty)$ is stable under ω, and hence we have a $\mathbb{Q}$-linear automorphism $\omega : \mathcal{L}(\infty)/q\mathcal{L}(\infty) \to \mathcal{L}(\infty)/q\mathcal{L}(\infty)$ induced from $\omega : \mathcal{L}(\infty) \to \mathcal{L}(\infty)$. It is obvious that $\mathcal{B}(\infty)$ is stable under ω. Moreover we deduce that $\omega(\mathcal{B}_w(\infty)) = \mathcal{B}_{\omega^* w (\omega^*)^{-1}}(\infty)$ for $w \in W$. Therefore, if $w \in \widetilde{W}$, then $\mathcal{B}_w(\infty)$ is stable under ω. We now set

$$\begin{aligned} \mathcal{B}^0(\infty) &:= \{ b \in \mathcal{B}(\infty) \mid \omega(b) = b \}, \\ \mathcal{B}_w^0(\infty) &:= \{ b \in \mathcal{B}_w(\infty) \mid \omega(b) = b \}. \end{aligned} \tag{3.4.1}$$

Theorem 3.4.1. (1) *The set $\mathcal{B}^0(\infty) \cup \{0\}$ is stable under the ω-Kashiwara operators $\widetilde{e}_i$ and $\widetilde{f}_i$, defined in the same way as (2.2.2).*

(2) *Each element* $b \in \mathcal{B}^0(\infty)$ *is of the form* $b = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\overline{v}_\infty$ *for some* $i_1, i_2, \ldots, i_k \in \breve{I}$.

(3) *There exists a canonical bijection* $P_\infty : \mathcal{B}^0(\infty) \xrightarrow{\sim} \breve{\mathcal{B}}(\infty)$ *such that*

$$(3.4.2)\quad \begin{aligned} &(P_\omega^*)^{-1}(\mathrm{wt}(b)) = \mathrm{wt}(P_\infty(b)) \quad \textit{for each} \;\; b \in \mathcal{B}^0(\infty), \\ &P_\infty \circ \widetilde{e}_i = \widehat{e}_i \circ P_\infty \quad \textit{and} \quad P_\infty \circ \widetilde{f}_i = \widehat{f}_i \circ P_\infty \quad \textit{for all} \;\; i \in \breve{I}. \end{aligned}$$

In addition, we have $P_\infty(\mathcal{B}_w^0(\lambda)) = \breve{\mathcal{B}}_{\widehat{w}}(\infty)$ *for each* $w \in \widetilde{W}$, *where* $\widehat{w} := \Theta^{-1}(w)$.

Proof. Because $\omega \circ \varphi_\lambda = \varphi_\lambda \circ \omega$ for $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$, we have the following commutative diagram (cf. Theorem 3.3.1):

$$(3.4.3)\quad \begin{array}{ccc} \mathcal{B}(\infty) & \xrightarrow{\overline{\varphi}_\lambda} & \mathcal{B}(\lambda) \cup \{0\} \\ {\scriptstyle \omega}\big\downarrow & & \big\downarrow{\scriptstyle \omega} \\ \mathcal{B}(\infty) & \xrightarrow{\overline{\varphi}_\lambda} & \mathcal{B}(\lambda) \cup \{0\}. \end{array}$$

Thus we obtain $\overline{\varphi}_\lambda(\mathcal{B}^0(\infty)) = \mathcal{B}^0(\lambda) \cup \{0\}$ and $\overline{\varphi}_\lambda(\mathcal{B}_w^0(\infty)) = \mathcal{B}_w^0(\lambda) \cup \{0\}$ for each $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$.

(1) Let $b \in \mathcal{B}^0(\infty)$. Assume that $\widetilde{e}_i b \neq 0$. Take $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ such that $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$. Then we deduce from Theorem 3.3.1 (2) and (3) that $\widetilde{e}_i \overline{\varphi}_\lambda(b) = \overline{\varphi}_\lambda(\widetilde{e}_i b) \neq 0$. Since $\widetilde{e}_i \overline{\varphi}_\lambda(b) \in \mathcal{B}^0(\lambda)$ by Proposition 3.2.1 (1), we conclude that $\widetilde{e}_i b \in \mathcal{B}^0(\infty)$. Similarly, we can show that $\widetilde{f}_i b \in \mathcal{B}^0(\infty) \cup \{0\}$.

(2) Let $b \in \mathcal{B}^0(\infty)$. Since $\overline{\varphi}_\lambda(b) \in \mathcal{B}^0(\lambda)$ if $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$, we see from Proposition 3.2.1 (2) that $\overline{\varphi}_\lambda(b) = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\overline{v}_\lambda$ for some $i_1, i_2, \ldots, i_k \in \breve{I}$. By Theorem 3.3.1 (1) and (2), we get $b = \widetilde{f}_{i_1}\widetilde{f}_{i_2}\cdots\widetilde{f}_{i_k}\overline{v}_\infty$. Thus we have proved part (2).

(3) Let $\xi \in Q_+ \cap (\mathfrak{h}^*)^0$, and set $\widehat{\xi} := (P_\omega^*)^{-1}(\xi)$. Take $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ such that $\lambda(\alpha_i^\vee) \gg 0$ for all $i \in I$, and set $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$. We define a bijection $P_{\infty,\xi} : \mathcal{B}(\infty)_{-\xi} \to \breve{\mathcal{B}}(\infty)_{-\widehat{\xi}}$ as in the following commutative diagram:

$$(3.4.4)\quad \begin{array}{ccc} \mathcal{B}^0(\infty)_{-\xi} & \xrightarrow{\sim} & \mathcal{B}^0(\lambda)_{\lambda-\xi} \\ {\scriptstyle P_{\infty,\xi}}\big\downarrow & & \big\downarrow{\scriptstyle P_\lambda} \\ \breve{\mathcal{B}}(\infty)_{-\widehat{\xi}} & \xleftarrow{\sim} & \breve{\mathcal{B}}(\widehat{\lambda})_{\widehat{\lambda}-\widehat{\xi}}. \end{array}$$

We can easily check that $P_{\infty,\xi}$ does not depend on the choice of λ. Now we define $P_\infty : \mathcal{B}^0(\infty) \to \breve{\mathcal{B}}(\infty)$ by $P_\infty(b) := P_{\infty,\xi}(b)$ for $b \in \mathcal{B}^0(\infty)_{-\xi}$.

We can easily show by Proposition 3.2.1 (3) and Theorem 3.3.1 that P_∞ has the desired properties (3.4.2). The equality $P_\infty(\mathcal{B}^0_w(\lambda)) = \breve{\mathcal{B}}_{\widehat{w}}(\infty)$ immediately follows from the definition of P_∞ and the equality $\overline{\varphi}_\lambda(\mathcal{B}^0_w(\infty)) = \mathcal{B}^0_w(\lambda) \cup \{0\}$. $\square$

Remark 3.4.2. It immediately follows from Theorem 3.4.1 that there exists an injection from the global base of $U_q^-(\breve{\mathfrak{g}})$ to the global base of $U_q^-(\mathfrak{g})$. Therefore we have an embedding $U_q^-(\breve{\mathfrak{g}}) \hookrightarrow U_q^-(\mathfrak{g})$ of vector spaces.

§4. Twining Character Formulas.

4.1. Definitions.

The twining character $\mathrm{ch}^\omega(U_q^-(\mathfrak{g}))$ of $U_q^-(\mathfrak{g})$ is defined to be the following formal sum:

$$\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) = \sum_{\chi\in(\mathfrak{h}^*)^0} \mathrm{tr}\big(\omega|_{(U_q^-(\mathfrak{g}))_\chi}\big)e(\chi). \tag{4.1.1}$$

For each $w \in \widetilde{W}$, we define the twining character $\mathrm{ch}^\omega((U_w^-)_q(\mathfrak{g}))$ of $(U_w^-)_q(\mathfrak{g})$ by

$$\mathrm{ch}^\omega((U_w^-)_q(\mathfrak{g})) := \sum_{\chi\in(\mathfrak{h}^*)^0} \mathrm{tr}\big(\omega|_{((U_w^-)_q(\mathfrak{g}))_\chi}\big)e(\chi). \tag{4.1.2}$$

4.2. Twining character formulas.

Corollary 4.2.1. *Let $w \in \widetilde{W}$, and set $\widehat{w} := \Theta^{-1}(w)$. Then we have*

$$\begin{aligned}\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) &= P_\omega^*(\mathrm{ch}\, U_q^-(\breve{\mathfrak{g}})),\\ \mathrm{ch}^\omega((U_w^-)_q(\mathfrak{g})) &= P_\omega^*(\mathrm{ch}\,(U_{\widehat{w}}^-)_q(\breve{\mathfrak{g}})).\end{aligned} \tag{4.2.1}$$

In order to prove this corollary, we need the following lemma, which can be shown in exactly the same way as [23, Lemma 3.4].

Lemma 4.2.2. *We have $\omega(G(b)) = G(\omega(b))$ for all $b \in \mathcal{B}(\infty)$. Therefore, we see that the global base $\{G(b) \mid b \in \mathcal{B}(\infty)\}$ of $U_q^-(\mathfrak{g})$ is stable under ω, and that $\omega(G(b)) = G(b)$ if and only if $b \in \mathcal{B}^0(\infty)$.*

Proof of Corollary 4.2.1. We give a proof only for the first equality of (4.2.1), since the proof for the second one is similar. Remark that for each $\chi \in (\mathfrak{h}^*)^0$, $\{G(b) \mid b \in \mathcal{B}(\infty)_\chi\}$ is a basis of $U_q^-(\mathfrak{g})_\chi$, which is stable under ω. Therefore we have

$$\mathrm{tr}\big(\omega|_{(U_q^-(\mathfrak{g}))_\chi}\big) = \#\{G(b) \mid \omega(G(b)) = G(b),\ b \in \mathcal{B}(\infty)_\chi\}$$

for each $\chi \in (\mathfrak{h}^*)^0$ (note that if an endomorphism f on a finite-dimensional vector space V stabilizes a basis of V, then the trace of f on V is equal to the number of the basis elements fixed by f). By Lemma 4.2.2, we get

$$\mathrm{tr}\big(\omega|_{(U_q^-(\mathfrak{g}))_\chi}\big) = \#\big(\mathcal{B}(\infty)_\chi \cap \mathcal{B}^0(\infty)\big),$$

and hence

$$\mathrm{ch}^\omega(U_q^-(\mathfrak{g})) = \sum_{b \in \mathcal{B}^0(\infty)} e(\mathrm{wt}(b)) \tag{4.2.2}$$

Therefore we obtain

$$\begin{aligned} \mathrm{ch}^\omega(U_q^-(\mathfrak{g})) &= \sum_{b \in \mathcal{B}^0(\infty)} e(\mathrm{wt}(b)) && \text{by (4.2.2)} \\ &= P_\omega^*\Big(\sum_{\breve{b} \in \breve{\mathcal{B}}(\infty)} e(\mathrm{wt}(\breve{b}))\Big) && \text{by Theorem 3.4.1 (3)} \\ &= P_\omega^*(\mathrm{ch}\, U_q^-(\breve{\mathfrak{g}})), \end{aligned}$$

as desired. □

Remark 4.2.3. Let $U_q^-(\mathfrak{g})_\mathbb{Z}$ be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q^-(\mathfrak{g})$ generated by the divided powers $\{y_i^{(n)} \mid i \in I,\, n \in \mathbb{Z}_{\geq 0}\}$ (see [6, §6.1]), and set

$$(U_w^-)_q(\mathfrak{g})_\mathbb{Z} := \sum_{m_j \geq 0} \mathbb{Z}[q, q^{-1}] y_{i_1}^{(m_1)} y_{i_2}^{(m_2)} \cdots y_{i_k}^{(m_k)} \subset (U_w^-)_q(\mathfrak{g}) \tag{4.2.3}$$

for $w \in W$ with $w = r_{i_1} r_{i_2} \cdots r_{i_k}$ its reduced expression. Now, for $\lambda \in P_+$ and $w \in \widetilde{W}$, we set $V(\lambda)_\mathbb{Z} := U_q^-(\mathfrak{g})_\mathbb{Z}\, v_\lambda \subset V(\lambda)$ and $V_w(\lambda)_\mathbb{Z} := (U_w^-)_q(\mathfrak{g})_\mathbb{Z}\, v_\lambda \subset V_w(\lambda)$ (cf. [8, Corollary 3.2.2]). Assume that $\lambda \in P_+ \cap (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. We can easily check that the $\mathbb{Z}[q, q^{-1}]$-forms $U_q^-(\mathfrak{g})_\mathbb{Z}$, $(U_w^-)_q(\mathfrak{g})_\mathbb{Z}$, $V(\lambda)_\mathbb{Z}$, and $V_w(\lambda)_\mathbb{Z}$ are stable under the action of ω. Hence we can define the twining characters of them in a way similar to (4.1.1) and (4.1.2). Using the fact that the global bases are $\mathbb{Z}[q, q^{-1}]$-bases of these $\mathbb{Z}[q, q^{-1}]$-forms, we can prove twining character formulas for these $\mathbb{Z}[q, q^{-1}]$-forms in exactly the same way as Corollary 4.2.1 (see also [23]).

Let $M(\lambda)$ be the Verma module of highest weight $\lambda \in \mathfrak{h}^*$ over $\mathfrak{g}$ with (nonzero) highest weight vector v_λ. For $w \in W$ with $w = r_{i_1} r_{i_2} \ldots r_{i_k}$

its reduced expression, we define a module $M_w(\lambda) \subset M(\lambda)$ of Demazure type by

$$(4.2.4) \qquad M_w(\lambda) := \sum_{m_j \in \mathbb{Z}} \mathbb{Q} y_{i_1}^{m_1} y_{i_2}^{m_2} \cdots y_{i_k}^{m_k} v_\lambda.$$

We see that $M_w(\lambda)$ does not depend on the choice of the reduced expression of w.

Assume that $\lambda \in (\mathfrak{h}^*)^0$. Then we have a $\mathbb{Q}$-linear automorphism $\omega : M(\lambda) \to M(\lambda)$ induced from the $\mathbb{Q}$-algebra automorphism $\omega \in \mathrm{Aut}(U(\mathfrak{g}))$ of the universal enveloping algebra of $\mathfrak{g}$ (cf. §1.1). We can easily check that $M_w(\lambda)$ is stable under ω if $w \in \widetilde{W}$. The twining characters $\mathrm{ch}^\omega(M(\lambda))$ and $\mathrm{ch}^\omega(M_w(\lambda))$ are defined in the same way as (4.1.1) and (4.1.2), respectively (see also [1, Definition 2.3]).

Corollary 4.2.4. *Let $\lambda \in (\mathfrak{h}^*)^0$, and $w \in \widetilde{W}$. Set $\widehat{\lambda} := (P_\omega^*)^{-1}(\lambda)$, and $\widehat{w} := \Theta^{-1}(w)$. Then we have*

$$(4.2.5) \qquad \begin{aligned} \mathrm{ch}^\omega(M(\lambda)) &= P_\omega^*(\mathrm{ch}\, \breve{M}(\widehat{\lambda})), \\ \mathrm{ch}^\omega(M_w(\lambda)) &= P_\omega^*(\mathrm{ch}\, \breve{M}_{\widehat{w}}(\widehat{\lambda})), \end{aligned}$$

where $\breve{M}(\widehat{\lambda})$ is the Verma module of highest weight $\widehat{\lambda}$ over the orbit Lie algebra $\breve{\mathfrak{g}}$, and $\breve{M}_{\widehat{w}}(\widehat{\lambda}) \subset \breve{M}(\widehat{\lambda})$ is the module of Demazure type for $\breve{\mathfrak{g}}$ corresponding to $\widehat{w}$.

Proof. We give a proof only for the first equality $\mathrm{ch}^\omega(M(\lambda)) = P_\omega^*(\mathrm{ch}\, \breve{M}(\widehat{\lambda}))$ of (4.2.5), since the proof of the second one is similar. We see easily that $\mathrm{ch}^\omega(M(\lambda)) = e(\lambda)\, \mathrm{ch}^\omega(M(0))$ and $\mathrm{ch}\, \breve{M}(\widehat{\lambda}) = e(\widehat{\lambda})\, \mathrm{ch}\, \breve{M}(0)$. Hence we need only show that $\mathrm{ch}^\omega(M(0)) = P_\omega^*(\mathrm{ch}\, \breve{M}(0))$.

As in [23, §2.2], we deduce that the specialization "$q = 1$" of $\mathrm{ch}^\omega(U_q^-(\mathfrak{g}))$ is equal to $\mathrm{ch}^\omega(M(0))$. On the other hand, the specialization "$q = 1$" of $\mathrm{ch}\, U_q^-(\breve{\mathfrak{g}})$ is equal to $\mathrm{ch}\, \breve{M}(0)$. By combining these facts with Corollary 4.2.1, we obtain $\mathrm{ch}^\omega(M(0)) = P_\omega^*(\mathrm{ch}\, \breve{M}(0))$. Thus we have proved the corollary. □

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Extremal weight modules of quantum affine algebras

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Abstract.

Let $\widehat{\mathfrak{g}}$ be an affine Lie algebra, and let $\mathbf{U}_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra introduced by Drinfeld and Jimbo. In [11] Kashiwara introduced a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module $V(\lambda)$, having a global crystal base for an integrable weight λ of level 0. We call it an *extremal weight module*. It is isomorphic to the Weyl module introduced by Chari-Pressley [6]. In [12, §13] Kashiwara gave a conjecture on the structure of extremal weight modules. We prove his conjecture when $\widehat{\mathfrak{g}}$ is an untwisted affine Lie algebra of a simple Lie algebra $\mathfrak{g}$ of type ADE, using a result of Beck-Chari-Pressley [5]. As a by-product, we also show that the extremal weight module is isomorphic to a universal standard module, defined via quiver varieties by the author [16, 18]. This result was conjectured by Varagnolo-Vasserot [19] and Chari-Pressley [6] in a less precise form. Furthermore, we give a characterization of global crystal bases by an almost orthogonality propery, as in the case of global crystal base of highest weight modules.

§1. Introduction

In the conference, I gave a survey on quiver varieties and finite dimensional representations of quantum affine algebras. Since I already wrote a survey article [17] on this subject, I will discuss a different one in this paper. But it is related to my talks since I will study extremal weight modules which turn out to be isomorphic to universal standard modules, which was one of the main objects in my talk.

Let us describe Kashiwara's conjecture [12, §13] on extremal weight modules when $\widehat{\mathfrak{g}}$ is the untwisted affine Lie algebra of a simple Lie algebra $\mathfrak{g}$ of type ADE. The notation will be explained in the next section.

Let λ be a dominant integral weight of $\mathfrak{g}$. We write $\lambda = \sum_{i\in I} m_i \varpi_i$, where ϖ_i is the i-th fundamental weight of $\mathfrak{g}$. We consider λ, ϖ_i as level 0 weights of $\widehat{\mathfrak{g}}$ by identifying them with $\lambda - \sum_i m_i a_i^\vee \Lambda_0$, $\Lambda_i - a_i^\vee \Lambda_0$,

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where $c = \sum_i a_i^\vee h_i$ is the central element, and Λ_i is the ith fundamental weight of $\widehat{\mathfrak{g}}$. Let $V(\lambda)$ be the extremal weight module of extremal weight λ with a global crystal base $(\mathcal{L}(\lambda), \mathcal{B}(\lambda), V^{\mathbb{Z}}(\lambda))$ (see §2.5 for definition). Let us define a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module

$$\widetilde{V}(\lambda) \stackrel{\text{def.}}{=} \bigotimes_{i\in I} V(\varpi_i)^{\otimes m_i},$$

where we take and fix any ordering of I to define the tensor product. It has $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-module automorphisms $z_{i,\nu}$ ($i \in I$, $\nu = 1, \dots, m_i$) (see §3.2).

Set $\widetilde{\mathcal{L}}(\lambda) \stackrel{\text{def.}}{=} \bigotimes_{i\in I} \mathcal{L}(\varpi_i)^{\otimes m_i}$, $\widetilde{u}_\lambda \stackrel{\text{def.}}{=} \bigotimes_{i\in I} u_{\varpi_i}^{\otimes m_i}$. Let $\widetilde{\mathcal{B}}_0(\lambda)$ be the connected component of the crystal $\bigotimes_{i\in I} \mathcal{B}(\varpi_i)^{\otimes m_i}$ containing $\widetilde{u}_\lambda$ mod $q\widetilde{\mathcal{L}}(\lambda)$. There is a (subset of) global base $\{G(b) \mid b \in \mathcal{B}_0(\lambda)\}$ (see §3.2). Let $\widetilde{\mathcal{B}}(\lambda) \stackrel{\text{def.}}{=} \{s(z)b \mid b \in \widetilde{\mathcal{B}}_0(\lambda),\ s \in (\mathbb{Z}^{\mathfrak{R}_0}_{\geq 0})(\lambda)\}$ where $s(z) = \prod_{i\in I} s_{\lambda^{(i)}}(z_{i,1}, \dots, z_{i,m_i})$ runs over the set $(\mathbb{Z}^{\mathfrak{R}_0}_{\geq 0})(\lambda)$ of products of Schur functions.

There exists a unique $\mathbf{U}_q(\widehat{\mathfrak{g}})$-linear homomorphism

$$\Phi_\lambda \colon V(\lambda) \to \widetilde{V}(\lambda)$$

sending u_λ to $\widetilde{u}_\lambda$ (see §3.2).

Theorem 1. (1) *Φ_λ is injective.*

(2) $\Phi_\lambda(\mathcal{L}(\lambda)) \subset \widetilde{\mathcal{L}}(\lambda)$.

Let Φ_λ^0 be the induced map $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \to \widetilde{\mathcal{L}}(\lambda)/q\widetilde{\mathcal{L}}(\lambda)$.

(3) *Φ_λ^0 gives a bijection between $\mathcal{B}(\lambda)$ and $\widetilde{\mathcal{B}}(\lambda)$.*

(4) *Φ_λ maps the global crystal base $\{G(b) \mid b \in \mathcal{B}(\lambda)\}$ to $\{s(z)G(b) \mid b \in \widetilde{\mathcal{B}}_0(\lambda),\ s \in (\mathbb{Z}^{\mathfrak{R}_0}_{\geq 0})(\lambda)\}$.*

While the author was preparing this article, he learned that Kashiwara also noticed that his conjecture follows from [5] when $\mathfrak{g}$ is of type ADE. In fact, some arguments (the proof of the injectivity of Φ_λ, the definition of (,), etc.) has been improved from the original form after the discussion with him. After the author posted the first version of this paper to the network archive, he was informed that Jonathan Beck also proved a part of Kashiwara's conjecture [4].

§2. Preliminaries

2.1. Affine Lie algebra

Let us fix notations for the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$. (For a moment we do not assume that $\mathfrak{g}$ is of type ADE.)

(1) $\widehat{I}$: the index set of simple roots,
(2) $\{\alpha_i\}_{i\in\widehat{I}}$: the set of simple roots; $\{h_i\}_{i\in\widehat{I}}$: the set of simple coroots,
(3) $\widehat{P}^* \overset{\text{def.}}{=} \bigoplus_{i\in\widehat{I}}\mathbb{Z}h_i\oplus\mathbb{Z}d$: the dual weight lattice; $\widehat{P} = \mathrm{Hom}_{\mathbb{Z}}(\widehat{P}^*, \mathbb{Z})$: the weight lattice,
(4) $\widehat{\mathfrak{h}} \overset{\text{def.}}{=} \widehat{P}^* \otimes_{\mathbb{Z}} \mathbb{Q}$: the Cartan subalgebra,
(5) the simple root $\alpha_i \in \widehat{P}$ defined by $\langle h_i, \alpha_j\rangle = a_{ij}, \langle d, \alpha_j\rangle = \delta_{0j}$, where a_{ij} is the Cartan matrix of $\widehat{\mathfrak{g}}$,
(6) the fundamental weight $\Lambda_i \in \widehat{P}$ defined by $\langle h_i, \Lambda_j\rangle = \delta_{ij}, \langle d, \Lambda_j\rangle = 0$.
(7) $\widehat{Q} \overset{\text{def.}}{=} \bigoplus_{i\in\widehat{I}}\mathbb{Z}\alpha_i$: the root lattice; $\widehat{Q}^\vee \overset{\text{def.}}{=} \bigoplus_{i\in\widehat{I}}\mathbb{Z}h_i$: the coroot lattice,
(8) $\widehat{Q}_+ \overset{\text{def.}}{=} \sum_{i\in\widehat{I}}\mathbb{Z}_{\geq 0}\alpha_i$; $\widehat{P}_+ \overset{\text{def.}}{=} \{\lambda \in \widehat{P} \mid \langle h_i, \lambda\rangle \geq 0 \text{ for all } i \in \widehat{I}\}$: the set of integral dominant weights,
(9) the unique element $c = \sum_{i\in\widehat{I}} a_i^\vee h_i$ $(a_i^\vee \in \mathbb{Z}_{\geq 0})$ satisfying
$\left\{h \in \widehat{Q}^\vee \,\middle|\, \langle h, \alpha_j\rangle = 0 \text{ for all } j \in \widehat{I}\right\} = \mathbb{Z}c$,
(10) the unique element $\delta = \sum_{i\in\widehat{I}} a_i\alpha_i$ $(a_i \in \mathbb{Z}_{\geq 0})$ satisfying
$\left\{\lambda \in \widehat{Q} \,\middle|\, \langle h_i, \lambda\rangle = 0 \text{ for all } i \in \widehat{I}\right\} = \mathbb{Z}\delta$,
(11) the symmetric bilinear form $(\ ,\ )$ on $\widehat{\mathfrak{h}}^*$, uniquely characterized by $\langle h_i, \lambda\rangle = \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$, $\langle c, \lambda\rangle = (\delta, \lambda)$, for $\lambda \in \widehat{\mathfrak{h}}^*$,
(12) $h \overset{\text{def.}}{=} \sum_{i\in\widehat{I}} a_i$: the Coxeter number; $h^\vee \overset{\text{def.}}{=} \sum_{i\in\widehat{I}} a_i^\vee$: the dual Coxeter number.

The symmetric bilinear form $(\ ,\ )$ is known to be nondegenerate, and defines an isomorphism $\nu\colon \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ by $\langle h, \lambda\rangle = (\nu(h), \lambda)$ for $\lambda \in \widehat{\mathfrak{h}}^*$. For example, $\nu(c) = \delta$. This coincides with one in [9, §6].

For $\beta \in \widehat{\mathfrak{h}}^*$ with $(\beta, \beta) \neq 0$, we set $\beta^\vee \overset{\text{def.}}{=} \frac{2\beta}{(\beta,\beta)}$. We have $\nu(h_i) = \alpha_i^\vee$.

We have an element $0 \in \widehat{I}$ such that $\{\alpha_i \mid i \neq 0\}$ is the set of simple roots of $\mathfrak{g}$. It is known $a_0^\vee = a_0 = 1$ for the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$. We denote $\widehat{I} \setminus \{0\}$ by I.

Let $\mathrm{cl}\colon \widehat{\mathfrak{h}}^* \to \widehat{\mathfrak{h}}^*/\mathbb{Q}\delta$ be the natural projection. Let $\widehat{\mathfrak{h}}^{*0} \overset{\text{def.}}{=} \{\lambda \in \widehat{\mathfrak{h}}^{*0} \mid \langle c, \lambda\rangle = 0\}$, $\widehat{P}^0 \overset{\text{def.}}{=} \widehat{P}\cap\widehat{\mathfrak{h}}^{*0}$ (level 0 weights). We identify $\mathrm{cl}(\widehat{\mathfrak{h}}^{*0}) \subset \widehat{\mathfrak{h}}^*/\mathbb{Q}\delta$ with the dual of the Cartan subalgebra $\mathfrak{h}$ of the finite dimensional Lie algebra $\mathfrak{g}$, which is $\bigoplus_{i\in I}\mathbb{Q}h_i$. Similarly we identify $\mathrm{cl}(\widehat{P}^0)$ with the weight lattice P of $\mathfrak{g}$. We define the root lattice of $\mathfrak{g}$ by $Q \overset{\text{def.}}{=} \bigoplus_{i\in I}\mathbb{Z}\alpha_i$. For $i \in I$, we set $\varpi_i \overset{\text{def.}}{=} \Lambda_i - a_i^\vee\Lambda_0 \in \widehat{P}^0$. Then $\mathrm{cl}(\varpi_i)$ is identified with the ith fundamental weight of $\mathfrak{g}$. Let $\widehat{P}^{0,+} \overset{\text{def.}}{=} \left\{\lambda \in \widehat{P}^0 \,\middle|\, \langle h_i, \lambda\rangle \geq 0 \text{ for } i \in I\right\}$.

Its projection $\mathrm{cl}(\widehat{P}^{0,+})$ is the set of dominant integrable weights of $\mathfrak{g}$. Let $P^\vee \stackrel{\text{def.}}{=} \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$. The fundamental coweights $\varpi_i^\vee$ are defined by $\langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}$ for $i, j \in I$. We extend $\varpi_i^\vee$ to a homomorphism $\widehat{Q} \to \mathbb{Z}$ by setting $\langle \varpi_i^\vee, \delta \rangle = 0$.

Let Δ (resp. Δ_+) be the set of roots (resp. positive roots) of $\mathfrak{g}$. The set of roots $\widehat{\mathcal{R}}$ of $\widehat{\mathfrak{g}}$ is given by $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}_+ \sqcup \widehat{\mathcal{R}}_-$, where

$$\widehat{\mathcal{R}}_+ = \begin{matrix} \{k\delta + \alpha \mid k \ge 0, \alpha \in \Delta_+\} \sqcup \{k\delta \mid k > 0\} \\ \sqcup \{k\delta - \alpha \mid k > 0, \alpha \in \Delta_+\}, \end{matrix}, \qquad \widehat{\mathcal{R}}_- = -\widehat{\mathcal{R}}_+.$$

The roots of the form $k\delta \pm \alpha$ ($k \in \mathbb{Z}$, $\alpha \in \Delta$) are called *real* roots, while roots $k\delta$ are called *imaginary* roots. The multiplicities of real roots are 1, and those of imaginary roots are equal to the rank of $\mathfrak{g}$, i.e., $\#I$.

Set

$$\mathcal{R}_> \stackrel{\text{def.}}{=} \{k\delta + \alpha \mid k \ge 0, \alpha \in \Delta^+\}, \quad \mathcal{R}_< \stackrel{\text{def.}}{=} \{k\delta - \alpha \mid k > 0, \alpha \in \Delta^+\},$$
$$\mathcal{R}_0 \stackrel{\text{def.}}{=} \{k\delta \mid k > 0\} \times I, \quad \mathcal{R} \stackrel{\text{def.}}{=} \mathcal{R}_> \sqcup \mathcal{R}_0 \sqcup \mathcal{R}_<.$$

These are sets of roots, counted with multiplicities.

For $i \in \widehat{I}$, we define the reflection s_i acting on $\widehat{\mathfrak{h}}^*$ by $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$. Moreover, s_i acts also on $\widehat{\mathfrak{h}}$ by $s_i(h) = h - \langle h, \alpha_i \rangle h_i$. The actions of s_i preserve $\widehat{P}$, $\widehat{Q}$, $\widehat{Q}^\vee$ and $\widehat{\mathfrak{h}}^{*0}$. We have $s_i\delta = \delta$, $s_i c = c$. If $i \in I$, the corresponding reflection s_i preserves $\mathfrak{h}$, P, $P^\vee$ and Q. The *Weyl group* W (resp. *affine Weyl group* $\widehat{W}$) of $\mathfrak{g}$ (resp. $\widehat{\mathfrak{g}}$) is the subgroups of $\mathrm{GL}(\mathfrak{h}^*)$ (resp. $\mathrm{GL}(\widehat{\mathfrak{h}}^*)$) generated by s_i for $i \in I$ (resp. $i \in \widehat{I}$). We define the *extended Weyl group* $\widetilde{W}$ as the semidirect product $\widetilde{W} \stackrel{\text{def.}}{=} W \ltimes P^\vee$, using the W-action on $P^\vee$. It is known that $\widehat{W}$ is a normal subgroup of $\widetilde{W}$, and the quotient $\mathcal{T} \stackrel{\text{def.}}{=} \widetilde{W}/\widehat{W}$ is a finite group isomorphic to a subgroup of the group of the diagram automorphisms of $\widehat{\mathfrak{g}}$, i.e., bijections $\tau \colon I \to I$. Moreover, $\widetilde{W}$ is isomorphic to $\mathcal{T} \ltimes \widehat{W}$.

When we consider $\xi \in P^\vee$ as an element of $\widetilde{W}$, we denote it by t_ξ. We have $t_\xi(\lambda) = \lambda - \langle \xi, \lambda \rangle \delta$ for $\xi \in P^\vee$, $\lambda \in \widehat{\mathfrak{h}}^{*0}$.

Lemma 2.1. *We have*

$$\sum_{\alpha \in \widehat{\mathcal{R}}_+ \cap t^{-1}_{\varpi_i^\vee}(\widehat{\mathcal{R}}_-)} (\alpha, \xi) = h^\vee \langle \varpi_i^\vee, \xi \rangle, \qquad \sum_{\alpha \in \widehat{\mathcal{R}}_+ \cap t^{-1}_{\varpi_i^\vee}(\widehat{\mathcal{R}}_-)} (\alpha^\vee, \xi) = h \langle \varpi_i^\vee, \xi \rangle.$$

Proof. From the above description of the root system $\widehat{\mathfrak{R}}_+$, we have

$$\widehat{\mathfrak{R}}_+ \cap t_{\varpi_i^\vee}^{-1}(\widehat{\mathfrak{R}}_-) = \{\beta + n\delta \mid \beta \in \Delta_+, 0 \le n < \langle \varpi_i^\vee, \beta\rangle\}.$$

Therefore

$$\sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap t_{\varpi_i^\vee}^{-1}(\widehat{\mathfrak{R}}_-)} (\alpha, \xi) = \sum_{\beta \in \Delta_+} (\beta, \xi)\langle \varpi_i^\vee, \beta\rangle = \sum_{\beta \in \Delta_+} \frac{a_i}{a_i^\vee}(\beta, \xi)(\beta, \varpi_i).$$

We consider the bilinear form on $\mathfrak{h}^*$ defined by

$$\Phi(\xi, \eta) \stackrel{\text{def.}}{=} \sum_{\beta \in \Delta_+} (\beta, \xi)(\beta, \eta).$$

By [9, Corollary 8.7] it is equal to $h^\vee(\xi, \eta)$ and we get the assertion. We give a proof since the corresponding equality for the second equation cannot be found there.

From the definition, it is invariant under the Weyl group W. So there is a constant c such that $\Phi(\xi, \eta) = c(\xi, \eta)$. Let $\theta = \delta - \alpha_0$ be the highest root of $\mathfrak{g}$. Then we have

$$(\theta, \theta) = (\alpha_0, \alpha_0) = 2.$$

On the other hand, we have

$$\Phi(\theta, \theta) = \sum_{\beta \in \Delta_+} (\beta, \theta)(\beta, \theta).$$

If $\beta = \sum_i n_i \alpha_i \in \Delta_+$, we have $0 \le n_i \le a_i$. So we have

$$(\beta, \theta) = -\sum_i n_i(\alpha_i, \alpha_0) > 0,$$

$$(\beta, \theta) = (\theta, \theta) - \sum_i (n_i - a_i)(\alpha_i, \alpha_0) \le 2,$$

where the equality holds when $\beta = \theta$. (Note that $(\alpha_i, \alpha_0) = a_{0i}$ is a negative integer.) Therefore

$$\begin{aligned}\Phi(\theta, \theta) &= \sum_{\beta \in \Delta_+} (\beta, \theta) + 2 = 2(\rho, \theta) + 2 \\ &= 2\sum_{i \in I} (\varpi_i, \theta) + 2 = 2\sum_{i \in I} a_i^\vee + 2 = 2h^\vee,\end{aligned}$$

where ρ is the half sum of the positive roots of $\mathfrak{h}$, which is known to be equal to $\sum_{i \in I} \varpi_i$. Therefore we have $c = h^\vee$ and get the first equation. A similar calculation shows the second equation. □

2.2. Quantum affine algebra

Let $\mathbf{U}_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra. We follow the notation in [1, 12]. We choose a positive integer d such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}d^{-1}$ for any $i \in \widehat{I}$. We set $q_s = q^{1/d}$. (Later we assume $\mathfrak{g}$ is of type ADE. Then $d = 1$ and $q_s = q$.) Then the quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ is the associative algebra over $\mathbb{Q}(q_s)$ with 1 generated by elements e_i, f_i $(i \in \widehat{I})$, q^h $(h \in d^{-1}\widehat{P}^*)$, $q^{\pm c/2}$ with certain defining relations. As customary, we set $q_i = q^{(\alpha_i,\alpha_i)/2}$, $t_i = q^{(\alpha_i,\alpha_i)h_i/2}$, $e_i^{(p)} = e_i^p/[p]_{q_i}!$, $f_i^{(p)} = f_i^p/[p]_{q_i}!$.

Let $\mathbf{U}'_q(\widehat{\mathfrak{g}})$ be the quantized enveloping algebra with $\mathrm{cl}(\widehat{P})$ as a weight lattice. It is the subalgebra of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ generated by e_i, f_i $(i \in \widehat{I})$, q^h $(h \in d^{-1}\bigoplus_i \mathbb{Z}h_i)$, $q^{\pm c/2}$. The quotient $\mathbf{U}'_q(\widehat{\mathfrak{g}})/(q^{\pm c/2} - 1)$ is denoted by $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ and called a *quantum loop algebra* in [16, 18].

Let $\mathbf{U}_q(\widehat{\mathfrak{g}})^+$ (resp. $\mathbf{U}_q(\widehat{\mathfrak{g}})^-$) be the $\mathbb{Q}(q_s)$-subalgebra of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ generated by elements e_i's (resp. f_i's). Let $\mathbf{U}_q(\widehat{\mathfrak{g}})^0$ be the $\mathbb{Q}(q_s)$-subalgebra generated by elements q^h $(h \in d^{-1}\widehat{P}^*)$. We have the triangular decomposition $\mathbf{U}_q(\widehat{\mathfrak{g}}) \cong \mathbf{U}_q(\widehat{\mathfrak{g}})^+ \otimes \mathbf{U}_q(\widehat{\mathfrak{g}})^0 \otimes \mathbf{U}_q(\widehat{\mathfrak{g}})^-$.

For $\xi \in \widehat{Q}$, we define the *root space* $\mathbf{U}_q(\widehat{\mathfrak{g}})_\xi$ by

$$\mathbf{U}_q(\widehat{\mathfrak{g}})_\xi \overset{\text{def.}}{=} \{x \in \mathbf{U}_q(\widehat{\mathfrak{g}}) \mid q^h x q^{-h} = q^{\langle h, \xi\rangle} x \text{ for all } h \in \widehat{P}^*\}.$$

Let $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$ be the $\mathbb{Z}[q_s, q_s^{-1}]$-subalgebra of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ generated by elements $e_i^{(n)}$, $f_i^{(n)}$, q^h for $i \in I$, $n \in \mathbb{Z}_{>0}$, $h \in d^{-1}\widehat{P}^*$.

Let us introduce a $\mathbb{Q}(q_s)$-algebra involutive automorphism $\vee$ and $\mathbb{Q}(q_s)$-algebra involutive anti-automorphisms $*$ and ψ of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ by

$$\begin{gathered} e_i^\vee = f_i, \quad f_i^\vee = e_i, \quad (q^h)^\vee = q^{-h}, \\ e_i^* = e_i, \quad f_i^* = f_i, \quad (q^h)^* = q^{-h}, \\ \psi(e_i) = q_i^{-1}t_i^{-1}f_i, \quad \psi(f_i) = q_i^{-1}t_i e_i, \quad \psi(q^h) = q^h. \end{gathered}$$

We define a $\mathbb{Q}$-algebra involutive automorphism $\overline{}$ of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ by

$$\begin{gathered} \overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q^h} = q^{-h}, \\ \overline{a(q_s)u} = a(q_s^{-1})\overline{u} \quad \text{for } a(q_s) \in \mathbb{Q}(q_s) \text{ and } u \in \mathbf{U}_q(\widehat{\mathfrak{g}}). \end{gathered}$$

In this article, we take the coproduct Δ on $\mathbf{U}_q(\widehat{\mathfrak{g}})$ given by

$$\begin{gathered} \Delta q^h = q^h \otimes q^h, \quad \Delta e_i = e_i \otimes t_i^{-1} + 1 \otimes e_i, \\ \Delta f_i = f_i \otimes 1 + t_i \otimes f_i. \end{gathered} \tag{2.2}$$

Let us denote by Ω the $\mathbb{Q}$-algebra anti-automorphism $* \circ \overline{} \circ \vee$ of $\mathbf{U}_q(\widehat{\mathfrak{g}})$. We have

$$\Omega(e_i) = f_i, \quad \Omega(f_i) = e_i, \quad \Omega(q^h) = q^{-h}, \quad \Omega(q_s) = q_s^{-1}.$$

A $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M is called *integrable* if

(1) all e_i, f_i $(i \in I)$ are locally nilpotent, and
(2) it admits a *weight space decomposition*:

$$M = \bigoplus_{\lambda \in P} M_\lambda, \quad \text{where } M_\lambda = \{u \in M \mid q^h u = q^{\langle h, \lambda \rangle} u \text{ for all } h \in \widehat{P}^*\}.$$

Let $\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})$ be the modified enveloping algebra [13, Part IV]. It is defined as

$$\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}}) \stackrel{\text{def.}}{=} \bigoplus_{\lambda \in \widehat{P}} \mathbf{U}_q(\widehat{\mathfrak{g}}) a_\lambda, \quad \mathbf{U}_q(\widehat{\mathfrak{g}}) a_\lambda \stackrel{\text{def.}}{=} \mathbf{U}_q(\widehat{\mathfrak{g}}) \Big/ \sum_{h \in \widehat{P}^*} \mathbf{U}_q(\widehat{\mathfrak{g}})(q^h - q^{\langle h, \lambda \rangle}).$$

Here the multiplication is given by

$$a_\lambda x = x a_{\lambda - \xi} \quad \text{for } \xi \in \mathbf{U}_q(\widehat{\mathfrak{g}})_\xi, \qquad a_\lambda a_\mu = \delta_{\lambda\mu} a_\lambda,$$

where a_λ is considered as the image of 1 in the above definition of $\mathbf{U}_q(\widehat{\mathfrak{g}}) a_\lambda$.

Let λ, $\mu \in \widehat{P}_+$. Let $V(\lambda)$ (resp. $V(-\mu)$) be the irreducible highest (resp. lowest) weight module of weight λ (resp. $-\mu$) [13, §3.5]. Then there is a surjective homomorphism

$$\mathbf{U}_q(\widehat{\mathfrak{g}}) a_{\lambda - \mu} \ni u \longmapsto u(u_\lambda \otimes u_{-\mu}) \in V(\lambda) \otimes V(-\mu), \tag{2.3}$$

where u_λ (resp. $u_{-\mu}$) is a highest (resp. lowest) weight vector of $V(\lambda)$ (resp. $V(-\mu)$).

2.3. Braid group action

For each $w \in \widehat{W}$, there exists an $\mathbb{Q}(q)$-algebra automorphism T_w [13, §39] (denoted there by $T''_{w,1}$). Also, for any integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M, there exists $\mathbb{Q}(q)$-linear map $T_w \colon M \to M$ satisfying $T_w(xu) = T_w(x) T_w(u)$ for $x \in \mathbf{U}_q(\widehat{\mathfrak{g}})$, $u \in M$ [13, §5]. We denote T_{s_i} by T_i hereafter. By [13, 39.4.5] we have

$$\Omega \circ T_w \circ \Omega = T_w. \tag{2.4}$$

Lemma 2.5. *We have*

$$(\psi \circ T_w \circ \psi)(x) = (-1)^{N^\vee} q^{-N} T_{w^{-1}}^{-1}(x) \quad \textit{for all } w \in \widehat{W},\ x \in \mathbf{U}_q(\widehat{\mathfrak{g}})_\xi,$$

where

$$N = \sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)} (\alpha, \xi), \qquad N^\vee = \sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_0)} (\alpha^\vee, \xi).$$

Proof. Let $T''_{i,-1}$ be the automorphism defined in [13, §37]. A direct calculation shows $\psi \circ T_i \circ \psi = T''_{i,-1}$. By [loc. cit., 37.2.4] we have $T''_{i,-1}(x) = (-1)^{\langle h_i, \xi\rangle} q^{-(\alpha_i, \xi)} T_i^{-1}(x)$ for $x \in \mathbf{U}_q(\widehat{\mathfrak{g}})_\xi$. Let $w = s_{i_m} \dots s_{i_1}$ be a reduced expression of w. Then

$$(\psi \circ T_w \circ \psi)(x) = (-1)^{N^\vee} q^{-N} \left(T_{i_m}^{-1} \dots T_{i_1}^{-1}\right)(x),$$

where

$$N^\vee = \langle h_{i_1} + s_{i_1} h_{i_2} + \dots + s_{i_1} \dots s_{i_{m-1}} h_{i_m}, \xi\rangle,$$
$$N = (\alpha_{i_1} + s_{i_1}\alpha_{i_2} + \dots + s_{i_1} \dots s_{i_{m-1}} \alpha_{i_m}, \xi).$$

Since we have $\widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-) = \left\{ s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k} \mid k = 1, \dots, m \right\}$, we are done. □

As in [2, 5], the definition of the automorphism T_w of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ can be extended to the case $w \in \widetilde{W}$ by setting

$$\tau e_i = e_{\tau(i)}, \quad \tau f_i = f_{\tau(i)}, \quad \tau q^{h_i} = q^{h_{\tau(i)}}, \quad \tau q^d = q^d.$$

2.4. Crystal base

We shall briefly recall the notion of crystal bases. For the notion of (abstract) crystals, we refer to [11, 1].

For $n \in \mathbb{Z}$ and $i \in \widehat{I}$, let us define an operator acting on any integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M by

$$\widetilde{F}_i^{(n)} \stackrel{\text{def.}}{=} \sum_{k \geq \max(0,-n)} f_i^{(n+k)} e_i^{(k)} a_k^n(t_i),$$

$$\text{where} \quad a_k^n(t_i) \stackrel{\text{def.}}{=} (-1)^k q_i^{k(1-n)} t_i^k \prod_{\nu=1}^{k-1} (1 - q_i^{n+2\nu}).$$

And we set $\widetilde{e}_i \stackrel{\text{def.}}{=} F_i^{(-1)}$, $\widetilde{f}_i \stackrel{\text{def.}}{=} F_i^{(1)}$.

These operators are different from those used for the definition of crystal bases in [10], but it gives us the same crystal bases by [12, Proposition 6.1].

A direct calculation shows

$$\psi(\widetilde{e}_i) = (1 - q_i)\widetilde{f}_i. \tag{2.6}$$

Let $\mathbf{A}_0 \overset{\text{def.}}{=} \{f(q_s) \in \mathbb{Q}(q_s) \mid f \text{ is regular at } q_s = 0\}$.

Definition 2.7. Let M be an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module. A pair $(\mathcal{L}, \mathcal{B})$ is called a *crystal base* of M if it satisfies

(1) $\mathcal{L}$ is a free $\mathbf{A}_0$-submodule of M such that $M \cong \mathbb{Q}(q_s) \otimes_{\mathbf{A}_0} \mathcal{L}$,
(2) $\mathcal{L} = \bigoplus_{\lambda \in \widehat{P}} \mathcal{L}_\lambda$ where $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$ for $\lambda \in \widehat{P}$,
(3) $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{L}/q\mathcal{L} \cong \mathbb{Q} \otimes_{\mathbf{A}_0} \mathcal{L}$,
(4) $\widetilde{e}_i\mathcal{L} \subset \mathcal{L}$, $\widetilde{f}_i\mathcal{L} \subset \mathcal{L}$ for all $i \in \widehat{I}$,
(5) if we denote operators on $\mathcal{L}/q\mathcal{L}$ induced by $\widetilde{e}_i$, $\widetilde{f}_i$ by the same symbols, we have $\widetilde{e}_i\mathcal{B} \subset \mathcal{B} \sqcup \{0\}$, $\widetilde{f}_i\mathcal{B} \subset \mathcal{B} \sqcup \{0\}$,
(6) for any $b, b' \in \mathcal{B}$ and $i \in \widehat{I}$, we have $b' = \widetilde{f}_i b$ if and only if $b = \widetilde{e}_i b'$.

We define functions $\varepsilon_i, \varphi_i \colon \mathcal{B} \to \mathbb{Z}_{\geq 0}$ by $\varepsilon_i(b) \overset{\text{def.}}{=} \max\{n \geq 0 \mid \widetilde{e}_i^n b \neq 0\}$, $\varphi_i(b) \overset{\text{def.}}{=} \max\{n \geq 0 \mid \widetilde{f}_i^n b \neq 0\}$. We set $\widetilde{e}_i^{\max} b \overset{\text{def.}}{=} \widetilde{e}_i^{\varepsilon_i(b)} b$, $\widetilde{f}_i^{\max} b \overset{\text{def.}}{=} \widetilde{f}_i^{\varphi_i(b)} b$.

Let $\overline{}$ be an automorphism of $\mathbb{Q}(q_s)$ sending q_s to q_s^{-1}. Let $\overline{\mathbf{A}_0}$ be the image of $\mathbf{A}_0$ under $\overline{}$, that is, the subring of $\mathbb{Q}(q_s)$ consisting of rational functions regular at $q_s = \infty$.

Definition 2.8. Let M be an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module with a crystale base $(\mathcal{L}, \mathcal{B})$. Let $\overline{}$ be an involution of an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M satisfying $\overline{xu} = \overline{x}\,\overline{u}$ for any $x \in \mathbf{U}_q(\widehat{\mathfrak{g}})$, $u \in M$. Let $M^{\mathbb{Z}}$ be a $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$-submodule of M such that $\overline{M^{\mathbb{Z}}} = M^{\mathbb{Z}}$, $u - \overline{u} \in (q_s - 1)M^{\mathbb{Z}}$ for $u \in M^{\mathbb{Z}}$. We say that M *has a global base* $(\mathcal{L}, \mathcal{B}, M^{\mathbb{Z}})$ if the following conditions are satisfied

(1) $M \cong \mathbb{Q}(q_s) \otimes_{\mathbb{Z}[q_s, q_s^{-1}]} M^{\mathbb{Z}} \cong \mathbb{Q}(q_s) \otimes_{\mathbf{A}_0} \mathcal{L} \cong \mathbb{Q}(q_s) \otimes_{\overline{\mathbf{A}_0}} \overline{\mathcal{L}}$,
(2) $\mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}} \to \mathcal{L}/q_s\mathcal{L}$ is an isomorphism.

As a consequence of the definition, natural homomorphisms

$$\mathbf{A}_0 \otimes_{\mathbb{Z}} (\mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}}) \to \mathcal{L}, \quad \overline{\mathbf{A}_0} \otimes_{\mathbb{Z}} (\mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}}) \to \overline{\mathcal{L}},$$
$$\mathbb{Z}[q_s, q_s^{-1}] \otimes_{\mathbb{Z}} (\mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}}) \to M^{\mathbb{Z}},$$

are isomorphisms.

Let G be the inverse isomorphism $\mathcal{L}/q_s\mathcal{L} \to \mathcal{L} \cap \overline{\mathcal{L}} \cap M^{\mathbb{Z}}$. Then $\{G(b) \mid b \in \mathcal{B}\}$ is a base of M. It is called a *global crystal base* of M. The above conditions imply $\overline{G(b)} = G(b)$.

For a dominant weight $\lambda \in \widehat{P}_+$, the irreducible highest weight module $V(\lambda)$ has a global crystal base [10]. If $\lambda, \mu \in \widehat{P}_+$, then the tensor product $V(\lambda) \otimes V(-\mu)$ also has a global crystal base. Moreover, $\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})$ has a global crystal base $\left(\mathcal{L}(\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})), \mathcal{B}(\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})), \tilde{\mathbf{U}}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})\right)$ such that the homomorphism (2.3) maps a global base of $\tilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})$ to the union of that of $V(\lambda) \otimes V(-\mu)$ and 0 [13, Part IV]. Furthermore, the global base is invariant under $*$ [11, 4.3.2].

2.5. Extremal vectors

A crystal $\mathcal{B}$ over $\mathbf{U}_q(\widehat{\mathfrak{g}})$ is called *regular* if, for any $J \subsetneq \widehat{I}$, $\mathcal{B}$ is isomorphic (as a crystal over $\mathbf{U}_q(\mathfrak{g}_J)$) to the crystal associated with an integrable $\mathbf{U}_q(\mathfrak{g}_J)$-module. (It was called *normal* in [11].) Here $\mathbf{U}_q(\mathfrak{g}_J)$ is the subalgebra generated by e_j, f_j $(j \in J)$, q^h $(h \in d^{-1}\widehat{P}^*)$. By [11], the affine Weyl group $\widehat{W}$ acts on any regular crystal. The action S is given by

$$S_{s_i} b = \begin{cases} \tilde{f}_i^{\langle h_i, \mathrm{wt}\, b\rangle} b & \text{if } \langle h_i, \mathrm{wt}\, b\rangle \geq 0, \\ \tilde{e}_i^{-\langle h_i, \mathrm{wt}\, b\rangle} b & \text{if } \langle h_i, \mathrm{wt}\, b\rangle \leq 0 \end{cases}$$

for the simple reflection s_i. We denote S_{s_i} by S_i hereafter.

Definition 2.9. Let M be an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module. A vector $u \in M$ with weight $\lambda \in \widehat{P}$ is called *extremal*, if the following holds for all $w \in \widehat{W}$:

$$\begin{cases} e_i T_w u = 0 & \text{if } \langle h_i, w\lambda\rangle \geq 0, \\ f_i T_w u = 0 & \text{if } \langle h_i, w\lambda\rangle \leq 0. \end{cases} \tag{2.10}$$

In this case, we define $S_w u$ so that

$$S_i S_w u = \begin{cases} f_i^{(\langle h_i, w\lambda\rangle)} S_w u & \text{if } \langle h_i, w\lambda\rangle \geq 0, \\ e_i^{(-\langle h_i, w\lambda\rangle)} S_w u & \text{if } \langle h_i, w\lambda\rangle \leq 0. \end{cases}$$

This is well-defined, i.e., $S_w u$ depends only on w.

Similarly, for a vector b of a regular crystal B with weight λ, we say that b is *extremal* if it satisfies

$$\begin{cases} \tilde{e}_i S_w b = 0 & \text{if } \langle h_i, w\lambda\rangle \geq 0, \\ \tilde{f}_i S_w b = 0 & \text{if } \langle h_i, w\lambda\rangle \leq 0. \end{cases}$$

Lemma 2.11. *Suppose that an integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module M has a crystal base $(\mathcal{L}, \mathcal{B})$. If $u \in \mathcal{L} \subset M$ is an extremal vector of weight λ*

satisfying $b \stackrel{\text{def.}}{=} u \bmod q\mathcal{L} \in \mathcal{B}$, *then* b *is an extremal vector, and we have*

$$S_w u = (-1)^{N_+^\vee} q^{-N_+} T_w u, \quad S_w b = S_w u \bmod q\mathcal{L} \quad \text{for all } w \in \widehat{W},$$

where $N_+ = \displaystyle\sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)} \max\left((\alpha, \lambda), 0\right)$, *and* $N_+^\vee$ *is given by replacing* α *by* $\alpha^\vee$.

Proof. The equation $S_w b = S_w u \bmod q\mathcal{L}$ follows from the definition of S_w.

If $v \in M_\xi$ satisfies $e_i v = 0$ (resp. $f_i v = 0$), we have

$$T_i v = (-q_i)^{\xi_i} f_i^{(\xi_i)} v \qquad \left(\text{resp. } T_i v = e_i^{(\xi_i)} v\right),$$

where $\xi_i = \langle h_i, \xi \rangle$. The rest of the proof is the same as that of Lemma 2.5. □

The following follows from a formula for the crystal $\mathcal{B}(\widetilde{\mathbf{U}}_q(\widehat{\mathfrak{g}}))$ (see [12, App. B]):

Lemma 2.12. *Let* $\lambda \in P^0$. *The followings hold for* $b = b_1 \otimes t_\lambda \otimes u_{-\infty} \in \mathcal{B}(\mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda) = \mathcal{B}(\infty) \otimes T_\lambda \otimes \mathcal{B}(-\infty)$ *with* $\mathrm{wt}\, b_1 \in \mathbb{Z}\delta$:
$\widetilde{e}_i b = 0$ *or* $\widetilde{f}_i b = 0$ *if and only if* $\varepsilon_i(b_1) \le \max(-\langle h_i, \lambda \rangle, 0)$.

For $\lambda \in P$, Kashiwara defined the $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module $V(\lambda)$ generated by u_λ with the defining relation that u_λ is an extremal vector of weight λ [11][1]. It is written as

$$V(\lambda) = \mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda / I_\lambda, \qquad I_\lambda \stackrel{\text{def.}}{=} \bigoplus_{b \in \mathcal{B}(\mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda) \setminus \mathcal{B}(\lambda)} \mathbb{Q}(q) G(b),$$

where $\mathcal{B}(\lambda) \stackrel{\text{def.}}{=} \{b \in \mathcal{B}(\mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda) \mid b^* \text{ is extremal}\}$. Thus $V(\lambda)$ has a crystal base $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ together with a $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$-submodule $V^{\mathbb{Z}}(\lambda)$ with a global crystal base, naturally induced from that of $\mathbf{U}_q(\widehat{\mathfrak{g}})a_\lambda$. If λ is dominant or anti-dominant, then $V(\lambda)$ is isomorphic to the highest weight module or the lowest weight module. So there is no fear of the confusion of the notation.

[1] He denoted it by $V^{\max}(\lambda)$.

2.6. Drinfeld realization

The quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ has another realization, due to [8, 2]. It is isomorphic to an associative algebra over $\mathbb{Q}(q_s)$ with generators $x_{i,r}^{\pm}$ ($i \in I$, $r \in \mathbb{Z}$), q^h ($h \in d^{-1}\widehat{P}^*$), $h_{i,m}^{\pm}$ ($i \in I$, $m \in \mathbb{Z} \setminus \{0\}$) with certain defining relations (see [2, §4]). The isomorphism depends on the choice of $o\colon I \to \{\pm 1\}$, and is given by

$$x_{i,r}^{+} = o(i)^r T_{\varpi_i^\vee}^{-r}(e_i), \quad x_{i,r}^{-} = o(i)^r T_{\varpi_i^\vee}^{r}(f_i),$$
$$\left[x_{i,r}^{+}, x_{j,s}^{-}\right] = \delta_{ij}\frac{q^{(r-s)c/2}\psi_{i,r+s}^{+} - q^{-(r-s)c/2}\psi_{i,r+s}^{-}}{q-q^{-1}},$$

where $\psi_i^{\pm}(u) \equiv \sum_{r=0}^{\infty}\psi_{i,\pm r}^{\pm}u^{\pm r} \overset{\text{def.}}{=} t_i^{\pm 1}\exp\left(\pm(q_i - q_i^{-1})\sum_{m=1}^{\infty} h_{i,\pm m}u^{\pm m}\right)$.

By (2.4) we have

$$\Omega(x_{i,r}^{\pm}) = x_{i,-r}^{\mp}, \quad \Omega(h_{i,m}) = h_{i,-m} \quad \text{for } i \in I,\ r \in \mathbb{Z},\ m \in \mathbb{Z}\setminus\{0\}.$$

2.7. The crystal base of $\mathbf{U}_q(\widehat{\mathfrak{g}})^+$

Let us recall results in [5]. We assume $\mathfrak{g}$ is of type ADE hereafter. We choose a reduced expression $s_{i_1}\cdots s_{i_N}$ of $2\rho = 2\sum_{i\in I}\varpi_i$ in a suitable way (see [loc. cit.] for detail), and consider a periodic doubly infinite sequence $(\dots, i_{-1}, i_0, i_1, \dots)$ of $\widehat{I}$ by setting $i_k = i_{k \bmod N}$. Let

$$\beta_k \overset{\text{def.}}{=} \begin{cases} s_{i_0}s_{i_{-1}}\cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \le 0, \\ s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k > 0. \end{cases}$$

We have

$$\mathcal{R}_{>} = \{\beta_k \mid k \le 0\}, \quad \mathcal{R}_{<} = \{\beta_k \mid k > 0\}. \tag{2.13}$$

We define

$$E_{\beta_k}^{(n)} \overset{\text{def.}}{=} \begin{cases} T_{i_0}^{-1}T_{i_{-1}}^{-1}\dots T_{i_{k+1}}^{-1}(e_{i_k}^{(n)}) & \text{if } k \le 0, \\ T_{i_1}T_{i_2}\dots T_{i_{k-1}}(e_{i_k}^{(n)}) & \text{if } k > 0. \end{cases}$$

We denote $E_{\beta_k}^{(1)}$ by E_{β_k}. These are *root vectors* for $\mathcal{R}_{>}$ and $\mathcal{R}_{<}$. By [13, 40.1.3] we have $E_{\beta_k}^{(n)} \in \mathbf{U}_q(\widehat{\mathfrak{g}})^+$. Explicit relations among $E_{\beta_k}^{(n)}$ and $x_{i,r}^{\pm}$ can be found in [5, Lemma 1.5].

We define $P_{m,i}$ ($m > 0$, $i \in I$) by

$$1 + \sum_{m>0} P_{m,i} u^m = \exp\left(- \sum_{m>0} \frac{(o(i) q^{c/2} u)^r h_{i,r}}{[r]_{q_i}} \right).$$

We also define $\widetilde{P}_{m,i} \in \mathbf{U}_q(\widehat{\mathfrak{g}})^+$ by replacing $h_{i,r}$ by $-h_{i,r}$. These are root vectors for $\mathcal{R}_0$. We also set $P_{-m,i} = \Omega(P_{m,i})$ $(m > 0, i \in I)$.

Let $\mathbf{c}\colon \mathcal{R} \to \mathbb{Z}_{\geq 0}$ be a map such that $\mathbf{c}(\alpha) = 0$ except for finitely many α. We denote its restrictions to $\mathcal{R}_>$, $\mathcal{R}_>$, $\mathcal{R}_0$ by $\mathbf{c}_>$, $\mathbf{c}_<$, $\mathbf{c}_0$ respectively. We define $E_{\mathbf{c}_>}, E_{\mathbf{c}_<} \in \mathbf{U}_q(\widehat{\mathfrak{g}})^+$ by

$$E_{\mathbf{c}_>} \stackrel{\text{def.}}{=} E_{\beta_0}^{(\mathbf{c}(\beta_0))} E_{\beta_{-1}}^{(\mathbf{c}(\beta_{-1}))} \cdots, \qquad E_{\mathbf{c}_<} \stackrel{\text{def.}}{=} \cdots E_{\beta_2}^{(\mathbf{c}(\beta_2))} E_{\beta_1}^{(\mathbf{c}(\beta_1))}.$$

Next, given $\mathbf{c}_0$, we associate an I-tuple of partitons $(\lambda^{(i)})_{i \in I}$ as

$$\lambda^{(i)} \stackrel{\text{def.}}{=} (1^{\mathbf{c}_0(\delta,i)} 2^{\mathbf{c}_0(2\delta,i)} \cdots k^{\mathbf{c}_0(k\delta,i)} \cdots).$$

As in [15] we denote it also in another notation:

$$\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots).$$

We define the corresponding Schur function

$$S_{\mathbf{c}_0} \stackrel{\text{def.}}{=} \prod_{i \in I} \det \left(\widetilde{P}_{{}^t\lambda_k^{(i)} - k + l, i} \right)_{1 \leq k, l \leq t},$$

where $t \geq l(\lambda^{(i)})$ and ${}^t\lambda^{(i)}$ means the transpose partition of $\lambda^{(i)}$. Note that $\widetilde{P}_{m,i}$ corresponds to an elementary symmetric function, while $P_{m,i}$ corresponds to a complete symmetric function, up to sign.

Now a main result of [5] says that

(1) $B_{\mathbf{c}} \stackrel{\text{def.}}{=} \overline{E_{\mathbf{c}_>} \cdot S_{\mathbf{c}_0} \cdot E_{\mathbf{c}_<}}$ is contained in $\mathcal{L}(\infty) \cap \mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})^+$,
(2) $\{B_{\mathbf{c}} \bmod q\mathcal{L}(\infty) \mid \mathbf{c} \in \mathbb{Z}_{\geq 0}^{\mathcal{R}}\}$ is the crystal base of $\mathbf{U}_q(\widehat{\mathfrak{g}})^+$.

Set $(\mathbb{Z}_{\geq 0}^{\mathcal{R}_0})(\lambda) \stackrel{\text{def.}}{=} \left\{ \mathbf{c}_0 \in \mathbb{Z}_{\geq 0}^{\mathcal{R}_0} \,\middle|\, l(\lambda^{(i)}) \leq \langle h_i, \lambda \rangle \text{ for all } i \in I \right\}$, where $(\lambda^{(i)})_{i \in I}$ is the I-tuple of partition corresponding to $\mathbf{c}_0$ as above.

We apply $\vee$ to the above crystal base to get

$$F_{\mathbf{c}_>} \stackrel{\text{def.}}{=} (E_{\mathbf{c}_>})^\vee, \quad F_{\mathbf{c}_<} \stackrel{\text{def.}}{=} (E_{\mathbf{c}_<})^\vee, \quad S_{\mathbf{c}_0}^- \stackrel{\text{def.}}{=} (S_{\mathbf{c}_0})^\vee.$$

2.8. Extremal weight modules and the Drinfeld realization

Extremal weight modules are defined in terms of Chevalley generators. We shall rewrite the definition in terms of Drinfeld generators, and derive several easy consequences in this subsection.

The following is a consequence of [12, Theorem 5.3].

Lemma 2.14. *Let u be a vector of an integrable $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-module M with weight $\lambda \in \widehat{P}^{0,+}$. Then the following conditions are equivalent:*

(1) *u is an extremal vector.*

(2) *$x^+_{i,r} u = 0$ for all $i \in I$, $r \in \mathbb{Z}$.*

Remark 2.15. The extremal weight module $V(\lambda)$ is isomorphic to the Weyl module $W_q(\lambda)$ introduced by Chari-Pressley [6]. This result was refered as 'an unpublished work' of Kashiwara in [loc. cit., Proposition 4.5]. Let us give Kashiwara's proof here. Let $\lambda = \sum_{i\in I} m_i \varpi_i \in \widehat{P}^{0,+}$. Then $W_q(\lambda)$ is integrable and contains a vector w_λ of weight λ which satisfies the above condition (2). Therefore, there is a unique $\mathbf{U}_q(\widehat{\mathfrak{g}})$-linear homomorphism $V(\lambda) \to W_q(\lambda)$, sending v_λ to w_λ. (The integrablity of $W_q(\lambda)$ was proved via the isomorphism $V(\lambda) \cong W_q(\lambda)$ in [loc. cit.]. So one must give another proof of the integrablity as sketched in [loc. cit.].) Since $W_q(\lambda)$ is generated by w_λ by definition, the homomorphism is surjective. On the other hand, any integrable $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module generated by a vector u of weight λ satisfying the above condition (2) is a quotient of $W_q(\lambda)$ [loc. cit., Proposition 4.6]. Therefore $V(\lambda)$ and $W_q(\lambda)$ are isomorphic.

Corollary 2.16. *Let u be an extremal vector with weight $\lambda \in \widehat{P}^{0,+}$. Then $S^-_{\mathbf{c}_0} u = \overline{S^*_{\mathbf{c}_0}} u = 0$ for $\mathbf{c}_0 \notin (\mathbb{Z}^{\mathfrak{R}_0}_{\geq 0})(\lambda)$.*

Proof. It is enough to show the assertion for $u = u_\lambda \in V(\lambda)$. We have a $\mathbb{Q}(q)$-vector space isomorphism

$$V(\lambda) \ni x u_\lambda \mapsto x^\vee u_{-\lambda} \in V(-\lambda).$$

Therefore it is enough to show $S_{\mathbf{c}_0} u_{-\lambda} = \Omega(S_{\mathbf{c}_0}) u_{-\lambda} = 0$. By [6, Proposition 4.3], which is applicable thanks to Lemma 2.14, we have

$$\widetilde{P}_{m,i} u_{-\lambda} = 0 \qquad \text{for } |m| > \langle h_i, \lambda\rangle.$$

(More precisely, we apply [loc. cit.] after composing an automorphism $x^\pm_{i,r} \mapsto -x^\mp_{i,-r}$, $h_{i,m} \mapsto -h_{i,-m}$.) Now the assertion follows from a standard result in the theory of symmetric polynomials. □

§3. A study of extremal weight modules

3.1. Fundamental representations

By [12, §5.2] there is a unique $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-linear automorphism z_i of $V(\varpi_i)$ with weight δ, which sends u_{ϖ_i} to $u_{\varpi_i+\delta}$. (Note that d_i in [12, §5.2] is equal to 1 for untwisted $\widehat{\mathfrak{g}}$.)

Proposition 3.1. $h_{i,1}u_{\varpi_i} = o(i)(-1)^{1-h}q^{-h^\vee} z_i u_{\varpi_i}$.

Proof. We have

$$h_{i,1}u_{\varpi_i} = t_i^{-1}\left[x_{i,1}^+, x_{i,0}^-\right]u_{\varpi_i} = o(i)t_i^{-1}T_{\varpi_i^\vee}^{-1}(e_i)f_i u_{\varpi_i}.$$

Let us write $T_{\varpi_i^\vee} = \tau T_w$ with $w \in \widehat{W}$. Then Lemma 2.11 implies

$$T_{\varpi_i^\vee}^{-1}(e_i)f_i u_{\varpi_i} = (-1)^{N_+'^\vee} q^{N_+'} S_w^{-1}\left(e_{\tau^{-1}(i)}S_w(f_i u_{\varpi_i})\right), \tag{3.2}$$

where $N'_+ = \sum_{\alpha\in\widehat{\mathfrak{R}}_+\cap w^{-1}(\widehat{\mathfrak{R}}_-)} \max((\alpha, s_i\varpi_i), 0) - \max((\alpha,\varpi_i),0)$, and $N_+'^\vee$ is given by replacing α by $\alpha^\vee$. Since $\widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-) = \widehat{\mathfrak{R}}_+ \cap t_{\varpi_i^\vee}^{-1}(\widehat{\mathfrak{R}}_-) = \{\beta + n\delta \mid \beta \in \Delta_+, 0 \le n < \langle \varpi_i, \alpha\rangle\}$, we have

$$\max((\alpha,\varpi_i),0) = (\alpha,\varpi_i), \quad \max((\alpha, s_i\varpi_i), 0) = \begin{cases} 0 & \text{if } \alpha = \alpha_i, \\ (\alpha, s_i\varpi_i) & \text{otherwise.}\end{cases}$$

Therefore

$$N'_+ = (\alpha_i,\varpi_i) - \sum_{\alpha\in\widehat{\mathfrak{R}}_+\cap w^{-1}(\widehat{\mathfrak{R}}_-)} (\alpha,\alpha_i) = (\alpha_i,\varpi_i) - h^\vee,$$

where we have used Lemma 2.1. Similarly we have $N_+'^\vee = 1 - h$. Now the assertion follows from the definition of the Weyl group action S. □

Remark 3.3. Let $W(\varpi_i) \stackrel{\text{def.}}{=} V(\varpi_i)/(z_i - 1)V(\varpi_i)$. This is a finite dimensional irreducible $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-module [12, §5.2]. The above proposition says that $W(\varpi_i)$ has the Drinfeld polynomial

$$P_j(u) = \begin{cases} 1 & \text{if } j \ne i, \\ 1 + o(i)(-1)^h q^{-h^\vee} u & \text{if } j = i.\end{cases}$$

Proposition 3.4. $(\tilde{P}_{\pm 1,i})^\vee u_{\varpi_i} = z_i^{\pm} u_{\varpi_i}$.

Proof. Let us endow a new $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module structure on $V(-\varpi_i)$ by

$$x \cdot u \overset{\text{def.}}{=} x^\vee \cdot u, \quad (x \in \mathbf{U}_q(\widehat{\mathfrak{g}}), u \in V(-\varpi_i)).$$

We denote it by $V(-\varpi_i)^\vee$. Then there is a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module isomorphism $V(\varpi_i) \cong V(-\varpi_i)^\vee$ sending u_{ϖ_i} to $u_{-\varpi_i}$. Using this isomorphism, we can calculate $(\widetilde{P}_{\pm 1,i})^\vee u_{\varpi_i}$ exactly as in the above proposition (in fact, more easily) to get the assertion. □

3.2. Tensor product modules

Let $\lambda = \sum_{i\in I} m_i\varpi_i \in \widehat{P}^{0,+}$. We define a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module $\widetilde{V}(\lambda)$, $\widetilde{\mathcal{L}}(\lambda)$, $\widetilde{\mathcal{B}}(\lambda)$, $\widetilde{u}_\lambda$ as in the introduction. Let $z_{i,\nu}$ ($i \in I$, $\nu = 1,\dots,m_i$) be the $\mathbf{U}_q'(\widehat{\mathfrak{g}})$-linear automorphism of $\widetilde{V}(\lambda)$ obtained by the action of $z_i \colon V(\varpi_i) \to V(\varpi_i)$ on the ν-th factor. Obviously they are commuting: $z_{i,\nu}z_{j,\mu} = z_{j,\mu}z_{i,\nu}$. Let

$$\breve{V}(\lambda) \overset{\text{def.}}{=} \mathbf{U}_q(\widehat{\mathfrak{g}})[z_{i,\nu}^{\pm}]_{i\in I,\nu=1,\dots,m_i} \cdot \widetilde{u}_\lambda, \quad \breve{\mathcal{L}}(\lambda) \overset{\text{def.}}{=} \widetilde{\mathcal{L}}(\lambda) \cap \breve{V}(\lambda),$$
$$\breve{\mathcal{B}}(\lambda) \overset{\text{def.}}{=} \bigotimes_{i\in I} \mathcal{B}(\varpi_i)^{\otimes m_i}, \quad \breve{V}^{\mathbb{Z}}(\lambda) \overset{\text{def.}}{=} \bigotimes_{i\in I} \left(V(\varpi_i)^{\mathbb{Z}}\right)^{\otimes m_i} \cap \breve{V}(\lambda).$$

By [12, §8], the submodule $\breve{V}(\lambda)$ has

(1) the unique bar involution $\overline{}$ satisfying $\overline{xu} = \overline{x}\,\overline{u}$ for $x \in \mathbf{U}_q(\widehat{\mathfrak{g}})[z_{i,\nu}^{\pm}]_{i\in I,\nu=1,\dots,m_i}$, $u \in \breve{V}(\lambda)$,
(2) the crystal base $(\breve{\mathcal{L}}(\lambda), \breve{\mathcal{B}}(\lambda))$, and
(3) the $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$-submodule $\breve{V}^{\mathbb{Z}}(\lambda)$ and the global crystal base $\{G(b) \mid b \in \breve{\mathcal{B}}(\lambda)\}$.

The module $\widetilde{V}(\lambda)$ contains the extremal vector $\widetilde{u}_\lambda$ of weight λ. Therefore there exists a unique $\mathbf{U}_q(\widehat{\mathfrak{g}})$-linear homomorphism $\Phi_\lambda \colon V(\lambda) \to \widetilde{V}(\lambda)$ sending u_λ to $\widetilde{u}_\lambda$. The image is contained in $\breve{V}(\lambda)$.

Recall that a function $\mathbf{c}_0 \in \mathcal{R}_0 \to \mathbb{Z}_{\geq 0}$ defines an I-tuple of partitions $(\lambda^{(i)})_{i\in I}$ as §2.7. We define an endomorphism of $\widetilde{V}(\lambda)$ by

$$s_{\mathbf{c}_0}(z^{\pm}) \overset{\text{def.}}{=} \prod_{i\in I} s_{\lambda^{(i)}}(z_{i,1}^{\pm}, \dots, z_{i,m_i}^{\pm}),$$

where $s_{\lambda^{(i)}}$ is the Schur polynomial corresponding to the partition $\lambda^{(i)}$. If $l(\lambda^{(i)}) > m_i$, it is understood as 0.

Proposition 3.5. $\Phi_\lambda(S_{\mathbf{c}_0}^{-} u_\lambda) = s_{\mathbf{c}_0}(z)\cdot\widetilde{u}_\lambda$, $\Phi_\lambda(\overline{S_{\mathbf{c}_0}^{*} u_\lambda}) = s_{\mathbf{c}_0}(z^{-1})\cdot \widetilde{u}_\lambda$.

Proof. On level 0 modules, we have

$$\Delta h_{i,\pm m} = h_{i,\pm m} \otimes 1 + 1 \otimes h_{i,\pm m} + \text{a nilpotent term}$$

by [7]. Up to sign, the transition between $h_{i,m}$'s and $P_{k,i}$'s is the same as that between power sums and elementary symmetric functions. The above equation means that Δ coincides with the standard coproduct on symmetric polynomials modulo nilpotent terms [15, Chap. I, §5, Ex. 25]. Therefore we have

$$\Delta P_{k,i} = \sum_{s=0}^{k} P_{s,i} \otimes P_{k-s,i} + \text{a nilpotent term.}$$

Using Corollary 2.16 and Proposition 3.4, we have the assertion. □

3.3. Detemination of extremal vectors

Proposition 3.6. *Suppose* $\lambda \in \widehat{P}^{0,+}$. *Consider* $B_{\mathbf{c}} = \overline{F_{\mathbf{c}_>} \cdot S^-_{\mathbf{c}_0} \cdot F_{\mathbf{c}_<}}$ *with* $\operatorname{wt} B_{\mathbf{c}} \in \mathbb{Z}\delta$, *and set* $b_1 \overset{\text{def.}}{=} B_{\mathbf{c}} \bmod q\mathcal{L}(\infty) \in \mathcal{B}(\infty)$ *and* $b \overset{\text{def.}}{=} b_1 \otimes t_\lambda \otimes u_{-\infty} \in \mathcal{B}(\widetilde{\mathbf{U}}_q(\widehat{\mathfrak{g}})a_\lambda)$. *If* b *and* b^* *are extremal, then we have* $\mathbf{c}_> \equiv 0 \equiv \mathbf{c}_<$ *and* $\mathbf{c}_0 \in (\mathbb{Z}^{\mathcal{R}_0}_{\geq 0})(\lambda)$.

Proof. Assume $\mathbf{c}_> \not\equiv 0$ and take the largest number $k \leq 0$ satisfying $\mathbf{c}(\beta_k) \neq 0$. Let $w = s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}}$.

Since b^* is extremal, we can consider b as an element of $\mathcal{B}(\lambda)$. We have

$$b = B_{\mathbf{c}} u_\lambda \bmod q\mathcal{L}(\lambda).$$

By Lemma 2.11, we have

$$S_w^{-1} b = (-1)^{N^\vee} q^N T_w^{-1}(B_{\mathbf{c}}) \cdot S_w^{-1}(u_\lambda) \bmod q\mathcal{L}(\lambda)$$

for some integers $N^\vee$, N. By [11, 8.2.2] there exists a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-linear isomorphism

$$V(\lambda) \to V(w^{-1}\lambda); \qquad S_w^{-1}(u_\lambda) \mapsto u_{w^{-1}\lambda},$$

respecting the crystal bases. Therefore we have

$$(-1)^{N^\vee} q^N T_w^{-1}(B_{\mathbf{c}}) u_{w^{-1}\lambda} \bmod q\mathcal{L}(w^{-1}\lambda) \in \mathcal{B}(w^{-1}\lambda).$$

(In fact, this is equal to $S^*_{w^{-1}} S_{w^{-1}} b$.) Let us denote this by $b'_1 \otimes t_{w^{-1}\lambda} \otimes b'_2$.

We have

$$T_w^{-1}(B_{\mathbf{c}}) = T_w^{-1}(\overline{F_{\mathbf{c}_>}}) \cdot T_w^{-1}(\overline{S^-_{\mathbf{c}_0}}) \cdot T_w^{-1}(\overline{F_{\mathbf{c}_<}}).$$

It is clear that $T_w^{-1}(\overline{F_{\mathbf{c}_<}}) \in \mathbf{U}_q(\widehat{\mathfrak{g}})^- \cap T_{i_k}\mathbf{U}_q(\widehat{\mathfrak{g}})^-$. We also have $T_w^{-1}(\overline{S_{\mathbf{c}_0}^-}) \in \mathbf{U}_q(\widehat{\mathfrak{g}})^- \cap T_{i_k}\mathbf{U}_q(\widehat{\mathfrak{g}})^-$ by [3, Lemma 2]. (More precisely, we apply [loc. cit.] after composing $\bar{\ } \circ \vee$. Note that $T_w^{-1} = \bar{\ } \circ \vee \circ T_w \circ \bar{\ } \circ \vee$ by [13, 39.4.5].) Moreover, by our choice of k, we have

$$T_w(\overline{F_{\mathbf{c}_>}}) = f_{i_k}^{(\mathbf{c}(\beta_k))} T_{i_k}(f_{i_{k-1}}^{(\mathbf{c}(\beta_{k-1}))}) \cdots \in f_{i_k}^{(\mathbf{c}(\beta_k))} \left(\mathbf{U}_q(\widehat{\mathfrak{g}})^- \cap T_{i_k}\mathbf{U}_q(\widehat{\mathfrak{g}})^-\right).$$

Therefore we have

$$b_2' = u_{-\infty}, \qquad b_1' = T_w^{-1}(B_{\mathbf{c}}) \bmod q\mathcal{L}(\infty), \qquad \varepsilon_{i_k}(b_1') = \mathbf{c}(\beta_k),$$

where the last equality follows from [13, 38.1.6]. Since $b_1' \otimes t_{w^{-1}\lambda} \otimes u_{-\infty}$ is extremal, Lemma 2.12 implies

$$\mathbf{c}(\beta_k) \le \max(-\langle h_{i_k}, w^{-1}\lambda\rangle, 0). \tag{3.7}$$

However, we have $\langle h_{i_k}, w^{-1}\lambda\rangle = (w\alpha_{i_k}^\vee, \lambda) \ge 0$ for $\lambda \in \widehat{P}^{0,+}$, because $w\alpha_{i_k} \in \widehat{\mathfrak{R}}_>$ by (2.13). So the right hand side of (3.7) is 0, and this contradicts with the choice of k. Therefore $\mathbf{c}_> \equiv 0$. Applying $*$, we similarly get $\mathbf{c}_< \equiv 0$. Now the last assertion is a consequence of Corollary 2.16. □

Proof of Theorem 1. We first prove (2), (3), (4) and then (1).

(2) Recall that any vector $b \in \mathcal{B}(\lambda)$ is connected to an extremal vector [11, 9.3.3]. Moreover, an extremal vector can be mapped by $\widetilde{f}_i^{\max}$ to an extremal vector of the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$. (See [12, Proof of Theorem 5.1]). Therefore

$$\mathcal{B}(\lambda) = \left\{X_l \cdots X_1\overline{S_{\mathbf{c}_0}^-} \bmod q\mathcal{L}(\Lambda) \,\middle|\, \mathbf{c}_0 \in (\mathbb{Z}_{\ge 0}^{\mathfrak{R}_0})(\lambda),\ X_\mu \text{ is } \widetilde{e}_i \text{ or } \widetilde{f}_i\right\} \setminus \{0\}$$

by Proposition 3.6. Then $\mathcal{L}(\lambda)$ is spanned by $\{X_l \cdots X_1\overline{S_{\mathbf{c}_0}^-}\}$ over $\mathbf{A}_0$, by Nakayama's lemma. Note that Φ_λ commutes with the operators $\widetilde{e}_i$, $\widetilde{f}_i$ and $\widetilde{\mathcal{L}}(\lambda)$ is invariant under $\widetilde{e}_i$, $\widetilde{f}_i$. Therefore it is enough to show that $\Phi_\lambda(\overline{S_{\mathbf{c}_0}^-}) \in \widetilde{\mathcal{L}}(\lambda)$. But this follows from Proposition 3.5.

(3) By Proposition 3.5, we have

$$\Phi_\lambda^0(\overline{S_{\mathbf{c}_0}^-} \bmod q\mathcal{L}(\lambda)) \in \widetilde{\mathcal{B}}(\lambda) \qquad \text{for } \mathbf{c}_0 \in (\mathbb{Z}_{\ge 0}^{\mathfrak{R}_0})(\lambda).$$

As in the proof of (1), we conclude that $\Phi_\lambda^0(\mathcal{B}(\lambda)) \subset \widetilde{\mathcal{B}}(\lambda) \sqcup \{0\}$. From the definition, it is obvious that the image contains $\widetilde{\mathcal{B}}(\lambda)$. Consider $\operatorname{Ker}\Phi_\lambda^0 \cap \mathcal{B}(\lambda)$. It is invariant under $\widetilde{e}_i$, $\widetilde{f}_i$. Since any vector is connected to an extremal vector, $\operatorname{Ker}\Phi^0_\lambda \cap \mathcal{B}(\lambda)$ contains an extremal vector if it is nonempty. But we already checked that every extremal vector is mapped to a nonzero vector. Hence $\operatorname{Ker}\Phi^0_\lambda \cap \mathcal{B}(\lambda) = \emptyset$. Now suppose b_1, $b_2 \in \mathcal{B}(\lambda)$ satisfy $\Phi^0_\lambda(b_1) = \Phi^0_\lambda(b_2)$. We want to show $b_1 = b_2$. Applying $\widetilde{e}_i$, $\widetilde{f}_i$'s, we may assume $b_1 = \overline{S^-_{\mathbf{c}_0}} \bmod q\mathcal{L}(\lambda)$. By [12, §5.1] b_2 is also extremal. Applying $\widetilde{f}_i^{\max}$'s if necessarily, we may assume b_2 is of form $b_2^- \otimes t_\lambda \otimes u_{-\infty}$, and hence $b_2 = \overline{S^-_{\mathbf{c}'_0}} \bmod q\mathcal{L}(\lambda)$. By this process, b_1 may be changed, but still is of form $b_1^- \otimes t_\lambda \otimes u_{-\infty}$, so we may assume $b_1 = \overline{S^-_{\mathbf{c}_0}} \bmod q\mathcal{L}(\lambda)$ after we change $\mathbf{c}_0$. By Proposition 3.5, we have $s_{\mathbf{c}_0}(z) \cdot \widetilde{u}_\lambda = \Phi^0_\lambda(b_1) = \Phi^0_\lambda(b_2) = s_{\mathbf{c}'_0}(z) \cdot \widetilde{u}_\lambda$. This implies $\mathbf{c}_0 = \mathbf{c}'_0$ and hence $b_1 = b_2$.

(4) By the uniqueness, Φ_λ respects the bar involutions on $V(\lambda)$ and $\widetilde{V}(\lambda)$. Since $V^{\mathbb{Z}}(\lambda) = \mathbf{U}^{\mathbb{Z}}_q(\widehat{\mathfrak{g}})u_\lambda$, we have $\Phi_\lambda(V^{\mathbb{Z}}(\lambda)) \subset \check{V}^{\mathbb{Z}}(\lambda)$. Therefore we have

$$\Phi_\lambda\left(\mathcal{L}(\lambda) \cap \overline{\mathcal{L}(\lambda)} \cap V^{\mathbb{Z}}(\lambda)\right) \subset \check{\mathcal{L}}(\lambda) \cap \overline{\check{\mathcal{L}}(\lambda)} \cap \check{V}^{\mathbb{Z}}(\lambda).$$

Now the assertion follows from (3).

(1) It is easy to see that $\widetilde{\mathcal{B}}(\lambda)$ is linearly independent. Therefore $\Phi^0_\lambda \colon \mathcal{L}(\lambda)/q\mathcal{L}(\lambda) \to \widetilde{\mathcal{L}}(\lambda)/q\widetilde{\mathcal{L}}(\lambda)$ is injective.

Let $\{G(b)\}$ be the global crystal base of $V(\lambda)$. Let $0 \neq \sum f_b(q)G(b) \in \operatorname{Ker}\Phi_\lambda$. Multiplying a power of q, we may assume $f_b(q) \in \mathbf{A}_0$ for all b and $f_{b_0}(0) \neq 0$ for some b_0. Then $\sum f_b(0)b \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ is mapped to 0 by Φ^0_λ. The injectivity of Φ^0_λ implies that $f_b(0) = 0$ for all b. This is a contradiction. □

Remark 3.8. Theorem 1 together with Proposition 3.5 implies that $S^-_{\mathbf{c}_0}u_\lambda$ and $\overline{S^*_{\mathbf{c}_0}}u_\lambda$ are elements of the global base.

3.4. Standard modules

Let us briefly recall the properties of the universal standard module $M(\lambda)$ with a weight $\lambda = \sum m_i\varpi_i \in \widehat{P}^{0,+}$ introduced in [16, 18]. (We do not review its definition, which is based on quiver varieties.) Let $G_\lambda \overset{\text{def.}}{=} \prod_i \mathrm{GL}_{m_i}(\mathbb{C})$. Its maximal torus consisting of diagonal matrices is denoted by H_λ. Their representation rings are denoted by $R(G_\lambda)$, $R(H_\lambda)$ respectively. They are isomrphic to $\bigotimes_i \mathbb{Z}[x^\pm_{i,1}, \dots, x^\pm_{i,m_i}]^{\mathfrak{S}_{m_i}}$ and $\bigotimes_i \mathbb{Z}[x^\pm_{i,1}, \dots, x^\pm_{i,m_i}]$ respectively. The universal standard module $M(\lambda)$ is a $\mathbf{U}'^{\mathbb{Z}}_q(\widehat{\mathfrak{g}}) \otimes_{\mathbb{Z}} R(G_\lambda)$-module which is integrable (in fact, it satisfies a stronger condition 'l–integrability') and contains a vector $[0]_\lambda$ with

$$x_{i,r}^+[0]_\lambda = 0 \quad \text{for any } i \in I,\ r \in \mathbb{Z}, \quad q^h[0]_\lambda = q^{\langle h,\lambda\rangle}[0]_\lambda,$$
$$M(\lambda) = \left(\mathbf{U}_q'^{\mathbb{Z}}(\widehat{\mathfrak{g}}) \otimes_{\mathbb{Z}} R(G_\lambda)\right)[0]_\lambda,$$
$$\psi_i^\pm(u)[0]_\lambda = q^{m_i}\left(\prod_{\nu=1}^{m_i} \frac{1-q^{-1}x_{i,\nu}u}{1-qx_{i,\nu}u}\right)^\pm [0]_\lambda,$$

where $(\ )^\pm$ denotes the expansion at $u = 0$ and ∞ respectively. (In fact, we have $M(\lambda) = \mathbf{U}_q'^{\mathbb{Z}}(\widehat{\mathfrak{g}})[0]_\lambda$ by the proof of Theorem 1.) Moreover, $M(\lambda)$ is free of finite rank as an $R(G_\lambda)$-module. And $M(\lambda)$ is simple if we tensor the quotient field of $\mathbb{Z}[q,q^{-1}] \otimes R(G_\lambda)$.

On the other hand, we have a $\bigotimes_{i\in I}\mathbb{Z}[z_{i,1}^\pm,\dots,z_{i,m_i}^\pm]^{\mathfrak{S}_{m_i}}$ -module structure on $V(\lambda)$ given by $s_{\mathbf{c}_0}(z)u_\lambda = S_{\mathbf{c}_0}^- u_\lambda$ and $s_{\mathbf{c}_0}(z^{-1})u_\lambda = \overline{S_{\mathbf{c}_0}^*}u_\lambda$ by the above discussion. We make it a $R(G_\lambda) = \bigotimes_{i\in I}\mathbb{Z}[x_{i,1}^\pm,\dots,x_{i,m_i}^\pm]^{\mathfrak{S}_{m_i}}$ -module structure by setting $x_{i,\nu} = o(i)(-1)^{1-h}q^{-h^\vee}z_{i,\nu}$.

Theorem 2. *There exists a unique $\mathbf{U}_q'^{\mathbb{Z}}(\widehat{\mathfrak{g}})\otimes_{\mathbb{Z}}R(G_\lambda)$ -isomorphism $V^{\mathbb{Z}}(\lambda) \to M(\lambda)$ sending u_λ to $[0]_\lambda$.*

This result follows from Theorem 1 as explained in [18, 1.23]. The calculation of Drinfeld polynomial, which was not given there, is done in Proposition 3.1.

Correction to [18]:

Delete $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}$ in Theorem 1.22.

Replace $R(G_\lambda)$ in page 411, line 5 by $R(H_\lambda)$.

Delete 'and forgetting the symmetric group invariance' in Remark 1.23.

Replace 'the submodule above' in line 8, 'the submodule $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})[x_{k,\nu}]_{k\in I,\nu=1,\dots,\lambda_k}\bigotimes_{k\in I}[0]_{\Lambda_k}^{\otimes\lambda_k}$.

§4. A bilinear form

Kashiwara proved that the crystal base $\mathcal{B}(\lambda)$ is an orthonomal base with respect to a natural bilinear form when λ is dominant [10, 5.1.1]. We prove a similar result for $\lambda \in \widehat{P}^{0,+}$ in this section. This generalizes a result of Varagnolo-Vasserot [20, Theorem A] from fundamental representations to arbitray λ.

Proposition 4.1 (Kashiwara). *The extremal weight module $V(\lambda)$ has a unique bilinear form $(\ ,\ )$ satisfying*

$$(u_\lambda, G(b)) = \begin{cases} 1 & \textit{if } G(b) = u_\lambda, \\ 0 & \textit{otherwise} \end{cases} \tag{4.2}$$

$$(xu, v) = (u, \psi(x)v) \quad \textit{for } x \in \mathbf{U}_q(\widehat{\mathfrak{g}}),\ u, v \in V(\lambda). \tag{4.3}$$

Proof. We define a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module structure on $\operatorname{Hom}(V(\lambda), \mathbb{Q}(q))$ by

$$\langle xf, u\rangle \overset{\text{def.}}{=} \langle f, \psi(x)u\rangle, \quad x \in \mathbf{U}_q(\widehat{\mathfrak{g}}), f \in \operatorname{Hom}(V(\lambda), \mathbb{Q}(q)), u \in V(\lambda).$$

This defines a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-module structure since $\psi\colon \mathbf{U}_q(\widehat{\mathfrak{g}}) \to \mathbf{U}_q(\widehat{\mathfrak{g}})^{\mathrm{opp}}$ is an algebra homomorphism. Let u^λ be the unique linear form such that

$$\langle u^\lambda, G(b)\rangle = \begin{cases} 1 & \text{if } G(b) = u_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then u^λ has a weight λ. We claim that u^λ is an extremal vector. From the definition all elements in a weight space $\operatorname{Hom}(V(\lambda), \mathbb{Q}(q))_\xi$ vanish on $V(\lambda)_\xi$. Since weights of $V(\lambda)$ are contained in the convex hull of $W\lambda$ [12, Theorem 5.3], the weights of $V'(\lambda)$ also have the same property. Therefore u^λ is an extremal vector. Now we have a $\mathbf{U}_q(\widehat{\mathfrak{g}})$-algebra homomorphism $V(\lambda) \to V'(\lambda) \subset \operatorname{Hom}(V(\lambda), \mathbb{Q}(q))$ sending u_λ to u^λ. This defines a bilinear form satisfying the desired properties. The uniqueness follows from the uniqueness of the above homomorphism. □

Remark 4.4. The uniqueness holds even if (4.3) holds only for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$. In fact, this condition together with (4.2) automatically implies (4.3) for $x = q^d$ as follows. When $u = u_\lambda$, (4.2) implies (4.3) for $x = q^d$. For a general case, we write $u = xu_\lambda$ with $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})_\xi$. Then

$$\begin{aligned}(q^d u, v) &= q^{\langle d,\xi\rangle}(xq^d u_\lambda, v) = q^{\langle d,\xi\rangle}(q^d u_\lambda, \psi(x)v) = q^{\langle d,\xi\rangle}(u_\lambda, q^d\psi(x)v) \\ &= (u_\lambda, \psi(x)q^d v) = (xu_\lambda, q^d v) = (u, q^d v),\end{aligned}$$

where we have used $\psi(x) \in \mathbf{U}'_q(\widehat{\mathfrak{g}})_{-\xi}$.

Lemma 4.5. *Let M be an integrable $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-module with a bilinear form $(\ ,\ )$ satisfying* (4.3) *for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$. Then*

$$(T_w u, v) = (-1)^{N^\vee} q^N (u, T_{w^{-1}} v) \quad \textit{for all } w \in \widehat{W},\ u \in M_\xi,\ v \in M,$$

where N and $N^\vee$ are as in Lemma 2.5.

Proof. Let $T'_{i,1}$ be the operator defined in [13, 5.2.1]. A direct calculation shows $(T_i u, v) = (u, T'_{i,1} v)$ for $u \in M_\xi$, $v \in M$. (We may assume that v is contained in a weight space. Thanks to (4.3) for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$, both hand sides are 0 unless the weight of v is $s_i\xi + m\delta$ for some $m \in \mathbb{Z}$.) By [loc. cit., 5.2.3], we have $T'_{i,1}v = (-1)^{\langle h_i,\xi\rangle} q^{(\alpha_i,\xi)} T_i v$. The rest of the proof is the same as that of Lemma 2.5. $\square$

Lemma 4.6. *Let M and $(\ ,\ )$ be as above. Let u, $v \in M$ be extremal vectors. Then*

$$(S_w u, v) = (u, S_{w^{-1}} v).$$

Proof. Let ξ be the weight of u. Using Lemmas 2.11, 4.5, we have

$$(S_w u, v) = (-1)^{N_+^\vee + N_+^{\vee\prime} + N^\vee} q^{-N_+ - N'_+ + N} (u, S_{w^{-1}} v),$$

where

$$N = \sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)} (\alpha, \xi), \qquad N_+ = \sum_{\alpha \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)} \max((\alpha,\xi), 0),$$
$$N'_+ = \sum_{\alpha' \in \widehat{\mathfrak{R}}_+ \cap w(\widehat{\mathfrak{R}}_-)} \max((\alpha', w\xi), 0),$$

and $N^\vee$, $N_+^\vee$, $N_+^{\vee\prime}$ are defined in similar ways. Noticing $\alpha' \in \widehat{\mathfrak{R}}_+ \cap w(\widehat{\mathfrak{R}}_-) \Leftrightarrow -w^{-1}\alpha' \in \widehat{\mathfrak{R}}_+ \cap w^{-1}(\widehat{\mathfrak{R}}_-)$, we have $N = N_+ + N'_+$. Similarly we have $N^\vee = N_+^\vee + N_+^{\vee\prime}$. Therefore we have the assertion. $\square$

In order to study $(\ ,\ )$ on $V(\lambda)$ we need to relate it to a bilinear form on the tensor product module $\widetilde{V}(\lambda)$.

Lemma 4.7. *We have $(z_i u, z_i v) = (u, v)$ for $u, v \in V(\varpi_i)$.*

Proof. By the uniqueness, it is enough to show that $(z_i u, z_i v)$ satisfies (4.2, 4.3). The property (4.3) is clear. If $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$, then it holds since z_i is $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-linear. It also holds for $x = q^d$ thanks to $z_i q^d z_i^{-1} = q^{-a_0 d_i} q^d$.

Let us check (4.2). Since $\dim V(\varpi_i)_{\varpi_i} = 1$ by [12, Proposition 5.10], it is enough to show that $(z_i u_{\varpi_i}, z_i u_{\varpi_i}) = 1$. But this follows from the previous lemma. $\square$

We define a $\mathbb{Q}(q)[z_i^{\pm}]$-valued bilinear form $(\!(\ , \ )\!)$ on $V(\varpi_i)$ by

$$(\!(u, v)\!) = \begin{cases} z_i^m (z_i^{-m} u, v) & \text{if } \mathrm{wt}(u) = \mathrm{wt}(v) + m d_i \delta \text{ for } m \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Since z_i is $\mathbf{U}'_q(\widehat{\mathfrak{g}})$-linear, we have

$$(\!(xu, v)\!) = (\!(u, \psi(x)v)\!) \quad \text{for } x \in \mathbf{U}'_q(\widehat{\mathfrak{g}}),\ u, v \in V(\varpi_i).$$

By Lemma 4.7 we have

$$(\!(z_i^m u_{\varpi_i}, z_i^n u_{\varpi_i})\!) = z_i^{m-n}. \tag{4.8}$$

We define a $\mathbb{Q}(q)[z_{i,\nu}^{\pm}]_{i \in I, \nu = 1, \dots, m_i}$-valued bilinear form $(\!(\ , \ )\!)$ on $\widetilde{V}(\lambda)$ by

$$(\!(u, v)\!) \stackrel{\text{def.}}{=} \prod_{i,\nu} (\!(u_{i,\nu}, v_{i,\nu})\!),$$

where $u_{i,\nu}$, $v_{i,\nu}$ is the ν-th $V(\varpi_i)$-factor of $u, v \in \widetilde{V}(\lambda)$. We define a bilinear form $(\ , \ )^{\sim}$ on $\widetilde{V}(\lambda)$ by

$$(u, v)^{\sim} \stackrel{\text{def.}}{=} \prod_{i \in I} \frac{1}{m_i!} \left[(\!(u, v)\!) \prod_{\mu \neq \nu} (1 - z_{i,\mu} z_{i,\nu}^{-1}) \right]_1,$$

where $[f]_1$ denote the constant term in f.

Lemma 4.9. *Let $\mathbf{c}_0$, $\mathbf{c}'_0 \in (\mathbb{Z}_{\geq 0}^{\mathfrak{R}_0})(\lambda)$. Then $(s_{\mathbf{c}_0}(z)\widetilde{u}_\lambda, s_{\mathbf{c}'_0}(z)\widetilde{u}_\lambda)^{\sim} = \delta_{\mathbf{c}_0, \mathbf{c}'_0}$.*

Proof. Let $f = f(z)$ and $g = g(z)$ be polynomials in $z_{i,\nu}$'s ($i \in I$, $\nu = 1, \dots, m_i$). By (4.8) we have

$$(f(z)\widetilde{u}_\lambda, g(z)\widetilde{u}_\lambda)^{\sim} = \prod_{i \in I} \frac{1}{m_i!} \left[f\overline{g} \prod_{\mu \neq \nu} (1 - z_{i,\mu} z_{i,\nu}^{-1}) \right]_1,$$

where $\overline{g} = g(\dots, z_{i,\nu}^{-1}, \dots)$. Considered as a bilinear form on the Laurent polynomial ring, it coincides with one in [15, Chap.VI, §9] with $q = t$. The Schur functions give an orthogonal base with respect to that bilinear form. Therefore we have the assertion. □

Proposition 4.10. *Let $u, v \in V(\lambda)$. Then $(u, v) = (\Phi_\lambda(u), \Phi_\lambda(v))^{\sim}$.*

Proof. It is enough to show that $(\Phi_\lambda(u), \Phi_\lambda(v))^\sim$ satisfies conditions in Proposition 4.1. It is clear that the condition (4.3) holds for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$. By Remark 4.4, it is enough to check (4.2). From (4.3) for $x \in \mathbf{U}'_q(\widehat{\mathfrak{g}})$, it is enough to check (4.2) when $\mathrm{cl}(\mathrm{wt}(b)) = \mathrm{cl}(\lambda)$, i.e., $\mathrm{wt}(b) = \lambda + m\delta$ for some $m \in \mathbb{Z}$. Since weights of $V(\lambda)$ is contained in the convex hull of $W\lambda$, b is an extremal vector. We have

$$(\Phi_\lambda(u_\lambda), \Phi_\lambda(G(b)))^\sim = (\Phi_\lambda(S_w u_\lambda), \Phi_\lambda(S_w G(b)))^\sim$$

by Lemma 4.6. We take S_w as sufficiently many compositions of $\widetilde{f}_i^{\max}$, we may assume $S_w u_\lambda = S^-_{\mathbf{c}_0} u_\lambda$, $S_w G(b) = S^-_{\mathbf{c}'_0} u_\lambda$. (Recall that $S^-_{\mathbf{c}_0} u_\lambda$ is an element of the global basis as we explained in Remark 3.8.) Then

$$(\Phi_\lambda(u_\lambda), \Phi_\lambda(G(b)))^\sim = (s_{\mathbf{c}_0}(z)\widetilde{u}_\lambda, s_{\mathbf{c}'_0}(z)\widetilde{u}_\lambda)^\sim = \delta_{\mathbf{c}_0,\mathbf{c}'_0} = \delta_{u_\lambda, G(b)},$$

where we have used Proposition 3.5 and Lemma 4.9. □

From the proof of Proposition 4.10 the bilinear form $(\ ,\ )$ on $V(\lambda)$ defined in Proposition 4.1 also has the following characterization: it satisfies (4.3) and $(S_{\mathbf{c}_0} u_\lambda, S_{\mathbf{c}'_0} u_\lambda) = \delta_{\mathbf{c}_0,\mathbf{c}'_0}$. Since these conditions are symmetric, we have the following:

Corollary 4.11. *The bilinear form $(\ ,\ )$ on $V(\lambda)$ is symmetric, i.e., $(u,v) = (v,u)$.*

Proposition 4.12. (1) $(\mathcal{L}(\lambda), \mathcal{L}(\lambda)) \subset \mathbf{A}_0$.
Let $(\ ,\ )_0$ be the $\mathbb{Q}$-valued bilinear form on $\mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$ induced by $(\ ,\ )|_{q=0}$ on $\mathcal{L}(\lambda)$.

(2) $(\widetilde{e}_i u, v)_0 = (u, \widetilde{f}_i v)_0$ *for $u, v \in \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$.*

(3) *$\mathcal{B}(\lambda)$ is an orthonormal base with respect to $(\ ,\ )_0$. In particular, $(\ ,\ )_0$ is positive definite.*

(4) $\mathcal{L}(\lambda) = \{u \in V \mid (u,u) \in \mathbf{A}_0\}$.

Proof. We shall prove

- there exist representatives $\widetilde{b}$ for all $b \in \mathcal{B}(\lambda)_\xi \subset \mathcal{L}(\lambda)_\xi / q\mathcal{L}(\lambda)_\xi$ such that $(\widetilde{b}, \widetilde{b'}) \equiv \delta_{bb'} \mod q\mathbf{A}_0$ for $b, b' \in \mathcal{B}(\lambda)_\xi$

by the induction on (ξ,ξ). Since $\mathcal{L}(\lambda)_\xi$ is spanned by $\widetilde{b}$'s over $\mathbf{A}_0$, this implies the above equations for *any* representatives $\widetilde{b}$. It also implies (1) and (3). Recall $(\widetilde{e}_i\widetilde{b}, \widetilde{b'}) = (1-q_i)(\widetilde{b}, \widetilde{f}_i\widetilde{b'})$ by (2.6). Therefore the above assertion also implies (2).

First suppose that b is extremal. Since we may assume that $\mathrm{wt}(b) = \mathrm{wt}(b')$ by (4.3), we may assume b' is also extremal by [12, 5.3]. Then we may assume $\widetilde{b} = S_{\mathbf{c}_0} u_\lambda$, $\widetilde{b}' = S_{\mathbf{c}'_0} u_\lambda$ by applying S_w for some $w \in \widehat{W}$. But, in this case, the assertion has been already shown in Lemma 4.9 and Proposition 4.10.

Now we start the induction. Recall that (ξ, ξ) is bounded from above and $b \in \mathcal{B}(\lambda)$ is extremal if $(\mathrm{wt}\, b, \mathrm{wt}\, b)$ is maximal ([11, §9.3]). Therefore when (ξ, ξ) is maximal, both b and b' are extremal. We have already proved the assertion this case.

Now assuming the above for ξ such that $(\xi, \xi) > a$, let us prove it for ξ with $(\xi, \xi) = a$. For $i \in I$, suppose that $\langle h_i, \xi \rangle \geq 0$. We consider $\widetilde{e}_i b$. If $\widetilde{e}_i b \neq 0$, then we have

$$(\mathrm{wt}(\widetilde{e}_i b), \mathrm{wt}(\widetilde{e}_i b)) = (\xi + \alpha_i, \xi + \alpha_i) > (\xi, \xi).$$

Therefore we have

$$\left(\widetilde{f}_i \widetilde{e}_i \widetilde{b}, \widetilde{b}'\right) = \frac{1}{1-q} \left(\widetilde{e}_i \widetilde{b}, \widetilde{e}_i \widetilde{b}'\right) \equiv \delta_{\widetilde{e}_i b, \widetilde{e}_i b'} \equiv \delta_{bb'} \mod q\mathbf{A}_0$$

by the induction hypotheis. Hence the assertion holds if we replace the representative $\widetilde{b}$ by another representative $\widetilde{f}_i \widetilde{e}_i \widetilde{b}$. Similarly, if $\langle h_i, \xi \rangle \leq 0$ and $\widetilde{f}_i b \neq 0$, we replace $\widetilde{b}$ by $\widetilde{e}_i \widetilde{f}_i \widetilde{b}$ to get the assertion.

Since we may suppose that b is not extremal, there exists $w \in \widehat{W}$ such that $S_w b$ satisfies $\widetilde{e}_i S_w b \neq 0$ if $\langle h_i, w\xi \rangle \geq 0$ and $\widetilde{f}_i S_w b \neq 0$ if $\langle h_i, w\xi \rangle \leq 0$. Then we have $(\widetilde{f}_i \widetilde{e}_i S_w \widetilde{b}, S_w \widetilde{b}')$ or $(\widetilde{e}_i \widetilde{f}_i S_w \widetilde{b}, S_w \widetilde{b}')$ is in $\delta_{bb'} + q\mathbf{A}_0$. Therefore we are done.

The statement (4) follows from [13, 14.2.2]. □

The following result generalizes [20, Theorem A] from fundamental representations to arbitrary λ:

Theorem 3. (1) $\{G(b)\}_{b \in \mathcal{B}(\lambda)}$ *is almost orthonormal for* $(\ ,\ )$, *that is,* $(G(b), G(b')) \equiv \delta_{bb'} \mod q\mathbb{Z}[q]$.
(2) $\{\pm G(b) \mid b \in \mathcal{B}(\lambda)\} = \{u \in V^{\mathbb{Z}}(\lambda) \mid \overline{u} = u,\ (u, u) \equiv 1 \mod q\mathbb{Z}[q]\}$.

Proof. We claim

$$(u, v) \in \mathbb{Z}[q, q^{-1}] \quad \text{for } u, v \in V^{\mathbb{Z}}(\lambda).$$

The assertion is obvious for the special case $u = u_\lambda$ by (4.2). For general case, we may assume $u = x u_\lambda$ for $x \in \mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$. Then $(x u_\lambda, v) = (u_\lambda, \psi(x) v)$. Since $\psi(x) \in \mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$ and $V^{\mathbb{Z}}(\lambda)$ is stable under the action of $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})$, the assertion follows from the special case.

Combining with Proposition 4.12, we have

$$(G(b), G(b')) - \delta_{bb'} \in \mathbb{Z}[q, q^{-1}] \cap q\mathbf{A}_0 = q\mathbb{Z}[q].$$

This is the statement (1). The statement (2) follows from the argument of [13, 14.2.3]. □

Remark 4.13. Lusztig conjectures that the universal standard module $M(\lambda)$, more precisely its tensor product of $\otimes_{R(G_\lambda)} R(H_\lambda)$, which is isomorphic to $\breve{V}^{\mathbb{Z}}(\lambda)$, has a signed base characterized by the almost orthogonality property Theorem 3(2), with respect to geometrically defined bilinear form and bar involution [14]. (See §3.4 for notations.) Recently Varagnolo-Vasserot [20] give a proof of the conjecture by showing that $\{G(b) \mid b \in \breve{\mathcal{B}}(\lambda)\}$ satisfies the property. They also conjecture that the global base $\{G(b) \mid b \in \mathcal{B}(\lambda)\}$ of $V(\lambda)$ satisfies the almost orthogonality property with respect to the geometric bilinear form and bar involution. Their conjecture follows from Theorem 3(2) since the geometric bilinear form and bar involution coincide with ones used in this paper, as Varangnolo and Vasserot proved that the formers satisfy the conditions in Proposition 4.1 (more precisely (4.3) and $(S_{\mathbf{c}_0} u_\lambda, S_{\mathbf{c}_0'} u_\lambda) = \delta_{\mathbf{c}_0, \mathbf{c}_0'}$) and the equality $\overline{xu} = \overline{x}\ \overline{u}$. Remark that these hold only after an *appropriate* normalization of universal standard modules so that we have $x_{i,\nu} = \pm z_{i,\nu}$. This is the normalization in [20] different from ours. This point is clarified during discussion with Varagnolo-Vasserot in Februrary 2002.

Added in Proof. Results of this paper are generalized to the case of arbitrary affine algebras in the paper "Crystal bases and two-sided cells of quantum affine algebras" by J. Beck and H. Nakajima, to appear in Duke Math. J.

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Tropical Robinson-Schensted-Knuth correspondence and birational Weyl group actions

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§ Introduction

This paper is an outcome of our attempt to understand internal connections among several appearances of the subtraction-free birational transformations.

There is a well-known procedure for passing from subtraction-free rational functions to piecewise linear functions. Roughly, this is the procedure of replacing the operations

$$a\,b \to a+b, \quad a/b \to a-b, \quad a+b \to \max\{a,b\} \quad (\text{or } \min\{a,b\}).$$

It can be applied consistently to an arbitrary rational function expressed as a ratio of two polynomials with positive real coefficients, in order to produce a combination of $+$, $-$ and max (or min), representing a piecewise linear function. In combinatorics, this procedure has been employed for the *algebraization* of combinatorial algorithms. A large class of combinatorial algorithms can be described as piecewise linear transformations among discrete variables which take integer values. For such a piecewise linear transformation, it is meaningful in many cases to find a good subtraction-free rational counterpart; algebraic computation of subtraction-free rational functions may possibly bring out unexpected solutions to combinatorial problems. For this *tropical approach* to combinatorics, we refer the reader to [1], [14] and the references therein.

In the context of discrete integrable systems, the same procedure is known as *ultra-discretization* [27]. A remarkable example is the ultra-discretization of discrete Toda equation which provides with soliton cellular automata, called the box-ball systems [28]. It is already recognized that the theory of box-ball systems is precisely the dynamics of

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crystal bases which arise as the $q \to 0$ limit of representations of quantum groups (see [6], for example). The ultra-discretization of certain q-Painlevé systems can also be understood as a non-autonomous deformation of box-ball systems [10, 11, 12]; the time evolution of such (ultra-)discrete systems arises from the translation lattice of affine Weyl groups.

Another important aspect is the connection with the theory of totally positive matrices. Totally positive matrices have been studied extensively from the viewpoint of geometric approach to canonical bases [19], [1], [4], [2]; they provide a basic tool for producing nice subtraction-free rational transformations.

The purpose of this paper is to develop a new, elementary approach to the application of subtraction-free birational transformations to combinatorial problems. Our method is based on the decomposition and exchange of matrices, and the path representation of minor determinants. We employ such techniques to construct both subtraction-free rational and piecewise linear transformations for typical combinatorial algorithms, such as the bumping procedure, the Schützenberger involution and the Robinson-Schensted-Knuth correspondence (RSK correspondence, for short). Our *matrix approach* can be regarded as an integration of the idea of totally positive matrices and the technique of discrete Toda equations. We also investigate certain birational and piecewise linear actions of (affine) Weyl groups on matrices and tableaux.

This work was motivated by the impressive paper [14] by A.N. Kirillov. It was a great surprise for the authors to find that many formulas in [14], arising from combinatorics, were essentially the same as what we had encountered with in the context of discrete Painlevé systems. The matrix approach, as we will develop below, was a natural consequence of our attempt to clarify the theoretical background of this remarkable coincidence.

In view of the elementary nature of our approach, we have tried to make this paper as self-contained as possible. Many of the explicit formulas discussed in this paper can already be found in the literature ([3], [14], [15]). Also, many of the statements on decomposition of matrices are essentially contained in a series of works [1], [4], [19] on totally positive matrices. We expect however that the results and techniques developed in this paper would be applicable to various problems both in combinatorics and in discrete integrable systems.

The authors would like to express their thanks to Professors S. Fomin, A.N. Kirillov, and A. Zelevinsky for valuable discussions.

Notes: The use of the phrase "tropical" comes originally from computer science; as in "tropical semirings", this word has been used in a restrictive way to refer to the semiring structure on various set of numbers with respect to the pair of operations $(\min, +)$. We thank Prof. Fomin for directing our attention to this point. In the combinatorial literature, the same phrase seems to be used in a broader sense, mostly in such a situation that subtraction-free rational functions and piecewise linear functions appear more or less exchangeably; it also depends on the author on which side emphasis is put. In this paper, following [14] we use the word "tropical" *tentatively* to refer to objects concerning subtraction-free rational functions (see Section 1.3). This may *not* be identical to the traditional usage, but we could not find a better alternative.

CONTENTS

§1. Preliminaries

In this section, we give some preliminary remarks on the matrix approach to nonintersecting paths. In the last part of this section, we also give a summary on a canonical procedure for passing from subtraction-free rational functions to piecewise linear functions. In what follows, we fix the ground field $\mathbb{K}$, and set $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. For a matrix $X = \left(x_j^i\right)_{i,j}$ given, we denote by

$$(1.1) \qquad X^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \left(x^{i_a}_{j_b}\right)^r_{a,b=1}, \quad \det X^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \det \left(x^{i_a}_{j_b}\right)^r_{a,b=1}$$

the $r \times r$ submatrix and the r-minor determinant of X with row indices $i_1, \dots, i_r$ and column indices $j_1, \dots, j_r$, respectively.

1.1. Path representation of minor determinants

For an n-vector $\boldsymbol{x} = (x_1, \dots, x_n) \in (\mathbb{K}^*)^n$ given, we introduce the following two matrices $E(\boldsymbol{x})$ and $H(\boldsymbol{x})$:

$$(1.2) \qquad E(\boldsymbol{x}) = \operatorname{diag}(\boldsymbol{x}) + \Lambda, \quad H(\boldsymbol{x}) = (\operatorname{diag}(\boldsymbol{x})^{-1} - \Lambda)^{-1},$$

where $\Lambda = \left(\delta_{j,i+1}\right)_{i,j=1}^{n}$ stands for the shift matrix. With the notation of matrix units $E_{ij} = \left(\delta_{a,i}\delta_{b,j}\right)_{a,b=1}^{n}$, these matrices can be written alternatively as

(1.3)
$$E(\boldsymbol{x}) = \sum_{i=1}^{n} x_i E_{ii} + \sum_{i=1}^{n-1} E_{i,i+1}, \quad H(\boldsymbol{x}) = \sum_{1 \le i \le j \le n} x_i x_{i+1} \cdots x_j \, E_{ij}.$$

For a given sequence of n-vectors $\boldsymbol{x}^1, \dots, \boldsymbol{x}^m$, $\boldsymbol{x}^i = (x_1^i, \dots, x_n^i) \in (\mathbb{K}^*)^n$, we define

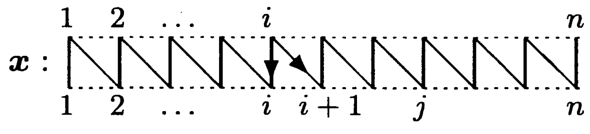
$$\begin{aligned} E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m) &= E(\boldsymbol{x}^1)E(\boldsymbol{x}^2)\cdots E(\boldsymbol{x}^m), \\ H(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m) &= H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m). \end{aligned} \tag{1.4}$$

Note that $H(\boldsymbol{x}) = DE(\overline{\boldsymbol{x}})^{-1}D^{-1}$, $D = \mathrm{diag}((-1)^{i-1})_{i=1}^{n}$, where $\overline{x} = (\overline{x}_1, \dots, \overline{x}_n)$, $\overline{x}_j = \frac{1}{x_j}$; we use the notation $\overline{x}$ for x^{-1} in order to avoid the conflict with that of upper indices. With this notation, the two matrices in (1.4) are related as

$$H(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m) = DE(\overline{\boldsymbol{x}}^m, \dots, \overline{\boldsymbol{x}}^1)D^{-1}. \tag{1.5}$$

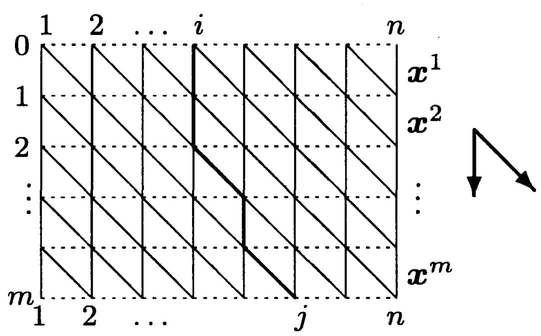
In the following, we propose graphical expressions for the minor determinants of $E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$ and $H(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$, in terms of nonintersecting paths.

We first consider the case of $E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$. We represent the matrix $E(\boldsymbol{x})$ by the diagram

(1.6)


with weight x_j attached to the j-th *vertical edge* for each $j = 1, \dots, n$, and weight 1 to each slanted edge. The (i,j)-component of $E(\boldsymbol{x})$ can then be read off by the weight of paths from i at the top to j at the bottom. Piling up the diagrams for $E(\boldsymbol{x}^1)$, $\dots$,$E(\boldsymbol{x}^m)$ all together, we obtain the following diagram for $E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$.

(1.7)



Here we make all the edges oriented downward, to the south or to the southeast. The (i, j)-component of $E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)$ is then given as the sum of weights over all paths from i at the top to j at the bottom. It can also be expressed as

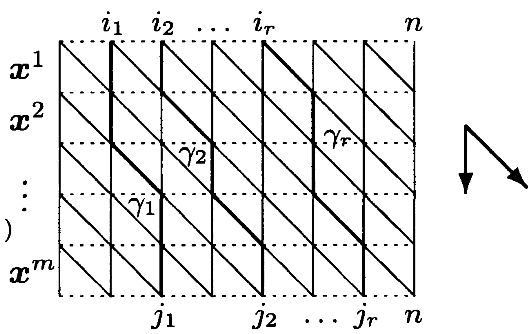
$$E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)^i_j = \sum_{1 \le k_i < k_{i+1} < \cdots < k_{j-1} \le m} \prod_{b=i}^{j} \prod_{a=k_{b-1}+1}^{k_b - 1} x^a_b, \tag{1.8}$$

where $k_{i-1} = 0$ and $k_j = m + 1$. Furthermore, we have

Proposition 1.1. *For any choice of row indices $i_1 < \cdots < i_r$ and column indices $j_1 < \dots < j_r$, the minor determinant $\det E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)^{i_1, \dots, i_r}_{j_1, \dots, j_r}$ is expressed as the sum of weights over all r-tuples $(\gamma_1, \dots, \gamma_r)$ of nonintersecting paths γ_k from i_k at the top to j_k at the bottom $(k = 1, \dots, r)$.*

(1.9)

$$\det E(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)^{i_1, \dots, i_r}_{j_1, \dots, j_r} = \sum_{(\gamma_1, \dots, \gamma_r)}$$



This proposition is an immediate consequence of the theorem of Gessel-Viennot [5]. In our context, however, it is also meaningful to understand this passage to nonintersecting paths through the multiplicative properties of minor determinants. Proposition 1.1 is essentially reduced to the multiplicative formula

$$\det(XY)^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum_{k_1<\cdots<k_r} \det X^{i_1,\dots,i_r}_{k_1,\dots,k_r} \det Y^{k_1,\dots,k_r}_{j_1,\dots,j_r} \tag{1.10}$$

for minor determinants of the product of matrices. A key step is the following simple lemma. Note that the matrix $E(\boldsymbol{x})$ is decomposed in the form

$$E(\boldsymbol{x}) = \operatorname{diag}(\boldsymbol{x})(1+\overline{x}_{n-1}E_{n-1,n})\cdots(1+\overline{x}_2E_{2,3})(1+\overline{x}_1E_{1,2}). \tag{1.11}$$

Also, the minor determinant

$$\det(1+aE_{k,k+1})^{i_1,\dots,i_r}_{j_1,\dots,j_r} \quad (i_1<\dots<i_r,\ j_1<\dots<j_r) \tag{1.12}$$

vanishes unless either the two index sets $I = \{i_1,\dots,i_r\}$ and $J = \{j_1,\dots,j_r\}$ are identical, or J is obtained from I by replacing $k \in I$ by $k+1$. From this remark, we have

Lemma 1.2. *For row indices $i_1 < \cdots < i_r$ and column indices $j_1 < \cdots < j_r$ given, the minor determinant $\det E(\boldsymbol{x})^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ vanishes unless*

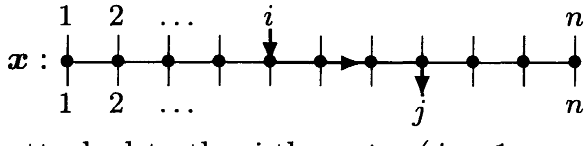
$$j_a = i_a \quad \text{or} \quad i_{a+1} \quad \text{for all} \quad a = 1,\dots,r. \tag{1.13}$$

If this is the case, $\det E(\boldsymbol{x})^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ is the product of x_{j_a} over all a such that $j_a = i_a$.

Proposition 1.1 is then obtained from Lemma 1.2 by applying the multiplicative formula (1.10) to the decomposition $E(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m) = E(\boldsymbol{x}^1)\cdots E(\boldsymbol{x}^m)$. Path representations as in (1.9) can also be translated into the language of tableaux; see for instance [21].

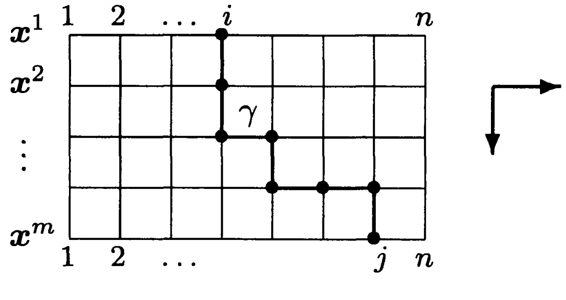
We now turn to the graphical representation of $H(\boldsymbol{x})$ and $H(\boldsymbol{x}^1,\dots,\boldsymbol{x}^m)$. We represent the matrix $H(\boldsymbol{x})$ by the diagram

(1.14)



with weight x_j attached to the j-th *vertex* ($j = 1,\dots,m$). Then piling up the diagrams for $H(\boldsymbol{x}_1),\dots,H(\boldsymbol{x}_m)$, we obtain the $m\times n$ rectangle.

(1.15)

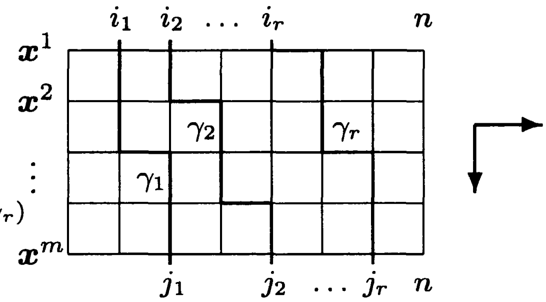


In this diagram for $H(\boldsymbol{x}_1, \dots, \boldsymbol{x}_m)$, for each $a = 1, \dots, m$ and $b = 1, \dots, n$, we attach the weight x_b^a to the vertex with coordinates (a, b). This time, the weight of a path γ is defined to be the product of all x_b^a's attached to the vertices on γ.

Proposition 1.3. *For any choice of row indices $i_1 < \cdots < i_r$ and column indices $j_1 < \dots < j_r$, the minor determinant $\det H(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)_{j_1, \dots, j_r}^{i_1, \dots, i_r}$ is expressed as the sum of weights over all r-tuples $(\gamma_1, \dots, \gamma_r)$ of nonintersecting paths $\gamma_k : (1, i_k) \to (m, j_k)$ $(k = 1, \dots, r)$.*

(1.16)

$$\det H(\boldsymbol{x}^1, \dots, \boldsymbol{x}^m)_{j_1, \dots, j_r}^{i_1, \dots, i_r} = \sum_{(\gamma_1, \dots, \gamma_r)}$$



The following corresponds to Lemma 1.2 for $E(\boldsymbol{x})$.

Lemma 1.4. *For row indices $i_1 < \cdots < i_r$ and column indices $j_1 < \cdots < j_r$ given, the minor determinant $\det H(\boldsymbol{x})_{j_1, \dots, j_r}^{i_1, \dots, i_r}$ vanishes unless*

$$i_1 \le j_1 < i_2 \le j_2 < \cdots < i_r \le j_r. \tag{1.17}$$

If this is the case, one has

$$\det H(\boldsymbol{x})_{j_1, \dots, j_r}^{i_1, \dots, i_r} = x_{i_1} \cdots x_{j_1} x_{i_2} \cdots x_{j_2} \cdots x_{i_r} \cdots x_{j_r}. \tag{1.18}$$

Note also

$$H(\boldsymbol{x}) = (1 + x_1 E_{1,2})(1 + x_2 E_{2,3}) \cdots (1 + x_{n-1} E_{n-1,n}) \operatorname{diag}(\boldsymbol{x}). \tag{1.19}$$

In the following, we apply the same idea to nonintersecting paths in triangles and trapezoids. For this purpose, we define

$$\Lambda_{\ge k} = \sum_{i=k}^{n-1} E_{i,i+1} \qquad (k = 1, \dots, n), \tag{1.20}$$

so that $\Lambda_{\geq 1} = \Lambda$ and $\Lambda_{\geq n} = 0$. With these *truncated shift matrices*, we introduce the following variations of $E(\boldsymbol{x})$ and $H(\boldsymbol{x})$:

$$E_k(\boldsymbol{x}) = \mathrm{diag}(\boldsymbol{x}) + \Lambda_{\geq k}, \quad H_k(\boldsymbol{x}) = (\mathrm{diag}(\overline{\boldsymbol{x}}) - \Lambda_{\geq k})^{-1} \tag{1.21}$$

for $k = 1, \ldots, n$. When we use these notations, we will tacitly assume that $\boldsymbol{x} = (1, \ldots, 1, x_k, \ldots, x_n)$, i.e., $x_j = 1$ $(j < k)$, unless otherwise mentioned. Under this convention, $E_k(\boldsymbol{x})$ and $H_k(\boldsymbol{x})$ are expressed as

$$E_k(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 \\ 0 & E(\boldsymbol{x}') \end{bmatrix}, \quad H_k(\boldsymbol{x}) = \begin{bmatrix} 1 & 0 \\ 0 & H(\boldsymbol{x}') \end{bmatrix}, \tag{1.22}$$

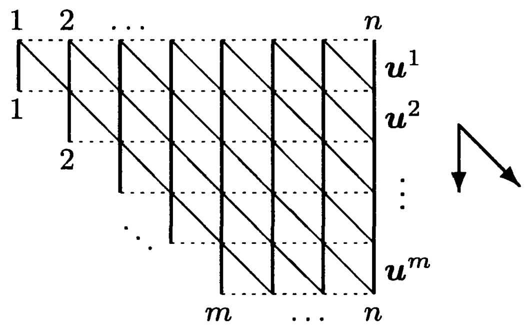
respectively, where $\boldsymbol{x}' = (x_k, x_{k+1}, \ldots, x_n)$; we will often identify the $(n-k-1)$-vector $(x_k, \ldots, x_n)$ with the n-vector $(1, \ldots, 1, x_k, \ldots, x_n)$, by putting 1's in front. Assuming that $m \leq n$, let us consider a sequence of n-vectors $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^m$, $\boldsymbol{u}^i = (u^i_i, \ldots, u^i_n)$, and arrange u^i_j $(i \leq j)$ in the form

$$U = \left(u^i_j\right)_{i \leq j} = \begin{bmatrix} u^1_1 & u^1_2 & \ldots & u^1_m & \ldots & u^1_n \\ & u^2_2 & \ldots & u^2_m & \ldots & u^2_n \\ & & \ddots & \vdots & & \vdots \\ & & & u^m_m & \ldots & u^m_n \end{bmatrix}. \tag{1.23}$$

For such a table U given, we define an $n \times n$ matrix E_U by

$$E_U = E_1(\boldsymbol{u}^1)E_2(\boldsymbol{u}^2) \cdots, E_m(\boldsymbol{u}^m). \tag{1.24}$$

The entries of this matrix can be represented by the diagram



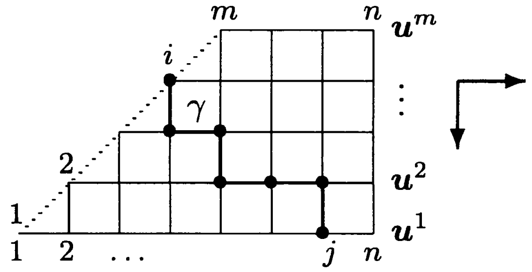
(1.25)

with the weights u^i_j attached to the vertical edges; for each (i,j) with $1 \leq i \leq j \leq n$, $(E_U)^i_j$ is the sum of weights over all paths γ from i at the top to j along the lower rim. The minor determinants of E_U are also represented by nonintersecting paths in diagram (1.25). We also introduce

$$H_U = H_m(\boldsymbol{u}^m) \cdots H_2(\boldsymbol{u}^2)H_1(\boldsymbol{u}^1), \tag{1.26}$$

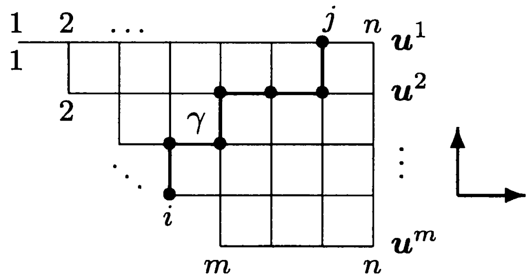
so that $H_U = DE_{\overline{U}}^{-1}D^{-1}$. The diagram for H_U is given by

(1.27)



or alternatively by

(1.28)



with edges oriented upward or rightward. The minor determinants of H_U are expressed in terms of nonintersecting paths in diagram (1.27) or (1.28).

1.2. Minor determinants of triangular matrices

The matrices E_U discussed above can be thought of as *canonical forms* of generic upper triangular matrices M of the form

$$(1.29)\qquad M = \begin{array}{c} \begin{array}{ccccc} 1 & & m & & n \end{array} \\ \begin{bmatrix} * \cdots * & 1 & & 0 \\ \ddots & a^i_j & \ddots & 1 \\ & \ddots & & * \\ 0 & & \ddots & \vdots \\ & & & * \end{bmatrix} \end{array}.$$

Let $M = \left(a^i_j\right)_{i,j=1}^n$ be an $n \times n$ upper triangular matrix satisfying the condition

$$(1.30)\qquad a^i_j = 0 \quad (j < i \text{ or } j > i + m), \qquad a^i_j = 1 \quad (j = i + m);$$

when $m = n$, this simply means that M is upper triangular. For each (i, j) with $1 \le i \le j \le n$, introduce the notation of minor determinants

$$(1.31)\qquad Q_{i,j} = Q_{i,j}(M) = \det M^{1,\dots,j-i+1}_{i,i+1,\dots,j};$$

when $j-i+1=0$, we set $Q_{i,j}=1$. We remark that the condition (1.30) for an upper triangular matrix M is equivalent to the condition

$$Q_{m+1,j}=1, \qquad Q_{i,j}=0 \quad (m+1<i\leq n), \tag{1.32}$$

for minor determinants. The following proposition is due to A. Berenstein, S. Fomin and A. Zelevinsky [1].

Proposition 1.5. *Let $M=\left(a^i_j\right)^n_{i,j=1}$ be an $n\times n$ upper triangular matrix satisfying the condition* (1.30) *for some m $(1\leq m\leq n)$. Suppose that $Q_{i,j}\neq 0$ for any (i,j) with $i\leq j$ and $i\leq m$. Then M can be decomposed uniquely in the form*

$$M=E_1(\boldsymbol{v}^1)E_2(\boldsymbol{v}^2)\cdots E_m(\boldsymbol{v}^m), \tag{1.33}$$

where $\boldsymbol{v}^i=(1,\ldots,1,v^i_i,\ldots,v^i_n)$, $v^i_j\neq 0$, for $i=1,\ldots,m$. Furthermore, v^i_j are determined by

$$v^i_i=Q_{i,i}, \quad v^i_j=\frac{Q_{i,j}\,Q_{i+1,j-1}}{Q_{i+1,j}\,Q_{i,j-1}} \qquad (i<j,\ i\leq m). \tag{1.34}$$

Proof. Assume first that M is decomposed as in (1.33). Then the minor determinants of M are expressed in terms of nonintersecting paths in diagram (1.25) for $V=\left(v^i_j\right)_{i\leq j}$. In particular we have

$$Q_{i,j}=\det M^{1,\ldots,j-i+1}_{i,\ldots,j}=\prod_{(a,b):\,a\geq i,b\leq j} v^i_j, \tag{1.35}$$

since there is only one $(j-i+1)$-tuple of nonintersecting paths relevant to the path representation of this case. Expression (1.34) follows immediately from (1.35), which also implies the uniqueness of decomposition (1.33). It remains to show that M has a decomposition of the form (1.33) under the condition on $Q_{i,j}$. We express M in the form

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.36}$$

so that B becomes a square matrix of size $n-m+1$: $B=M^{1,\ldots,n-m+1}_{m,m+1,\ldots,n}$. We can apply the Gauss decomposition to the matrix

$$B=\begin{array}{r} 1 \\ \\ m \\ \\ \\ \end{array}\begin{bmatrix} * & 1 & & & \\ \vdots & \ddots & \ddots & 0 & \\ * & & \ddots & \ddots & \\ & \ddots & & \ddots & 1 \\ 0 & & * & \cdots & * \end{bmatrix} \tag{1.37}$$

since

$$\det B^{1,\dots,r}_{1,\dots,r} = \det M^{1,\dots,r}_{m,\dots,m+r-1} = Q_{m,m+r-1}(M) \neq 0 \tag{1.38}$$

for $r = 1,\dots,n-m+1$. It also turns out that the lower and the upper triangular components of the Gauss decomposition $B = B_{<0}B_{\geq 0}$ are in the form

$$B_{<0} = \begin{array}{c} 1 \\ \\ m \\ \\ \end{array}\begin{bmatrix} 1 & & & & \\ * & \ddots & & 0 & \\ \vdots & \ddots & \ddots & & \\ * & & \ddots & \ddots & \\ 0 & & * \cdots & * & 1 \end{bmatrix}, \quad B_{\geq 0} = \begin{bmatrix} * & 1 & & \\ & \ddots & \ddots & 0 \\ & & \ddots & \ddots \\ 0 & & & \ddots & 1 \\ & & & & * \end{bmatrix}. \tag{1.39}$$

Denoting the diagonal entries of $B_{\geq 0}$ by $v^m_m,\dots,v^m_n$, we introduce the vector $\boldsymbol{v}^m = (1,\dots,1,v^m_m,\dots,v^m_n)$. Then we have the following decomposition of M:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B_{<0} \\ C & D' \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & B_{\geq 0} \end{bmatrix} = M'E_m(\boldsymbol{v}^m), \tag{1.40}$$

where $D' = DB^{-1}_{\geq 0}$. The matrix M' thus obtained satisfies the condition (1.30) with m replaced by $m-1$, as can be seen from the decomposition above. Also, if $i \leq m-1$ and $i \leq j$, from $M = M'E_m(\boldsymbol{v})$ it follows that

$$\begin{aligned} Q_{i,j}(M) &= \det M^{1,\dots,j-i+1}_{i,\dots,j} \\ &= \det(M')^{1,\dots,j-i+1}_{i,\dots,j} \det E_m(\boldsymbol{v}^m)^{i,\dots,j}_{i,\dots,j} \\ &= Q_{i,j}(M')\, v^m_m \cdots v^m_j \neq 0. \end{aligned} \tag{1.41}$$

Hence we can apply the descending induction on m to obtain the decomposition (1.33). □

We now translate Proposition 1.5 into a statement concerning the decomposition of type H_U. Let H be an $n\times n$ invertible upper triangular matrix. For the decomposition of type $H = H_U$, it is convenient to use the following notation

$$\tau^i_j = \tau^i_j(H) = \det H^{1,\dots,i}_{j-i+1,\dots,j} \qquad (1 \leq i \leq j \leq n). \tag{1.42}$$

We also define $\tau^0_j = 1$ for any j. Setting $M = DH^{-1}D^{-1}$, $D = \operatorname{diag}((-1)^{i-1})^n_{i=1}$, we compare the minor determinants $\tau^i_j = \tau^i_j(H)$ and $Q_{i,j} = Q_{i,j}(M)$. We remark that

$$\det H^{1,\dots,i}_{j-i+1,\dots,j} = \frac{\det M^{1,\dots,j-i,j+1,\dots,n}_{i+1,\dots,n}}{\det M} = \frac{\det M^{1,\dots,j-i}_{i+1,\dots,j}}{\det M^{1,\dots,j}_{1,\dots,j}}; \tag{1.43}$$

for these particular minor determinants, we have no minus sign. This implies

$$(1.44) \qquad \tau_j^i = \frac{Q_{i+1,j}}{Q_{1,j}}, \qquad Q_{i,j} = \frac{\tau_j^{i-1}}{\tau_j^j} \qquad (i \le j).$$

Note also that $\tau_j^j = Q_{1,j}^{-1}$. Hence we see that the condition (1.30) is equivalent to

$$(1.45) \qquad \tau_j^i = \delta_{i,j}\tau_j^m \qquad (m < i \le j \le n).$$

Proposition 1.6. *Let H be an $n \times n$ upper triangular matrix, and suppose that the minor determinants $\tau_j^i = \tau_j^i(H)$ $(1 \le i \le j \le n)$ satisfy the condition*

$$(1.46) \qquad \tau_j^i \ne 0 \quad (1 \le i \le m), \quad \tau_j^i = \delta_{i,j}\tau_j^m \quad (m < i \le n),$$

for some m $(1 \le m \le n)$. Then the matrix H can be decomposed uniquely in the form

$$(1.47) \qquad H = H_m(\boldsymbol{u}^m) \cdots H_2(\boldsymbol{u}^2) H_1(\boldsymbol{u}^1),$$

where $\boldsymbol{u}^i = (1, \dots, 1, u_i^i, \dots, u_n^i)$, $u_j^i \ne 0$, for $i = 1, \dots, m$. Furthermore u_j^i are determined by

$$(1.48) \qquad u_i^i = \frac{\tau_i^i}{\tau_i^{i-1}}, \qquad u_j^i = \frac{\tau_j^i\,\tau_{j-1}^{i-1}}{\tau_j^{i-1}\tau_{j-1}^i} \quad (i < j,\ i \le m).$$

Under the condition (1.46), from Proposition 1.5 we have

$$(1.49) \qquad DH^{-1}D^{-1} = M = E_1(\overline{\boldsymbol{u}}^1) E_2(\overline{\boldsymbol{u}}^2) \cdots E_m(\overline{\boldsymbol{u}}^m),$$

where we have set $\boldsymbol{u}^i = \overline{\boldsymbol{v}}^i$ $(i = 1, \dots, m)$. Once we have the decomposition (1.47), the minor determinants of H is expressed in terms of nonintersecting paths in diagram (1.28) for $U = \left(u_j^i\right)_{i,j}$. In particular, each τ_j^i is expressed as

$$(1.50) \qquad \tau_j^i = \det H_{j-i+1,\dots,j}^{1,\dots,i} = \prod_{(a,b);\, a \le i,\, b \le j} u_b^a,$$

since there is only one i-tuple of nonintersecting paths relevant to this minor determinant. Expression (1.48) for u_j^i follows immediately from (1.50).

Proposition 1.6 implies the following theorem concerning the path representation of minor determinants of a triangular matrix.

Theorem 1.7. *Let H be an $n \times n$ upper triangular matrix, and suppose that the minor determinants $\tau^i_j = \tau^i_j(H)$ $(i \le j)$ satisfy the condition*

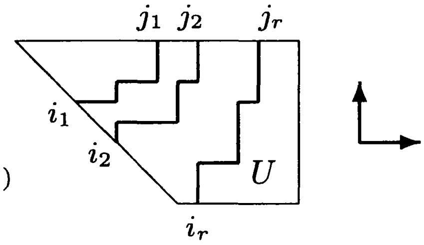
$$\tau^i_j \neq 0 \quad (1 \le i \le m), \qquad \tau^i_j = \delta_{i,j}\tau^m_j \quad (m < i \le n) \tag{1.51}$$

for some m $(1 \le m \le n)$. For each (i,j) with $1 \le i \le j \le n$, $i \le m$, define

$$u^i_i = \frac{\tau^i_i}{\tau^{i-1}_i}, \qquad u^i_j = \frac{\tau^i_j\,\tau^{i-1}_{j-1}}{\tau^{i-1}_j\tau^i_{j-1}} \quad (i < j,\ i \le m). \tag{1.52}$$

Then, for any choice of row indices $i_1 < \dots < i_r$ and column indices $j_1 < \dots < j_r$, the minor determinant $\det H^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ is expressed as a sum

$$\det H^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum_{(\gamma_1,\dots,\gamma_r)}$$



(1.53)

of weights associated with $U = \left(u^i_j\right)_{i,j}$, over all r-tuples of nonintersecting paths $\gamma_k : (\min\{i_k, m\}, i_k) \to (1, j_k)$ from i_k along the lower rim to j_k at the top $(k = 1, \dots, r)$, in diagram (1.28).

Remark 1.8. Proposition 1.6 for $m = n$ can be reformulated as follows. Let us denote

$$\mathcal{B} = \{B = \left(b^i_j\right)^n_{i,j=1} \in GL_n(\mathbb{K}) \mid b^i_j = 0 \quad (i > j)\} \tag{1.54}$$

the group of all $n \times n$ invertible upper triangular matrices. For each $U = \left(u^i_j\right)^n_{i,j=1} \in \mathcal{B}$, we define $H = \left(h^i_j\right)^n_{i,j=1} \in \mathcal{B}$ by setting

$$h^i_j = \sum_{\gamma:(i,i)\to(1,j)} u_\gamma \quad (i \le j), \quad h^i_j = 0 \quad (i > j). \tag{1.55}$$

We now define two open subsets of $\mathcal{B}$ as follows:

$$\begin{aligned} \mathcal{B}_0 &= \{U = \left(u^i_j\right)^n_{i,j=1} \in \mathcal{B} \mid u^i_j \neq 0 \quad (i \le j)\}, \\ \mathcal{B}_\tau &= \{H = \left(h^i_j\right) \in \mathcal{B} \mid \tau^i_j(H) \neq 0 \quad (i \le j)\}. \end{aligned} \tag{1.56}$$

Then the correspondence $U \mapsto H$ induces the isomorphism of affine varieties $h : \mathcal{B}_0 \xrightarrow{\sim} \mathcal{B}_\tau$. The inverse mapping $H \mapsto U$ is given by

$$u^i_i = \frac{\tau^i_i}{\tau^{i-1}_i}, \qquad u^i_j = \frac{\tau^i_i\,\tau^{i-1}_{j-1}}{\tau^{i-1}_i\tau^i_{j-1}} \quad (i < j), \tag{1.57}$$

where $\tau^i_j = \tau^i_j(H)$ for $i \leq j$. Under this correspondence $U \leftrightarrow H$, for any choice of row indices $i_1 < \ldots < i_r$ and column indices $j_1 < \ldots < j_r$, the minor determinant $\det H^{i_1,\ldots,i_r}_{j_1,\ldots,j_r}$ of H is expressed as the sum of weights, associated with U, over all r-tuples of nonintersecting paths $\gamma_k : (i_k, i_k) \to (1, j_k)$ $(k = 1, \ldots, r)$.

1.3. Passage from tropical to combinatorial variables

We now assume that $\mathbb{K}$ is a field of characteristic 0. Consider the ring of polynomials $\mathbb{K}[x] = \mathbb{K}[\,x_i\ (i \in I)]$ in a set of variables $x = (x_i)_{i\in I}$. Denoting by

$$\text{(1.58)}\quad \mathbb{N}^{(I)} = \{\alpha = (\alpha_i)_{i\in I} \mid \alpha_i = 0 \ \text{ except for a finite number of } i\text{'s}\}$$

the set of multi-indices, we use the notation of multi-indices $x^\alpha = \prod_{i\in I} x_i^{\alpha_i}$ for the monomials in the x-variables. Note that any polynomial $a(x) \in \mathbb{K}[x]$ is expressed uniquely in the form

$$\text{(1.59)}\qquad a(x) = \sum_{\alpha\in A} a_\alpha\, x^\alpha \qquad (a_\alpha \in \mathbb{K}^*),$$

as a sum of monomials over a finite subset $A \subset \mathbb{N}^{(I)}$ of multi-indices, with nonzero coefficients. In this way, a polynomial $a(x)$ is identified with a pair (A, a) of a finite subset $A \subset \mathbb{N}^{(I)}$ and a mapping $a : A \to \mathbb{K}$. Note that $0 \in \mathbb{K}[x]$ and $c \in \mathbb{K}[x]$ $(c \in \mathbb{K}^*)$ correspond to (ϕ, ϕ) and $(\{0\}, c)$, respectively.

In the following, we fix a multiplicative subgroup $\mathbb{K}_{>0}$ of $\mathbb{K}^*$ such that $c, c' \in \mathbb{K}_{>0} \quad\Rightarrow\quad c + c' \in \mathbb{K}_{>0}$. We say that a nonzero rational function $f(x) \in \mathbb{K}(x) = \mathbb{K}(x_i\ (i \in I))$ in the x-variables is *subtraction free* (or *tropical*) with respect to the cone $\mathbb{K}_{>0}$ if it is expressed as a ratio

$$\text{(1.60)}\qquad f(x) = \frac{a(x)}{b(x)} \qquad (\,a(x), b(x) \in \mathbb{K}_{>0}[x]\,)$$

of two polynomials with coefficients in $\mathbb{K}_{>0}$. We denote by $\mathbb{K}(x)_{>0}$ the set of all subtraction-free rational functions with respect to $\mathbb{K}_{>0}$. It is clear that $\mathbb{K}(x)_{>0}$ forms again a multiplicative subgroup of $\mathbb{K}(x)^*$ closed under the addition. It is worthwhile to note that all the coefficients of a *polynomial* $f(x) \in \mathbb{K}(x)_{>0}$ may *not* necessarily belong to $\mathbb{K}_{>0}$: Observe the example

$$\text{(1.61)}\qquad f(x, y) = \frac{x^3 + y^3}{x + y} = x^2 - xy + y^2.$$

For a subtraction-free rational function $f = f(x) \in \mathbb{K}(x)_{>0}$ given, choose an expression as (1.60). Expressing $a(x)$ and $b(x)$ as

$$a(x) = \sum_{\alpha \in A} a_\alpha \, x^\alpha, \quad b(x) = \sum_{\beta \in B} b_\beta \, x^\beta \tag{1.62}$$

with coefficients in $\mathbb{K}_{>0}$, we define two piecewise linear functions $M(f)$ and $m(f)$ on $\mathbb{R}^I$ by

$$\begin{aligned} M(f) &= \max\{\langle \alpha, x\rangle \mid \alpha \in A\} - \max\{\langle \beta, x\rangle \mid \beta \in B\}, \\ m(f) &= \min\{\langle \alpha, x\rangle \mid \alpha \in A\} - \min\{\langle \beta, x\rangle \mid \beta \in B\}, \end{aligned} \tag{1.63}$$

where $\langle \alpha, x\rangle = \sum_{i\in I} \alpha_i x_i$. In this definition, we have identified $x = (x_i)_{i\in I}$ with the canonical coordinates of $\mathbb{R}^I$. It is easily shown that the definition of $M(f)$ and $m(f)$ does not depend on the choice of expression (1.60). Note also that $M(c) = m(c) = 0$ for any $c \in \mathbb{K}_{>0}$.

Proposition 1.9. (1) *For any subtraction-free rational functions $f, g \in \mathbb{K}(x)_{>0}$, one has*

$$\begin{gathered} M(fg) = M(f) + M(g), \quad M(\frac{f}{g}) = M(f) - M(g), \\ M(f+g) = \max\{M(f), M(g)\}, \end{gathered} \tag{1.64}$$

and

$$\begin{gathered} m(fg) = m(f) + m(g), \quad m(\frac{f}{g}) = m(f) - m(g), \\ m(f+g) = \min\{m(f), m(g)\}. \end{gathered} \tag{1.65}$$

(2) *Let $\iota : \mathbb{K}(x) \to \mathbb{K}(x)$ be the isomorphism defined by $\iota(x_i) = x_i^{-1}$ $(i \in I)$. Then one has*

$$M(f) = m(\iota(f)^{-1}), \quad m(f) = M(\iota(f)^{-1}) \tag{1.66}$$

for any $f \in \mathbb{K}(x)_{>0}$.

This proposition means that the correspondence $f \mapsto M(f)$ is nothing but the simple procedure of replacing the operations

$$a\, b \to a + b, \quad \frac{a}{b} \to a - b, \quad a + b \to \max\{a, b\}. \tag{1.67}$$

Similarly, the correspondence $f \mapsto m(f)$ is the procedure

$$a\, b \to a + b, \quad \frac{a}{b} \to a - b, \quad a + b \to \min\{a, b\}. \tag{1.68}$$

The second part of the proposition implies that one can interchange "max" and "min" freely with each other, by using the operation $f(x) \to \iota(f)^{-1} = f(x^{-1})^{-1}$.

Proposition 1.9 guarantees that these procedures can be applied consistently to arbitrary subtraction-free rational functions to obtain piecewise linear functions. This passage from the subtraction-free rational functions to piecewise linear functions, either by max or min, is called in several ways in the literature; it is called the *ultra-discretization* in the context of discrete integrable systems, and also the *tropicalization* in the context of totally positive matrices. In this paper, we will use the adjective "tropical" for objects and notions concerning subtraction-free rational functions, and "combinatorial" for those concerning piecewise linear functions. It should be noted that there is no *canonical* procedure in the opposite direction; when a combinatorial expression is given, it becomes an interesting problem in many occasions to find a *good* counterpart in the tropical setting.

The passage from the tropical side to the combinatorial side is *functorial* in the following sense. Consider two fields of rational functions $\mathbb{K}(x)$ in the variables $x = (x_i)_{i\in I}$ and $\mathbb{K}(y)$ in the variables $y = (y_j)_{j\in J}$. We say that an isomorphism φ form $\mathbb{K}(y)$ into $\mathbb{K}(x)$ is subtraction free if $\varphi(y_j) \in \mathbb{K}(x)_{>0}$ for all $j \in J$. The set of subtraction-free rational functions $f_j(x) = \varphi(y_j)$ $(j \in J)$ then defines a subtraction-free rational mapping

$$F : y_j = f_j(x) \quad (j \in J) \tag{1.69}$$

from the affine space $\mathbb{K}^I$ with coordinates $x = (x_i)_{i\in I}$ to $\mathbb{K}^J$ with coordinates $y = (y_j)_{j\in J}$. For such a rational mapping F given, we define two piecewise linear mappings $M(F), m(F) : \mathbb{R}^I \to \mathbb{R}^J$ by setting

$$M(F) : y_j = M(f_j(x)) \quad (j \in J), \qquad m(F) : y_j = m(f_j(x)) \quad (j \in J), \tag{1.70}$$

respectively. Then Proposition 1.9 implies

Proposition 1.10. *Consider the two subtraction-free rational mappings*

$$F : y_j = f_j(x) \quad (j \in J), \qquad G : z_k = g_k(y) \quad (k \in K). \tag{1.71}$$

Then the piecewise linear mappings corresponding to the composition $G \circ F$ are given by

$$M(G \circ F) = M(G) \circ M(F), \quad m(G \circ F) = m(G) \circ m(F). \tag{1.72}$$

§2. Tropical row insertion and tropical tableaux

In this section we introduce a tropical analogue of row insertion by clarifying the internal structure of bumping. Combining this with the matrix approach of Section 1, we give explicit tropical and combinatorial formulas for describing the tableau obtained from a word by row insertion, and for the Schützenberger involution on the set of column strict tableaux.

2.1. Row insertion

Taking the set of letters $\{1,\dots,n\}$, we consider a column strict tableau T of shape λ, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ is a partition with $l(\lambda) \le m$. For each $i = 1,\dots,m$, we define a weakly increasing word

$$w_i = i^{x^i_i}(i+1)^{x^i_{i+1}} \dots n^{x^i_n} \qquad (x^i_i + \cdots + x^i_n = \lambda_i) \tag{2.1}$$

by reading the i-th row of T from left to right, where x^i_j stands for the number of j's appearing in the i-th row of T for $i \le j$. For a weakly increasing word $v = 1^{a_1}2^{a_2}\dots n^{a_n}$ given, consider the tableau $T' = T \leftarrow v$ obtained by the row insertion of v into T; we denote by $w'_i = i^{y^i_i}(i+1)^{y^i_{i+1}} \dots n^{y^i_n}$ the weakly increasing word representing the i-th row of T' for $i = 1,\dots,m'$. Our question is : *How can one describe y^i_j explicitly in terms of x^i_j's and a_j's ?*

The bumping procedure $T \leftarrow v$ can be decomposed as follows.

$$T \overset{v}{\underset{\phi}{\longrightarrow}} T' \quad : \quad \begin{array}{ccc} & v = v_1 & \\ w_1 & \overset{}{\underset{\downarrow}{\longrightarrow}} & w'_1 \\ & v_2 & \\ w_2 & \underset{\downarrow}{\longrightarrow} & w'_2 \\ & v_3 & \\ & \vdots & \end{array} \tag{2.2}$$

Here $v_1 = v$, and for $i = 2, 3, \dots,$ $v_i = i^{a^i_i}(i+1)^{a^i_{i+1}} \dots n^{a^i_n}$ stands for the weakly increasing word consisting of the letters that have bumped out from w_{i-1} by the row insertion of v_{i-1}. In what follows, we use the diagram

(2.3)

$$w \overset{v}{\underset{v'}{\longrightarrow}} w' \qquad \left(\begin{array}{ll} w = 1^{x_1}2^{x_2}\dots n^{x_n}, & v = 1^{a_1}2^{a_2}\dots n^{a_n} \\ w' = 1^{y_1}2^{y_2}\dots n^{y_n}, & v' = 1^{b_1}2^{b_2}\dots n^{b_n} \end{array} \right)$$

consisting of four weakly increasing words w, v, w', v', to indicate a procedure of inserting a word v into w; $w' = w \leftarrow v$ denotes the resulting word, and v' is the word of the letters bumped out from w. (We always have $b_1 = 0$ in this setting.) Our question is thus reduced to the problem of describing y_j and b_j in terms of x_j and a_j in this diagram. We also use the diagram of row insertion for the corresponding vectors of integers:

$$\text{(2.4)} \qquad \boldsymbol{x} \xrightarrow[\boldsymbol{b}]{\boldsymbol{a}} \boldsymbol{y} \qquad \left(\begin{array}{ll} \boldsymbol{x} = (x_1, \dots, x_n), & \boldsymbol{a} = (a_1, \dots, a_n) \\ \boldsymbol{y} = (y_1, \dots, y_n), & \boldsymbol{b} = (b_1, \dots, b_n) \end{array} \right).$$

We now consider the procedure of row insertion as in (2.3). It is convenient to use the variables

$$\text{(2.5)} \qquad \xi_j = x_1 + x_2 + \cdots + x_j, \quad \eta_j = y_1 + y_2 + \cdots + y_j \quad (j = 1, \dots, n).$$

Assume first that $v = k^a$ $(k = 1, \dots, n)$; in this case, it is easy to see

$$\text{(2.6)} \qquad \begin{aligned} &\eta_j = \xi_j \quad (j < k), \qquad \eta_k = \xi_k + a, \\ &\eta_j = \max\{\xi_k + a, \xi_j\} = \max\{\eta_k, \xi_j\} \quad (j > k). \end{aligned}$$

Applying this result repeatedly for $k = 1, \dots, n$, we obtain following recurrence relations for the general case $v = 1^{a_1} 2^{a_2} \dots n^{a_n}$:

$$\text{(2.7)} \qquad \begin{aligned} &\eta_1 = \xi_1 + a_1, \quad \eta_2 = \max\{\eta_1, \xi_2\} + a_2, \\ &\eta_3 = \max\{\eta_1, \eta_2, \xi_3\} + a_3, \quad \dots . \end{aligned}$$

Since $\eta_1 \le \eta_2 \le \cdots \le \eta_n$, it is equivalent to

$$\text{(2.8)} \qquad \begin{aligned} \eta_1 &= \xi_1 + a_1, \\ \eta_j &= \max\{\eta_{j-1}, \xi_j\} + a_j \\ &= \max\{\eta_{j-1} + a_j, \xi_j + a_j\} \quad (j = 2, \dots, n). \end{aligned}$$

Hence we have

$$\text{(2.9)} \qquad \begin{aligned} \eta_j &= \max\{\xi_1 + a_1 + \cdots + a_j, \xi_2 + a_2 + \cdots + a_j, \dots, \xi_j + a_j\} \\ &= \max_{1 \le k \le j} \{x_1 + \cdots + x_k + a_k + \cdots + a_j\} \end{aligned}$$

for $j = 1, \dots, n$. Note that

$$\text{(2.10)} \qquad y_1 = \eta_1, \quad y_j = \eta_j - \eta_{j-1} \quad (j = 2, \dots, n),$$

and that b_j are determined as $b_1 = 0$ and

(2.11)
$$b_j = a_j + x_j - y_j = a_j + \xi_j - \xi_{j-1} - \eta_j + \eta_{j-1} \quad (j = 2, \ldots, n),$$

since the number of j's is conserved during the process.

Example 2.1. Let us consider an example of row insertion

$$w = \underline{2}2\underline{3}4\underline{5}\_ \quad \overset{v\,=\,1245}{\underset{v'\,=\,235}{\longrightarrow}} \quad w' = 122445 \ . \tag{2.12}$$

In terms of the vectors of integers, this procedure is expressed as

$$\boldsymbol{x} = (0,2,1,1,1) \quad \overset{\boldsymbol{a}\,=\,(1,1,0,1,1)}{\underset{\boldsymbol{b}\,=\,(0,1,1,0,1)}{\longrightarrow}} \quad \boldsymbol{y} = (1,2,0,2,1) \ . \tag{2.13}$$

The numbers

$$\eta_j = \max_{1 \le k \le j} (x_1 + \cdots + x_k + a_k + \cdots + a_j) \tag{2.14}$$

can be read off from the table

(2.15)

	1	2	3	4	5
$\boldsymbol{x}$:	0	2	1	1	1
$\boldsymbol{a}$:	1	1	0	1	1

as

$$\eta_1 = 1, \quad \eta_2 = 3, \quad \eta_3 = 3, \quad \eta_4 = 5, \quad \eta_5 = 6. \tag{2.16}$$

By taking the first difference of this sequence, we have $\boldsymbol{y} = (1,2,0,2,1)$.

The general procedure $T' = T \leftarrow v$ of inserting a weakly increasing word v into a column strict tableau T should be described as a superposition of row insertions of type (2.3). We will make use of the tropical analogue of combinatorial (piecewise linear) formulas above in order to systematize the superposition of row insertions.

2.2. Tropical row insertion

We introduce a tropical analogue of combinatorial formulas for the row insertion (2.3). We use the same symbols x_j, a_j, y_j, b_j as in (2.4) for the *tropical variables* (indeterminates). Introducing the auxiliary variables

$$\xi_j = x_1 \cdots x_j, \quad \eta_j = y_1 \cdots y_j \quad (j = 1, \ldots, n), \tag{2.17}$$

we define the transformation $(\boldsymbol{x}, \boldsymbol{a}) \mapsto (\boldsymbol{y}, \boldsymbol{b})$ by

$$\begin{aligned} &\eta_1 = \xi_1 a_1, && \eta_j = (\eta_{j-1} + \xi_j) a_j && (j = 2, \dots, n), \\ &y_1 = \eta_1, && y_j = \frac{\eta_j}{\eta_{j-1}} && (j = 2, \dots, n), \\ &b_1 = 1, && b_j = a_j \frac{x_j}{y_j} = a_j \frac{\xi_j \eta_{j-1}}{\xi_{j-1}\eta_j} && (j = 2, \dots, n). \end{aligned} \tag{2.18}$$

We have made these formulas from the combinatorial formulas (2.8), (2.10), (2.11) by the simple rule of replacement

$$\max\{a, b\} \to a + b, \quad a + b \to a\, b \quad a - b \to \frac{a}{b}, \tag{2.19}$$

which is sometimes called the *tropical variable change*. From the recurrence relations for η_j above, we easily obtain

$$\begin{aligned} \eta_j &= \xi_1 a_1 \cdots a_j + \xi_2 a_2 \cdots a_j + \cdots + \xi_j a_j \\ &= x_1 a_1 a_2 \cdots a_j + x_1 x_2 a_2 \cdots a_j + \cdots + x_1 \cdots x_j a_j. \end{aligned} \tag{2.20}$$

From this formula, we can recover the combinatorial formula (2.9) by the standard procedure as we discussed in Section 1.3.

The tropical transformation $(\boldsymbol{x}, \boldsymbol{a}) \mapsto (\boldsymbol{y}, \boldsymbol{b})$ we have discussed above arises also from the system of algebraic equations *of discrete Toda type*

$$\begin{aligned} &a_1 x_1 = y_1, && a_j x_j = y_j b_j && (j = 2, \dots, n), \\ &\frac{1}{a_1} + \frac{1}{x_2} = \frac{1}{b_2}, && \frac{1}{a_j} + \frac{1}{x_{j+1}} = \frac{1}{y_j} + \frac{1}{b_{j+1}} && (j = 2, \dots, n), \end{aligned} \tag{2.21}$$

for $(y_1, \dots, y_n)$ and $(b_2, \dots, b_n)$, where we regard x_j, a_j as given variables, and y_j, b_j as unknown functions. (For the relationship between (2.21) and the discrete Toda equation, see Remark 2.3 below.)

Lemma 2.2. *The system of algebraic equations* (2.21) *is equivalent to the recurrence formulas* (2.18) *together with* (2.17).

Proof. In fact, by eliminating b_j $(j = 2, \dots, n)$ from (2.21), and by rewriting the equations in terms of ξ_j and η_j, we obtain

(2.22)
$$\frac{\eta_n - \eta_{n-1} a_n}{\xi_n a_n} = \frac{\eta_{n-1} - \eta_{n-2} a_{n-1}}{\xi_{n-1} a_{n-1}} = \cdots = \frac{\eta_2 - \eta_1 a_2}{\xi_2 a_2} = \frac{\eta_1}{\xi_1 a_1} = 1,$$

which is equivalent to (2.18). □

This fact is a key to our matrix approach to tropical combinatorics. Note that (2.21) is written as a matrix equation:

$$\begin{bmatrix} \overline{a}_1 & 1 & & & \\ & \overline{a}_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \overline{a}_{n-1} & 1 \\ & & & & \overline{a}_n \end{bmatrix} \begin{bmatrix} \overline{x}_1 & 1 & & & \\ & \overline{x}_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \overline{x}_{n-1} & 1 \\ & & & & \overline{x}_n \end{bmatrix}$$

$$(2.23) \qquad = \begin{bmatrix} \overline{y}_1 & 1 & & & \\ & \overline{y}_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \overline{y}_{n-1} & 1 \\ & & & & \overline{y}_n \end{bmatrix} \begin{bmatrix} 1 & 0 & & & \\ & \overline{b}_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \overline{b}_{n-1} & 1 \\ & & & & \overline{b}_n \end{bmatrix},$$

where we have used the notation $\overline{x} = \frac{1}{x}$. By using the notation of Section 1, this equation can be expressed as

$$(2.24) \qquad E(\overline{\boldsymbol{a}})\, E(\overline{\boldsymbol{x}}) = E_1(\overline{\boldsymbol{y}})\, E_2(\overline{\boldsymbol{b}}).$$

It is also equivalent to

$$(2.25) \qquad H(\boldsymbol{x})\, H(\boldsymbol{a}) = H_2(\boldsymbol{b}) H_1(\boldsymbol{y}).$$

Each of these two equations (2.24) and (2.25) can be thought of as a tropical expression of the row insertion

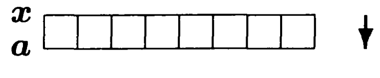
(2.26)

$$\boldsymbol{x} \xrightarrow[\boldsymbol{b}]{\boldsymbol{a}} \boldsymbol{y} \qquad \left(\begin{array}{ll} \boldsymbol{x} = (x_1, \dots, x_n), & \boldsymbol{a} = (a_1, \dots, a_n) \\ \boldsymbol{y} = (y_1, \dots, y_n), & \boldsymbol{b} = (b_1, \dots, b_n) \end{array} \right),$$

where $b_1 = 1$.

We show how the matrix equation (2.24) is solved by using the path representations of Section 1. Setting $H = H(\boldsymbol{x})H(\boldsymbol{a})$, we look at the minor determinants $\tau^i_j(H)$ $(i \leq j)$. By the argument of Section 1, the minor determinants of H are read off from the following diagram.

(2.27)



The result for $\tau^i_j = \tau^i_j(H)$ is:

$$
\begin{aligned}
\tau^1_j &= \sum_{k=1}^{j} x_1 \cdots x_k \, a_k \cdots a_j && (j \geq 1), \\
\tau^2_j &= \tau^j_j = x_1 \cdots x_j a_1 \cdots a_j && (j \geq 2), \\
\tau^i_j &= 0 && (3 \leq i < j \leq n).
\end{aligned} \tag{2.28}
$$

By Proposition 1.6, we already know that equation (2.25) has a unique solution such that

(2.29)
$$\tau^1_j = y_1 \dots y_j \quad (j = 1, \dots, n), \quad \tau^2_j = y_1 \dots y_j b_1 \dots b_j \quad (j = 2, \dots, n).$$

Namely, y_j and b_j are determined as

$$y_j = \frac{\tau^1_j}{\tau^1_{j-1}} \quad (j = 1, \dots, n), \quad b_j = \frac{x_j a_j}{y_j} \quad (j = 2, \dots, n), \tag{2.30}$$

consistently with what we have seen before.

Remark 2.3. The system of algebraic equations (2.21) is closely related to the *discrete Toda equation* ([9], [26]):

$$I^{t+1}_i V^{t+1}_i = I^t_{i+1} V^t_i, \quad I^{t+1}_i + V^{t+1}_{i-1} = I^t_i + V^t_i \tag{2.31}$$

where $i \in \mathbb{Z}$ and $t \in \mathbb{Z}$ stand for the discrete coordinates of space and time, respectively, and I^t_i, V^t_i are the dependent variables. If we set

(2.32)
$$a_i = (I^t_{i+1})^{-1}, \quad x_i = (V^t_i)^{-1}, \quad y_i = (V^{t+1}_i)^{-1}, \quad b_i = (I^{t+1}_i)^{-1},$$

we have

$$a_i x_i = y_i b_i, \quad \frac{1}{a_i} + \frac{1}{x_{i+1}} = \frac{1}{y_i} + \frac{1}{b_{i+1}} \quad (i \in \mathbb{Z}). \tag{2.33}$$

Note also that the discrete Toda equation can be expressed as the matrix equation

$$L(t+1)R(t+1) = R(t)L(t) \tag{2.34}$$

for the $\mathbb{Z} \times \mathbb{Z}$ matrices

$$L(t) = \sum_{i \in \mathbb{Z}} E_{ii} + \sum_{j \in \mathbb{Z}} V^t_i E_{i+1,i}, \quad R(t) = \sum_{i \in \mathbb{Z}} I^t_i E_{ii} + \sum_{i \in \mathbb{Z}} E_{i,i+1}. \tag{2.35}$$

2.3. Tropical tableaux

In the following we discuss the following question both in the tropical and the combinatorial setting.

Question: For a sequence of weakly increasing words $w_1, \dots, w_m$ given, find an explicit formula for the column strict tableau

$$P = P(w) = (\cdots(w_1 \leftarrow w_2)\cdots \leftarrow w_m) \tag{2.36}$$

obtained from the word $w = w_1 \dots w_m$ by the row insertion.

We can employ (2.25) as building blocks for the tropical analogue of various combinatorial algorithms. Let us consider the procedure of successive row insertion

$$P = (\cdots(w_1 \leftarrow w_2) \leftarrow \cdots \leftarrow w_m) \tag{2.37}$$

of weakly increasing word $w_i = 1^{x_1^i} \cdots n^{x_n^i}$ $(i = 1, \dots, n)$ to obtain a column strict tableau P. This procedure can be described by the following diagram.

$$\begin{array}{ccccccc} & \boldsymbol{x}^1 = \boldsymbol{x}^{1,1} & & \boldsymbol{x}^2 = \boldsymbol{x}^{2,1} & & \boldsymbol{x}^3 = \boldsymbol{x}^{3,1} & \\ \phi & \longrightarrow\!\!\!\!\!\!\!\downarrow\;\; & \boldsymbol{y}^{1,1} & \longrightarrow\!\!\!\!\!\!\!\downarrow\;\; & \boldsymbol{y}^{2,1} & \longrightarrow\!\!\!\!\!\!\!\downarrow\;\; & \cdots \\ & \phi & & \boldsymbol{x}^{2,2} & & \boldsymbol{x}^{3,2} & \\ & & \phi & \longrightarrow\!\!\!\!\!\!\!\downarrow\;\; & \boldsymbol{y}^{2,2} & \longrightarrow\!\!\!\!\!\!\!\downarrow\;\; & \cdots \\ & & & \phi & & \boldsymbol{x}^{3,3} & \\ & & & & \phi & \longrightarrow\!\!\!\!\!\!\!\downarrow\;\; & \cdots \end{array} \tag{2.38}$$

Passing the tropical variables $\boldsymbol{x}^i = (x_1^i, \dots, x_n^i)$ $(i = 1, \dots, m)$, we can compute the row insertion above as

$$\begin{aligned} H(\boldsymbol{x}^1) &= H_1(\boldsymbol{y}^{1,1}) \\ H(\boldsymbol{x}^1)H(\boldsymbol{x}^2) &= H_1(\boldsymbol{y}^{1,1})H(\boldsymbol{x}^{2,1}) \\ &= H_2(\boldsymbol{y}^{2,2})H_1(\boldsymbol{y}^{2,1}) \\ H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)H(\boldsymbol{x}^3) &= H_2(\boldsymbol{y}^{2,2})H_1(\boldsymbol{y}^{2,1})H(\boldsymbol{x}^3) \\ &= H_2(\boldsymbol{y}^{2,2})H_2(\boldsymbol{x}^{3,2})H_1(\boldsymbol{y}^{3,1}) \\ &= H_3(\boldsymbol{y}^{3,3})H_2(\boldsymbol{y}^{3,2})H_1(\boldsymbol{y}^{3,1}) \\ &\cdots, \end{aligned} \tag{2.39}$$

where $\boldsymbol{y}^{k,i} = (1, \dots, 1, y_i^{k,i}, \dots, y_n^{k,i})$. When $m \le n$, by setting $\boldsymbol{y}^{m,i} = \boldsymbol{p}^i$, $\boldsymbol{p}^i = (1, \dots, 1, p_i^i, \dots, p_n^i)$ $(i = 1, \dots, m)$, we finally obtain

$$H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m) = H_m(\boldsymbol{p}^m)\cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1), \tag{2.40}$$

In this formula, each p^i_j $(i \le j)$ denotes the tropical variable corresponding to the number of j's in the i-th row of the tableau P. Namely, we can regard the expression

$$H_P = H_m(\boldsymbol{p}^m) \cdots H_2(\boldsymbol{p}^2) H_1(\boldsymbol{p}^1) \tag{2.41}$$

as representing the *tropical tableau* $P = \left(p^i_j\right)_{i \le j}$; it provides the tropical analogue of a general column strict tableau whose shape is a partition of length m.

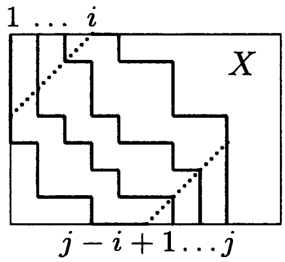
The argument above shows that our question can be answered by considering the matrix equation

$$\begin{aligned} H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m) &= H_m(\boldsymbol{p}^m)\cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1) \quad (m \le n), \\ H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m) &= H_n(\boldsymbol{p}^n)\cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1) \quad (m \ge n) \end{aligned} \tag{2.42}$$

for the unknowns $\boldsymbol{p}^i = (1, \dots, 1, p^i_i, \dots, p^i_n)$ $(i = 1, \dots, \min\{m, n\})$. In the following we regard x^i_j $(i = 1, \dots,\ j = 1, \dots, n)$ as indeterminates, and look for solutions p^i_j of (2.42) in the filed of rational functions in the x-variables.

Denoting the left-hand side by H, consider the minor determinants $\tau^i_j(H) = \det H^{1,\dots,i}_{j-i+1,\dots,j}$ for $1 \le i \le j \le n$. By Proposition 1.3, $\tau^i_j(H)$ is expressed in terms of the nonintersecting paths in the $m \times n$ rectangle associated with the matrix $X = \left(x^i_j\right)_{i,j}$:

(2.43) $\qquad \tau^i_j(H) = \sum$



When $m < i \le j \le n$, we have $\tau^i_j(H) = \delta_{i,j}\tau^m_j(H)$, $\tau^m_j(H) = \prod_{(a,b);\, b \le j} x^a_b$. Hence, by Theorem 1.7, we see that the matrix equation (2.42) has a unique rational solution in the x-variables; the solution is expressed by $\tau^i_j(H)$ above. To summarize, we have

Theorem 2.4. *For $m \times n$ indeterminates x^i_j $(1 \le i \le m,\ 1 \le j \le n)$ given, consider the following matrix equation for unknown variables p^i_j $(1 \le i \le l, 1 \le j \le n, i \le j)$, $l = \min\{m, n\}$:*

$$H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m) = H_l(\boldsymbol{p}^l)\cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1), \tag{2.44}$$

where $\boldsymbol{x}^i = (x_1^i, \dots, x_n^i)$ *and* $\boldsymbol{p}^i = (1, \dots, 1, p_i^i, \dots, p_n^i)$. *This equation has a unique rational solution in the* x*-variables ; it is given explicitly as*

$$p_i^i = \frac{\tau_i^i}{\tau_i^{i-1}}, \qquad p_j^i = \frac{\tau_j^i\,\tau_{j-1}^{i-1}}{\tau_j^{i-1}\tau_{j-1}^i} \quad (i < j) \tag{2.45}$$

for $1 \le i \le l$, $1 \le j \le n$, *with* τ_j^i $(i \le j)$ *defined as the sum*

$$\tau_j^i = \sum_{(\gamma_1, \dots, \gamma_i)} x_{\gamma_1} \cdots x_{\gamma_i} \tag{2.46}$$

of monomials over all i*-tuples of nonintersecting paths* $\gamma_k : (1, k) \to (m, j - i + k)$ $(k = 1, \dots, i)$ *in the* $m \times n$ *rectangle. Here, the weight* x_γ *of a path* γ *is the product*

$$x_\gamma = \prod_{(a,b)\in\gamma} x_b^a \tag{2.47}$$

of all x_b^a*'s corresponding to the vertices on* γ.

The explicit formula for p_j^i above is formulated by A.N. Kirillov [14], Theorem 4.23.

By the standard passage from subtraction-free rational functions to piecewise-linear functions, we obtain the following combinatorial formula ([14], Theorem 3.5) for the column strict tableaux P obtained from a word $w = w_1 w_2 \dots w_m$ by the row insertion.

Theorem 2.5. *Taking the set of letters* $\{1, \dots, n\}$, *let* $w_1, \dots, w_m$ *be a sequence of weakly increasing words* $w^i = 1^{x_1^i} \cdots n^{x_n^i}$ $(i = 1, \dots, m)$. *Consider the column strict tableau*

$$P = P(w) = (\cdots (w_1 \leftarrow w_2) \leftarrow \cdots \leftarrow w_m) \tag{2.48}$$

obtained from the word $w = w_1 w_2 \cdots w_m$ *by row insertion, and denote by* $i^{p_i^i} \cdots n^{p_n^i}$ *the weakly increasing word representing the* i*-th row of* P, *for* $i = 1, \dots, l$, $l = \min\{m, n\}$. *Then, for each* (i, j), *the number* p_j^i *of the letter* j *in the* i*-th row of* P *is determined explicitly as*

$$p_i^i = \tau_i^i - \tau_i^{i-1}, \quad p_j^i = \tau_j^i - \tau_j^{i-1} - \tau_{j-1}^i + \tau_{j-1}^{i-1} \quad (i < j) \tag{2.49}$$

with τ_j^i $(i \le j)$ *defined as the maximum*

$$\tau_j^i = \max_{(\gamma_1, \dots, \gamma_i)} (x_{\gamma_1} + \cdots + x_{\gamma_i}) \tag{2.50}$$

of weights over all i-tuples of nonintersecting paths $\gamma_k : (1,k) \to (m, j-i+k)$ $(k = 1, \dots, i)$ in the $m \times n$ rectangle; we set $\tau_j^0 = 0$ for all j. Here, the weight x_γ of a path γ is the sum

$$x_\gamma = \sum_{(a,b)\in\gamma} x_b^a \tag{2.51}$$

of all x_b^a's corresponding to the vertices on γ.

Note also that each τ_j^i represents the sum of p_b^a's in the region $a \le i$, $b \le j$:

$$\tau_j^i = \sum_{(a,b):\, a\le i,\, b\le j} p_b^a \qquad (i \le j). \tag{2.52}$$

Example 2.6. We give an example with $m = 3$, $n = 4$ by taking the word

$$w = 2234134411224 = 2234|1344|11224, \tag{2.53}$$

which corresponds to the column strict tableau

$$P = P(w) = \begin{array}{l} 1\,1\,1\,2\,2\,4\,4 \\ 2\,2\,3\,3\,4 \\ 4 \end{array}. \tag{2.54}$$

From w, we first construct the 3×4 matrix $X = \left(x_j^i\right)_{i,j}$ by counting the number of j's in the i-th block of w:

$$X = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{bmatrix}. \tag{2.55}$$

The numbers τ_j^i of (2.50) are determined from X by statistics of nonintersecting paths:

$$\boldsymbol{\tau} = \left(\tau_j^i\right)_{i,j} = \begin{bmatrix} 3 & 5 & 5 & 7 \\ & 7 & 9 & 12 \\ & & 9 & 13 \end{bmatrix}. \tag{2.56}$$

For instance, τ_3^2 is computed as the maximum of weights over three pairs of nonintersecting paths (γ_1, γ_2) such that $\gamma_1 : (1,1) \to (3,2)$ and $\gamma_2 : (1,2) \to (3,3)$:

$$X = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{bmatrix}. \tag{2.57}$$

From

$$(0+1+2+2)+(2+0+1+0)=8$$
$$(0+1+2+2)+(2+1+1+0)=9 \tag{2.58}$$
$$(0+1+0+2)+(2+1+1+0)=7$$

we get

$$\tau_3^2 = \max\{8,9,7\} = 9. \tag{2.59}$$

According to (2.49), we compute p_j^i by taking the *discrete Laplacian* of τ_j^i:

$$\boldsymbol{p} = \left(p_j^i\right)_{i,j} = \begin{bmatrix} 3 & 2 & 0 & 2 \\ & 2 & 2 & 1 \\ & & 0 & 1 \end{bmatrix}. \tag{2.60}$$

Then p_j^i gives the number of j's in the i-th row of the tableau P above.

2.4. Tropical Schützenberger involution

We now recall the Schützenberger involution on the set of column strict tableaux. Taking the set $\{1,2\dots,n\}$ of letters as before, we define an involution $k \mapsto k^*$ on $\{1,2\dots,n\}$ by $k^* = n-k+1$ for $k=1,\dots,n$. For a word $w = k_1\, k_2 \dots k_l$ consisting of letters in $\{1,\dots,n\}$ given, we define the word w^* by

$$w^* = k_l^* \dots k_2^*\, k_1^*, \tag{2.61}$$

by applying $k \mapsto k^*$ to each letter, and then by reversing the order. Let us denote by $P = P(w)$ the column strict tableau obtained from a word w. Since the involution $w \mapsto w^*$ on the set of words preserve the Knuth equivalence, it induces an involution $P \mapsto P^{\mathrm{s}}$ on the set of column strict tableaux such that

$$P(w^*) = P(w)^{\mathrm{s}} \tag{2.62}$$

for any word w; we call this involution $P \mapsto P^{\mathrm{s}}$ the *Schützenberger involution.* It is well known that P and P^{s} has the same shape, and that the column strict tableau P^{s} is obtained from P as the evacuation tableau by a successive application of *jeu de taquin.* We remark that, when the word $w = k_1 \dots k_n$ represents a permutation in $\boldsymbol{S}_n$, w^* is the conjugation of w by the longest element of $\boldsymbol{S}_n$. (Schützenberger's algorithm for column strict tableaux can be obtained essentially from [25], Theorem 3.9.4, for instance. In [25], Schützenberger's algorithm is formulated for permutations, but it is not difficult to extend it to that for words.)

Example 2.7. Consider the word $w = 42213132$ with $n = 4$. In this case, we have $w^* = 32424331$.

$$P = P(w) = \begin{matrix} 1\,1\,2\,3 \\ 2\,2\,3 \\ 4 \end{matrix}\ , \quad P^{\mathrm{s}} = P(w^*) = \begin{matrix} 1\,2\,3\,3 \\ 2\,4\,4 \\ 3 \end{matrix}\ . \tag{2.63}$$

Let us decompose a given word w into a chain of weakly increasing words:

$$w = w_1 w_2 \dots w_m, \qquad w_i = 1^{x_1^i} 2^{x_2^i} \dots n^{x_n^i} \quad (i = 1, \dots, m). \tag{2.64}$$

Then we have

$$w^* = w_n^* \dots w_2^* w_1^*, \qquad w_i^* = 1^{x_n^i} 2^{x_{n-1}^i} \dots n^{x_1^i} \quad (i = 1, \dots, m). \tag{2.65}$$

Setting $P = P(w)$, $P^{\mathrm{s}} = P(w^*)$, we denote by p_j^i and by $\widetilde{p}_j^i$ the number of j's in the i-th row of P and P^{s}, respectively. Passing to the tropical variables, we have the matrix equation

$$\begin{aligned} &H(\boldsymbol{x}^1)H(\boldsymbol{x}^2)\cdots H(\boldsymbol{x}^m) = H_P = H_l(\boldsymbol{p}^n)\cdots H_2(\boldsymbol{p}^2)H_1(\boldsymbol{p}^1), \\ &H(\boldsymbol{x}_*^m)\cdots H(\boldsymbol{x}_*^2)H(\boldsymbol{x}_*^1) = H_{P^{\mathrm{s}}} = H_l(\widetilde{\boldsymbol{p}}^n)\cdots H_2(\widetilde{\boldsymbol{p}}^2)H_1(\widetilde{\boldsymbol{p}}^1), \end{aligned} \tag{2.66}$$

where $l = \min\{m, n\}$ and, for a vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ given, $\boldsymbol{x}_* = (x_n, \dots, x_2, x_1)$ denotes the vector obtained by reversing the order. In the following, we denote by

$$J_n = \left(\delta_{i+j,n+1}\right)_{i,j=1}^n \tag{2.67}$$

the permutation matrix representing the longest element of $\boldsymbol{S}_n$. Since $J_n \Lambda J_n = \Lambda^{\mathrm{t}}$, from

$$H(\boldsymbol{x}) = (\mathrm{diag}(\overline{\boldsymbol{x}}) - \Lambda)^{-1}, \quad \boldsymbol{x} = (x_1, x_2, \dots, x_n) \tag{2.68}$$

we have

$$J_n H(\boldsymbol{x})^{\mathrm{t}} J_n = (\mathrm{diag}(\overline{\boldsymbol{x}}_*) - \Lambda)^{-1}, \quad \boldsymbol{x}_* = (x_n, \dots, x_2, x_1), \tag{2.69}$$

namely, $J_n H(\boldsymbol{x})^{\mathrm{t}} J_n = H(\boldsymbol{x}_*)$. Hence, (2.66) implies $J_n H_P^{\mathrm{t}} J_n = H_{P^{\mathrm{s}}}$, namely,

$$J_n H_1(\boldsymbol{p}^1)^{\mathrm{t}} H_2(\boldsymbol{p}^2)^{\mathrm{t}} \cdots H_n(\boldsymbol{p}^n)^{\mathrm{t}} J_n = H_n(\widetilde{\boldsymbol{p}}^n)\cdots H_2(\widetilde{\boldsymbol{p}}^2)H_1(\widetilde{\boldsymbol{p}}^1). \tag{2.70}$$

Supposing that $m \leq n$, we take two general tropical tableaux

$$H_U = H_m(\boldsymbol{u}^m)\cdots H_1(\boldsymbol{u}^1), \quad H_V = H_m(\boldsymbol{v}^m)\cdots H_1(\boldsymbol{v}^1), \tag{2.71}$$

where $\boldsymbol{u}^i = (u^i_i, \dots, u^i_n)$ and $\boldsymbol{v}^i = (v^i_i, \dots, v^i_n)$ for $i = 1, \dots, m$. In view of (2.70), we define the *tropical Schützenberger involution* to be the birational transformation $U \mapsto V$ defined through the matrix equation $J_n H_U^{\mathrm{t}} J_n = H_V$:

$$J_n H_1(\boldsymbol{u}^1)^{\mathrm{t}} \cdots H_m(\boldsymbol{u}^m)^{\mathrm{t}} J_n = H_m(\boldsymbol{v}^m) \cdots H_1(\boldsymbol{v}^1). \tag{2.72}$$

We now propose to solve this matrix equation for v^i_j $(i \le j)$, regarding u^i_j $(i \le j)$ as indeterminates. Note that

$$J_n \Lambda^{\mathrm{t}}_{\ge k} J_n = \Lambda_{\le n-k+1}, \quad \Lambda_{\le k} = \sum_{i=1}^{k-1} E_{i,i+1}, \tag{2.73}$$

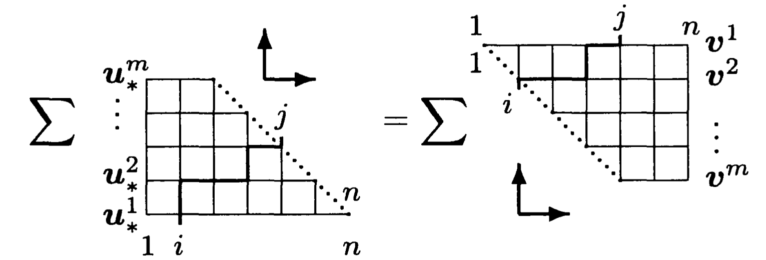
for $k = 1, \dots, n$. Hence, (2.72) can be written as

(2.74)

$$(\mathrm{diag}(\overline{\boldsymbol{u}}^1_*) - \Lambda_{\le n})^{-1} (\mathrm{diag}(\overline{\boldsymbol{u}}^2_*) - \Lambda_{\le n-1})^{-1} \cdots (\mathrm{diag}(\overline{\boldsymbol{u}}^m_*) - \Lambda_{\le n-m+1})^{-1}$$
$$= (\mathrm{diag}(\overline{\boldsymbol{v}}^m) - \Lambda_{\ge m})^{-1} \cdots (\mathrm{diag}(\overline{\boldsymbol{v}}^2) - \Lambda_{\ge 2})^{-1} (\mathrm{diag}(\overline{\boldsymbol{v}}^1) - \Lambda_{\ge 1})^{-1}.$$

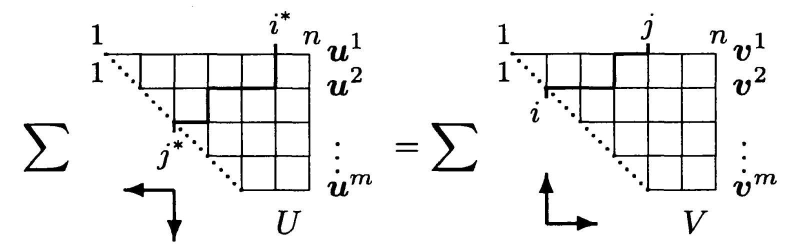
This equality is expressed graphically as follows. For each $1 \le i \le j \le n$,

(2.75)



or equivalently,

(2.76)



where $i^* = n - i + 1$ and $j^* = n - j + 1$. Hence we have

Theorem 2.8. *For the indeterminates u^i_j $(1 \le i \le j \le n)$ given, consider the matrix equation*

(2.77)

$$J_n H_1(\boldsymbol{u}^1)^{\mathrm{t}}\, H_2(\boldsymbol{u}^2)^{\mathrm{t}} \cdots H_m(\boldsymbol{u}^m)^{\mathrm{t}} J_n = H_m(\boldsymbol{v}^m) \cdots H_2(\boldsymbol{v}^2) H_1(\boldsymbol{v}^1)$$

for v^i_j $(1 \le i \le j \le n)$, *where* $m \le n$, *and* $\boldsymbol{u}^i = (u^i_i, \dots, u^i_n)$, $\boldsymbol{v}^i = (v^i_i, \dots, v^i_n)$ *for* $i = 1, \dots, m$. *This equation has a unique rational solution; it is given by*

$$v^i_i = \frac{\sigma^i_i}{\sigma^{i-1}_i}, \quad v^i_j = \frac{\sigma^i_j\, \sigma^{i-1}_{j-1}}{\sigma^{i-1}_j \sigma^i_{j-1}} \quad (i < j), \tag{2.78}$$

with σ^i_j $(i \le j)$ *defined as the sum*

$$\sigma^i_j = \sum_{(\gamma_1, \dots, \gamma_i)} u_{\gamma_1} \cdots u_{\gamma_i} \tag{2.79}$$

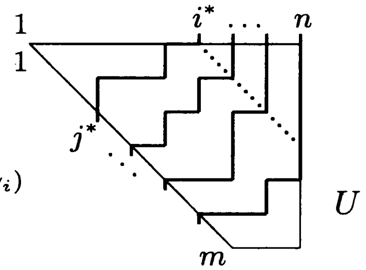
of weights over all i*-tuples of nonintersecting paths*

$$\gamma_k : (1, n-i+k) \to (\min\{m, n-j+k\}, n-j+k) \tag{2.80}$$

$(k = 1, \dots, i)$, *where the weight* u_γ *of a path* γ *is the product of all* u^a_b*'s corresponding to the vertices of* γ.

Graphically, σ^i_j $(i \le j)$ in the explicit formula above is expressed as follows.

(2.81) $$\sigma^i_j = \sum_{(\gamma_1, \dots, \gamma_i)}$$



The explicit formula above for the tropical Schützenberger involution is proposed by A.N. Kirillov [14], Theorem 4.18.

By returning to the combinatorial variables, we obtain the following explicit formula for the Schützenberger involution (with $m = n$), due to H. Knight and A. Zelevinsky [16] (see also [1], [30]).

Theorem 2.9. *Let* P *be a column strict tableau and denote by* p^i_j *the number of* j*'s in the* i*-th row of* P *for* $1 \le i \le j \le n$. *Let* P^{s} *be the column strict tableau obtained from* P *by applying the Schützenberger involution, and denote by* $\widetilde{p}^i_j$ *the number of* j*'s in the* i*-th row of* P^{s} *for* $1 \le i \le j \le n$. *Then* $\widetilde{p}^i_j$ *are determined from* p^i_j *by the following explicit formula:*

$$\widetilde{p}^i_i = \sigma^i_i - \sigma^{i-1}_i, \quad \widetilde{p}^i_j = \sigma^i_j - \sigma^{i-1}_j - \sigma^i_{j-1} + \sigma^{i-1}_{j-1} \quad (i < j), \tag{2.82}$$

with σ^i_j $(1 \le i \le j \le n)$ defined as the maximum

$$\sigma^i_j = \max_{(\gamma_1, \dots, \gamma_i)} (p_{\gamma_1} + \dots + p_{\gamma_i}) \tag{2.83}$$

of weights over all i-tuples of nonintersecting paths $\gamma_k : (1, n-i+k) \to (n-j+k, n-j+k)$ $(k = 1, \dots, i)$, where the weight p_γ of a path γ is the sum of all p^a_b's corresponding to the vertices of γ.

Example 2.10. In the case of P and P^{s} of Example 2.7, $\boldsymbol{p} = \left(p^i_j\right)_{i \le j}$ and $\widetilde{\boldsymbol{p}} = \left(\widetilde{p}^i_j\right)_{i \le j}$ are given by

$$\boldsymbol{p} = \begin{bmatrix} 2 & 1 & 1 & 0 \\ & 2 & 1 & 0 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}, \quad \widetilde{\boldsymbol{p}} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ & 1 & 0 & 2 \\ & & 1 & 0 \\ & & & 0 \end{bmatrix}. \tag{2.84}$$

The table $\widetilde{\boldsymbol{p}}$ can be determined through $\boldsymbol{\sigma} = \left(\sigma^i_j\right)_{i \le j}$:

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 2 & 4 & 4 \\ & 3 & 5 & 7 \\ & & 6 & 8 \\ & & & 8 \end{bmatrix}. \tag{2.85}$$

Remark 2.11. With the notation of Remark 1.8, the tropical Schützenberger involution can be formulated as follows. We define the involution $\theta : \mathcal{B} \to \mathcal{B}$ by setting

$$\theta(H) = J_n H^{\mathrm{t}} J_n \qquad (H \in \mathcal{B}). \tag{2.86}$$

Then the isomorphism $h : \mathcal{B}_0 \xrightarrow{\sim} \mathcal{B}_\tau$ induces a birational involution $U \mapsto U^{\mathrm{s}}$ on $\mathcal{B}_0$ such that

$$h(U^{\mathrm{s}}) = \theta(h(U)) = J_n h(U)^{\mathrm{t}} J_n \tag{2.87}$$

for generic $U \in \mathcal{B}_0$. We already gave the explicit formula for $V = U^{\mathrm{s}}$ in Theorem 2.8. Note also that the inverse correspondence $V \mapsto U$ is given by the same formula.

§3. Tropical RSK correspondence

3.1. Variations of RSK correspondence

Let $A = \left(a^i_j\right)_{i,j} \in \mathrm{Mat}_{m,n}(\mathbb{N})$ be an $m \times n$ matrix of nonnegative integers.

$$A = \begin{bmatrix} a^1_1 & a^1_2 & \dots & a^1_n \\ a^2_1 & a^2_2 & \dots & a^2_n \\ \vdots & \vdots & & \vdots \\ a^m_1 & a^m_2 & \dots & a^m_n \end{bmatrix} \tag{3.1}$$

Setting $w_i = 1^{a^i_1} \dots n^{a^i_n}$ for $i = 1, \dots, m$, we denote by $P = P(w)$ the column strict tableau obtained from the word $w = w_1 \dots w_m$ by row insertion. Similarly, setting $w'_j = 1^{a^1_j} \dots m^{a^m_j}$ for $j = 1, \dots, n$, we denote by $Q = P(w')$ the column strict tableau obtained from the word $w' = w'_1 \dots w'_n$ by row insertion. The two tableaux P and Q have the same shape, and the correspondence $A \mapsto (P, Q)$ induces a bijection between the set of all $m \times n$ matrices of nonnegative integers and the set of pairs (P, Q) of column strict tableaux of a same shape, P with contents in $\{1, \dots, n\}$ and Q with contents in $\{1, \dots, m\}$. This bijection $A \mapsto (P, Q)$ is called the *Robinson-Schensted-Knuth correspondence* (*RSK correspondence*, for short). In this context, the matrix A is sometimes called the *transportation matrix.* By combining this *standard* RSK correspondence with the Schützenberger involution, we have the following four variations of RSK correspondences:

$$\begin{array}{llll} A \mapsto (P, Q), & & A \mapsto (P, Q^{\mathrm{s}}), \\ A \mapsto (P^{\mathrm{s}}, Q), & & A \mapsto (P^{\mathrm{s}}, Q^{\mathrm{s}}). \end{array} \tag{3.2}$$

Let us denote by p^i_j, q^i_j, $\widetilde{p}^i_j$ and $\widetilde{q}^i_j$ the number of j's in the i-th row of the tableaux P, Q, P^{s} and Q^{s}, respectively. The common shape $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, $l = \min\{m, n\}$, of these four tableaux is given by

(3.3)
$$\lambda_i = p^i_i + \dots + p^i_n = q^i_i + \dots + q^i_m = \widetilde{p}^i_i + \dots + \widetilde{p}^i_n = \widetilde{q}^i_i + \dots + \widetilde{q}^i_m$$

for $i = 1, \dots, l$.

Example 3.1. Consider the transportation matrix

$$A = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 \end{bmatrix}. \tag{3.4}$$

The tableaux P, Q for A, and their counterparts under the Schützenberger involution are determined as follows:

$$P = P(2234|1344|11224) = \begin{matrix} 1\,1\,1\,2\,2\,4\,4 \\ 2\,2\,3\,3\,4 \\ 4 \end{matrix}, \quad \boldsymbol{p} = \begin{bmatrix} 3\,2\,0\,2 \\ 2\,2\,1 \\ 0\,1 \end{bmatrix}$$

$$Q = P(233|1133|12|1223) = \begin{matrix} 1\,1\,1\,1\,2\,2\,3 \\ 2\,2\,3\,3\,3 \\ 3 \end{matrix}, \quad \boldsymbol{q} = \begin{bmatrix} 4\,2\,1 \\ 2\,3 \\ 1 \end{bmatrix}$$

$$P^{\mathrm{s}} = P(13344|1124|1233) = \begin{matrix} 1\,1\,1\,1\,2\,3\,3 \\ 2\,3\,4\,4\,4 \\ 3 \end{matrix}, \quad \widetilde{\boldsymbol{p}} = \begin{bmatrix} 4\,1\,2\,0 \\ 1\,1\,3 \\ 1\,0 \end{bmatrix}$$

$$Q^{\mathrm{s}} = P(1223|23|1133|112) = \begin{matrix} 1\,1\,1\,1\,1\,2\,3 \\ 2\,2\,2\,3\,3 \\ 3 \end{matrix}, \quad \widetilde{\boldsymbol{q}} = \begin{bmatrix} 5\,1\,1 \\ 3\,2 \\ 1 \end{bmatrix} \tag{3.5}$$

In the arguments of this section, the correspondence $A \mapsto (P, Q^{\mathrm{s}})$ will play the essential role, rather than the ordinary RSK correspondence. For this reason, we use the following convention. Let $X = \left(x_j^i\right)_{i,j}$ be an $m \times n$ matrix of nonnegative integers. We denote the i-th row of X by $\boldsymbol{x}^i$, and the j-th column of X by $\boldsymbol{x}_j$:

$$X = \begin{bmatrix} x_1^1 & x_2^1 & \dots & x_n^1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^m & x_2^m & \dots & x_n^m \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \\ \vdots \\ \boldsymbol{x}^m \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \; \boldsymbol{x}_2 \; \dots \; \boldsymbol{x}_n \end{bmatrix}. \tag{3.6}$$

From X, we construct the column strict tableau

$$U = P(w_m w_{m-1} \cdots w_1), \quad w_i = 1^{x_1^i} 2^{x_2^i} \cdots n^{x_n^i} \tag{3.7}$$

by reading the rows $\boldsymbol{x}^m, \dots, \boldsymbol{x}^1$ of X *from bottom to top*, and

$$V = P(w'_n w'_{n-1} \cdots w'_1), \quad w'_j = 1^{x_j^1} 2^{x_j^2} \cdots m^{x_j^m}, \tag{3.8}$$

by reading the columns $\boldsymbol{x}_n, \dots, \boldsymbol{x}_1$ *from right to left*. If we set $X = J_m A$, these U and V correspond to P and Q^{s}, determined from A, so that

$$U = P, \quad V = Q^{\mathrm{s}}, \quad U^{\mathrm{s}} = P^{\mathrm{s}}, \quad V^{\mathrm{s}} = Q. \tag{3.9}$$

In what follows, we refer to this particular correspondence $X \mapsto (U, V)$ as the *RSK* correspondence*.

Before the discussion of tropical RSK correspondences, we formulate an isomorphism theorem concerning the path representation of generic matrices.

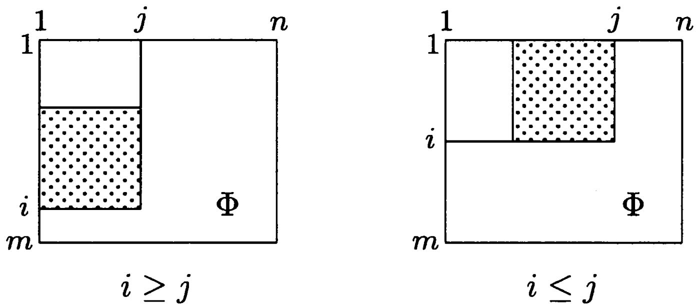
3.2. A fundamental isomorphism

We first fix a notation of *special* minor determinants. For an $m \times n$ matrix $\Phi = \left(\varphi_j^i\right)_{i,j}$ given, we introduce the notation

$$\tau_j^i(\Phi) = \begin{cases} \det \Phi_{1,\dots,j}^{i-j+1,\dots,i} & (i \geq j), \\ \det \Phi_{j-i+1,\dots,j}^{1,\dots,i} & (i \leq j), \end{cases} \tag{3.10}$$

of the minor determinant of Φ corresponding to the largest square in the rectangle $\{1, \dots, m\} \times \{1, \dots, n\}$ whose right-bottom corner is located at (i, j).

$\tau_j^i(\Phi)$:

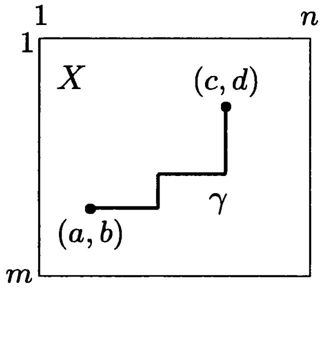


(3.11)

For convenience, we define $\tau_k^0(\Phi) = \tau_0^k(\Phi) = 0$ for any k. We define the subset $\mathrm{Mat}_{m,n}(\mathbb{K})_\tau$ of $\mathrm{Mat}_{m,n}(\mathbb{K})$ by

$$\mathrm{Mat}_{m,n}(\mathbb{K})_\tau = \{\Phi \in \mathrm{Mat}_{m,n}(\mathbb{K}) \mid \tau_j^i(\Phi) \neq 0 \ \text{ for all } \ (i,j)\}. \tag{3.12}$$

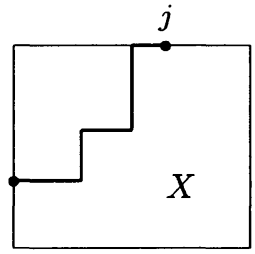
For an $m \times n$ matrix $X = \left(x_j^i\right)_{i,j}$ given, we now construct an $m \times n$ matrix $\Phi = \left(\varphi_j^i\right)_{i,j}$ by using the paths on the lattice $\{1, \dots m\} \times \{1, \dots, n\}$. When we refer to a path in the rectangular lattice, we mean a shortest path joining two vertices, without specifying the orientation of edges. As before, for each path $\gamma : (a, b) \to (c, d)$, we define the weight x_γ of γ, associated with X, to be the product of all x_j^i's corresponding the vertices on γ.



(3.13)

With this definition of weight, we define the matrix $\Phi = \left(\varphi^i_j\right)_{i,j}$ by setting

$$\varphi^i_j = \varphi^i_j(X) = \sum_{\gamma:(i,1)\to(1,j)} x_\gamma = \sum_\gamma$$



(3.14)

for each $(i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}$, where the summation is taken over all paths from $(i,1)$ to $(1,j)$. The mapping $X \mapsto \Phi$ thus obtained will be denoted by

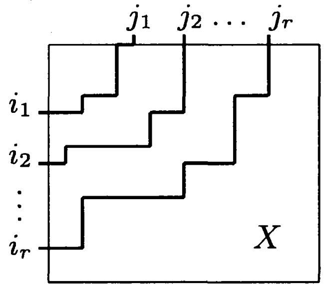
$$\varphi \ : \ \mathrm{Mat}_{m,n}(\mathbb{K}) \to \mathrm{Mat}_{m,n}(\mathbb{K}), \quad \varphi(X) = \left(\varphi^i_j(X)\right)_{i,j}. \tag{3.15}$$

This mapping φ provides us with a device for generating nonintersecting paths on the lattice $\{1,\dots,m\} \times \{1,\dots,n\}$. In fact, from the theorem of Gessel-Viennot [5], it follows that, for any choice of column indices $i_1 < \cdots < i_r$ and row indices $j_1 < \cdots < j_r$, the corresponding minor determinant of Φ is expressed as a sum

$$\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum_{(\gamma_1,\dots,\gamma_r)} x_{\gamma_1} \cdots x_{\gamma_r} \tag{3.16}$$

of the product of weights over all r-tuples $(\gamma_1,\dots,\gamma_r)$ of nonintersecting paths $\gamma_k : (i_k, 1) \to (1, j_k)$ $(k = 1,\dots,r)$. Graphically, this summation can be expressed as follows.

$$\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum$$



(3.17)

Let us look at the special minor determinants $\tau^i_j(\Phi)$ introduced above. Notice that, for each (i,j), there is only one r-tuple $(r = \min\{i,j\})$ of nonintersecting paths relevant to the summation, so that the minor determinant $\tau^i_j(\Phi)$ reduces to the product

$$\tau^i_j(\Phi) = \prod_{(a,b);\, a\le i,\, b\le j} x^a_b. \tag{3.18}$$

This implies that, if $x^i_j \neq 0$ for all i, j, then one has $\tau^i_j(\Phi) \neq 0$ for all i, j. From formula (3.18), it is also clear that the entries x^i_j of the matrix X are recovered as the ratios of minor determinants of Φ

$$x^i_j = \frac{\tau^i_j\,\tau^{i-1}_{j-1}}{\tau^{i-1}_j\,\tau^i_{j-1}}, \qquad \tau^i_j = \tau^i_j(\Phi), \tag{3.19}$$

provided that $x^i_j \neq 0$ for all i, j. The correspondence $X \mapsto \Phi$ defined by (3.14) induces a mapping

$$\varphi \;:\; \mathrm{Mat}_{m,n}(\mathbb{K}^*) \;\to\; \mathrm{Mat}_{m,n}(\mathbb{K})_\tau. \tag{3.20}$$

As we have seen above, if $\Phi = \varphi(X)$, then the matrix X is recovered by the formula (3.19).

Theorem 3.2. *The correspondence $X \mapsto \Phi$ defined by* (3.14) *induces an isomorphism of affine varieties*

$$\varphi \;:\; \mathrm{Mat}_{m,n}(\mathbb{K}^*) \;\xrightarrow{\sim}\; \mathrm{Mat}_{m,n}(\mathbb{K})_\tau. \tag{3.21}$$

Proof. Since φ has a left inverse defined by (3.19), we have only to show that φ is surjective. For each $\Phi \in \mathrm{Mat}_{m,n}(\mathbb{K})_\tau$, we construct an $X \in \mathrm{Mat}_{m,n}(\mathbb{K}^*)$ such that $\varphi(X) = \Phi$, by the induction on m. For this purpose, we first investigate the inductive structure of the mapping φ. Assuming that $\varphi(X) = \Phi$, set

$$\psi^i_j = \sum_{\gamma:(i,1)\to(2,j)} x_\gamma \qquad (2 \le i \le m, 1 \le j \le n). \tag{3.22}$$

Then φ^i_j are determined as

(3.23)

$$\varphi^1_j = x^1_1\,x^1_2 \cdots x^1_j, \qquad \varphi^i_j = \sum_{k=1}^{j} \psi^i_k\,x^1_k\,x^1_{k+1} \cdots x^1_j \quad (i = 2, \ldots, m),$$

for all j. In view of this, we consider the $n \times n$ upper triangular matrix $H(\boldsymbol{x}^1)$, $\boldsymbol{x}^1 = \left(x^1_1, \ldots, x^1_n\right)$, associated with the first row of X. Then the condition (3.23) is equivalent to the matrix equation

$$\Phi = \Psi\,H(\boldsymbol{x}^1), \quad \Psi = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \psi^2_1 & \psi^2_2 & \ldots & \psi^2_n \\ \vdots & \vdots & & \vdots \\ \psi^m_1 & \psi^m_2 & \ldots & \psi^m_n \end{bmatrix}. \tag{3.24}$$

Let us show that any $\Phi \in \mathrm{Mat}_{m,n}(\mathbb{K})_\tau$ can be decomposed in this form by choosing x^1_j and ψ^i_j appropriately. The condition to be satisfied by the first row $(x^1_1, \dots, x^1_n)$ of X is:

$$\left(\Phi H(\boldsymbol{x}^1)^{-1}\right)^1_j = \delta_{1,j} \qquad (j = 1, \dots, n). \tag{3.25}$$

Since $H(\boldsymbol{x}^1)^{-1} = \mathrm{diag}(\overline{\boldsymbol{x}}^1) - \Lambda$, it is easily seen that (3.25) is equivalent to

$$x^1_1 = \varphi^1_1, \quad x^1_j = \frac{\varphi^1_j}{\varphi^1_{j-1}} \ (j = 2, \dots, n). \tag{3.26}$$

Since $\varphi^1_j = \tau^1_j(\Phi) \neq 0$, we can define x^1_j $(j = 1, \dots, n)$ as above. Then the matrix $\Psi = \Phi H(\boldsymbol{x}^1)^{-1}$ has the first row $(1, 0, \dots, 0)$; hence, $H(\boldsymbol{x}^1)$ and Ψ satisfy the condition (3.24). Define Φ' to be the $(m-1) \times n$ matrix obtained from Ψ by removing the first row. We will verify that $\Phi' \in \mathrm{Mat}_{m-1,n}(\mathbb{K})_\tau$ so that Φ' can be expressed as $\Phi' = \varphi(X')$ by the induction hypothesis. Then, setting

$$X = \begin{bmatrix} x^1_1 & \dots & x^1_n \\ & X' & \end{bmatrix}, \quad X' = \begin{bmatrix} x^2_1 & \dots & x^2_n \\ \vdots & & \vdots \\ x^m_1 & \dots & x^m_n \end{bmatrix}, \tag{3.27}$$

we must have $\varphi(X) = \Phi$, which will complete the proof of Theorem 3.2. We now examine the minor determinants of $\Phi = \Psi H(\boldsymbol{x}^1)$. Let $(i,j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ and assume $i > j$. Then it is clear

$$\tau^i_j(\Phi) = \det \Phi^{i-j+1,\dots,i}_{1,\dots,j} = \det \Psi^{i-j+1,\dots,i}_{1,\dots,j} x^1_1 \dots x^1_j, \tag{3.28}$$

since H is upper triangular. Hence we have

$$\tau^{i-1}_j(\Phi') = \det \Psi^{i-j+1,\dots,i}_{1,\dots,j} = \frac{1}{x^1_1 \dots x^1_j} \tau^i_j(\Phi) \neq 0 \quad (i > j). \tag{3.29}$$

Next assume $i \leq j$. In this case, we have

(3.30)
$$\tau^i_j(\Phi) = \det \Phi^{1,\dots,i}_{j-i+1,\dots,j} = \sum_{k_1 < \dots < k_i} \det \Psi^{1,\dots,i}_{k_1,\dots,k_i} \det H(\boldsymbol{x}^1)^{k_1,\dots,k_i}_{j-i+1,\dots,j}.$$

Since the first row of Ψ is $(1, 0, \dots, 0)$, we have $\det \Psi^{1,\dots,i}_{k_1,\dots,k_i} = 0$ unless $k_1 = 1$. When $k_1 = 1$, from Lemma 1.4 it follows that $\det H(\boldsymbol{x}^1)^{1,k_2,\dots,k_i}_{j-i+1,\dots,j}$

$= 0$ unless $(k_2, \dots, k_i) = (j-i+2, \dots, j)$. Since $\det H(\boldsymbol{x}^1)^{1,j-i+2,\dots,j}_{j-i+1,\dots,j} = x^1_1 x^1_2 \cdots x^1_j$, we finally obtain

$$\tau^i_j(\Phi) = \det \Psi^{1,\dots,i}_{1,j-i+2,\dots,j}\, x^1_1 \cdots x^1_j. \tag{3.31}$$

Hence we have

$$\tau^{i-1}_j(\Phi') = \det \Psi^{1,\dots,i}_{1,j-i+2,\dots,j} = \frac{1}{x^1_1 \cdots x^1_j} \tau^i_j(\Phi) \neq 0 \quad (i \leq j). \tag{3.32}$$

This argument implies $\tau^i_j(\Phi') \neq 0$ for all (i,j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n$ as desired. This completes the proof of Theorem 3.2. □

It is convenient for our purpose to restate Theorem 3.2 as follows.

Theorem 3.3. *Let $\Phi = \left(\varphi^i_j\right)_{i,j}$ be an $m \times n$ matrix with coefficients in $\mathbb{K}$ such that $\tau^i_j(\Phi) \neq 0$ for all (i,j). For such a matrix Φ given, define the $m \times n$ matrix $X = \left(x^i_j\right)_{i,j}$ by setting*

$$x^i_j = \frac{\tau^i_j\, \tau^{i-1}_{j-1}}{\tau^{i-1}_j\, \tau^i_{j-1}}, \qquad \tau^i_j = \tau^i_j(\Phi). \tag{3.33}$$

Then, for any choice of column indices $i_1 < \cdots < i_r$ and row indices $j_1 < \cdots < j_r$, the corresponding minor determinant of Φ is expressed as a sum

$$\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r} = \sum_{(\gamma_1,\dots,\gamma_r)} x_{\gamma_1} \cdots x_{\gamma_r} \tag{3.34}$$

of the product of weights over all r-tuples $(\gamma_1, \dots, \gamma_r)$ of nonintersecting paths $\gamma_k : (i_k, 1) \to (1, j_k)$ $(k = 1, \dots, r)$ in the $m \times n$ rectangle.

Remark 3.4. An $m \times n$ real matrix Φ ($\mathbb{K} = \mathbb{R}$) is said to be *totally positive* if $\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r} > 0$ for any choice of row indices $i_1 < \cdots < i_r$ and column indices $j_1 < \cdots < j_r$. Theorem 3.2 implies that, if $\tau^i_j(\Phi) > 0$ for all (i,j), then Φ is already totally positive. In fact, if this condition is satisfied, all the x^i_j's are positive; hence, any minor determinant $\det \Phi^{i_1,\dots,i_r}_{j_1,\dots,j_r}$ is positive since it is expressed as a sum of weights associated with X over r-tuples of nonintersecting paths as in (3.34). Let us denote by $\mathrm{Mat}_{m,n}(\mathbb{R})_{\mathrm{tot.pos.}}$ the open subset of $\mathrm{Mat}_{m,n}(\mathbb{R})$ consisting of all totally positive matrices. Then Theorem 3.2 also implies that φ induces the isomorphism

$$\varphi \;:\; \mathrm{Mat}_{m,n}(\mathbb{R}_{>0}) \xrightarrow{\sim} \mathrm{Mat}_{m,n}(\mathbb{R})_{\mathrm{tot.pos.}} \tag{3.35}$$

In particular, $\mathrm{Mat}_{m,n}(\mathbb{R})_{\mathrm{tot.pos}}$ is isomorphic to $\mathbb{R}^{mn}_{>0}$ as a real analytic manifold. For the theory of totally positive matrices, we refer the reader to [1].

We apply the fundamental isomorphism of Theorem 3.2 to formulating a prototype of subtraction-free birational involution on the space $\mathrm{Mat}_{m,n}(\mathbb{K}^*)$ of matrices.

For each $\Phi \in \mathrm{Mat}_{m,n}(\mathbb{K})$, we define the matrix $\Phi^\vee$ by setting

$$\Phi^\vee = J_m \,\Phi\, J_n, \tag{3.36}$$

where $J_m = \left(\delta_{i+j,m+1}\right)_{i,j=1}^{m}$ and $J_n = \left(\delta_{i+j,n+1}\right)_{i,j=1}^{n}$ are the permutation matrices representing to the longest element of $\boldsymbol{S}_m$ and $\boldsymbol{S}_n$, respectively. This correspondence $\Phi \mapsto \Phi^\vee$ defines an involution on the space $\mathrm{Mat}_{m,n}(\mathbb{K})$ of $m \times n$ matrices. Then, via the isomorphism

$$\varphi:\ \mathrm{Mat}_{m,n}(\mathbb{K}^*) \ \overset{\sim}{\to}\ \mathrm{Mat}_{m,n}(\mathbb{K})_\tau, \tag{3.37}$$

we obtain a birational involution $X \mapsto \iota(X)$ on $\mathrm{Mat}_{m,n}(\mathbb{K}^*)$ such that

$$\varphi(\iota(X)) = \varphi(X)^\vee = J_m\, \varphi(X)\, J_n \tag{3.38}$$

for *generic* $X \in \mathrm{Mat}_{m,n}(\mathbb{K}^*)$.

To be more explicit, let us consider two matrices $X, Y \in \mathrm{Mat}_{m,n}(\mathbb{K}^*)$, and set $\Phi = \varphi(X)$ and $\Psi = \varphi(Y)$. If we impose the relation $\Psi = \Phi^\vee$ between Φ and Ψ, it induces a birational correspondence between X and $Y = \iota(X)$. As we will see below, this correspondence $X \leftrightarrow Y$ provides the essential ingredient of the RSK* correspondence. Recall that $X = \left(x^i_j\right)_{i,j}$ is recovered from $\Phi = \left(\varphi^i_j\right)_{i,j}$ by the formula

$$x^i_j = \frac{\tau^i_j\, \tau^{i-1}_{j-1}}{\tau^{i-1}_j \tau^i_{j-1}}, \qquad \tau^i_j = \tau^i_j(\Phi), \tag{3.39}$$

and $Y = \left(y^i_j\right)_{i,j}$ from $\Psi = \left(\psi^i_j\right)_{i,j}$ by

$$y^i_j = \frac{\sigma^i_j\, \sigma^{i-1}_{j-1}}{\sigma^{i-1}_j \sigma^i_{j-1}}, \qquad \sigma^i_j = \tau^i_j(\Psi). \tag{3.40}$$

We now look at the determinant σ^i_j. Since $\Psi = \Phi^\vee$, we have

$$\sigma^i_j = \tau^i_j(\Psi) = \det \Psi^{i-r+1,\dots,i}_{j-r+1,\dots,j} = \det \Phi^{m-i+1,\dots,m-i+r}_{n-j+1,\dots,n-j+r}\,, \tag{3.41}$$

where $r = \min\{i, j\}$. Hence, each σ^i_j is expressed as the sum

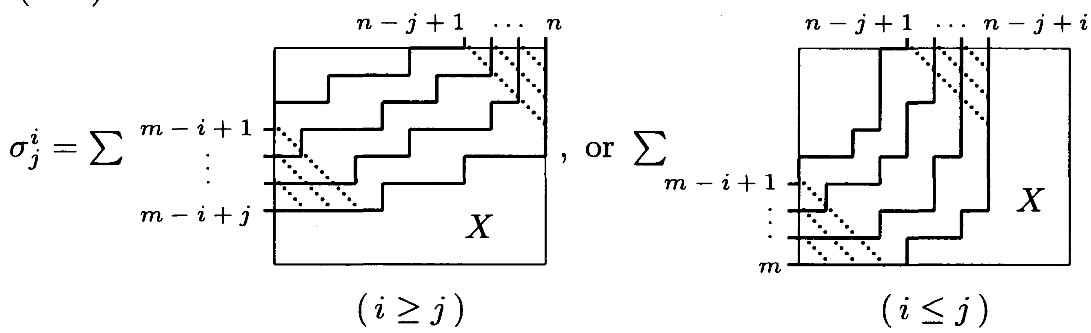
$$\sigma^i_j = \sum_{(\gamma_1,\dots,\gamma_r)} x_{\gamma_1} \cdots x_{\gamma_r} \tag{3.42}$$

of weights associated with X, over all r-tuples of nonintersecting paths

$$\gamma_k : (m - i + k, 1) \to (1, n - j + k) \quad (l = 1, \dots, r). \tag{3.43}$$

Graphically, σ^i_j can be expressed as follows.

(3.44)



From symmetry of the construction, x^i_j are recovered from y^i_j by the same procedure.

Theorem 3.5. *Let $X = \left(x^i_j\right)_{i,j}$, $Y = \left(y^i_j\right)_{i,j}$ be two $m \times n$ matrices such that $x^i_j \neq 0$, $y^i_j \neq 0$ for all i, j. Setting $\Phi = \varphi(X)$, $\Psi = \varphi(Y)$, suppose that Φ and Ψ are related as $\Psi = \Phi^\vee$:*

$$X \overset{\varphi}{\to} \Phi \overset{\vee}{\leftrightarrow} \Psi \overset{\varphi}{\leftarrow} Y. \tag{3.45}$$

Then, for each (i, j), y^i_j is expressed as follows in terms of X :

$$y^i_j = \frac{\sigma^i_j\, \sigma^{i-1}_{j-1}}{\sigma^{i-1}_j \sigma^i_{j-1}}, \qquad \sigma^i_j = \sum_{(\gamma_1,\dots,\gamma_r)} x_{\gamma_1} \cdots x_{\gamma_r}, \tag{3.46}$$

where $r = \min\{i, j\}$, and the summation is taken over all r-tuples of nonintersecting paths $\gamma_k : (m - i + k, 1) \to (1, n - j + k)$ $(k = 1, \dots, r)$. Conversely, each x^i_j is expressed as follows in terms of Y :

$$x^i_j = \frac{\tau^i_j\, \tau^{i-1}_{j-1}}{\tau^{i-1}_j \tau^i_{j-1}}, \qquad \tau^i_j = \sum_{(\gamma_1,\dots,\gamma_r)} y_{\gamma_1} \cdots y_{\gamma_r}, \tag{3.47}$$

summed over the same set of r-tuples of nonintersecting paths as above.

Note that the transformation from $X = \left(x_j^i\right)_{i,j}$ to $Y = \left(y_j^i\right)_{i,j}$ in Theorem 3.5 is realized as a subtraction-free birational mapping from $\mathrm{Mat}_{m,n}(\mathbb{K}^*)$ to itself; this birational mapping is in fact an involution on $\mathrm{Mat}_{m,n}(\mathbb{K}^*)$. Passing to the piecewise linear functions, we obtain

Theorem 3.6. *For each $m \times n$ matrix $X = \left(x_j^i\right)_{i,j} \in \mathrm{Mat}_{m,n}(\mathbb{R})$, define an $m \times n$ matrix $Y = \left(y_j^i\right)_{i,j}$ by*

(3.48)
$$y_j^i = \sigma_j^i - \sigma_j^{i-1} - \sigma_{j-1}^i + \sigma_{j-1}^{i-1}, \qquad \sigma_j^i = \max_{(\gamma_1,\dots,\gamma_r)} (x_{\gamma_1} + \cdots + x_{\gamma_r}),$$

where $r = \min\{i, j\}$, and the maximum is taken over all r-tuples of nonintersecting paths $\gamma_k : (m-i+k, 1) \to (1, n-j+k)$ $(k = 1, \dots, r)$; the weight of a path γ is the sum of all x_b^a's corresponding to the vertices of γ. Then the piecewise linear mapping $X \mapsto Y$ is an involution on $\mathrm{Mat}_{m,n}(\mathbb{R})$.

3.3. Tropical RSK* correspondence

Theorem 3.5 is an essential ingredient of the tropical RSK correspondences. Regarding x_j^i as indeterminates, we now work within the field of rational functions $\mathbb{K}(x)$ in mn variables x_j^i $(1 \le i \le m, 1 \le j \le n)$. In what follows, we assume that $m \le n$ to fix the idea.

Consider the $m \times n$ matrix $X = \left(x_j^i\right)_{i,j}$ regarding x_j^i as indeterminates. We denote the i-th row of X by $\boldsymbol{x}^i$, and the j-th column of X by $\boldsymbol{x}_j$:

$$X = \begin{bmatrix} x_1^1 & x_2^1 & \dots & x_n^1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^m & x_2^m & \dots & x_n^m \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^1 \\ \boldsymbol{x}^2 \\ \vdots \\ \boldsymbol{x}^m \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \ \boldsymbol{x}_2 \ \dots \ \boldsymbol{x}_n \end{bmatrix}. \tag{3.49}$$

From the matrix $X = \left(x_j^i\right)_{i,j}$, we construct four tropical tableaux

$$U = \left(u_j^i\right)_{i \le j}, \quad V = \left(v_j^i\right)_{i \le j}, \quad U^{\mathsf{s}} = \left(\widetilde{u}_j^i\right)_{i \le j}, \quad V^{\mathsf{s}} = \left(\widetilde{v}_j^i\right)_{i \le j} \tag{3.50}$$

as follows:

$$\begin{aligned} H(\boldsymbol{x}^m) \cdots H(\boldsymbol{x}^2) H(\boldsymbol{x}^1) &= H_m(\boldsymbol{u}^m) \cdots H_2(\boldsymbol{u}^2) H_1(\boldsymbol{u}^1) = H_U, \\ H(\boldsymbol{x}_n) \cdots H(\boldsymbol{x}_2) H(\boldsymbol{x}_1) &= H_m(\boldsymbol{v}^m) \cdots H_2(\boldsymbol{v}^2) H_1(\boldsymbol{v}^1) = H_V, \\ H(\boldsymbol{x}_*^1) H(\boldsymbol{x}_*^2) \cdots H(\boldsymbol{x}_*^m) &= H_m(\widetilde{\boldsymbol{u}}^m) \cdots H_2(\widetilde{\boldsymbol{u}}^2) H_1(\widetilde{\boldsymbol{u}}^1) = H_{U^{\mathsf{s}}}, \\ (3.51) \quad H(\boldsymbol{x}_1^*) H(\boldsymbol{x}_2^*) \cdots H(\boldsymbol{x}_n^*) &= H_m(\widetilde{\boldsymbol{v}}^m) \cdots H_2(\widetilde{\boldsymbol{v}}^2) H_1(\widetilde{\boldsymbol{v}}^1) = H_{V^{\mathsf{s}}}, \end{aligned}$$

where $\boldsymbol{x}^i_* = (x^i_n, \dots, x^i_2, x^i_1)$ and $\boldsymbol{x}^*_j = (x^m_j, \dots, x^2_j, x^1_j)$; H_U, $H_{U^{\mathrm{s}}}$ are $n \times n$ matrices, and H_V, $H_{V^{\mathrm{s}}}$ are $m \times m$ matrices. Note that U and U^{s} (resp. V and V^{s}) are transformed into each other by the tropical Schützenberger involution. We also introduce the *tropical Gelfand-Tsetlin pattern* $\boldsymbol{\mu}$ associated with the tropical tableau U as

$$\boldsymbol{\mu} = \begin{bmatrix} \mu^{(n)}_1 \quad \mu^{(n)}_2 \quad \cdots \quad \mu^{(n)}_n \\ \mu^{(n-1)}_1 \quad \mu^{(n-1)}_2 \quad \cdots \quad \mu^{(n-1)}_{n-1} \\ \cdots \\ \mu^{(1)}_1 \end{bmatrix}, \tag{3.52}$$

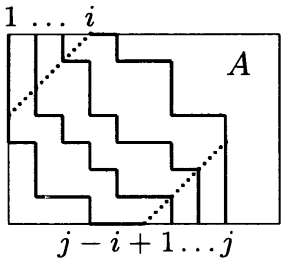
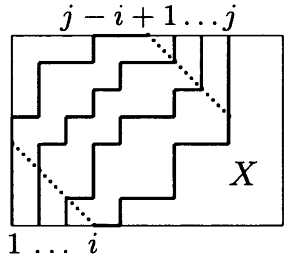
where, for $i \le j$, we define $\mu^{(j)}_i = u^i_i \cdots u^i_j$ $(i \le m)$ and $\mu^{(j)}_i = 1$ $(i > m)$.

Applying Theorem 2.4 to $A = J_m X$, we already know that the variables u^i_j $(i \le j)$ are determined by

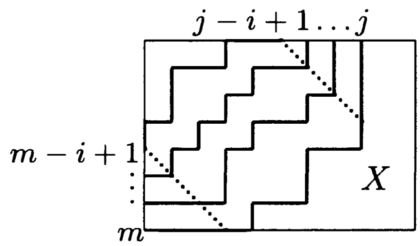
$$u^i_i = \frac{\tau^i_i(H_U)}{\tau^{i-1}_i(H_U)}, \quad u^i_j = \frac{\tau^i_j(H_U)\, \tau^{i-1}_{j-1}(H_U)}{\tau^{i-1}_j(H_U)\tau^i_{j-1}(H_U)} \quad (i < j), \tag{3.53}$$

with

(3.54)

$\tau^i_j(H_U) = \sum$  $= \sum$  .

Notice that $\tau^i_j(H_U)$ for $i \le j$ coincides with

$\det \Phi^{m-i+1,\dots,m}_{j-i+1,\dots,j} = \sum$  .

(3.55)

Hence we have

$$\tau^i_j(H_U) = \det \Phi^{m-i+1,\dots,m}_{j-i+1,\dots,j} = \det \Psi^{1,\dots,i}_{n-j+1,\dots,n-j+i} = \tau^i_{n-j+i}(\Psi). \tag{3.56}$$

This implies

$$\sigma^i_j = \tau^i_j(\Psi) = \tau^i_{n-j+i}(H_U) = \prod_{(a,b);\, a \le i,\, b \le n-j+i} u^a_b \qquad (i \le j). \tag{3.57}$$

Similarly, for $i \le j$, we have

(3.58)
$$\tau^i_j(H_V) = \det \Phi^{j-i+1,\dots,j}_{n-i+1,\dots,i} = \det \Psi^{m-j+1,\dots,m-j+i}_{1,\dots,i} = \tau^{m-j+i}_i(\Psi),$$

hence, for $i \ge j$,

$$\sigma^i_j = \tau^i_j(\Psi) = \tau^j_{m-i+j}(H_V) = \prod_{(a,b);\, a\le j,\, b\le m-i+j} v^a_b \qquad (i \ge j). \tag{3.59}$$

Summarizing the argument above, we have

$$\sigma^i_j = \tau^i_j(\Psi) = \begin{cases} \displaystyle\prod_{(a,b);\, a\le i,\, b\le n-j+i} u^a_b & (i \le j), \\ \displaystyle\prod_{(a,b);\, a\le j,\, b\le m-i+j} v^a_b & (i \ge j), \end{cases} \tag{3.60}$$

for all (i,j) with $1 \le i \le m$ and $1 \le j \le n$. Conversely, u^i_j and v^i_j are determined as

$$u^i_i = \frac{\sigma^i_n}{\sigma^{i-1}_{n-1}}, \quad u^i_j = \frac{\sigma^i_{n-j+i}\,\sigma^{i-1}_{n-j+i}}{\sigma^{i-1}_{n-j+i-1}\sigma^i_{n-j+i+1}} \quad (i < j) \tag{3.61}$$

and

$$v^i_i = \frac{\sigma^m_i}{\sigma^{m-1}_{i-1}}, \quad v^i_j = \frac{\sigma^{m-j+i}_i\,\sigma^{m-j+i}_{i-1}}{\sigma^{m-j+i-1}_{i-1}\sigma^{m-j+i-1}_i} \quad (i < j). \tag{3.62}$$

Remark 3.7. It should be noted that the upper (resp. lower) triangular components of the $m \times n$ matrix $S = \left(\sigma^i_j\right)_{i,j}$ are determined from $U = \left(u^i_j\right)_{i\le j}$ (resp. $V = \left(v^i_j\right)_{i\le j}$), and *vice versa*. Formula (3.60) is also equivalent to

$$\frac{\sigma^i_j}{\sigma^{i-1}_{j-1}} = \begin{cases} u^i_i \cdots u^i_{n-j+i} = \mu^{(n-j+i)}_i & (i \le j), \\ v^j_j \cdots v^j_{m-i+j} = \nu^{(m-i+j)}_j & (i \ge j), \end{cases} \tag{3.63}$$

where $\mu^{(j)}_i$ and $\nu^{(j)}_i$ are the tropical variables representing the Gelfand-Tsetlin pattern of U and V, respectively. Namely, the $m \times n$ matrix

$$\left(\frac{\sigma^i_j}{\sigma^{i-1}_{j-1}}\right)_{i,j} = \begin{bmatrix} \lambda_1 & \mu^{(n-1)}_1 & \cdots\cdots & \mu^{(n-m)}_1 & \cdots & \mu^{(1)}_1 \\ \nu^{(m-1)}_1 & \lambda_2 & \ddots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \vdots & & \vdots \\ \nu^{(1)}_1 & \cdots & \nu^{(m-1)}_{m-1} & \lambda_m & \mu^{(n-1)}_m & \cdots & \mu^{(m)}_m \end{bmatrix}, \tag{3.64}$$

defined by the ratios of σ^i_j, is obtained by glueing the two Gelfand-Tsetlin patterns $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ at the main diagonal, where the diagonal entries

$$\lambda_i = \mu^{(n)}_i = \nu^{(m)}_i \quad (i = 1, \dots, m) \tag{3.65}$$

are the tropical variables representing the common shape of U and V.

From (3.60), we obtain the following expression for y^i_j:

$$y^i_j = \frac{\sigma^i_j\,\sigma^{i-1}_{j-1}}{\sigma^{i-1}_j\sigma^i_{j-1}} = \begin{cases} \dfrac{u^1_{n-j+i}\cdots u^{i-1}_{n-j+i}}{u^1_{n-j+i+1}\cdots u^i_{n-j+i+1}} & (i<j), \\ \dfrac{\lambda_i}{u^1_n\cdots u^{i-1}_n v^1\cdots v^{i-1}_m} & (i=j), \\ \dfrac{v^1_{m-i+j}\cdots v^{j-1}_{m-i+j}}{v^1_{m-i+j+1}\cdots u^j_{m-i+j+1}} & (i>j). \end{cases} \tag{3.66}$$

Hence we have

Theorem 3.8. *Under the assumption of Theorem 3.5, let* $U = \left(u^i_j\right)_{i\le j}$, $V = \left(v^i_j\right)_{i\le j}$ *be the tropical tableaux defined by the tropical row insertions*

(3.67)
$$H_U = H(\boldsymbol{x}^m)\cdots H(\boldsymbol{x}^2)H(\boldsymbol{x}^1), \quad H_V = H(\boldsymbol{x}_n)\cdots H(\boldsymbol{x}_2)H(\boldsymbol{x}_1),$$

respectively. Then u^i_j *and* v^i_j *are expressed as* (3.61) *and* (3.62), *respectively, in terms of* σ^i_j *defined in Theorem 3.5. Conversely, the matrix* $X = \left(x^i_j\right)_{i,j}$ *is recovered from the tropical tableaux* U *and* V *by the formula* (3.47) *with* y^i_j *defined by* (3.66).

Passing to the combinatorial variables, we obtain the explicit inversion formula for the RSK* correspondence.

Theorem 3.9. *Let* $X = \left(x^i_j\right)_{i,j}$ *be an* $m\times n$ *matrix of nonnegative integers. Consider the two column strict tableaux* U *and* V *obtained by the row insertion*

$$\begin{aligned} U &= (\cdots(w_m \leftarrow w_{m-1}) \leftarrow \cdots \leftarrow w_1), \quad w_i = 1^{x^i_1}2^{x^i_2}\cdots n^{x^i_n} \\ V &= (\cdots(w'_n \leftarrow w'_{n-1}) \leftarrow \cdots \leftarrow w'_1), \quad w'_j = 1^{x^1_j}2^{x^2_j}\cdots m^{x^m_j}. \end{aligned} \tag{3.68}$$

Denote by u^i_j (resp. v^i_j) be the number of j's in the i-th row of U (resp. V). Then u^i_j and v^i_j are expressed as

$$u^i_i = \sigma^i_n - \sigma^{i-1}_{n-1},$$
$$u^i_j = \sigma^i_{n-j+i} - \sigma^{i-1}_{n-j+i-1} - \sigma^i_{n-j+i+1} + \sigma^{i-1}_{n-j+i} \quad (i < j), \tag{3.69}$$
$$v^i_i = \sigma^m_i - \sigma^{m-1}_{i-1},$$
$$v^i_j = \sigma^{m-j+i}_i - \sigma^{m-j+i-1}_{i-1} - \sigma^{m-j+i-1}_i + \sigma^{m-j+i}_{i-1} \quad (i < j) \tag{3.70}$$

in terms of σ^i_j defined in Theorem 3.6. Let $\lambda = (\lambda_1, \dots, \lambda_l)$, $l = \min\{m, n\}$, be the common shape of U and V, and set

$$y^i_j = \begin{cases} \sum_{k=1}^{i-1} u^k_{n-j+i} - \sum_{k=1}^{i} u^k_{n-j+i+1} & (i < j), \\ \lambda_i - \sum_{k=1}^{i-1} u^k_n - \sum_{k=1}^{i-1} v^k_m & (i = j), \\ \sum_{k=1}^{j-1} v^k_{m-i+j} - \sum_{k=1}^{j} v^k_{m-i+j+1} & (i > j), \end{cases} \tag{3.71}$$

for each $i = 1, \dots, m$ and $j = 1, \dots, n$. Then the matrix X is recovered from U and V by the formulas

$$x^i_j = \tau^i_j - \tau^{i-1}_j - \tau^i_{j-1} + \tau^{i-1}_{j-1}, \qquad \tau^i_j = \max_{(\gamma_1, \dots, \gamma_r)} (y_{\gamma_1} + \cdots + y_{\gamma_r}), \tag{3.72}$$

where $r = \min\{i, j\}$, and the maximum is taken over all r-tuples of nonintersecting paths $\gamma_k : (m - i + k, 1) \to (1, n - j + k)$ $(k = 1, \dots, r)$ in the $m \times n$ rectangle; the weight of a path γ is the sum of all y^a_b's corresponding to the vertices of γ.

An explicit inversion formula for the usual RSK correspondence is obtained by combining Theorem 3.9 and the Schützenberger involution. The inversion formula discussed above has an obvious theoretical meaning, but is somewhat indirect. We will discuss in the following subsection a different type of inversion formulas for the four variations of RSK correspondences which recovers the transportation matrix directly from the corresponding tableaux.

3.4. Inverse tropical RSK via the Gauss decomposition

Keeping the notations X, Φ, Ψ as before, we now consider the "Gauss decomposition" of the $m \times n$ matrix Ψ $(m \leq n)$:

$$\Psi = \Psi_- \, \Psi_0 \, \Psi_+, \tag{3.73}$$

where Ψ_+, Ψ_0 and Ψ_- are a $m\times n$ upper unitriangular matrix, a $m\times m$ diagonal matrix, and a $m\times m$ lower unitriangular matrix, respectively:

(3.74)

$$(\Psi_+)^i_j=\delta_{i,j}\quad(i\geq j),\qquad(\Psi_0)^i_j=0\quad(i\neq j),\qquad(\Psi_-)^i_j=\delta_{i,j}\quad(i\leq j).$$

Nontrivial entries of these matrices are given explicitly as follows:

$$(3.75)\qquad\begin{aligned}(\Psi_+)^i_j&=\frac{\det\Psi^{1,\dots,i}_{1,\dots,i-1,j}}{\det\Psi^{1,\dots,i}_{1,\dots,i}}=\frac{\det\Phi^{m-i+1,\dots,m}_{n-j+1,n-i+2,\dots,n}}{\det\Phi^{m-i+1,\dots,m}_{n-i+1,\dots,n}}&&(i\leq j),\\(\Psi_0)^i_i&=\frac{\det\Psi^{1,\dots,i}_{1,\dots,i}}{\det\Psi^{1,\dots,i-1}_{1,\dots,i-1}}=\frac{\det\Phi^{m-i+1,\dots,m}_{n-i+1,\dots,n}}{\det\Phi^{m-i+2,\dots,m}_{n-i+2,\dots,n}}&&(i=j),\\(\Psi_-)^i_j&=\frac{\det\Psi^{1,\dots,j-1,i}_{1,\dots,j}}{\det\Psi^{1,\dots,j}_{1,\dots,j}}=\frac{\det\Phi^{m-i+1,m-j+2,\dots,m}_{n-j+1,\dots,n}}{\det\Phi^{m-j+1,\dots,m}_{n-j+1,\dots,n}}&&(i\geq j).\end{aligned}$$

Comparing the path representations of Φ and H_U, $H_{U^{\mathrm{s}}}$, H_V, $H_{V^{\mathrm{s}}}$, we have

$$(3.76)\qquad\det\Phi^{m-r+1,\dots,m}_{l_1,\dots,l_r}=\det(H_U)^{1,\dots,r}_{l_1,\dots,l_r}=\det(H_{U^{\mathrm{s}}})^{l_1^*,\dots,l_r^*}_{n-r+1,\dots,n},$$

where $1\leq l_1<\ldots<l_r\leq n$ and $l_i^*=n-l_i+1$, and

$$(3.77)\qquad\det\Phi^{k_1,\dots,k_r}_{n-r+1,\dots,n}=\det(H_V)^{1,\dots,r}_{k_1,\dots,k_r}=\det(H_{V^{\mathrm{s}}})^{k_1^*,\dots,k_r^*}_{m-r+1,\dots,m},$$

for $1\leq k_1<\ldots<k_r\leq m$ with $k_i^*=m-k_i+1$. From these formulas, we see for instance,

$$(3.78)\qquad(\Psi_0)^i_i=u^i_i\cdots u^i_n=\widetilde{u}^i_i\cdots\widetilde{u}^i_n=v^i_i\cdots v^i_m=\widetilde{v}^i_i\cdots\widetilde{v}^i_m$$

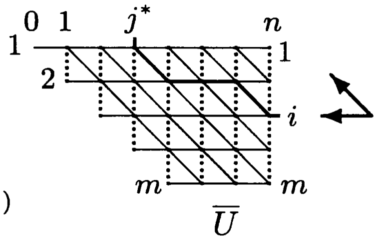
for $i=1,\dots,m$. We set $\lambda_i=(\Psi_0)^i_i$, so that the vector $\lambda=(\lambda_1,\dots,\lambda_m)$ represents the common shape of U, U^{s}, V and V^{s}. By using formulas (3.76), (3.77), we can represent Ψ_+, Ψ_0, Ψ_- in terms of the tropical tableaux.

Proposition 3.10. (0) *For $i=1,\dots,m$,*

$$(3.79)\qquad(\Psi_0)^i_i=\lambda_i=u^i_i\cdots u^i_n=\widetilde{u}^i_i\cdots\widetilde{u}^i_n=v^i_i\cdots v^i_m=\widetilde{v}^i_i\cdots\widetilde{v}^i_m.$$

(1) *For $i\leq j$, $(\Psi_+)^i_j$ is expressed in terms of U as the sum*

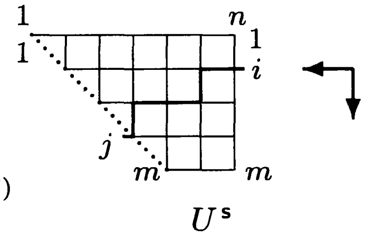
$$(\Psi_+)^i_j=\sum_{\gamma:(i,n)\to(1,n-j+1)}$$



(3.80)

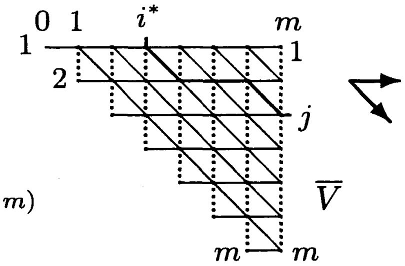
of weights over all path $\gamma : (i, n) \to (1, n-j+1)$, *with weight* $\overline{u}^a_b = \frac{1}{u^a_b}$ *assigned to the horizontal edge connecting* $(a, b-1)$ *and* (a, b), *for each* $a \le b$. *In terms of* U^{s}, $(\Psi_+)^i_j$ *is expressed as*

$$(\Psi_+)^i_j = \lambda_i^{-1} \sum_{\gamma:(i,n)\to(\min\{j,m\},j)} \tag{3.81}$$



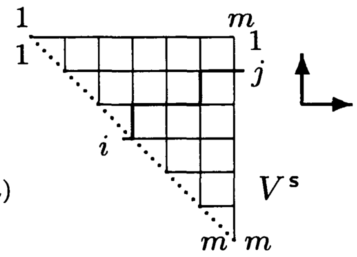
(2) *For* $i \ge j$, $(\Psi_-)^i_j$ *is expressed in terms of* V *as the sum*

$$(\Psi_-)^i_j = \sum_{\gamma:(1,m-i+1)\to(j,m)} \tag{3.82}$$



of weights over all path $\gamma : (m-i+1, 1) \to (j, m)$, *with weight* $\overline{v}^a_b = \frac{1}{v^a_b}$ *assigned to the horizontal edge connecting* $(a, b-1)$ *and* (a, b), *for each* $a \le b$. *In terms of* V^{s}, $(\Psi_-)^i_j$ *is expressed as*

$$(\Psi_-)^i_j = \lambda_j^{-1} \sum_{\gamma:(i,i)\to(j,m)} \tag{3.83}$$



This proposition implies that the matrix $\Psi = \Psi_- \Psi_0 \Psi_+$, as well as $\Phi = \Psi^\vee$, is completely recovered from each of the four pairs of tropical tableaux

$$(U, V), \quad (U, V^{\mathrm{s}}), \quad (U^{\mathrm{s}}, V), \quad (U^{\mathrm{s}}, V^{\mathrm{s}}). \tag{3.84}$$

By combining the graphical representations in Proposition 3.10, we can construct a path representation of Ψ, associated with each pair in (3.84). According to

$$\Psi^i_j = \sum_{k=1}^{m} (\Psi_-)^i_k \, (\Psi_0)^k_k \, (\Psi_+)^k_j, \tag{3.85}$$

we glue the diagrams of (3.80) or (3.81) for Ψ_+ and (3.82) or (3.83) for Ψ_-. We show in Figure 1 the diagrams

$$\Gamma = \Gamma_{U,V}, \quad \Gamma_{U,V^{\mathrm{s}}}, \quad \Gamma_{U^{\mathrm{s}},V}, \quad \Gamma_{U^{\mathrm{s}},V^{\mathrm{s}}} \tag{3.86}$$

obtained in this way. In diagram Γ, the orientation of the edges are indicated by arrows. We assign the weights, associated with the pair of tropical tableaux, to the thick edges and the vertices marked by $\bullet$. For each path γ in Γ, we define the weight $\mathrm{wt}(\gamma)$ to be the product of weights attached to all the edges and the vertices. In Figure 1, $a_1, \dots, a_m$ and $b_1, \dots, b_n$ indicate the entrances and the exits for the path representation of Ψ, respectively. Namely, for each (i, j), ψ^i_j is expressed as the sum

$$\psi^i_j = \sum_{\gamma: a_i \to b_j} \mathrm{wt}(\gamma) \tag{3.87}$$

of weights defined as above, over all paths $\gamma : a_i \to b_j$ in Γ. Recall that the matrix $X = \left(x^i_j\right)$ is determined from Φ through the minor determinants $\tau^i_j = \tau^i_j(\Phi)$ of $\Phi = \Psi^\vee$. Hence each x^i_j is determined as

$$x^i_j = \frac{\tau^i_j \, \tau^{i-1}_{j-1}}{\tau^{i-1}_j \tau^i_{j-1}}, \qquad \tau^i_j = \sum_{(\gamma_1, \dots, \gamma_r)} \mathrm{wt}(\gamma_1) \cdots \mathrm{wt}(\gamma_r), \tag{3.88}$$

where the summation is taken over all r-tuples ($r = \min\{i, j\}$) of nonintersecting paths

$$\gamma_k : a_{m-i+k} \to b_{n-j+k} \qquad (k = 1, \dots, r) \tag{3.89}$$

in Γ. For each pair in (3.84), we have thus obtained an explicit inversion formula of the corresponding tropical RSK correspondence in terms of nonintersecting paths. The corresponding combinatorial formula for the inverse RSK correspondence is obtained simply by the standard procedure:

(3.90)
$$x^i_j = \tau^i_j - \tau^{i-1}_j - \tau^i_{j-1} + \tau^{i-1}_{j-1}, \qquad \tau^i_j = \max_{(\gamma_1, \dots, \gamma_r)} (\mathrm{wt}(\gamma_1) + \cdots + \mathrm{wt}(\gamma_r)),$$

where the weight of a path is defined as the sum of weights attached to the edges and the vertices; read the weights $\bar{u}^i_j$ and $\bar{v}^i_j$ in Γ as $-u^i_j$ and $-v^i_j$ in the combinatorial setting.

In the case of the pair (U, V), we can give a rectangular diagram as well, by deforming the diagram $\Gamma_{U,V}$. Note first that the diagram $\Gamma_{U,V}$

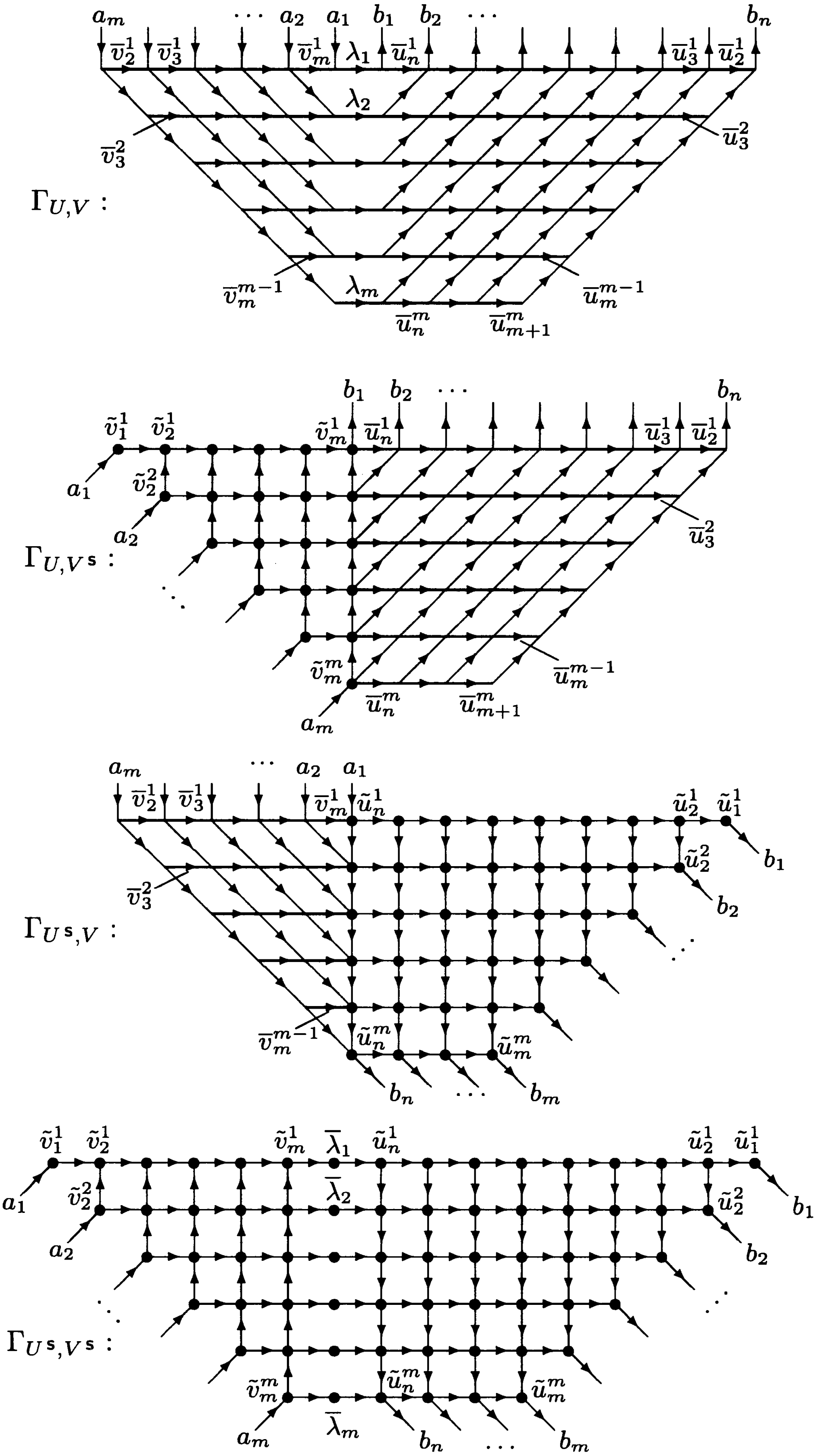
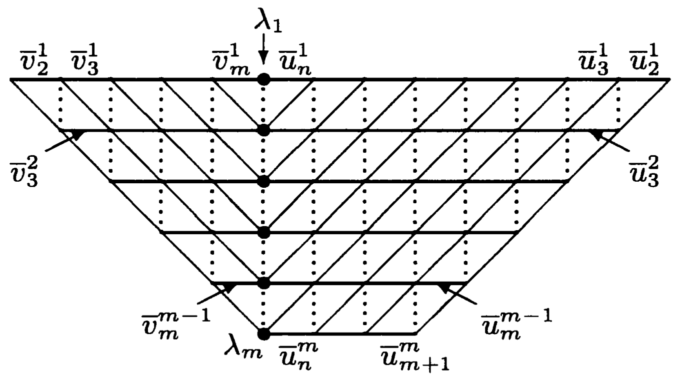


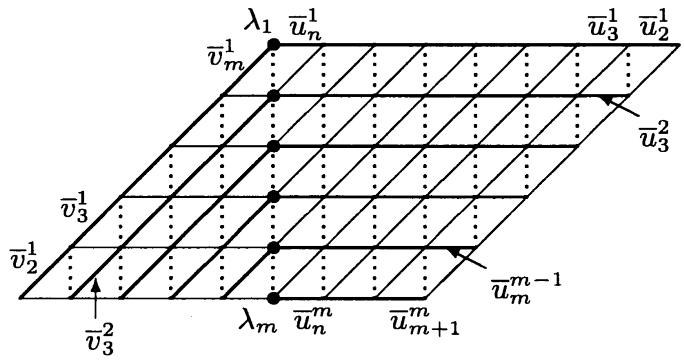
Fig. 1. Path representations of Ψ

is equivalent to the following.



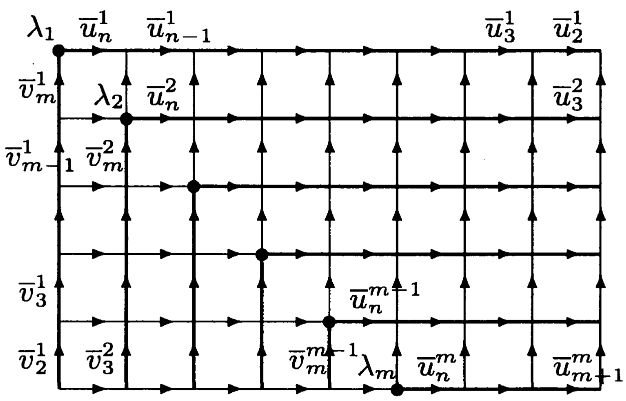
(3.91)

We deform this diagram to



(3.92)

and finally to the $m \times n$ rectangle:



(3.93)

By this rectangle, ψ_j^i is expressed as the sum

$$\psi_j^i = \sum_{\gamma:(i,1)\to\gamma(1,j)} \mathrm{wt}(\gamma) \tag{3.94}$$

of weights defined as above, over all paths $\gamma : (i,1) \to (1,j)$. (This representation is similar to that by $Y = \left(y_j^i\right)_{i,j}$, although the weights are defined in a different way.) Hence we have

$$x^i_j = \frac{\tau^i_j\,\tau^{i-1}_{j-1}}{\tau^{i-1}_j\,\tau^i_{j-1}}, \quad \tau^i_j = \sum_{(\gamma_1,\dots,\gamma_r)} \mathrm{wt}(\gamma_1)\cdots\mathrm{wt}(\gamma_r), \tag{3.95}$$

where $r = \min\{i,j\}$ and the summation is taken over all r-tuples of nonintersecting paths $\gamma_k : (m-i+k,1) \to (1,n-j+k)$ $(k=1,\dots,r)$ in the $m\times n$ rectangle. This inversion formula is essentially equivalent to the inverse RSK* correspondence discussed in the previous subsection.

Remark 3.11. As we have seen above, the RSK correspondence can be thought of as the Gauss (or LR) decomposition of *ultra-discretized* matrices with respect to the product defined by

$$(XY)^i_j = \max_k\,(X^i_k + Y^k_j). \tag{3.96}$$

§4. Birational Weyl group actions

In this section, we introduce a subtraction-free birational affine Weyl group action on the space of tropical transportation matrices. It induces an action of the symmetric group on the space of tropical tableaux through the tropical RSK correspondence. In this section, we work with the generic $m\times n$ matrix $X = \left(x^i_j\right)_{i,j}$, regarding x^i_j as indeterminates.

4.1. Affine Weyl group action on the matrix space

In what follows, we consider the following two (extended) affine Weyl groups $\widetilde{W}^m$ and $\widetilde{W}_n$ of type $A^{(1)}_{m-1}$ and $A^{(1)}_{n-1}$, respectively. We denote by

$$\widetilde{W}^m = \langle r_0, r_1, \dots, r_{m-1}, \omega\rangle \tag{4.1}$$

the group generated by the *simple reflections* $r_0, r_1, \dots, r_{m-1}$ and the *diagram rotation* ω subject to the fundamental relations

$$\begin{aligned} &r_i^2 = 1, \\ &r_i r_j = r_j r_i && (j \not\equiv i, i\pm 1 \mod m), \\ &r_i r_j r_i = r_j r_i r_j && (j \equiv i\pm 1 \mod m), \\ &\omega r_i = r_{i+1}\omega, \end{aligned} \tag{4.2}$$

where we understand the indices for r_i as elements of $\mathbb{Z}/m\mathbb{Z}$. Notice that we have not imposed the relation $\omega^m = 1$. This version of extended affine Weyl group is isomorphic to the semidirect product of the lattice $\mathbb{Z}^m$ of rank m (*not* of rank $m-1$) and the symmetric group $\boldsymbol{S}_m$ acting on it; the subgroup $\langle r_1, \dots, r_{m-1}\rangle$ of $\widetilde{W}^m$ is identified with $\boldsymbol{S}_m$ by mapping each r_i to the adjacent transposition $\sigma_i = (i, i+1)$ $(i = 1, \dots, m-1)$. We define $\widetilde{W}_n = \langle s_0, s_1, \dots, s_{n-1}, \pi\rangle$ similarly to be the group generated by simple reflections $s_0, s_1, \dots, s_{n-1}$ and the diagram rotation π:

$$
\begin{aligned}
&s_i^2 = 1, \qquad s_i s_j = s_j s_i \quad (j \not\equiv i, i \pm 1 \mod n),\\
&s_i s_j s_i = s_j s_i s_j \quad (j \equiv i \pm 1 \mod n), \quad \pi s_i = s_{i+1}\pi.
\end{aligned}
\tag{4.3}
$$

The subgroup $\langle s_1, \dots, s_{n-1}\rangle$ of $\widetilde{W}_n$ is identified with the symmetric group $\boldsymbol{S}_n$.

We now propose to realize these two affine Weyl groups as a group of automorphisms of the field of rational functions $\mathbb{K}(x)$ in mn variables $x = (x_j^i)_{i,j}$. With two extra parameters p, q, we take the field of rational functions $\mathbb{K} = \mathbb{Q}(p, q)$ in (p, q) as the ground field. In our realization, the groups $\widetilde{W}^m$ and $\widetilde{W}_n$ concern the *nontrivial* permutation of rows and columns of the matrix $X = \left(x_j^i\right)_{i,j}$, respectively. We first extend the indexing set $\{1, \dots, m\} \times \{1, \dots, n\}$ for the matrix $X = \left(x_j^i\right)_{i,j}$ to $\mathbb{Z} \times \mathbb{Z}$ by imposing the periodicity condition

$$
x_j^{i+m} = q^{-1} x_j^i, \quad x_{j+n}^i = p^{-1} x_j^i \qquad (i, j \in \mathbb{Z}).
\tag{4.4}
$$

We define the automorphism r_k $(k \in \mathbb{Z}/m\mathbb{Z})$ and ω of $\mathbb{K}(x)$ by

$$
\begin{aligned}
&r_k(x_j^i) = p\, x_j^{i+1} \frac{P_j^i}{P_{j-1}^i}, \quad r_k(x_j^{i+1}) = p^{-1} x_j^i \frac{P_{j-1}^i}{P_j^i} \quad (i \equiv k \mod m),\\
&r_k(x_j^i) = x_j^i \quad (i \not\equiv k, k+1 \mod m), \quad \omega(x_j^i) = x_j^{i+1}
\end{aligned}
\tag{4.5}
$$

for $i, j \in \mathbb{Z}$, where P_j^i is the sum

$$
P_j^i = \sum_{k=1}^{n} x_{j+1}^{i+1} x_{j+2}^{i+1} \cdots x_{j+k}^{i+1} x_{j+k}^i x_{j+k+1}^i \cdots x_{j+n}^i
\tag{4.6}
$$

over all paths $\gamma : (i+1, j+1) \to (i, j+n)$ in the lattice $\mathbb{Z} \times \mathbb{Z}$. We define s_l $(l \in \mathbb{Z}/n\mathbb{Z})$ and π by interchanging the roles of rows and columns, and of p and q:

$$
\begin{aligned}
&s_l(x_j^i) = q\, x_{j+1}^i \frac{Q_j^i}{Q_j^{i-1}}, \quad s_l(x_{j+1}^i) = q^{-1} x_j^i \frac{Q_j^{i-1}}{Q_j^i} \quad (j \equiv l \mod n),\\
&s_l(x_j^i) = x_j^i \quad (j \not\equiv l, l+1 \mod n), \quad \pi(x_j^i) = x_{j+1}^i
\end{aligned}
\tag{4.7}
$$

for $i, j \in \mathbb{Z}$, where

$$(4.8) \qquad Q_j^i = \sum_{k=1}^{m} x_{j+1}^{i+1} x_{j+1}^{i+2} \cdots x_{j+1}^{i+k} x_j^{i+k} x_j^{i+k+1} \cdots x_j^{i+m}$$

summed over all paths $\gamma : (i+1, j+1) \to (i+m, j)$. It is directly seen that these definitions are consistent with the periodicity conditions on x_j^i. Also, it is clear that r_k and s_l have rotational symmetry

$$(4.9) \qquad \begin{array}{lll} \omega\, r_k = r_{k+1}\omega, & \pi\, r_k = r_k\pi & (k \in \mathbb{Z}/m\mathbb{Z}), \\ \omega\, s_l = s_l\omega, & \pi\, s_l = s_{l+1}\pi & (l \in \mathbb{Z}/n\mathbb{Z}), \end{array}$$

respectively.

Remark 4.1. The polynomials P_j^i and Q_j^i above are characterized by the recurrence relations

(4.10)
$$\begin{aligned} x_j^{i+1} P_j^i - P_{j-1}^i x_{j+n}^i &= x_j^{i+1}(x_{j+1}^{i+1} \cdots x_{j+n}^{i+1} - x_j^i \cdots x_{j+n-1}^i) x_{j+n}^{i+1}, \\ x_{j+1}^i Q_j^i - Q_j^{i-1} x_j^{i+m} &= x_{j+1}^i (x_{j+1}^{i+1} \cdots x_{j+1}^{i+m} - x_j^i \cdots x_j^{i+m-1}) x_j^{i+m}, \end{aligned}$$

and the periodicity conditions $P_{j+n}^i = p^{-n-1} P_j^i$, $Q_j^{i+m} = q^{-m-1} Q_j^i$.

Theorem 4.2. *The automorphisms r_k ($k \in \mathbb{Z}/m\mathbb{Z}$), ω, s_l ($l \in \mathbb{Z}/n\mathbb{Z}$), π of $\mathbb{K}(x)$ defined as above give a realization of the direct product $\widetilde{W}^m \times \widetilde{W}_n$ of two extended affine Weyl groups. In particular, the actions of $\widetilde{W}^m = \langle r_0, \dots, r_{m-1}, \omega \rangle$ and $\widetilde{W}_n = \langle s_0, \dots, s_{n-1}, \pi \rangle$ commute with each other.*

In the next two subsections, we give a proof of this theorem by using two characterizations of birational actions of r_k and s_l.

Remark 4.3. The realization of $\widetilde{W}^m \times \widetilde{W}_n$ mentioned above is the same as the one we gave in [11] ($p = q = 1$), and [12]; the variables x_j^i above correspond to x_{ij}^{-1} in [12]. When $p = q = 1$, it coincides with the birational realization of $\widetilde{W}^m \times \widetilde{W}_n$ constructed in [14], Theorem 4.12.

4.2. First characterization

By introducing the spectral parameter z, for an n-vector $\boldsymbol{x} = (x_1, \dots, x_n)$ with $x_i \neq 0$ given, we introduce the following two matrices:

$$(4.11) \qquad E(\boldsymbol{x}; z) = \operatorname{diag}(\boldsymbol{x}) + \Lambda(z), \quad H(\boldsymbol{x}; z) = (\operatorname{diag}(\overline{\boldsymbol{x}}) - \Lambda(z))^{-1},$$

where

$$(4.12)\qquad \Lambda(z) = \sum_{k=1}^{n-1} E_{k,k+1} + zE_{n,1}.$$

Note that the definition of $H(\boldsymbol{x}; z)$ makes sense since

$$(4.13)\qquad \det(\mathrm{diag}(\overline{\boldsymbol{x}}) - \Lambda(z)) = \overline{x}_1 \cdots \overline{x}_n - z.$$

When $z = 0$, these matrices reduce to $E(\boldsymbol{x})$ and $H(\boldsymbol{x})$ used in previous sections. Note also that $H(\boldsymbol{x}; z) = DE(\overline{\boldsymbol{x}}; z)^{-1}D^{-1}$, $D = \mathrm{diag}((-1)^{i-1})_{i=1}^{n}$. We remark that the entries of the matrix $H(\boldsymbol{x}; z)$ are expressed explicitly as

$$(4.14)\qquad H(\boldsymbol{x}; z)^i_j = \begin{cases} \dfrac{x_i x_{i+1} \cdots x_j}{1 - x_1 \cdots x_n z} & (i \le j), \\ \dfrac{x_1 \cdots x_j x_i \cdots x_n z}{1 - x_1 \cdots x_n z} & (i > j). \end{cases}$$

For two n-vectors $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{y} = (y_1, \dots, y_n)$ of indeterminates given, we consider the following matrix equation for unknown vectors $\boldsymbol{u} = (u_1, \dots, u_n)$, $\boldsymbol{v} = (v_1, \dots, v_n)$ such that $u_j \neq 0$, $v_j \neq 0$:

$$(4.15)\qquad H(\boldsymbol{y}; z)H(\boldsymbol{x}; pz) = H(\boldsymbol{v}; z)H(\boldsymbol{u}; pz),$$

or equivalently,

$$(4.16)\qquad E(\overline{\boldsymbol{x}}; pz)E(\overline{\boldsymbol{y}}; z) = E(\overline{\boldsymbol{u}}; pz)E(\overline{\boldsymbol{v}}; z).$$

As before we extend the indexing set for $x_j, y_j, \dots$ to $\mathbb{Z}$ by setting $x_{j+n} = p^{-1}x_j$, $y_{j+n} = p^{-1}y_j, \dots$. Then the matrix equation (4.16) is equivalent to the system of algebraic equations of *discrete Toda type*

$$(4.17)\qquad x_j y_j = u_j v_j, \quad \frac{1}{x_j} + \frac{1}{y_{j+1}} = \frac{1}{u_j} + \frac{1}{v_{j+1}} \quad (j \in \mathbb{Z}).$$

(For the discrete Toda equation, see Remark 2.3.) The next lemma is fundamental in the following argument.

Lemma 4.4. *The matrix equation* (4.15) *has the following two solutions:*

(4.18)

$$(1)\qquad u_j = x_j, \qquad v_j = y_j \qquad (j = 1, \dots, n),$$

$$(2)\qquad u_j = p y_j \frac{P_j}{P_{j-1}}, \qquad v_j = p^{-1} x_j \frac{P_{j-1}}{P_j} \qquad (j = 1, \dots, n),$$

where

$$P_j = \sum_{k=1}^{n} y_{j+1} \cdots y_{j+k} x_{j+k} \cdots x_{j+n} \quad (j = 0, 1, \ldots, n). \tag{4.19}$$

Proof. If $v_{j+1} = y_{j+1}$ for some $j \in \mathbb{Z}$, form (4.17) it follows that $u_j = x_j$, and $v_j = y_j$. Hence we have $u_j = x_j$, $v_j = y_j$ for all $j \in \mathbb{Z}$. Assuming that $v_j \neq y_j$ for any $j \in \mathbb{Z}$, we introduce the variable h_j ($j \in \mathbb{Z}$) such that

$$\frac{1}{v_{j+1}} = \frac{1}{y_{j+1}} + \frac{1}{h_j} \quad (j \in \mathbb{Z}), \tag{4.20}$$

so that $h_{j+n} = p^{-1} h_j$. Then by eliminating u_j in (4.17), we obtain the recurrence relations

$$\frac{h_j}{x_j} = 1 + \frac{h_{j-1}}{y_j}, \quad \text{i.e.,} \quad h_j = x_j + \frac{x_j}{y_j} h_{j-1} \quad (j \in \mathbb{Z}) \tag{4.21}$$

for h_j. Hence we have

(4.22)

$$\begin{aligned} h_{j+n} &= x_{j+n} + \frac{x_{j+n-1} x_{j+n}}{y_{j+n}} + \cdots + \frac{x_{j+1} \cdots x_{j+n}}{y_{j+2} \cdots y_{j+n}} + \frac{x_{j+1} \cdots x_{j+n}}{y_{j+1} \cdots y_{j+n}} h_j \\ &= \frac{P_j + x_{j+1} \cdots x_{j+n} h_j}{y_{j+1} \cdots y_{j+n}}. \end{aligned}$$

Since $h_{j+n} = p^{-1} h_j$, this equation determines h_j as

$$h_j = \frac{P_j}{p^{-1} y_{j+1} \cdots y_{j+n} - x_{j+1} \cdots x_{j+n}}. \tag{4.23}$$

In fact, these h_j satisfy the recurrence relations above, since

$$y_j P_j - P_{j-1} x_{j+n} = y_j y_{j+1} \cdots y_{j+n} x_{j+n} - y_j x_j x_{j+1} \cdots x_{j+n}, \tag{4.24}$$

Hence we have

$$\frac{1}{v_j} = \frac{1}{y_j} + \frac{1}{h_{j-1}} = \frac{h_j}{x_j h_{j-1}} = \frac{p P_j}{x_j P_{j-1}}, \tag{4.25}$$

and

$$\frac{1}{u_j} = \frac{1}{x_j} - \frac{1}{h_j} = \frac{h_{j-1}}{y_j h_j} = \frac{P_{j-1}}{y_j p P_j}, \tag{4.26}$$

which gives the solution (2). □

We remark that the two solutions above are characterized by the conditions

$$
\begin{array}{llll}
(4.27) & (1) & u_1 \cdots u_n = x_1 \cdots x_n, & v_1 \cdots v_n = y_1 \cdots y_n, \\
 & (2) & u_1 \cdots u_n = p^{-1} y_1 \cdots y_n, & v_1 \cdots v_n = p x_1 \cdots x_n,
\end{array}
$$

respectively. Note here that $P_n = p^{-n-1} P_0$.

Returning to the setting of the previous subsection, we consider the matrix $X = \left(x_j^i\right)_{i,j}$. We denote the row vectors and the column vectors of $X = \left(x_j^i\right)_{i,j}$ by $\boldsymbol{x}^i = (x_1^i, \ldots, x_n^i)$ and $\boldsymbol{x}_j = (x_j^1, \ldots, x_j^m)$, respectively. Then Lemma 4.4 implies

$$
H(\boldsymbol{x}^{k+1}; z) H(\boldsymbol{x}^k; pz) = H(r_k(\boldsymbol{x}^{k+1}); z) H(r_k(\boldsymbol{x}^k); pz), \tag{4.28}
$$

where we have used the notation $r_k(\boldsymbol{x}) = (r_k(x_1), \ldots, r_k(x_n))$ for $\boldsymbol{x} = (x_1, \ldots, x_n)$. Since $r_k(\boldsymbol{x}^i) = \boldsymbol{x}^i$ for $i \not\equiv k \mod m$, we have

$$
\begin{aligned}
& H(\boldsymbol{x}^m; z) H(\boldsymbol{x}^{m-1}; pz) \cdots H(\boldsymbol{x}^1; p^{m-1} z) \\
& = H(r_k(\boldsymbol{x}^m); z) H(r_k(\boldsymbol{x}^{m-1}); pz) \cdots H(r_k(\boldsymbol{x}^1); p^{m-1} z)
\end{aligned} \tag{4.29}
$$

for $k = 1, \ldots, m-1$. Namely, the product of matrices in the left-hand side is invariant under the action of r_k $(k = 1, \ldots, m-1)$. Hence we see that

$$
\begin{aligned}
& H(\boldsymbol{x}^m; z) H(\boldsymbol{x}^{m-1}; pz) \cdots H(\boldsymbol{x}^1; p^{m-1} z) \\
& = H(w(\boldsymbol{x}^m); z) H(w(\boldsymbol{x}^{m-1}); pz) \cdots H(w(\boldsymbol{x}^1); p^{m-1} z)
\end{aligned} \tag{4.30}
$$

for any composition $w = r_{k_1} r_{k_2} \cdots r_{k_l}$ with $k_1, \ldots, k_l \in \{1, \ldots, m-1\}$. In the following, we set

$$
H(X; z) = H(\boldsymbol{x}^m; z) H(\boldsymbol{x}^{m-1}; pz) \cdots H(\boldsymbol{x}^1; p^{m-1} z) \tag{4.31}
$$

and

$$
M(X; z) = E(\overline{\boldsymbol{x}}^1; p^{m-1} z) E(\overline{\boldsymbol{x}}^2; p^{m-2} z) \cdots E(\overline{\boldsymbol{x}}^m; z) \tag{4.32}
$$

so that $H(X; z) = D M(X; z)^{-1} D^{-1}$. Then we have

$$
H(X; z) = H(w(X); z), \qquad M(X; z) = M(w(X); z) \tag{4.33}
$$

for any $w = r_{k_1} r_{k_2} \cdots r_{k_l}$ $(k_1, \ldots, k_l \in \{1, \ldots, m-1\})$, where $w(X) = \left(w(x_j^i)\right)_{i,j}$ denotes the matrix obtained from X by applying w to its entries.

Proposition 4.5. *All the entries of the matrices $H(X;z)$ and $M(X;z)$ are invariant under the action of $r_1,\dots,r_{m-1}$.*

Considering $X=\left(x^i_j\right)_{i,j}$ as given, we now investigate in general the matrix equation $H(X;z)=H(Y;z)$ for an $m\times n$ unknown matrix $Y=(y^i_j)_{i,j}$, $y^i_j\neq 0$:

$$\begin{aligned}&H(\boldsymbol{x}^m;z)H(\boldsymbol{x}^{m-1};pz)\cdots H(\boldsymbol{x}^1;p^{m-1}z)\\ &=H(\boldsymbol{y}^m;z)H(\boldsymbol{y}^{m-1};pz)\cdots H(\boldsymbol{y}^1;p^{m-1}z).\end{aligned}\tag{4.34}$$

Note that this equation is equivalent to $M(X;z)=M(Y;z)$:

$$\begin{aligned}&E(\overline{\boldsymbol{x}}^1;p^{m-1}z)E(\overline{\boldsymbol{x}}^2;p^{m-2}z)\cdots E(\overline{\boldsymbol{x}}^m;z)\\ &=E(\overline{\boldsymbol{y}}^1;p^{m-1}z)E(\overline{\boldsymbol{y}}^2;p^{m-2}z)\cdots E(\overline{\boldsymbol{y}}^m;z).\end{aligned}\tag{4.35}$$

Since $\det H(\boldsymbol{x};z)=(\overline{x}_1\cdots\overline{x}_n-z)^{-1}$, by comparing the determinants of the both sides of (4.34), we see that, for any solution of (4.34), there exists a unique permutation $\sigma\in\boldsymbol{S}_m$ such that

$$p^{m-i}y^i_1\cdots y^i_n=p^{m-\sigma(i)}x^{\sigma(i)}_1\cdots x^{\sigma(i)}_n\qquad(i=1,\dots,m).\tag{4.36}$$

Theorem 4.6. *For each permutation $\sigma\in\boldsymbol{S}_m$, the matrix equation (4.34) has a unique solution satisfying the condition (4.36). For any choice of expression $\sigma=\sigma_{k_1}\cdots\sigma_{k_l}$ of σ as a product of adjacent transpositions $\sigma_k=(k,k+1)$ $(k=1,\dots,m-1)$, the solution corresponding to σ is given by*

$$y^i_j=w(x^i_j)\qquad(i=1,\dots,m;\ j=1,\dots,n),\tag{4.37}$$

where $w=r_{k_1}\cdots r_{k_l}$.

Proof. Since $P^i_n=p^{-n-1}P^i_0$ for any i, we have

(4.38)
$$\begin{aligned}&r_k(x^k_1\dots x^k_n)=p^{-1}x^{k+1}_1\dots x^{k+1}_n,\quad r_k(x^{k+1}_1\dots x^{k+1}_n)=p\,x^k_1\dots x^k_n\\ &r_k(x^i_1\dots x^i_n)=x^i_1\dots x^i_n\qquad(i=1,\dots,k-1,k+2,\dots,n)\end{aligned}$$

for $k=1,\dots,m-1$. Hence,

$$r_k(x^i_1\dots x^i_n)=p^{i-\sigma_k(i)}x^{\sigma_k(i)}\dots x^{\sigma_k(i)}_m\qquad(i=1,\dots,m).\tag{4.39}$$

This implies furthermore that, for any composition $w=r_{k_1}\dots r_{k_l}$ of r_k $(k=1,\dots,m-1)$, we have

$$w(x^i_1\dots x^i_n)=p^{i-\sigma(i)}x^{\sigma(i)}\dots x^{\sigma(i)}_n\qquad(i=1,\dots,m).\tag{4.40}$$

where $\sigma = \sigma_{k_1} \cdots \sigma_{k_l}$. Namely, $w(\boldsymbol{x}^1), \dots, w(\boldsymbol{x}^m)$ give a solution of (4.34) satisfying the condition (4.36). In order to complete the proof of the theorem, we show that any solution $\boldsymbol{y}^1, \dots, \boldsymbol{y}^m$ satisfying (4.36) must coincide with this solution. In the following we denote by $\xi_i = p^{-m+i}\,\overline{x}_1^i \cdots \overline{x}_n^i$ the pole of $H(\boldsymbol{x}^i; p^{m-i}z)$. Consider the equality

$$\begin{aligned} &H(w(\boldsymbol{x}^m); z) H(w(\boldsymbol{x}^{m-1}); pz) \cdots H(w(\boldsymbol{x}^1); p^{m-1}z) \\ &= H(\boldsymbol{y}^m; z) H(\boldsymbol{y}^{m-1}; pz) \cdots H(\boldsymbol{y}^1; p^{m-1}z), \end{aligned} \tag{4.41}$$

and multiply the both sides by $H(\boldsymbol{y}^m; z)^{-1}$ from the left to get

$$\begin{aligned} &H(\boldsymbol{y}^m; z)^{-1} H(w(\boldsymbol{x}^m); z) H(w(\boldsymbol{x}^{m-1}); pz) \cdots H(w(\boldsymbol{x}^1); p^{m-1}z) \\ &= H(\boldsymbol{y}^{m-1}; pz) \cdots H(\boldsymbol{y}^1; p^{m-1}z). \end{aligned} \tag{4.42}$$

Since the right-hand side is regular at $z = \xi_{\sigma(m)} = \overline{y}_1^m \cdots \overline{y}_n^m$, the residue of the left-hand side at $z = \xi_{\sigma(m)}$ must vanish. It implies

$$\big(\mathrm{diag}(\overline{\boldsymbol{y}}^m) - \Lambda(\xi_{\sigma(m)})\big)\, \mathrm{Res}_{z=\xi_{\sigma(m)}} (H(w(\boldsymbol{x}^m); z) dz) = 0 \tag{4.43}$$

since the matrices $H(w(\boldsymbol{x}^i); \xi_{\sigma(m)})$ $(i = 1, \dots, m-1)$ are all invertible. If we set $\widetilde{H}(\boldsymbol{x}; z) = (\overline{x}_1 \cdots \overline{x}_n - z) H(\boldsymbol{x}; z)$, it is equivalent to

$$(\mathrm{diag}(\overline{\boldsymbol{y}}^m) - \Lambda(\xi_{\sigma(m)})) \widetilde{H}(w(\boldsymbol{x}^m); \xi_{\sigma(m)}) = 0. \tag{4.44}$$

This equation determines $\boldsymbol{y}^m$ uniquely since $\widetilde{H}(w(\boldsymbol{x}^m); \xi_{\sigma(m)})^i_j \neq 0$ for any i, j. Since $(\mathrm{diag}(w(\overline{\boldsymbol{x}}^m)) - \Lambda(\xi_{\sigma(m)}))\widetilde{H}(w(\boldsymbol{x}^m); \xi_{\sigma(m)}) = 0$, we have $\boldsymbol{y}^m = w(\boldsymbol{x}^m)$, and also

$$H(w(\boldsymbol{x}^{m-1}); pz) \cdots H(w(\boldsymbol{x}^1); p^{m-1}z) = H(\boldsymbol{y}^{m-1}; pz) \cdots H(\boldsymbol{y}^1; p^{m-1}z). \tag{4.45}$$

By repeating the same procedure, we finally obtain $\boldsymbol{y}^i = w(\boldsymbol{x}^i)$ for all $i = 1, \dots, m$, as desired. □

Corollary 4.7. *Let $i_1, \dots, i_k$ and $j_1, \dots, j_l$ be two sequences of elements of $\{1, \dots, m-1\}$ such that*

$$\sigma_{i_1} \cdots \sigma_{i_k} = \sigma_{j_1} \cdots \sigma_{j_l}. \tag{4.46}$$

Then the automorphisms $r_1, \dots, r_{m-1}$ satisfies the relation

$$r_{i_1} \cdots r_{i_k} = r_{j_1} \cdots r_{j_l}. \tag{4.47}$$

By this corollary and the rotational symmetry of r_k, we see that the automorphisms $r_0, r_1, \dots, r_{m-1}, \omega$ satisfy the fundamental relations for the generators of $\widetilde{W}^m$. The same statement is valid for $s_0, s_1, \dots, s_{n-1}$, π and $\widetilde{W}_n$ by the symmetry under the transposition of the matrix X.

Remark 4.8. Lemma 4.4 implies that the action of r_k ($k = 1, \dots, m-1$) is characterized by the system of algebraic equations of discrete Toda type

$$x_j^i x_j^{i+1} = y_j^i y_j^{i+1}, \quad \frac{1}{x_j^i} + \frac{1}{x_{j+1}^{i+1}} = \frac{1}{y_j^i} + \frac{1}{y_{j+1}^{i+1}} \tag{4.48}$$

for $y_j^i = r_k(x_j^i)$ $(i, j \in \mathbb{Z})$ with periodicity condition $y_j^{i+m} = q^{-1} y_j^i$, $y_{j+n}^i = p^{-1} y_j^i$, and an extra constraint

$$y_1^i \cdots y_n^i = p^{i-\sigma_k(i)} x_1^{\sigma_k(i)} \cdots x_n^{\sigma_k(i)} \quad (i = 1, \dots, m). \tag{4.49}$$

4.3. Second characterization

We give another characterization of r_k and s_l, and use it for proving the commutativity of the actions of $\langle r_0, r_1, \dots, r_{m-1}, \omega \rangle$ and $\langle s_0, s_1, \dots, s_{n-1}, \pi \rangle$.

We define the $n \times n$ matrices $G_l(u; z)$, depending on a parameter u, by setting

(4.50)
$$G_0(u; z) = 1 + \frac{1}{u} E_{1,n} z^{-1}, \quad G_l(u; z) = 1 + \frac{1}{u} E_{l+1,l} \quad (l = 1, \dots, n-1).$$

For $l = 1, \dots, n-1$, we also use the notation $G_l(u) = G_l(u; z)$ since they do *not* depend on z. Fix an index $l = 0, 1, \dots, n-1$, and consider the system of matrix equations

$$G_l(g_{i-1}; pz) E(\overline{\boldsymbol{x}}^i; z) = E(\overline{\boldsymbol{y}}^i; z) G_l(g_i; z) \quad (i = 1, \dots, m) \tag{4.51}$$

for unknown variables $\boldsymbol{y}^i$ $(i = 1, \dots, m)$ and g_i $(i = 0, 1, \dots, m)$.

Theorem 4.9. *Under the periodicity condition $g_m = q^{-1} g_0$, the system of algebraic equations (4.51) has a unique solution. It is given explicitly as*

$$\begin{aligned} g_i &= \frac{Q_l^i}{q^{-1} x_{l+1}^{i+1} \cdots x_{l+1}^{i+m} - x_l^{i+1} \cdots x_l^{i+m}} \quad (i = 0, 1, \dots, m), \\ y_j^i &= s_l(x_j^i) \quad (i = 1, \dots, m;\ j = 1, \dots, n). \end{aligned} \tag{4.52}$$

Proof. It is easily seen that the matrix equation (4.51) is equivalent to the recurrence relations

$$g_i = x_l^i + \frac{x_l^i}{x_{l+1}^i} g_{i-1} \qquad (i = 1, \dots, m), \tag{4.53}$$

together with

$$\begin{gathered} \frac{1}{y_l^i} = \frac{1}{x_l^i} - \frac{1}{g_i}, \quad \frac{1}{y_{l+1}^i} = \frac{1}{x_{l+1}^i} + \frac{1}{g_{i-1}}, \\ y_j^i = x_j^i \qquad (j \not\equiv l, l+1 \mod n). \end{gathered} \tag{4.54}$$

These are the same recurrence relations as we have discussed in Lemma 4.4. As we already know, (4.53) determines g_i, and (4.54) gives rise to the expressions we have used in defining s_l. □

Note that the matrix equation (4.51) implies

$$G_l(g_0; p^m z) M(X; z) = M(Y; z) G_l(q^{-1} g_0; z). \tag{4.55}$$

Hence, by Theorem 4.9, we see that the action of s_l on $M(X; z)$ is described by

$$M(s_l(X); z) = G_l(g_0; p^m z) M(X; z) G_l(q^{-1} g_0; z)^{-1}. \tag{4.56}$$

In terms of the matrix $H(X; z)$, this formula can be written as

$$H(s_l(X); z) = G_l(q^{-1} g_0; (-1)^n z)^{-1} H(X; z) G_l(g_0; (-1)^n z). \tag{4.57}$$

We remark that the rational function g_0 can be determined only from $M(X; z)$. It is an easy exercise to show

Lemma 4.10. *Let $\mathfrak{b}$ be the space of all $n \times n$ matrix $M(z)$ with coefficients in $\mathbb{K}(x)[z]$ such that $M(0)$ is upper triangular. For a matrix*

$$M(z) = M_0 + M_1 z + \cdots + M_d z^d \in \mathfrak{b} \tag{4.58}$$

given, set

$$\begin{aligned} \varepsilon_i &= (M_0)_i^i, & (i = 1, \dots, n), \\ \varphi_0 = (M_1)_1^n, \quad \varphi_i &= (M_0)_{i+1}^i & (i = 1, \dots, n-1). \end{aligned} \tag{4.59}$$

Then we have

(4.60)

$$G_0(u; az) M(z) G_0(q^{-1} u; z)^{-1} \in \mathfrak{b} \iff (q^{-1} \varepsilon_n - a \varepsilon_1) u = \varphi_0,$$

and

$$(4.61) \qquad G_l(u)M(z)G_l(q^{-1}u)^{-1} \in \mathfrak{b} \iff (q^{-1}\varepsilon_l - \varepsilon_{l+1})u = \varphi_l,$$

for $l = 1, \dots, n-1$. *In particular, the parameter* u *is determined uniquely from* $M(z)$ *if* ε_i *and* φ_i *are generic.*

This lemma implies that g_0 is expressed as a rational function of entries of $M(X; z)$. Hence, by Proposition 4.5, we see that g_0 in invariant under the action of $r_1, \dots, r_{m-1}$.

We now prove the commutativity of the actions of $\langle r_0, r_1, \dots, r_{m-1}\rangle$ and $\langle s_0, s_1, \dots, s_{n-1}\rangle$. By the rotational symmetry of r_k, it suffices to prove $s_l w = w s_l$ ($l = 0, 1, \dots, n-1$), assuming that $w \in \langle r_1, \dots, r_{m-1}\rangle$. Applying w to (4.56), we have

$$(4.62) \quad M(ws_l(X); z) = G_l(w(g_0); p^m z)M(w(X); z)G_l(q^{-1}w(g_0); z)^{-1}.$$

By Proposition 4.5, we have $M(w(X); z) = M(X; z)$, and also, $w(g_0) = g_0$ as we remarked above. This implies that $M(ws_l(X); z) = M(s_l(X); z)$. By applying s_l again, we obtain

$$(4.63) \qquad M(s_l w s_l(X); z) = M(s_l^2(X); z) = M(X; z).$$

Note that $s_l(x_1^i \dots x_n^i) = x_1^i \dots x_n^i$ for any $i = 1, \dots, m$. Hence, for $Y = s_l w s_l(X)$, we have

$$(4.64) \qquad y_1^i \cdots y_n^i = p^{i-\sigma(i)} x_1^{\sigma(i)} \dots x_n^{\sigma(i)} \quad (i = 1, \dots, m),$$

where $\sigma \in \boldsymbol{S}_m$ is the permutation corresponding to w. Then, by Theorem 4.6, we obtain $Y = w(X)$. This means that $s_l w s_l(X) = w(X)$, namely, $s_l w s_l = w$. This completes the proof of Theorem 4.2.

Recall that the roles of r_k, s_l are interchanged with each other by the transposition of the matrix $X = \left(x_j^i\right)_{i,j}$. Accordingly, the two characterization we have discussed so far can be applied to both r_k and s_l.

4.4. Passage to the tropical tableaux

In what follows we set $\boldsymbol{S}^m = \langle r_1, \dots, r_{m-1}\rangle$ and $\boldsymbol{S}_n = \langle s_1, \dots, s_{n-1}\rangle$.

Let us consider the tropical RSK* correspondence $X \mapsto (U, V)$ with the notation as in the previous section:

$$\begin{aligned} &H(\boldsymbol{x}^m) \cdots H(\boldsymbol{x}^2)H(\boldsymbol{x}^1) = H_m(\boldsymbol{u}^m) \cdots H_2(\boldsymbol{u}^2)H_1(\boldsymbol{u}^1) = H_U, \\ (4.65)\, &H(\boldsymbol{x}_n) \cdots H(\boldsymbol{x}_2)H(\boldsymbol{x}_1) = H_m(\boldsymbol{v}^m) \cdots H_2(\boldsymbol{v}^2)H_1(\boldsymbol{v}^1) = H_V. \end{aligned}$$

As before we assume that $m \leq n$. We regard now the variables u^i_j and v^i_j $(i \leq j)$ as elements of $\mathbb{K}(x)$. Note that by specializing the spectral parameter z to zero, we have

$$H(X;0) = H(\boldsymbol{x}^m)\cdots H(\boldsymbol{x}^1) = H_U. \tag{4.66}$$

By Proposition 4.5, we already know that $H(X;z)$, hence $H(X;0)$ is invariant under the action of the symmetric group $\boldsymbol{S}^m = \langle r_1, \dots, r_{m-1}\rangle$. Since the variables u^i_j are determined uniquely from the matrix $H_U = H(X;0)$, we conclude that all u^i_j are invariant under the action of $\boldsymbol{S}^m = \langle r_1, \dots, r_{m-1}\rangle$.

We now consider the action of $\boldsymbol{S}_n = \langle s_1, \dots, s_{n-1}\rangle$. For $l = 1, \dots, n-1$, from (4.57), we have

$$H(s_l(X);0) = G_l(q^{-1}g_0)^{-1}H(X;0)G_l(g_0), \tag{4.67}$$

hence

$$s_l(H_U) = G_l(q^{-1}g_0)^{-1}H_U\, G_l(g_0), \tag{4.68}$$

where

$$g_0 = \frac{\displaystyle\sum_{k=1}^{m} x^1_{l+1}\cdots x^k_{l+1}x^k_l\cdots x^m_l}{q^{-1}x^1_{l+1}\cdots x^m_{l+1} - x^1_l\cdots x^m_l}. \tag{4.69}$$

This formula is equivalent to

$$s_l(M_U) = G_l(g_0)M_U G_l(q^{-1}g_0), \tag{4.70}$$

where

$$M_U = DH_U^{-1}D^{-1} = E_1(\overline{\boldsymbol{u}}^1)\cdots E_m(\overline{\boldsymbol{u}}^m). \tag{4.71}$$

By applying Lemma 4.10 to (4.70), we see that g_0 is expressed as follows in terms of the u-variables:

$$g_0 = \begin{cases} \dfrac{\displaystyle\sum_{k=1}^{l} u^1_{l+1}\cdots u^k_{l+1}u^k_l\cdots u^l_l}{q^{-1}u^1_{l+1}\cdots u^{l+1}_{l+1} - u^1_l\cdots u^l_l} & (l = 1, \dots, m-1), \\[2ex] \dfrac{\displaystyle\sum_{k=1}^{m} u^1_{l+1}\cdots u^k_{l+1}u^k_l\cdots u^m_l}{q^{-1}u^1_{l+1}\cdots u^m_{l+1} - u^1_l\cdots u^m_l} & (l = m, \dots, n-1). \end{cases} \tag{4.72}$$

Hence, formula (4.70) as well as (4.68) determines completely the action of s_l on the u-variables.

In order to describe the action of s_l on the u-variables, for each $0 \leq i \leq \min\{l, m\}$, we define

(4.73)

$$\begin{aligned} A_l^i &= u_l^1 \cdots u_l^i \sum_{k=i+1}^{l} u_{l+1}^{i+1} \cdots u_{l+1}^k u_l^k \cdots u_l^l \\ &\quad + q^{-1} u_{l+1}^{i+1} \cdots u_{l+1}^{l+1} \sum_{k=1}^{i} u_{l+1}^1 \cdots u_{l+1}^k u_l^k \cdots u_l^i \qquad (1 \leq l \leq m-1), \\ A_l^i &= u_l^1 \cdots u_l^i \sum_{k=i+1}^{m} u_{l+1}^{i+1} \cdots u_{l+1}^k u_l^k \cdots u_l^m \\ &\quad + q^{-1} u_{l+1}^{i+1} \cdots u_{l+1}^{m} \sum_{k=1}^{i} u_{l+1}^1 \cdots u_{l+1}^k u_l^k \cdots u_l^i \qquad (m \leq l \leq n-1). \end{aligned}$$

Theorem 4.11. *Under the tropical RSK* correspondence $X \mapsto (U, V)$, the variables u_j^i $(1 \leq i \leq m;\ i \leq j \leq n)$ are invariant with respect to the action of $\boldsymbol{S}^m = \langle r_1, \ldots, r_{m-1} \rangle$. The action of s_l $(l = 1, \ldots, n-1)$ on u_j^i is described as follows:*

$$(4.74) \qquad \begin{gathered} s_l(u_l^i) = u_{l+1}^i \frac{A_l^i}{A_l^{i-1}}, \quad s_l(u_{l+1}^i) = u_l^i \frac{A_l^{i-1}}{A_l^i}, \\ s_l(u_j^i) = u_j^i \quad (j \neq l, l+1) \end{gathered}$$

for $1 \leq i \leq \min\{l, m\}$ and $s_l(u_j^i) = u_j^i$ for $\min\{l, m\} + 1 \leq i \leq m$.

Proof. Fixing the index $l = 1, \ldots, n-1$, we consider the system of matrix equations

$$(4.75) \qquad G_l(a_{i-1}) E_i(\overline{\boldsymbol{u}}^i) = E_i(\overline{\boldsymbol{t}}^i) G_l(a_i) \quad (i = 1, \ldots, m)$$

for unknown variables $\boldsymbol{t}^i = (1, \ldots, 1, t_i^i, \ldots, t_n^i)$ $(i = 1, \ldots, m)$ and a_i $(i = 0, 1, \ldots, m)$. We will construct below a solution of this system such that $a_m = q^{-1} a_0$, so that

$$(4.76) \qquad G_l(a_0) M_U = M_T\, G_l(q^{-1} a_0);$$

this equation must imply $a_0 = g_0$ and $T = s_l(U)$. The system of matrix equations (4.75) gives the recurrence relations

$$
\begin{aligned}
&a_i = u_l^i + \frac{u_l^i}{u_{l+1}^i} a_{i-1} \qquad (1 \le i \le l), \\
&a_{l+1} = \frac{1}{u_{l+1}^{l+1}} a_l, \quad a_i = a_{i-1} \quad (l+2 \le i \le m).
\end{aligned} \tag{4.77}
$$

for a_i, and also

$$
\frac{1}{t_l^i} = \frac{1}{u_l^i} - \frac{1}{a_i}, \quad \frac{1}{t_{l+1}^i} = \frac{1}{u_{l+1}^i} + \frac{1}{a_{i-1}}, \quad t_j^i = u_j^i \quad (j \ne l, l+1). \tag{4.78}
$$

for $1 \le i \le l$ and $t_j^i = u_j^i$ for $l+1 \le i \le m$. Under the condition $a_m = q^{-1} a_0$, the recurrence relations (4.77) for a_i are solved by

$$
a_i = \frac{A_l^i}{q^{-1} u_{l+1}^1 \cdots u_{l+1}^{\min\{l+1,m\}} - u_l^1 \cdots u_l^l} \qquad (0 \le i \le \min\{l, m\}). \tag{4.79}
$$

Hence we obtain the expression for $t_j^i = s_l(u_j^i)$ as (4.74). □

By eliminating a_i in (4.77), (4.78), we obtain

Proposition 4.12. *The action of s_l $(l = 1, \dots, n-1)$ on the tropical tableau $U = \left(u_j^i\right)_{i \le j}$ is characterized by the following system of algebraic equations of discrete Toda type for $t_j^i = s_l(u_j^i)$:*

$$
\begin{aligned}
&t_l^i t_{l+1}^i = u_l^i u_{l+1}^i \quad (i = 1, \dots, l), \qquad t_{l+1}^{l+1} = u_{l+1}^{l+1}, \\
&\frac{1}{t_l^i} + \frac{1}{t_{l+1}^{i+1}} = \frac{1}{u_l^i} + \frac{1}{u_{l+1}^{i+1}} \qquad (i = 1, \dots, l-1), \\
&\frac{1}{t_l^l} + \frac{q}{t_{l+1}^{l+1} t_{l+1}^1} = \frac{1}{u_l^l} + \frac{q}{u_{l+1}^{l+1} u_{l+1}^1}
\end{aligned} \tag{4.80}
$$

for $l = 1, \dots, m-1$, and

$$
\begin{aligned}
&t_l^i t_{l+1}^i = u_l^i u_{l+1}^i \qquad (i = 1, \dots, m), \\
&\frac{1}{t_l^i} + \frac{1}{t_{l+1}^{i+1}} = \frac{1}{u_l^i} + \frac{1}{u_{l+1}^{i+1}} \qquad (i = 1, \dots, m-1), \\
&\frac{1}{t_l^m} + \frac{q}{t_{l+1}^1} = \frac{1}{u_l^m} + \frac{q}{u_{l+1}^1}
\end{aligned} \tag{4.81}
$$

for $l = m, \dots, n-1$, together with the constraint

$$
\begin{aligned}
&t_l^1 \cdots t_l^{\min\{l,m\}} = u_{l+1}^1 \cdots u_{l+1}^{\min\{l+1,m\}}, \\
&t_{l+1}^1 \cdots t_{l+1}^{\min\{l+1,m\}} = u_l^1 \cdots u_l^{\min\{l,m\}}.
\end{aligned} \tag{4.82}
$$

The action of the tropical Schützenberger involution on the tropical tableau $U = \big(u^i_j\big)_{i\leq j}$ plays the role of reversing the indices of the transformations $s_1, \dots, s_{n-1}$ and interchanging q and q^{-1}. Denoting by $\mathbb{K}(u)$ the field of rational functions in the variables $u = (u^i_j)_{i,j}$, we define the involutive automorphism $\mathsf{s} : \mathbb{K}(u) \to \mathbb{K}(u)$ by using the tropical Schützenberger involution of Theorem 2.8:

$$\mathsf{s}(u^i_i) = \frac{\sigma^i_i}{\sigma^{i-1}_i}, \quad \mathsf{s}(u^i_j) = \frac{\sigma^i_j\,\sigma^{i-1}_{j-1}}{\sigma^{i-1}_j\sigma^i_{j-1}} \quad (i<j), \tag{4.83}$$

where

$$\sigma^i_j = \sum_{(\gamma_1,\dots,\gamma_i)} u_{\gamma_1}\cdots u_{\gamma_i} \tag{4.84}$$

is the sum of weights associated with U, over all i-tuples of nonintersecting paths $\gamma_k : (1, n-i+k) \to (\min\{m, n-j+k\}, n-j+k)$ $(k = 1, \dots, i)$.

Theorem 4.13. *For each $l = 1, \dots, n-1$, let $s^q_l : \mathbb{K}(u) \to \mathbb{K}(u)$ the automorphisms defined as in Theorem 4.11. Then we have $\mathsf{s}\, s^q_l = s^{q^{-1}}_{n-l}\,\mathsf{s}$ for $(l = 1, \dots, n-1)$.*

Proof. The tropical tableau $\mathsf{s}(U) = \big(\mathsf{s}(u^i_j)\big)_{i\leq j}$ is characterized by the condition

$$H_{\mathsf{s}(U)} = \theta(H_U) = J_n H^{\mathrm{t}}_U J_n, \tag{4.85}$$

or equivalently, by

$$M_{\mathsf{s}(U)} = \theta(M_U) = J_n M^{\mathrm{t}}_U J_n. \tag{4.86}$$

Hence, by applying s^q_l to this equality, we have

$$M_{s^q_l\,\mathsf{s}(U)} = \theta(M_{s^q_l(U)}). \tag{4.87}$$

Recall that the action of s_l $(l = 1, \dots, n-1)$ is characterized by

$$M_{s^q_l(U)} = G_l(a_0)\; M_U\; G_l(q^{-1}a_0)^{-1}. \tag{4.88}$$

Since $\theta(G_l(a)) = G_{n-l}(-a) = G_{n-l}(a)^{-1}$, we obtain

$$\begin{aligned}\theta(M_{s^q_l(U)}) &= G_{n-l}(q^{-1}a_0)\; \theta(M_U)\; G_{n-l}(a_0)^{-1}\\ &= G_{n-l}(q^{-1}a_0)\; M_{\mathsf{s}(U)}\; G_{n-l}(a_0)^{-1}\end{aligned} \tag{4.89}$$

By combining this with (4.87), we obtain

$$M_{s_l^q \mathsf{s}(U)} = G_{m-l}(q^{-1}a_0)\ M_{\mathsf{s}(U)}\ G_{n-l}(a_0)^{-1}. \tag{4.90}$$

By applying s again, we have

$$\begin{aligned} M_{\mathsf{s}\, s_l^q \mathsf{s}(U)} &= G_{m-l}(q^{-1}\mathsf{s}(a_0))\ M_{\mathsf{s}^2(U)}\ G_{n-l}(\mathsf{s}(a_0))^{-1} \\ &= G_{m-l}(q^{-1}\mathsf{s}(a_0))\ M_U\ G_{n-l}(\mathsf{s}(a_0))^{-1} \\ &= M_{s_l^{q^{-1}}(U)}. \end{aligned} \tag{4.91}$$

The last equality is a consequence of Lemma 4.10. This implies $\mathsf{s}\, s_l^q \mathsf{s}(u_j^i) = s_l^{q^{-1}}(u_j^i)$ for all $i \le j$, as desired. □

4.5. Combinatorial formulas for the Weyl group action

By the standard procedure of Section 1.3, we can derive the piecewise linear action of the direct product $\widetilde{W}^m \times \widetilde{W}_n$ of affine Weyl groups on the space of transportation matrices X. Also, via the RSK* correspondence, we obtain the piecewise linear action of $\boldsymbol{S}_n = \langle s_1, \dots, s_{n-1} \rangle$ (resp. $\boldsymbol{S}^m = \langle r_1, \dots, r_{m-1} \rangle$) on the space of tableaux $U = \left(u_j^i\right)_{i \le j}$ (resp. $V = \left(v_j^i\right)_{i \le j}$).

Consider the space $\mathrm{Mat}_{m,n}(\mathbb{R})$ of real $m \times n$ matrices $X = \left(x_j^i\right)_{i,j}$, regarding $x = (x_j^i)_{i,j}$ as the canonical coordinates. For each multi-index $\alpha = (\alpha_j^i)_{i,j} \in \mathbb{N}^{mn}$ and $a, b \in \mathbb{N}$, we define the linear function $\ell_{\alpha,a,b}(x)$ on $\mathrm{Mat}_{m,n}(\mathbb{R})$ by

$$\ell_{\alpha,a,b}(x) = \sum_{i,j} \alpha_j^i x_j^i + ap + bq, \tag{4.92}$$

where p, q are parameters. Note that $\ell_{\alpha,a,b}(x) = M(x^\alpha p^a q^b)$ in the notation of Section 1.3. We denote by $\mathcal{F}_X$ the set of all piecewise linear functions $f = f(x)$ in the form

(4.93)
$$f(x) = \max\{\ell_{\alpha,a,b}(x) \mid (\alpha, a, b) \in A\} - \max\{\ell_{\beta,c,d}(x) \mid (\beta, c, d) \in B\},$$

where A, B are nonempty finite sets of triples (α, a, b) of $\alpha \in \mathbb{N}^{mn}$ and $a, b \in \mathbb{N}$. Since $\mathcal{F}_X = M(\mathbb{Q}(x, p, q)_{>0})$, $\mathcal{F}_X$ is closed under the addition, the subtraction and "max"; it is also closed under "min" since $\min\{f, g\} = f + g - \max\{f, g\}$. We say that an isomorphism $w : \mathcal{F}_X \to \mathcal{F}_X$ of $\mathbb{Z}$-modules is *combinatorial* if $w(\max\{f, g\}) = \max\{w(f), w(g)\}$ for any $f, g \in \mathcal{F}_X$.

We first extend the indexing set for x^i_j by setting

$$x^{i+m}_j = x^i_j - q, \quad x^i_{j+n} = x^i_j - p \qquad (i, j \in \mathbb{Z}). \tag{4.94}$$

We define the action of r_k $(k = 0, 1, \dots, m)$, ω and s_l $(l = 0, 1, \dots, n)$, π on the variables x^i_j as follows:

$$\begin{aligned}
&\left.\begin{aligned} r_k(x^i_j) &= x^{i+1}_j + P^i_j - P^i_{j-1} + p \\ r_k(x^{i+1}_j) &= x^i_j + P^i_{j-1} - P^i_j - p \end{aligned}\right. \quad (i \equiv k \mod m), \\
&r_k(x^i_j) = x^i_j \quad (i \not\equiv k, k+1 \mod m), \quad \omega(x^i_j) = x^{i+1}_j, \\
&\left.\begin{aligned} s_l(x^i_j) &= x^i_{j+1} + Q^i_j - Q^{i-1}_j + q \\ s_l(x^i_{j+1}) &= x^i_j + Q^{i-1}_j - Q^i_j - q \end{aligned}\right. \quad (j \equiv l \mod n), \\
&s_l(x^i_j) = x^i_j \quad (j \not\equiv l, l+1 \mod n), \quad \pi(x^i_j) = x^i_{j+1},
\end{aligned} \tag{4.95}$$

for $i, j \in \mathbb{Z}$, where

$$\begin{aligned}
P^i_j &= \max_{1 \le k \le n} \Big(\sum_{a=1}^{k} x^{i+1}_{j+a} + \sum_{a=k}^{n} x^i_{j+a} \Big), \\
Q^i_j &= \max_{1 \le k \le m} \Big(\sum_{a=1}^{k} x^{i+a}_{j+1} + \sum_{a=k}^{m} x^{i+a}_j \Big).
\end{aligned} \tag{4.96}$$

These formulas can also be written in terms of "min"; for instance,

$$\begin{aligned} r_k(x^i_j) &= x^{i+1}_j - R^i_j + R^i_{j-1} + p \\ r_k(x^{i+1}_j) &= x^i_j - R^i_{j-1} + R^i_j - p \end{aligned} \quad (i \equiv k \mod m), \tag{4.97}$$

where

$$R^i_j = \min_{1 \le k \le n} \Big(\sum_{a=1}^{k-1} x^i_{j+a} + \sum_{a=k+1}^{n} x^{i+1}_{j+a} \Big). \tag{4.98}$$

Theorem 4.14. *Define the mappings r_k $(k \in \mathbb{Z}/m\mathbb{Z})$, ω and s_l $(l \in \mathbb{Z}/n\mathbb{Z})$, π from the set of variables x^i_j to $\mathcal{F}_X$ as above. Then each of them extends uniquely to a combinatorial isomorphism $\mathcal{F}_X \to \mathcal{F}_X$. Furthermore, they give a realization of the direct product $\widetilde{W}^m \times \widetilde{W}_n$ of affine Weyl groups as a group of combinatorial isomorphisms of $\mathcal{F}_X$.*

Proposition 4.15. *By using the action of r_k $(k = 1, \dots, m-1)$, set $y^i_j = r_k(x^i_j)$ for all $i, j \in \mathbb{Z}$. Then we obtain a solution to the ultra-discrete equation of Toda type*

$$x^i_j + x^{i+1}_j = y^i_j + y^{i+1}_j, \quad \min\{x^i_j, x^{i+1}_{j+1}\} = \min\{y^i_j, y^{i+1}_{j+1}\} \tag{4.99}$$

with periodicity condition $y_j^{i+m} = y_j^i - q$, $y_{j+n}^i = y_j^i - p$, *satisfying the constraint*

(4.100)
$$y_1^i + \cdots + y_n^i = x_1^{\sigma_k(i)} + \cdots + x_n^{\sigma_k(i)} + (i - \sigma_k(i))p \quad (i = 1, \ldots, m).$$

Remark 4.16. Let $B = \bigcup_{l=0}^{\infty} B_l$ the crystal basis of the symmetric tensor representation $S(V) = \bigoplus_{l=0}^{\infty} S_l(V)$ of $\mathfrak{gl}(n)$ associated with the vector representation $V = \mathbb{C}^n$. Then B is identified with the set of n-vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ of nonnegative integers. The crystal basis $B^{\otimes m} = B \otimes \cdots \otimes B$ (m times) for the m-th tensor product $S(V)^{\otimes m}$ is parametrized by $\mathbb{N}^{mn}$. We identify the matrix $X = \left(x_j^i\right)_{i,j}$ with the coordinates of $B^{\otimes m} = \mathbb{N}^{mn}$, regarding $\boldsymbol{x}^i = (x_1^i, \ldots, x_n^i)$ as corresponding to the i-th component. When $p = q = 0$, the actions of r_k ($k = 1, \ldots, m-1$) and s_l ($l = 1, \ldots, n-1$) on the variables x_j^i coincide with the combinatorial R-matrix acting on the k-th and $(k+1)$-st components of $B^{\otimes m}$, and Kashiwara's Weyl group actions, respectively (see [6], [29]).

Assuming that $m \leq n$, we consider the variables u_j^i ($1 \leq i \leq m$; $i \leq j \leq n$) and v_j^i ($1 \leq i \leq j \leq m$), associated with the column strict tableaux U and V, through the RSK* correspondence $X \mapsto (U, V)$. Recall that each u_j^i (resp. v_j^i) denotes the number of j's in the i-th row of U (resp. V). Then, by the explicit piecewise linear formulas described in Theorem 3.9, u_j^i and v_j^i are regarded as elements in $\mathcal{F}_X$. We thus obtain a combinatorial action of $\widetilde{W}^m \times \widetilde{W}_n$ on the variables u_j^i and v_j^i. We describe below the action of the subgroups $\boldsymbol{S}^m = \langle r_1, \ldots, r_{m-1} \rangle$ and $\boldsymbol{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ on u_j^i; their action on v_j^i is given by an obvious modification.

In view of (4.73), we define A_l^i ($0 \leq i \leq \min\{l, m\}$) for the combinatorial version by

(4.101)
$$\begin{aligned} A_l^i = \max\Big\{ &\max_{i+1 \leq k \leq l} (u_l^1 + \cdots + u_l^i + u_{l+1}^{i+1} + \cdots + u_{l+1}^k + u_l^k \cdots + u_l^l), \\ &\max_{1 \leq k \leq i} (u_{l+1}^{i+1} + \cdots + u_{l+1}^{l+1} + u_{l+1}^1 + \cdots + u_{l+1}^k + u_l^k + \cdots + u_l^i - q) \Big\} \end{aligned}$$

for $1 \leq l \leq m-1$, and by

(4.102)
$$\begin{aligned} A_l^i = \max\Big\{ &\max_{i+1 \leq k \leq m} (u_l^1 + \cdots + u_l^i + u_{l+1}^{i+1} + \cdots + u_{l+1}^k + u_l^k \cdots + u_l^m), \\ &\max_{1 \leq k \leq i} (u_{l+1}^{i+1} + \cdots + u_{l+1}^m + u_{l+1}^1 + \cdots + u_{l+1}^k + u_l^k + \cdots + u_l^i - q) \Big\} \end{aligned}$$

for $m \le l \le n-1$. Then, from Theorem 4.11 we obtain

Theorem 4.17. *The variables u^i_j are invariant under the action of the symmetric group $\boldsymbol{S}^m = \langle r_1, \dots, r_{m-1}\rangle$ induced via the RSK* correspondence. The action of s_l $(l = 1, \dots, n-1)$ is given explicitly as follows:*

$$\begin{gathered} s_l(u^i_l) = u^i_{l+1} + A^i_l - A^{i-1}_l, \quad s_l(u^i_{l+1}) = u^i_l + A^{i-1}_l - A^i_l, \\ s_l(u^i_j) = u^i_j \quad (j \neq l, l+1) \end{gathered} \tag{4.103}$$

for $1 \le i \le \min\{l, m\}$ and $s_l(u^i_j) = u^i_j$ for $\min\{l, m\} + 1 \le i \le m$.

Note that v^i_j are invariant under the action of $\boldsymbol{S}_n = \{s_1, \dots, s_{n-1}\}$, and that r_k $(k = 1, \dots, m-1)$ act on v^i_j by explicit piecewise linear formulas similar to those described above.

Proposition 4.18. *By using the action of s_l $(l = 1, \dots, n-1)$, set $t^i_j = s_l(u^i_j)$ for $1 \le i \le m$, $i \le j \le n$. Then we obtain a solution to the ultra-discrete equation of Toda type*

$$\begin{aligned} & t^i_l + t^i_{l+1} = u^i_l + u^i_{l+1} \quad (i = 1, \dots, l), \qquad t^{l+1}_{l+1} = u^{l+1}_{l+1}, \\ & \min\{t^i_l, t^{i+1}_{l+1}\} = \min\{u^i_l, u^{i+1}_{l+1}\} \qquad (i = 1, \dots, l-1), \\ & \min\{t^l_l, t^{l+1}_{l+1} + t^1_{l+1} - q\} = \min\{u^l_l, u^{l+1}_{l+1} + u^1_{l+1} - q\} \end{aligned} \tag{4.104}$$

for $l = 1, \dots, m-1$, and

$$\begin{aligned} & t^i_l + t^i_{l+1} = u^i_l + u^i_{l+1} \qquad\qquad (i = 1, \dots, m), \\ & \min\{t^i_l, t^{i+1}_{l+1}\} = \min\{u^i_l, u^{i+1}_{l+1}\} \qquad (i = 1, \dots, m-1), \\ & \qquad \min\{t^m_l, t^1_{l+1} - q\} = \min\{u^m_l, u^1_{l+1} - q\} \end{aligned} \tag{4.105}$$

for $l = m, \dots, n-1$, satisfying the constraint

$$\begin{aligned} & t^1_l + \cdots + t^{\min\{l,m\}}_l = u^1_{l+1} + \cdots + u^{\min\{l+1,m\}}_{l+1}, \\ & t^1_{l+1} + \cdots + t^{\min\{l+1,m\}}_{l+1} = u^1_l + \cdots + u^{\min\{l,m\}}_l. \end{aligned} \tag{4.106}$$

Remark 4.19. The variables u^i_j are identified with the coordinates of crystal bases for general finite dimensional irreducible representations of $\mathfrak{gl}(n)$ as in [13]. Then, by using an argument as in [6], it can be shown that the combinatorial action of $\boldsymbol{S}_n = \langle s_1, \dots, s_{n-1}\rangle$ on u^i_j with $q = 0$ provides with the description of Kashiwara's Weyl group actions on the crystal bases. This symmetric group action on the set of column strict tableaux thus coincides with the one introduced earlier by A. Lascoux and M.P. Schützenberger [18](see also [17]). The corresponding piecewise linear action is discussed in [15].

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Green functions attached to limit symbols

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To Ryoshi Hotta on his 60th birthday

Abstract.

Hall-Littlewood functions and Green functions associated to complex reflection groups $W = G(r,1,n)$ were constructed in [S1] by means of symbols, which are a generalization of partitions. In this paper, we consider such functions in the case where the symbols are of very special type, called "limit symbols". The situation becomes simple, and is close to the case of symmetric groups when the symbols tends to the "limit". In the case where W is a Weyl group of type B_n, we give a closed formula for Hall-Littlewood functions, and verify some of the conjectures stated in [S1] for the case of Green functions attached to limit symbols.

§0. Introduction

Green functions associated to symmetric groups $\mathfrak{S}_n$ were originally introduced by Green [G] in connection with the representation theory of finite general linear groups $GL_n(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$. Later Deligne and Lusztig [DL] constructed Green functions for any finite reductive groups. The algorithm of computing Green functions, in particular in the case of classical groups, shows that Green functions are determined by the information on Weyl groups and some combinatorial data centering u-symbols. Note that u-symbols are combinatorial objects introduced by Lusztig [L] describing unipotent classes in $G(\mathbb{F}_q)$, which is a natural generalization of the notion of partitions in the case of $GL_n(\mathbb{F}_q)$.

In [S1], Green functions associated to complex reflection groups $W = G(e,1,n)$ were introduced, and it was shown that there exists a combinatorial framework for such Green functions based on the theory of symmetric functions as in the case of symmetric groups. In particular, the notion of u-symbols were generalized to a various type of symbols, and Hall-Littlewood functions associated to such symbols were

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constructed. Green functions are essentially given by the matrix $K_\pm(t)$ of Kostka functions, which is defined as the transition matrix between the set of Schur functions and Hall-Littlewood functions (both are associated to W). The set of symbols are divided into similarity classes, and accordingly, $K_\pm(t)$ is regarded as a block matrix. Then $K_\pm(t)$ has the lower triangular shape as a block matrix, with the identity matrix on each diagonal block. These results were generalized in [S2] to the case of complex reflection groups $G(e,p,n)$.

In this paper, we consider the case of limit symbols (see section 1 for the precise definition). The limit symbol is, in some sense, a limit of the symbols discussed in [S1], and Hall-Littlewood functions turn out to be independent of the choice of symbols when it tends to the limit. In this limit, the situation becomes drastically simple, and is close to the case of symmetric groups. For example, each similarity class consists of one element, and so $K_\pm(t)$ is just a lower unitriangular matrix. We further restricts ourselves to the case where $e = 2$ (i.e., W is the Weyl group of type B_n), and give a closed formula for Hall-Littlewood functions, just as in the case of $\mathfrak{S}_n$. This enables us (in the case where $e = 2$) to show that Hall-Littlewood functions and Green functions are polynomials with integral coefficients, which verifies some conjectures in [S1] in this case. Note that even in the case where $e = 2$, Green functions given here (associated to limit symbols) are different from Green functions associated to $Sp_{2n}(\mathbb{F}_q)$ or $SO_{2n+1}(\mathbb{F}_q)$.

It is likely that Green functions associated to limit symbols have rich structures from geometric and combinatorial point of view. For example, one can expect that they are described in terms of Poincaré polynomials of the quotient of the coinvariant algebras of W, just as in the case of symmetric groups (see 3.14 for details). Yamada [Y] has computed such Poincaré polynomials in some small rank cases, which supports our conjecture.

This paper grew up from the discussion with H.-F. Yamada. The author is very grateful to him.

§1. Limit symbols

1.1. We review some notations from [S1]. We denote by $\mathcal{P}_{n,e}$ the set of e-partitions $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$ such that $|\boldsymbol{\alpha}| = \sum_{k=0}^{e-1} |\alpha^{(k)}| = n$. Let W be the complex reflection group $G(e,1,n) \simeq \mathfrak{S}_n \ltimes (\mathbb{Z}/e\mathbb{Z})^n$. The set of irreducible characters of W is in bijection with $\mathcal{P}_{n,e}$. We denote by $\chi^{\boldsymbol{\alpha}}$ the irreducible character of W corresponding to $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$. In particular, the unit character corresponds to $(n;-;\dots;-)$ and $\overline{\det}_V$

corresponds to $(-;\dots;-;1^n)$, where $\overline{\det}_V$ denotes the one dimensional representation arising from the complex conjugate of the determinant of the reflection representation V of W.

Let $m_0,\dots,m_{e-1}$ be positive integers such that $m_k \ge n$, and put $\mathbf{m} = (m_0,\dots,m_{e-1})$. We denote by $Z_n^{0,0} = Z_n^{0,0}(\mathbf{m})$ the set of e-partitions $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$ such that each $\alpha^{(k)}$ is regarded as an element in $\mathbb{Z}^{m_k}$, written in the form $\alpha^{(k)} : \alpha_1^{(k)} \ge \cdots \ge \alpha_{m_k}^{(k)} \ge 0$. We fix an integer $r > 0$ and an e-tuple of non-negative integers $\mathbf{s} = (s_0,\dots,s_{e-1})$ such that $s_k \le r$. Let us define an e-partition $\boldsymbol{\Lambda}^0 = \boldsymbol{\Lambda}^0(\mathbf{m},\mathbf{s},r) = (\Lambda^{(0)},\dots,\Lambda^{(e-1)})$ as follows.

$$\Lambda^{(k)} : s_k + (m_i-1)r \ge \cdots \ge s_k + 2r \ge s_k + r \ge s_k \tag{1.1.1}$$

for $k = 0,\dots,e-1$. We denote by $Z_n^{r,\mathbf{s}} = Z_n^{r,\mathbf{s}}(\mathbf{m})$ the set of e-partitions of the form $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, where $\boldsymbol{\alpha} \in Z_n^{0,0}$ and the sum is taken entry-wise. We denote by $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$ if $\boldsymbol{\Lambda} = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0$, and call it the e-symbol of type $(r,\mathbf{s})$ corresponding to $\boldsymbol{\alpha}$. We often denote the symbol $\boldsymbol{\Lambda} = (\Lambda^{(0)},\dots,\Lambda^{(e-1)})$ in the form $\boldsymbol{\Lambda} = (\Lambda_j^{(k)})$ with $\Lambda^{(k)} : \Lambda_1^{(k)} > \cdots > \Lambda_{m_k}^{(k)}$ for $k = 0,\dots,e-1$.

Put $\mathbf{m}' = (m_0+1,\dots,m_{e-1}+1)$, and we define a shift operation $Z_n^{r,\mathbf{s}}(\mathbf{m}) \to Z_n^{r,\mathbf{s}}(\mathbf{m}')$ by associating $\boldsymbol{\Lambda}' = (\Lambda'_0,\dots,\Lambda'_{e-1}) \in Z_n^{r,\mathbf{s}}(\mathbf{m})$ to $\boldsymbol{\Lambda} = (\Lambda_0,\dots,\Lambda_{e-1}) \in Z_n^{r,\mathbf{s}}(\mathbf{m})$, where $\Lambda'_k = (\Lambda_k + r) \cup \{s_k\}$ for $k = 0,\dots,e-1$. In other words, for $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, $\boldsymbol{\Lambda}'$ is obtained as $\boldsymbol{\Lambda}' = \boldsymbol{\alpha} + \boldsymbol{\Lambda}^0(\mathbf{m}',\mathbf{s},r)$, where $\boldsymbol{\alpha}$ is regarded as an element of $Z_n^{0,0}(\mathbf{m}')$ by adding 0 in the entries of $\boldsymbol{\alpha}$. We denote by $\bar{Z}_n^{r,\mathbf{s}}$ the set of classes in $\coprod_{\mathbf{m}'} Z_n^{r,\mathbf{s}}(\mathbf{m}')$ under the equivalence relation generated by shift operations. Note that $\mathcal{P}_{n,e}$ coincides with the set $\bar{Z}_n^{0,0}$. Also note that $\boldsymbol{\Lambda}^0$ is regarded as a symbol in $Z_n^{r,\mathbf{s}}$ with $n = 0$.

Two elements $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$ in $\bar{Z}_n^{r,\mathbf{s}}$ ares said to be similar if there exist representatives in $Z_n^{r,\mathbf{s}}(\mathbf{m})$ such that all the entries of them coincide each other with multiplicities. The set of symbols which are similar to a fixed symbol is called a similarity class in $Z_n^{r,\mathbf{s}}$.

We shall define a function $a : \bar{Z}_n^{r,\mathbf{s}} \to \mathbb{Z}_{\ge 0}$. For $\boldsymbol{\Lambda} \in Z_n^{r,\mathbf{s}}$, we put

$$a(\boldsymbol{\Lambda}) = \sum_{\lambda,\lambda' \in \boldsymbol{\Lambda}} \min(\lambda,\lambda') - \sum_{\mu,\mu' \in \boldsymbol{\Lambda}^0} \min(\mu,\mu'). \tag{1.1.2}$$

The function a on $Z_n^{r,\mathbf{s}}$ is invariant under the shift operation, and it induces a function a on $\bar{Z}_n^{r,\mathbf{s}}$. Clearly, the a-function takes a constant value on each similarity class in $Z_n^{r,\mathbf{s}}$.

Remark 1.2. The definition of symbols given here is slightly more general than the one in [S1], where it is assumed that $\mathbf{s}$ is of the form

$(0, s, \dots, s)$. The symbols of this type appear in [Ma] in parameterizing unipotent characters associated to W. However, the arguments in [S1] can be applied without change to the setting as above (except the last paragraph of section 1, see Remark 3.2), and we shall refer to the results in [S1] freely.

1.3. From now on we assume that $\mathbf{m}$ is of the type

$$m_0 = \cdots = m_a = m + 1, m_{a+1} = \cdots = m_{e-1} = m$$

for some integers $m \ge n$ and $0 \le a \le e - 1$. A symbol $\boldsymbol{\Lambda} = (\Lambda_j^{(k)})$ is called special if it satisfies the relation

$$\begin{aligned} \Lambda_j^{(k)} &\ge \Lambda_j^{(k+1)} \qquad && \text{for } 1 \le j \le m, 0 \le k \le e - 2, \\ \Lambda_j^{(e-1)} &\ge \Lambda_{j+1}^{(0)} \qquad && \text{for } 1 \le j \le m. \end{aligned}$$

If $\boldsymbol{\Lambda}^0$ is special, each similarity class in $Z_n^{r,\mathbf{s}}$ contains a unique special symbol, and the set of special symbols is in a bijective correspondence with the set of similarity classes in $Z_n^{r,\mathbf{s}}$.

We now assume that $\boldsymbol{\Lambda}^0 = (\Lambda_j^{(k)})$ itself is special and satisfies the condition that

$$\Lambda_j^{(k)} - \Lambda_j^{(k+1)} \ge n, \qquad \Lambda_j^{(e-1)} - \Lambda_{j+1}^{(0)} \ge n. \tag{1.3.1}$$

For example, we may choose that $r = en, s_0 = 0, s_k = (e - k)n$ for $k = 1, \dots, e - 1$. Symbols in $Z_n^{r,\mathbf{s}}$ determined by $\boldsymbol{\Lambda}^0$ satisfying (1.3.1) are called limit symbols. In this case any symbol is special, and so each similarity class consists of one element. The combinatorics concerning Hall-Littlewood functions and Green functions turn out to be drastically simple, and the situation becomes quite similar to the case of symmetric groups, though it is related to W. In the remainder of this section, we shall discuss Hall-Littlewood functions and Green functions associated to limit symbols.

1.4. From now on, we assume that $Z_n^{r,\mathbf{s}}$ is the set of limit symbols. One can identify a symbol $\boldsymbol{\Lambda} \in Z_n^{r,\mathbf{s}}$ (resp. an e-partition $\boldsymbol{\alpha} \in Z_n^{0,0}$) with an element in $\mathbb{Z}_{\ge 0}^M$, where $M = \sum m_i$, by arranging $\boldsymbol{\Lambda} = (\Lambda_j^{(k)})$ as in 1.3,

$$\Lambda_1^{(0)}, \cdots, \Lambda_1^{(e-1)}, \Lambda_2^{(0)}, \cdots, \Lambda_2^{(e-1)}, \Lambda_3^{(0)}, \cdots \tag{1.4.1}$$

and similarly for $\boldsymbol{\alpha} = (\alpha_j^{(k)})$. In particular, symbols give rise to partitions of M by this identification. For $\lambda = (\lambda_i) \in \mathbb{Z}^M$, we define an integer $n(\lambda)$ by

$$n(\lambda) = \sum_i (i-1)\lambda_i.$$

If λ is a partition, we have $n(\lambda) = \sum_{i \neq j} \min(\lambda_i, \lambda_j)$. Then it is easy to see, by (1.1.2), that

$$a(\boldsymbol{\Lambda}) = n(\boldsymbol{\Lambda}) - n(\boldsymbol{\Lambda}^0) = n(\boldsymbol{\alpha}), \tag{1.4.2}$$

where $\boldsymbol{\Lambda}, \boldsymbol{\Lambda}^0, \boldsymbol{\alpha}$ are regarded as elements in $\mathbb{Z}^M$ under the above identification. Let us introduce a partial order $\boldsymbol{\Lambda} \geq \boldsymbol{\Lambda}'$ on $Z_n^{r,\mathbf{s}}$ by using the dominance order $\geq$ on $\mathbb{Z}^M$, i.e., for $\lambda = (\lambda_i), \mu = (\mu_i) \in \mathbb{Z}^M$, we define $\lambda \geq \mu$ if

$$\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$$

for $k = 1, \dots, M$. By (1.4.2), one can check that the partial order on $Z_n^{r,\mathbf{s}}$ is compatible with the a-function, i.e., we have

$$a(\boldsymbol{\Lambda}) > a(\boldsymbol{\Lambda}') \quad \text{if} \quad \boldsymbol{\Lambda} < \boldsymbol{\Lambda}'. \tag{1.4.3}$$

Under the bijection $Z_n^{0,0} \simeq Z_n^{r,\mathbf{s}}$ by $\boldsymbol{\alpha} \leftrightarrow \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, the partial order on $Z_n^{r,\mathbf{s}}$ and a-function on it are inherited to $Z_n^{0,0} \simeq \mathcal{P}_{n,e}$. This partial order on $Z_n^{0,0}$ is nothing but the order obtained from the dominance order on $\mathbb{Z}^M$ under the embedding $Z_n^{0,0} \subset \mathbb{Z}^M$. Combining this with (1.4.2), we see that

(1.4.4) a-functions and the partial orders on $Z_n^{0,0}$ defined by limit symbols are independent of the choice of $\boldsymbol{\Lambda}^0$ as far as $\boldsymbol{\Lambda}^0$ satisfies (1.3.1).

In the rest of the paper, we express the set of limit symbols $Z_n^{r,\mathbf{s}}$ as Z_n^∞, and always consider the a-functions and partial orders on $Z_n^{0,0}$ inherited from Z_n^∞.

§2. Hall-Littlewood functions attached to limit symbols

2.1 For a given $\mathbf{m} = (m_0, \dots, m_{e-1})$, we introduce a set of indeterminate $x_j^{(k)}$ $(0 \leq k \leq e-1, 1 \leq m_k)$. We denote by x the whole variables $(x_j^{(k)})$, and also denote by $x^{(k)}$ the variables $x_1^{(k)}, \dots, x_{m_k}^{(k)}$. For an e-partition $\boldsymbol{\alpha} = (\alpha^{(0)}, \dots, \alpha^{(e-1)})$, one can define the Schur function $s_{\boldsymbol{\alpha}}(x)$ and monomial symmetric function $m_{\boldsymbol{\alpha}}(x)$ by

$$s_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} s_{\alpha^{(k)}}(x^{(k)}), \qquad m_{\boldsymbol{\alpha}}(x) = \prod_{k=0}^{e-1} m_{\alpha^{(k)}}(x^{(k)})$$

where $s_{\alpha^{(k)}}$ (resp. $m_{\alpha^{(k)}}$) denotes the usual Schur function (resp. the monomial symmetric function) associated to the partition $\alpha^{(k)}$ with respect to the variables $x^{(k)}$.

In what follows we regard the variables $x_i^{(k)}$ defined for $k \in \mathbb{Z}/e\mathbb{Z} \simeq \{0, 1, \dots, e-1\}$. We now introduce a new variable t, and define a function $\widetilde{q}_{r,\pm}^{(k)}(x;t)$ associated to $+$ or $-$, for each $0 \le k \le e-1$ and an integer $r \ge 0$, by

$$\widetilde{q}_{r,\pm}^{(k)}(x;t) = \sum_{i\ge 1} (x_i^{(k)})^{r+\delta} \frac{\prod_j x_i^{(k)} - t x_j^{(k\mp 1)}}{\prod_{j\ne i} x_i^{(k)} - x_j^{(k)}} \qquad (r \ge 1), \tag{2.1.1}$$

where $\delta = m_k - 1 - m_{k\pm 1}$, In the product of the denominator, $x_j^{(k)}$ runs over all the variables in $x^{(k)}$ except $x_i^{(k)}$, while in the numerator, $x_j^{(k\pm 1)}$ runs over all the variables in $x^{(k\pm 1)}$. $\widetilde{q}_{r,\pm}^{(k)}$ is a polynomial in $\mathbb{Z}[x;t]$ if $\delta \ge 0$, and lies in $\mathbb{Z}[x, x^{-1}; t]$ in general. We define $q_{r,\pm}^{(k)}(x;t)$ as follows. If $\delta \ge 0$ i.e., $m_k \ge m_{k\pm 1} + 1$, put $q_{r,\pm}^{(k)} = \widetilde{q}_{r,\pm}^{(k)}$. If $\delta < 0$, we add $m_{k\pm 1} + 1 - m_k$ variables $\mathbf{x}' = x_{m_k+1}^{(k)}, \dots, x_{m_{k\pm 1}+1}^{(k)}$ to $x^{(k)}$, and consider the polynomial $\widetilde{q}_{r,\pm}^{(k)}$ for such variables with $\delta = 0$, and put $q_{r,\pm}^{(k)} = \widetilde{q}_{r,\pm}^{(k)}|_{\mathbf{x}'=0}$. Hence $q_{r,\pm}^{(k)} \in \mathbb{Z}[x;t]$ in all cases, and we have $q_{0,\pm}^{(k)} = 1$.

For an e-partition $\boldsymbol{\alpha} \in \mathcal{P}_{n,e}$, we define a function $q_{\boldsymbol{\alpha},\pm}(x;t)$ by

$$q_{\boldsymbol{\alpha},\pm}(x;t) = \prod_{k=0}^{e-1} \prod_{j=1}^{m_k} q_{\alpha_j^{(k)},\pm}^{(k)}(x;t). \tag{2.1.2}$$

Remark 2.2. In [S1, 2.2], the function $q_{r,\pm}^{(k)}$ was defined by the formula (2.1.1). But since it is not a polynomial, its definition should be modified as above. Then this $q_{r,\pm}^{(k)}$ coincides with the polynomial obtained from the generating function (2.3.1) in [S1], and the properties stated in Lemma 2.3 in [S1] holds for $q_{r,\pm}^{(k)}$. Accordingly, the definition of $R_{\boldsymbol{\alpha},<}^{\pm}(x;t)$, etc. in [S1, 3.2] must be modified appropriately. (However, the notations below have some discrepancies with [S1]. See Reamrk 5.7 in [S2] for details.)

2.3 We denote by $\Xi_{\mathbf{m}} = \bigotimes_{k=0}^{e-1} \mathbb{Z}[x_1^{(k)}, \dots, x_{m_k}^{(k)}]^{\mathfrak{S}_{m_k}}$ the ring of symmetric polynomials (with respect to $\mathfrak{S}_{\mathbf{m}} = \mathfrak{S}_{m_0} \times \cdots \times \mathfrak{S}_{m_{e-1}}$) with variables $x = (x_j^{(k)})$. $\Xi_{\mathbf{m}}$ has a structure of a graded ring $\Xi_{\mathbf{m}} = \bigoplus_{i \ge 0} \Xi_{\mathbf{m}}^i$, where $\Xi_{\mathbf{m}}^i$ consists of homogeneous symmetric polynomials of degree i, together with the zero polynomial. As given in [S1, 3.15], one can define the ring of symmetric functions $\Xi = \oplus_{i \ge 0} \Xi^i$ as the direct sum of the inverse limit Ξ^i of $\Xi_{\mathbf{m}}^i$. The Schur function $s_{\boldsymbol{\alpha}}(x)$ with infinitely many variables $x_1^{(k)}, x_2^{(k)}, \dots$ is regarded as an element in Ξ^n with $n = |\boldsymbol{\alpha}|$, and the set $\{s_{\boldsymbol{\alpha}}(x)\}$ with $\boldsymbol{\alpha} \in Z_n^{0,0}$ forms a $\mathbb{Z}$-basis of Ξ^n. It is also shown (see [S1, 3.15]) that $\{q_{\boldsymbol{\alpha},\pm} \mid \boldsymbol{\alpha} \in Z_n^{0,0}\}$ gives rise to a basis of the $\mathbb{Q}(t)$-space $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$ (according to $+$ or $-$, respectively). A similar property holds if one replaces Ξ^n by $\Xi_{\mathbf{m}}^n$.

We now define a scalar product $\langle \, , \, \rangle$ on $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$ by the property that

$$\langle q_{\boldsymbol{\alpha},+}(x;t), m_{\boldsymbol{\beta}}(x) \rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$$

for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{P}_{n,e}$. Then we have $\langle m_{\boldsymbol{\alpha}}(x), q_{\boldsymbol{\beta},+}(x;t) \rangle = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ by [S1, (4.7.2)] (But there are some discrepancies with the formulas in [S1] in the discussion below because of some errors in [S1]. For this see Remarks 5.7 in [S2]).

Hall-Littlewood functions $P_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ and $Q_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ associated to symbols were constructed in [S1]. $\{P_{\boldsymbol{\Lambda}}^{\pm}\}, \{Q_{\boldsymbol{\Lambda}}^{\pm}\}$ give bases of $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$. In the case of Z_n^{∞}, $\{P_{\boldsymbol{\Lambda}}^{\pm}\}$ are characterized by the following two properties (cf. [S1, Proposition 4.7]).

(2.3.1) For $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z_n^{\infty}$, $P_{\boldsymbol{\Lambda}}^{\pm}(x;t)$ can be expressed in terms of Schur functions $s_{\boldsymbol{\beta}}(x)$ as

$$P_{\boldsymbol{\Lambda}}^{\pm}(x;t) = s_{\boldsymbol{\alpha}}(x) + \sum_{\boldsymbol{\beta} < \boldsymbol{\alpha}} u_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\pm}(t) s_{\boldsymbol{\beta}}(x) \qquad (u_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\pm}(t) \in \mathbb{Q}(t)),$$

(2.3.2) $\langle P_{\boldsymbol{\Lambda}}^{+}, P_{\boldsymbol{\Lambda}'}^{-} \rangle = 0$ for $\boldsymbol{\Lambda} \ne \boldsymbol{\Lambda}'$,

Then $Q_{\boldsymbol{\Lambda}}^{\pm}$ are determined as the dual of $P_{\boldsymbol{\Lambda}}^{\pm}$, i.e., we have

(2.3.3) $\langle P_{\boldsymbol{\Lambda}}^{+}, Q_{\boldsymbol{\Lambda}'}^{-} \rangle = \langle Q_{\boldsymbol{\Lambda}}^{+}, P_{\boldsymbol{\Lambda}'}^{-} \rangle = \delta_{\boldsymbol{\Lambda},\boldsymbol{\Lambda}'}$.

Here the partial order $\boldsymbol{\beta} < \boldsymbol{\alpha}$ in $Z_n^{0,0}$ is the one given in 1.4. We note that $P_{\boldsymbol{\Lambda}}^{\pm}$ coincides with $Q_{\boldsymbol{\Lambda}}^{\pm}$ up to scalar by (2.3.2) and (2.3.3). So one can write it, for $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, as

$$Q_{\boldsymbol{\Lambda}}^{\pm}(x;t) = b_{\boldsymbol{\alpha}}^{\pm}(t) P_{\boldsymbol{\Lambda}}^{\pm}(x;t) \tag{2.3.4}$$

for some $b^{\pm}_{\boldsymbol{\alpha}}(t) \in \mathbb{Q}(t)$.

Let $\mathcal{A}$ be the subring of $\mathbb{Q}(t)$ consisting of functions which has no pole at $t = 0$. Then $\mathcal{A}$ is the local ring with the unique maximal ideal $t\mathcal{A}$, and $\mathcal{A}^* = \mathcal{A} - t\mathcal{A}$ is the set of units in $\mathcal{A}$. $P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ and $Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ are also characterized by the expansions in terms of $s_{\boldsymbol{\beta}}(x)$ and $q_{\boldsymbol{\beta},\pm}(x)$ as follows.

Theorem 2.4 ([S1, Th. 4.4]). (i) *$P^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ are the unique functions having the following expansions.*

$$P^{\pm}_{\boldsymbol{\Lambda}}(x;t) = \sum_{\boldsymbol{\beta} \ge \boldsymbol{\alpha}} c^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta},\pm}(x;t)$$

$$P^{\pm}_{\boldsymbol{\Lambda}}(x;t) = s_{\boldsymbol{\alpha}}(x) + \sum_{\boldsymbol{\beta} < \boldsymbol{\alpha}} u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) s_{\boldsymbol{\beta}}(x),$$

where $c^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Q}(t)$ in the first formula, and $u^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in t\mathcal{A}$ in the second formula.

(ii) *$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t)$ are the unique functions having the following expansions.*

$$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t) = q_{\boldsymbol{\alpha},\pm}(x;t) + \sum_{\boldsymbol{\beta} > \boldsymbol{\alpha}} d^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta},\pm}(x;t),$$

$$Q^{\pm}_{\boldsymbol{\Lambda}}(x;t) = \sum_{\boldsymbol{\beta} \le \boldsymbol{\alpha}} w^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) s_{\boldsymbol{\beta}}(x),$$

where $d_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Q}(t)$ in the first formula, and $w^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in t\mathcal{A}$ for $\boldsymbol{\beta} \ne \boldsymbol{\alpha}$ and $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}} \in \mathcal{A}^*$ in the second formula.

2.5 We shall give a closed formula for $Q^{\pm}_{\boldsymbol{\Lambda}}$ and $P^{\pm}_{\boldsymbol{\Lambda}}$ in the special case where $e = 2$. So, in what follows we assume that $e = 2$. In this case, $Q^{+}_{\boldsymbol{\Lambda}}$, $P^{+}_{\boldsymbol{\Lambda}}$, $q^{+}_{\boldsymbol{\alpha}}$, etc. coincide with $Q^{-}_{\boldsymbol{\Lambda}}$, $P^{-}_{\boldsymbol{\Lambda}}$, $q^{-}_{\boldsymbol{\alpha}}$, etc., and so we omit the signature $\pm$ and express them simply as $Q_{\boldsymbol{\Lambda}}$, $P_{\boldsymbol{\Lambda}}$, $q_{\boldsymbol{\alpha}}$, etc. In order to obtain the closed forms of $P_{\boldsymbol{\Lambda}}$ and $Q_{\boldsymbol{\Lambda}}$, we recall here another type of symmetric functions $R_{\boldsymbol{\Lambda}} = R^{\pm}_{\boldsymbol{\Lambda}}$ introduced in [S1]. Let $\mathcal{M} = \{(i,k) \mid 0 \le k \le 1, 1 \le i \le m_k\}$ be the set of pairs (i,k) corresponding to $x^{(k)}_i$. We define a total order on $\mathcal{M}$ compatible with the embedding $Z^{\infty}_n \subset \mathbb{Z}^M$, as in (1.4.1), i.e.,

$$(1,0) < (1,1) < (2,0) < \cdots < (m,1) < (m+1,0) < \cdots .$$

For a fixed $\boldsymbol{\alpha} = (\alpha^{(k)}_j) \in Z^{0,0}_n$, we denote by $\nu_0 = (i_0, k_0)$ the largest element in $\mathcal{M}$ such that $\alpha^{(k_0)}_{i_0} \ne 0$. We assume that $m \ge i_0 + 1$. Put

$$\delta_k = \sharp\{j \mid (i,k) < (j,k+1)\} - \sharp\{j \mid (i,k) < (j,k)\}, \tag{2.5.1}$$

which is independent of the choice of i. We define a function $I_i^{(k)}(x;t)$ attached to $\boldsymbol{\alpha}$ for $0 \le k \le e-1, 1 \le i \le m_k$ by

$$(2.5.2)\quad I_i^{(k)}(x;t) = \begin{cases} \displaystyle\prod_{\substack{1\le j\le m_{k+1}-\delta_k \\ (i,k)<(j,k+1)}} (x_i^{(k)} - tx_j^{(k+1)}) & \text{if } (i,k) \le \nu_0, \\ \displaystyle\prod_{\substack{1\le j\le m_k \\ (i,k)<(j,k)}} (x_i^{(k)} - tx_j^{(k)}) & \text{if } (i,k) > \nu_0. \end{cases}$$

For $\boldsymbol{\alpha} \in Z_n^{0,0}$, we define a polynomial $v_{\boldsymbol{\alpha}}(t)$ by

$$v_{\boldsymbol{\alpha}}(t) = \prod_{k=0}^{e-1} v_{\mu_k}(t),$$

where $\mu_k = \sharp\{j \mid (j,k) > \nu_0\}$ and

$$v_r(t) = \prod_{i=1}^{r} \frac{1-t^i}{1-t}$$

for each $r \ge 1$. For a sequence $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{m_k})$, we write as

$$(x^{(k)})^{\beta} = (x_1^{(k)})^{\beta_1}(x_2^{(k)})^{\beta_2}\cdots(x_{m_k}^{(k)})^{\beta_{m_k}}.$$

Finally, put $\mathfrak{S}_{\mathbf{m}} = \mathfrak{S}_{m_0} \times \mathfrak{S}_{m_1}$ as before. We now define a function $R_{\boldsymbol{\alpha}}(x;t)$ associated to $\boldsymbol{\alpha}$ by

(2.5.3)

$$R_{\boldsymbol{\alpha}}(x;t) = v_{\boldsymbol{\alpha}}(t)^{-1} \times$$
$$\sum_{w\in\mathfrak{S}_{\mathbf{m}}} w\Big\{\prod_k (x^{(k)})^{\alpha^{(k)}} \prod_{k,i} I_{i,\pm}^{(k)}(x;t) \Big/ \prod_k \prod_{(i,k)<(j,k)} (x_i^{(k)} - x_j^{(k)})\Big\}.$$

Since $R_{\boldsymbol{\alpha}}$ can be expressed as

$$R_{\boldsymbol{\alpha}}(x;t) = v_{\boldsymbol{\alpha}}(t)^{-1} \prod_k \prod_{(i,k)<(j,k)} (x_i^{(k)} - x_j^{(k)})^{-1} \times$$
$$\sum_{w\in\mathfrak{S}_{\mathbf{m}}} \varepsilon(w) w\Big\{\prod_k (x^{(k)})^{\alpha^{(k)}} \prod_{k,i} I_{i,\pm}^{(k)}(x;t)\Big\},$$

it is a polynomial in x.

Remark 2.6. $R_{\boldsymbol{\alpha}}^{\pm}$ was defined in [S1, (3.1.2)] by using a slightly different formula. But the function defined there is a Laurent polynomial in general, and not necessarily a polynomial in x. So, it should be modified to the above form. The results in [S1, 3] remain valid for this $R^{\pm}_{\boldsymbol{\alpha}}$ under an appropriate modification.

2.7. We regard $\mathfrak{S}_{\mu_k}$ as a subgroup of $\mathfrak{S}_{m_k}$ as a permutation group with respect to the letters $\{1 \le j \le m_k \mid (j,k) > \nu_0\}$. In this way, we regard $\mathfrak{S}^0_{\boldsymbol{\alpha}} = \mathfrak{S}_{\mu_0} \times \mathfrak{S}_{\mu_1}$ as a subgroup of $\mathfrak{S}_{\mathbf{m}}$. It is shown in [S1, (3.2.1)] that $R_{\boldsymbol{\alpha}}(x;t)$ can be expressed in the following form also.

(2.7.1)

$$R_{\boldsymbol{\alpha}}(x;t) =$$

$$\sum_{w \in \mathfrak{S}_{\mathbf{m}}/\mathfrak{S}^0_{\boldsymbol{\alpha}}} w\Biggl\{\prod_k (x^{(k)})^{\alpha^{(k)}} \prod_{\substack{k,i \\ (i,k)\le\nu_0}} I^{(k)}_{i,\pm}(x;t) \Big/ \prod_k \prod_{\substack{(i,k)<(j,k) \\ (i,k)\le\nu_0}} (x^{(k)}_i - x^{(k)}_j)\Biggr\}.$$

We denote by $\mathfrak{S}_{\boldsymbol{\alpha}}$ the stabilizer of $\boldsymbol{\alpha}$ in $\mathfrak{S}_{\mathbf{m}}$. Let us define a function $n : \mathfrak{S}_{\boldsymbol{\alpha}} \to \mathbb{Z}_{\ge 0}$ by

$$n(w) = \sharp\{(\nu,\nu') \in \mathcal{M}^2 \mid \nu < \nu', w^{-1}(\nu) > w^{-1}(\nu'), b(\nu') \ne b(\nu)\},$$

and define a polynomial $b_{\boldsymbol{\alpha}}(t)$ by

$$b_{\boldsymbol{\alpha}}(t) = \sum_{w \in \mathfrak{S}_{\boldsymbol{\alpha}}} \varepsilon(w)(-t)^{n(w)}.$$

The following result gives an explicit description of Hall-Littlewood functions $Q_{\boldsymbol{\Lambda}}$ and $P_{\boldsymbol{\Lambda}}$.

Theorem 2.8. *Assume that $e = 2$. The for each $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha}) \in Z^{\infty}_n$, we have*

$$Q_{\boldsymbol{\Lambda}}(x;t) = R_{\boldsymbol{\alpha}}(x;t), \qquad P_{\boldsymbol{\Lambda}}(x;t) = b_{\boldsymbol{\alpha}}(t)^{-1} R_{\boldsymbol{\alpha}}(x;t).$$

2.9. The theorem will be proved in 2.12 after some preliminaries. We define an operator R_{ij} on the set $\mathbb{Z}^M$ by $R_{ij}(\lambda) = \lambda'$, where if $\lambda = (\lambda_1, \dots, \lambda_M) \in \mathbb{Z}^M$, then $\lambda' \in \mathbb{Z}^M$ is given by

$$\lambda'_i = \lambda_i + 1, \qquad \lambda'_j = \lambda_j - 1$$

and $\lambda'_l = \lambda_l$ for $l \ne i,j$. A raising operator (resp. a lowering operator) R on $\mathbb{Z}^M$ is defined as a product of various R_{ij} with $i < j$ (resp. $i > j$). In the following, we identify $\mathcal{M}$ with the set $\{1, 2, \dots, M\}$ via the total order on $\mathcal{M}$ and express the operator R_{ij} as $R_{\nu,\nu'}$ for $\nu, \nu' \in \mathcal{M}$. We define the action of raising operators R on $q_{\boldsymbol{\alpha},\pm}$ by $R(q_{\boldsymbol{\alpha},\pm}) = q_{R(\boldsymbol{\alpha}),\pm}$. Note that $q_{\boldsymbol{\beta},\pm}$ makes sense for $\boldsymbol{\beta} \in \mathbb{Z}^M_{\geq 0}$, and we regard $q_{\boldsymbol{\beta},\pm} = 0$ if $\boldsymbol{\beta}$ contains a negative factor. For $\nu = (i,k) \in \mathcal{M}$, we put $b(\nu) = k$. By [S1, Cor. 3.7], $R_{\boldsymbol{\alpha}}$ can be expressed in terms of raising operators as follows.

$$\text{(2.9.1)} \qquad R_{\boldsymbol{\alpha}} = \Big\{\prod_{(*)}(1 - R_{\nu\nu'})\Big/\prod_{(**)}(1 - tR_{\nu\nu'})\Big\}q_{\boldsymbol{\alpha},\pm},$$

where the conditions (*) and (**) are given by

(*) $\qquad \nu = (i,k), \nu' = (j,k), \nu < \nu'$,

(**) $\qquad \nu = (i,k), \nu' = (j,k+1), \nu < \nu', 1 \leq j \leq m_{k+1} - \delta_k$.

Using this, we have

Lemma 2.10. *$R_{\boldsymbol{\alpha}}(x;t)$ can be expressed as*

$$R_{\boldsymbol{\alpha}}(x;t) = q_{\boldsymbol{\alpha}}(x;t) + \sum_{\boldsymbol{\beta} > \boldsymbol{\alpha}} d_{\boldsymbol{\beta}}(t) q_{\boldsymbol{\beta}}(x;t)$$

with $d_{\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$ for $\boldsymbol{\beta} \in Z_n^{0,0}$.

Proof. By (2.9.1), $R_{\boldsymbol{\alpha}}(x;t)$ can be written as a $\mathbb{Z}[t]$-linear combination of $R(q_{\boldsymbol{\alpha}})$ by various raising operators R. It is known (e.g. [M, I]) that $R(\boldsymbol{\alpha}) \geq \boldsymbol{\alpha}$ for a raising operator R and $\boldsymbol{\alpha} \in Z_n^{0,0} \subset \mathbb{Z}^M$. If $R(\boldsymbol{\alpha})$ is not an e-partition, we must replace $R(\boldsymbol{\alpha})$ by $\boldsymbol{\beta} \in Z_n^{0,0}$ by permuting the entries of $R(\boldsymbol{\alpha})$. But then $\boldsymbol{\beta} \geq R(\boldsymbol{\alpha})$, and so we can write $R(q_{\boldsymbol{\alpha}}) = q_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta} \in Z_n^{0,0}$ such that $\boldsymbol{\beta} \geq \boldsymbol{\alpha}$. It is clear that $\boldsymbol{\beta} = \boldsymbol{\alpha}$ if and only if $R = 1$. The lemma follows from this. □

Next we show that

Lemma 2.11. *$R_{\boldsymbol{\alpha}}(x;t)$ can be expressed as*

$$R_{\boldsymbol{\alpha}}(x;t) = \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) s_{\boldsymbol{\beta}}(x),$$

where $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in t\mathbb{Z}[t]$ for $\boldsymbol{\alpha} \neq \boldsymbol{\beta} \in Z_n^{0,0}$, and $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(t) = b_{\boldsymbol{\alpha}}(t)$. Moreover $b_{\boldsymbol{\alpha}}(0) = 1$.

Proof. We shall prove the lemma by using a similar argument as in the case of usual Hall-Littlewood functions ([M, III, 1]). First note that the definition of Schur functions $s_{\boldsymbol{\alpha}}$ given in 1.5 can be extended to the case where $\boldsymbol{\alpha}$ is not necessary an e-partition. If $\alpha_j^{(k)} + (m_k - j)$ are positive and all distinct for $j = 1, \dots, m_k$ (for a fixed k), then $s_{\boldsymbol{\alpha}}$

coincides with the usual Schur function $s_{\boldsymbol{\beta}}(x)$ up to sign, where $\boldsymbol{\beta} = (\beta_j^{(k)})$ is obtained by permuting the sequence $\{\alpha_j^{(k)} + (m_k - j)\}$ (for a fixed k) in the decreasing order and by writing it as $\{\beta_j^{(k)} + (m_k - j)\}$. If $\alpha_j^{(k)} + (m_k - j)$ are not all distinct for a fixed k, then $s_{\boldsymbol{\alpha}} = 0$.

In the description of $R_{\boldsymbol{\alpha}}$ by (2.7.1), the product $\prod_{i,k} I_i^{(k)}(x;t)$ gives a contribution

$$\prod_{\nu_0 \geq \nu, \nu' > \nu} (x_i^{(k)})^{r_{\nu,\nu'}} (-t x_j^{(k+1)})^{r_{\nu',\nu}}$$

where $\nu = (i,k), \nu' = (j,k+1)$. Here $(r_{\nu,\nu'})$ is an integral matrix indexed by $\mathcal{M}$ consisting of 0 and 1 satisfying the relation

$$r_{\nu,\nu'} + r_{\nu',\nu} = \begin{cases} 1 & \text{if } \nu \leq \nu_0, \nu' \in \mathcal{M}_\nu \\ 0 & \text{otherwise.} \end{cases} \tag{2.11.1}$$

where for each $\nu = (i,k)$, $\mathcal{M}_\nu$ is defined by

$$\mathcal{M}_\nu = \{\nu' = (j, k+1) \mid \nu < \nu', 1 \leq j \leq m_{k+1} - \delta_k\}.$$

Put, for a fixed choice of the matrix $(r_{\nu,\nu'})$ as above,

$$\lambda_i^{(k)} = \alpha_i^{(k)} + \sum_{\nu' \in \mathcal{M}} r_{\nu,\nu'} \tag{2.11.2}$$

for $\nu = (i,k)$. Then the e-composition $\boldsymbol{\lambda} = (\lambda_i^{(k)})$ yields the "Schur function" $a_{\boldsymbol{\lambda}}/a_{\boldsymbol{\delta}}$, where $a_{\boldsymbol{\lambda}} = \sum_{w \in \mathfrak{S}_{\mathbf{m}}} \varepsilon(w) w(x^{\boldsymbol{\lambda}})$ and $\boldsymbol{\delta} = (\delta^{(0)}, \delta^{(1)})$ with $\delta^{(k)} = (m_k - 1, \dots, 1, 0)$. $R_{\boldsymbol{\alpha}}$ can be written as a $\mathbb{Z}$-linear combination of $(-t)^d a_{\boldsymbol{\lambda}}/a_{\boldsymbol{\delta}}$ attached to various matrices $(r_{\nu,\nu'})$, where $d = \sum_{\nu < \nu'} r_{\nu',\nu}$.

Now $a_{\boldsymbol{\lambda}}(x) = 0$ if the composition $\lambda^{(k)}$ is not all distinct for some k. Hence we may assume that all the entries of $\lambda^{(k)}$ are distinct. Then by rearranging its entries in the descending order, we can write as

$$\lambda_{w_k(i)}^{(k)} = \beta_i^{(k)} + (m_k - i) \qquad (1 \leq j \leq m_k)$$

with some $w_k \in \mathfrak{S}_{m_k}$ for $0 \leq k \leq 1$. Then $\boldsymbol{\beta} = (\beta_i^{(k)}) \in Z_n^{0,0}$ and $a_{\boldsymbol{\lambda}}/a_{\boldsymbol{\delta}}$ coincides with $\varepsilon(w) s_{\boldsymbol{\beta}}$. Thus $R_{\boldsymbol{\alpha}}$ is written as a sum of $\varepsilon(w)(-t)^d s_{\boldsymbol{\beta}}$ for such $\boldsymbol{\beta}$. We shall show that

$$\boldsymbol{\beta} \leq \boldsymbol{\alpha} \tag{2.11.3}$$

Let us define a matrix $(s_{\nu,\nu'})$ by $s_{\nu,\nu'} = r_{w(\nu),w(\nu')}$, where $w(\nu) = (w_k(i), k)$ for $\nu = (i,k) \in \mathcal{M}$. Hence $w(\nu) \in \mathcal{M}$, and the matrix $(s_{\nu,\nu'})$ satisfies a similar condition as in (2.11.1). We can write

$$\beta_i^{(k)} + (m_k - i) = \alpha_{w_k(i)}^{(k)} + \sum_{\nu' \in \mathcal{M}} s_{\nu,\nu'} \tag{2.11.4}$$

We want to show that

$$\sum_{k=0}^{1}\sum_{i=1}^{t} \beta_i^{(k)} + \sum_{k=0}^{p} \beta_{t+1}^{(k)} \le \sum_{k=0}^{1}\sum_{i=1}^{t} \alpha_{w_k(i)}^{(k)} + \sum_{k=0}^{p} \alpha_{w_k(t+1)}^{(k)} \tag{2.11.5}$$

for $0 \le p \le 1$ and $1 \le t \le m$. Note that (2.11.5) implies (2.11.3) since $w(\boldsymbol{\alpha}) \le \boldsymbol{\alpha}$ for any $w \in \mathfrak{S}_{\mathbf{m}}$. Now by (2.11.4) we have

$$\begin{aligned}
\sum_{k=0}^{1}\sum_{i=1}^{t} \beta_i^{(k)} &+ \sum_{k=0}^{p} \beta_{t+1}^{(k)} \\
&= \sum_{k=0}^{1}\sum_{i=1}^{t} \alpha_{w_k(i)}^{(k)} + \sum_{k=0}^{p} \alpha_{w_k(t+1)}^{(k)} \\
&- \sum_{k=0}^{1}\sum_{i=1}^{t} (m_k - i) - \sum_{k=0}^{p} (m_k - (t+1)) \\
&+ \sum_{\nu \in \mathcal{B}, \nu' \in \mathcal{M}} s_{\nu,\nu'}
\end{aligned}$$

where

$$\mathcal{B} = \{(i,k) \mid 1 \le i \le t, 0 \le k \le 1\} \cup \{(t+1,k) \mid 0 \le k \le p\}.$$

Hence, in order to show (2.11.5), it is enough to see that

$$\begin{aligned}
\sum_{\nu \in \mathcal{B}, \nu' \in \mathcal{M}} s_{\nu,\nu'} &\le \sum_{k=0}^{1}\sum_{i=1}^{t} (m_k - i) + \sum_{k=0}^{p} (m_k - (t+1)) \\
&= tM + \sum_{k=0}^{p} m_k - (t+1)(t+1+p),
\end{aligned} \tag{2.11.6}$$

where $M = m_0 + m_1$ as before. First we note that

$$\delta_k = \begin{cases} m_1 - m_0 + 1 & \text{if } k = 0, \\ m_0 - m_1 & \text{if } k = 1. \end{cases} \tag{2.11.7}$$

One can write

$$\sum_{\nu\in\mathcal{B},\nu'\in\mathcal{M}} s_{\nu,\nu'} = \sum_{\nu,\nu'\in\mathcal{B}} s_{\nu,\nu'} + \sum_{\nu\in\mathcal{B},\nu'\in\mathcal{M}-\mathcal{B}} s_{\nu,\nu'}. \tag{2.11.8}$$

We shall compute the right hand side of (2.11.8). On the one hand, we have

$$\begin{aligned}\sum_{\nu,\nu'\in\mathcal{B}} s_{\nu,\nu'} &= \sharp\{(\nu,\nu')\in\mathcal{B}^2 \mid b(\nu)=0, b(\nu')=1\} \\ &= (t+1)(t+p)\end{aligned} \tag{2.11.9}$$

by (2.11.1). On the other hand, if $p=0$, we have

(2.11.10)
$$\begin{aligned}\sum_{\nu\in\mathcal{B},\nu'\in\mathcal{M}-\mathcal{B}} s_{n,\nu'} &\le (m_1-\delta_0-t)(t+1) + (m_0-\delta_1-(t+1))t \\ &= tM + m_0 - (t+1)(2t+1)\end{aligned}$$

by (2.11.7). Then it is easy to see that the sum of (2.11.9) and the right hand side of (2.11.10) coincides with the right hand side of (2.11.6). If $p=1$, we have

$$\begin{aligned}\sum_{\nu\in\mathcal{B},\nu'\in\mathcal{M}-\mathcal{B}} s_{\nu,\nu'} &\le \sum_{k=0}^{1}(m_k-\delta_{k-1}-(t+1))(t+1) \\ &= (t+1)M-(t+1)-2(t+1)^2,\end{aligned} \tag{2.11.11}$$

and again the sum of (2.11.9) and the right hand side of (2.11.11) coincides with the right hand side of (2.11.6). Hence (2.11.6) holds and we have proved (2.11.3).

The above computation shows that $\boldsymbol{\beta}=\boldsymbol{\alpha}$ if and only if $w=(w_0,w_1)\in\mathfrak{S}_{\boldsymbol{\alpha}}$ and that $s_{\nu,\nu'}=1$ for all $\nu<\nu'$ such that $b(\nu')\ne b(\nu)$. Then we have $d=n(w)$ since

$$d = \sum_{\nu<\nu'} r_{\nu',\nu} = \sum_{\nu<\nu'} s_{w^{-1}(\nu'),w^{-1}(\nu)},$$

and $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(t)$ is given as $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(t)=\sum_{w\in\mathfrak{S}_{\boldsymbol{\alpha}}}\varepsilon(w)(-t)^d = b_{\boldsymbol{\alpha}}(t)$. Now it is easily checked by the definition that $R_{\boldsymbol{\alpha}}(x;0)=s_{\boldsymbol{\alpha}}(x)$ (see [S1, (3.13.2)]). This implies that $w_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(0)=b_{\boldsymbol{\alpha}}(0)=1$ and that $w_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)\in t\mathbb{Z}[t]$ for $\boldsymbol{\beta}\ne\boldsymbol{\alpha}$. The lemma is now proved. □

2.12. We are now ready to prove Theorem 2.8. By Lemma 2.10 and Lemma 2.11, $R_{\boldsymbol{\alpha}}(x;t)$ satisfies the condition in Theorem 2.4 for $Q_{\boldsymbol{\alpha}}(x;t)$. Also $(b_{\boldsymbol{\alpha}})^{-1}R_{\boldsymbol{\alpha}}(x;t)$ satisfies the condition for $P_{\boldsymbol{\alpha}}(x;t)$. Hence Theorem 2.8 holds.

Remarks 2.13. (i) Theorem 2.8 together with Lemma 2.10 implies that $Q_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$. It is shown in the next section that $P_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$ also.

(ii) It is known by [S1, (3.13.1)] that the expansion of $R^{\pm}_{\boldsymbol{\alpha}}$ by Schur functions has an interpretation in terms of lowering operators. Hence in view of Theorem 2.8, we have

$$Q_{\boldsymbol{\Lambda}} = v_{\boldsymbol{\alpha}}(t)^{-1} \prod_{\substack{\nu<\nu',\nu\le\nu_0 \\ b(\nu')\ne b(\nu)}} (1 - tR_{\nu'\nu}) \prod_{\substack{\nu<\nu',\nu>\nu_0 \\ b(\nu')=b(\nu)}} (1 - tR_{\nu'\nu}) s_{\boldsymbol{\alpha}}$$

for $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$, where $R(s_{\boldsymbol{\alpha}})$ is defined as $a_{R(\boldsymbol{\alpha}+\boldsymbol{\delta})}/a_{\boldsymbol{\delta}}$ for a lowering operator R.

(iii) Lemma 2.11 (i.e. the property that $\boldsymbol{\beta} \le \boldsymbol{\alpha}$) does not hold in general for $R^{\pm}_{\boldsymbol{\alpha}}$ if $e \ge 3$. For example, assume that $e = 3$, and consdier $W = G(3,1,2)$. Then for $\boldsymbol{\alpha} = (1^2;-;-) \in \mathcal{P}_{2,3}$, we have

$$R^{+}_{\boldsymbol{\alpha}} = s_{(1^2;-;-)} - t^2 s_{(1;-;1)} - t^2 s_{(-;1^2;-)} - t^3 s_{(-;1;1)},$$

and $(1;-;1) > (1^2;-;-) = \boldsymbol{\alpha}$. Hence $R^{+}_{\boldsymbol{\alpha}}$ does note coincide with $Q^{+}_{\boldsymbol{\Lambda}}$ in this case.

§3. Green functions attached to limit symbols

3.1. Although we shall treat the case where $e = 2$ in later discussions, first we review some results from [S1, 5] for general e. Let us define $K_{\pm}(t) = (K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t))$ as the transition matrix $M(s, P^{\pm})$ between the basis $s = \{s_{\boldsymbol{\alpha}}(x)\}$ of Schur functions and the basis $P^{\pm} = \{P^{\pm}_{\boldsymbol{\Lambda}}(x;t)\}$ of Hall-Littlewood functions in $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$, i.e.,

$$s_{\boldsymbol{\alpha}}(x) = \sum_{\boldsymbol{\beta}} K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) P^{\pm}_{\boldsymbol{\Lambda}(\boldsymbol{\beta})}(x;t). \tag{3.1.1}$$

We fix a total order on $Z_n^{0,0} \simeq \mathcal{P}_{n,e}$ which is compatible with the partial order $\boldsymbol{\beta} \le \boldsymbol{\alpha}$ on it. Then $K_{\pm}(t)$ is a lower unitriangular matrix with entries $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Q}(t)$, and $K_{\pm}(0)$ is the identity matrix. We define the matrix $\widetilde{K}_{\pm}(t) = (\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t))$ by

$$\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) = t^{n(\boldsymbol{\beta})} K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t^{-1}),$$

where $n(\boldsymbol{\beta}) = a(\boldsymbol{\Lambda}(\boldsymbol{\beta}))$ is the function given in (1.4.2). $K^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ (resp. $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$) are called Kostka functions (resp. modified Kostka functions). Green functions are defined as a linear combination of modified Kostka functions. The determination of Green functions is equivalent to that of Kostka functions once we know the character table of W. Here we concentrate ourselves to (modified) Kostka functions rather than Green functions themselves.

Let

$$\Omega(x,y;t) = \prod_{k=0}^{e-1}\prod_{i,j}\frac{1-tx_i^{(k)}y_j^{(k+1)}}{1-x_i^{(k)}y_j^{(k)}}. \tag{3.1.2}$$

By Corollary 4.6 in [S1], combined with (2.3.4), $\Omega(x,y;t)$ has the following expansion in terms of Hall-Littlewood functions.

$$\begin{aligned}\Omega(x,y;t) &= \sum_{\boldsymbol{\alpha}} b^-_{\boldsymbol{\alpha}}(t)P^+_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(x;t)P^-_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(y;t)\\ &= \sum_{\boldsymbol{\alpha}} P^+_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(x;t)Q^-_{\boldsymbol{\Lambda}(\boldsymbol{\alpha})}(y;t),\end{aligned} \tag{3.1.3}$$

where $\boldsymbol{\alpha}$ runs over e-partitions of any size. Let N^* be the number of complex reflections in W. We define a polynomial $\mathbb{G}(t) \in \mathbb{Z}[t]$ by $\mathbb{G}(t) = (t-1)^n t^{N^*}P_W(t)$, where $P_W(t)$ is the Poincaré polynomial associated to W (see. [S1, 1.1]). We denote by $\Lambda'(t)$ the diagonal matrix indexed by $Z_n^{0,0}$, whose $\boldsymbol{\alpha}\boldsymbol{\alpha}$-entry is given by $b^+_{\boldsymbol{\alpha}}(t^{-1})$. We put $\widetilde{\Lambda}(t) = t^{-n}\mathbb{G}(t)\Lambda'(t)$. Let $\Omega' = (\omega'_{{}_Ba,\boldsymbol{\beta}})$ be the matrix defined by

$$\omega'_{\boldsymbol{\alpha},\boldsymbol{\beta}} = t^{N^*}R(\chi^{\boldsymbol{\alpha}}\otimes\overline{\chi^{\boldsymbol{\beta}}}\otimes\overline{\det}_V),$$

where $\chi^{\boldsymbol{\alpha}}$ is the irreducible character of W associated to $\boldsymbol{\alpha}$. In general, we denote by $R(f)$, for a class function f of W, the graded multiplicity of f in the coinvariant algebra $R = \oplus R_i$ of W, i.e.,

$$R(f) = \sum_{i\geq 0}\langle f, R_i\rangle_W\, t^i \tag{3.1.4}$$

(see [S1, 1.1]). Then it is known by [S1, Th. 5.4] that $\widetilde{K}^{\pm}(t)$ and $\widetilde{\Lambda}(t)$ are determined as a unique solution for the following matrix equation.

$$\widetilde{K}_-(t)\widetilde{\Lambda}(t)\,{}^t\widetilde{K}_+(t) = \Omega'. \tag{3.1.5}$$

Remark 3.2. Let $\Omega = (\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ be the matrix defined by

$$\omega_{\boldsymbol{\alpha},\boldsymbol{\beta}} = t^{N^*}R(\chi^{\boldsymbol{\alpha}}\otimes\chi^{\boldsymbol{\beta}}\otimes\overline{\det}_V).$$

In [S1, 1.4, 1.5] it is shown that the equation $P'\Lambda'\,{}^tP'' = \Omega'$ such as (3.1.5) is equivalent to the equation $P\Lambda\,{}^tP = \Omega$ with $P' = P, P'' = \sigma P\sigma$, where σ is a permutation matrix arising from the complex conjugates of irreducible characters of W. Although it is not written explicitly there, we note that this equivalence works only when $\mathbf{m} = (m+1, m, \dots, m)$ and $s_1 = \cdots = s_{e-1}$ for $\mathbf{s} = (s_0, \dots, s_{e-1})$ in 1.1. So a simple relation between $\widetilde{K}_+(t)$ and $\widetilde{K}_-(t)$ as given in [S1, 1.5] can not be found in our situation.

We now restrict ourselves to the case where $e = 2$, and write $\widetilde{K}_\pm(t)$, etc. as $\widetilde{K}(t)$, etc. as before by omitting the signature $\pm$. The following fact holds.

Proposition 3.3. *Assume that $e = 2$. Then $K_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$, which is a monic of degree $n(\boldsymbol{\beta}) - n(\boldsymbol{\alpha})$, and so $\widetilde{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t) \in \mathbb{Z}[t]$. Moreover, we have $P_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$.*

Proof. We remark that $q = \{q_{\boldsymbol{\alpha},\pm}(x;t)\}$ and $m = \{m_{\boldsymbol{\alpha}}(x)\}$, $Q = \{Q_{\boldsymbol{\Lambda}}(x;t)\}$ and $P = \{P(x;t)\}$ are dual bases of each other with respect to the scalar product $\langle\, , \rangle$ on $\mathbb{Q}(t) \otimes_{\mathbb{Z}} \Xi^n$. It follows that

$$M(Q, q) = M(P, m)^* = (K(t)^{-1}K)^* = {}^tK(t)K^*,$$

where $K = M(s, m)$ is the Kostka matrix, and K^* denotes the transposed inverse of K. In view of Lemma 2.10 and Theorem 2.8, $M(Q, q)$ are the matrices with entries in $\mathbb{Z}[t]$. Since K is a matrix with entries in $\mathbb{Z}$, we see that $K(t)$ is a matrix with entries in $\mathbb{Z}[t]$. Since $K(t)$ is unitriangular, $K(t)^{-1}$ is also a matrix with entries in $\mathbb{Z}[t]$. This implies that $P_{\boldsymbol{\Lambda}}(x;t) \in \mathbb{Z}[x;t]$.

It remains to show the formula for $\deg K_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. The following argument is similar to [S1, Cor. 6.8]. By [S1, (6.7.3)], K^* coincides with the matrix of the operator $\prod_{\nu<\nu'}\prod_{b(\nu')=b(\nu)}(1 - R_{\nu,\nu'})$. This fact together with (2.9.1) implies, by a similar argument as in [M, III, (6.3)], that $K_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ is the coefficient of $s_{\boldsymbol{\alpha}}$ in

$$\begin{aligned}(3.3.1)\qquad &\prod_{\nu<\nu'}\prod_{b(\nu')\neq b(\nu)}(1 - tR_{\nu,\nu'})^{-1}s_{\boldsymbol{\beta}} \\ &= \prod_{\nu<\nu'}\prod_{b(\nu')\neq b(\nu)}(1 + tR_{\nu,\nu'} + t^2R_{\nu,\nu'}^2 + \cdots)s_{\boldsymbol{\beta}}.\end{aligned}$$

Let $\varepsilon_1, \dots, \varepsilon_M$ be the standard basis of $\mathbb{Z}^M$. We denote by R^+ the set of positive roots of type A_{M-1}, i.e., $R^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le M\}$. For any $\xi = (\xi_1, \dots, \xi_M) \in \mathbb{Z}^M$ such that $\sum \xi_i = 0$, we define a polynomial $P(\xi;t)$ in t by

$$P(\xi;t) = \sum_{(m_\gamma)} t^{\sum m_\gamma},$$

where (m_γ) runs over all the choices such that $\xi = \sum_{\gamma\in R^+} m_\gamma\gamma$ with $m_\gamma \in \mathbb{Z}_{\geq 0}$. We also define $P^*(\xi;t)$ by a similar formula as $P(\xi;t)$, but this time, (m_γ) runs over only the expression such that $\xi = \sum_\gamma m_\gamma\gamma$ and that $\gamma = \varepsilon_i - \varepsilon_j$ corresponds to the raising operator R_{ν_i,ν_j} occurring in the expression in (3.3.1). Then $P(\xi;t)$ is non-zero only when $\xi = \sum \eta_i(\varepsilon_i - \varepsilon_{i+1})$ with $\eta_i \geq 0$, and in that case, $P(\xi;t)$ is a monic of degree $\sum \eta_i = \langle \xi, \delta\rangle$. (See [M, III, 6, Ex.4], here $\delta = (M,\dots,1,0) \in \mathbb{Z}^M$ and $\langle\, ,\rangle$ denotes the standard inner product in $\mathbb{Z}^M$.) Clearly $\deg P^*(\xi;t) \leq \deg P(\xi;t)$ and since the choice $(m_\gamma) = (\eta_i)$ is allowed, we see that $\deg P^*(\xi;t) = \deg P(\xi;t)$.

Hence, by a similar argument as in [loc. cit.], we see that $K_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ coincides with

$$\sum_{w\in\mathfrak{S}_{\mathbf{m}}} \varepsilon(w)P^*(w^{-1}(\boldsymbol{\alpha}+\delta) - (\boldsymbol{\beta}+\delta);t),$$

where $\boldsymbol{\alpha}+\delta, \boldsymbol{\beta}+\delta$ are sums as elements in $\mathbb{Z}^M$. We have

$$\begin{aligned}\langle w^{-1}(\boldsymbol{\alpha}+\delta) - (\boldsymbol{\beta}+\delta), \delta\rangle &= \langle \boldsymbol{\alpha}+\delta, w(\delta)\rangle - \langle \boldsymbol{\beta}+\delta,\delta\rangle \\ &\leq \langle \boldsymbol{\alpha}+\delta,\delta\rangle - \langle\boldsymbol{\beta}+\delta,\delta\rangle \\ &= n(\boldsymbol{\beta}) - n(\boldsymbol{\alpha}).\end{aligned}$$

The equality holds only when $w = 1$. This proves the proposition.

□

We shall now compute certain values of $\widetilde{K}^{\pm}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$. The following fact holds for any $e \geq 1$.

Proposition 3.4. *Let $\boldsymbol{\beta}_0 = (-;\cdots;-;1^n)$ be the smallest element in $Z_n^{0,0}$. (Hence $\chi^{\boldsymbol{\beta}_0}$ coincides with the character $\overline{\det}_V$ of W.) Then we have*

$$\widetilde{K}^{-}_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t) = R(\chi^{\boldsymbol{\alpha}}), \qquad \widetilde{K}^{+}_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t) = R(\overline{\chi}^{\boldsymbol{\alpha}} \otimes (\overline{\det}_V)^2).$$

(See (3.1.4) for the definition of $R(\ )$. Note that $R(\chi^{\boldsymbol{\alpha}})$ coincides with the fake degree of $\boldsymbol{\alpha}$).

Proof. Although the argument is similar to, and much simpler than the proof of Lemma 7.2 in [S1], we give it below for the sake of completeness. We consider the equation (3.1.5). Let $\tilde{b}_{\boldsymbol{\beta}_0}$ be the first entry of the diagonal matrix $\widetilde{A}(t)$. Since $n(\boldsymbol{\beta}_0) = \sum_{i=1}^{n}(ei-1) = N^*$, the equation (3.1.5) implies that $\tilde{b}_{\boldsymbol{\beta}_0} = 1$. Again by (3.1.5), we have

$$\widetilde{K}^-_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t)\tilde{b}_{\boldsymbol{\beta}_0}t^{n(\boldsymbol{\beta}_0)} = \omega'_{\boldsymbol{\alpha},\boldsymbol{\beta}_0} = t^{N^*}R(\chi^{\boldsymbol{\alpha}}),$$

and so $\widetilde{K}^-_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t) = R(\chi^{\boldsymbol{\alpha}})$. Similarly, we have

$$t^{n(\boldsymbol{\beta}_0)}\tilde{b}_{\boldsymbol{\beta}_0}\widetilde{K}^+_{\boldsymbol{\alpha},\boldsymbol{\beta}_0}(t) = \omega'_{\boldsymbol{\beta}_0,\boldsymbol{\alpha}} = t^{N^*}R(\overline{\chi}^{\boldsymbol{\alpha}} \otimes (\overline{\det}_V)^2).$$

□

Next we shall show that

Proposition 3.5. *Assume that $e = 2$, and let $\boldsymbol{\beta}_1 = (n; -)$ be the largest element in $Z_n^{0,0}$. (Hence $\chi^{\boldsymbol{\beta}_1}$ is the unit character of W). Then for any $\boldsymbol{\alpha} \in Z_n^{0,0}$, we have*

$$K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t) = t^{n(\boldsymbol{\alpha})}.$$

In particular, we have $\widetilde{K}_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t) = 1$.

3.6. The proof of Proposition 3.5 will be done in 3.9. We consider the substitution of $\mathbf{t} = (1, t, t^2, \cdots)$ into the variables $y = \{y_j^{(k)} \mid 1 \le j \le m_k, 0 \le k \le 1\}$ by $y_j^{(k)} = t^{2(j-1)+k}$. Then we have

Lemma 3.7. *$R_{\boldsymbol{\alpha}}(y;t)|_{y=\mathbf{t}}$ is a polynomial in t of the form $t^{n(\boldsymbol{\alpha})}+$ higher degree terms.*

Proof. We consider $\mathfrak{S}_{\mathbf{m}} = \mathfrak{S}_{m_0} \times \mathfrak{S}_{m_1}$ as a subgroup of $\mathfrak{S}_M$ along the total order in $\mathcal{M}$ in 2.5. Suppose that $\nu_0 = (i_0, k_0) \in \mathcal{M}$ corresponds to a number b $(1 \le b \le M)$. We define a subset X of $\mathfrak{S}_{\mathbf{m}}$ as follows. If $m_0 = m_1 + 1$, put $X = \mathfrak{S}^0_{\boldsymbol{\alpha}}$ (see 2.7). If $m_0 = m_1$, put

$$X = \left\{ w = \begin{pmatrix} 1 & 2 & \cdots & b & \cdots & M \\ 2a+1 & 2a+2 & \cdots & 2a+b & * & 2a \end{pmatrix} \right.$$
$$\left. \Big| w \in \mathfrak{S}_{\mathbf{m}}, 1 \le 2a+b \le M \right\} \cup \mathfrak{S}^0_{\boldsymbol{\alpha}}.$$

Then it is easy to check by the definition (2.5.2) that

$$w\Big\{\prod_{i,k} I_i^{(k)}(y;t)\Big\}\Big|_{y=\mathbf{t}} = 0$$

unless $w \in X$. If $w \in X$, we see that

$$w\Big\{\prod_k (y^{(k)})^{a^{(k)}}\Big\}\Big|_{y=\mathbf{t}} = t^{n(w(\boldsymbol{\alpha}))}. \tag{3.7.1}$$

Moreover, $w(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ if $w \in \mathfrak{S}^0_{\boldsymbol{\alpha}}$. If $w \notin \mathfrak{S}^0_{\boldsymbol{\alpha}}$, then $w(\boldsymbol{\alpha}) < \boldsymbol{\alpha}$, and so $n(w(\boldsymbol{\alpha})) > n(\boldsymbol{\alpha})$. We also note that

$$w\Big\{\prod_{\substack{k,i\\(i,k)\le\nu_0}} I^{(k)}_{i,\pm}(x;t)\Big/\prod_k \prod_{\substack{(i,k)<(j,k)\\(i,k)\le\nu_0}} (x^{(k)}_i - x^{(k)}_j)\Big\}\Big|_{y=\mathbf{t}} = 1$$

for $w \in X$. It follows, by [M, III, 1.4], that

$$\sum_{w\in\mathfrak{S}^0_{\boldsymbol{\alpha}}} w\Big\{\prod_{k,i} I^{(k)}_i(y;t)\Big/\prod_k \prod_{(i,k)<(j,k)} (y^{(k)}_i - y^{(k)}_j)\Big\}\Big|_{y=\mathbf{t}} = v_{\boldsymbol{\alpha}}(t).$$

Then one can write as

$$R_{\boldsymbol{\alpha}}(y;t)\ |_{y=\mathbf{t}} = t^{n(\boldsymbol{\alpha})} + v_{\boldsymbol{\alpha}}(t)^{-1}A(t),$$

where

$$A(t) = \sum_{w\in X\setminus\mathfrak{S}^0_{\boldsymbol{\alpha}}} w\Big\{\prod_k (y^{(k)})^{\alpha^{(k)}}\prod_{k,i} I^{(k)}_i(y;t)\Big/\prod_k \prod_{(i,k)<(j,k)} (y^{(k)}_i - y^{(k)}_j)\Big\}\Big|_{y=\mathbf{t}}.$$

One can check that $A(t)$ has an expansion as a formal power series of t whose initial term is strictly bigger than $t^{n(\boldsymbol{\alpha})}$ by (3.7.1). Since $R_{\boldsymbol{\alpha}}(y;t)|_{y=\mathbf{t}}$ is a polynomial in t, $A(t)$ is a polynomial divisible by $v_{\boldsymbol{\alpha}}(t)$. This implies that $v_{\boldsymbol{\alpha}}(t)^{-1}A(t)$ is a polynomial in t whose lowest degree term is strictly bigger than $t^{n(\boldsymbol{\alpha})}$. The lemma is proved. □

We now consider the substitution of $\mathbf{t} = (1, t, t^2, \dots)$ into the infinitely many variables $y = \{y^{(k)}_j \mid j = 1, 2, \dots\}$. Then we have

Lemma 3.8. $\Omega(x, y; t)\ |_{y=\mathbf{t}} = \prod_j \dfrac{1}{1 - x^{(0)}_j}$.

Proof. We consider the second expression of $\Omega(x, y; t)$ in (3.1.2). For each $i, j \ge 1$, we have

$$\frac{1 - ty^{(0)}_j x^{(1)}_i}{1 - y^{(1)}_j x^{(1)}_i}\cdot\frac{1 - ty^{(1)}_j x^{(0)}_i}{1 - y^{(0)}_j x^{(0)}_i} = \frac{1 - t^{2j}x^{(0)}_i}{1 - t^{2(j-1)}x^{(0)}_i}$$

by substituting $y = \mathbf{t}$. It follows that

$$\begin{aligned}\Omega(x,y;t)|_{y=\mathbf{t}} &= \prod_j \left\{ \frac{1-t^2x_i^{(0)}}{1-x_i^{(0)}} \cdot \frac{1-t^4x_i^{(0)}}{1-t^2x_i^{(0)}} \cdots \right\} \\ &= \prod_j \frac{1}{1-x_i^{(0)}}.\end{aligned}$$

□

3.9. We shall prove Proposition 3.5. By substituting $y = \mathbf{t}$ in the both sides of (3.1.3), and by using Lemma 3.8, we have

$$\prod_j \frac{1}{1-x_j^{(0)}} = \sum_{\boldsymbol{\alpha}} P_{\boldsymbol{\Lambda}}(x;t) Q_{\boldsymbol{\Lambda}}(y;t) \,|_{y=\mathbf{t}},$$

where $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\alpha})$. By taking the degree n parts on both sides (cf. [M, I, (2.5)]),

$$h_n(x^{(0)}) = \sum_{|\boldsymbol{\alpha}|=n} Q_{\boldsymbol{\Lambda}}(y;t)|_{y=\mathbf{t}} P_{\boldsymbol{\Lambda}}(x;t), \tag{3.9.1}$$

where $h_n(x^{(0)})$ is a complete symmetric function of degree n with respect to the variables $x^{(0)}$. Since $h_n(x^{(0)}) = s_{(n)}(x^{(0)})$, we see that $h_n(x^{(0)})$ coincides with $s_{\boldsymbol{\beta}_1}(x)$. Comparing (3.9.1) with (3.1.1), we see that $K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t)$ is obtained as the limit of the polynomials $Q_{\boldsymbol{\Lambda}}(y;t)$ with finitely many variables $y = (y_j^{(k)})$ under the substitution $y = \mathbf{t}$. (Here the limit of $Q_{\boldsymbol{\Lambda}}(y;t)$ is taken in the sense of [S1, 3.15].) On the other hand, $Q_{\boldsymbol{\Lambda}} = R_{\boldsymbol{\alpha}}$ by Theorem 2.8. Hence by Lemma 3.7, we see that $K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t)$ is obtained as the limit of the polynomials of the form $t^{n(\boldsymbol{\alpha})}+$ higher terms. But Proposition 3.3 implies that $\deg K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}} = n(\boldsymbol{\alpha}) - n(\boldsymbol{\beta}_1) = n(\boldsymbol{\alpha})$. This shows that $K_{\boldsymbol{\beta}_1,\boldsymbol{\alpha}}(t) = t^{n(\boldsymbol{\alpha})}$, and Proposition 3.5 follows.

As a corollary, we have

Corollary 3.10. *Assume that $e = 2$. Let the $\boldsymbol{\alpha}\boldsymbol{\alpha}$-entry of the diagonal matrix $\widetilde{\Lambda}(t)$ in 3.1 be $\widetilde{b}_{\boldsymbol{\alpha}}(t)$. Then we have*

$$\sum_{\boldsymbol{\alpha} \in \mathcal{P}_{n,r}} \widetilde{b}_{\boldsymbol{\alpha}}(t) = t^{2N^*}.$$

Proof. The equation (3.1.4) together with Proposition 3.5 implies that

$$\begin{aligned}\sum_{\boldsymbol{\alpha}} \widetilde{b}_{\boldsymbol{\alpha}}(t) &= \omega_{\boldsymbol{\beta}_1,\boldsymbol{\beta}_1} \\ &= t^{N^*} R(\chi^{\boldsymbol{\beta}_1} \otimes \chi^{\boldsymbol{\beta}_1} \otimes \varepsilon) \\ &= t^{N^*} R(\varepsilon) \\ &= t^{2N^*}.\end{aligned}$$

□

Remark 3.11. Let $G(\mathbb{F}_q)$ be a (split) finite reductive group over a finite field of q elements and W its Weyl group. To each irreducible character χ of W, Green function $Q_\chi(q)$ of $G(\mathbb{F}_q)$ is associated by Deligne-Lusztig [DL]. They are determined as a solution of the matrix equation of the form $P\Lambda\,{}^tP = \Omega$. It is known that Λ is a block diagonal matrix, and the sum of the 11-entries of each block is equal to q^{2N^*}, which coincides with the number of unipotent elements in $G(\mathbb{F}_q)$. In the case of $GL_n(\mathbb{F}_q)$, the matrix Λ is a diagonal matrix indexed by partitions of n, and the $\lambda\lambda$-entry of Λ coincides with the number of elements in the unipotent class in $G(\mathbb{F}_q)$ corresponding to λ.

In [GM], Geck and Malle formulated a different matrix equation $P\Lambda\,{}^tP = \Omega$ for each $G(\mathbb{F}_q)$ by making use of parameter set of unipotent characters of $G(\mathbb{F}_q)$ instead of unipotent classes. They conjectured that the sum of 11-entries (which correspond to special characters of W) of each block of Λ is again equal to q^{2N^*}, and verified it in the case of exceptional groups.

In our situation, limit symbols are related neither to unipotent classes nor to unipotent characters. Corollary 3.10 shows that, even so, a similar fact holds in our case.

3.12. Here we give some examples of Green functions associated to limit symbols in the case where $e = 2$. Below is the tables of modified Kostka functions $\widetilde{K}(t) = (\widetilde{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t))$. In each of the tables, first column denotes double partitions $\boldsymbol{\beta} \in \mathcal{P}_{n,2}$, under the order compatible with the values of a functions.

Our Green functions associated to limit symbols are different from original Green functions associated to u-symbols, even in the case of Weyl groups of type B_n. We give below the table of modified Kostka functions associated to u-symbols in $W(B_2)$ which is related to the original Green functions of $SO_5(\mathbb{F}_q)$ for the sake of comparison.

Table 1. $\widetilde{K}(t)$ for $W(B_2)$

$(-;1^2)$	t^4				
$(1^2;-)$	t^2	t^2			
$(-;2)$	t^2		t^2		
$(1;1)$	t^3+t	t	t	t	
$(2;-)$	1	1	1	1	1

Table 2. $\widetilde{K}(t)$ for $W(B_3)$

$(-;1^3)$	t^9									
$(1^3;-)$	t^6	t^6								
$(-;21)$	t^7+t^5		t^5							
$(1;1^2)$	$t^8+t^6+t^4$	t^4	t^4	t^4						
$(-;3)$	t^3		t^3		t^3					
$(1^2;1)$	$t^7+t^5+t^3$	t^5+t^3	t^3	t^3		t^3				
$(1;2)$	$t^6+t^4+t^2$	t^2	t^4+t^2	t^2	t^2	t^2	t^2			
$(21;-)$	t^4+t^2	t^4+t^2	t^2	t^2		t^2		t^2		
$(2;1)$	t^5+t^3+t	t^3+t	t^3+t	t^3+t	t	t	t	t	t	
$(3;-)$	1	1	1	1	1	1	1	1	1	1

Table 3. $\widetilde{K}(t)$ for $W(B_2)$, the case of u-symbols

$(-;1^2)$	t^4				
$(1^2;-)$	t^2	t^2			
$(1;1)$	t^3+t	t	t		
$(-;2)$	t^2			t	
$(2;-)$	1	1	1		1

3.13. Let $\mathfrak{gl}_n$ be the Lie algebra of $GL_n(\mathbb{C})$, and $\mathfrak{t}$ the Cartan subalgebra of $\mathfrak{gl}_n$ consisting of diagonal matrices. Let $\mathfrak{o}_\lambda$ be the nilpotent orbit in $\mathfrak{gl}_n$ corresponding to a partition λ of n. We consider the scheme theoretic intersection $\mathfrak{t} \cap \bar{\mathfrak{o}}_\lambda$ of $\mathfrak{t}$ with the closure $\bar{\mathfrak{o}}_\lambda$ of $\mathfrak{o}_\lambda$. Then the coordinate ring $\mathbb{C}[\mathfrak{t}\cap\bar{\mathfrak{o}}_\lambda]$ is a finite dimensional $\mathbb{C}$-algebra, equipped with a structure of graded $\mathfrak{S}_n$-modules. We denote it by $R^\lambda = \bigoplus_i R_i^\lambda$. De Concini and Procesi [DP], and Tanisaki [T] showed that the polynomial

$$R^\beta(\chi^\alpha) = \sum_i \langle \chi^\alpha, R_i^\beta \rangle_{\mathfrak{S}_n} t^i$$

coincides with the modified Kostka polynomial $\widetilde{K}_{\alpha,\beta}(t)$ associated to $\mathfrak{S}_n$. R^λ is also interpreted as the quotient ring of $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$ by I_λ, where I_λ is the ideal generated by $p(x) \in \mathbb{C}[x]$ such that $p(\partial)f = 0$ for any $f \in V^\lambda$, (here V^λ is the Specht module of $\mathfrak{S}_n$ realized in $\mathbb{C}[x]$). Note that the map $\mathbb{C}[x] \to R^\lambda$ factors through the surjection $\mathbb{C}[x] \to R$ (R is the coinvariant algebra of $\mathfrak{S}_n$) and we have a surjective algebra homomorphism $R \to R^\lambda$.

This latter construction of R^λ makes sense even in the case of complex reflection groups $W = G(e, 1, n)$, and we get the graded W-module $R^{\boldsymbol{\beta}}$ for $\boldsymbol{\beta} \in \mathcal{P}_{n,e}$. One might expect that $R^{\boldsymbol{\beta}}(\chi^{\boldsymbol{\alpha}})$ coincides with our modified Kostka function $\widetilde{K}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(t)$ associated to limit symbols. (Note that this does not hold in the case of original Green functions of type B_n since the counter part of the map $R \to R^{\boldsymbol{\alpha}}$ for Green functions is no longer surjective). H.-F. Yamada [Y] has computed some examples of $R^{\boldsymbol{\beta}}(\chi^{\boldsymbol{\alpha}})$ for small rank cases, which supports our conjecture.

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Cells for a Hecke Algebra Representation

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Abstract.

If Y is an affine symmetric variety for the reductive group G with Weyl group W, there exists by Lusztig and Vogan a representation of the Hecke algebra of W in a module which has a basis indexed by the set Λ of pairs (v, ξ), where v is an orbit in Y of a Borel group B and ξ is a B-equivariant rank one local system on v. We introduce cells in Λ and associate with a cell a two-sided cell in W.

Introduction.

Let G be a connected reductive group over an algebraically closed field of characteristic $\neq 2$. Let θ be an automorphism of G of order 2, with fixed point group K. In [LV] Lusztig and Vogan introduced a module $\mathcal{M}$ over the Hecke algebra of the Weyl group W of G, coming from the action of K on the flag manifold $\mathcal{F}$ of G. In the present note we introduce cells for that situation. Instead of K-orbits on $\mathcal{F}$ we prefer to work with orbits of a Borel group B on the affine symmetric variety $Y = G/K$. The module $\mathcal{M}$ then has a basis indexed by the set of pairs $\Lambda = (v, \xi)$, where v is a B-orbit on Y and ξ a B-equivariant rank one local system on v.

After the introductory Section 1 we define the cells of Λ in Section 2, in much the same way as the cells of W. In Section 3 we attach to a cell in Λ a representation of Lusztig's asymptotic ring $\mathcal{J}$. We also attach in 3.5 (ii) to a cell in Λ a two-sided cell in W. The final Section 4 discusses some complements and also two examples for $G = SL_3$.

The results of this note about cells in Λ are more or less well-known. In the case $k = \mathbb{C}$ they can probably be extracted from the literature on representations of real reductive groups.

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The definition of cells is also given in [G, 3.1] and [H, no. 4]. In [G, loc. cit.] it is also stated without proof that one can attach to a cell in Λ a two-sided cell in W (see also [H, 4.6]).
The examples at the end of Section 4 are counterparts of the examples of [V, 16.2, 16.3], which are discussed in the context of representation theory of real Lie groups (viz. $SL_3(\mathbb{R})$ and $SU(2,1)$). But cells do not occur there.

I am indebted to A. Henderson for a useful remark.

1. Notations and recollections.

1.1. Let B be a θ-stable Borel subgroup of G and T a θ-stable torus contained in B. The root system of (G,T) is R, the system of positive roots in R defined by B is R^+. The Weyl group of R is W and S is the set of simple reflections defined by B. The associated length function is l.

Denote by $\mathcal{H}$ the generic Hecke algebra defined by (W,S) (see [Cu, p. 16]). It is a free module over $\mathbb{Z}[t,t^{-1}]$, with basis $(e_x)_{x\in W}$. The multiplication is described in [loc. cit.]. In particular, $e_s^2 = (t^2-1)e_s + t^2$ $(s \in S)$.

1.2. Denote by V the set of B-orbits on Y. The results to be used about these orbits can be found in [RS1] and [S1].
For $v \in V$ denote by $\mathcal{L}_v$ the group of isomorphism classes of B-equivariant rank one local systems on v. Let Λ be the set of pairs $l = (v,\xi)$ with $v \in V$, $\xi \in \mathcal{L}_v$. Let $\mathcal{M}$ be the free module over $\mathbb{Z}[t,t^{-1}]$ with basis ϵ_l indexed by the elements $l \in \Lambda$. Then $\mathcal{M}$ has a left module structure over the Hecke algebra $\mathcal{H}$. The products $e_s\epsilon_l$ $(s \in S,\ l \in \Lambda)$ are described in [MS2, 4.3.1]. We shall not write down the formulas of [loc. cit.], as we shall not need them. (Notice that in the present case we have, with the notations of [loc. cit.], $\hat{\phi}_v\xi = 0$ since we are dealing with B-equivariant local systems. Moreover in the cases IIIb and IVb, $2a_v(\xi) = 0$, see [loc. cit., 6.7].)

The construction of $\mathcal{M}$ is sheaf-theoretic. One works over the algebraic closure of a sufficiently large finite field. The elements of $\mathcal{M}$ lie in a Grothendieck group built out of B-equivariant l-adic sheaves on Y with Frobenius action. In the general situation of [loc. cit.], $\mathcal{M}$ appears as a module over a large ring, which can in the present case be cut down to $\mathbb{Z}[t,t^{-1}]$.

Let $l = (v, \xi) \in \Lambda$. The basis element ϵ_l of $\mathcal{M}$ is the element in the appropriate Grothendieck group defined by the sheaf on Y extending ξ by zero.
Denote by $A_{\xi,v}$ the irreducible perverse sheaf on Y supported by the closure $\bar{v}$ whose restriction to v is $\xi[\dim v]$ (the "perverse extension" of ξ). It defines an element γ_l of $\mathcal{M}$ (see [loc. cit., 3.1.2, p. 42]).
For $l = (v, \xi) \in \Lambda$ put $d(l) = \dim(v)$.

We next quote some results of Lusztig and Vogan, established in [LV] (see also [MS2, no. 7]).

1.3. Lemma. *There exists an additive duality map D of $\mathcal{M}$ such that for $\mu \in \mathcal{M}, s \in S, l \in \Lambda$*
(a) $D(t\mu) = t^{-1}D(\mu)$,
(b) $D(e_s\mu) = e_s^{-1}D(\mu)$,
(c) $D(\epsilon_l) = t^{-2d(l)}(\epsilon_l + \sum_{d(m)<d(l)} R_{m,l}(t^2)\epsilon_m)$, where $R_{m,l} \in \mathbb{Z}[T]$ has degree $\leq d(l) - d(m)$.
D is an algebraic reflection of Verdier duality.

1.4. Lemma. *γ_l is the unique element of $\mathcal{M}$ satisfying $D(\gamma_l) = \gamma_l$, of the form*

$$t^{-d(l)}(\epsilon_l + \sum_{d(m)<d(l)} P_{m,l}(t^2)\epsilon_m), \tag{1}$$

where $P_{m,l} \in \mathbb{Z}[T]$ has degree $\leq \frac{1}{2}(d(l) - d(m) - 1)$ and has positive coefficients.

If $l = (v, \xi)$, $m = (w, \eta)$ and $P_{m,l} \neq 0$ then w is contained in the closure of v.
For $d(m) < d(l)$ we denote by $\mu(m, l)$ the coefficient of $T^{\frac{1}{2}(d(l)-d(m)-1)}$ in $P_{m,l}$. If $d(l) < d(m)$ we put $\mu(m, l) = \mu(l, m)$.

Denote by b_x ($x \in W$) the Kazhdan-Lusztig elements of $\mathcal{H}$ (see [Cu, p. 30]). They can also be viewed as the elements $[A_{0,x}]$ of [MS2, 3.2]).

Let $l = (v, \xi)$. For $s \in S$ let $P_s = B \cup BsB$ be the parabolic subgroup defined by s. Denote by $\tau(l) \subset S$ the set of simple reflections s such that $\dim P_s v = \dim v$ and, moreover, ξ extends to a sheaf on $P_s v$. (In the notations of [MS2, 4.3.1] the $s \in \tau(l)$ are the simple reflections for which we have one of the cases I, IIb, IIIb or IVb with $a(\xi) = 0$.)

1.5. Proposition. *$b_s\gamma_l$ equals*

$$\sum_{s\in\tau(m)} \mu(m,l)\gamma_m \quad \text{if} \quad s \notin \tau(l), \tag{2}$$

$$(t+t^{-1})\gamma_l \quad \text{if} \quad s \in \tau(l). \tag{3}$$

Proof. (2) is proved in the same way as [LV, 5.3], using [loc. cit., 5.4]. For (3) see [loc. cit., 5.2].

1.6. Corollary. *Assume that $\gamma = \sum_{l\in\Gamma} f_l\gamma_l$, where the f_l are Laurent polynomials. If $b_s\gamma = (t+t^{-1})\gamma$ then $s \in \tau(l)$ if $f_l \neq 0$.*
Proof. Using (3) we see that it suffices to prove that if $f_l = 0$ for all l with $s \in \tau(l)$ then $f_l = 0$ for all l. This follows from (2).

1.7. Proposition. *Let $x \in W, l \in \Lambda$. Then*

$$b_x\gamma_l = \sum_{m\in\Lambda} g_{x,l,m}\gamma_m,$$

where the $g_{x,l,m}$ lie in $\mathbb{Z}[t,t^{-1}]$ and have non-negative coefficients. Moreover, they are invariant under the map $t \mapsto t^{-1}$.
Proof. The first part follows from the sheaf-theoretic construction of the product, using the decomposition theorem and the fact that the eigenvalues of Frobenius on the stalks of the cohomology sheaves of the perverse sheaves $A_{\xi,v}$ are powers of q (see [MS2, 7.1.2]). For a similar result see [MS1, 4.2.6]. The last point is a consequence of the relative hard Lefschetz theorem.

2. Cells.

2.1. We define a preorder relation $\leq$ on Λ as follows: $m \leq l$ if $g_{x,l,m} \neq 0$ for some $x \in W$, where $g_{x,l,m}$ is as in 1.7. An equivalent definition is: γ_m occurs with a non-zero coefficient in some element of $\mathcal{H}\gamma_l$.
Since the b_s $(s \in S)$ generate $\mathcal{H}$, it follows that the relation can also be defined to be the one generated by the elementary relations $\leq_s$ $(s \in S)$, where $m \leq_s l$ if $s \notin \tau(l)$ and γ_m occurs in $b_s\gamma_l$ with a non-zero coefficient. By (2) the latter condition is equivalent with: $s \in \tau(m)$ and $\mu(m,l) \neq 0$.
We define an equivalence relation $\sim$ on Λ by $l \sim m$ if $l \leq m$ and $m \leq l$. The equivalence classes are the *cells* of Λ. These definitions are similar to the well-known definition of cells in W, due to Kazhdan and Lusztig. For the results about such cells in W we refer to [Cu, Ch. II, III].

Let Γ be a cell in Λ. Write $m \leq \Gamma$ ($m < \Gamma$) if $m \leq l$ (respectively, $m \leq l$ and $m \not\sim l$) for some $l \in \Gamma$. The γ_l with $l \leq \Gamma$ (respectively, $l < \Gamma$) span a submodule $\mathcal{M}_\Gamma$ (respectively, $\mathcal{M}'_\Gamma$) of $\mathcal{M}$. Put

$$\mathcal{N} = \mathcal{N}_\Gamma = \mathcal{M}_\Gamma / \mathcal{M}'_\Gamma.$$

This is a free $\mathcal{H}$-module, with basis $\delta_l = \gamma_l + \mathcal{M}'_\Gamma$ ($l \in \Gamma$). We define an integer $a = a(\Gamma)$ by

$$a = \max_{x \in W; l,m \in \Gamma} (\deg g_{x,l,m}).$$

Clearly $a \geq 0$. For $x \in W$, $l, m \in \Lambda$ all Laurent polynomials $g_{x,l,m}$ have degree $\leq a$. Let $c_{x,l,m}$ be the coefficient of t^a in $g_{x,l,m}$. It is an integer ≥ 0.

In the proof of the next lemma the notations are as in [loc. cit., no. 6]: the $h_{x,y,z}$ are the structure constants of $\mathcal{H}$ for the Kazhdan-Lusztig basis (b_x), $a(z) = \max_{x,y} (\deg h_{x,y,z})$ is Lusztig's cell invariant and $\gamma_{x,y,z}$ is the coefficient of $t^{a(z)}$ in $h_{x,y,z}$.
We shall also use Lusztig's asymptotic ring, which we denote by $\mathcal{J}$, see [loc. cit., no. 9]. It is a free abelian group with basis j_z ($z \in W$), the $\gamma_{x,y,z}$ being the corresponding structure constants. By [loc. cit., 9.2] we may view $\mathcal{J}$ as a subring of $\mathbb{Q}(t) \otimes_{\mathbb{Z}[t]} \mathcal{H}$, such that for $x \in W$

$$b_x = \sum_{d \in \mathcal{D}, a(d) = a(z)} h_{x,d,z} j_z, \tag{4}$$

where $\mathcal{D} \subset W$ is the set of Duflo involutions in W (introduced in [loc. cit., 6.8 (ii)]).

2.2. Lemma. *If* $a(x) > a$ *then* $b_x \mathcal{N} = 0$.

Proof. If j_z occurs in the right-hand side of (4) with a non-zero coefficient then $z \leq_R x$, whence $a(z) \geq a(x)$ by [loc. cit., 6.9 (ii)]. Hence in order to prove the lemma it suffices to show that $j_x \mathcal{N} = 0$ for $a(x) > a$ ($\mathcal{N}$ being viewed as a subset of $\mathbb{Q}(t) \otimes \mathcal{N}$). Putting

$$b = \max\{a(x) \mid j_x \mathcal{N} \neq 0\},$$

this amounts to proving that $b \leq a$.
Let $j_x \mathcal{N} \neq 0$ and $a(x) = b$. Let d be the Duflo involution in the left cell of x (see [loc. cit., 6.11]). Then by [loc. cit., 9.5 (i)] we have $j_x = j_x j_d$,

whence $j_d\mathcal{N} \neq 0$, and $a(d) = a(x) = b$. So we may assume that $x = d$. Let tr be the trace function on $\mathbb{Q}(t) \otimes \mathcal{H}$, acting on $\mathbb{Q}(t) \otimes \mathcal{N}$. By (4)

$$\mathrm{tr}(b_d) = \sum_{e \in \mathcal{D}, a(e)=a(z)} h_{d,e,z}\mathrm{tr}(j_z).$$

The non-zero $h_{d,e,z}$ in the right-hand side are such that $a(z) \geq a(d) = b$. Our assumption implies that we can restrict the summation to the z with $a(z) = b$. Then $\deg(h_{d,e,z}) \leq b$. If equality holds we must have $\gamma_{d,e,z} \neq 0$, which can only be if $d = e = z$, by [loc. cit., 6.10 (i), 6.8 (ii)]. This implies that $\mathrm{tr}(b_d) - h_{d,d,d}\mathrm{tr}(j_d)$ is a Laurent polynomial of degree $< b$ (notice that all $\mathrm{tr}(j_z)$ are algebraic integers). Since $j_d^2 = j_d$ we have $\mathrm{tr}(j_d) > 0$. Hence $\mathrm{tr}(b_d)$ is a Laurent polynomial of degree b. Now

$$\mathrm{tr}(b_d) = \sum_{l \in \Gamma} g_{d,l,l},$$

from which we see that there is $l \in \Gamma$ with deg $(g_{d,l,l}) \geq b$. This implies that $b \leq a$, which we had to prove.

For $x \in W$ define $\tau(x) = \{s \in S \mid sx < x\}$.

2.3. Lemma. *Let $x \in W$, $l, m \in \Gamma$ and assume that $c_{x,m,l} \neq 0$.*
(i) $\tau(x) = \tau(l)$;
(ii) For any $l' \in \Gamma$ there exists $x' \in W$ such that $x' \leq_L x$ and $c_{x',m,l'} \neq 0$.

Proof. Assume that $l, l' \in \Gamma$ and $l' \leq_s l$ for some $s \in S$. Then $s \notin \tau(l)$. We have the associativity relation

$$(b_s b_x)\gamma_m = b_s(b_x\gamma_m).$$

If $sx < x$ we have $b_s b_x = (t + t^{-1})b_x$ by [Cu, 5.1], whence $b_s(b_x\delta_m) = (t+t^{-1})(b_x\delta_m)$. From 1.6 we infer that this is impossible, since δ_l occurs in $b_x\delta_m$ with a non-zero coefficient. It follows that $\tau(x) \subset \tau(l)$.
Now assume that $sx > x$. Writing out the associativity relation and comparing coefficients of l' on both sides we obtain, using [loc. cit.] and 1.5,

$$(5) \qquad \sum_{sx'<x'} \mu(x', x)g_{x',m,l'} = (t + t^{-1})g_{x,m,l'} + \sum_{n,\ s\notin\tau(n)} g_{x,m,n}\mu(l', n).$$

In the left-hand side of (5), $\mu(x', x)$ is the usual Kazhdan-Lusztig coefficient.
If $c_{x,m,l} \neq 0$ and $\mu(l', l) \neq 0$, the right-hand side contains a non-zero

multiple of t^a. Since all Laurent polynomials occurring in (5) have coefficients ≥ 0, the left-hand side also contains a non-zero multiple of t^a. We conclude that there is $x' <_{L,s} x$ with $c_{x',m,l'} \neq 0$ (where $<_{L,s}$ is the elementary preorder relation on W defined by s, i.e. $sx' < x', sx > x$ and $\mu(x', x) \neq 0$, cf. [Cu, 5.2]). (ii) follows in the case that $l' \leq_s l$. The general case is a consequence.
Again, let $sx > x$ and consider (5) with l' arbitrary such that $s \notin \tau(l')$. Since the left-hand side has degree $\leq a$ we must have $c_{x,m,l'} = 0$. This implies that $\tau(l) \subset \tau(x)$ if $c_{x,m,l} \neq 0$ and (i) follows.

2.4. Lemma. *Let $l, m \in \Gamma$ and $x \in W$ be such that $c_{x,l,m} \neq 0$.*
(i) $a(x) = a$;
(ii) In 2.3 (ii) we have $x' \sim_L x$.

Proof. From the asssociativity relation $(b_x b_y)\gamma_n = b_x(b_y \gamma_n)$ $(x, y \in W, n \in \Gamma)$ we obtain for $m \in \Gamma$

$$\sum_{z \in W} h_{x,y,z} g_{z,n,m} = \sum_{p \in \Gamma} g_{x,p,m} g_{y,n,p}. \tag{6}$$

Let $c_{x,l,m} \neq 0$, then $g_{x,l,m}$ has degree a. By 2.3 (ii) there exists $y \leq_L x$ such that $\deg(g_{y,l,l}) = a$. Take $n = l$ in (6). The right-hand side has degree $2a$. If in the left-hand side of (6) we have $g_{z,l,m} \neq 0$ then $b_z \mathcal{N} \neq 0$ and $a(z) \leq a$, whence $\deg(h_{x,y,z}) \leq a$. Since the right-hand side has degree $2a$ there is $z \in W$ with $\deg(h_{x,y,z}) = a(z) = a$. Then $\gamma_{x,y,z} \neq 0$. By [Cu, 6.10] we have $x \sim_R z$ and $a(x) = a(z) = a$, proving (i).
(ii) is a consequence of (i) and [loc. cit.].

2.5. Lemma. *For $x, y \in W$ and $m, n \in \Gamma$ we have $\sum_{z \in W} \gamma_{x,y,z} c_{z,n,m} = \sum_{l \in \Gamma} c_{x,l,m} c_{y,n,l}$.*

Proof. We use (6). From the proof of 2.4 we see that all structure constant occurring in (6) are Laurent polynomials of degree $\leq a$. The asserted identity then follows by comparing coefficients of t^{2a} in both sides of (6).

3. A $\mathcal{J}$-module.

3.1. Let $\mathcal{K} = \mathcal{K}_\Gamma$ the free abelian group with basis k_l indexed by the elements of Γ. For $x \in W$, $l \in \Lambda$ define

$$j_x k_l = \sum_{m \in \Gamma} c_{x,l,m} k_m,$$

and extend this to an additive map $\mathcal{J} \otimes_{\mathbf{Z}} \mathcal{K} \to \mathcal{K}$. By 2.5 we have

$$j_x(j_y k_l) = (j_x j_y) k_l.$$

This shows that we have defined a $\mathcal{J}$-module structure on $\mathcal{K}$. We have not yet established that $\mathcal{K}$ is a unital module, i.e that the identity element

$$\mathbf{1} = \sum_{d \in \mathcal{D}} j_d$$

of $\mathcal{J}$ acts as the identity on $\mathcal{K}$. We shall do this presently.

3.2. Proposition. *For $z \in \mathcal{J}$ the traces $tr(j_z, \mathcal{K})$ and $tr(j_z, \mathcal{N})$ are equal.*

Proof. It follows from (4) that for $z \in W$

$$j_z = \sum_{w \in W} \xi_{z,w} t^{a(z)} b_w,$$

where $(\xi_{z,w})$ is a matrix with entries in $\mathbb{Q}(t)$. Also, $\xi_{z,w}$ is defined at $t = 0$ and $\xi_{z,w}(0) = \delta_{z,w}$ (cf. [Cu, p. 54]). Hence

$$j_z \delta_l = \sum_{m \in \Gamma} \eta_{z,l,m} \delta_m,$$

with

$$\eta_{z,l,m} = \sum_{w} \xi_{z,w} t^{a(z)} g_{w,l,m}.$$

By 2.4 we may assume that $a(z) = a$. Since $g_{w,l,m}$ is invariant under the map $t \mapsto t^{-1}$ (see 1.7), it follows that $t^{a(z)} g_{w,l,m} \in \mathbb{Z}[t]$ and has value $c_{w,l,m}$ for $t = 0$. We conclude that $\eta_{z,m,l}$ is a rational function in t which is defined at $t = 0$ with value $c_{z,m,l}$.
We have

$$\operatorname{tr}(j_z, \mathcal{N}_{\mathbb{Q}(t)}) = \sum_{l \in \Gamma} \eta_{z,l,l},$$

a rational function of t which is defined at $t = 0$. Since $\operatorname{tr}(j_z)$ is an algebraic integer for all z, this rational function must be constant and its value is the value at 0, which is $\operatorname{tr}(j_z, \mathcal{K})$. The proposition follows.

3.3. Corollary. *$\mathcal{K}$ is unital.*

Proof. Put

$$\mathcal{K}_0 = \{k \in \mathcal{K} \mid \mathbf{1}.k = 0\},$$

this is a direct summand of $\mathcal{K}$. We have a structure of unital $\mathcal{J}$-module on $\mathcal{K}/\mathcal{K}_0$, whence $\mathrm{tr}(\mathbf{1},\mathcal{K}) = |\Gamma| - \mathrm{rank}(\mathcal{K}_0)$. The proposition shows that $\mathrm{tr}(\mathbf{1},\mathcal{K}) = |\Gamma|$ and it follows that $\mathcal{K}_0 = \{0\}$, i.e. that $\mathcal{K}$ is unital.

We write $\mathcal{H}_{\mathbb{Q}(t)} = \mathbb{Q}(t) \otimes_{\mathbb{Z}[t]} \mathcal{H}$, and similarly for other objects obtained by extending coefficients. We know that $\mathcal{H}_{\mathbb{Q}(t)} = \mathcal{J}_{\mathbb{Q}(t)}$ (recall that $\mathcal{J}$ is a subring of $\mathcal{H}_{\mathbb{Q}(t)}$). From 3.3 we see that $\mathcal{K}_{\mathbb{Q}(t)}$ is a $\mathcal{H}_{\mathbb{Q}(t)}$-module.

3.4. Proposition. *The $\mathcal{H}_{\mathbb{Q}(t)}$-modules $\mathcal{N}_{\mathbb{Q}(t)}$ and $\mathcal{K}_{\mathbb{Q}(t)}$ are isomorphic.*

Proof. The algebra $\mathcal{H}_{\mathbb{Q}(t)}$ is split semi-simple (see [Cu, 8.3]). Using the orthogonality relations for its irreducible representations (cf. [MS1, 11.1.4]) it follows from 3.2 that the multiplicities of an irreducible representation of $\mathcal{H}_{\mathbb{Q}(t)}$ in $\mathcal{N}_{\mathbb{Q}(t)}$ and $\mathcal{K}_{\mathbb{Q}(t)}$ are the same. This proves 3.4.

3.5. Proposition. *(i) For every $l \in \Gamma$ there is a unique Duflo involution d with $j_d k_l = k_l$;*
(ii) The involutions of (i) lie in a unique two-sided cell of W.

Proof. By 3.3 we have

$$\sum_{d \in \mathcal{D}} j_d k_l = k_l.$$

Now any product $j_d k_l$ is a positive integral linear combination of k_m's. (i) follows from the observation that the left-hand side of the formula can contain only one non-zero term.
Let $d, e \in \mathcal{D}$ and $l, m \in \Gamma$ be such that $j_d k_l = k_l$ and $j_e k_m = k_m$. It follows from 2.3 (ii) and 2.4 (ii) that there is $x \sim_L d$ such that $j_x k_l$ contains k_m with a non-zero coefficient. Then $j_e j_x k_l \neq 0$, in particular $j_e j_x \neq 0$. This implies that $e \sim_L x^{-1}$ (see [Cu, 9.5 (ii)]). Then $e \sim_R x \sim_L d$, whence $d \sim_{LR} e$, proving the proposition.

4. Complements and examples.

4.1. A bilinear form. Let $v \in V$. For $s \in S$ denote by $P_s \supset B$ the parabolic subgroup of semi-simple rank 1 associated to s. Let $m(s)v$ be the unique open orbit of B in $P_s v$. We recall the notion of a reduced decomposition $\mathbf{v} = ((v_0, ..., v_l), \mathbf{s} = (s_1, ..., s_l))$ of v: the v_i lie in V and the s_j in S, v_0 is a closed orbit, $v_l = v$ and $v_i = m(s_i)v_{i-1} \neq v_{i-1}$ (see [RS1, 5.7, no. 7]). Then $d(v_i) = d(v_{i-1}) + 1$ $(1 \leq i \leq l)$. Let $\lambda_c(v)$ $(\lambda_i(v))$ be the number of i such that s_i is complex (respectively, imaginary) for v_{i-1}. See [loc.cit., 4.3], the cases correspond

to the cases IIa (respectively, IIIa or IVa) of [MS2, 4.1.4]. It follows from the definitions, using [RS1, 3.7], that these numbers depend only on v, and not on the choice of the reduced decomposition $(\mathbf{v}, \mathbf{s})$.
Denote by U the unipotent part of B, so $B = TU$. It is known that $B \cap K$ $(T \cap K)$ is a Borel subgroup (respectively, a maximal torus) of K. Let $v_0 \in V$ be the closed orbit BK/K. It is isomorphic to $B/B \cap K$. In fact, this is true for any closed orbit $v_0 \in V$, as follows from [S1, 6.6]. If $v \in V$ we put

$$d_c(v) = \lambda_c(v) + \dim U/U \cap K,\ d_i(v) = \lambda_i(v) + \dim T/T \cap K.$$

Then

$$d(v) = d_c(v) + d_i(v).$$

If $(x, \xi) \in \Lambda$ we put $d_i(l) = d_i(v)$, $d_c(l) = d_l(v)$.
Denote by N be the normalizer of T. Let $v \in V$. There exists $x \in G$ with $xK \in v$ such that $x(\theta x)^{-1} \in N$ (see [loc.cit., 4.2]). Denote isotropy subgroups of x by a suffix x.

4.2. Lemma. *(i) v is isomorphic as a variety to B/B_x;*
(ii) $B_x = T_x U_x$.
(iii) $\dim T/T_x = d_i(v)$, $\dim U/U_x = d_c(v)$.

Proof. It is clear that there is a bijective morphism of homogeneous spaces $B/B_x \to v$. It is separable (see [MS2, 6.3]), hence is an isomorphism. This proves (i). (ii) and (iii) follow from [S1, 4.7].

We introduce the $\mathbf{Z}[t, t^{-1}]$-bilinear form β on $\mathcal{M}$ with

$$\beta(\epsilon_l, \epsilon_m) = \delta_{l,m}(t^2 - 1)^{d_i(l)} t^{2d_c(l)}.$$

Clearly, it is symmetric and nondegenerate.

4.3. Proposition. *For $x \in W$, $\mu, \nu \in \mathcal{M}$ we have*

$$\beta(e_x \mu, \nu) = \beta(\mu, e_{x^{-1}} \nu).$$

Proof. It suffices to prove this in the case that x is a simple reflection s and $\mu = \epsilon_l, \nu = \epsilon_m$ $(l, m \in \Lambda)$. Using the explicit formulas of [MS2, 4.3.1] for the products $e_s \epsilon_l$, the verification of the asserted formula is straightforward. It is left to the reader. (The explicit formulas in our special case are also described, somewhat differently, in [RS2, 7.3]).

4.4. Corollary. *Let $l, m \in \Lambda$.*
(i) $\beta(\gamma_l, \gamma_m) - \delta_{l,m} \in t^{-1}\mathbb{Z}[t^{-1}]$;
(ii) For $x \in W$, $l, m \in \Lambda$ we have $\deg g_{x,m,l} = \deg g_{x^{-1},l,m}$*;*
(iii) Let Γ be a cell in Λ. For $x \in W$, $l, m \in \Gamma$ we have $c_{x,m,l} = c_{x^{-1},l,m}$.

Proof. Inserting the expressions of (1) for γ_l and γ_m and using the degree estimates for the polynomials $P_{m,l}$ of (1), (i) readily follows.
Let M be the matrix $(\beta(\epsilon_l, \epsilon_m))_{l,m\in\Lambda}$. For $x \in W$, multiplication by b_x in $\mathcal{M}$ is given (relative to the basis (ϵ_l)) by the matrix $M_x = (h_{x,m,l})$. By the proposition, the matrix of $b_{x^{-1}}$ is given by the transpose of M_x relative to β, which is $M^{-1}({}^tM_x)M$. Using (i) we see that $M^{-1} - I$ is a matrix with entries in $t^{-1}\mathbb{Z}[[t^{-1}]]$. (ii) then follows from (i) and this observation. (iii) also follows.

4.5. Corollary. *Let $x \in W$, $l, m \in \Gamma$ and assume that $c_{x,m,l} \neq 0$.*
(i) $\tau(x^{-1}) = \tau(m)$;
(ii) For any $m' \in \Gamma$ there exists $x' \in W$ such that $x' \leq_R x$ and $c_{x',m',l} \neq 0$.

Proof. This follows from 4.4 (iii) and 2.3.

4.6. Examples. We briefly discuss two examples with $G = SL_3$. We take B and T to be the subgroups of upper triangular, respectively diagonal, matrices. The Weyl group is S_3.
The simple roots are the characters α_1, α_2 of T sending $(a_1, a_2, a_3) \in T$ to $a_1a_2^{-1}$, respectively $a_2a_3^{-1}$. The corresponding simple reflections are the transpositions (12) and (23). The corresponding generators of the Hecke algebra $\mathcal{H}$ are denoted by e_1 and e_2.

(a) $\theta(g) = a({}^tg)^{-1}a^{-1}$ where a is such that θ stabilizes B and T. Then $K \simeq SO_3$.
The set V of B-orbits in G/K has 4 elements v_0, v_1, v_1', v_2, of respective dimensions $3, 4, 4, 5$, as follows from [RS1, p. 432-433]. One checks that the group $\mathcal{L}_v$ of B-equivariant local systems on the orbit v is trivial except if $v \neq v_2$, in which case it is the character group of the subgroup of T of elements of order ≤ 2.
We abbreviate $\epsilon_{v_0,0}$ to ϵ_0. Similarly, we have ϵ_1 and ϵ_1'. We have 4 basis elements $\epsilon_{v_2,\xi}$ denoted by $\epsilon_{20}, \epsilon_{21}, \epsilon_{22}, \epsilon_{23}$, where ϵ_{20} corresponds to the constant sheaf on v_2. We use similar notations for the Kazhdan-Lusztig elements γ_l.
The action of e_1 and e_2 on the basis elements is described in [RS2, p. 141] (in the first formula of line 6 of that page f_1 should be replaced by

f_1').
We now deal with the duality operator D. It follows from 1.3 (c) that $D(\epsilon_0) = t^{-6}\epsilon_0$. By [loc. cit.] we have $e_1\epsilon_0 = \epsilon_1'$, $e_2\epsilon_0 = \epsilon_1$. Then $D(\epsilon_1), D(\epsilon_1')$ can be determined from 1.3 (b). One checks that

$$\gamma_0 = t^{-3}\epsilon_0,\ \gamma_1 = t^{-4}(\epsilon_1 + \epsilon_0),\ \gamma_1' = t^{-4}(\epsilon_1' + \epsilon_0)$$

have the properties of 1.4 and thus are the correct Kazhdan-Lusztig elements. Next, since G/K is smooth its intersection cohomology complex is the shifted constant sheaf $E[5]$, from which it follows that

$$\gamma_{20} = t^{-5}(\epsilon_{20} + \epsilon_1 + \epsilon_1' + \epsilon_0).$$

Since γ_{20} is D-invariant, this formula determines $D(\epsilon_{20})$.
We have

$$e_2\epsilon_1' = \epsilon_{20} + \epsilon_{22} + \epsilon_1'.$$

By 1.3 (b) one knows how D acts on the right-hand side. Using what is already known one finds $D(\epsilon_{22})$, and similarly $D(\epsilon_{21})$. Then

$$\gamma_{21} = t^{-5}(\epsilon_{21} + \epsilon_1),\ \gamma_{22} = t^{-5}(\epsilon_{22} + \epsilon_1')$$

satisfy the requirements of 1.4.
Finally, we claim that

$$\gamma_{23} = t^{-10}\epsilon_{23}.$$

To see this it suffices to show that $D(\epsilon_{23}) = t^{-5}\epsilon_{23}$. Now by the formulas of [loc. cit.], ϵ_{23} is annihilated by $e_1 + 1$ and $e_2 + 1$. By 1.3 (b) the same must be true of $\mu = D(\epsilon_{23})$. Then μ must be orthogonal, with respect to the bilinear form β of 4.3, to $(e_1+1)\mathcal{M}$ and $(e_2+1)\mathcal{M}$. The formulas of [loc. cit.] show that this can only be if μ is a multiple of ϵ_{23}. By 1.3 (c) we then must have $\mu = t^{-10}\epsilon_{23}$.
Let $c_i = c_{s_i}$ $(i = 1, 2)$. The products $c_i\gamma_l$ which are not 0 or $(t + t^{-1})\gamma_l$ are the following: $c_1\gamma_0 = \gamma_1'$, $c_2\gamma_0 = \gamma_1$, $c_1\gamma_1 = \gamma_{20} + \gamma_{21}$, $c_2\gamma_1' = \gamma_{20} + \gamma_{22}$, $c_1\gamma_{22} = \gamma_1'$, $c_2\gamma_{21} = \gamma_1$. Using these formulas we see that the cells are: $\Gamma_0 = \{v_0\}$, $\Gamma_1 = \{v_1,\ \gamma_{21}\}$, $\Gamma_1' = \{v_1', v_{22}\}$, $\Gamma_3 = \{v_{20}\}$, $\Gamma_0' = \{v_{23}\}$. The index denote the a-value on the cell.
The two-sided cells in S_3 are $\Delta_0 = \{1\}$, $\Delta_1 = \{(12), (23), (123), (132)\}$, $\Delta_3 = \{(13)\}$. The two-sided cell in S_3 attached to a cell in Λ is the one with the same suffix.

(b) $\theta(g) = aga^{-1}$, where $a = \mathrm{diag}(-\zeta, \zeta, \zeta)$ with $\zeta^3 = -1$. Now $K \simeq GL_2$.
This case is discussed (more generally, for SL_n) in [RS1, 10.5]. We have three closed orbits v_1, v_2, v_3 of dimension 2, two orbits v_{12}, v_{23} of

dimension 3 and the open orbit v_{13} of dimension 4. The numbering is such that $v_i \leq v_{jk}$ if and only if $i = j$ or $i = k$.
One checks that all groups $\mathcal{L}_v$ are trivial. Using [loc. cit.] and the formulas of [MS2, 4.3.1] or [RS2, 7.3] it is straightforward to determine the products $e_i \epsilon_v$. Proceeding as in the previous example one determines the various $D(\epsilon_v)$ and the Kazhdan-Lusztig elements γ_v.
It turns out that for all $v \in V$

$$\gamma_v = t^{-\dim v} \sum_{w \leq v} \epsilon_w$$

(which means that all orbit closures $\bar{v}$ are rationally smooth). We can then determine the products $c_i \gamma_v$. The upshot is that the cells are $\Gamma_0 = \{v_2\}$, $\Gamma_1 = \{v_1, v_{12}\}$, $\Gamma_1' = \{v_2, v_{23}\}$, $\Gamma_3 = \{v_{13}\}$. Again, the suffixes denote the a-values.

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On the Characterization of the Set $\mathcal{D}_1$ of the Affine Weyl Group of Type $\tilde{A}_n$

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Dedicated to N. Iwahori

Abstract.

In this paper we show that a conjecture of Lusztig on distinguished involutions is true for the affine Weyl group of type $\tilde{A}_n$.

§1. Springer's formula and Lusztig's Conjecture

1.1. Let (W, S) be a Coxeter group with S the set of simple reflections. Let H be the Hecke algebra of (W, S) over $\mathcal{A} = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ ($q^{\frac{1}{2}}$ an indeterminate). Then H is a free $\mathcal{A}$-module with a basis $\{T_w\}_{w\in W}$ and multiplication relations

$$(T_s - q)(T_s + 1) = 0 \qquad \text{if} \quad s \in S,$$

$$T_w T_u = T_{wu} \qquad \text{if} \quad l(wu) = l(w) + l(u),$$

where $l : W \to \mathbf{N}$ is the length function.

In 1979, Kazhdan and Lusztig published a paper [KL] in which the famous Kazhdan-Lusztig polynomials are introduced, which are uniquely defined by the following properties. For each x in W, there exists a unique element

$$C_x = q^{-\frac{l(x)}{2}} \sum_{y \leq x} P_{y,x} T_y$$

(here $\leq$ is the Bruhat order on (W, S)) such that
(1) C_x is invariant under the ring involution $H \to H$ defined by $q^{\frac{1}{2}} \to q^{-\frac{1}{2}}$, $T_w \to T_{w^{-1}}^{-1}$,
(2) $P_{y,x}$ are polynomials in q with degree less than or equal to $\frac{1}{2}(l(x) - l(y) - 1)$ if $y \leq x$ and $y \neq x$, and $P_{x,x} = 1$.

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$P_{y,x}$ are the famous Kazhdan-Lusztig polynomials, which play a great role in Lie Theory.

1.2. Suppose that $y \leq x$ and $y \neq x$. We then can write

$$P_{y,x} = \mu(y,x)q^{\frac{1}{2}(l(x)-l(y)-1)} + \text{lower degree terms}.$$

The coefficients $\mu(y,x)$ are important for understanding the Kazhdan-Lusztig polynomials and Kazhdan-Lusztig cells. We are interested in properties of the coefficients $\mu(y,x)$. We set $\mu(x,y) = \mu(y,x)$ if $\mu(y,x)$ is defined.

Assume that (W,S) is an affine Weyl group or a Weyl group. The following formula is due to Springer (see [Sp, S2]),

(a) $$\mu(y,x) = \sum_{d\in\mathcal{D}_0} \delta_{y^{-1},x,d} + \sum_{f\in\mathcal{D}_1} \gamma_{y^{-1},x,f}\pi(f).$$

We need explain the notations. Write

$$C_xC_y = \sum_{z\in W} h_{x,y,z}C_z, \qquad h_{x,y,z} \in \mathcal{A} = \mathbf{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

Define

$$a(z) = \min\{i \in \mathbf{N} \mid q^{-\frac{i}{2}}h_{x,y,z} \in \mathbf{Z}[q^{-\frac{1}{2}}] \text{ for all } x,y \in W\}.$$

If for any i, $q^{-\frac{i}{2}}h_{x,y,z} \notin \mathbf{Z}[q^{-\frac{1}{2}}]$ for some $x,y \in W$, we set $a(z) = \infty$. It is not clear that whether there exists a Coxeter group (W,S) such that $a(z) = \infty$ for some $z \in W$.

From now on, we assume that the function $a: W \to \mathbf{N}$ is bounded and (W,S) is crystallographic. Obviously, when W is finite, the function a is bounded. Lusztig showed that the function a is bounded for all affine Weyl groups (see [L1]). Following Lusztig and Springer, we define $\delta_{x,y,z}$ and $\gamma_{x,y,z}$ by the following formula,

$$h_{x,y,z} = \gamma_{x,y,z}q^{\frac{a(z)}{2}} + \delta_{x,y,z}q^{\frac{a(z)-1}{2}} + \text{ lower degree terms}.$$

Springer showed that $l(z) \geq a(z)$ (see [L2]). Let $\delta(z)$ be the degree of $P_{e,z}$, where e is the neutral element of W. Then actually one has $l(z) - a(z) - 2\delta(z) \geq 0$ (see [L2]). Set

$$\mathcal{D}_i = \{z \in W \mid l(z) - a(z) - 2\delta(z) = i\}.$$

The number $\pi(z)$ is defined by $P_{e,z} = \pi(z)q^{\delta(z)} +$ lower degree terms.

The elements of $\mathcal{D}_0$ are involutions, called distinguished involutions of (W,S) (see [L2]). Moreover, in a Weyl group or an affine Weyl group, each left cell (resp. right cell) contains exactly one element of $\mathcal{D}_0$, see [L2].

Example: Let S' be a subset of S such that the subgroup W' of W generated by S' is finite. Then the longest element of W' is in $\mathcal{D}_0$.

1.3. Comparing with the set $\mathcal{D}_0$, we know very little about the set $\mathcal{D}_1$. From the formula of Springer, we see that the set $\mathcal{D}_1$ is important for understanding the coefficients $\mu(y, x)$. Lusztig has an interesting conjecture for describing the set $\mathcal{D}_1$. For stating the conjecture we need the concept of cell.

We refer to [KL] for the definition of left cell, right cell and two-sided cell. For elements w, u in W we shall write $w \underset{L}{\sim} u$ (resp. $w \underset{R}{\sim} u$; $w \underset{LR}{\sim} u$) if w and u are in the same left (resp. right; two-sided) cell of W. Now we can state the conjecture of Lusztig (see[L3, S2]).

Conjecture: Let $z \in W$. Then z is in $\mathcal{D}_1$ if and only if there exists some d in $\mathcal{D}_0$ such that $z \underset{LR}{\sim} d$ and $\mu(z, d) \neq 0$.

To go further, we need some properties of $\gamma_{x,y,z}$ and $\delta_{x,y,z}$. The following are some properties of $\gamma_{w,u,v}$ (see [L2] for (a)-(d) and [L1] for (e)).

(a) If $\gamma_{w,u,v}$ is not equal to 0, then $w \underset{L}{\sim} u^{-1}$, $u \underset{L}{\sim} v$ and $w \underset{R}{\sim} v$. In particular we have $w \underset{LR}{\sim} u \underset{LR}{\sim} v$ if $\gamma_{w,u,v}$ is not equal to 0.

(b) $\gamma_{w,u,v} = \gamma_{u,v^{-1},w^{-1}}$ and $\gamma_{u^{-1},w^{-1},v^{-1}} = \gamma_{w,u,v}$.

(c) Let d be in $\mathcal{D}_0$. Then $\gamma_{w,d,u} \neq 0$ if and only if $w = u$ and $w \underset{L}{\sim} d$. Moreover $\gamma_{w,d,w} = \gamma_{d,w^{-1},w^{-1}} = \gamma_{w^{-1},w,d} = 1$.

(d) $w \underset{L}{\sim} u^{-1}$ if and only if $\gamma_{w,u,v}$ is not equal to 0 for some v.

(e) The positivity: as a Laurent polynomial in $q^{\frac{1}{2}}$, the coefficients of $h_{w,u,v}$ are non-negative. In particular, $\gamma_{w,u,v}$ and $\delta_{w,u,v}$ are non-negative for all w, u, v in W.

The following property is due to Springer, see [Sp] or [S2].

(f) If $w \underset{LR}{\sim} u$ and $\delta_{w,u,v} \neq 0$, then $w \underset{R}{\sim} v$ and $u \underset{L}{\sim} v$.

1.4. The observations of this subsection are due to Shi, Springer and Lusztig.

We can see easily that the "only if" part of the conjecture in subsection 1.3 is true. If z is in $\mathcal{D}_1$, then we can find some d in $\mathcal{D}_0$ such that $z \underset{R}{\sim} d$ (or equivalently, $z^{-1} \underset{L}{\sim} d$, note that $d = d^{-1}$). By 1.3 (c), we have

$\gamma_{z^{-1},d,z^{-1}} = 1$. By the positivity (see 1.3 (e)) and Springer's formula (see 1.2 (a)), we see that $\mu(z,d) \neq 0$. This observation and argument are due to Shi, see [S2]

Moreover, for $z \in W$ and $d \in \mathcal{D}_0$, if $\mu(z,d) \neq 0$, $z \underset{LR}{\sim} d$ and $z \underset{L}{\not\sim} z^{-1}$, it is easy to prove that z is in $\mathcal{D}_1$. Now we argue for this. We have

$$\mu(z,d) = \sum_{d' \in \mathcal{D}_0} \delta_{z^{-1},d,d'} + \sum_{f \in \mathcal{D}_1} \gamma_{z^{-1},d,f}\pi(f).$$

By 1.3 (c), $\gamma_{z^{-1},d,f} \neq 0$ implies that $f = z^{-1}$. Note that z is in $\mathcal{D}_1$ if and only if z^{-1} is in $\mathcal{D}_1$. If z is not in $\mathcal{D}_1$, then we have $\mu(z,d) = \sum_{d' \in \mathcal{D}_0} \delta_{z^{-1},d,d'}$. But if $\delta_{z^{-1},d,d'} \neq 0$, by 1.3 (f), we then have $z^{-1} \underset{R}{\sim} d'$ and $d \underset{L}{\sim} d'$. Thus we must have $d' = d$ since each left cell contains only one element of $\mathcal{D}_0$. So we must have $z^{-1} \underset{R}{\sim} d$. Since $\mu(z,d) = \mu(z^{-1},d^{-1}) = \mu(z^{-1},d)$, applying the same argument to z^{-1} we can see that $z \underset{R}{\sim} d$. In conclusion, we have $z \underset{R}{\sim} z^{-1}$, i.e., $z \underset{L}{\sim} z^{-1}$. This contradicts $z \underset{L}{\not\sim} z^{-1}$. Therefore we must have $z \in \mathcal{D}_1$. Essentially this observation and argument are due to Springer, see [S2] and [Sp].

The conjecture was also proved for Weyl groups by Lusztig with the following two exceptions (1) W is of type E_7 and $a(z) = 512$, (2) W is of type E_8 and $a(z) = 4096$ (see [L3]). In this paper we shall prove the following result.

Theorem 1.5. *Let (W,S) be an affine Weyl group of type $\tilde{A}_n$, then the conjecture of Lusztig in subsection 1.3 is true.*

We need some preparation to prove the theorem.

§2. Proof of Theorem 1.5

We shall need the star operations introduced by Kazhdan and Lusztig in [KL].

2.1. For w in W, set $L(w) = \{s \in S \mid sw \leq w\}$ and $R(w) = \{s \in S \mid ws \leq w\}$. Let s and t be in S such that st has order 3, i.e. $sts = tst$. Define

$$D_L(s,t) = \{w \in W \mid L(w) \cap \{s,t\} \text{ has exactly one element}\},$$

$$D_R(s,t) = \{w \in W \mid R(w) \cap \{s,t\} \text{ has exactly one element}\}.$$

If w is in $D_L(s,t)$, then $\{sw, tw\}$ contains exactly one element in $D_L(s,t)$, denoted by ${}^*w$, here $* = \{s,t\}$. The map: $D_L(s,t) \to D_L(s,t)$,

$w \to {}^*w$, is an involution and is called a **left star operation**. Similarly if $w \in D_R(s,t)$ we can define the **right star operation** $w \to w^* = \{ws, wt\} \cap D_R(s,t)$ on $D_R(s,t)$, where $* = \{s,t\}$. The following are some properties proved in [KL].

Let s and t be in S such that st has order 3 and set $* = \{s,t\}$. Assume that y, w are in $D_L(s,t)$. We have

(a) $\mu(y,w) = \mu({}^*y, {}^*w)$.

(b) $y \underset{R}{\sim} w$ if and only if ${}^*y \underset{R}{\sim} {}^*w$.

(c) $w \underset{L}{\sim} {}^*w$.

Let $* = \{s,t\}$. Assume that y, w are in $D_R(s,t)$. We have

(d) $\mu(y,w) = \mu(y^*, w^*)$.

(e) $y \underset{L}{\sim} w$ if and only if $w^* \underset{L}{\sim} y^*$.

(f) $w \underset{R}{\sim} w^*$.

2.2. Recall that we have assumed that (W,S) is crystallographic and the function $a : W \to \mathbf{N}$ is bounded. The following results (a-d) are proved in [X, section 1.4] and the assertion (e) is due to Shi (see [C, Theorem 1.10]).

Let s, t, s', t' be in S such that both st and $s't'$ have order 3. Assume that w is in $D_L(s,t) \cap D_R(s',t')$. Set $* = \{s,t\}$ and $\star = \{s',t'\}$. Then

(a) ${}^*w$ is in $D_R(s',t')$ and $w^\star$ is in $D_L(s,t)$. Moreover, we have ${}^*(w^\star) = ({}^*w)^\star$. We shall write ${}^*w^\star$ for ${}^*(w^\star) = ({}^*w)^\star$.

(b) $({}^*w)^{-1} = (w^{-1})^*$ and $({}^*w^\star)^{-1} = {}^\star(w^{-1})^*$

Let s, t, s', t' be as above and $* = \{s,t\}$ and $\star = \{s',t'\}$. Suppose that w is in $D_L(s,t)$ and u is in $D_R(s',t')$. Let v be in W such that $v \underset{L}{\sim} u$ and $v \underset{R}{\sim} w$. Then

(c) We have $v \in D_L(s,t) \cap D_R(s',t')$, so ${}^*v^\star$ is well defined and we have $h_{w,u,v} = h_{{}^*w,u^\star,{}^*v^\star}$, see subsection 1.2 for the definition of $h_{w,u,v}$.

Let s, t, s', t', s'', t'' be in S such that all st, $s't'$ and $s''t''$ have order 3. Suppose that w is in $D_L(s,t) \cap D_R(s'',t'')$ and u is in $D_L(s'',t'') \cap D_R(s',t')$. Set $* = \{s,t\}, \# = \{s'',t''\}$, and $\star = \{s',t'\}$. Let v be in W such that v is in $D_L(s,t) \cap D_R(s',t')$. Then we have

(d) $\gamma_{w,u,v} = \gamma_{{}^*w^\#,{}^\#u^\star,{}^*v^\star}$.

(e) Let w be in W such that $w = w^{-1}$. If w is in $D_L(s,t)$ or in $D_R(s,t)$, then ${}^*w^*$ is well defined for $* = \{s,t\}$. Moreover, if w is in $\mathcal{D}_0$, then ${}^*w^*$ is also in $\mathcal{D}_0$.

Lemma 2.3. *Let s,t be in S such that st has order 3, z in W and d in $\mathcal{D}_0$ such that $z \underset{LR}{\sim} d$. Assume that $\delta_{z^{-1},d,d} = 0$ and $\mu(z,d) \neq 0$. Then*
(a) z is in $\mathcal{D}_1$ and $\mu(z,d) = \pi(z)$.
(b) If $z \underset{L}{\sim} z^{-1} \underset{L}{\sim} d$ and ${}^*z^*$ is well defined for $* = \{s,t\}$, then ${}^*z^*$ is in $\mathcal{D}_1$ and $\mu({}^*z^*, {}^*d^*) = \pi({}^*z^*) = \pi(z) = \mu(z,d)$.

Proof. (a) Using Springer's formula 1.2 (a) and 1.3 (f), we see

$$\mu(z,d) = \delta_{z^{-1},d,d} + \sum_{f \in \mathcal{D}_1} \gamma_{z^{-1},d,f}\pi(f).$$

Now $\delta_{z^{-1},d,d} = 0$, so we get

$$\mu(z,d) = \sum_{f \in \mathcal{D}_1} \gamma_{z^{-1},d,f}\pi(f).$$

By 1.3 (c), $\gamma_{z^{-1},d,f} \neq 0$ implies that $f = z^{-1}$ and $\gamma_{z^{-1},d,f} = 1$. Hence z^{-1} is in $\mathcal{D}_1$, or equivalently z is in $\mathcal{D}_1$, and $\mu(z,d) = \pi(z^{-1}) = \pi(z)$.

(b) According to 2.1 (a) and 2.1 (d), we have $\mu({}^*z^*, {}^*d^*) = \mu(z,d) \neq 0$. By 2.2 (e), ${}^*d^*$ is in $\mathcal{D}_0$. By Springer's formula 1.2 (a), 1.3 (f) and 2.2 (b), we have

$$\mu({}^*z^*, {}^*d^*) = \delta_{{}^*(z^{-1})^*,{}^*d^*,{}^*d^*} + \sum_{f \in \mathcal{D}_1} \gamma_{{}^*(z^{-1})^*,{}^*d^*,f}\pi(f).$$

We need prove $\delta_{{}^*(z^{-1})^*,{}^*d^*,{}^*d^*} = 0$. By 2.1 (b-c), 2.1 (e-f) and 2.2 (c), $h_{{}^*(z^{-1})^*,{}^*d^*,{}^*d^*} = h_{(z^{-1})^*,{}^*d,d}$, so

$$\delta_{{}^*(z^{-1})^*,{}^*d^*,{}^*d^*} = \delta_{(z^{-1})^*,{}^*d,d}.$$

By Springer's formula 1.2 (a) and 2.2 (b), we have

$$\mu({}^*z, {}^*d) = \sum_{d' \in \mathcal{D}_0} \delta_{(z^{-1})^*,{}^*d,d'} + \sum_{f \in \mathcal{D}_1} \gamma_{(z^{-1})^*,{}^*d,f}\pi(f).$$

By 2.1 (a), $\mu({}^*z, {}^*d) = \mu(z,d)$, so

$$\mu({}^*z, {}^*d) = \mu(z,d) = \sum_{d' \in \mathcal{D}_0} \delta_{(z^{-1})^*,{}^*d,d'} + \sum_{f \in \mathcal{D}_1} \gamma_{(z^{-1})^*,{}^*d,f}\pi(f).$$

By 2.2 (d), we know that $\gamma_{(z^{-1})^*,{}^*d,f} = \gamma_{z^{-1},d,f}$. Thus, $\gamma_{(z^{-1})^*,{}^*d,f} \neq 0$ implies that $f = z^{-1}$ and $\gamma_{(z^{-1})^*,{}^*d,f} = 1$. By (a), we also have $\mu(z,d) = \pi(z) = \pi(z^{-1})$. As a consequence, we must have $\delta_{(z^{-1})^*,{}^*d,d'} = 0$ for any d' in $\mathcal{D}_0$. Therefore, $\delta_{{}^*(z^{-1})^*,{}^*d^*,{}^*d^*} = 0$. Since ${}^*d^*$ is in $\mathcal{D}_0$ (see 2.2 (e)), by 1.3 (c), we know $\gamma_{{}^*(z^{-1})^*,{}^*d^*,f} \neq 0$ implies that $f = {}^*(z^{-1})^*$ and $\gamma_{{}^*(z^{-1})^*,{}^*d^*,f} = 1$. Therefore ${}^*(z^{-1})^* = ({}^*z^*)^{-1}$ and ${}^*z^*$ are in $\mathcal{D}_1$. Moreover we have $\mu(z,d) = \mu({}^*z^*,{}^*d^*) = \pi({}^*(z^{-1})^*) = \pi({}^*z^*)$. The lemma is proved. □

2.4. Now we can prove the theorem. When $n = 1$, Theorem 1.5 is clearly true. Now assume that $n \geq 2$. By the discussion in 1.4, we only need to prove that for some $d \in \mathcal{D}_0$, if $z \underset{L}{\sim} z^{-1} \underset{L}{\sim} d$ and $\mu(z,d) \neq 0$, then $z \in \mathcal{D}_1$. According to [S1, 18.3.2], there exists a sequence of right star operations such that its composition sends z (resp. d) to some $z'w$ (resp. yw) for the longest element w of a parabolic subgroup of W and such that $z'w \underset{L}{\sim} w$ (resp. $yw \underset{L}{\sim} w$). Using 2.2 (a), we can apply the corresponding (in the same order) left star operations to $z'w$ (resp. yw). Then we obtain some element xw (resp. w, here 2.2 (e) is needed) such that $xw \underset{L}{\sim} (xw)^{-1} \underset{L}{\sim} w$. By 2.1 (a) and 2.1 (d), clearly we have $\mu(xw,w) = \mu(z,d) \neq 0$. Note that for x_1 and x_2 in a Coxeter group, we have $R(x_1) = R(x_2)$ (resp. $L(x_1) = L(x_2)$) if x_1 and x_2 are in the same left (resp. right) cell of the Coxeter group, see [KL]. By 2.1 (b) and 2.1 (e), we have $R(z'w) = R(yw) = R(xw) = L(xw) = R(w) = L(w)$. Thus it is obvious that $\delta_{xw,w,w} = 0$. By Lemma 2.3 (a), we see that xw is in $\mathcal{D}_1$. By Lemma 2.3 (b) we know that z is in $\mathcal{D}_1$. The theorem is proved.

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