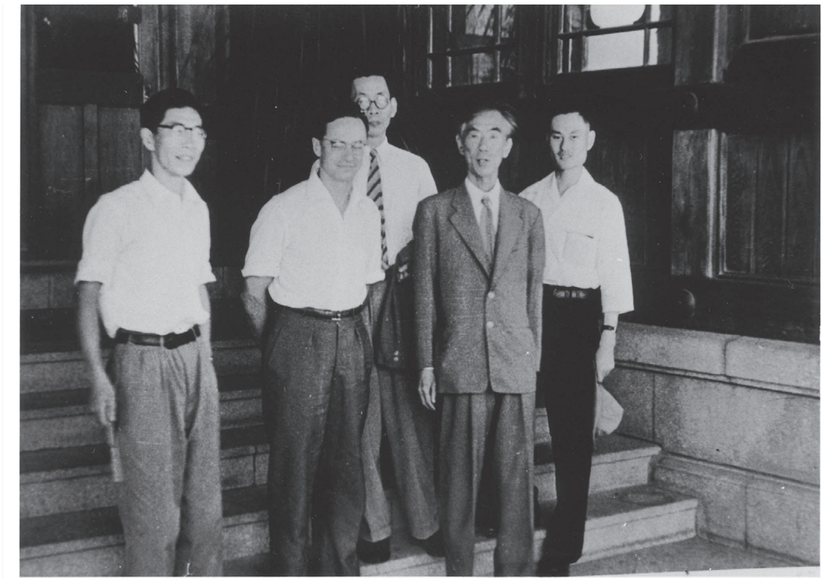
Part I



From left to right, S. Nakano, J.-P. Serre, Y. Akizuki, K. Oka and S. Hitotsumatsu at Nara, 1955[1]

K. Oka and T. Nishino, 1965[2]

[1]By courtesy of Mrs. S. Kujiraoka.
[2]By courtery of Asahi Shimbun and Asahi Journal.

岡 潔
Oka, Kiyoshi

Curriculum Vitae

1901 April 19	:	Born in Osaka
1919 – 1921	:	The Third High School (Dai San Kou) in Kyoto
1922 – 1925	:	Kyoto Imperial University
1925 – 1929	:	Lecturer, Kyoto Imperial University
1929 – 1932	:	Assistant Professor, Kyoto Imperial University
1929 – 1932	:	Sabbatical stay in Paris
1932 – 1940	:	Assistant Professor, Hiroshima University
1941 – 1942	:	Research fellow, Hokkaido University
1949 – 1964	:	Professor, Nara Women's University
1960	:	Cultural Medal (Bunka-kunsyō)
1969 –	:	Professor, Kyoto Sangyo University
1978 March 1	:	Died in Nara

Published Papers

Note sur les familles de fonctions multiformes etc.,
J. Sci. Hiroshima Univ. 4 (1934), p.93-98.

Sur les fonctions analytiques de plusieurs variables

I Domaines convexes par rapport aux fonctions rationnelles,
J. Sci. Hiroshima Univ. 6 (1936), 245-255.

II Domaines d'holomorphie,
J. Sci. Hiroshima Univ. 7 (1937), 115-130.

III Deuxieme problème de Cousin,
J. Sci. Hiroshima Univ. 9 (1939), 7-19.

IV Domaines d'holomorphie et domaines rationnellement convexes,
Japanese J. Math.17 (1941), 517-521.

V L'intégrale de Cauchy,
Japanese J. Math.17 (1941), 523-531.

VI Domaines pseudoconvexes.
Tôhoku Math. J. 49 (1942), 15-52.

VII Sur quelques notions arithmétiques,
Bul. Soc. Math. France 78 (1950), 1-27.

VIII Lemme fondamental,
J. Math. Soc. Japan 3 (1951) 204-214;259-278.

IX Domaines finis sans point critique intérieur,
Japanese J. Math. 27 (1953), 97-155.

X Une mode nouvelle engendrant les domaines pseudoconvexes,
Japanese J. Math. 32 (1962), 1-12.

Sur les domaines pseudoconvexes,
Proceedings of the Imperial Academy, Tokyo,(1941) 7-10.

Note sur les fonctions analytiques de plusieurs variables,
Kōdai Math. Sem. Rep., (1949). no. 5-6, 15–18.

Collected Papers

(i) Sur les Fonctions Analytiques de Plusieurs Variables,
Iwanami Shoten, Tokyo (1961), including papers I – IX.

(ii) Kiyoshi Oka Collected Papers,
translated by R. Narasimhan, commentaries by H.Cartan, edited by R. Remmert, Springer-Verlag, (1984), including papers I–X, “Note sur les familles de fonctions multiformes etc.” and “Sur les domaines pseudoconvexes”

(iii) Oka Kiyoshi Sensei ikou shuu (Posthumous Papers of Kiyoshi Oka),
7 Vols (1980–1983), edited by T. Nishino and A. Takeuchi.

Manuscripts included in ‘Posthumous Papers’

(i) Fonctions algébriques permutables avec une fonction rationnelle non–linéaire, 1930, (French)

(ii) Sur les ensembles de points à 4 dimensions engendrés analytiquement, 1934, vol. (French)

(iii) III– Exemple.

(iv) Rapport fragmentaire 1, 1942/8/7.

(v) Rapport 2 1942/10/8.

(vi) VII Deux lemmes sur la congruence de fonctions holomorphes, 1943/9/5

(vii) VIII Premier lemme fondamental pour domaines finis sans point critique intérieur, 1943/9/5,

(viii) IX Fonctions pseudoconvexes, 1943/8/24,

(ix) X Deuxième lemme fondamental, 1943/11/12.

(x) XI Domaines pseudoconvexes et domaines d’holomorphie finis, quelque theoremes pour les domaines d’holomorphie finis, 1943/12/12.

(xi) XII Representation d’ensembles caractéristiqes 1944/5/26.

(xii) XII Une extension de deuxieme problème de Cousin, 1945/2/28.

(xiii) XIII Sur une condition pour le théorème preliminaire de Weierstrass, 1945/11.

(xiv) Lemme de Picard, 1948/2/20.
(xv) Etude quantitative de congruence, 1948/8/6.
(xvi) VIII Un problème d'existence intérieure, 1948/12/7.
(xvii) XI Rappelées du printemps. 1949/4/9.
(xviii) Note sur les fonctions analytiques de plusieurs variables, 1949/12/1, (French).
(xix) Lemme fondamental, 1950/11/4

岡潔生誕百年記念多変数複素解析国際会議 京都/奈良 2001
2001 年 10 月 30 日〜11 月 5 日/11 月 6 日〜11 月 8 日
Memorial Coneference of Kiyoshi Oka's Centennial Birthday on Complex Analysis in Several Variables Kyoto/Nara 2001
October 30–November 5, Kyoto/ November 6-8, Nara

1. **Organizing Committee**

Honorary Chair: Toshio Nishino
(Professor Emeritus, Kyushu University)

Kyoto Programm (at RIMS Kyoto University):
Junjiro Noguchi (Chair, University of Tokyo)
Hirotaka Fujimoto (Kanazawa University)
Mikio Furushima (Kumamoto University)
Hideaki Kazama (Kyushu University)
Akio Kodama (Kanazawa University)
Kimio Miyajima (Kagoshima University)
Takeo Ohsawa (Nagoya University)
Hajime Tsuji (Tokyo Institute of Technology)
Tetsuo Ueda (Kyoto University)
Eric Bedford (Indiana)
Jean Pierre Demailly (Grenoble)
Klas Diederich (Wuppertal)
John Erik Fornaess (Michigan)
Bernard Shiffman (Johns Hopkins)
Nessim Sibony (Paris Sud)
Yum-Tong Siu (Harvard)

Nara Programm (joint with Nara Women's Library):
Hiroshi Yamaguchi (Chair, Nara Woemn's University)
Akira Takeuchi (Professor Emeritus Kyoto University)
Junjiro Noguchi (University of Tokyo)
Tetsuo Ueda (Kyoto University)
Takeo Ohsawa (Nagoya University)
Kazuko Matsumoto (Osaka Women's University)

2. Programms

Programm at Kyoto (October 30–November 5):

October 30 (Tuesday)

9:30 – Opening
10:30 – 11:30 **Toshio Nishino** (Kyushu Univ., Japan)
Mathematics of Professor Oka – a landscape in his mind –
13:30 – 14:30 **Kyoji Saito** (Kyoto Univ. RIMS, Japan)
Holomorphic invariant theory for the elliptic Lie group (the inversion map to the period map for elliptic primitive forms)
14:50 – 15:20 **Hirotaka Fujimoto** (Kanazawa Univ., Japan)
Some constructions of hyperbolic hypersurfaces in $P^n(C)$
15:50 – 16:20 **Franc Forstneric** (Univ. of Ljubljana, Slovenia)
The Oka principle for sub-elliptic mappings
16:30 – 17:00 **R. Michael Range** (State Univ. of New York, USA)
On the decomposition of holomorphic functions by integrals and the local CR extension theorem

October 31 (Wednesday)

9:00 – 10:00 **Hans Grauert** (Univ. Göttingen, Germany)
A simple way to perform the Levi-Oka-Theory
10:30 – 11:30 **Mitsuhiro Shishikura** (Kyoto Univ., Japan)
The rigidity in complex dynamics and the Teichmüller theory
13:30 – 14:00 **Jurgen Leiterer** (Humboldt-Univ., Germany)
On Hausdorffness of the Dolbeault cohomology and an open problem concerning the Oka-Grauert principle for Banach space bundles
14:10 – 14:40 **Junjiro Noguchi** (Univ. of Tokyo, Japan)
Intersection multiplicities of holomorphic and algebraic curves with divisors
14:50 – 15:20 **Satoru Shimizu** (Tohoku Univ., Japan)
Prolongation of holomorphic vector fields on a tube domain and its applications
15:50 – 16:20 **Katsutoshi Yamanoi** (Kyoto Univ. RIMS, Japan)
On Nevanlinna theory for holomorphic curves in Abelian varieties
16:30 – 17:00 **Seiki Mori** (Yamagata Univ., Japan)
Holomorphic mappings and deficiencies
17:10 – 17:40 **Jawher El Goul** (Univ. Toulouse III, France)
Demailly's 2-jet negativity of certain hyperbolic fibrations

November 1 (Thursday)

9:00 – 10:00 **Masatake Kuranishi** (Columbia Univ., USA)
Cartan geometry and complex analysis

10:30 – 11:30 **John Erik Fornaess** (Univ. of Michigan, USA)
Short $\mathbf{C}^2$
13:30 – 14:00 **Tetsuo Ueda** (Kyoto Univ., Japan)
Fixed points of polynomial automorphisms of C^n
14:10 – 14:40 **Stephanie Nivoche** (Univ. Toulouse III, France)
Proof of a Zahariuta's conjecture about a Kolmogorov's problem
14:50 – 15:20 **Takao Akahori** (Himeji Inst. of Tech., Japan)
Flat connection over the parameter space of the versal family of CR structure
15:50 – 16:20 **John M. Lee** (Univ. of Washington, USA)
Analysis on domains in strictly pseudoconvex CR manifolds
16:30 – 17:00 **Gen Komatsu** (Osaka Univ., Japan)
Singularity and invariance of Sobolev-Bergman kernels of strictly pseudo-convex domains
17:10 – 17:40 **Kang-Tae Kim** (Pohang Univ. of Sci. and Tech., Korea)
Two dimensional analytic polyhedra with non-compact automorphism group

November 2 (Friday)

9:00 – 10:00 **Joseph J. Kohn** (Princeton Univ., USA)
Ideals of multipliers
10:30 – 11:30 **Eric Bedford** (Indiana Univ., USA)
Invariant measures for bimeromorphic mappings of complex surfaces
13:30 – 14:00 **Bernard Shiffman** (Johns Hopkins Univ., USA)
Random polynomials with a given Newton polytope: A probabilistic form of Kouchnerenko's theorem
14:10 – 14:40 **Steven Lu** (Osaka Univ., Japan)
Infinitesimal Kobayashi metrics on algebraic manifold
14:50 – 15:20 **Oswald Riemenschneider** (Univ. Hamburg, Germany)
The monodromy covering of the versal deformation of cyclic quotient surface singularities
15:50 – 16:20 **Hajime Tsuji** (Tokyo Inst. of Tech., Japan)
Inductive structure of flips
16:30 – 17:00 **Shigeharu Takayama** (Kyushu Univ., Japan)
Seshadri constants and a criterion for bigness of pseudo-effective line bundles
17:10 – 17:40 **Su-Jen Kan** (Academia Sinica, Taiwan)
Complexifications and the rigidity

November 3 (Saturday)

9:00 – 10:00 **T. Kawai - T. Aoki - Y. Takei** (Kyoto Univ. RIMS, Japan)

Exact steepest descent method — a newly found missing link between microlocal analysis and exact WKB analysis

10:30 – 11:30 **Yum-Tong Siu** (Harvard Univ., USA)

Some problems and results in the interface of complex Neumann estimates and complex geometry

13:30 – 14:00 **Shoshichi Kobayashi** (Univ. of California, USA)

Almost complex manifolds and hyperbolicity

14:10 – 14:40 **Mikael Passare** (Stockholm Univ., Sweden)

Amoebas, Monge-Ampère measure, and triangulations of the Newton polytope

14:50 – 15:20 **Nessim Sibony** (Univ. Paris-Sud, France)

$\bar{\partial}$-equation on harmonic and positive currents

15:50 – 16:20 **Kazuko Matsumoto** (Osaka Women's Univ., Japan)

On the convexity of complements of analytic subsets

16:30 – 17:00 **Yoshihiro Aihara** (Numazu Col. of Tech., Japan)

Uniqueness problems for meromorphic mappings under condition on the preimages of divisors

17:10 – 17:40 **Manabu Shirosaki** (Osaka Prefecture Univ., Japan)

Hypersurfaces and uniqueness of holomorphic mappings

November 4 (Sunday)

9:00 – 10:00 **Yoichi Miyaoka** (Univ. of Tokyo, Japan)

Rational curves and complex symplectic geometry

10:30 – 11:30 **Georg Schumacher** (Philipps-Univ. Marburg, Germany)

Quasi-projectivity of moduli spaces

14:10 – 14:40 **Ken-Ichi Yoshikawa** (Univ. of Tokyo, Japan)

K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space

14:50 – 15:20 **Stephen S.-T. Yau** (Univ. of Illinois at Chicago, USA)

Numerical characterization of affine variety to be a cone over non-singular projective variety

15:50 – 16:20 **Andrei Iordan** (Univ. Paris VI, France)

Cohomology of forms with Sobolev coefficients on pseudoconcave domains in the projective space

16:30 – 17:00 **Kengo Hirachi** (Univ. of Tokyo, Japan)

Explicit relation between the asymptotic expansions of the Bergman and the Szegö kernels

17:10 – 17:40 **Takahiro Nakata** (Nagoya Univ., Japan)
Some convexity properties of covering spaces of pseudoconvex manifolds

November 5 (Monday)

9:00 – 10:00 **László Lempert** (Purdue Univ., USA)
Analytic cohomology in infinite dimensional spaces
10:30 – 11:30 **Takeo Ohsawa** (Nagoya Univ., Japan)
L^2 division theorems on manifolds

Programm at Nara (November 6-8):

Talks

November 7 (Wednesday) – in Japanese –

10:00 – 11:00 **Toshio Nishino** (Kyushu Univ., Japan)
My teacher Oka as a mathematician
— background of his creative thinking
11:15 – 12:15 **Hironori Shiga** (Chiba Univ., Japan)
Honka-dori as a methodology of mathematics
— Luminescence of what I heard from late professor Oka
14:00 – 15:00 **Tohru Morimoto** (Nara Women's Univ., Japan)
Streams and Encounters in Mathematics
— Kiyoshi Oka and a theorem of Clifford

November 8 (Thurssday) – in English –

10:00 – 11:00 **Pierre Dolbeault** (Univ. Paris VI, France)
On the Influence of Oka's Work
11:15 – 12:15 **John Wermer** (Brown Univ., USA)
Boundaries of the Analytic Varieties

Exhibition of Oka Collection
November 1 – 8 (10:00 - 16:00)

Notes for the establishing his papers, especially Memoire VIII.

Letters to or from Prof. T. Takagi, H. Behnke, H. Cartan, A. Weil, R. Thom.

Lecture notes for the lectures in Department of Nara Women's University.

His favorite books through his life.

His (mathematical) diaries.

His books and essays written in Japanese, some of which attracted broad interest of the public, and were read by people from the young to the senior.

His calligraphies and photoes.

Talk between Kiyoshi Oka and Yasuo Akizuki
— 45 minuites NHK film, 1965 —

November 7 – 8 (10:00 –15:00)

List of Participants

The number of registered participants was one hundred forty one, and among them there were thirty five participants from oversea.

Abe, Makoto (Kumamoto Univ.)
Adachi, Yukinobu
Adachi, Kenzo (Nagasaki Univ.)
Aihara, Yoshiiro (Numazu Col. of Tech.)
Akahori, Takao (Himeji Inst. of Tech.)
Aoki, Takashi (Kinki Univ.)
Azukawa, Kazuo (Toyama Univ.)
Bedford, Eric (Indiana Univ.)
Bhupal, Mohan (Hokkaido Univ.)
Bland, John (Univ. of Toronto)
Cerne, Miran (Univ. of Ljubljana)
Cho, Koji (Kyushu Univ.)
Choi, Seon A. (Kyushu Univ.)
Dini, Gilberto (Univ. of Florence)
Dolbeault, Pierre (Univ. of Paris VI)
El Goul, Jawher (Univ. Paul Sabatier)
Enoki, Ichiro (Osaka Univ.)
Fornaess, John E. (Univ. of Michigan)
Forstneric, Franc (Univ. of Ljubljana)
Fujiki, Akira (Osaka Univ.)
Fujimoto, Hirotaka (Kanazawa Univ.)
Fujita, Keiko (Saga Univ.)
Fujita, Osamu (Nara Women's Univ.)
Fujita, Takao (Tokyo Inst. of Tech.)
Fukuda, Shigetaka (Gifu Shotoku Gakuen Univ.)
Fukushima, Yukio (Fukuoka Univ.)
Furushima, Mikio (Kumamoto Univ.)
Garfield, Peter (Univ. of Washington)
Grauert, Hans (Univ. Goettingen)
Harada, Tomoyo (Nara Women's Univ.)
Hasegawa, Keizo (Niigata Univ.)
Hayashimoto, Atsushi (Nagano Col. of Tech.)
Hirachi, Kengo (Univ. of Tokyo)
Hirai, Etsuko (Kyoto Sangyo Univ.)
Hiraoka, Yoshiko (Toyo Univ.)
Hisamatsu, Makoto (Tokyo Inst. of Tech.)
Honda, Tatsuhiro (Ariake Col. of Tech.)
Hyuga, Takayuki (Nagoya Univ.)
Iordan, Andrei (Univ. of Paris VI)
Ishii, Yutaka (Kyushu Univ.)
Izumi, Shuzo (Kinki Univ.)
Jimbo, Toshiya (Nara Univ. of Education)
Jin, Teisuke (Univ. of Tokyo)

Kan, Su-Jen (Academia Sinica)
Kaneko, Akira (Ochanomizu Univ.)
Kanemaru, Tadayoshi (Kumamoto Univ.)
Kashiwara, Hiroko (Osaka Prefecture Univ.)
Kasuga, Kazuhiro (Niigata Univ.)
Kato, Kazuko (Ryukoku Univ.)
Kato, Mitsuo (Univ. of the Ryukyus)
Katsumi, Masaaki (Ishikawa Col. of Tech.)
Kawai, Takahiro (R.I.M.S., Kyoto Univ.)
Kazama, Hideaki (Kyushu Univ.)
Kim, Kang-Tae (Pohang Univ. of Sci. and Tech.)
Kim, Kyung N. (Kyushu Univ.)
Kizuka, Takashi (Kyushu Univ.)
Kobayashi, Shoshichi (Univ. of California)
Kobayashi, Masashi (Univ. of Tokyo)
Kodama, Akio (Kanazawa Univ.)
Kodama, Mitsuru (Kagoshima Univ.)
Kohn, Joseph J. (Princeton Univ.)
Koizumi, Eisuke (Tohoku Univ.)
Komatsu, Gen (Osaka Univ.)
Kota, Osamu (Rikkyo Univ.)
Kuranishi, Masatake (Columbia Univ. and Tokyo Univ.)
Lee, John M. (Univ. of Washington)
Leiterer, Juergen (Humboldt-Univ. zu Berlin)
Lempert, Laszlo (Purdue Univ.)
Lu, Steven (Osaka Univ.)
Mabuchi, Toshiki (Osaka Univ.)
Maeda, Hidetoshi (Waseda Univ.)
Maegawa, Kazutoshi (Kyoto Univ.)
Matsugu, Yasuo (Shinshu Univ.)
Matsumoto, Kazuko (Osaka Women's Univ.)
Matsushima, Toshio (Ishikawa Col. of Tech.)
Matsushita, Daisuke (Hokkaido Univ.)
Miyajima, Kimio (Kagoshima Univ.)
Miyaoka, Yoichi (Univ. of Tokyo)
Miyatake, Motoko
Miyazawa, Kazuhisa (Nagoya Univ.)
Mori, Seiki (Yamagata Univ.)
Murakami, Masaaki (Kyoto Univ.)
Nakagawa, Yasuhiro (Tohoku Univ.)
Nakamura, Yayoi (Ochanomizu Univ.)
Nakata, Takahiro (Nagoya Univ.)
Namba, Makoto (Osaka Univ.)
Nishihara, Masaru (Fukuoka Inst. of Tech.)
Nishimura, Yasuichiro (Osaka Medical College)
Nishino, Toshio (Kyushu Univ.)
Nivoche, Stephanie (Univ. Toulouse)
Noguchi, Junjiro (Univ. of Tokyo)
Nono, Kiyoharu (Fukuoka Univ. of Education)
Obitsu, Kunio (Kagoshima Univ.)
Ohsawa, Takeo (Nagoya Univ.)
Ohta, Tomoaki (Kyushu Kyoritsu Unv.)
Oh'uchi, Shigeki (Univ. of Tokyo)
Okai, Takayuki
Okuyama, Yusuke (Shizuoka Univ.)
Passare, Mikael (Stockholm Univ.)
Range, R. Michael (State Univ. of

New York)
Riemenschneider, Oswald (Univ. Hamburg)
Saito, Kyoji (R.I.M.S., Kyoto Univ.)
Sakai, Akira (Univ. of Osaka Prefecture)
Sakai, Makoto (Tokyo Metropolitan Univ.)
Schumacher, Georg (Univ. Marburg)
Shiffman, Bernard (Johns Hopkins Univ.)
Shiga, Hironori (Chiba Univ.)
Shiga, Kiyoshi (Gifu Univ.)
Shimizu, Satoru (Tohoku Univ.)
Shimomura, Katsunori (Ibaraki Univ.)
Shinohara, Tomoko (Kanazawa Univ.)
Shirosaki, Manabu (Osaka Prefecture Univ.)
Shishikura, Mitsuhiro (Kyoto Univ.)
Sibony, Nessim (Univ. Paris-Sud)
Siu, Yum-Tong (Harvard Univ.)
Sumi, Hiroki (Tokyo Inst. of Tech.)
Suzuki, Masakazu (Kyushu Univ.)
Tajima, Shinichi (Niigata Univ.)
Takahashi, Sechiko (Nara Women's Univ.)
Takamura, Masakazu (Kanazawa Univ.)
Takayama, Shigeharu (Kyushu Univ.)
Takei, Yoshitsugu (R.I.M.S., Kyoto Univ.)
Takeuchi, Akira (Kyoto Univ.)
Takeuchi, Shigeru (Gifu Univ.)
Terada, Toshiaki (Shiga Univ. of Medical Sci.)
Toda, Nobushige (Nagoya Inst. of Tech.)
Tomaru, Tadashi (Gunma Univ.)
Tsuboi, Shoji (Kagoshima Univ.)
Tsuji, Hajime (Tokyo Inst. of Tech.)
Uchimura, Keisuke (Tokai Univ.)
Ueda, Tetsuo (Kyoto Univ.)
Umeno, Takashi (Kyushu Sangyo Univ.)
Watanabe, Kiyoshi (Kobe Univ.)
Wermer, John (Brown Univ.)
Winkelmann, Joerg (Korean Inst. for Advanced Studies)
Yamagishi, Yoshikazu (Ryukoku Univ.)
Yamaguchi, Hiroshi (Nara Women's Univ.)
Yamanoi, Katsutoshi (R.I.M.S., Kyoto Univ.)
Yau, Stephen S.-T. (Univ. of Illinois)
Yoshida, Mamoru (Fukuoka Univ.)
Yoshikawa, Ken-Ichi (Univ. of Tokyo)

Message from Professor Henri Cartan

Je suis heureux de m'associer à l'hommage qui sera rendu à Kiyoshi OKA à l'occasion du centième anniversaire de sa naissance. C'est en 1934 que nos relations épistolaire commencèrent. Oka avait lu le livre de Behnke–Thullen où étaient exposés une série de problèmes non résolus de la théorie des fonctions analytiques de plusieurs variable complexes. Il se mit alors en devoir de les résoudre et c'est ce qu'il fit entre 1934 et 1961, dans une série de 9 Mémoires écrits en français (à vrai dire dans une langue française qui lui était personelle). Nos relation épistolaires furent interrompues en 1940 par la guerre, et ne furent reprises qu'en 1948. En cette année-là, Oka m'envoya le manuscrit de son Mémoire VII, où il entroduisait la notion d'idéaux de domaines indéterminés. Ce Mémoire fut publié au Bulletin de la Société Mathématique de France, juste après un article où j'introduisais les même notions sous d'autre formes.

En 1963, au cours d'un long séjour au Japon, j'eus le privilège de passer une journée entière ans la ville de Nara, où enseignait alors OKA. Il me fit visiter les principaux temples de cette ville. Ce fut mon dernier contact avec cet homme exceptionnel.

Les Œuvres Complètes d'OKA avaient été publiées au Japon de son vivant. Lorsque, en 1982, les Editions Springer-Verlag décidèrent d'en publier une nouvelle édition, sous la direction de R. Remmert et R. Narashimhan, je fus heureux d'écrire un commentaire détaillé de chacun des Mémoires d'OKA et de mettre ainsi en évidence ce qu'on peut bien appeler son génie.

Henri Cartan
(Paris, Juillet 2001)

Part II

Advanced Studies in Pure Mathematics 42, 2004
Complex Analysis in Several Variables
pp. 17–30

Mathematics of Professor Oka – a landscape in his mind –

Toshio Nishino

Preface

As a mathematician working in several complex variables, I would like to share with you the pleasure of being here to attend this marvelous symposium, held in honor of late Professor Kiyoshi Oka's 100th birthday. As one of his students, I would like to express my sincere thanks to you all for its success.

He was born on 1901/04/19 in Osaka and passed away on 1978/03/01 in Nara. The day of his last breath was, to my memory, a calm day of early spring. I remember that a thin veil of mist was wandering on the hills of Saki that day. More than two decades have slipped by since then.

In his carrier Oka published only ten papers, except for three summaries. However, in contrast to this apparent scarceness, influence of his work is immense. His idea does not stay within the realm of several complex variables, but extends far beyond, contributing to the development of whole mathematics. Such a state of art will surely be credited by many of the talks that are going to be given here.

On my side, I was first allowed to call him as my teacher in 1956. Thanks to this opportunity, from the next year I could spend a few years with him in Nara Women's University. Actually we sat together desk to desk in a humble office and I could hear many things then from horse's mouth.

Further, by courtesy of his bereaved family, I could read his many unpublished papers and materials prepared for research. These are mainly gathered in the volume 'Posthumous Papers', which is exhibited in the homepage of the library of Nara Women's University (http://www.lib.nara-wu.ac.jp/oka/). So there are materials enough for drawing up a complete picture of Oka's research style, namely we can see how he approached each question and from where he viewed the whole landscape of the research field.

Received March 16, 2002.

To my great pleasure I was asked to give the first talk in this meeting. What is anticipated for in the talk is, I guess, to describe a sort of landscape which Oka had in mind. Therefore the purpose of my talk will be to give you a suggestion how a theory of mathematics grew up in his mind. I will do it by pursuing the process how he established "the lifting principle" which I regard, without so much prejudice I hope, his ultimate achievement.

Such a trial may be justified because, as I have already mentioned, Oka left many things behind about this topic. For instance, a paper titled with

"Rappelées du printemps"

written around 1949 presents his research activity historically as a reminiscence. (The word 'printemps' ought to mean here 'the beginning'.) In the following, many materials are without mentioning attributed to this article. Nevertheless my apology is for letting some speculation to creep in, which is, admittedly I hope, inevitable to accomplish my task.

Oka's main research purposes

Let me first show you, just as it is, how Oka began the introduction of Paper I.

> Malgré le progrès récént de la théorie des fonctions analytiques de plusieurs variables complexes, diverses choses importantes restent plus ou moins obscures, notamment : le type de domaines dans lesquels le théorème de Runge ou ceux de M. P. Cousin subsistent, la relation entre la convexité de M. F. Hartogs et celle de MM. H. Cartan et P. Thullen; parmi eux il y a des relations intimes. C'est à traiter ces problèmes que le présent mémoire et ceux qui suivront, sont destinés.

This is supplemented in the footnote:

> Voir l'Ouvrage de MM. H. Behnke et P. Thullen : *Theorie der Funktionen mehrerer komplexer Veränderlichen*, spécialement aux pages 54, 68, 79.

To solve the problems that had been raised in this book, Oka devoted most of his life.

Keeping this in mind, let me briefly describe some of the steps made during his youth, when he had not yet confronted with this task.

§1. Steps to the principal questions

1. In 1919, Oka entered the Third High School in Kyoto, which was one of the most prestigious high school in Japan at that time. This is today a part of Kyoto University. I was told that he was then fascinated by Poincaré's essays 'La Science et l'Hypothèse' and 'Science et Méthode' which were both widely read in those days. Later he recalls Poincaré's influence on his thought as follows.

> "I was moved very much by Poincaré's question 'How do mathematical discoveries come up?' Since then I have intended to solve it to the best of my ability."

It seems that he considered this question not to be pursued just from curiosity, but to be exactly solved by exploiting all his research activity as basic data. The same thing can be said for another question 'How do mathematical researches start?'

The point is that Oka, as a mathematician, sincerely asked how mathematical researches should be done.

Probably I must admit then that he regarded me as one of the marmots to test his method. Of course you cannot conclude from this miserable example that his method is useless, for the results of experiments usually depend on the materials.

2. Let me come back to Oka's mathematics. It was around 1927 that he started his own research. Really nothing had been done before. He wrote down the result in French under the title

Fonctions algébriques permutables avec une fonction rationnelle non-linéaire

and left it as a typewritten manuscript. This paper was never published although it was once submitted for publication through G. Julia. Let me show you only the statement of the result:

> Let $R(x)$ be a rational function of degree at least 2, and let $A(x)$ be an algebraic function. If one substitutes $A(x)$ into $R(x)$, the composite $R[A(x)]$ is well-defined as an algebraic function. Conversely, if one substitutes $R(x)$ into $A(x)$, the composite $A[R(x)]$ splits in general into several different algebraic functions corresponding to the branches of A. If one of these algebraic functions happens to coincide with $R[A(x)]$, we say that $R(x)$ and $A(x)$ are permutable.

In this situation the following holds true.

≪If a given algebraic function (of a reduced form in a suitable sense) is permutable with a rational function of degree at least 2, it must satisfy an algebraic equation arising from the multiplication formulae of e^z, $\cos z$ or elliptic functions.≫

Clearly this is a continuation of Julia's work[1].

As a paper of Oka, this article is, to my impression, most involved and most intriguing. Probably he enjoyed himself by writing a puzzling paper.

Later he wrote that this article might be published as well because of the following reasons.

1. This is the very 'melody' of the current of our mathematical research (indicating something more permanent than the mathematical result itself).
2. Without this paper, mutual consistency (or harmony) of later works would not be clear enough (because it shows the original form).

It had, however, never been published before his death.

3. To say some words about his surroundings at that time, Oka got a sabbatical in 1929 to visit France and stayed there for three years. The work on iteration just mentioned above was carried over in France for one year, more or less, and written down around 1930.

The reason why Oka wanted to stay in France was probably because he wanted to get acquainted with Julia, but the biggest motivation for going abroad was, according to his own word, because he thought "I will never be able to see a new ground of mathematics that deserves to be cultivated, as long as I stick to a life in Japan".

As he expected, soon after he moved to France he found the desired new ground, the field of several complex variables.

He used to talk about the feeling when he first saw Goursat's brief introduction to this field which was written in the middle part of the second volume of his 'Cours d'Analyse'. He explained it to me by quoting a haiku:

Kiri nagara Ookina machini Idenikeri (Ichiku)

(The meaning is; I have just arrived at a place, where I see a huge city in a dense fog.)

---

[1]G. Julia, Mémoire sur la permutabilité des fractions rationnelles, Annale de l'Ecole Normale Superieure, 1922.

After that he set forth to work in the field of several complex variables, by throwing away at first the affection to the subject of iteration, on which he had been working for almost 3 years.

There was, according to him, another difficulty in changing the research field. He told me that in several variables, compared to the case of one variable, things were much more complicated. However, he could overcome this difficulty by reading Julia's famous paper[2] again and again. Soon he could obtain several results before going back to Japan in 1932. These results were published as a summary without proofs under the title

Note sur les familles de fonctions multiformes etc., 1934

which became accordingly his first published paper. Its detail is left as an unfinished paper, whose manuscript is handwritten in French, and is approximately 150 pages long. Nevertheless, when it is combined with another manuscript, which was not totally translated into French, the paper is more or less in a complete form.

Let me show you main results of this Note. I have chosen two simple ones among three theorems.

A. Normal family of analytic sets. Let $\mathcal{F} = \{S\}$ be a family of analytic surfaces in a domain of two complex variables x, y. If the area of S is uniformly bounded, then $\mathcal{F}$ is a normal family, (normality of $\mathcal{F}$ is defined in terms of the defining functions of S). In general, the hull of normality for $\mathcal{F}$ is pseudoconvex.

B. Generalization of Hartogs theorem. Let E be a closed subset of a domain $\mathcal{D}$ of (x, y)-variables. E is called an H-set if E is locally the complement of a pseudoconvex domain. Let $\mathcal{D}$ be the product of a domain in the x-plane and the y-plane, and let E be an H-set in $\mathcal{D}$. Suppose that $E(x') := E \cap \{x = x'\}$ is uniformly bounded in x'. In this situation, if the set of points x for which $E(x)$ consists of finitely many points has positive logarithmic capacity, then $E(x)$ consists of finitely many points for all x, and E is then an analytic surface of $\mathcal{D}$. A similar statement is true when $E(x)$ consists of countably many points.

Some explanation seems to be needed about the motivation of this research. As the title shows, the purpose of this paper was to study the convergence of multivalued analytic functions from the viewpoint of function theory of two variables. As in the first work on iteration,

[2] G. Julia, Sur les familles de fonctions analytiques de plusieurs variables, Acta Math., **47**, 1927.

the problem had been raised by Julia. However, this time, the work is related to Oka's later accomplishment in many aspects.

4. Before describing how Oka did his full-scale works, I would like to draw your attention, as well as aforementioned Pincaré's problem, to the following episode.

Oka says "When I need a method to solve a problem, and cannot get anything in mathematics, I try to find a key to the riddle in other places". To carry out this somewhat paradoxical way, he started to study Basho's poetry deeply, already when he was staying in France, because he thought "I must absorb the time-honored culture of Japan that may help mathematical researches", according to his words. If I am allowed to say it in other words, Oka's style of doing mathematics is first to bring things in mind closer to his inmost emotion, so that he can recognize the spiritual melody they play, and next to give some forms to it, which is the stage of creation.

§2. The rise of problems

1. In 1932, soon after the return from Paris, Oka got a position as an associate professor of Hiroshima Bun-Rika Daigaku (= Hiroshima University).

Although he was quite eager to work in the field of several complex variables, it was still necessary for him to grasp its whole picture before getting down to a full-scale research.

However, the library of that university, which was still in its infancy, was useless for that purpose. The lack of information should have disappointed him very much. It was just at that moment that the book of Behnke-Thullen

Theorie der Funktionen mehrerer komplexer Veränderlichen

was published and brought to his desk.

As he wrote in the introduction of Paper I, Oka could grasp the central current questions of this field in virtue of Behnke-Thullen's book. Later he repeatedly wrote about the benefit of this book, showing his sincere gratitude.

2. That book, which Oka kept long at hand, finds now its place in the library of Nara Women's University. From the notes in the blanks one knows that he began to read it on January 2 in 1935. When he had read it through, he said, the problems written in the introduction of Paper I emerged as 'mountains that separate the future and the past of several complex variables'.

By the words that the problems 'separate the future and the past', Oka meant that it is impossible to go ahead without conquering them. From the beginning Oka bore a plan "to cross this mountain pass and to open a flower garden beyond it". Exploiting this opportunity I would like to add that Oka's Paper X was written as "an example of flower garden that would be opened beyond the pass". Once he told me, "Contrary to my expectation, it took me very long before crossing it."

§3. The lifting principle

1. Soon after the problems were clearly caught in his eyes, Oka set forth to work on them. What he did was, however, not to attack Cousin's first problem on analytic polyhedra. I can say it because, this 'goal'was almost achieved in the footnote of Oka's Paper I. Namely Cousin's first problem was solved for rationally convex domains by using Weil's integral formula similarly as Cousin's integral. For Oka, this discovery must not have been too small. In fact, an integral of this kind is used in the integral equation which plays the key role in solving the inverse problem of Hartogs.

In Oka's mind, the problems were connected to each other into one piece. For instance, in order to see the relation between the convexity notions of Hartogs and Cartan-Thullen, it is necessary to prove how the validity of Runge's theorem and Cousin's theorem depend on the types of the domains. In addition, if a solution of the latter question is useless to solve the former, that solution is meaningless, to put it extremely.

2. Oka read through Behnke-Thullen's book in a couple of months, and soon afterwards began to "look for the first move". In three months from then, however, he was stuck. He says "After such a period of no progress, no plan came into my mind any more, no matter how ridiculous it is". It is after this point that Oka showed marvelous originality.

Let me explain more concretely how he continued. Oka's method was to pursue the problem to a single point. Moreover he did it with a "pin-point accuracy". In the above case the point may be described as follows.

Inside the space of two complex variables x, y, a general domain D can be visualized in the following way.

First, on a sheet of paper you draw the x-plane and, to its right hand side, the y-plane. Then you take one point x' in the x-plane, and draw on the y-plane the slice $D(x')$ of D by $x = x'$. Since $D(x')$ changes as x' moves around, you may imagine a family of $D(x')$ when x' runs through a part of the x-plane.

The point is then

≪To provide a general circumstance for the domains of this kind, in which one can solve some questions, Cousin's first problem for instance.≫

(To my impression, there was in Oka's mind influence of Riemann's idea, of introducing Riemann surfaces to study multivalued functions.)

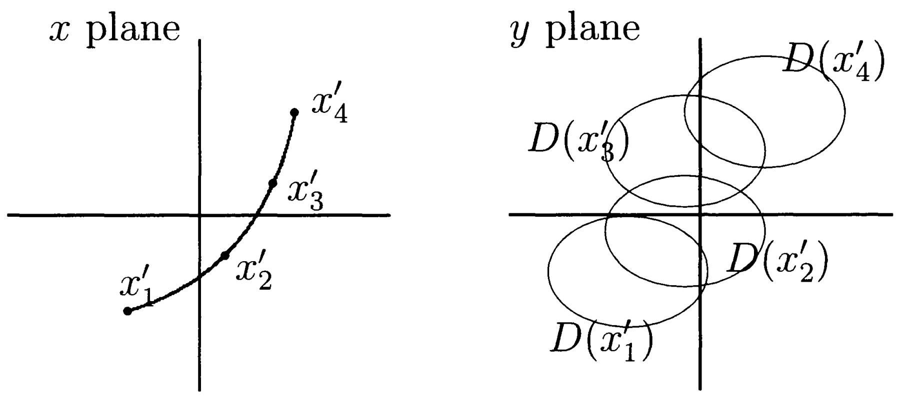


Fig. 1

Anyway, if you take several points on the x-plane and draw the corresponding domains in the y-plane, the picture looks, as above, like the leaves of trees reflected on the surface of water. Oka called such a figure "Ryoku-in-zu (projection of green leaves)" and looked them carefully day after day. He once told me that he had felt like easily walking on thin ice, which might mean that these days were for him a period of delightful devotion to a heavenly mission. This 'delight' is the melody of his heartstring. (Nothing can be understood by knowledge without ethos.) In such a situation, where he felt that his idea was exhausted, it seems that he was grasping something in such a way.

3. This question was settled in the following form:

Let $R_j(x)\,(j = 1,\ldots,m)$ be rational functions in n complex variables $x_1,\ldots,x_n$, and let (Δ) be a bounded closed region in the (x)-space defined by

$$(\Delta) \qquad |x_i| \leq r_i \;\; (i = 1,2,\ldots,n), \quad |R_j(x)| \leq 1 \;\; (j = 1,2,\ldots,m).$$

Then, by adding m complex variables $y_1,\ldots,y_m$ to (x), we consider in the product of the (x)-space and the (y)-space a closed polycylinder

$$(\mathrm{C}) \qquad |x_i| \leq r_i\,(i = 1,2,\ldots,n), \quad |y_j| \leq 1\,(j = 1,2,\ldots,m)$$

and an analytic subset

$$(\Sigma) \qquad y_j = R_j(x)\,(j = 1, 2, \ldots, m)$$

of (C).

The following expresses the miniatuarized relationship between (Δ), (Σ) and (C).

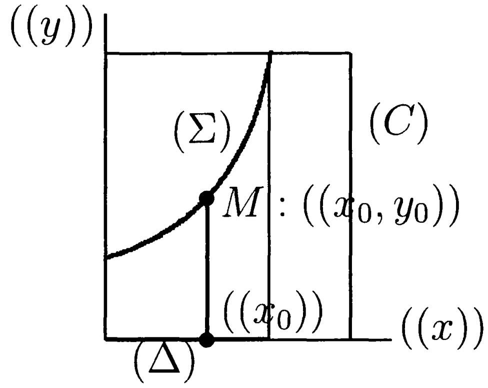


Fig. 2

In such a geometric situation the result is:

Theorem . *Given any holomorphic function $f(x)$ on (Δ), one can find a holomorphic function $F(x, y)$ on (C) satisfying*

$$f(x) = F[x_1, x_2, \ldots, x_n, R_1(x), \ldots, R_m(x)].$$

Oka used to call this theorem 'Jôku Ikô no Genri' (the lifting principle or, more literally, the hovering principle). A foundation was laid by this principle to study the types of domains on which the theorems of Cousin and Runge hold true.

August of that year was nearly going when Oka discovered this theorem. There is a widespread episode that someone close to Oka nicknamed him "encephalitis lethargica (= sleeping sickness)" because he nearly continued to doze every day during the end of that period.

As for the proof of this theorem, it was done by double induction on the dimensions, and, as a result, Cousin's first problem was settled at the same time. Oka wrote about this method in a footnote of Paper I as

> ≪ Je dois l'idée à M. H. Cartan pour ce mode d'application
> du théorème de M. Cousin.≫

and quoted Cartan's paper

Sur les fonctions de deux variables complexes. Bull. Sci. math. 1930.

When you look at this solution, you will see that the overlapping leaves in Fig. 1 are lifted to Σ, and holomorphic functions on Δ are extended holomorphically to the polycylinder above. Probably the lifting principle was already emotionally grasped when he was drawing the overlapping leaves.

§4. Subsequent questions

After 'the first move' was followed by the discovery of the lifting principle for rationally convex domains, several questions naturally arose.

1. In Paper I, the problem was solved for rationally convex domains, i.e. for those domains which are convex with respect to the family of rational functions. The gap between the rationally convex domains and the domains of holomorphy was filled by Paper II:

When $R_j(x)$ are holomorphic functions defined only on some neighbourhood of (Δ) , the proof for the rational case does not extend, although the geometric relation between (Δ), (C) and (Σ) is preserved. However, you draw now an analytic polyhedron, defined by polynomials, in any neighbourhood of (Σ). Then, by lifting it again you can solve the problem.

Paper II, which contains this, has many subtle points for reading. According to Oka, however, not so much difficulty was left before this work was finished, because he had already a stock of researches in the former Note.

2. It became a problem to find a distinction, if any, between the domains of holomorphy and rationally convex ones. As is well known, there is no such problem in the one variable case. This was also an important question on which Oka said "It's impossible to proceed further without knowing its solution".

Oka started from Gronwall's example which shows that Cousin's second problem is not necessarily solvable even on a domain of holomorphy.

Like Gronwall's example, take a domain of holomorphy, say $\mathcal{D}$, such that there exists an analytic hypersurface S for which there is no holomorphic function on $\mathcal{D}$ whose zero locus coincides with S, but one has a holomorphic function f on a neighbourhood of S satisfying $S = f^{-1}(0)$. In this situation, there exists a meromorphic function on $\mathcal{D}$ whose principal part is $1/f$. It is expected then, that this meromorphic function cannot be approximated by a sequence of rational functions uniformly on compact subsets of $\mathcal{D} \setminus S$, because if it were not the case the denominators of those rational functions would define analytic hypersurfaces that 'converge' to S.

To clarify this situation, Oka constructed another simple example of a domain of holomorphy on which Cousin's second problem is not necessarily solvable. This shows in particular that a domain of holomorphy is not necessarily rationally convex.

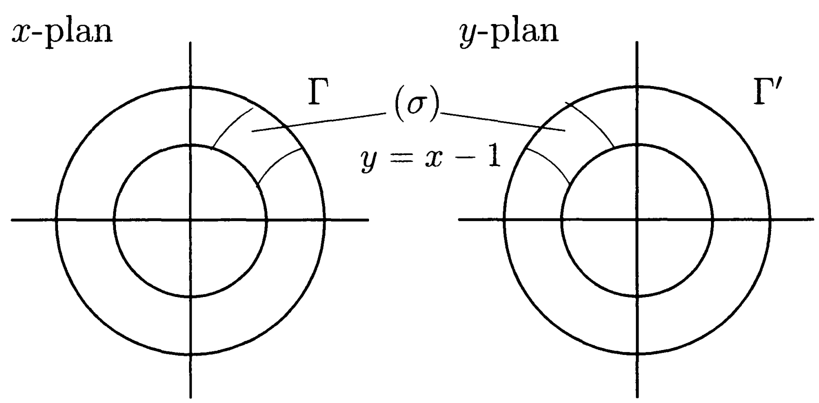


Fig. 3

III-Example, in Japanese, was written only to describe this example. The published Paper IV amounts to it.

There he classifies the domains into four species:

polycylinders, rationally convex domains,
domains of holomorphy, pseudoconvex domains.

According to this classification, things are stated more in order;

1. Paper I established a method of reducing rationally convex domains to polycylinders.
2. Paper II established a method of reducing domains of holomorphy to rationally convex ones.

From this viewpoint, it is clearly seen that the fruit of Oka's researches was establishment of a method of reducing the domains of arbitrary shape to standard ones as in the case of Riemann's mapping theorem.

3. It must be noted here that, in advance of IV, Cousin's second problem had been solved in Paper III. Oka had an opinion that Cousin's second problem itself should have been worried about much later if the above mentioned classification results had nothing to do with it. The reason is that it was not directly related to the principal questions for Oka. Nevertheless, solving Cousin's second problem after the first one may well be regarded as a quite natural procedure.

4. Once it was known that the domains of holomorphy are not necessarily rationally convex, the condition for the validity of Weil's integral formula was to be examined next. It was done as follows, to put it concisely.

Let $\mathcal{D}$ be a domain in the space of two variables x, y, and let $X_1, X_2, \dots, X_N$ be N holomorphic functions on $\mathcal{D}$ such that the (closed) domain

$$\Delta \quad : \quad |X_i(x,y)| \leq 1 \quad (i = 1, 2, \dots, N)$$

is a compact subset of $\mathcal{D}$. Then, letting S_i be the set $|X_i| = 1$ on the boundary of $\mathcal{D}$, we put $\sigma_{ij} = S_i \cap S_j$.

Suppose that one can associate, to each $X_i(x,y)$, two holomorphic functions $P_i(x, y; x_0, y_0)$, $Q_i(x, y; x_0, y_0)$ in $(x, y) \in \mathcal{D}$ and $(x_0, y_0) \in \mathcal{D}$, in such a way that

$$\text{(W)} \qquad X_i(x,y) - X_i(x_0, y_0) = (x - x_0)P_i + (y - y_0)Q_i$$

holds true.

Then, if we put

$$K_{ij}(x, y, x_0, y_0) = \frac{(P_i Q_j - P_j Q_i)}{[X_i(x,y) - X_i(x_0,y_0)][X_j(x,y) - X_j(x_0,y_0)]}$$

any holomorphic function $f(x,y)$ has an integral representation

$$f(x_0, y_0) = \frac{-1}{4\pi^2} \sum_{(i,j)} \int_{\sigma_{ij}} K_{ij}(x, y, x_0, y_0) f(x,y) dx dy,$$

which is the celebrated Weil's integral formula.

At that time, however, the condition (W) was known to hold only when $X_i(x,y)$ are rational functions. Concerning this question, Oka made a breakthrough by first discovering a fact that

≪Every holomorphic function on a domain of holomorphy can be uniformly approximated on the compact subsets by the branches of algebraic functions≫

and deduced from it that Weil's integral formula holds true on the domains of holomorphy without any essential changes. This is Paper V. Later it was simplified by H. Hefer and was generalized further after the introduction of the theory of systems of ideals of undetermined domains. They are stuffs that arose subsequently after the establishment of the lifting principle.

He has told me that these things came out very smoothly after the discovery of the lifting principle. In fact, around October of that year, the manuscript of Paper I (written in Japanese) was written up almost in the final form. As for the solvability of Cousin's second problem, the so called Oka principle, the discovery, belongs to later periods, to my speculation.

§5. Hartogs inverse problem

The inverse problem of Hartogs was a really difficult question even after such a preparation, so that Oka had to wait until some point around 1940 to get a solution. This problem was pursued to the following point:

Let $\mathcal{D}$ be a bounded domain in the space of two complex variables x, y, and let $\mathcal{D}_1, \mathcal{D}_2$ be respectively the subsets of $\mathcal{D}$ defined by Im $x > a_1$ and Im $x < a_2$. We assume here that $\mathcal{D}_1$ and $\mathcal{D}_2$ are both holomorphically convex.

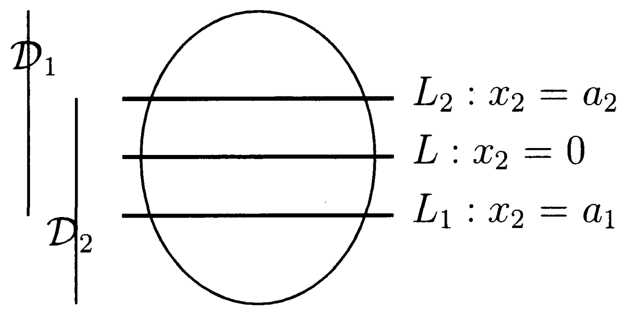


Fig. 4

In this geometric setting, the point is

$\ll$to solve Cousin's first problem on $\mathcal{D}\gg$.

This was settled in Paper VI by solving an integral equation including the Weil integral. He told me that when he was consulting Goursat's book, Cours d'Analyse, Volume III, certain integral equation caught his eyes, which finally led him to the solution of the problem.

Paper VI is written by restricting the situation to the space of two complex variables. This is probably because Oka planned to replace the Weil integral by the Cauchy integral, by using the lifting principle as before.

Oka wanted to do this not just because it is troublesome to apply Weil's formula in higher dimensions. He intended, from the beginning, to generalize the principal questions to the many-sheeted domains. This plan was completely realized after two years in 1943.

Concluding remarks

Oka called his work through Paper VI 'examination of the shore'. By this word one may understand that he had encouraged himself further to cross the river. As you know, a splendid bridge was built later, after the invention of the theory of systems of ideals of undetermined domains.

The lifting principle, together with its generalization remained Oka's lifelong research subject, whereas I could just describe how its most naïve form was created. However I must be contented with it if I succeeded in describing how a mathematical part of nature had grown in Oka's mind.

Thank you very much for your attention.

(Originally written in Japanese and translated by T. Ohsawa.)

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Uniqueness problem for meromorphic mappings under conditions on the preimages of divisors

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Abstract.

We first give a finiteness theorem for meromorphic mappings. Next, we give conditions under which two holomorphic mappings from a finite analytic covering space over the complex m-space into a smooth elliptic curve are algebraically related.

Introduction.

The uniqueness problem of meromorphic mappings under condition on the preimages of divisors was first studied by G. Pólya and R. Nevanlinna. They proved the following famous five point theorem: Let f and g be nonconstant meromorphic functions on $\mathbb{C}$. If $f^{-1}(a_j) = g^{-1}(a_j)$ for distinct five points $a_1, \cdots, a_5$ in $\mathbb{P}_1(\mathbb{C})$, then f and g are identical. So far, many researchers have studied unicity theorems for meromorphic functions on $\mathbb{C}$, as well in the multidimensional case. Among these, H. Fujimoto has proved a number of remarkable unicity theorems. For example, he proved the following excellent theorem ([4]):

Theorem (Fujimoto). *Let $f, g : \mathbb{C}^m \to \mathbb{P}_n(\mathbb{C})$ be nonconstant meromorphic mappings with the same inverse images of q hyperplanes in general position.*

(1) *If $q = 3n+1$, then there exists an automorphism L of $\mathbb{P}_n(\mathbb{C})$ such that $f = L \cdot g$.*

(2) *If $q = 3n+2$ and either f or g is linearly nondegenerate, then f and g are identical.*

The finiteness theorem for meromorphic mappings was also studied by H. Cartan and R. Nevanlinna in 1920's. The finiteness theorem of Cartan-Nevanlinna states that there exist at most two meromorphic functions on $\mathbb{C}$ that have the same inverse images with multiplicities

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for distinct three values in $\mathbb{P}_1(\mathbb{C})$. In 1981, H. Fujimoto generalized the theorem of Cartan-Nevanlinna to the case of meromorphic mappings of $\mathbb{C}^m$ into complex projective spaces $\mathbb{P}_n(\mathbb{C})$ by making use of Borel's identity ([5]). He proved the finiteness of families of linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}_n(\mathbb{C})$ with the same inverse images for some hyperplanes. In his results, the number of hyperplanes in general position is essential and must be larger than a certain number depending on the dimension of the projective spaces. Note that an essential problem in the multidimensional case exists in this point. Namely, in the case where a given divisor is irreducible, what kind of condition yields the finiteness of families of meromorphic mappings ? In this paper, we first give a finiteness theorem for meromorphic mappings f of $\mathbb{C}^m$ into a compact complex manifold M and for an irreducible divisor D on M. Next, we give some theorems on uniqueness problems of holomorphic mappings into smooth elliptic curves.

§1. Finiteness theorem for meromorphic mappings.

In this section, we give a finiteness theorem. For details, see [1]. To state our results, we give some definitions. Let $L \to M$ be a fixed line bundle over M, and let $\sigma_1, \cdots, \sigma_s$ be linearly independent holomorphic sections of $L \to M$ with $s \geq 2$. Throughout this paper, we assume that $(\sigma_j) = dD_j$ $(1 \leq j \leq s)$ for some positive integer d, where D_j are effective divisors on M. Set

$$\varpi = c_1\sigma_1 + \cdots + c_s\sigma_s,$$

where $c_j \in \mathbb{C}^*$. Let D be a divisor defined by $\varpi = 0$. We define a meromorphic mapping $\Psi : M \to \mathbb{P}_{s-1}(\mathbb{C})$ by $\Psi = (\sigma_1, \cdots, \sigma_s)$.

Definition 1.1. Let p be a nonnegative integer. For divisors Z_1 and Z_2 on $\mathbb{C}^m$, we write

$$Z_1 \equiv Z_2 \pmod{p}$$

if there exists a divisor Z' on $\mathbf{C}^m$ such that $Z_1 - Z_2 = pZ'$; in the special case of $p = 0$, $Z_1 \equiv Z_2 \pmod{0}$ if and only if $Z_1 = Z_2$.

Let Z be a nonzero effective divisor on $\mathbb{C}^m$. We denote by

$$\mathcal{F}(p; (\mathbb{C}^m, Z), (M, D))$$

the set of all meromorphic mappings $f : \mathbb{C}^m \to M$ such that

$$f^*D \equiv Z \pmod{p}.$$

Definition 1.2. We say that a meromorphic mapping $f : \mathbb{C}^m \to M$ has *the Zariski dense image* if $f(\mathbb{C}^m)$ is not included in any proper analytic subset of M.

Let

$$\mathcal{F}^*(p; (\mathbb{C}^m, Z), (M, D))$$

denote the subset of all $f \in \mathcal{F}(p; (\mathbb{C}^m, Z), (M, D))$ with *the Zariski dense image*. The main result of the present article is as follows ([1, Theorem 2.1]):

Theorem 1.3. *If* rank $\Psi = \dim\ M$ *and* $d > (s+1)!\ \{(s+1)! - 2\}$, *then the number of mappings in* $\mathcal{F}^*(d; (\mathbb{C}^m, Z), (M, D))$ *is bounded by a constant depending only on* D.

§2. Holomorphic curves into smooth elliptic curves.

In this section, we give some theorems on the uniqueness of holomorphic mappings into smooth elliptic curves E. In particular, we consider the problem to determine the condition which yields $f = \varphi(g)$ for an endomorphism φ of the abelian group E. For details, see [2]. The uniqueness problem of holomorphic mappings into elliptic curves was first studied by E. M. Schmid (Math. Z. **23** (1971)). Schmid's unicity theorem is the following: Let $f,\ g : R \to E$ be nonconstant holomorphic mappings, where R is an open Riemann surface of a certain type. Then there exists a nonnegative integer d depending only on R such that, if $f^{-1}(a_j) = g^{-1}(a_j)$ for distinct $d + 5$ points $a_1, \cdots, a_{d+5}$ in E, then f and g are identical. In the special case $R = \mathbb{C}$, we have $d = 0$. However, there have been only few studies on the uniqueness problem of holomorphic mappings into elliptic curves (cf. [3]).

Let $\pi : X \to \mathbb{C}^m$ be a finite analytic covering space and s_0 its sheet number. We denote by $[p]$ the point bundle determined by $p \in E$ and set $\tilde{F} = \pi_1^*[p] \otimes \pi_2^*[p]$, where $\pi_j : E \times E \to E$ are the natural projections. Let $f,\ g : X \to E$ be nonconstant holomorphic mappings. We denote by $\mathrm{End}(E)$ the ring of endomorphisms of E. If E has no complex multiplication, it is well-known that $\mathrm{End}(E) \cong \mathbb{Z}$. Hence $\varphi(x) = nx$ for some integer n. We now seek conditions which yield $g = \varphi(f)$ for some $\varphi \in \mathrm{End}(E)$. Let $\varphi \in \mathrm{End}(E)$ and consider a curve

$$\tilde{S} = \{(x,\ y) \in E \times E;\ y = \varphi(x)\}$$

in $E \times E$. Let $[\tilde{S}]$ be the line bundle determined by $\tilde{S}$. Denote by $\tilde{\gamma}$ the infimum of rational numbers γ such that $\gamma\tilde{F} \otimes [\tilde{S}]^{-1}$ is ample.

Then we have $\tilde{\gamma} = \deg \varphi + 1$ which is proved by T. Katsura (see [2]). Hence, if $\varphi \in \mathrm{End}(E)$ is an endomorphism defined by $\varphi(x) = nx$, then $\tilde{\gamma} = n^2 + 1$. Let Z be an effective divisor on X, and let k be either a positive integer or $+\infty$. If $Z = \sum_j \nu_j Z_j$ for distinct irreducible hypersurfaces Z_j in X and for nonnegative integers ν_j, then we define the support of Z with order at most k by $\mathrm{Supp}_k Z = \bigcup_{0<\nu_j\leq k} Z_j$. We now have the following:

Theorem 2.1. *Let f and g be as above. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and φ a endomorphism of E. Set $D_2 = \varphi(D_1)$. Assume that the number of points in D_2 is also d. Suppose that $\mathrm{Supp}_k\ f^*D_1 = \mathrm{Supp}_k\ g^*D_2$ for some k. If $d > 2(\deg\varphi + 1) + 8(s_0 - 1)(1 + k^{-1})$, then $g = \varphi(f)$.*

In the above theorem, we assume that the cardinality $\sharp D_2$ of the point set D_2 equals d. However, it may happen that $\sharp D_2 < d$. For example, if $\varphi(x) = nx$ $(n \in \mathbb{Z})$ and there exists at least one pair $(i,\ j)$ such that $a_i - a_j$ is n-torsion point, then $\sharp D_2 < d$. In this case, we have the following:

Theorem 2.2. *Let $f,\ g : \mathbb{C}^m \to E$ be nonconstant holomorphic mappings. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and $\varphi \in \mathrm{End}(E)$. Set $D_2 = \varphi(D_1)$. Assume that the number of points in D_2 is d'. Suppose that $\mathrm{Supp}_1\ f^*D_1 = \mathrm{Supp}_1\ g^*D_2$. If $dd' > (d + d')(\deg\varphi + 1)$, then $g = \varphi(f)$.*

Corollary 2.3. *Let f and g be as in Theorem 2.2. Let $D_1 = \{a_1, \cdots, a_d\}$ be a set of d points and set $D_2 = \{na_1, \cdots, na_d\}$ for some integer n. Assume that the number of points in D_2 is d'. Suppose that $\mathrm{Supp}_1\ f^*D_1 = \mathrm{Supp}_1\ g^*D_2$. If $dd' > (d+d')(n^2+1)$, then $g = nf$.*

We do not know whether Theorem 2.2 is sharp or not. However, if the condition $dd' > (d + d')(\deg\varphi + 1)$ is not satisfied, then it is not necessarily true that $g = \varphi(f)$.

Example 2.4. Let φ be an endomorphism defined by $\varphi(x) = 2x$. Define $f, g : \mathbb{C} \to E$ by $f(z) = \overline{\pi}(x)$ and $g(z) = -2\overline{\pi}(x)$, where $\overline{\pi} : \mathbb{C} \to E$ be the universal covering mapping. Let $D_1 = \{x \in E; 4x = 0\}$. Then $D_2 = \varphi(D_1) = 2D_1$. It is clear that $\mathrm{Supp}_1\ f^*D_1 = \mathrm{Supp}_1\ g^*D_2$. In this case, $d = 16$, $d' = 4$ and $\deg\varphi + 1 = 5$. Thus we have

$$dd' - (d + d')(\deg\varphi + 1) = -36 < 0$$

and $g \neq \varphi(f)$.

For nonconstant holomorphic mappings $f,\ g : X \to E$, we have the following unicity theorem, which is a direct conclusion of Theorem 2.1:

Theorem 2.5. *Let $a_1, \cdots, a_d$ be distinct points in E. Suppose that $\mathrm{Supp}_k\ f^* a_j = \mathrm{Supp}_k\ g^* a_j$ for all j, where $1 \leq k \leq +\infty$. If $d > 8s_0 - 4 + 8k^{-1}(s_0 - 1)$, then f and g are identical.*

In the case of $X = \mathbb{C}^m$, we have the following:

Theorem 2.6. *Let $a_1, \cdots, a_d$ be distinct points in E. Suppose that $X = \mathbb{C}^m$ and $\mathrm{Supp}_1\ f^* a_j = \mathrm{Supp}_1\ g^* a_j$ for all j. If $d \geq 5$, then f and g are identical.*

We give here the concluding remark. If we choose special points of E, we obtain an example which yields that Theorem 2.6 is sharp. Indeed, let $a_1, \cdots, a_4$ be two-torsion points in E and let $\wp$ be the Weierstrass $\wp$ function. If $f_1^* a_j = f_2^* a_j$ for $j = 1, \cdots, 4$, it is easy to see that $\wp \circ f_1 = \wp \circ f_2$ by Nevanlinna's four points theorem. Hence $f_1 = f_2$ or $f_1 = -f_2$. Since $p \mapsto -p$ $(p \in E)$ is an automorphism of E, it is acceptable that f_1 and f_2 are essentially identical. In this example, it seems that the structure of the function field of E affects strongly the uniqueness problem for holomorphic mappings.

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On the middle dimension cohomology of A_l singularity

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Abstract.

Let (V, o) be a normal isolated singularity in a complex Euclidean space (C^N, o). Let M be the intersection of this singularity and the real hypersphere $S_\epsilon^{2N-1}(o)$, centered at the origin o with an ϵ radius. Then, naturally, this link M admits a CR structure, induced from V, and the deformation theory of this CR structures has been studied in [1], [2],[3]. Especially in [3], a particular subspace of the infinitesimal deformation space is found, and we propose to study the relation between this subspace and simultaneous deformation. We note that: if the canonical line bundle of the CR structure is trivial, then the infinitesimal space of the deformation of CR structures is a part of the middle dimension cohomology. And in this line, we conjecture that Z^1, introduced in [3], might be related to the simultaneous deformation of isolated singularity (V, o)(see also [2]). We discuss this problem for A_l singularities.

§1. Motivation and Z^1 - space

Let $(V^{(n)}, o)$ be an isolated singularity in a complex eucliean space (C^N, o). We consider the intersection

$$M = S_\epsilon^{2N-1}(o) \cap V.$$

Then M is a compact non-singular real $2n-1$ dimensional C^∞ manifold, and a CR structure $(M, {}^0T'')$ is induced from V, by ;

$${}^0T'' = C \otimes TM \cap T''(V - o).$$

Here $T''(V-o)$ means the space consisting of type $(1,0)$ vectors on $V-o$. This pair $(M, {}^0T'')$ is called a CR structure(or a CR manifold). For this CR structure, the deformation theory, related to the deformation theory

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of isolated singularities (V, o), is successfully developed by Kuranishi. After the great work of Kuranishi, we are interested in the mixed Hodge structure of CR manifolds. We take a supplement vector field ζ to ${}^0T'' + {}^0T'$, here ${}^0T' = \overline{{}^0T''}$. For this CR structure with the supplement vector field $\{(M, {}^0T''), \zeta\}$, we can introduce a mixed Hodge structure which should correspond to the mixed Hodge structure on a tubular neighborhood U of M in V. Here, we assume that there is a real vector field ζ satisfying:

$$\zeta_p \notin {}^0T''_p + {}^0T'_p \tag{1}$$

$$[\zeta, \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T''). \tag{2}$$

While, during our studying deformation theory of CR structures, we learn that: for Calabi-Yau manifolds, the Kuranishi family is unobstructed. So, in order to obtain the analogy to isolated singularities, Z^1 space is found(see [3]).

$$Z^1 = \{u : u \in F^{n-1,1}, d''u = 0, d'u = 0\}. \tag{3}$$

In the case complex manifolds, Z^1 might be translated as follows. For a tubular neighborhood U of M in V, we set

$$\{u : u \in \Gamma(U, \wedge^{n-1}(T'U)^* \wedge (T''U)^*), \overline{\partial}u = 0, \partial u = 0\}. \tag{4}$$

If $X^{(n)}$ is a compact n-dimensional Kaehler manifold, then

$$\{u : u \in \Gamma(X^{(n)}, \wedge^{n-1}(T'X^{(n)})^* \wedge (T''X^{(n)})^*), \overline{\partial}u = 0, \partial u = 0\}. \tag{5}$$

includes the $\overline{\partial}$-harmonic space consisting of $(n-1, 1)$ forms. While, here, we are treating an open manifold U(tubular neighborhood of M). So even if the $(n-1, 1)$ Kohn-Rossi cohomology does not vanish(the existence of a non- trivial $\overline{\partial}$-harmonic space consisting of $(n-1, 1)$ forms), the above space might be 0. Here we give a program to obtain a non-trivial element of (4) from a non- trivial simultaneous deformation.

Let $\tilde{V}$ be the resolution of the isolated singularity with complex dimension n in C^N, V, and π is the resolution map π; $\tilde{V} \to V$. And consider non-trivial deformations of isolated singularity (V, o) with this resolution. Namely, π_t is a resolution map of V_t in C^N , π_t; $\tilde{V}_t \to V_t$, $t \in T$, where V_t is a deformation of V, $\tilde{V}_t$ is a deformation of $\tilde{V}$, T is an analytic space with the origin, and at the origin, $\pi_o = \pi$, $\tilde{V}_o = \tilde{V}$, $V_o = V$. Furthermore, we assume that $V_t \subset C^N$. Now we take a C^∞ trivialization i_t : a tubular neighborhood of $M_o \to$ a tubular neighborhood of M_t, which satisfies $i_t(M_o) = M_t$. In this setting, our program is as follows.

- (First Step) By using the simultaneous resolution, we construct a non-trivial $(n,0)$ form ω_t, which is not d exact on $\tilde{V}_t$ for a generic t, and depends on t complex analytically. In general, "to give an $(n,0)$ form, satisfying a certain condition", might be easier than "to give an $(n-1,1)$ form with the corresponding condition".
- (Second Step) By choosing a proper C^∞ trivialization of the simultaneous deformation, i_t,

  $$i_t^*\omega_t = \omega_0 + \omega_1 t + \cdots, \quad \text{(expansion with respect to } t).$$

- (Third Step) From $d\omega_t = 0$, it follows that: $d\omega_1 = 0$. By the definition, ω_1 is a form of type $(n,0)+(n-1,1)$ on $\tilde{V}_o - \pi^{-1}(o)$, we write it by;

  $$\omega_1 = \omega_1^{(n,0)} + \omega_1^{(n-1,1)}.$$

  As $d\omega_1 = 0$, this is equivalent to

  $$\overline{\partial}\omega_1^{(n-1,1)} = 0,$$

  $$\overline{\partial}\omega_1^{(n,0)} + \partial\omega_1^{(n-1,1)} = 0.$$

  The $\overline{\partial}$-cohomology class, determined by $\omega_1^{(n-1,1)}$, is the induced one by the Kodaira-Spencer class of deformations. So, this must be non-trivial. In this setting, we would like to construct a non-trivial element of (4), associated with the given simultaneous deformation.

For the Third Step, we have to comment on a crucial point. The naive answer is that:

$$\partial\omega_1^{(n-1,1)} = 0 \quad ?$$

This is too strong. There is an ambiguity to choose the C^∞ trivialization, i_t. By changing the C^∞ trivialization, ω_1(resp. $\omega_1^{(n-1,1)}$) is replaced by $\omega_1 - du$(resp. $\omega_1^{(n-1,1)} - \overline{\partial}u$), where u is an $(n-1,1)$ form. Hence our problem(to obtain a non-trivial element of (4)) is reduced to that; is there any C^∞ $(n-1,1)$ form u, satisfying: $\overline{\partial}\omega_1^{(n-1,1)} - \partial\overline{\partial}u = 0$? This is so called "$\partial\overline{\partial}$ lemma". For a compact Kaehler manifold, by taking the harmonic part, this is always solvable. However, for an open manifold, this is not an easy problem. One of our conjecture is that; if $\omega_1^{(n-1,1)}$ is induced by the simultaneous deformation, then this might be solvable. In the next section, we study this conjecture in A_l singularities.

§2. A_l singularities

Let

$$X = \{(z_1, \dots, z_{n+1}) : (z_1, \dots, z_{n+1}) \in C^{n+1}, z_1^2 + \cdots + z_{n+1}^{l+1} = 0\},$$

where l is a positive integer. We call this isolated singularity A_l singularity. Consider a family of deformations of X,

$$X_t = \{(z_1, \dots, z_{n+1}) : (z_1, \dots, z_{n+1}) \in C^{n+1}, z_1^2 + \cdots + z_{n+1}^{l+1} = t\}.$$

Let $M = X \cap \{(z_1, \dots, z_{n+1}) :\mid z_1 \mid^2 + \cdots + \mid z_{n+1} \mid^2 = 1\}$. And consider a C^∞ trivialization of this deformation over a neighborhood of M in X. Let $i_t : (z_1, \dots, z_{n+1}) \to (z_1(t), \dots, z_{n+1}(t))$, where

$$\begin{aligned}
z_1(t) &= z_1 + \frac{1}{2k(z,\overline{z})}\overline{z}_1(1+ \mid z_{n+1} \mid^2 + \cdots + \mid z_{n+1} \mid^{2l})t, \\
&\dots \\
z_n(t) &= z_n + \frac{1}{2k(z,\overline{z})}\overline{z}_n(1+ \mid z_{n+1} \mid^2 + \cdots + \mid z_{n+1} \mid^{2l})t \\
z_{n+1}(t) &= z_{n+1} + \frac{1}{(l+1)k(z,\overline{z})}\overline{z}_{n+1}^{l}t
\end{aligned}$$

Here

$$k(z,\overline{z}) = (1+\mid z_{n+1}\mid^2+\cdots+\mid z_{n+1}\mid^{2(l-1)})(\mid z_1\mid^2+\cdots+\mid z_n\mid^2)+\mid z_{n+1}\mid^2 l.$$

So, on M, because of $\mid z_1 \mid^2 + \cdots + \mid z_n \mid^2 = 1- \mid z_{n+1} \mid^2$, $k(z,\overline{z}) = 1$ holds. And,

$$\begin{aligned}
&z_1(t)^2 + \cdots + z_n(t)^2 + z_{n+1}(t)^{l+1} \\
&\quad = z_1^2 + \cdots + z_n^2 + z_{n+1}^{l+1} \\
&\qquad + \frac{1}{k(z,\overline{z})}\{(1+\mid z_{n+1}\mid^2+\cdots+\mid z_{n+1}\mid^{2(l-1)})(\mid z_1\mid^2+\cdots+\mid z_n\mid^2) \\
&\qquad + \mid z_{n+1} \mid^{2l}\}t + \text{higher order term of } t \\
&\quad \equiv t \mod t^2
\end{aligned}$$

By adjusting higher order term, we have a C^∞ trivialization $i_t : X \to X_t$ over a neighborhood of M. However, in this paper, we discuss only differential forms of type $(n-1,1)$. So the above map is enough.

§3. An approach to the First Step

In this section, we give a non-trivial holomorphic $(n,0)$ form on $X_t \cap$ (a neighborhood of M in C^{n+1}), which depends on t, complex analytically. Let $f = z_1^2 + \cdots + z_n^2 + z_{n+1}^{l+1}$. Like in [2], we, first, set a type $(1,0)$ vector field Z_f, defined on a neighborhood of M in the C^{n+1}, as follows. Let Ω be the standard symplectic form.

$$\Omega = \sum_{i=1}^{n+1} \sqrt{-1} dz_i \wedge d\overline{z}_i.$$

By using this metric, we define a $(1,0)$ vector field Z_f on a neighborhood of M by;

$$df(X) = \Omega(X, \overline{Z}_f), \quad \text{for all } (1,0) \text{ vector field } X.$$

This Z_f is easily written down as follows.

$$\begin{aligned} Z_f &= \sqrt{-1} \sum_{i=1}^{n+1} \overline{(\frac{\partial f}{\partial z_i})} \frac{\partial}{\partial z_i} \\ &= \sqrt{-1} \{ \sum_{i=1}^{n} 2\overline{z}_i \frac{\partial}{\partial z_i} + (l+1)\overline{z}_{n+1}^l \frac{\partial}{\partial z_{n+1}} \}. \end{aligned}$$

So,

$$\begin{aligned} Z_f(f) &= \sqrt{-1}(2^2 \sum_{i=1}^{n} | z_i |^2 + (l+1)^2 | z_{n+1} |^{2l}) \\ &\neq 0 \quad \text{on a neighborhood of } M. \end{aligned}$$

Let $\omega = dz_1 \wedge \cdots \wedge dz_{n+1}$. For X_t, we set a holomorphic $(n,0)$ form $\omega'(t)$, which depends on t, complex analytically by ;

$$\omega'(t) = Z_f \rfloor \omega \quad \text{on} \quad X_t \text{ (inner product with vector field } Z_f).$$

And set

$$\omega'_t = \frac{1}{\sum_{i=1}^{n} 2^2 | z_i |^2 + (l+1)^2 | z_{n+1} |^{2l}} \omega'(t).$$

By the type of ω, our ω'_t is of type $(n,0)$ on X_t. We must show that our ω'_t is holomorphic on X_t. For this, we recall the following lemma.

Lemma 3.1. $\omega = -\sqrt{-1} df \wedge \omega'_t$ *on a neighborhood of* M.

We sketch the proof of this lemma. For a point p of a neighborhood of M in C^{n+1}, $T'_p C^{n+1}$ is spanned by Z_f and $\{X_i(p)\}_{1 \leq i \leq n}$, which satisfy $X_i(p)f = 0$. So, with these vector fields, just by a direct computation, we have our lemma.

By this lemma, on X_t,

$$d\omega_t' = 0.$$

We have to see that our ω_o' is not a d-exactn on $X_o = X$. But if we restric ω_t to

$$\{(z_1, \cdots, z_n, z_{n+1}) : z_1^2 + \cdots + z_n^2 + z_{n+1}^{l+1} = 0, z_{n+1} = 0\}$$

a complex $n-1$ dimensional A_1 singularity, then it gives a non-trivial $n-1$ dimensional cohomology(by the definition of our ω_t', it coincides with nontrivial element, constructed in [2]). So, we have a non trivial form.

§4. An approach to the Third Step

By the C^∞ trivialization of the simultaneous deformations, i_t, constructed in Section 2, on a tubular neighborhood of M,

$$i_t^*\omega_t = \omega_0 + \omega_1 t + \cdots, \quad \text{(expansion with respect to } t\text{)}.$$

We explain a difficulty about this part. For example, we take A_1 singularity (in our notations, $l = 1$). Then, in the C^∞ isomorphism map, i_t, as a denominator, $k(z, \overline{z})$ appears. Only on the boundary case(CR case)

$$k(z, \overline{z}) = 1 \text{ on the boundary.}$$

But we are treating the tubular neighborhood case. So, it is not so valid that there is no extra non-trivial $(n, 0)$ term of ω_1 (we write it by $\omega_1^{(n,0)}$). Fortunately, for the case $l = 1$ (the case of an ordinary double point), $(n, 0)$ term doesn't appear(this means that it is not necessary to change the C^∞ trivialization i_t, constructed in Section 2). So, in this case, $d\omega_1 = 0$ means that; $\partial\omega_1 = 0$ and $\overline{\partial}\omega_1 = 0$. For the other l, we have to control the difficulty which arises from the term $k(z, \overline{})$. In another paper, we discuss the other case.

For the case $l = 1$, the C^∞ isomorphism map is as follows.

$$z_i(t) = z_i + \frac{1}{2\sum_{i=1}^{n+1} |z_i|^2}\overline{z}_i t, \quad i = 1, \ldots, n+1.$$

And

$$Z_f = 2\left(\sum_{i=1}^{n+1} \overline{z}_i \frac{\partial}{\partial z_i}\right).$$

In order to simplify the sketch, we assume $n = 2$. Then,

$$Z_f = 2(\overline{z}_1 \frac{\partial}{\partial z_1} + \overline{z}_2 \frac{\partial}{\partial z_2} + \overline{z}_3 \frac{\partial}{\partial z_3})$$

And so,

$$Z_f \rfloor \omega = 2(\overline{z}_1 dz_2 \wedge dz_3 - \overline{z}_2 dz_1 \wedge dz_3 + \overline{z}_3 dz_1 \wedge dz_2),$$

$$\begin{aligned} Z_f(f) &= 4(\mid z_1 \mid^2 + \mid z_2 \mid^2 + \mid z_3 \mid^2) \\ &= 4r^2. \end{aligned}$$

Here $r^2 = \mid z_1 \mid^2 + \mid z_2 \mid^2 + \mid z_3 \mid^2$. And

$$\begin{aligned} z_1(t) &= z_1 + \frac{1}{2}\frac{1}{r^2}\overline{z}_1 t, \\ z_2(t) &= z_2 + \frac{1}{2}\frac{1}{r^2}\overline{z}_2 t, \\ z_3(t) &= z_3 + \frac{1}{2}\frac{1}{r^2}\overline{z}_3 t. \end{aligned}$$

Now we compute ω_1.

$$\begin{aligned} i_t^*(\frac{1}{4r^2} Z_f \rfloor \omega) &= \frac{1}{2} i_t^*(\frac{1}{r^2}(\overline{z}_1 dz_2 \wedge dz_3 - \overline{z}_2 dz_1 \wedge dz_3 + \overline{z}_3 dz_1 \wedge dz_2)) \\ &= \frac{1}{2}(\frac{\overline{z}_1(t) dz_2(t) \wedge dz_3(t) - \overline{z}_2(t) dz_1(t) \wedge dz_3(t) + \overline{z}_3(t) dz_1(t) \wedge dz_2(t))}{z_1(t)\overline{z}_1(t) + z_2(t)\overline{z}_2(t) + z_3(t)\overline{z}_3(t)}) \\ &\equiv \frac{1}{2}(\frac{\overline{z}_1 dz_2(t) \wedge dz_3(t) - \overline{z}_2 dz_1(t) \wedge dz_3(t) + \overline{z}_3 dz_1(t) \wedge dz_2(t))}{z_1(t)\overline{z}_1 + z_2(t)\overline{z}_2 + z_3(t)\overline{z}_3}) \mod(t^2, \overline{t}) \\ &= \frac{1}{2}(\frac{\overline{z}_1 dz_2(t) \wedge dz_3(t) - \overline{z}_2 dz_1(t) \wedge dz_3(t) + \overline{z}_3 dz_1(t) \wedge dz_2(t))}{z_1\overline{z}_1 + z_2\overline{z}_2 + z_3\overline{z}_3}) \end{aligned}$$

because of $z_1^2 + z_2^2 + z_3^2 = 0$.

While

$$\begin{aligned} \overline{z}_1 dz_2(t) \wedge dz_3(t) &= \overline{z}_1(dz_2 + \frac{1}{2}(d(\frac{1}{r^2}))\overline{z}_2 t + \frac{1}{2}\frac{1}{r^2} d\overline{z}_2 t) \wedge (dz_3 + \frac{1}{2}(d(\frac{1}{r^2}))\overline{z}_3 t + \frac{1}{2}\frac{1}{r^2} d\overline{z}_3 t) \\ &\equiv \overline{z}_1 dz_2 \wedge dz_3 + \{\overline{z}_1 \frac{1}{2}(d(\frac{1}{r^2}))\overline{z}_2 \wedge dz_3 + \overline{z}_1 \frac{1}{2}\frac{1}{r^2} d\overline{z}_2 \wedge dz_3 \\ &+ \overline{z}_1 dz_2 \wedge \frac{1}{2}(d(\frac{1}{r^2}))\overline{z}_3 + \overline{z}_1 dz_2 \frac{1}{2}\frac{1}{r^2} d\overline{z}_3\} t \mod t^2. \end{aligned}$$

Therefore from this term, $(2,0)$ part is

$$\frac{1}{2}\overline{z}_1\overline{z}_2 \partial(\frac{1}{r^2}) \wedge dz_3 + \frac{1}{2}\overline{z}_1\overline{z}_3 dz_2 \wedge \partial(\frac{1}{r^2}).$$

By the same way, from $-\bar{z}_2 dz_1(t) \wedge dz_3(t)$, as a $(2,0)$ part,

$$-\frac{1}{2}\bar{z}_1\bar{z}_2\partial(\frac{1}{r^2}) \wedge dz_3 - \frac{1}{2}\bar{z}_2\bar{z}_3 dz_1 \wedge \partial(\frac{1}{r^2}).$$

And from $\bar{z}_3 dz_1(t) \wedge dz_2(t)$, $(2,0)$ part is

$$\frac{1}{2}\bar{z}_1\bar{z}_3\partial(\frac{1}{r^2}) \wedge dz_2 + \frac{1}{2}\bar{z}_2\bar{z}_3 dz_1 \wedge \partial(\frac{1}{r^2}).$$

So summing up these three terms, in this case, we see that $(2,0)$ part does not appear.

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The exact steepest descent method — a new steepest descent method based on the exact WKB analysis

Takashi Aoki[1], Takahiro Kawai[2] and Yoshitsugu Takei[3]

Abstract.

Introducing a new notion of the exact steepest descent path, we develop a new steepest descent method applicable to general ordinary differential equations with polynomial coefficients. Its application to the connection problem for solutions is also discussed.

§1. Introduction

In [2] we proposed a new method called the "exact steepest descent method". It is designed to enlarge the scope of applicability of the steepest descent method by making use of the exact WKB analysis, i.e., WKB analysis based on the Borel resummation technique (cf. [15], [7], [9] and references cited there). It sheds a new light on some missing link between microlocal analysis (cf., e.g., [11], [8]) and exact WKB analysis and, at the same time, it provides us with a new tool in global analysis of ordinary differential equations with polynomial coefficients. In this paper we explain what the exact steepest descent method is and how it is related with microlocal analysis, and discuss its application to the connection problem of ordinary differential equations.

To help the reader's understanding of the theory, we first give an overview of the exact steepest descent method. Let us consider an ordinary differential equation with polynomial coefficients of the following

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form:

$$P\psi = \sum_{\substack{0\le j\le m\\ 0\le k\le n}} a_{jk}x^k\eta^{m-j}\frac{d^j\psi}{dx^j} = 0, \tag{1}$$

where a_{jk} is a complex constant and $\eta > 0$ is a large parameter. If all the coefficients are linear polynomials (i.e., $n = 1$), the Laplace transformation with respect to an independent variable x transforms (1) into a first order equation. Hence, by solving it explicitly, we can readily obtain an integral representation of solutions. In this case every Borel resummed WKB solution of (1) is represented as an integral along a steepest descent path passing through a saddle point and various connection problems for solutions (such as determination of the monodromy group, computation of Stokes multipliers, etc.) can be solved by tracing the configuration of such steepest descent paths ("steepest descent method"; cf., e.g., [12], [13], [14]). The exact steepest descent method allows us to apply this approach to more general equations. That is, to study (1) when $n \ge 2$, we consider the inverse Laplace transform

$$\int e^{\eta x\xi}\hat{\psi}_k(\xi,\eta)d\xi \tag{2}$$

of a WKB solution $\hat{\psi}_k$ $(1 \le k \le n)$ of the Laplace transformed equation $\hat{P}\hat{\psi} = 0$, using the idea of Berk et al. ([5]). We can then observe that the Borel transform $\hat{\psi}_{k,B}$ of $\hat{\psi}_k$ is related with the Borel transform of a WKB solution of (1) by the quantized Legendre transformation near a saddle point of the integral (2). Furthermore, if we introduce a sophisticated notion of the "exact steepest descent path" (which reflects the connection formula for Borel resummed WKB solutions of $\hat{P}\hat{\psi} = 0$; cf. §3 below for its precise definition), the Borel sum of a WKB solution of the original equation $P\psi = 0$ is represented as the integral (2) along an exact steepest descent path passing through a saddle point. Hence the global behavior of solutions of (1) can be analyzed by tracing the configuration of exact steepest descent paths; this is the "exact steepest descent method".

This paper is organized as follows: After reviewing the ordinary steepest descent method briefly in §2, we recall in §3 the notion of the exact steepest descent paths introduced in [2], emphasizing its relevance to microlocal analysis. In §4 we then show how to apply the exact steepest descent method to the computation of Stokes multipliers (which is a typical connection problem). Finally in §5 we give a summary and present some open problems.

§2. Review of the ordinary steepest descent method

As is mentioned in §1, when all the coefficients are linear polynomials (i.e., $n = 1$), an integral representation of solutions of (1) can be readily obtained by employing the Laplace transformation (with a large parameter η) $\psi(x) \mapsto \hat{\psi}(\xi)$, i.e.,

$$\psi(x) = \int e^{\eta x\xi}\hat{\psi}(\xi)d\xi. \tag{3}$$

Then the steepest descent method applied to the integral representation provides us with a powerful tool in global analysis of solutions of (1). Let us illustrate it by the following well-known example.

Example 1. Let us consider the Airy equation:

$$P\psi = \left(\frac{d^2}{dx^2} - \eta^2 x\right)\psi = 0. \tag{4}$$

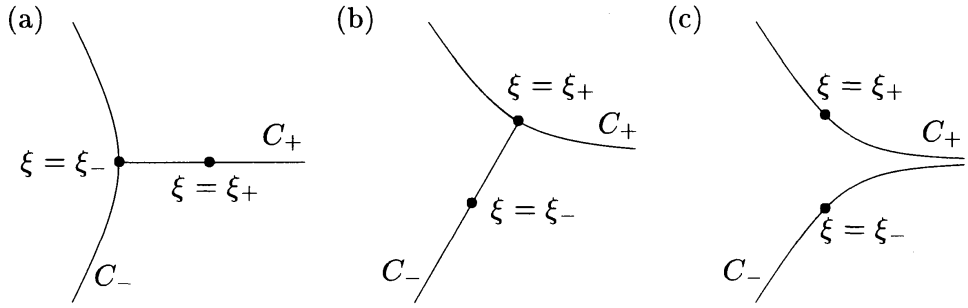
For (4) the integral representation obtained through the Laplace transformation is given by the following:

$$\psi(x,\eta) = \int \exp\left(\eta\left(x\xi - \frac{\xi^3}{3}\right)\right)d\xi. \tag{5}$$

Let $f(x,\xi)$ denote the phase function $x\xi - \xi^3/3$ of (5). To study the analytic continuation of a solution of (4), we trace the configuration of steepest descent paths of $\operatorname{Re} f$ passing through saddle points of f. (Recall that, by definition, a saddle point of f is a point satisfying $\partial f/\partial\xi = 0$ and a steepest descent path of $\operatorname{Re} f$ is a level curve of $\operatorname{Im} f$ on which $\operatorname{Re} f$ decreases monotonically.) In this case there exist two saddle points $\xi = \xi_\pm = \pm\sqrt{x}$ and Fig. 1 describes the configuration of the steepest descent paths passing through these two saddle points for $\arg x = 0$, $\arg x = 2\pi/3$ and $\arg x = \pi$.

From Fig. 1 we can perceive that the integral (5) along a steepest descent path C_- for $\arg x = 0$ is analytically continued through the upper half plane to the sum of the integral along C_- and that along C_+ for $\arg x = \pi$. As scaling of the integration variable ξ ensures the equivalence between the asymptotics of (5) for $\eta \to \infty$ and that for $|x| \to \infty$ and further the asymptotics for $\eta \to \infty$ can be readily computed by the saddle point method, this implies that the asymptotic solution

$$\int_{C_-} \exp\left(\eta\left(x\xi - \frac{\xi^3}{3}\right)\right)d\xi \sim \frac{i\sqrt{\pi}}{\sqrt{\eta}}x^{-1/4}\exp\left(-\frac{2}{3}\eta x^{3/2}\right)(1 + \cdots) \tag{6}$$



(for $\arg x = 0$) (for $\arg x = 2\pi/3$) (for $\arg x = \pi$)

Fig. 1. Steepest descent paths of (5).

of (4) for $x \to +\infty$ is analytically continued to

$$(7) \qquad \int_{C_+} \exp\left(\eta\left(x\xi - \frac{\xi^3}{3}\right)\right) d\xi + \int_{C_-} \exp\left(\eta\left(x\xi - \frac{\xi^3}{3}\right)\right) d\xi$$
$$\sim \frac{2i\sqrt{\pi}}{\sqrt{\eta}}(-x)^{-1/4} \sin\left(\frac{2}{3}\eta(-x)^{3/2} + \frac{\pi}{4}\right)(1 + \cdots)$$

for $x \to -\infty$. This is the well-known "Stokes phenomenon" for the Airy equation. In this way, by the steepest descent method, that is, by tracing the configuration of steepest descent paths for the integral representation, we can solve connection problems for ordinary differential equations with linear coefficients.

In [13] and [14] some other interesting examples are discussed from this viewpoint. Note that the steepest descent method is, in a sense, equivalent to the exact WKB analysis for ordinary differential equations with linear coefficients. See [12] for the precise description of the relationship between these two methods. This approach is also related to the "hyperasymptotic analysis" of Berry and Howls ([6]).

Our goal is to generalize this method so that it may be applicable to ordinary differential equations with polynomial coefficients.

§3. Exact steepest descent method

In this section we explain the framework of the exact steepest descent method. For the details of the theory we refer the reader to [2].

Now, to generalize the steepest descent method so that it may be applied to an equation of the form (1), we again apply the Laplace transformation (3) to (1) and consider its Laplace transformed equation

$$\hat{P}\hat{\psi} = \sum_{\substack{0\le j\le m\\ 0\le k\le n}} a_{jk}\eta^{m-k}\left(-\frac{d}{d\xi}\right)^k (\xi^j\hat{\psi}) = 0. \tag{8}$$

In case $n \geq 2$ it is difficult to solve (8) explicitly. Instead we use a WKB solution

$$\hat{\psi}_k = \eta^{-1/2}\exp\left(\eta\int^{\xi}(-x_k(\xi))d\xi + \cdots\right) \tag{9}$$

of (8) and consider its inverse Laplace transform

$$\int e^{\eta x\xi}\hat{\psi}_k d\xi = \eta^{-1/2}\int \exp\left(\eta\left(x\xi - \int^{\xi} x_k(\xi)d\xi\right) + \cdots\right)d\xi, \tag{10}$$

where $x_k(\xi)$ $(k = 1,\dots,n)$ denotes a root (with respect to x) of the characteristic equation

$$p(x,\xi) \stackrel{\mathrm{def}}{=} \sum a_{jk}x^k\xi^j = 0, \tag{11}$$

and $\eta^{-1/2}$ is added to (9) for the sake of convenience in defining its Borel transform. (Throughout this paper we frequently use the terminologies in the exact WKB analysis. For their precise meaning see [9] or [1].)

Let $f_k(x,\xi)$ denote $x\xi - \int^{\xi} x_k(\xi)d\xi$. Roughly speaking, we apply the steepest descent method to the integral (10) with regarding $f_k(x,\xi)$ as its phase function. This idea was first presented by Berk et al. ([5]). In what follows we polish up their idea by examining it from the viewpoint of the exact WKB analysis.

Let us first fix the path of integration for (10). In parallel with the case of the Airy equation, we take a steepest descent path of $\operatorname{Re} f_k$ passing through a saddle point of f_k as the path of integration for (10). Since a saddle point $\tilde{\xi}$ of f_k satisfies $x = x_k(\tilde{\xi})$, $\tilde{\xi} = \xi_j(x)$ holds for some j $(j = 1,\dots,m)$, where $\xi_j(x)$ denotes a root of (11) with respect to ξ. If we let $C_k^{(j)}$ denote the steepest descent path of $\operatorname{Re} f_k$ passing through $\xi_j(x)$, our task is then to relate the integral (10) along $C_k^{(j)}$ with a WKB solution of the original equation (1) of the form

$$\psi_j = \eta^{-1}\exp\left(\eta\int^{x}\xi_j(x)dx + \cdots\right), \tag{12}$$

where another normalization factor η^{-1} is used for later convenience.

Local correspondence of Borel transformed WKB solutions

In the exact WKB analysis a WKB solution is given its analytic meaning by the Borel resummation. Hence it follows from the definition of the Borel sum that the integral we are interested in is

$$\int_{C_k^{(j)}} e^{\eta x\xi}\left(\int e^{-\eta z}\hat{\psi}_{k,B}(\xi,z)dz\right)d\xi, \tag{13}$$

where $\hat{\psi}_{k,B}$ denotes the Borel transform of $\hat{\psi}_k$ and the integration in z-space is performed along the path $z=\int^{\xi} x_k(\xi)d\xi+v$, $v\geq 0$. Note that, if we write (9) as $(\exp(-\eta\int^{\xi} x_k(\xi)d\xi))\eta^{-1/2}(c_0+c_1\eta^{-1}+\cdots)$ after applying the Taylor expansion, the Borel transform $\hat{\psi}_{k,B}$ is, by definition, given by

$$\sum_{l=0}^{\infty}\frac{c_l}{\Gamma(l+1/2)}\left(z-\int^{\xi} x_k(\xi)d\xi\right)^{l-1/2}. \tag{14}$$

Furthermore, introducing a new integration variable $y=z-x\xi$, we find that the integral (13) can be rewritten as follows:

$$\iint \exp(-\eta y)\hat{\psi}_{k,B}(\xi,y+x\xi)d\xi dy. \tag{15}$$

Here the path of integration in y-space is described by $y=-\int^{x}\xi_j(x)dx+w$, $w\geq 0$, and the integration in ξ-space is performed on the portion $[\xi^{(-)},\xi^{(+)}]$ of the steepest descent path $C_k^{(j)}$, where $\xi^{(\pm)}$ is the two different points on $C_k^{(j)}$ satisfying $\operatorname{Re} f_k(x,\xi^{(\pm)})-\operatorname{Re} f_k(x,\xi_j(x))=-w$ for a fixed pair (x,w) $(w\geq 0)$. Therefore, the integral (13) can be written also as

$$\int_{y=-\int^{x}\xi_j(x)dx+w,w\geq 0} e^{-\eta y}\chi(x,y)dy \tag{16}$$

with

$$\chi(x,y)\overset{\text{def}}{=}\int_{[\xi^{(-)},\xi^{(+)}]}\hat{\psi}_{k,B}(\xi,y+x\xi)d\xi. \tag{17}$$

The form of the integral (16) is the same as that of the Borel sum of a WKB solution (12), provided that $\chi(x,y)$ is its Borel transform. To confirm that $\chi(x,y)$ is the Borel transform of (12), we should note that the corresondence (17) between χ and $\hat{\psi}_{k,B}$ is given by

$$(T\varphi)(x,y)\overset{\text{def}}{=}\int\varphi(\xi,y+x\xi)d\xi=\iint\delta(y-z+x\xi)\varphi(\xi,z)d\xi dz, \tag{18}$$

which is the so-called "quantized Legendre transformation", that is, a quantization of the canonical transformation from $T^*\mathbb{C}^2_{(\xi,z)}$ to $T^*\mathbb{C}^2_{(x,y)}$ with a generating function $\Omega(\xi,z,x,y)=y-z+x\xi$ (cf., e.g., [8, Example 4.2.5]). Through the transformation T operators in (ξ,z)-space and those in (x,y)-space correspond in the following manner:

$$\begin{aligned}\xi &\longmapsto \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)^{-1}, \quad z \longmapsto y + x\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)^{-1}, \\ \frac{\partial}{\partial \xi} &\longmapsto -x\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \longmapsto \frac{\partial}{\partial y}.\end{aligned} \tag{19}$$

Having this correspondence in mind, we find

$$\begin{aligned}&\sum a_{jk}x^k\left(\frac{\partial}{\partial y}\right)^{m-j}\left(\frac{\partial}{\partial x}\right)^j \chi(x,y) \\ &= \sum a_{jk}\left(\frac{\partial}{\partial y}\right)^{m-k}\left(x\frac{\partial}{\partial y}\right)^k\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right)^{-1}\right)^j T(\hat{\psi}_{k,B}(\xi,z)) \\ &= T\left(\sum a_{jk}\left(\frac{\partial}{\partial z}\right)^{m-k}\left(-\frac{\partial}{\partial \xi}\right)^k \xi^j\hat{\psi}_{k,B}(\xi,z)\right) = 0.\end{aligned} \tag{20}$$

(The final equality follows from the definition of the Borel transform.) The differential equation (20) combined with the study of the local behavior of $\chi(x,y)$ near its singular point $y=-\int^x \xi_j(x)dx$ (which can be done by applying Prop. 4.2.4 in [11, p.422]) then entails that $\chi(x,y)$ is the Borel transform of (12).

Summing up, the quantized Legendre transformation relates $\hat{\psi}_{k,B}$ to $\psi_{j,B}$. This correspondence is valid near the saddle point $\xi_j(x)$, or as far as no extra singularities appear in the domain of integration of (15). In this manner the local aspect of the exact steepest descent method, i.e., local correspondence of Borel transformed WKB solutions, is governed by microlocal analysis. (Strictly speaking, the above proof of (20) is somewhat heuristic as we have not specified the meaning of $(\partial/\partial y)^{-1}$. See [2, Sect. III] for its rigorous proof based on the integration by parts.)

Global correspondence of Borel resummed WKB solutions

We have observed so far the local correspondence between $\hat{\psi}_{k,B}$ and $\psi_{j,B}$. However, this correspondence is violated when the steepest descent path $C_k^{(j)}$ crosses a Stokes curve of type $(k>k')$ for $\hat{P}$ given by

$$\operatorname{Im}\int_{\hat{a}}^{\xi}(x_k(\xi)-x_{k'}(\xi))d\xi = 0 \quad (k'\neq k) \tag{21}$$

at, say, $\xi = \xi_0$. (Cf. Fig. 2. Here $\hat{a}$ denotes a turning point for $\hat{P}$ from which the Stokes curve in question emanates. Note that "of type $(k > k')$" means that $\hat{\psi}_k$ is dominant over $\hat{\psi}_{k'}$ along the Stokes curve.) As a matter of fact, at $\xi = \xi_0$ the singularity of $\hat{\psi}_{k,B}(\xi, z)$ located at

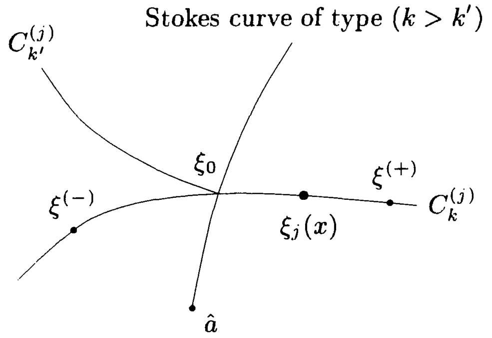


Fig. 2. Crossing of a steepest descent path and a Stokes curve.

$z = \int^{\xi} x_{k'}(\xi)d\xi$ hits the path of integration in z-space for the integral (13) by the definition of a Stokes curve in the exact WKB analysis (cf., e.g., [15]), and consequently a singular point $\xi = \xi_*$ of $\hat{\psi}_{k,B}(\xi, y + x\xi)$ corresponding to the above singularity hits the path of integration in ξ-space for the integral (15) (cf. Fig. 3). This observation implies that,

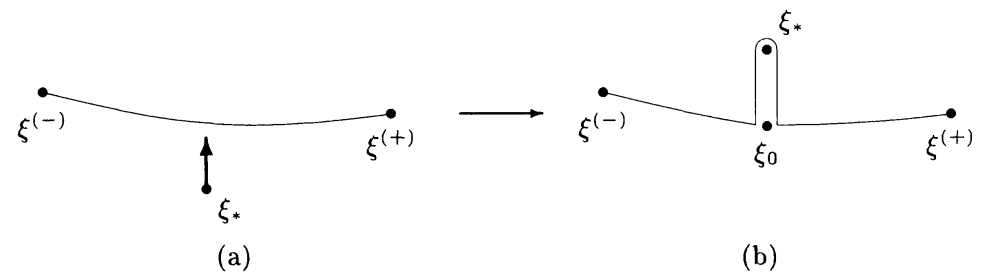


Fig. 3. Singular point ξ_* hitting the path of integration.

to get the analytic continuation of $\chi(x, y)$ defined by (17) beyond the crossing point, we have to take into account the effect of the integral I_* obtained by the integration from ξ_0 to ξ_* in Fig. 3 (b).

Then a natural question arises: Where does the integral I_* come from? The answer is quite simple: I_* is coincident with the integral

$$(22) \quad \alpha_{k'} \int_{C_{k'}^{(j)}} e^{\eta x\xi} \hat{\psi}_{k'} d\xi = \alpha_{k'} \int_{C_{k'}^{(j)}} e^{\eta x\xi} \left(\int e^{-\eta z} \hat{\psi}_{k',B}(\xi, z) dz \right) d\xi,$$

where $C^{(j)}_{k'}$ is a steepest descent path of $\operatorname{Re} f_{k'}$ emanating from the crossing point ξ_0 (cf. Fig. 2) and $\alpha_{k'}$ is a constant which appears in the connection formula

$$\hat{\psi}_k \longrightarrow \hat{\psi}_k + \alpha_{k'}\hat{\psi}_{k'} \tag{23}$$

that the dominant Borel resummed WKB solution $\hat{\psi}_k$ satisfies when crossing the Stokes curve in question. This leads to the conclusion that

$$\psi_j^\dagger \overset{\text{def}}{=} \int_{C^{(j)}_k} e^{\eta x\xi}\hat{\psi}_k d\xi + \alpha_{k'} \int_{C^{(j)}_{k'}} e^{\eta x\xi}\hat{\psi}_{k'} d\xi \tag{24}$$

gives the Borel sum of a WKB solution (12) of $P\psi = 0$ unless the steepest descent paths $C^{(j)}_k$ and $C^{(j)}_{k'}$ cross any other Stokes curves for $\hat{P}$. See [2, Sect. IV] for the proof of the coincidence of I_* and (22). (In [2] an additional assumption $n = 2$ is imposed. See also §5 below.)

We are thus forced to consider not only the steepest descent path $C^{(j)}_k$ of $\operatorname{Re} f_k$ passing through a saddle point $\xi_j(x)$ but also another steepest descent path $C^{(j)}_{k'}$ of $\operatorname{Re} f_{k'}$ bifurcated from $C^{(j)}_k$ at its crossing point with a Stokes curve for $\hat{P}$. In more general situations we should consider an "exact steepest descent path" which is defined as follows:

Definition. Let f_k denote $x\xi - \int^\xi x_k(\xi)d\xi$, i.e., the phase function of the inverse Laplace transform (10). An exact steepest descent path passing through a saddle point $\xi = \xi_j(x)$ is, by definition, the union of portions of steepest descent paths obtained by the following procedure:

Start with a steepest descent path $C^{(j)}_k$ of $\operatorname{Re} f_k$ for some k that passes through $\xi_j(x)$. If $C^{(j)}_k$ crosses a Stokes curve of type $(k > k')$ for $\hat{P}$, consider the steepest descent path $C^{(j)}_{k'}$ for $\operatorname{Re} f_{k'}$ which starts from the crossing point. If $C^{(j)}_{k'}$ (or $C^{(j)}_k$) crosses another Stokes curve of type $(k' > k'')$ (or $(k > k'')$), consider another steepest descent path $C^{(j)}_{k''}$ for $\operatorname{Re} f_{k''}$ in the same manner, and so on.

Letting $C^{(j)} = C^{(j)}_k \cup C^{(j)}_{k'} \cup C^{(j)}_{k''} \cup \cdots$ denote an exact steepest descent path in the above sense, we can then expect that

(25)
$$\psi_j^\dagger = \int_{C^{(j)}_k} e^{\eta x\xi}\hat{\psi}_k d\xi + \alpha_{k'} \int_{C^{(j)}_{k'}} e^{\eta x\xi}\hat{\psi}_{k'} d\xi + \alpha_{k''} \int_{C^{(j)}_{k''}} e^{\eta x\xi}\hat{\psi}_{k''} d\xi + \cdots$$

(where $\alpha_{k'}$ etc. are constants determined by the connection formula) should coincide with a Borel resummed WKB solution ψ_j of (1). In other words, (25) should give an integral representation of solutions of (1). Hence, it can be further expected that we can analyze global behavior of solutions of (1) by tracing the configuration of exact steepest descent paths. To use exact steepest descent paths instead of ordinary steepest descent paths is a key idea of the exact steepest descent method. The necessity of introducing bifurcated steepest descent paths is an effect of the connection formula for Borel resummed WKB solutions of $\hat{P}$. In this manner the global aspect of the method is governed by the exact WKB analysis.

Remark 1. As was observed by Berk et al. ([5]), the configuration of a steepest descent path abruptly changes when it hits a turning point for $\hat{P}$. But introduction of exact steepest descent paths resolves this trouble. For example, when a steepest descent path $C_k^{(j)}$ hits a simple turning point $\hat{a}$, no topological change occurs for the configuration of the exact steepest descent path $C_k^{(j)} \cup C_{k'}^{(j)}$ as is shown in Fig. 4. (In Fig. 4 a lightfaced line and a wiggly line respectively designate a Stokes curve and a cut defining the Riemann surface of $x_k(\xi)$.) Furthermore,

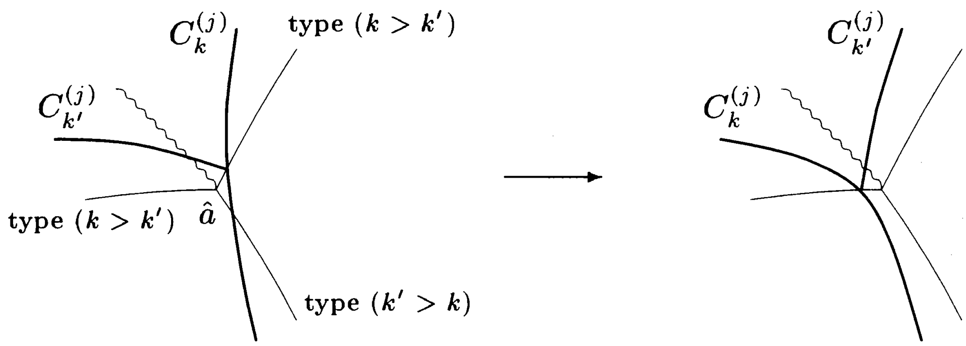


Fig. 4. Change of the configuration of an exact steepest descent path when it hits a simple turning point $\hat{a}$.

the integral

$$\int_{C_k^{(j)}} e^{\eta x\xi}\hat{\psi}_k d\xi + \alpha_{k'}\int_{C_{k'}^{(j)}} e^{\eta x\xi}\hat{\psi}_{k'} d\xi \tag{26}$$

in Fig. 4 (a) is analytically continued to

$$\int_{C_k^{(j)}} e^{\eta x\xi}\hat{\psi}_k d\xi + \tilde{\alpha}_{k'}\int_{C_{k'}^{(j)}} e^{\eta x\xi}\hat{\psi}_{k'} d\xi \tag{27}$$

in Fig. 4 (b), since the connection formula near a simple turning point guarantees that the analytic continuation of $\hat{\psi}_k$ on $C_k^{(j)}$ (resp. $\alpha_{k'}\hat{\psi}_{k'}$

on $C_{k'}^{(j)}$) in Fig. 4 (a) is equal to $\tilde{\alpha}_{k'}\hat{\psi}_{k'}$ on $C_{k'}^{(j)}$ (resp. $\hat{\psi}_k$ on $C_k^{(j)}$) in Fig. 4 (b) (cf. [15, p.245–p.246]). Hence, in general, the integral (25) is expected to be analytic even when a steepest descent path hits a turning point for $\hat{P}$.

Remark 2. At a crossing point ξ_0 of $C_k^{(j)}$ with a Stokes curve the steepest descent direction of $\mathrm{Re}\, f_k$ and that of $\mathrm{Re}\, f_{k'}$ always lie on the same side of the Stokes curve. To confirm this it suffices to note that the steepest descent direction of $\mathrm{Re}\, f_k$ and that of $\mathrm{Re}\, f_{k'}$ at $\xi = \xi_0$ are respectively given by $\vec{v}_k = -\mathrm{grad}_{(\mathrm{Re}\,\xi,\ \mathrm{Im}\,\xi)}\,\mathrm{Re} f_k = -\overline{(x - x_k(\xi_0))}$ and $\vec{v}_{k'} = -\overline{(x - x_{k'}(\xi_0))}$ and that they satisfy

$$i\,\overline{(x_k(\xi_0) - x_{k'}(\xi_0))}\,\overline{(\vec{v}_k - \vec{v}_{k'})} = i\,|x_k(\xi_0) - x_{k'}(\xi_0)|^2 \in i\mathbb{R}, \tag{28}$$

where $i\,\overline{(x_k(\xi_0) - x_{k'}(\xi_0))}$ is a normal vector of the Stokes curve. Thanks to this fact the orientation of the integral along $C_{k'}^{(j)}$ in (24) (or (25)) is naturally determined by that along $C_k^{(j)}$, that is, if the orientation along $C_k^{(j)}$ is the receding one from (resp. approaching one to) the saddle point, the orientation along $C_{k'}^{(j)}$ is also chosen to be receding (resp. approaching).

§4. An application to the computation of Stokes multipliers

In this section we examine the effectiveness of the exact steepest descent method by applying it to the computation of Stokes multipliers of a concrete example.

Example 2. Let us discuss the following equation

$$P\psi = \left(\frac{d^3}{dx^3} + \eta^2\frac{d}{dx} + x^2\eta^3\right)\psi = 0 \tag{29}$$

with its Laplace transform

$$\hat{P}\hat{\psi} = \eta\left(\frac{d^2}{d\xi^2} + (\xi^3 + \xi)\eta^2\right)\hat{\psi} = 0. \tag{30}$$

In this case the characteristic equation is given by

$$p(x,\xi) = \xi^3 + \xi + x^2 = 0 \tag{31}$$

and we label its roots $\xi = \xi_j(x)$ $(j = 0, 1, 2)$ and $x = x_\pm(\xi)$ as follows:

$$
(32) \qquad \begin{aligned} &\xi_j(x) \sim -\omega^j x^{2/3} \quad (\text{as } x \to +\infty, \text{ where } \omega = e^{2\pi i/3}), \\ &x_\pm(\xi) = \pm i\sqrt{\xi^3 + \xi} \quad (\text{where } \sqrt{\xi^3 + \xi} > 0 \text{ for } \xi > 0). \end{aligned}
$$

Using the exact steepest descent method, we now discuss the connection problem for (29) with the aid of a computer. As a path of analytic continuation we take Γ obtained by slightly deforming the real axis (cf. Fig. 5, where the ordinary Stokes curves for (29) are also included for the reference of the reader familiar with the exact WKB analysis). Let

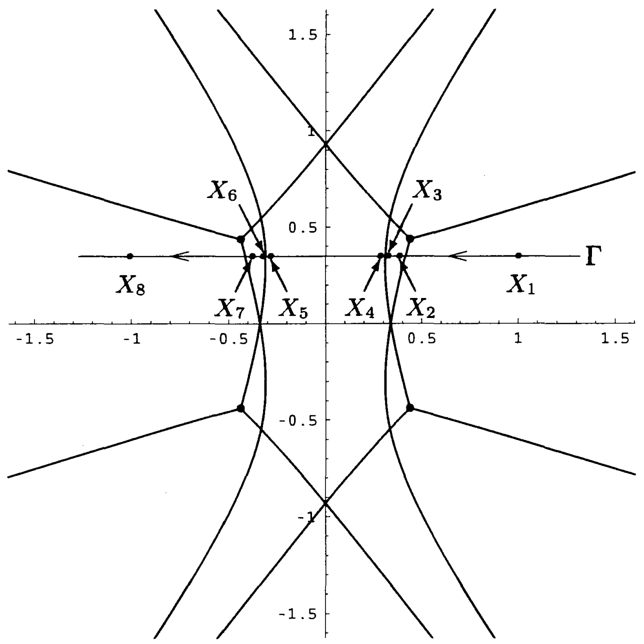


Fig. 5. The path Γ of analytic continuation.

us start with $\psi_0^\dagger$, a solution with integral representation (25) along an exact steepest descent path $C^{(0)}$ passing through a saddle point ξ_0, for a point $x = X_1$ on Γ (cf. Fig. 6 (a)). First, as is clear from the comparison between Fig. 6 (a) and (b), the exact steepest descent path $C^{(0)}$ hits another saddle point ξ_1 between $x = X_1$ and $x = X_2$. Hence the analytic continuation of $\psi_0^\dagger$ becomes the sum of $\psi_0^\dagger$ and $\psi_1^\dagger$ at $x = X_2$, where $\psi_1^\dagger$ is a solution with integral representation along $C^{(1)}$. Next, between $x = X_2$ and $x = X_3$ $C^{(0)}$ hits a turning point for $\hat{P}$ and consequently the role of the ordinary steepest descent path and that of a bifurcated one are interchanged. However, as is noted in Remark 1 in §3, the solution $\psi_0^\dagger$ is analytic and no abrupt change occurs with $\psi_0^\dagger$ there. Instead $\psi_0^\dagger$ acquires $\psi_2^\dagger$, a solution with integral representation along $C^{(2)}$, between

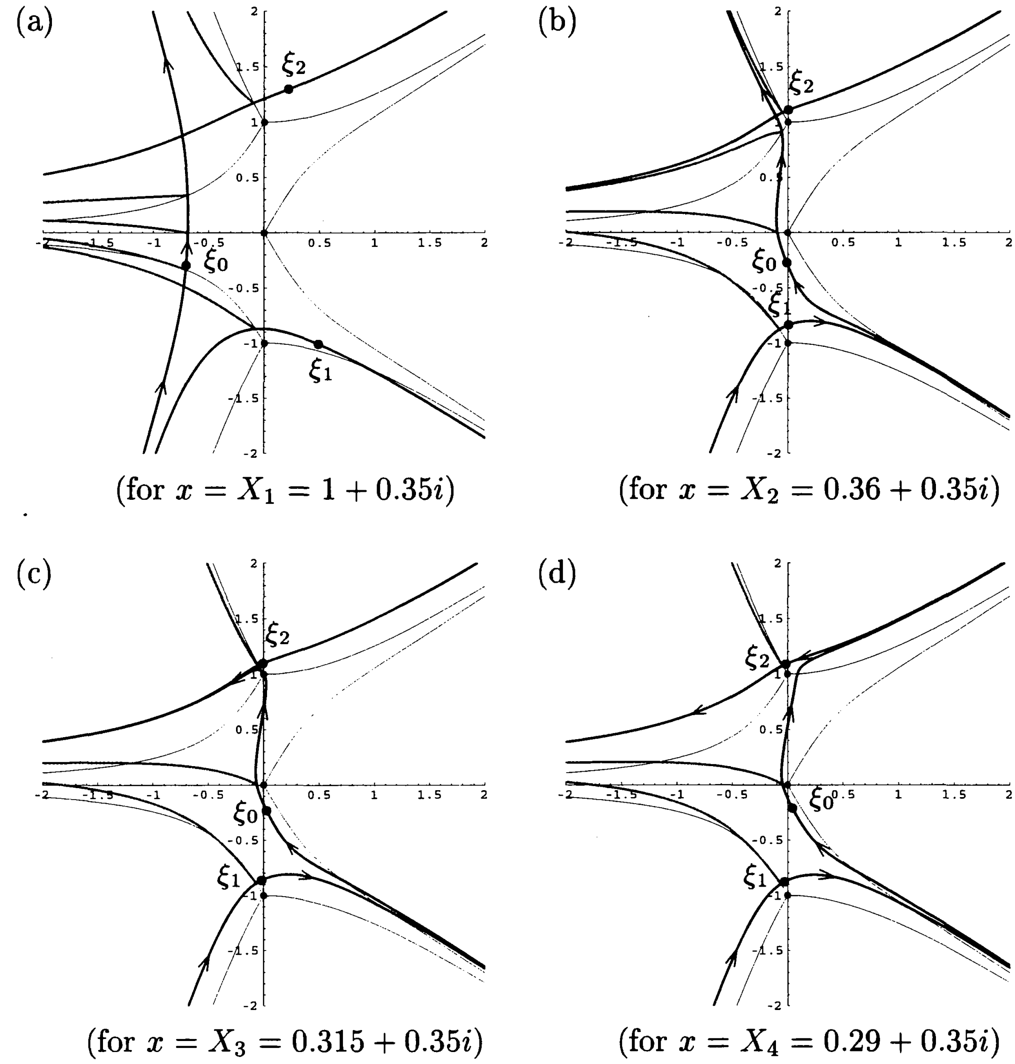
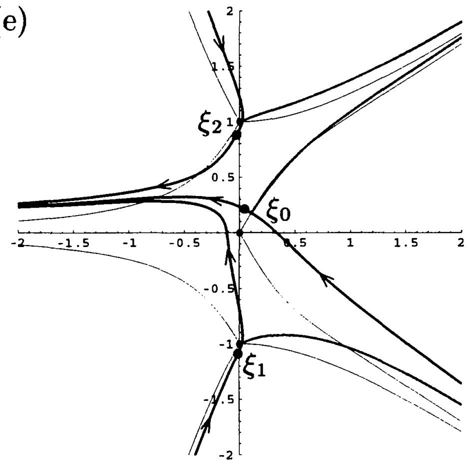


Fig. 6. Exact steepest descent paths (designated by boldfaced lines) for (29); lightfaced lines designate Stokes curves for $\hat{P}$.

$x = X_3$ and $x = X_4$ and the solution in question becomes $\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$ at $x = X_4$. This procedure can be easily repeated until we reach the point $x = X_8$; our solution is changed into $2\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$ at $x = X_6$ and finally into $3\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$ at $x = X_8$. Note that an exact steepest descent path hits a saddle point exactly on a Stokes curve for P. We thus conclude that the analytic continuation of $\psi_0^\dagger$ along the real axis is given by $3\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$ for $x \to -\infty$.

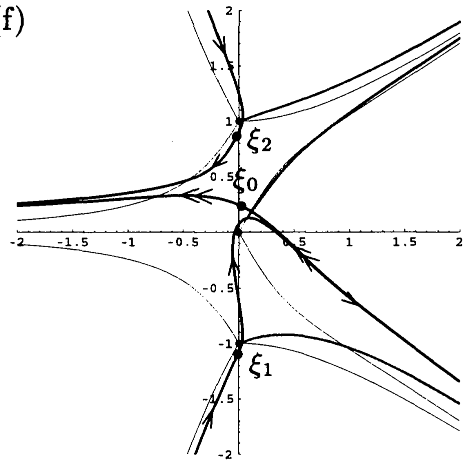
As the asymptotics of each $\psi_j^\dagger$ for $\eta \to \infty$ can be readily computed by the saddle point method (note that, except for exponentially small terms, the contribution to the $\eta \to \infty$ asymptotics comes from the saddle point

(e)



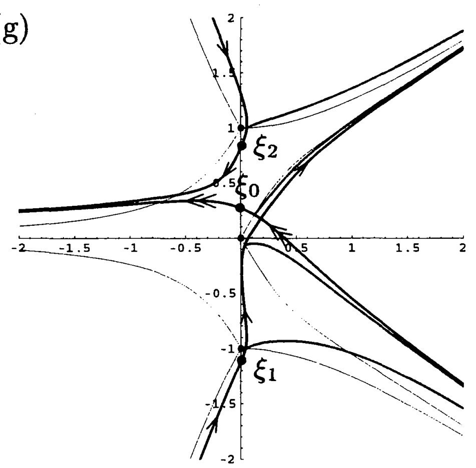
(for $x = X_5 = -0.29 + 0.35i$)

(f)



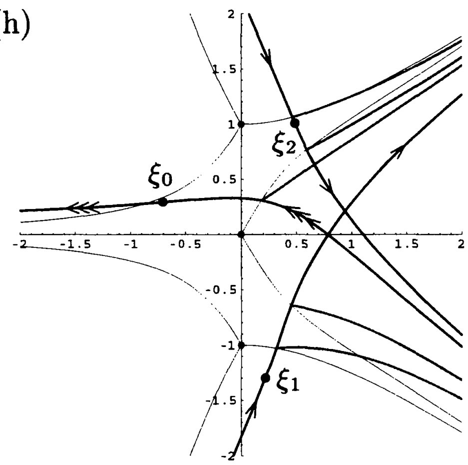
(for $x = X_6 = -0.315 + 0.35i$)

(g)



(for $x = X_7 = -0.36 + 0.35i$)

(h)



(for $x = X_8 = -1 + 0.35i$)

Fig. 6. (*Continued.*)

only) and, similarly to the case of the Airy equation, the asymptotics of (29) for $\eta \to \infty$ is consistent with that for $|x| \to \infty$, the above conclusion implies that the asymptotic solution

$$(33)\quad \psi_0^\dagger \sim \frac{2\sqrt{\pi}}{\sqrt{3\eta}} e^{3\pi i/4} x^{-2/3} \exp\left(\eta\left(-\frac{3}{5}x^{5/3} + x^{1/3} + \cdots\right)\right)(1 + \cdots)$$

of (29) for $x \to +\infty$ is analytically continued to

$$(34)\qquad 3\psi_0^\dagger + \psi_1^\dagger + \psi_2^\dagger$$

with

$$(35)\quad \psi_j^\dagger \sim \frac{2\sqrt{\pi}}{\sqrt{3}\eta} e^{(9-8j)\pi i/12}(-x)^{-2/3} \times \exp\left(\eta\left(\frac{3}{5}\omega^j(-x)^{5/3} - \omega^{2j}(-x)^{1/3} + \cdots\right)\right)(1+\cdots)$$

$(j = 0, 1, 2)$ for $x \to -\infty$. Thus by virtue of the exact steepest descent method we have succeeded in computing a Stokes multiplier of (29) explicitly.

§5. Summary and discussion

As we have observed so far, it is possible to develop a new steepest descent method applicable to ordinary differential equations with polynomial coefficients. A key point is the introduction of exact steepest descent paths; not only ordinary steepest descent paths passing through a saddle point but also bifurcated ones emanating from a crossing point of a steepest descent path and a Stokes curve for $\hat{P}$ should be taken into account. The theoretical background of the method is provided by microlocal analysis for its local aspect and by the exact WKB analysis for its global aspect.

In ending the paper, we present some open problems. The argument in [2] is based on the proviso that $\hat{P}$ is of the second order (i.e., $n = 2$). This proviso is imposed just because the exact WKB analysis is complete only for second-order operators; for higher-order operators we have to introduce new Stokes curves and virtual turning points (cf. [5], [1], [3]). To clarify the effect of new Stokes curves for $\hat{P}$ in the exact steepest descent method is the first step toward the complete understanding of the exact steepest descent method when $n > 2$. See [10] for some case study of this problem. As a related problem, we also note that finding out an algorithm of describing the complete Stokes geometry for a higher-order operator P is one of the most important open problems in the exact WKB analysis. Since the exact steepest descent method explained in this paper is useful to locate the Stokes curves of P as is emphasized in [2], it will turn out to be a powerful tool to attack this problem. See [4] also for some related problems.

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Excursions of a complex analyst into the realm of dynamical systems

Eric Bedford

§0. Introduction

The purpose of this talk is to discuss some connections between dynamics and complex analysis, especially the aspects of dynamical systems that were sufficiently interesting to me to make me drop what I was doing several years ago and enter into a long collaboration with John Smillie. One of the motivations for the work with Smillie has been to consider the dynamics of a polynomial diffeomorphism f of $\mathbf{C}^2$ which is the complexification of a map of $\mathbf{R}^2$. In general, the dynamical systems induced by f on $\mathbf{R}^2$ and $\mathbf{C}^2$ can be considerably different. However, if the complex Julia set $J \subset \mathbf{C}^2$ also happens to be a subset of $\mathbf{R}^2$, then in addition to the usual tools of real dynamics, we may also use complex methods. In the following talk, we will present the approach developed with Smillie in [BS1–5]. In §3, we outline the work [BD1,2] with Jeff Diller in which this same approach has been applied to a family of birational maps of the plane. In §4 we describe the Hénon attractor in $\mathbf{R}^2$, for which it has been difficult to actually prove anything. Although it is speculative, we present the suggestion that this phenomenon might profitably be investigated in the complex domain, a suggestion that I think Oka might have found intriguing. Given the constraints of space and time, I have limited myself to expounding a point of view and have made no attempt to survey the literature.[1]

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[1]We recommend [MNTU] for an extended introduction to the dynamics of polynomial diffeomorphisms of $\mathbf{C}^2$ and Sibony [S] for a unified treatment of the iteration of rational mappings of $\mathbf{P}^k$.

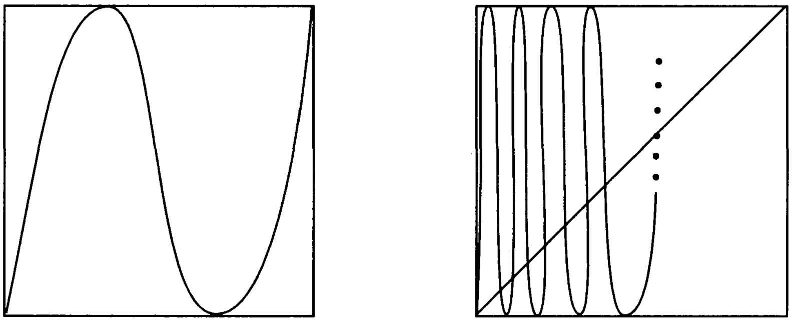

Figure 1. Graph of Γ_f (on left) and Γ_{f^n} intersected with the diagonal

In the study of dynamical systems, we consider a self mapping $f : X \to X$ of a space X and describe the behavior of its iterates $f^n = f\circ\cdots\circ f$ as $n \to \infty$. It is not clear at the outset exactly which connections this might have with the questions and techniques of analysis. If $f : X \to X$ is a d-to-1 mapping, then the graph Γ_f might look like something going up and down d times; the real analogy to what the complex case is like is shown on the left hand side of Figure 1. The graph Γ_{f^n}, corresponding to f^n will oscillate rapidly, going up and down d^n times as in the right hand side of Figure 1. If we focus on the graph of the map, we are led to consider properties of Γ_{f^n} as $n \to \infty$. Such properties might be: (1) the area (volume) growth of Γ_{f^n} as $n \to \infty$, or (2) the number of intersection points in $\Delta \cap \Gamma_{f^n}$, where $\Delta \subset X \times X$ is the diagonal, which yields the number of fixed points of f^n. Both of these have appeal for analysts (or algebraic geometers) in the complex case because (1) the volume of a variety is easy to compute, and (2) intersection theory is well developed. Here we consider the case where $X = \mathbf{R}^2$ is the real plane, and $f : X \to X$ is rational, i.e. the coordinate functions of f are rational functions. We let $\tilde{f} : \mathbf{C}^2 \to \mathbf{C}^2$ denote the complexification of f. In fact, for a compactification $\tilde{X}$ of $\mathbf{C}^2$, $\tilde{f}$ induces a meromorphic map $\tilde{f} : \tilde{X} \to \tilde{X}$. The nth iterate of a meromorphic map is again meromorphic, and the complexification commutes with iteration, i.e. $\widetilde{(f^n)} = (\tilde{f})^n$.

In dimension 2, there is a difference between the dynamics of invertible and non-invertible maps. It is typical that a bimeromorphic map of a complex surface exhibits "saddle type" behavior, whereas a holomorphic mapping of $\mathbf{P}^2$ of topological degree $d \geq 2$ typically exhibits "expanding" behavior. Throughout this talk, we assume that f is invertible.

The map $\tilde{f}$ is holomorphic outside a finite set $I = I(f)$ of points of indeterminacy. The forward iterates f^n, $n \geq 1$ are well-defined (single-valued) only off of the set $\bigcup_{n\geq 0} f^{-n}I$. However, the pull-back

$$f^* : H^{1,1}(\tilde{X}) \to H^{1,1}(\tilde{X})$$

is well-defined. Birational maps have a lot in common with invertible maps. They can fail to be invertible mappings in specific ways: they can blow curves down and blow up points. We will consider mappings with the property that if a curve is blown down to a point, the forward orbit of that point never gets blown up. In other words, $\bigcup_{n\geq 0} f^{-n}I$ contains no curve. In this case, $(\tilde{f}^*)^n = (\tilde{f}^n)^*$ for all $n \geq 1$, in which case we say that this compactification is natural for $H^{1,1}$. It follows that the area of $\Gamma_{\tilde{f}^n} \subset \tilde{X} \times \tilde{X}$ grows like ρ^n, where ρ is the spectral radius of f^*. Diller and Favre [DF] have shown that for any birational map $\tilde{f} : \tilde{X} \to \tilde{X}$ there is a birational equivalence $h : \hat{X} \to \tilde{X}$ and an induced birational map $\hat{f} = h^{-1}\tilde{f}h : \hat{X} \to \hat{X}$ such that $(\hat{f}, \hat{X})$ is natural for $H^{1,1}$.

The rate of volume growth of the graph of a complex mapping is closely related to the entropy of f. It is elementary that the entropy of the real map is no greater than the entropy of its complexification. By the estimate of Friedland [F], it follows that

$$\mathit{entropy}(f) \leq \mathit{entropy}(\tilde{f}) \leq \log \rho$$

where ρ denotes the spectral radius of f^*. Thus one strategy for bounding the entropy of a real rational mapping is to find a complex compactification which is natural for $H^{1,1}$.

In the sequel we focus on two families of maps. The first is

$$h_{a,b}(x, y) = (a - x^2 - by, x).$$

for $a, b \in \mathbf{R}$, $b \neq 0$, which are polynomial automorphisms. This family can take many forms under affine conjugacy: under the conjugacy $(x, y) \mapsto (-y, -x)$, it becomes

$$(x, y) \mapsto (y, y^2 - a - bx).$$

Conjugated by $(x, y) \mapsto (ax, ay)$, $h_{a,b}$ becomes

$$(x, y) \mapsto (1 - ax^2 - by, x),$$

which is the family introduced and studied numerically by Hénon [H]. The compactification $\tilde{X} = \mathbf{P}^2$ is natural for h, but $\tilde{X} = \mathbf{P}^1 \times \mathbf{P}^1$ is not. The dimension of $H^{1,1}(\mathbf{P}^2)$ is one, and $\tilde{h}^*$ acts on $H^{1,1}(\mathbf{P}^2)$ as multiplication by 2. Thus the entropy of the real map $h_{a,b}$ is bounded above by $\log 2$ (see [FM]). The second family is

$$f_a(x,y) = (y\frac{x+a}{x-1}, x+a-1)$$

for $a \in \mathbf{R}$. This family was studied extensively from the computational point of view in [AABHM1,2,3] and [AABM1,2]. The compactification $\tilde{X} = \mathbf{P}^1 \times \mathbf{P}^1$ is natural for f_a if $a \neq 1/n$, for $n \geq 1$ and $a \neq n/(n+2)$, for $n \geq -1$. (On the other hand, the compactification $\tilde{X} = \mathbf{P}^2$ is not natural for f_a.) The action of f^* on $H^{1,1}(\mathbf{P}^1 \times \mathbf{P}^1)$ is given by $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. The spectral radius of f^* is $(1+\sqrt{5})/2$, and so the entropy of f_a is no greater than $\log((1+\sqrt{5})/2)$.

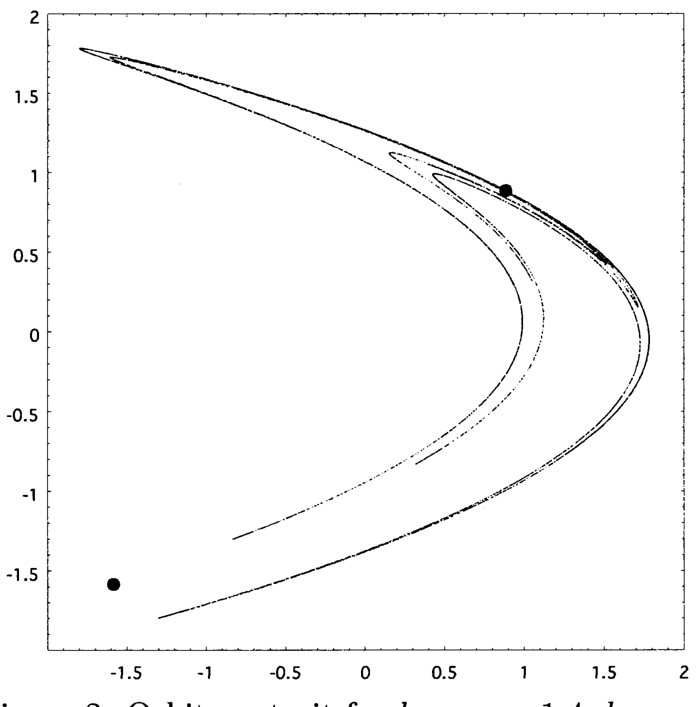

Figure 2. Orbit portrait for $h_{a,b}$: $a = 1.4$, $b = -.3$

§1. Polynomial Diffeomorphisms of $\mathbf{R}^2$

We will compare the real and complex points of view on dynamical systems by contrasting the sorts of computer pictures that may be drawn. Computer pictures that are well planned and executed have been a powerful tool for the development of dynamical systems. One frequently drawn picture is that of an orbit portrait: given a point $p \in \mathbf{R}^2$, one plots the forward orbit $O^+(p) = \{f^n p : n \geq 0\}$. Sometimes an orbit portrait $O^+(p)$ is interesting and sometimes it is not, as in the case when $f^n p$ converges to a sink (or to infinity) as $n \to \infty$. The pictures in [H], [AABHM1,2,3] and [AABM1,2] are point orbits, as well as Figure 2, which plots the first 10000 iterates of a point. The large dots are the two (fixed) saddle points of h. This map will be discussed further in §4.

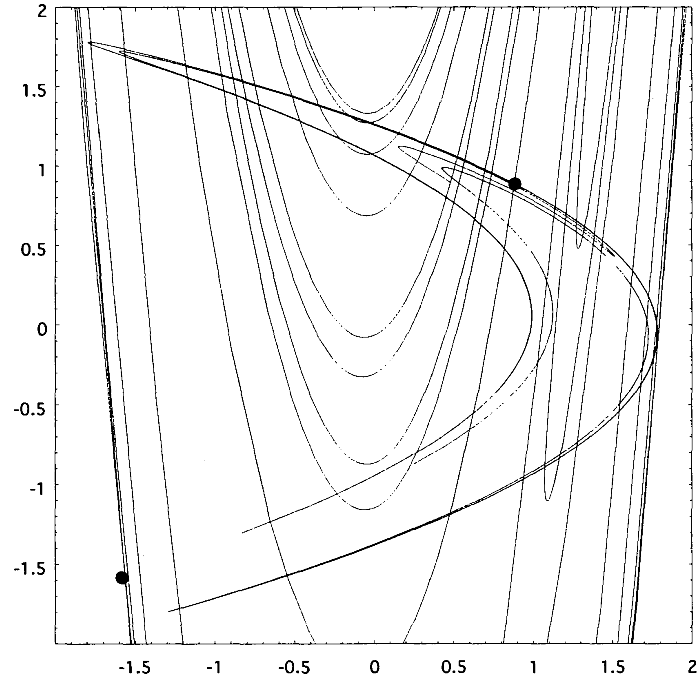

Figure 3. Stable/unstable manifolds for upper right hand fixed point: $a = 1.4$, $b = -.3$

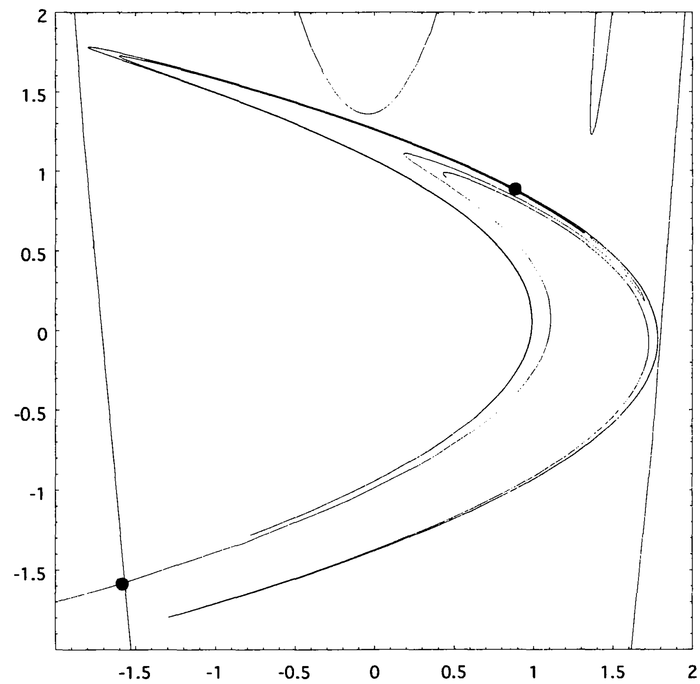

Figure 4. Stable/unstable manifolds for lower left hand fixed point: $a = 1.4$, $b = -.3$

A point p is a saddle fixed point if $h(p) = p$ and the eigenvalues λ^s and λ^u of $h'(p)$ satisfy $0 < |\lambda^s| < 1 < |\lambda^u|$. Then there are stable and unstable manifolds $W^s(p)$ and $W^u(p)$ passing through p. A useful computer picture is to show $W^{s/u}(p)$ directly, or more precisely, to draw a large arc inside $W^s(p)$ and a large arc inside $W^u(p)$.

Figure 3 corresponds to the same mapping as Figure 2. We have chosen p to be the upper right hand saddle point (indicated by the dot), and we have drawn an arc of length about 40 inside $W^u(p)$. This unstable arc closely resembles the orbit portrait in Figure 2. We have also drawn a considerably longer subarc of $W^s(p)$. While this arc is connected, it is cut off by the viewbox, and the resulting picture resembles a number of parabolas opening upward.

Figure 5 arises from a horseshoe mapping. The saddle point $p = (-4, -4)$ is indicated by the dot. The pieces of curves which look like parabolas opening to the left are all part of a (connected) arc of $W^u(p)$, which is clipped off by the viewbox. The pieces which resemble parabolas opening downwards are portions of an arc inside $W^s(p)$. One property of the horseshoe is that it is hyperbolic; the apparent transverse intersection of $W^s(p)$ and $W^u(p)$ is consistent with this. Another property is that if p_1 and p_2 are saddles, then $W^s(p_1)$ and $W^s(p_2)$ have the same closures. Thus we do not need to draw $W^{s/u}$ for the other saddle fixed point.

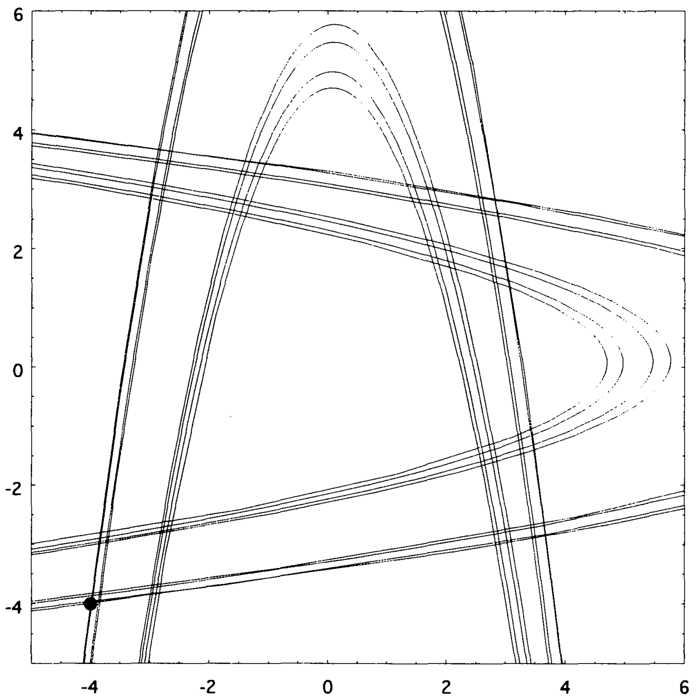

Figure 5. Stable and unstable manifolds of $h_{a,b}$: $a = 8$, $b = 1$

In order to describe pictures in $\mathbf{C}^2$ we need to first develop some knowledge about the situation in the complex domain and what we might hope to see of dynamical significance. Let us define the rate of escape functions

$$G^{\pm}(x,y) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ |\tilde{h}^{\pm n}(x,y)|$$

which give the super-exponential rate of escape to infinity of the orbit of a point (x,y) in forward/backward time. These functions are psh and continuous on $\mathbf{C}^2$, and $G^{\pm}$ is pluriharmonic on $\{G^{\pm} > 0\}$. It is evident that $G^{\pm}$ satisfies $G^+ \circ \tilde{h} = 2G^+$ and $G^- \circ \tilde{h} = \frac{1}{2}G^-$. We set $K^{\pm} = \{G^{\pm} = 0\}$ and $K = K^+ \cap K^-$ and $J^{\pm} = \partial K^{\pm}$, and $J = J^+ \cap J^-$.

Figure 6[2] uses the same real viewbox $[-2,2] \times [-2,2]$ as Figures 2, 3 and 4. The white/gray/black regions are the sets $\{c_1 < G^- < c_2\}$, and under h such a region is mapped in to $\{\frac{c_1}{2} < G^- < \frac{c_2}{2}\}$. The set K^- has zero area and is not directly visible. It is detected indirectly: to reach a point of K^-, it is necessary to pass through an infinite number of color transitions.

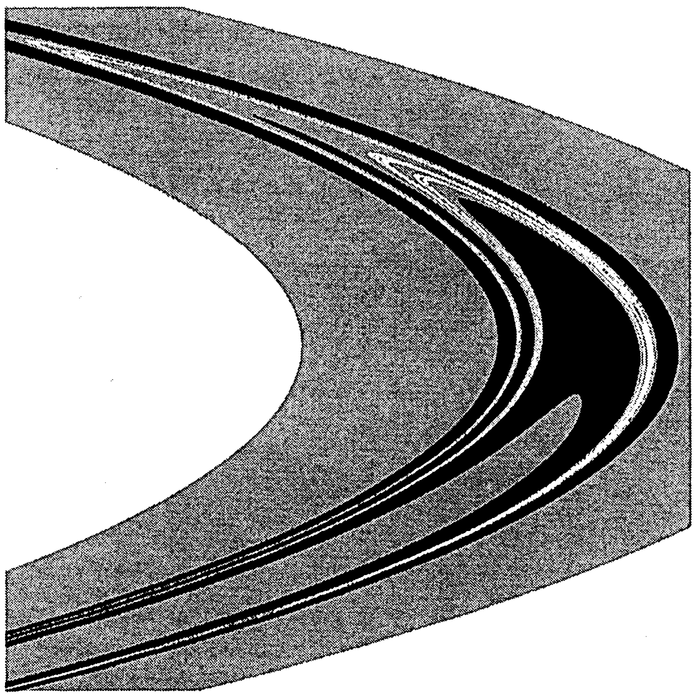

Figure 6. Level sets of G^- in $\mathbf{R}^2$ closing down on $K^- \cap \mathbf{R}^2$: $a = 1.4$, $b = -.3$

[2]Figure 6 and all the unstable slice pictures were made using the program FractalAsm which was developed by J.H. Hubbard and K. Papadantonakis. This program and other useful dynamical software is freely available at `http://www.math.cornell.edu/~dynamics`. The algorithms are explained in detail in [HP].

We define the currents $\mu^{\pm} = \frac{1}{2\pi} dd^c G^{\pm}$; they satisfy $\tilde{h}^* \mu^+ = 2\mu^+$ and $\tilde{h}^* \mu^- = \frac{1}{2}\mu^-$, and $J^{\pm} = \mathrm{supp}(\mu^{\pm})$. Ruelle and Sullivan [RS] showed generally that if h is Axiom A, then there are invariant currents $T^{\pm}$ in $\mathbf{R}^2$. The family of stable manifolds forms a lamination $\mathcal{W}^s$, and the current T^+ is constructed from this laminar structure: it involves currents of integration over pieces of stable manifolds and a family of transversal measures to $\mathcal{W}^s$. In order to turn a manifold into a current of integration, it is necessary to choose an orientation. In the absence of some condition like hyperbolicity, the stable/unstable manifolds may do a lot of "folding", and choosing an orientation presents a problem.

The stable/unstable manifolds $W^{s/u}(p, \tilde{h})$ of $\tilde{h}$ in $\mathbf{C}^2$ are Riemann surfaces which are complexifications of $W^{s/u}(p, h) \subset \mathbf{R}^2$. When $\tilde{h}$ is Axiom A, and $K \subset \mathbf{R}^2$, it follows that $\mu^{\pm}$ is the complexification of $T^{\pm}$, that is the currents $\mu^{\pm}$ may be constructed the same as $T^{\pm}$, except that the currents of integration over pieces of the laminations $\mathcal{W}^{s/u}$ are replaced by their complexifications.

The following result, which applies to all mappings $\tilde{h}_{a,b}$, shows that $\mu^{\pm}$ may be considered in some sense to be the current of integration defined by $W^{s/u}(p, \tilde{h})$. It also shows that $W^{s/u}(p, \tilde{h})$ are imbedded in $\mathbf{C}^2$ in a complicated, non-proper way.

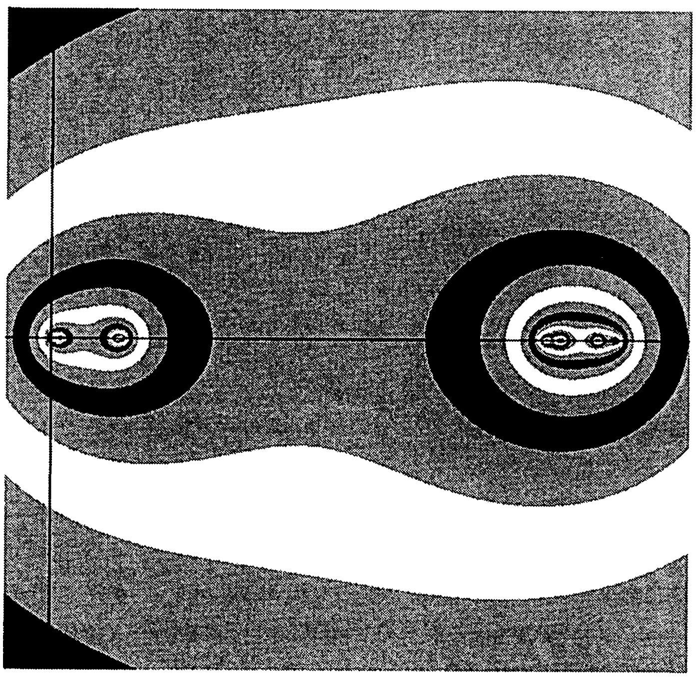

Figure 7. Unstable (complex) slice of K^+ for the horseshoe $a = 8$, $b = 1$

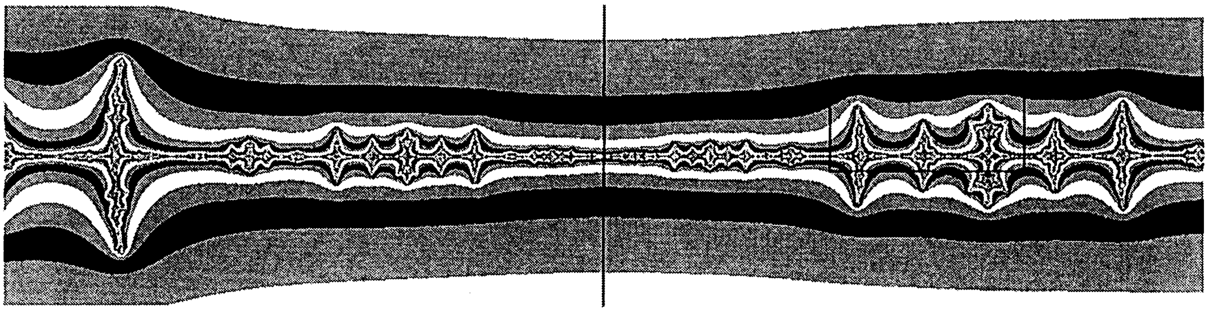

Figure 8. Level sets for G^+ for an unstable slice:[3] $a = 1.4$, $b = -.3$

Theorem ([BS1]). *Let $D^u \subset W^u(p, \tilde{h})$ denote an open disk containing p such that the current μ^+ puts no mass on ∂D^u. Then there is a constant $c > 0$ such that the normalized currents of integration $\frac{1}{2^n}[f^n D^u]$ converge to $c\mu^-$ as $n \to \infty$. It follows that for any saddle point p, the closure of $W^u(p, \tilde{h})$ is exactly J^-.*

$W^u(p, \tilde{h})$ is parametrized by an entire mapping $\psi : \mathbf{C} \to W^u(p, \tilde{h}) \subset \mathbf{C}^2$ such that $\psi(0) = p$ and $\tilde{h}(\psi(\zeta)) = \psi(\lambda^u \zeta)$ for all $\zeta \in \mathbf{C}$. (See [MNTU, §6.4].) The mapping ψ may be generated as follows. Let $v \in \mathbf{R}^2$ be an unstable eigenvector for $h'(p)$, and let $L(\zeta) = p + \zeta v$ be a parametrization of the complex line passing through p in the direction v, and

$$\psi(\zeta) = \lim_{n \to \infty} \tilde{h}^n(L((\lambda^u)^{-n}\zeta),$$

which may be used as a naive algorithm for computing ψ. An object which is dynamically meaningful is the "unstable slice" $W^u(p) \cap K^+$. Let us define $g = G^+ \circ \psi$ which is subharmonic on $\mathbf{C}$. The most useful computer pictures have been those following an idea suggested by Hubbard: Plot the level surfaces of g and its harmonic conjugate g^* inside the plane $\mathbf{C}$.[4] Note that the set $\{g = 0\}$ corresponds to the unstable slice $W^u(p) \cap K^+$. The coloring in the Hubbard picture may be chosen so that it is self-similar under the multiplication $\zeta \mapsto \lambda^u \zeta$, since $\{g = c\}$ is taken to $\{g = 2c\}$. The picture is also symmetric under complex conjugation $\zeta \mapsto \bar{\zeta}$ because h has real coefficients. By the minimum principle for harmonic functions, every compact component of the sub-level set $\{g \leq c\}$ must intersect $\{g = 0\}$. In contrast, a linear slice such as Figure

[3]We are grateful to S. Ushiki for giving us a number of pictures of complex slices of this mapping in 1993.

[4]While we do not use g^* in the pictures below, the use of g^* is related to other interesting structures, in particular "external rays," see [BS6] and [O].

6 is not self similar. And for a real linear slice such as Figure 6, g is not subharmonic.

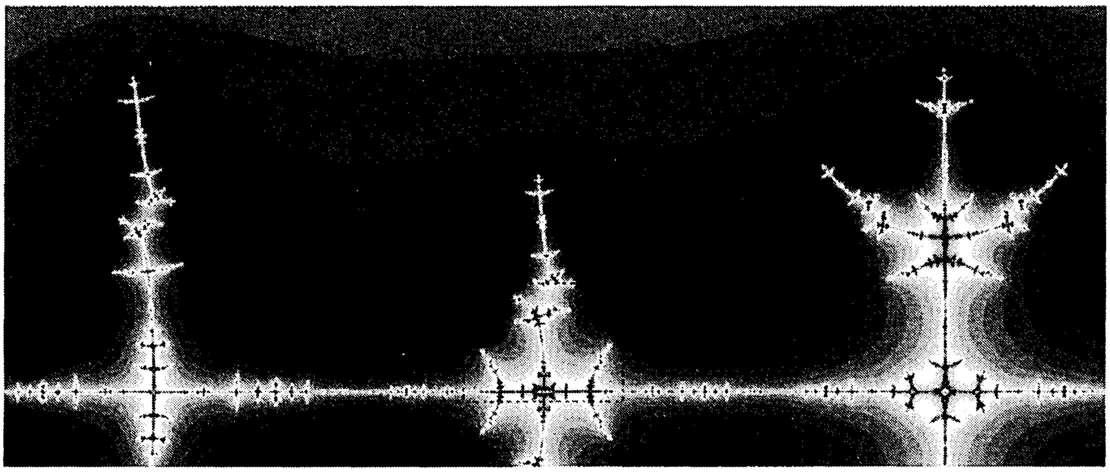

Figure 9. Detail of Figure 8, with different coloring

An example of such a picture is given in Figure 7. This is the complexification of Figure 5 inside a (square) complex disk D with $p \in D \subset W^u(p, \tilde{h})$, where p is located at the large dot in Figure 5. The origin corresponds to p under the map ψ, and the x-axis corresponds to the interval of $W^u(p)$ running from about $x = -4.5$ to about $x = -2$. That is, D cuts through the left hand legs of all the downward-opening parabolas. It appears that the points of $K^+ \cap W^u(p)$ lie on the x-axis, which corresponds to the property of the real horseshoe that $K \subset \mathbf{R}^2$.

Figure 8 gives the complex slice by the complexification of the unstable manifold in Figure 3. The origin is at the exact center of the picture (the imaginary axis has been drawn in) and the origin corresponds (under ψ) to p. The x-axis in Figure 8 corresponds in Figure 3 to a relatively short arc of $W^u(p)$ containing p. An interesting feature here is that there are "limbs" which rise off of the x-axis. This corresponds to the intersection between the Riemann surface $W^u(p, \tilde{h})$ and points of K^+ lying outside of $\mathbf{R}^2$. It is intriguing to know whether there is any connection between the "limbs" and the stable/unstable intersections in Figure 3.

Figures 9 and 10 give successively more detailed blow-ups of Figure 8. Note that the scheme for coloring the level sets $\{2^{-n-1} < G^+ < 2^{-n}\}$ has been changed, giving a different visual impression of $W^u(p) \cap K^+$. Figures 9 and 10 make it clear that the unstable slice contains a compact component. By [BS2] this implies that $J \subset \mathbf{C}^2$ is disconnected.

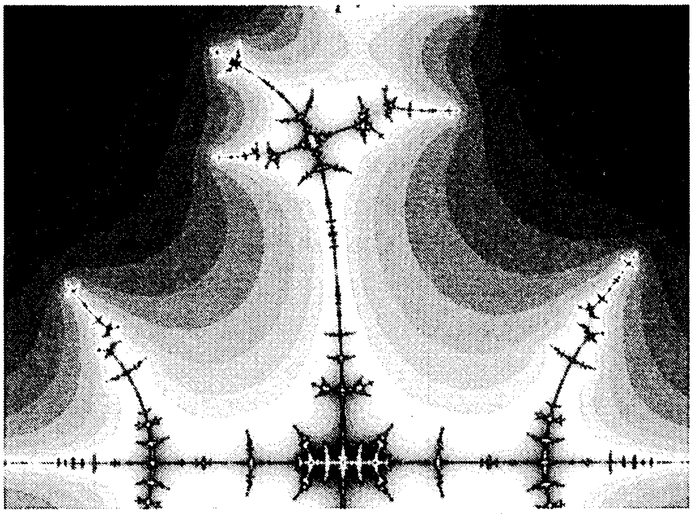

Figure 10. Detail of Figure 9

§2. Horseshoes and Maps of Maximal Entropy

We say that h has maximal entropy if $entropy(h) = \log 2$. Real mappings of maximal entropy are especially well suited for treatment by complex methods. Recall that by [FM] the entropy is equal to the exponential rate of growth of periodic points:

$$entropy(h) = \lim_{n\to\infty} \left(\#\{p \in \mathbf{R}^2 : h^n(p) = p\}\right)^{\frac{1}{n}}.$$

The situation is simpler, or at least more complete, in the complex domain. By the Bezout Theorem (see [FM]), we have that $\#\{p \in \mathbf{C}^2 : h^n(p) = p\} = 2^n$, counting multiplicity. It was shown in [BLS] that h has maximal entropy if and only if $\{p \in \mathbf{C}^2 : h^n(p) = p\} \subset \mathbf{R}^2$ holds for all $n \geq 1$.

The Smale horseshoe mapping is an important mapping which arises in many situations. Let us describe the horseshoe from the topological point of view. We start with a homeomorphism $f : \mathbf{R}^2 \to \mathbf{R}^2$, and we suppose that there is a topological box $B \subset \mathbf{R}^2$ which is mapped across itself as in the left hand side of Figure 11.

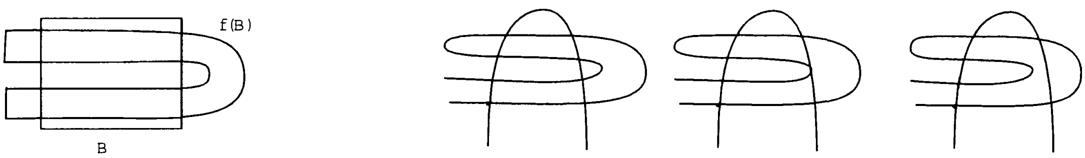

Figure 11. Topological horseshoe and degeneration

Let us define

$$B_\infty = \{p \in B : f^n p \in B, \ \forall n \in \mathbf{Z}\}.$$

Thus $f : B_\infty \to B_\infty$ is the dynamical system within B that is induced by f. Let us choose an arbitrary labeling of the the components of $B \cap fB$ by "0" and "1". We give the set $\{0,1\}$ the discrete topology and we give the sequence space $\Sigma = \{0,1\}^{\mathbf{Z}}$ the infinite product topology. We let $c : B_\infty \to \Sigma$ be the coding map where $c(p)$ is the itinerary of the orbit of p. That is, $c(p) = \cdots c_{-1}c_0c_1\cdots$ is an infinite sequence of 0's and 1's, where the nth symbol is determined by the condition that $c_n = 0$ if $f^n(p) \in B_0$, and $c_n = 1$ if $f^n(p) \in B_1$. We define the shift map as $\sigma(c) = c'$, where $c' = \cdots c'_{-1}c'_0c'_1\cdots$ is given by $c'_n = c_{n+1}$. Thus σ induces a dynamical system on Σ. It is evident, then, that the map c induces a semi-conjugacy from the dynamical system (f, B_∞) to (σ, Σ).

The standard treatment of the horseshoe is to assume at this stage that f is hyperbolic on B_∞, which is to say that there is a splitting of the tangent space $T_p\mathbf{R}^2 = E^s_p + E^u_p$, $p \in B_\infty$ into subspaces which are uniformly contracted/expanded under f'. It follows from the contraction/expansion, that the connected components of $B \cap f^{-n}B \cap f^n B$ shrink to points as $n \to \infty$. Thus the coding map $c : (f, B_\infty) \to (\sigma, \Sigma)$ is in fact a conjugacy. The horseshoe map (f, B_∞) has an interesting geometry arising from its imbedding in $\mathbf{R}^2$ (see Figure 5) and it has a simple symbolic model, which is the topological analogue of the Bernoulli shift model of coin flipping.

It was discovered by Hubbard and Oberste-Vorth [HO] (see also [MNTU, §7.4]) that if the map $f = h_{a,b}$ and a square $B = \{(x,y) \in \mathbf{R}^2 : |x|, |y| < R\}$ generate a topological horseshoe, then hyperbolicity follows automatically (by use of the Poincaré metric on a complex neighborhood). The definition of horseshoe given above specifies a method of construction. Let us give a more general definition in terms of dynamical properties alone. We say that a mapping $h_{a,b}$ is a complex horseshoe if $\tilde{h}_{a,b}$ is hyperbolic on K, and $(\tilde{h}_{a,b}, K)$ is topologically conjugate to the 2-shift (σ, Σ). The complex horseshoes are widespread: Hubbard and Oberste-Vorth (see [MNTU, §7.4]) showed that $\tilde{h}_{a,b}$ generates a complex horseshoe if $|a| > 2(1 + |b|)^2$.

We would like to use the horseshoes as a starting place to explore what is happening in parameter space $\mathcal{P} = \{(a,b) \in \mathbf{R}^2 : b \neq 0\}$. A complex horseshoe is said to be a real horseshoe (or simply a horseshoe) if $K \subset \mathbf{R}^2$. We define the horseshoe locus $\mathcal{H} \subset \mathcal{P}$ to be the set of parameters such that $h_{a,b}$ is a (real) horseshoe. Note that the entropy of a horseshoe is $\log 2$, and $(a,b) \mapsto \mathit{entropy}(h_{a,b})$ is a continuous function, so that the closure $\bar{\mathcal{H}}$ of $\mathcal{H}$ in $\mathcal{P}$ consists of mappings of maximal entropy. If $(a_0, b_0) \in \mathcal{P}$ is a parameter for which f_{a_0,b_0} is hyperbolic, then $(f_{a,b}, J_{a,b})$ is conjugate to $(f_{a_0,b_0}, J_{a_0,b_0})$ for (a,b) sufficiently close to (a_0, b_0). If f_{a_0,b_0} is not hyperbolic, then there is no general statement about nearby maps.

Let us consider $(a_0, b_0) \in \partial\mathcal{H}$ in the boundary of the horseshoe locus. This mapping is not hyperbolic. There are several possible ways that hyperbolicity might break down. One of them is that the uniformity of the expansion or contraction is lost. However, if h is a mapping of maximal entropy, then every saddle point p of period n is uniformly hyperbolic: the multipliers of $Dh^n(p)$ satisfy $|\lambda^s| \le 2^{-n}$ and $|\lambda^u| \ge 2^n$. (See [BS3] for details.) Thus the uniformity of expansion and contraction is maintained to the boundary of $\mathcal{H}$. The way that horseshoes of the form $h_{a,b}$ can degenerate is pictured in the right hand side of Figure 11: a loop of unstable manifold "pulls through" to create a tangency. That is, the picture on the left hand side of the triplet corresponds to a horseshoe; the central picture corresponds to $\partial\mathcal{H}$; and by the right hand picture we have completely left $\bar{\mathcal{H}}$. This picture is summarized in the following:

Theorem [BS4]. *Suppose that $b > 0$ and $h_{a,b}$ is a mapping of maximal entropy. Then either $h_{a,b}$ is hyperbolic, or there is a point of tangential intersection between $W^u(p_+)$ and $W^s(p_+)$, where p_+ denotes the unique fixed point such that the eigenvalues of $Df(p_+)$ are both positive.*

We can extend this result to a more global description of $\mathcal{H}$.

Theorem [BS5]. *There are real analytic functions κ^+ and κ^- defined on the interval $[-.085, .085]$ such that*
(1) $\mathcal{H} \cap \{|b| < .085\} = \{(a,b) \in \mathcal{P} : a > \max(\kappa^+(b), \kappa^-(b))\}$.
(2) If $a < \max(\kappa^+(b), \kappa^-(b))$, $0 < |b| < .085$, *then entropy*$(h_{a,b}) < \log 2$.

This shows that the horseshoe locus is nicely bounded by two real analytic curves, at least in the region $|b| < .085$. The bifurcation situation on the other side of $\partial\mathcal{H}$ is very complicated: some of this complexity is suggested by the computer picture in El Hamouly and Mira [EM].

§3. A Family of Birational Maps

Let us describe the birational map f_a of $\mathbf{R}^2$ to itself. We consider the complex compactification $\mathbf{P}^1 \times \mathbf{P}^1$. Intersecting this with the real points, we have f_a on the torus $S^1 \times S^1$. The mapping f_a has a rational inverse. In fact, f_a is conjugate to f_a^{-1} via the involution $\tau(x, y) = (-y, -x)$. However, if $a \neq -1$, f_a is not a diffeomorphism: its critical locus is $\mathcal{C} = \{x = 1\} \cup \{x = -a\}$. The line $\{x = 1\}$ is mapped to the point (∞, a). The points of indeterminacy are $(1, 0)$ and $(-a, \infty)$. One of the fixed points is (∞, ∞), which is parabolic, and so f_a is not hyperbolic. The other fixed point is $p_a = ((1-a)/2, (a-1)/2)$. If $a < 0$, $a \neq -1$, then p_a is a saddle point. If $a = -2$, then the saddle point is $(3/2, -3/2)$; an arc inside its unstable manifold is given in Figure 12. We are working on the torus, so the bands of curves that exit to the right through the vertical line $x = 20$ continue on in $S^1 \times S^1$ through $\{x = \infty\} \times S^1$ and then re-enter the picture from the left through $x = -20$. The two critical lines are indicated as dotted vertical lines. Let Γ denote the set of all of the arcs which pass between the components of $\mathcal{C}$. Since $f_a(\{x = 1\}) = (\infty, a)$, and $f_a(\{x = -a\}) = (0, -1)$, it follows that $f(\Gamma)$ is the topmost band of arcs in Figure 12, running from $(0, -1)$ (where they are pinched together) to $(\infty, a) = (\infty, -2)$ (where they are also pinched together). This pinching phenomenon does not occur for diffeomorphisms.

The orbit of $(0, -1)$ marches off to infinity: $f^n(0, -1) = (na, na-1)$. The "pinched" point $(0, -1)$ and its orbit of pinches $(-2, -3)$, $(-4, -5)$, etc., are visible in Figure 12. Similarly, the orbit of (∞, a) goes to (∞, ∞) along pinch points, alternating between $\{x = \infty\}$ and $\{y = \infty\}$.

Since f_a is conjugate to its inverse via the involution τ, we could obtain a picture of the stable manifold by applying τ to Figure 12, i.e., flipping it about the line $y = -x$.

The approach we have taken in [BD1] is first to work in the category of complex dynamics. The map induced on $H^{1,1}$ is given by $f^* = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, which has $\lambda = (1 + \sqrt{5})/2$ as its spectral radius. There are invariant currents $\mu^\pm$ with the property that $\tilde{f}^* \mu^\pm = \lambda^{\pm 1} \mu^\pm$. The current μ^- is given by the unstable manifold, and Figure 12 gives a good picture of what the real slice of μ^- looks like for $a = -2$.

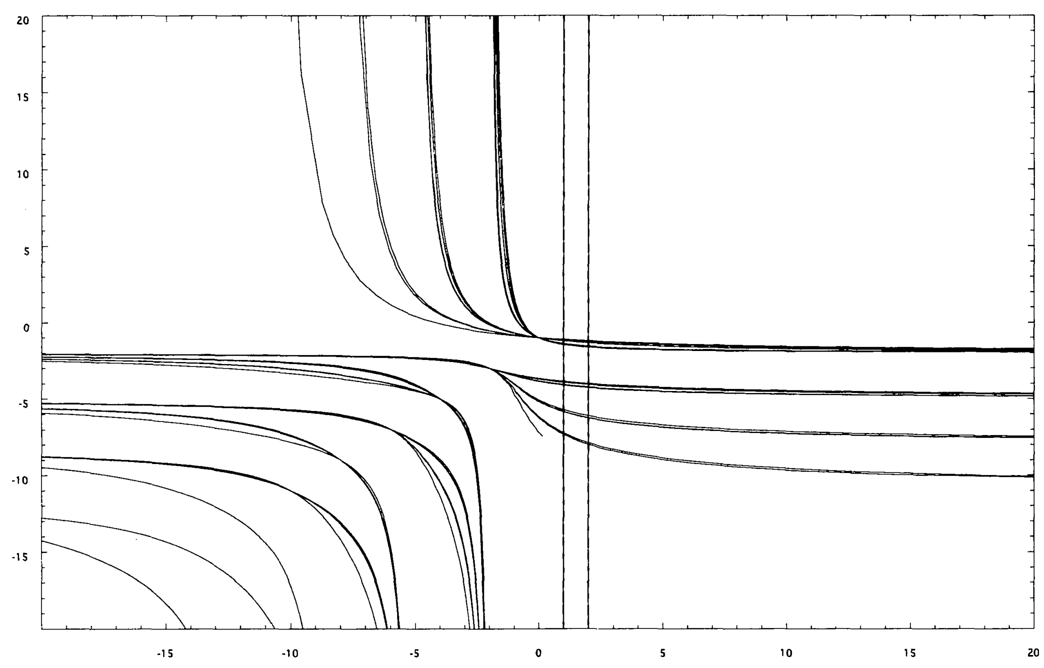

Figure 12. Unstable manifold for f_a: $a = -2$

We would also like to take the wedge product $\mu := \mu^+ \wedge \mu^-$ and obtain an invariant measure. Although the local potentials of μ^+ and μ^- are both unbounded near (∞, ∞), the wedge product may be defined (see [BD2]). Given the laminar structure of μ^+ and μ^-, the wedge product coincides with the intersection product. To help visualize the measure μ in the case $a = -2$, we have re-drawn $W^u(p_a)$ in Figure 13, together with $W^s(p_a)$, which is its "flip" under τ: μ is a measure carried by the intersection of these two sets. Further general properties of μ are that it is mixing, and the larger Lyapunov exponent is bounded below by $\frac{1}{8}\log\lambda > 0$.

The measure μ plays the same basic role in the dynamics of $\tilde{f}_a$ that the measure μ plays for the mappings $\tilde{h}_{a,b}$. Our plan is to show that if $a < 0$, $a \neq -1$, then μ puts no mass on $\mathbf{P}^1 \times \mathbf{P}^1 - \mathbf{R}^2$. We may obtain μ^+ (resp. μ^-) by pulling back the current of integration over a vertical line L (resp. pushing forward the current of integration over a horizontal line L):

$$\mu^\pm = \lim_{n\to\infty} \frac{1}{\lambda^n} f^{\pm n *}[L] = \lim_{n\to\infty} \frac{1}{\lambda^n}[f^{\mp n} L]$$

We know that f^* is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, and the entries of the nth power of this matrix are the Fibonacci numbers. Thus we know that the number of complex intersections $f^n\{y = \mathit{const}\} \cap f^{-m}\{x = \mathit{const}\}$ is given by Fibonacci numbers.

Now we require that $a < 0$, $a \neq -1$. We show by a combinatorial/geometric argument that for $c_1 > 1$, $c_2 < -1$, the number of points in $f^n\{y = c_2\} \cap f^{-m}\{x = c_1\} \cap \mathbf{R}^2$ is given by these same Fibonacci numbers. Thus all of the complex intersections $f^n\{y = c_2\} \cap f^{-m}\{x = c_1\}$ are simple and occur already in $\mathbf{R}^2$. Using the Lefschetz Index Theorem, we get an exact count of the points of period n, and we conclude that (except for the point (∞, ∞)) the fixed points all lie in $\mathbf{R}^2$ and are saddles.

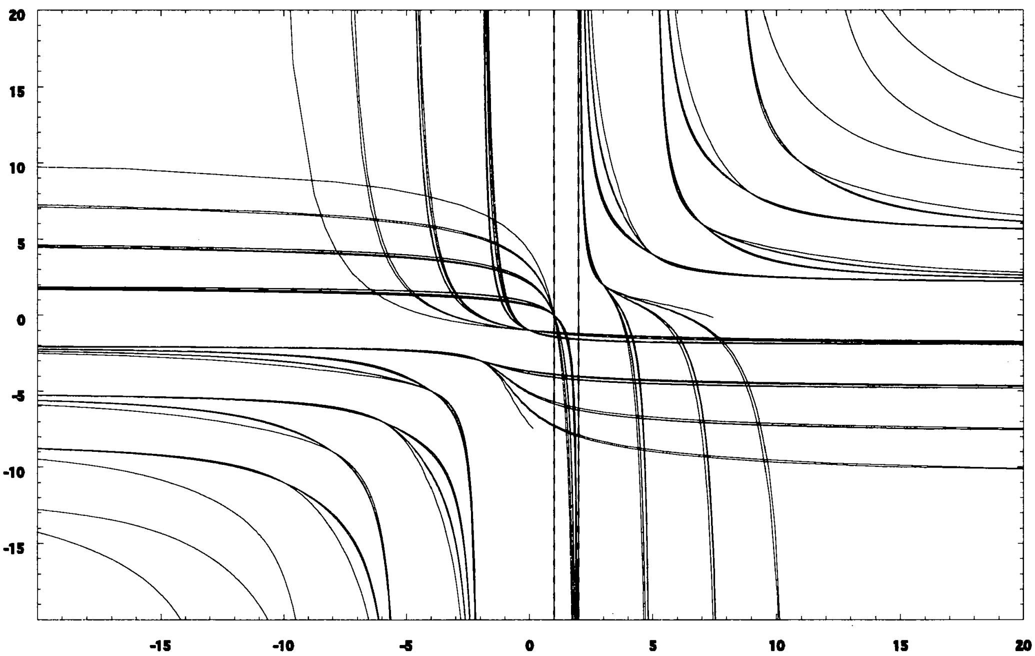

Figure 13. Intersection of stable/unstable laminations produces invariant measure

Further, for almost every $c_1 > 1$ and $c_2 < -1$, we have

$$\mu = \lim_{n \to \infty} \frac{1}{\lambda^{2n}} [f^n\{y = c_2\} \cap f^{-m}\{x = c_1\}]$$

where the right hand side means that the measure is defined by the sum of the point masses at the intersection points.

Now let $\Omega \subset \mathbf{R}^2$ denote the support of μ. Let R_0 denote the fourth quadrant and R_1 denote the second quadrant. We show that $\Omega \subset R_0 \cup R_1$ and $f_a(R_1 \cap \Omega) \cap R_1 = \emptyset$. Let Σ_G denote the so-called "golden mean" subshift of Σ, which consists of the set of bi-infinite symbol sequences of 0's and 1's such that 1 is always followed by 0, which is to say that the word "11" does not appear anywhere in the sequence. The shift $\sigma : \Sigma_G \to \Sigma_G$ defines a dynamical system, and there is a unique invariant measure η of entropy $\log \lambda$.

We let $c : \Omega \to \Sigma_G$ be the coding map, which sends a point p to its itinerary, just as we did in the case of the horseshoe. This coding map is well-defined outside the zero measure set of points whose orbits contain the point of indeterminacy $(-a, \infty)$ for f or the point of indeterminacy (∞, a) for f^{-1}. It follows that c gives a measure-theoretic equivalence between (Ω, μ) and (Σ_G, η). Thus the real map f_a has maximal entropy since (Σ_G, η) has entropy $\log \lambda$.

§4. The Hénon Attractor

Hénon [H] performed numerical explorations of the family $\{h_{a,b}\}$ from the point of view of finding dynamical phenomena. One mapping he focused on is $h = h_{a,b}$ with $a = 1.4$ and $b = -.3$. The study of this mapping has led to "computer phenomena" which have given rise to an area rich with questions and conjectures. Deep results have been obtained (see Benedicks and Carleson [BC]), but interestingly enough they seem not to be applicable to these parameters. In fact, it seems unclear what the phenomena might actually be. Perhaps a reconsideration of these questions in the complex domain will lead to formulations which can be understood and proved.

In Figures 2, 3, and 4, we have seen an orbit portrait and pairs of stable/unstable manifolds for h. Figure 3 suggests that $W^s(p)$ and $W^u(p)$ have transverse intersection points, and thus h has positive entropy. Since h has a nonreal periodic point $p \in \mathbf{C}^2 - \mathbf{R}^2$ (for instance, there is one of period 3), it follows from [BLS] that the entropy of h is strictly less than $\log 2$.

Hénon showed that there is a quadrilateral $Q \subset \mathbf{R}^2$ with the property that $h(\bar{Q}) \subset int(Q)$. Thus $A := \bigcap_{n \geq 0} h^n(Q)$ is an attractor in the sense of Conley. (There are several reasonable definitions of "attractor.") If we set $B := \bigcup_{n \geq 0} h^{-n}(Q)$, then the orbit of every point of B approaches A in forward time in the sense that $\lim_{n \to \infty} dist(h^n q, A) = 0$ for all $q \in B$. An additional feature that one would like to ask for the attractor A would be minimality.

Let us use p_R (resp. p_L) to denote the upper right hand (resp. lower left hand) fixed point of h. All unstable manifolds (and their complexifications) are contained in K^-, so $\overline{W^u(p_L)} \subset K^- \cap \mathbf{R}^2$. It is easy to imagine that $\partial B = W^s(p_L)$. Since p_R is contained in the quadrilateral Q, it follows that

$$\overline{W^u(p_R)} \subset A \subset K \cap \mathbf{R}^2.$$

Thus $W^u(p_R)$ is contained in the basin B, so $W^u(p_R)$ is contained in the interior of K^+ inside $\mathbf{R}^2$. On the other hand, saddle points (and in particular p_R) never belong to the $\mathbf{C}^2$-interior of K^+. In Figure 8 we see that 0 is not even in the interior of the unstable slice (because the picture is self-similar about 0).

If we remove p_L, the unstable manifold splits into two pieces $W^u(p_L) - \{p_L\} = \gamma' \cup \gamma''$, where γ' (resp. γ'') is the part which leaves p_L in the lower left (resp. upper right) direction. Figure 14, the complex unstable slice, can be used to prove that $\gamma' \cap K^+ = \emptyset$, by showing that G^+ is strictly positive in a fundamental region of the negative x-axis. On the other hand Figure 14 is consistent with the idea that $\gamma'' \subset K^+$. Since $p_L \notin Q$ and p_L is fixed, we have $p_L \notin B$, so

$$W^u(p_L) \cap A = \emptyset.$$

Let us define the ω-limit set, written $\omega(q)$, of a point q to be the set of accumulation points of the forward orbit $O^+(q)$. One device for plotting a computer picture of $\omega(q)$ is to plot the set $\{h^j(q) : n_1 \le j \le n_2\}$. Choosing n_1 large would remove extraneous points of $O^+(q)$ and allow $h^j(q)$ time to get close to $\omega(q)$. Choosing n_2 large would give enough points to "fill out" $\omega(q)$. In Figure 2, we chose q rather close to $\omega(q)$, so that $O^+(q)$ is close to $\omega(q)$.

An intriguing computer phenomenon discovered in [H] is that for "every" point $p \in Q$, the computer picture of $\omega(p)$ looks the same. This gives computer evidence that there is a compact set $\Omega \subset \mathbf{R}^2$ such that $\omega(p) = \Omega$ for almost every $p \in B$. For the sake of discussion, let us suppose that such an Ω exists, and let us write $B' = \{p \in B : \omega(p) = \Omega\}$. It is easy to see that

$$\Omega \subset A, \text{ and } W^s(p_R) \cap B' = \emptyset.$$

In fact, if we draw more and more of $W^s(p_R)$ it starts to look as though the stable manifold could be dense in B.

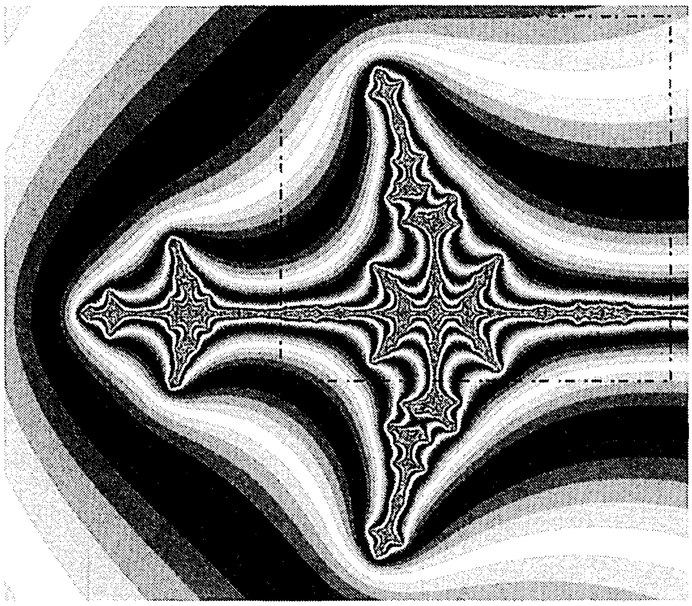

Figure 14. Unstable (complex) slice for lower left saddle point: $a = 1.4$, $b = -.3$

An inspection of Figures 2 and 3 seems to suggest that $\overline{W^u(p_R)} = \Omega$. On the other hand, this may be a case where the computer picture is deceptive: any differences between them may only become visible at a very small scale. Let us point out that $W^u(p_L)$ also looks very much like Ω, but we have $W^u(p_L) \cap \Omega \subset W^u(p_L) \cap A = \emptyset$.

One question that has been studied is whether A contains a sink orbit (a situation in which A would not be minimal). In fact, it has not been possible to prove that Ω itself is not a sink orbit, which would necessarily have a high period. A celebrated result of Newhouse and Robinson states that if f_t is a family of mappings with a nondegenerate tangency, then there is an interval $[\alpha, \beta]$ and a residual set $T \subset [\alpha, \beta]$ such that for $t \in T$, f_t has infinitely many sinks. (The possibility of a tangency is consistent with Figure 3.) Fornæss and Gavosto [FG1,2] have shown that the family $g_t = h_{1.395,t}$ is nondegenerate in a neighborhood of $t = -.3$, and so there is such an interval $[\alpha, \beta]$ containing $-.3$ in its interior.

Let us note that this region of parameter space is rich with bifurcations, and the point $(1.395, -.3)$ would not be considered to be "close" to $(1.4, -.3)$, in the sense that the dynamical system $h_{1.395,-.3}$ is separated from h by infinite cascades of bifurcations. On the other hand, the computer phenomenon of the Hénon attractor is robust in the sense that $h_{1.395,-.3}$ generates an attractor with the same appearance as A.

Now suppose that h has an attracting periodic point q. Let $\mathcal{B}$ denote the basin of attraction of q in $\mathbf{C}^2$. It is a theorem that $\partial\mathcal{B} = \partial K^+$, and $\mathcal{B}$ intersects any (complex) algebraic curve. In particular, it intersects any complex line. We do not know whether in fact $\mathcal{B}$ can exist, but the various unstable slice pictures indicate that the intersection $W^u \cap \mathcal{B}$ must be small.

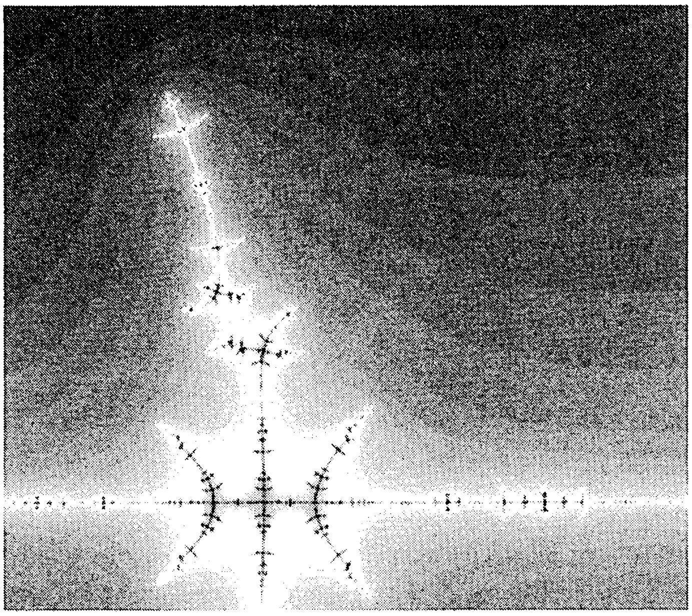

Figure 15. Detail of Figure 14

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Demailly's 2-jet negativity of certain hyperbolic fibrations

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Abstract.

We prove here a weak negativity property on Demailly's 2-jet bundles of hyperbolic (singular) fibrations on hyperbolic curves with some restrictions on the singularities of special fibres.

§1. Introduction

The concept of "k-jet negativity" was introduced by Demailly in [2] as a generalization, to higher jets, of the negativity of holomorphic sectional curvature of Finsler metrics on the tangent bundle. He conjectured that the existence of such a metric of negative curvature (in a weak sense) on a k-jet bundle he constructed should characterize Kobayashi's hyperbolicity for compact manifolds. This notion of negativity, with some appropriate non-degeneracy conditions, implies the hyperbolicity by an Ahlfors-Schwarz type lemma. In our case, we consider this conjecture only for fibrations on a hyperbolic curve with certain conditions on the singularities of special fibres. In fact, our method so far only works up to the 2-jet stage and thus imposes our restrictions on the singularities. The method is carried out as follows. We use some algebro-geometric arguments to obtain sections of the jet tautological bundle. This allows us to construct metrics of negative curvatures with some degeneracy sets. Then we do the same constructions by considering the restriction of bundles on the degeneracy sets of metrics and we continue this process. In this way one obtains a collection of metrics of negative curvatures which we piece-together to get the desired global metric. This is done by Demailly's technique of piecing together plurisubharmonic functions.

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§2. Demailly's k-jet negativity

Let X be a compact complex manifold. For a holomorphic vector bundle E on X, we denote by $P(E)$ the associated projective bundle of lines of E. Recall that a Finsler metric on E is a homogenous continuous function on its total space, smooth outside the zero section. Alternatively we can define a Finsler metric as a hermitian semi-norm on the tautological line bundle $\mathcal{O}_{P(E)}(-1)$ on $P(E)$.

Now we have the following classical theorem in [7]

Theorem 2.1 (Kobayashi 70). *Suppose that T_X admits a Finsler metric of negative holomorphic sectional curvature. Then X is hyperbolic.*

Remark that the ampleness of $T^\star_X$, i.e., the ampleness of $\mathcal{O}_{P(T_X)}(1)$ is equivalent to the existence of a hermitian metric of negative curvature on its dual $\mathcal{O}_{P(T_X)}(-1)$. This implies the hypothesis of Kobayashi's theorem but we actually need negativity only in "some important directions" in this theorem. For this reason, Demailly in [2] introduced the bundle

$$V_1 := ((\pi_1)_\star)^{-1}(\mathcal{O}_{X_1}(-1)) \subset T_{X_1},$$

where $X_1 := P(T_X)$ and $\pi_1 : X_1 \to X$ is the natural projection, and gave the following definition.

Definition 2.2. *We say that X has (or more precisely, can be given a metric of) negative 1-jet curvature, if, for some smooth hermitian metric h on $\mathcal{O}_{X_1}(-1)$, there exist $\epsilon > 0$ and a smooth hermitian metric ω such that,*

$$\Theta_h(\mathcal{O}_{X_1}(-1))(\xi) \leq -\epsilon\|\xi\|^2_\omega,\ \forall\, \xi \in V_1.$$

Remark that if the metric h in the definition above come from an hermitian metric on T_X, then this negativity is equivalent to the negativity of holomorphic sectional curvature of X.

Now, we iterate the construction $(X, T_X) \to (X_1, V_1)$ to

$$(X_1, V_1) \to (X_2 := P(V_1), V_2 := ((\pi_2)_\star)^{-1}(\mathcal{O}_{X_2}(-1)) \subset T_{X_1}),$$

where $\pi_2 : X_2 \to X_1$ is the natural projection and $\mathcal{O}_{X_2}(-1)$ is the tautological line bundle associated to V_1. We obtain a tower

$$X_k \to X_{k-1} \to \ldots \to X_2 \to X_1 \to X,$$

with the important propriety that every holomorphic germ $f : (\mathbb{C}, 0) \to X$ can be lifted to a germ $f_{[k]} : (\mathbb{C}, 0) \to X_k$ with $f'_{[k]}(0) \in V_k$. Such a holomorphic germ of curve is said to be regular if $f'(0) \neq 0$. We define two sets contained in X_k:

- $X_k^{\text{reg}} :=$ the set of liftings of regular germs of curves which is an open set in X_k.
- $X_k^{\text{sing}} := X_k \setminus X_k^{\text{reg}}$ called the set of singular jets of curves.

If we let $D_j := P(TX_{j-1}/X_{j-2}) \subset X_j$ then it was proved in [2] that $X_k^{\text{sing}} = \cup_{j=2}^{k} \pi_{k,j}^{-1}(D_j)$, where $\pi_{k,j} : X_k \to X_j$ is the projection map.

We can define now the negativity of Demailly's k-jets for $k \geq 2$.

Definition 2.3. *Let h_k be a metric on $\mathcal{O}_{X_k}(-1)$ (possibly singular with L^1_{loc} weight). We say that h_k has negative curvature in the sense of Demailly if there exist $\epsilon > 0$ and ω_k a smooth metric on X_k such that,*

$$\Theta_{h_k}(\mathcal{O}_{X_k}(-1))(\xi) \leq -\epsilon \|\xi\|_{\omega_k^2}, \ \forall \xi \in V_k.$$

Remark that for $k \geq 2$, $\mathcal{O}_{X_k}(1)$ is not relatively ample with respect to $X_k \to X$. Hence we need to allow singularities in the metric h_k in the above definition. This notion of negativity implies Kobayashi's hyperbolicity as stated in the following theorem in [2].

Theorem 2.4 (Demailly 95). *If X has a k-jet metric h_k with negative curvature in the sense of Demailly, then every entire non-constant curve $f : \mathbb{C} \to X$ has an image $f_{[k]}(\mathbb{C}) \subset \Sigma_{h_k}$, where Σ_{h_k} is the degeneracy set of h_k. In particular, if $\Sigma_{h_k} \subset X_k^{\text{sing}}$ (in this case we say that X has nondegenerate negative Demailly's k-jet curvature), then X is Kobayashi hyperbolic.*

Now we have the following conjecture this paper is concerned with.

Conjecture 2.5. *A compact complex manifold X is hyperbolic if and only if X has nondegenerate negative Demailly's k-jet curvature for k sufficiently large.*

Using a hyperbolic surface fibred over a hyperbolic base, J.-P. Demailly showed in [2] that for each $k_0 > 0$ there exists a surface which has not nondegenerate negative Demailly's k_0-jet curvature. Consequently the sought jet metric can not be absolutely bounded.

We will now study Conjecture 2.5 for a fibered surface. In the following X will be a compact complex hyperbolic surface fibred over a hyperbolic base $X \to B$. In other words the genera of all components of fibres and of the base B are at least 2. When the fibres are all smooth, we can easily construct a hermitian metric of negative holomorphic sectional curvature and then X has a nondegenerate negative Demailly's

1-jet curvature. Therefore, in the sequel, we will consider fibrations which have at least one singular fibre and nonisotrivial, i.e., not locally trivial outside singular fibres.

§3. Almost ampleness on k-jets

In this section we prove an algebraic statement for our fibration, which, in the general case of a projective general type manifold, would imply the celebrated Green-Griffiths conjecture on degeneration of entire curves and provides an important step toward the resolution of Conjecture 2.5. We begin with the following definition introduced by S. Lu in [8] in the 1-jet case (the terminology comes from Miyaoka's almost everywhere ampleness in [9]).

Definition 3.1. *Let d be an integer with $1 \leq d \leq \dim X_k$. We say that $T_X^\star$ is almost ample on k-jets in all dimensions $\geq d$, if the restriction $\mathcal{O}_{X_k}(1)_{|Y}$ is big for every subvariety $Y \subset X_k$ such that*

$$\dim Y \geq d \quad \text{and} \quad \dim \pi_k(Y) \geq \inf\{d, \dim X\},$$

where $\pi_k : X_k \to X$ is the projection map.

Remark that if in the definition above we take $d = \dim X_k$, then this means the same as supposing the tautological line bundle $\mathcal{O}_{X_k}(1)$ to be big. Also, for $d = 1$ and $k = 1$, this is equivalent to the ampleness of the cotangent bundle. Now we have the following fact which is an application of Theorem 2.4 (or more precisely of its proof).

Fact 3.2. *Almost ampleness in all dimensions $\geq d = \dim X$ for a manifold X of general type implies the degeneration of entire curves in X.*

This motivates the following conjecture.

Almost ampleness conjecture 3.3. *Let X be a projective manifold of general type with stable tangent bundle. Then there exists k such that the cotangent bundle is almost ample on k-jets in all dimensions $\geq \dim X$.*

We remark that the additional hypothesis of "stable" is necessary in the above conjecture. In fact, exceptional examples like smooth quotients of the bidisk can not have almost ample cotangent bundle on k-jets in dimensions ≥ 2 for any $k > 0$. An important example supporting this conjecture is the class of surfaces of general type with positive indices, i.e., with $\frac{c_1^2 - 2\,c_2}{3} > 0$. This is due to the work of Y. Miyaoka

[9] cited above. Another support for this almost ampleness conjecture is the following.

Theorem 3.4. *Let $f : X \to B$ be a surface of general type fibred over a hyperbolic curve. Suppose that f is not isotrivial, then $T^\star_X$ is almost ample on 1-jets in all dimensions ≥ 2.*

Proof . — To see that $\mathcal{O}_{X_1}(1)$ is big, we observe that there is a generically injective rational map $\mathcal{O}_{P(T^\star_X)}(1) \to \mathcal{O}_{P(T^\star_B)}(1)$. We have also that $T^\star_{X|F}$, where F is a generic fibre, is ample by a criterion of Gieseker [6]. Then, applying the additivity of Kodaira dimensions of T. Fujita in [5], we are done. In fact, this is a particular case of Sakai's additivity of $\lambda-$dimensions [10].

Now, let Y be a surface in X_1 with $\pi_1(Y) = X$. Let $X^{(1)} \subset X_1$ the surface containing the liftings of all the fibres of f. We have to distinguish two cases:

The first is when $Y \neq X^{(1)}$. Then we have a generically injective morphism $(\pi_1 \circ f)^\star : T^\star_B \to \mathcal{O}_{X_1}(1)_{|Y}$ and we conclude the above (applying the additivity of Kodaira dimensions).

It remains to consider the case $Y = X^{(1)}$. Here, $(\pi_1)_\star(\mathcal{O}_{X_1}(1)_{|Y}) = \Omega_{X/B}$, where $\Omega_{X/B}$ is the sheaf of relative differentials with respect to f. We will prove that this sheaf is big. Remark that it suffices to prove this for a semi-stable fibration using the semi-stability reduction theorem. We assume that X is semi-stable. Then, we blow-up the singularities of each fiber so that we have an exceptional curve through each singular point. We use the same notation X for the surface obtained. It suffices to prove that $\Omega_{X/B}$ is big which is equivalent to proving that $\mathcal{O}_{X_1}(1)_{|Y}$ is big with Y is defined as above. Let $\mathcal{O}_{\overline{X}_1}(1)$ be the tautological line bundle associated with the logarithmic tangent bundle along the exceptional curves in X (which form a finite number). Actually, it suffices to prove that $\mathcal{O}_{\overline{X}_1}(1)_{|\overline{X}^{(1)}}$ is big with $\overline{X}^{(1)}$ the associated surface in $\overline{X}_1$. For simplicity, suppose that we have only one singular point. Let $\overline{X}_k$ the k-th logarithmic jet-bundle and $\mathcal{O}_{\overline{X}_k}(1)$ the tautological line bundle on it (see [4] for the definitions). Then we have $\mathcal{O}_{\overline{X}_k}(1)_{|\overline{X}^{(k)}} = \pi_k^\star(\omega_{X/B}) \otimes \mathcal{O}_{\overline{X}_k}(-E_k)$, where E_k is an exceptional curve of the first kind and $\omega_{X/B} = K_X \otimes K_B^{-1}$ the relative dualizing sheaf. By a result in [1], this latter bundle is big. Now we can verify easily that, for large k, the self intersection of the following bundle $\mathcal{O}_{\overline{X}_k}(1) \otimes \mathcal{O}_{\overline{X}_{k-1}}(1) \otimes \ldots \otimes \mathcal{O}_{\overline{X}_1}(1)_{|\overline{X}^{(k)}}$ is positive. This implies that $\mathcal{O}_{\overline{X}_k}(1)_{|\overline{X}^{(k)}}$ is big and then $\mathcal{O}_{\overline{X}_1}(1)_{|\overline{X}^{(1)}}$ is (actually those last bundles have the same sections on $\overline{X}^{(k)}$). $\square$

Theorem 3.5. *Let $f : X \to B$ be a hyperbolic surface fibred over a hyperbolic base. Suppose that f is not isotrivial, then there exists a positive integer k_0 such that $\mathcal{O}_{X_k}(-1)_{|Y}$ has a (singular) metric of negative k-jets curvature for all $Y \subset X_k$ not contained in X_k^{sing} for every $k \geq k_0$.*

Proof . — Let k_0 be the stage where all the liftings of the fibres of f become smooth (this is possible by Proposition 5.11 in [2]). By Theorem 3.4, $T_X^\star$ is almost ample on 1-jets in all dimensions ≥ 2. This implies (see Lemma 7.6 in [2]) that it is also almost ample on k-jets in all dimensions ≥ 2. Then, for $k \geq 2$, we have that $\mathcal{O}_{X_k}(1)_{|Y}$ is big for all $Y \subset X_k$ with $\pi_k(Y) = X$. In addition, using an easy Riemann-Roch calculation, this is also true if $\pi_k(Y)$ is a curve. This implies, using sections, that we have a metric of negative curvature on $\mathcal{O}_{X_k}(-1)_{|Y}$ for all such Y.

Now take $k \geq k_0$. It remains to consider the case when Y is a curve (the case when Y projects to a point to X is treated similarly). If a curve $Y \subset X_k$ is not tangent to V_k except at a finite set of points, we obtain a metric of negative k-jet curvature on $\mathcal{O}_{X_1}(-1)_{|Y}$ just by taking a smooth metric which is equal to the Poincaré metric in the neighbourhood of each of those points. If the curve is a lifting of some fibre, then, as the fibres are hyperbolic and the tangent bundle of such lifting (being smooth) is negative and isomorphic to $\mathcal{O}_{X_1}(-1)_{|Y}$, we are done. The final case is when Y is a lifting of a curve in X which is not a fibre of f. In this case the existence of a nontrivial sheaf morphism shows that the negativity of T_B implies the negativity of $\mathcal{O}_{X_1}(-1)_{|Y}$. □

§4. Application to Demailly's conjecture

Let L be a line bundle on a compact complex manifold and h_0 a fixed smooth metric on it. Consider a singular metric h on L. We write $h = h_0 \exp(-\Phi)$, where Φ is a smooth function outside the singularities of h. Then we obtain the following relation between curvatures

$$\Theta_h(L) = \Theta_{h_0}(L) + i\partial\overline{\partial}\Phi.$$

This relation permits us to reduce the problem of piecing together metrics to piecing together quasi-psh functions (a terminology of J.-P. Demailly which means functions locally a sum of plurisubharmonic functions and smooth functions). Now, we have the following two lemmas needed for piecing together quasi-psh functions which can be easily proved using techniques from [3].

Lemma 4.1. *Let Y and Z be two subvarieties of a compact complex manifold X. Let V be a subbundle of T_X and let ω be a smooth metric on X. Suppose there exist a smooth $(1,1)$-form α on X and a smooth function Φ_Y (resp. Φ_Z) on Y (resp. on Z) such that*

$$\alpha + i\partial\overline{\partial}\Phi_Y \geq \epsilon\omega \text{ on } V \cap T_{Y_{\mathrm{reg}}},$$

and

$$\alpha + i\partial\overline{\partial}\Phi_Z \geq \epsilon\omega \text{ on } V \cap T_{Z_{\mathrm{reg}}}.$$

Then, there exists a smooth function $\Phi_{Y\cup Z}$ on a neighbourhood U of $Y \cup Z$ such that

$$\alpha + i\partial\overline{\partial}\Phi_{Y\cup Z} \geq \frac{\epsilon}{4}\omega \text{ on } V_{|U}.$$

Lemma 4.2. *Let $Y \subset Z$ be two subvarieties of a compact complex manifold X. Let V be a subbundle of T_X and ω a smooth metric on X. Suppose there exist a smooth $(1,1)$-form α on X and a smooth function Φ_Y (resp. $\Phi_{Z\setminus Y}$) on Y (resp. on $Z \setminus Y$ locally bounded from above on Y) such that*

$$\alpha + i\partial\overline{\partial}\Phi_Y \geq \epsilon\omega \text{ on } V \cap T_{Y_{\mathrm{reg}}},$$

and

$$\alpha + i\partial\overline{\partial}\Phi_{Z\setminus Y} \geq \epsilon\omega \text{ on } V \cap T_{Z_{\mathrm{reg}}}.$$

Then, there exists a smooth function Φ_Z on Z such that

$$\alpha + i\partial\overline{\partial}\Phi_Z \geq \frac{\epsilon}{2}\omega \text{ on } V_{|U}.$$

By Theorem 3.5 we obtain a collection of metrics of negative curvatures on X_k for $k \geq k_0$: We start from a metric on X_k of negative k-jet curvature and we consider its base locus which is a finite union of irreducible proper subvarieties. By the same theorem, the restriction of $\mathcal{O}_{X_k}(-1)$ to each of those components (not contained in X_k^{sing}) has a metric with negative k-jet curvature with smaller base locus and so on. In order to obtain a global metric with non degenerate k-jet curvature we should piece together these metrics. For a stable fibration (where singularities of fibres are nodal), we can take $k_0 = 1$ in Theorem 3.5. As X_1^{sing} is empty, we can thus do this piecing together easily using lemmas 4.1 and 4.2 above. We obtain:

Theorem 4.3. *Let $X \to B$ be a stable fibration as in Theorem 3.5. Then X has nondegenerate Demailly's 1-jet negative curvature.*

For $k \geq 2$ the piecing together procedure is complicated because, in this case, $\mathcal{O}_{X_k}(1)$ is not relatively nef with respect to π_2 and X_k^{sing} is nonempty. Nevertheless, using a weaker condition on singularities than stability, we can achieve the construction for the 2-jet stage. In fact, for this stage, we have a good alternative tautological bundle $L_2 := \mathcal{O}_{X_2}(1) \otimes \pi_{2,1}^\star(\mathcal{O}_{X_1}(2))$ which is relatively nef with respect to the projection to X by Proposition 6.16 in [2]. We have the following:

Theorem 4.4. *Let $f : X \to B$ be a fibration as in Theorem 3.5. Suppose that L_2 has positive degree on every lifting to X_2 of the singular fibres of f. Then X has nondegenerate Demailly's 2-jet negative curvature.*

Proof . — By theorem 3.4, the restriction $\mathcal{O}_{X_1}(1)_{|Y}$ is big for all $Y \subset X_1$ which projects surjectively onto X. As $X_2^{\text{sing}} = D_2$ is equal to $\mathcal{O}_{X_2}(1) \otimes \mathcal{O}_{X_1}(-1)$, we have $L_2 = \pi_{2,1}^\star \mathcal{O}_{X_1}(3) \otimes \mathcal{O}(D_2)$. Consequently, $L_{2|Y}$ is big for all $Y \subset X_2$ not contained in D_2 and which projects surjectively onto X. In particular L_2 is big on X_2. This implies that the line bundle $L_2^\epsilon := \mathcal{O}_{X_2}(1) \otimes \pi_{2,1}^\star(\mathcal{O}_{X_1}(2+\epsilon))$, which is relatively ample with respect to $X_2 \to X$, is also big for small ϵ.

Let h_0 be a metric of negative curvature on $(L_2^\epsilon)^\star$, and Σ_{h_0} its singular set. Then Σ_{h_0} is a finite union of subvarieties of X_2 of dimensions at most 3. Now, for components Y of Σ_{h_0} not contained in $X_2^{\text{sing}} = D_2$ and which projects onto X, the same argument as above shows that $(L_2^\epsilon)^\star{}_{|Y}$ has also a metric of negative curvature (though perhaps for a smaller ϵ). This is also true when $Y = D_2$ by a Riemann-Roch calculation. This gives a collection of metrics $h_j, j = 1, .., s$ (for some integer s) with a finite union $\cup_{j=1,..,s}\Sigma_j$ of subvarieties of dimensions at most 2 as singular loci.

It remains to study the restrictions of $(L_2^\epsilon)^\star$ to components Y of $\cup_{j=0,..,s}\Sigma_j$ which projects to a curve in X (the case $Y \subset D_2$ projects on X surjectively is treated similarly). If Y is a curve, the hypothesis and a similar argument as in the proof of Theorem 3.5 show that $(L_2^\epsilon)^\star{}_{|Y}$ has a metric of negative 2-jet curvature. Suppose now that Y is not a curve and $\pi_2(Y)$ is a curve C in X. Then the intersection of the tangent sheaf to Y and V_2 consists of the tangent sheaf of curves contained in the fibres and of the lifting of C to two jets. As L_2^ϵ is relatively ample and $(L_2^\epsilon)^\star$ has negative 2-jet curvature on C, we obtain, using lemma 4.2, that $(L_2^\epsilon)^\star{}_{|Y}$ has a smooth metric of negative 2-jet curvature.

Finally, using lemmas 4.1 and 4.2, we glue together all the metrics we have now to obtain a smooth metric of negative 2-jet curvature on $(L_2^\epsilon)^\star$. This gives a non degenerate metric of negative 2-jet curvature on $\mathcal{O}_{X_2}(-1)$ by the identity $L_2^\epsilon = \pi_{2,1}^\star(\mathcal{O}_{X_1}(3+\epsilon)) \otimes \mathcal{O}(D_2)$. □

Remark 4.5. *Suppose we have a sequence $(L_k)_{k \in \mathbf{N}}$ of relatively nef line bundles on X_k such that, for k sufficiently large, L_k has positive degree on the lifting to X_k of each singular fibres of f. Then, using the same proof as for Theorem 4.4, we can prove Demailly's conjecture without additional hypothesis for our fibration.*

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Short $\mathbb{C}^k$

John Erik Fornæss

§1. Introduction

One of Oka's main contributions was to solve the Levi problem.

There are various ways to generalize the Levi Problem. The Union problem is one: Let $\Omega_0 \subset \Omega_1 \subset \cdots \subset \cup\Omega_n = \Omega$. Suppose that each Ω_j is Stein. Is Ω Stein? To approach the Union Problem, one can try at first to understand the simplest cases of Ω.

Example 1.1. *Long $\mathbb{C}^2$. Suppose that each Ω_j is biholomorphic to $\mathbb{C}^2$. Then we call Ω a long $\mathbb{C}^2$. It is an open question whether all long $\mathbb{C}^2$ are actually biholomorphic to $\mathbb{C}^2$.*

Example 1.2. (*Fornæss*, ([*F*, 1976])) *In dimension 3 and higher it can happen that Ω is not Stein and that each Ω_n is biholomorphic to a ball.*

This left open the question in dimension 2.

Theorem 1.3. (*Fornæss-Sibony*, ([*FS*, 1981])) *Suppose that each Ω_j is biholomorphic to the unit ball in $\mathbb{C}^2$. If the (infinitesimal) Kobayashi metric of Ω is not identically zero, then Ω is biholomorphic to the ball or to $\Delta \times \mathbb{C}$, where Δ is the unit disc.*

Recall that the (infinitesimal) Kobayashi metric of Ω vanishes identically if and only if for all $p \in \Omega$ and any tangent vector ξ to Ω at p and for any $R > 0$, there exists a holomorphic map $f : \Delta = \{z \in \mathbb{C}; |z| < 1\} \to \Omega$ so that $f(0) = p$ and $f'(0) = R\xi$.

This theorem left still open the case when the Kobayashi metric vanishes identically. The most obvious example of such a case is when $\Omega = \mathbb{C}^2$. However, the question remaining was whether there was any

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other possibility (Diederich-Sibony ([DS,1979]). In this paper we show that indeed there are other such Ω. In fact, such Ωs occur quite naturally in dynamics. In random iteration, basins of attraction can be such domains. Under iteration of fixed maps, they occur as sublevel sets of Green functions.

Fix an integer $d \geq 2$. For any $\eta > 0$, let $\mathrm{Aut}_{d,\eta}$ denote the set of polynomial automorphisms F of $\mathbb{C}^k, k \geq 2$, of the form $F(z_1, \dots, z_k) = (z_1^d + P_1(z_1, \dots, z_k), P_2(z_1, \dots, z_k), \dots, P_k(z_1, \dots, z_k))$ where each P_j is a polynomial of degree at most $d-1$ and where each coefficient is at most η in modulus.

An example is $F(z) = (z_1^d + \eta z_k, \eta z_1, \dots, \eta z_{k-1})$.

Suppose that $F_n \in \mathrm{Aut}_{d,\eta_n}, \eta_n = a_n^{d^n}, n = 1, 2, \dots, 1 > a_1 \geq a_2 \geq \cdots \lim_{n\to\infty} a_n = a_\infty \geq 0$, and set $F(n) = F_n \circ \cdots \circ F_1$. Let Ω denote the set of points $z = (z_1, \dots, z_k) \in \mathbb{C}^k$ such that $F(n)(z) \to 0$ as $n \to \infty$.

Theorem 1.4. *The set Ω has the following properties:*
(i) Ω is a nonempty, open, connected set in $\mathbb{C}^k$,
(ii) $\Omega = \cup_{j=1}^\infty \Omega_j \supset \cdots \supset \Omega_\ell \supset \cdots \supset \Omega_1$. Each Ω_j is biholomorphic to the unit ball $B^k(0,1)$.
(iii) The infinitesimal Kobayashi metric of Ω vanishes identically.
(iv) There is a plurisubharmonic function $\psi : \mathbb{C}^k \to [\log a_\infty, \infty)$ such that $\Omega = \{\psi < 0\}$ and ψ is nonconstant on Ω.

The reason that Ω fails to be biholomorphic to $\mathbb{C}^k$ is that there is a nonconstant bounded plurisubharmonic function on Ω. In some sense this means that Ω is "too small" to be all of $\mathbb{C}^k$. So we might call such an Ω a short $\mathbb{C}^k$.

Next, we mention some more details about the function ψ.

Theorem 1.5 (=Theorem 3.4). *The set $U = \{z \in \mathbb{C}^k; \psi(z) > \log a_\infty\}$ is open and ψ is pluriharmonic on U.*

Theorem 1.6. *The function ψ has no critical points on $\{\psi > \log a_\infty\}$.*

This shows that the level sets of ψ are foliated by complex hypersurfaces Σ.

Theorem 1.7. *Any leaf $\Sigma(z^0)$ of $\{\psi = c > \log a_\infty\}$ can be exhausted by relatively open sets U biholomorphic to $B^{k-1}(0,1)$. Each such U is Runge in $\mathbb{C}^k$. Moreover the intrinsic infinitesimal Kobayashi metric of $\Sigma(z^0)$ vanishes identically.*

Theorem 1.8. *Each leaf* $\Sigma(z^0)$ *of* $\{\psi = c > \log a_\infty\}$ *is dense in* $\{\psi = c\}$.

It is a little harder to get good control on the set $\{\psi = \log a_\infty\}$. We investigate here only a special case with very rapidly decreasing coefficients where one can get pluripolar sets with only one singular point.

Theorem 1.9 (=Theorem 3.10). *Let* $F_n(z,w) = (z^2 + a_n w, a_n z)$. *Suppose that* $|a_n| \searrow 0$ *sufficiently rapidly. Then* $\{\psi = -\infty\} =: P$ *has the following shape:* $P \setminus (0)$ *is closed in* $\mathbb{C}^2 \setminus (0)$ *and is laminated by Riemann surfaces.*

On the contrary, when one lets the coefficients decrease at a slightly slower pace than in Theorem 1.4, Ω is biholomorphic to $\mathbb{C}^2$.

Theorem 1.10 (=Theorem 3.11). *Let* $F_n = (z^2 + a_n w, a_n z)$. *Suppose that* $0 < |a_n| < c < 1$ *and* $|a_{n+1}| \geq |a_n|^t$ *for some* $1 < t < 2$. *Then the basin of attraction of* 0 *is biholomorphic to* $\mathbb{C}^2$.

Theorem 1.11 (=Theorem 3.7). *For every* $c > \log a_\infty$, *the sublevel sets* $\{\psi < c\}$ *is a short* $\mathbb{C}^k$.

The same result is valid for other maps, such as for example iterations of any given fixed Hénon map. The next result shows also that "short" $\mathbb{C}^k$, Ω, might contain subsets biholomorphic to $\mathbb{C}^k$.

Theorem 1.12 (=Theorem 3.8). *Let* H *be a Hénon map, and let* G^+ *be the pluricomplex Green function,* $G^+(z) = \lim_{n\to\infty} \frac{\log^+ \|H^n(z)\|}{d^n}$, $d =$ *degree* H. *Then for every* $c > 0$, $\{G^+ < c\}$ *is a "short"* $\mathbb{C}^2$.

The plan of the paper is to first prove Theorem 1.4 in Section 2. Then in Section 3 we prove some of the other results above. Due to lack of space the remaining theorems and also other results in this direction will be published elsewhere.

The author would like to thank Dror Varolin for many discussions during the preparation of this talk.

§2. Proof of Theorem 1.4.

Proof of Theorem 1.4.
(i): Let τ denote the maximum possible number of terms of a polynomial of degree $d-1$ in k variables. Let $\Delta^k(0,c)$ denote the polydisc in $\mathbb{C}^k$ with center at the origin and polyradius $(c,c,\dots,c), c > 0$. Suppose that $z \in \Delta^k(0,c), 0 < c < 1$, and assume that $F \in \mathrm{Aut}_{d,\eta}, F = (z_1^d +$

$P_1, \ldots, P_k)$. Then $|P_i(z)| \leq \tau\eta, i = 1, \ldots k$ and $|z_1^d| \leq c^d$. It follows that $F(\Delta^k(0,c)) \subset \Delta^k(0, c^d + \tau\eta)$. Pick $c, c', 0 < c < c' < 1$ and set $c_\ell = c(c')^\ell$. If $\ell \geq 0$, we have that $d^\ell \geq \ell + 1$. We show that if $n \geq n_0$, n_0 large enough and $\ell \geq 0$, then $F_{n+\ell}(\Delta^k(0, c_\ell)) \subset \Delta^k(0, c_{\ell+1})$:

$$\begin{aligned}
\log(\tau\eta_{n+\ell}) &= \log\tau + d^{n+\ell}\log a_{n+\ell} \\
&\leq \log\tau + (\ell+1)d^n \log a_1 \\
&= [\log\tau + (\ell+1)\frac{d^n}{2}\log a_1] + (\ell+1)\frac{d^n}{2}\log a_1. \\
&\quad \text{If } n \geq n_0 : \\
\log(\tau\eta_{n+\ell}) &< \log c(1-c) + (\ell+1)\log c'. \\
\tau\eta_{n+\ell} &< c(1-c)(c')^{\ell+1}. \\
\tau\eta_{n+\ell} &< c(c')^{\ell+1} - (c(c')^\ell)^d. \\
c_\ell^d + \tau\eta_{n+\ell} &< c_{\ell+1}.
\end{aligned}$$

It follows that if $n \geq n_0$ and $\ell \geq 0$, then $F_{n+\ell}(\Delta^k(0, c_\ell)) \subset \Delta^k(0, c_{\ell+1})$.

Set $\Omega_n := \{z \in \mathbb{C}^k; F(n)(z) \in \Delta^k(0,c)\}$. It follows that if $n \geq n_0$, $\Omega_n \subset \Omega_{n+1}$ and that $F(n+\ell)(z) \to 0$ uniformly on Ω_n when $\ell \to \infty$. Hence we have that $\Omega \supset \cup_{n \geq n_0}\Omega_n$ and the union is increasing. Suppose next that $z \in \Omega$. Then $F(n)(z) \to 0$ and hence $F(n)(z) \in \Delta^k(0,c)$ for some $n \geq n_0$. Hence $z \in \cup_{n \geq n_0}\Omega_n$. This proves (i).

(ii): We set $U_n = \{z \in \mathbb{C}^k; \|F(n)(z)\| < c\}$. Then $U_n \subset \Omega_n$. If $z \in \Omega_n, n \geq n_0$, then $F(n+\ell)(z) \in \Delta^k(0, c(c')^\ell) \subset B(0,c)$ for a fixed $\ell \geq 1, \forall n \geq n_0$. Hence $\Omega_n \subset U_{n+\ell}$. Therefore $\Omega = \cup_{m \geq 0} U_{n_0 + m\ell} \supset \cdots \supset U_{n_0}$, writing Ω as an increasing union of balls. This proves (ii).

(iii): Fix (p, ξ), $p \in \Omega$ and ξ a tangent vector to Ω at p. Pick $R > 0$. Then $p_n = F(n)(p) \to 0$. Set $F'(n)(\xi) = \xi_n$. Then $\xi_n \to 0$ also. Define $\zeta_n : \Delta = \{\tau \in \mathbb{C}; |\tau| < 1\} \to \mathbb{C}^2$, $\zeta_n(\tau) = p_n + \tau R\xi_n$. If n is large enough, $\zeta_n(\Delta) \subset \Delta^k(0,c)$. This implies that $\zeta = F(n)^{-1} \circ \zeta_n : \Delta \to \Omega_n \subset \Omega$. Moreover $\zeta(0) = p, \zeta'(0) = F^{-1}(n)'(R\xi_n) = R\xi$. Hence (iii) is proved.

(iv): We use a modification of the Green function construction introduced by Hubbard ([H]). Write $F(n) = (f_1^n, \ldots, f_k^n)$.

We define $\phi_n : \mathbb{C}^k \to \mathbb{R}$ by $\phi_n(z) = \max\{|f_1^n|, \dots, |f_k^n|, \eta_n\}$. Each ϕ_n is a continuous function on $\mathbb{C}^k$.

Lemma 2.1. $\psi_n := \frac{\log \phi_n}{d^n} \to \psi$, *$\psi$ plurisubharmonic on* $\mathbb{C}^k$.

Proof: We show first that $\phi_{n+1} \leq (\tau + 1)\phi_n^d$.

(a): $\phi_n(z) \leq 1$:

$$\begin{aligned} \phi_{n+1}(z) &= \max\{|(f_1^n)^d + P_1(f_1^n, \dots, f_k^n)|, |P_2|, \dots, |P_k|, \eta_{n+1}\} \\ &\leq \max\{\phi_n^d + \tau\eta_{n+1}, \tau\eta_{n+1}, \eta_{n+1}\} \\ &\leq \max\{\phi_n^d + \tau\eta_n^d, \eta_n^d\} \\ &\leq (\tau+1)\phi_n^d. \end{aligned}$$

(b): $\phi_n(z) > 1$:

$$\begin{aligned} \phi_{n+1}(z) &= \max\{|(f_1^n)^d + P_1(f_1^n, \dots, f_k^n)|, |P_2|, \dots, |P_k|, \eta_{n+1}\} \\ &\leq \max\{\phi_n^d + \tau\eta_{n+1}\phi_n^{d-1}, \tau\eta_{n+1}\phi_n^{d-1}, \eta_{n+1}\} \\ &\leq \max\{\phi_n^d + \tau\phi_n^{d-1}, 1\} \\ &\leq (\tau+1)\phi_n^d. \end{aligned}$$

Hence $\frac{\log \phi_{n+1}}{d^{n+1}} \leq \frac{\log(\tau+1)}{d^{n+1}} + \frac{\log \phi_n}{d^n}$ which implies that the sequence

$$\{\frac{\log \phi_n}{d^n} + \sum_{j>n} \frac{\log(\tau+1)}{d^{j+1}}\}$$

is monotonically decreasing and the limit is a plurisubharmonic function $\psi \geq \log a_\infty$, (a priori it is possible that $\psi \equiv -\infty$ but we will show that this cannot happen). For simplicity we say that $\{\frac{\log \phi_n}{d^n}\}$ is <u>*almost*</u> monotonically decreasing to the limit ψ.

Lemma 2.2. $\Omega = \{\psi < 0\}$.

Proof of the Lemma: Assume that $\psi(z) < 0$. Then for all large n and some constant $s < 0$,

$$\frac{\log \phi_n(z)}{d^n} < s < 0.$$

Hence $\phi_n(z) < e^{d^n s}$ which implies that $|f_j^n(z)| < e^{d^n s}, j = 1, \ldots, k$ and hence $F(n)(z) \to 0$, so $z \in \Omega$.

Next assume that $z \in \Omega$. Then $F(n)(z) \in \Delta(0, c)$ for all large n. This implies that $\psi_n(z) < 0$ for all large n and hence that $\psi(z) \leq 0$.

Next, let $z^n = F^{-1}(n)(0)$. So for $n \geq n_0, z^n \in \Omega$. Then $\phi_n(z^n) = \eta_n$. Therefore

$$\begin{aligned}
\psi(z^n) &\leq \frac{\log \phi_n(z^n)}{d^n} + \sum_{j>n} \frac{\log(\tau+1)}{d^{j+1}} \\
&= \frac{\log \eta_n}{d^n} + \sum_{j>n} \frac{\log(\tau+1)}{d^{j+1}} \\
&= \log a_n + \sum_{j>n} \frac{\log(\tau+1)}{d^{j+1}} \\
&\leq \log a_1 + \sum_{j>n} \frac{\log(\tau+1)}{d^{j+1}} \\
&\leq \log a_1 + \frac{\log(\tau+1)}{d^n} \\
&< 0
\end{aligned}$$

for all large enough n. Since $\psi \leq 0$ on Ω and $\psi(z) < 0$ at some point in Ω, it follows from the subaveraging principle that $\psi < 0$ everywhere on Ω.

It remains only to show the ψ is not constant on Ω. Suppose that $\psi_{|\Omega} \equiv \alpha < 0$. First note that Ω is not all of $\mathbb{C}^2$. For example, it is easy to estimate that $F(n)(z)$ goes to infinity for any $z = (x, 0, \ldots, 0), x >> 1$ since the z_1 coordinate of the iterates grows much faster than any of the other coordinates. Pick a point $z^0 \in \Omega$. Then there exists a number $R > 0$ so that the ball $B(z^0, R) \subset \Omega$ while there is a point $p \in \partial B(z^0, R) \cap \partial\Omega$. By the above lemma we know that $\psi(p) \geq 0$. By the subaveraging property of plurisubharmonic functions, $\psi(p)$ is bounded above by the average on any small ball $B(p, \epsilon)$. Since $\psi = \alpha < 0$ on almost half the ball and since ψ is upper semicontinuous, this leads to a contradiction when ϵ is small enough. This contradiction shows that ψ is nonconstant on Ω. This proves (iv). (We have also ruled out here that $\psi \equiv -\infty$, as promised.)

§3. Proofs of further results.

In this section we study in more detail the properties of Ω and its defining function ψ as given in Theorem 1.4. Hubbard ([H]) introduced a filtration of $\mathbb{C}^2$ which has proved very useful in the investigation of Hénon maps. We use the natural generalization of this filtration to $\mathbb{C}^k$.

Definition 3.1 (Filtration). *Set* $R := 2\tau + 2$.

$$\begin{aligned} V &:= \Delta(0,R) \\ V^+ &:= \{z \in \mathbb{C}^k; |z_1| \geq R, \max\{|z_2|,\dots,|z_k|\} \leq |z_1|\} \\ V^- &:= \{z \in \mathbb{C}^k; \max\{|z_2|,\dots,|z_k|\} \geq R, |z_1| \leq \max\{|z_2|,\dots,|z_k|\}\}. \end{aligned}$$

The basic properties of this filtration is given in the following Lemma. Fix any integer $n > n_0$, where n_0 is large enough. (More precisely, we will need $3\tau\eta_{n_0-1}^{d-1} \leq 1, \tau R^{d-2}\eta_{n_0} \leq 1, (*)$.) For $z = (z_1,\dots,z_k)$, set $z' = (z_1',\dots,z_k') = F_n(z)$. Set $m = \max\{|z_2|,\dots,|z_k|\}, m' = \max\{|z_2'|,\dots,|z_k'|\}$.

Lemma 3.2. *Assume* $n > n_0$.
(i) Suppose $z \in V^+$. *Then* $z' = F_n(z) \in int(V^+)$ *and* $|z_1'| > 2|z_1|$.
(ii) If $z \in V$, *then* $z' \in V \cup V^+$.
(iii) If $z \in V^-$, *then* $m' \leq \tau m^{d-1}\eta_n$.
(iv) Suppose that $|z_1| \geq 2\tau\sigma \max\{m, \eta_{n-1}\}$ *for some* $\sigma \geq 1$. *Then* $|z_1'| \geq 3\tau\sigma \max\{m', \eta_n\}$ *and* $||z_1'| - |z_1|^d| \leq \tau\eta_n \max\{1, |z_1|^{d-1}\}$.

The proof is straightforward and will be omitted.
We show next that no orbit can stay in V^- forever.

Lemma 3.3. *Suppose that* $z \in \mathbb{C}^k$. *Then there exists an integer* $n = n(z)$ *so that* $F(n)(z) \in V \cup V^+$ *for all* $n \geq n(z)$.

Proof: By Lemma 3.2, (i) and (ii), it suffices to show that for some $n > n_0$, $F(n)(z) \in V \cup V^+$. Suppose to the contrary that $F(n_0 + 1)(z) =: z^1, \dots, F(n_0 + \ell)(z) =: z^\ell, \dots \in V^-$ for all $\ell \geq 1$. Let $m_\ell := \max\{|z_2^\ell|,\dots,|z_k^\ell|\}$. Then applying Lemma 3.2, (iii), we obtain

$$\begin{aligned} m_{\ell+1} &\leq \tau m_\ell^{d-1}\eta_{n_0+\ell} \\ &\leq \tau m_\ell^{d-1} a_1^{d^\ell} \\ \frac{\log m_{\ell+1}}{d^{\ell+1}} &\leq \frac{\log \tau}{d^{\ell+1}} + \frac{\log m_\ell}{d^\ell} + \log a_1 \end{aligned}$$

It follows that the sequence $\frac{\log m_\ell}{d^\ell}$ will eventually decrease by at least $\frac{\log a_1}{2}$ each step. This implies that eventually $\frac{\log m_\ell}{d^\ell} < 0$ which implies that $m_\ell < 1$, contradicting that $m_\ell \geq R$. This proves the Lemma.

■

Theorem 3.4. *The set $U = \{z \in \mathbb{C}^k; \psi(z) > \log a_\infty\}$ is open and ψ is pluriharmonic on U.*

Lemma 3.5. *If $\psi(z) > \log a_\infty$ then there exists n arbitrarily large so that $|z_1^n| > 2\tau \max\{|z_2^n|, \ldots, |z_k^n|, \eta_n\}$.*

Proof of the Lemma: Pick two constants α, β, $\min\{\psi(z), 0\} > \log\alpha > \log\beta > \log a_\infty$. There exists a large integer n_1 so that if $n \geq n_1$ then $a_n < \beta$. Suppose that for some $n_2 \geq n_1$,

$$|z_1^n| \leq 2\tau \max\{|z_2^n|, \ldots, |z_k^n|, \eta_n\},\ \forall\, n \geq n_2.$$

Set $m_n := \max\{|z_2^n|, \ldots, |z_k^n|\}$. Suppose that for some $n \geq n_2$, we have that $m_n \geq 1$. Then $|P_j(z^n)| \leq \tau\eta_{n+1}(2\tau m_n)^{d-1}$. Hence $m_{n+1} \leq \tau^d \beta^{d^{n+1}} m_n^d 2^{d-1}$. Therefore,

$$\begin{aligned}
\frac{\log m_{n+1}}{d^{n+1}} &\leq \frac{\log\tau}{d^n} + \log\beta + \frac{\log m_n}{d^n} + (d-1)\frac{\log 2}{d^n} \\
&\leq \frac{1}{2}\log\beta + \frac{\log m_n}{d^n} \text{ if } n_2 \text{ is large enough}
\end{aligned}$$

This easily implies that for some large n, $\log m_n < 0$ so $m_n < 1$. Now suppose that $n \geq n_2$ and that $m_n < 1$. Then $|P_j(z^n)| \leq \tau\eta_{n+1}$. Hence

$$\begin{aligned}
\phi_{n+1} &= \max\{|z_1^{n+1}|, \ldots, |z_k^{n+1}|, \eta_{n+1}\} \\
&\leq 2\tau \max\{|z_2^{n+1}|, \ldots, |z_k^{n+1}|, \eta_{n+1}\} \\
&\leq 2\tau \max\{\tau\eta_{n+1}, \eta_{n+1}\} \\
&= 2\tau^2 \eta_{n+1}
\end{aligned}$$

But then $\frac{\log\phi_{n+1}}{d^{n+1}} \leq \frac{\log(2\tau^2)}{d^{n+1}} + \log a_{n+1} \leq \frac{\log(2\tau^2)}{d^{n+1}} + \log\beta$. This contradicts that $\psi(z) > \log\alpha$ if n_2 is chosen even larger.

■

Proof of the Theorem: Suppose that $\psi(z) > \log a_\infty$. Then by Lemma 3.5 there exists an arbitrarily large integer n_1 so that

$$|z_1^{n_1}| > 2\tau \max\{|z_2^{n_1}|, \dots, |z_k^{n_1}|, \eta_{n_1}\}.$$

By continuity this inequality holds for all w in some neighborhood V of z. But then by Lemma 3.2(iv) this inequality is still true for all $n \geq n_2$ on V. Hence $\psi_n \equiv \log|f_1^n|$ on V. Therefore the ψ_n are pluriharmonic on V. Moreover, they converge (almost) monotonically to a limit ψ which has a finite value at z. Hence the limit is pluriharmonic on V. In particular, ψ is continuous on V so $\{\zeta \in \mathbb{C}^k; \psi(\zeta) > \log a_\infty\}$ contains an open neighborhood of z.

■

Lemma 3.6. *Let* $K^{compact} \subset \{\psi < c_1\}, \log a_\infty < c_1 < c_2$. *Then there exists for any* $\epsilon > 0$ *an open set* $U \subset \{\psi < c_2\}$ *and an automorphism* Φ *of* $\mathbb{C}^k$ *so that* $\Phi(U) = B^k(0,1), \Phi(K) \subset B^k(0,\epsilon)$.

Proof: Since $\psi < c_1$ on K, there exists an integer N so that $\psi_n < c_1$ for all $n \geq N$. Hence $\frac{\log \phi_n}{d^n} < c_1$ on K $\forall$ $n \geq N$. This implies that $|f_j^n| < e^{c_1 d^n}$ on $K, n \geq N, j = 1, \dots, k$. Suppose next that $w \in \mathbb{C}^k$ and $|f_j^n(w)| < Re^{c_1 d^n}$ for some $n \geq N, j = 1, \dots, k$. Then $\phi_n(w) = \max\{|f_1^n|(w), \dots, |f_k^n|(w), \eta_n\}$. Hence $\psi_n = \frac{\log \phi_n(w)}{d^n} < \max\{\frac{\log Re^{c_1 d^n}}{d^n}, \frac{\log a_n^{d^n}}{d^n}\}$. We can assume that $\log a_n < c_1$. Hence $\psi_n(w) < \max\{\frac{\log R}{d^n} + c_1, c_1\} < c_2$, n large. This completes the proof of the Lemma.

■

Theorem 3.7. *For any* $c > \log a_\infty$ *the sublevel set* $\{\psi < c\}$ *is connected and is a short* $\mathbb{C}^k$.

Proof: We can write $\{\psi < c\} = \cup_{n=1}^\infty K_n^{\text{compact}}, K_n \subset \text{int}(K_{n+1})$. We next find a sequence of open sets $U_n, K_n \subset U_n \subset\subset U_{n+1} \subset\subset \{\psi < c\}$ and biholomorphic maps $\Phi_n : \mathbb{C}^k \to \mathbb{C}^k$ so that $\Phi_n(U_n) = B(0,1), \Phi_{n+1}(U_n) \subset B(0, \frac{1}{n})$. If we have found U_n, set $K = \overline{U_n} \cup K_{n+1}$. Then there exists $\log a_\infty < c_1 < c_2 < c$ so that $K \subset \{\psi < c_1\}$. We apply Lemma 3.6 to find $U_{n+1} \subset \{\psi < c_2\}$.

■

In the same way we get:

Theorem 3.8. *Let H be a Hénon map, and let G^+ ([H]) be the pluricomplex Green function, $G^+(z) = \lim_{n\to\infty} \frac{\log^+ \|H^n(z)\|}{d^n}$, $d =$ degree H. Then for every $c > 0$, $\{G^+ < c\}$ is a "short" $\mathbb{C}^2$.*

Next we discuss the nature of the set $\psi = \log a_\infty$ when $a_\infty = 0$. Notice that any $\mathbb{C}^k$ contained in a sublevel set $\{\psi < c\}$ must be contained in $\{\psi = \log a_\infty\}$. Since $\{\psi = -\infty\}$ is a pluripolar set, there is no $\mathbb{C}^k$ contained in any sublevel set of ψ in this case.

Lemma 3.9. *Let $\delta, \epsilon, R > 0$ be given. If $a \in \mathbb{C}$, $0 < |a|$ small enough, then $\left|\frac{\log|z^2+aw|}{2} - \log|z|\right| < \epsilon$ if $|z|, |w| \leq R, |z| \geq \delta$. Moreover $\frac{\log|z^2+aw|}{2} < \log\delta + 1$ on $\{|z| \leq \delta, |w| \leq R\}$.*

Proof: Let $|a| < \frac{\delta^2}{R}$. Then if $(z,w) \in K := \{\delta \leq |z| \leq R, |w| \leq R\}$, $|z^2 + aw| \geq |z|^2 - |aw| > \delta^2 - |a|R > 0$. We get, for $(z,w) \in K$, $\frac{\log|z^2+aw|}{2} - \log|z| = \frac{1}{2}\log\left|1 + a\frac{w}{z^2}\right|$. Since $\left|\frac{aw}{z^2}\right| \leq |a|\frac{R}{\delta^2}$ we can clearly choose $|a|$ small enough that $|\log|1 + \frac{aw}{z^2}|| < \epsilon$. The last part is obvious.

■

Theorem 3.10. *Let $F_n(z,w) = (z^2 + a_n w, a_n z)$. Suppose that $|a_n| \searrow 0$ sufficiently rapidly. Then $\{\psi = -\infty\} =: P$ has the following shape: $P \setminus (0)$ is closed in $\mathbb{C}^2 \setminus (0)$ and is foliated by Riemann surfaces.*

Proof: Suppose that $\{a_j\}_{j\leq n}$ have been chosen. Set $F(n) = (f_1^n, f_2^n)$. Let $X_n = \{f_1^n = 0\}$. Then X_n is the pole set of $\tilde{\psi}_n = \frac{\log|f_1^n|}{2^n}$. Set $U_n = \{(z,w); \frac{1}{n} \leq \max\{|z|,|w|\} \leq n\}$. Set $V_n = \{(z,w); \tilde{\psi}_n < -n\}$. Let $\hat{\psi} = \max\{\tilde{\psi}_n, -n\}$. Then $X_n \subset V_n$. $F(n)(X_n) = \{z = 0\}$. If $\delta > 0$ is small enough, then $F(n)(\{\max\{|z|,|w|\} \leq n\}) \subset \Delta^2(0,R)$ for some $R > 0$ and $F(n)(V_n) \supset \{|z| \leq \delta, |w| \leq R\}$. Set $\epsilon = 1$. We apply Lemma 3.9 to find a constant $a = a_{n+1}$, $0 < |a_{n+1}| << |a_n|^2$ so that if $F_{n+1}(z,w) = (z^2 + a_{n+1}w, a_{n+1}z)$ then

$$\left|\frac{\log|z^2 + a_{n+1}w|}{2} - \log|z|\right| < 1, \delta \leq |z| \leq R, |w| \leq R.$$

Moreover, $\log|z^2 + a_{n+1}w| < \log\delta + 1$ on $\{|z| \leq \delta, |w| \leq R\}$. It follows that

$$\left|\frac{\log|f_1^{n+1}|}{2^{n+1}} - \frac{\log|f_1^n|}{2^n}\right| < \frac{1}{2^n} \text{ if } \delta \le |f_1^n| \le R, |f_2^n| \le R.$$

and

$$\tilde{\psi}_{n+1} < \frac{\log\delta + 1}{2^{n+1}} \text{ on } \{|f_1^n| \le \delta, |f_2^n| \le R\}.$$

Choosing δ even smaller, we may assume that $\frac{\log\delta+1}{2^{n+1}} < -n-1$.

Suppose that $|z|, |w| \le n$. Then $|f_1^n(z,w)|, |f_2^n(z,w)| \le R$.

(i) $|f_1^n(z,w)| \le \delta$. Then $(z,w) \in V_n$ and hence $\hat{\psi}_n(z,w) = -n$. Moreover, $\hat{\psi}_{n+1}(z,w) = \max\{\tilde{\psi}_{n+1}(z,w), -n-1\} = -n-1$.

(ii) $|f_1^n(z,w)| \ge \delta$. Then $|\tilde{\psi}_n(z,w) - \tilde{\psi}_{n+1}(z,w)| \le \frac{1}{2^n}$. Hence,

$$|\max\{\tilde{\psi}_n, -n-1\} - \max\{\tilde{\psi}_{n+1}, -n-1\}| \le \frac{1}{2^n}.$$

Suppose $\hat{\psi}_n > -n$. Then $\tilde{\psi}_n > -n$ so $\tilde{\psi}_{n+1} > -n-1$, so $|\hat{\psi}_n - \hat{\psi}_{n+1}| \le \frac{1}{2^n}$ whenever $\hat{\psi}_n > -n$.

Next observe that $\{z^2 + a_{n+1}w = 0\}$ is a parabola of the form $w = -\frac{z^2}{a_{n+1}}$. Hence on U_n, X_{n+1} consists locally of two graphs over X_n and these can be chosen arbitrarily close to X_n.

The above shows that $\psi = -\infty$ is the limit in the Hausdorff metric of $\{X_n\}$ and this has the desired laminar structure.

■

Theorem 3.11. *Let* $F_n = (z^2 + a_n w, a_n z)$. *Suppose that* $0 < |a_n| < c < 1$ *and* $|a_{n+1}| \ge |a_n|^t$ *for some* $1 < t < 2$. *Then the basin of attraction of* 0 *is biholomorphic to* $\mathbb{C}^2$.

Proof: We first estimate the rate of convergence towards the origin. So assume that $(z_0,w_0) \in K^{\text{compact}} \subset \Omega$. Set $(z_n,w_n) = F(n)(z_0,w_0)$, $\delta_n = \sup_{(z_0,w_0)\in K} \max\{|z_n|,|w_n|\}$.

Lemma 3.12. *There exists an* $n_0 = n_0(z_0, z_0) > 0$ *and a constant* $\alpha > 0$ *so that if* $n > n_0$, $\delta_{n+k} \le |a_{n+k+1}|c^{\alpha k}$ *for all* $k \ge 0$.

We omit the details of the proof.

Set $A_n := F_n'(0)$. Then $A_n(z,w) = (a_n w, a_n z)$ and $A_n^{-1}(z,w) = (w/a_n, z/a_n)$. Hence, $A_n^{-1} \circ F_n - Id = \left(\frac{a_n z}{a_n}, \frac{z^2+a_n w}{a_n}\right) - (z,w) = (0, z^2/a_n)$. Next estimate $A_1^{-1} \circ \cdots A_{n+1}^{-1} \circ F_{n+1} \circ \cdots F_1$ for large n.

We get $\|A_1^{-1} \circ \cdots A_{n+1}^{-1} \circ F_{n+1} \circ \cdots F_1 - A_1^{-1} \circ \cdots A_n^{-1} \circ F_n \circ \cdots F_1\| = \|A_1^{-1} \circ \cdots A_n^{-1}(A_{n+1}^{-1} \circ F_{n+1} - Id) \circ F_n \cdots F_1\| \leq \frac{1}{|a_1|\cdots|a_n|} \frac{|\delta_n|^2}{|a_{n+1}|}$

Now observe that for $k \geq 0$ we have $\delta_{n+k+1} \leq \delta_{n+k}^2 + |a_{n+k+1}|\delta_{n+k} \leq \delta_{n+k}|a_{n+k+1}|(c^{\alpha k}+1)$. Hence, inductively, we have

$$\delta_{n+k+1} \leq \delta_n |a_{n+k+1}||a_{n+k}| \cdots |a_{n+1}| \pi_{j=0}^k (1 + c^{\alpha j}).$$

We can also rewrite this estimate as $\delta_\ell \leq C_1 |a_1 a_2 \cdots a_\ell|$ for a large constant C_1 and for all ℓ. Notice that by Lemma 3.12 we also have $\delta_\ell \leq C_2 |a_{\ell+1}| e^{\alpha \ell}$ for all $\ell \geq 0$.

$$\begin{aligned}
& \|A_1^{-1} \circ \cdots A_{n+1}^{-1} \circ F_{n+1} \circ \cdots F_1 - A_1^{-1} \circ \cdots A_n^{-1} \circ F_n \circ \cdots F_1\| \\
\leq\ & \frac{\delta_n^2}{|a_1 \cdots a_{n+1}|} \\
\leq\ & \frac{\delta_n}{|a_1 \cdots a_n|} \frac{\delta_n}{|a_{n+1}|} \\
\leq\ & C_1 C_2 e^{\alpha n}
\end{aligned}$$

This implies uniform convergence on compact subsets of Ω. Next we show that the limit map is a biholomorphic map from Ω onto $\mathbb{C}^2$.

First it is clear, since the Jacobian determinant at the origin is always equal to one and never vanishes for any $\Phi_n := A_1^{-1} \circ \cdots A_n^{-1} \circ F_n \circ \cdots F_1$, that the limit map $\Phi : \Omega \to \mathbb{C}^2$ has a Jacobian which never vanishes. Hence Φ is locally one-to-one. To show that Φ is globally one-to-one assume to the contrary that $\Phi(p) = \Phi(q), q \neq p$. Then two small neighborhoods of p, q are mapped onto the same neighborhood of $\Phi(p)$. By the open mapping theorem it follows that the same holds for small perturbations of Φ and hence for Φ_n for large n. This contradicts that each Φ_n is one-to-one.

It remains to show that Φ is onto $\mathbb{C}^2$.

Lemma 3.13. *Let $R_0 > 0$. Then there exists a number n_0 large enough so that if $0 < R \leq R_0$ and $n \geq n_0$, then*

$$F_{n+1}(\Delta^2(0, R|a_1 \cdots a_n|)) \supset \Delta^2(0, R|a_1 \cdots a_{n+1}|e^{-2R_0 c^{n/2}}).$$

Proof: Let $j \geq 0$ be the integer for which $c^{t^{j+1}} < |a_{n+1}| \leq c^{t^j}$. One shows at first with a short calculation that $|a_1 \cdots a_n| \leq c^{n/2}|a_{n+1}|$ if $n \geq n_0$.

To complete the proof of the Lemma, we show that if

$$(z, w) \in \partial\Delta^2(0, R|a_1 \cdots a_n|)$$

then $(z', w') \in \mathbb{C}^2 \setminus \Delta^2(0, R|a_1 \cdots a_{n+1}|e^{-2R_0 c^{n/2}})$.

Assume at first that $|z| = R|a_1 \cdots a_n|$. Then

$$|w'| = |a_{n+1}||z| \geq R|a_1 \cdots a_{n+1}| > R|a_1 \cdots a_{n+1}|e^{-2R_0 c^{n/2}}$$

so we are done. Assume next that $|w| = R|a_1 \cdots a_n|, |z| \leq R|a_1 \cdots a_n|$. Then

$$\begin{aligned}
|z'| &= |z^2 + a_{n+1}w| \\
&\geq |a_{n+1}||w| - (R|a_1 \cdots a_n|)^2 \\
&\geq R|a_1 \cdots a_{n+1}| - R|a_1 \cdots a_n|Rc^{\frac{n}{2}}|a_{n+1}| \\
&\geq R|a_1 \cdots a_{n+1}|(1 - Rc^{\frac{n}{2}}) \\
&\geq R|a_1 \cdots a_{n+1}|e^{-2Rc^{\frac{n}{2}}}, n \geq n_0 \\
&\geq R|a_1 \cdots a_{n+1}|e^{-2R_0 c^{\frac{n}{2}}}.
\end{aligned}$$

■

Next, fix a number $R_0 > 0$. We want to prove that $\Phi(\Omega) \supset \Delta^2(0, \frac{R_0}{2})$. Fix n_0 large as in the above Lemma and define $U := \{(z, w) \in \mathbb{C}^2; F(n_0)(z, w) \in \Delta^2(0, R_0|a_1 \cdots a_{n_0}|)$. Then $\overline{U}$ is compact in Ω. Using the above Lemma, it follows for any $n \geq n_0$ that $F(n)(U) \supset \Delta^2(0, R_0|a_1 \cdots a_n|e^{-2R_0 \sum_{n \geq n_0} c^{\frac{n}{2}}}) \supset \Delta^2(0, \frac{R_0}{2}|a_1 \cdots |a_n|)$ Hence it follows that $\Phi_n(U) \supset \Delta^2(0, \frac{R_0}{2})$ for all $n \geq n_0$. Hence $\Phi(\Omega) \supset \Phi(\overline{U}) \supset \Delta^2(0, \frac{R_0}{2})$. Since R_0 was arbitrary it follows that $\Phi(\Omega) = \mathbb{C}^2$.

■

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Some constructions of hyperbolic hypersurfaces in $\mathbf{P}^n(\mathbf{C})$

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Abstract.

We show some methods of constructing hyperbolic hypersurfaces in the complex projective space, which gives a hyperbolic hypersurface of degree 2^n in $P^n(\mathbf{C})$ for every $n \geq 2$. Moreover, we show that there are some hyperbolic hypersurfaces of degree d in $P^n(\mathbf{C})$ for every $d \geq 2 \times 6^n$ for each $n \geq 3$.

§1. Introcution

Since S. Kobayashi asked whether a generic hypersurface of large degree in $P^n(\mathbf{C})$ is hyperbolic or not in [8], many papers were devoted to constructing various examples of hypersurfaces in $P^n(\mathbf{C})$. In [2], R. Brody and M. Green gave an example of hyperbolic hypersurface in $P^3(\mathbf{C})$ of even degree ≥ 50. Afterwards, new types of hyperbolic hypersurfaces of degree d in $P^3(\mathbf{C})$ were given by A. Nadel in the case of $d = 6p + 3$ for $p \geq 3$ in [10], by J. El Goul for $d \geq 14$ in [7], by J. P. Demailly and by Y. T. Siu–S. K. Yeung for $d \geq 11$ in 1997 respectively. Moreover, J. P. Demailly-J. El Goul proved that a very generic hypersurface of degree at least 21 in $P^3(\mathbf{C})$ is hyperbolic in [4] and M. Shirosaki constructed a hyperbolic hypersurface of degree 10 in [11]. On the other hand, in [9], K. Masuda and J. Noguchi proved that there exists a hyperbolic hypersurface of every degree $d \geq d(n)$ for a positive integer $d(n)$ depending only on n and some concrete examples of hyperbolic hypersurfaces in $P^n(\mathbf{C})$ for $n \leq 5$.

Recently, the author constructed a family of hyperbolic hypersurfaces of degree 2^n in $P^n(\mathbf{C})$ for $n \geq 3$ in [6]. The purpose of this note is to explain the results in [6] and to give some lower estimate of $d(n)$ in the above-mentioned results given by Masuda-Noguchi. The author would like to thank J. Noguchi for useful suggestions to this work.

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§2. Construction of H-polynomials

For convenience' sake, we introduce the following terminology.

Definition 2.1. We call a homogeneous polynomial $Q(w)$ of degree d in $w = (w_0, w_1, \dots, w_n)$ an *H-polynomial* if it satisfies the conditions:

(H1) If a holomorphic map $f := (f_0 : f_1 : \cdots : f_n)$ of $\mathbf{C}$ into $P^n(\mathbf{C})$ satisfies the identity $Q(f_0, f_1, \dots, f_n) = cf_0^d$ for some $c \in \mathbf{C}$, then f is a constant.

(H2) If a holomorphic map $f := (f_1 : \cdots : f_n)$ of $\mathbf{C}$ into $P^{n-1}(\mathbf{C})$ satisfies the identity $Q(0, f_1, \dots, f_n) = cf_{n+1}^d$ for some $c \in \mathbf{C}$ and entire function f_{n+1}, then f is a constant.

Definition 2.2. We say a complex space M to be Brody hyperbolic if there is no nonconstant holomorphic map of $\mathbf{C}$ into M.

As was shown by R. Brody in [1], a compact complex manifold is Brody hyperbolic if and only if it is hyperbolic in the sense of S. Kobayashi. In the following, a compact hyperbolic space means a compact Brody hyperbolic space.

Proposition 2.3. *Let Q be an H-polynomial. Then,*

(i) $V := \{(w_0 : \cdots : w_n); Q(w_0, \dots, w_n) = 0\}$ *is hyperbolic and*

(ii) *for* $W := \{(w_1 : \cdots : w_n); Q(0, w_1, \dots, w_n) = 0\} \subset P^{n-1}(\mathbf{C})$, $P^{n-1}(\mathbf{C}) \setminus W$ *is Brody hyperbolic.*

In fact, (i) is nothing but the case $c = 0$ of (H1), and (ii) is a result of (H2) because we can find an entire function f_{n+1} such that $Q(0, f_1, \dots, f_n) = f_{n+1}^d$ if $Q(0, f_1, \dots, f_n)$ has no zeros.

For the case where $n = 2$ we have the following:

Theorem 2.4. *Let $Q(u_0, u_1, u_2)$ be a homogeneous polynomial of degree $d \geq 4$ and consider the associated inhomogeneous polynomial $\tilde{Q}(v, w) := Q(1, v, w)$. Assume that*

(C1) *the simultaneous equations $\tilde{Q}_v(v, w) = \tilde{Q}_w(v, w) = 0$ have only finitely many solutions, say $P_k := (v_k, w_k)$ $(1 \leq k \leq N)$,*

(C2) $\tilde{Q}(P_k) \neq \tilde{Q}(P_\ell)$ *for* $1 \leq k < \ell \leq N$,

(C3) $Q_{u_0}(1, v_k, w_k) \neq 0$ *for* $1 \leq k \leq N$,

(C4) $\{(u_1, u_2); Q_{u_i}(0, u_1, u_2) = 0, i = 0, 1, 2\} = \{(0, 0)\}$.

(C5) *Hessian* $\varphi := \tilde{Q}_{vv}\tilde{Q}_{ww} - \tilde{Q}_{vw}^2 \neq 0$ *at* (v_k, w_k) $(1 \leq k \leq N)$.

Then, Q is an H-polynomial.

For the proof, refer to [6].

Remark. We can show that generic homogeneous polynomials of degree $d \geq 4$ satisfy the conditions in Theorem 2.4. Here, generic homogeneous polynomials mean all polynomials in some nonempty Zariski open set in the space of all homogeneous polynomials of degree d.

For the case $n \geq 3$, we can prove the following:

Theorem 2.5. *Let $Q(u_0, u_1, \ldots, u_n)$ be an H-polynomial of degree d_0 and $P(u_0, u_{n+1})$ a homogeneous polynomial of degree $d_1 (\geq 3)$ such that $P(u_0, u_{n+1})$ and $\tilde{P}(w) := P(1, w)$ satisfies the conditions;*
(P1) $P(0, u_{n+1}) \not\equiv 0$,
(P2) *$\tilde{P}'(w)$ has only simple zeros $\alpha_1, \alpha_2, \ldots, \alpha_{d_1-1}$,*
(P3) *$\tilde{P}(\alpha_k) \neq \tilde{P}(\alpha_\ell)$ for $1 \leq k < \ell \leq d_1 - 1$.*
For $m \geq 2$, if $d_1 := md_0$ and $2/(d_1 - 2) + 1/m < 1$, then

$$R(u_0, u_1, \ldots, u_n, u_{n+1}) := P(u_0, u_{n+1}) - Q(u_0, u_1, \ldots, u_n)^m$$

is an H-polynomial.

This is a slight improvement of [6, Theorem II]. We state the outline of the proof. Consider holomorphic functions f_j, some of which are nonzero, such that $R(f_0, \ldots, f_{n+1}) = cf_0^{d_1}$. If $f_0 \equiv 0$, then

$$Q(0, f_1, \ldots, f_n) = ef_{n+1}^{d_0}$$

for some constant e and hence f is a constant by (H2). Otherwise, setting $\varphi := f_{n+1}/f_0$ and $\tilde{Q} := Q(1, f_1/f_0, \ldots, f_n/f_0)$, we have $\tilde{P}(\varphi) - c = \tilde{Q}^m$. By the assumption, $\tilde{P}(w) - c$ has at least $d_1 - 2$ simple zeros β_j and φ takes the values β_j with multiplicities at least m, whence $\Theta_\varphi(\beta_j) \geq 1 - 1/m$, where $\Theta_\varphi(\beta_j)$ denote the truncated defects of β_j. By virtue of the defect relation for meromorphic functions, we can conclude from the assumption that f is a constant. We can prove that R satisfies (H2) by the same argument as in the proof of [6, Theorem II]. We omit the details.

By Theorem 2.4 and by using Theorem 2.5 repeatedly, we can easily conclude the following:

Theorem 2.6. *For each $n \geq 2$ there is a hyperbolic hypersurfaces of degree 2^n in $P^n(\mathbf{C})$ and a hypersurface W of degree 2^n in $P^{n-1}(\mathbf{C})$ such that $P^{n-1}(\mathbf{C}) \setminus W$ is Brody hyperbolic.*

We can also construct many hyperbolic hypersurfaces in the complex projective space. For example, by Theorem 2.4, we can construct a hyperbolic hypersurface of degree 5 in $P^2(\mathbf{C})$ and, by the use of the case $m = 3$ of Theorem 2.5 repeatedly, hyperbolic hypersurfaces of degree $5 \times 3^{n-2}$ in $P^n(\mathbf{C})$, which are used later.

§3. Hyperbolic hypersurfaces of high degree

In this section, we construct some examples of hyperbolic hypersurfaces of high degrees. We first give the following:

Theorem 3.1. *Take a polynomial* $F := \sum_{i_1,\dots,i_m} a_{i_1\cdots i_m} x_1^{i_1}\cdots x_m^{i_m}$ *and consider the associated weighted homogeneous polynomial*

$$F^*(x_0, x_1, \dots, x_m) := \sum_{i_1,\dots,i_m} a_{i_1\cdots i_m} x_0^{d-i_1d_1-\cdots-i_md_m} x_1^{i_1}\cdots x_m^{i_m}$$

in $(x_0, x_1, \dots, x_n)$ *with weights* $(1, d_1, \dots, d_m)$ *for some positive integers* d_i*, where* $d := \max\{i_1d_1 + \cdots + i_md_m; a_{i_1\cdots i_m} \neq 0\}$. *Assume that*

(i) $F^*(0, x_1, \dots, x_m)$ *consits of only one monomial,*

(ii) *if* $F(\varphi_1, \dots, \varphi_m) = 0$ *for meromorphic functions* φ_i *on* $\mathbf{C}$*, then at least one of* φ_i*'s is a constant.*

Then, for arbitrary H-polynomials $Q_i(w_0, \dots, w_n)$ *of degree* d_i $(1 \leq i \leq m)$*, the hypersurface*

$$V := \left\{w = (w_0 : \dots : w_n); w_0^d F\left(Q_1(w)/w_0^{d_1}, \dots, Q_m(w)/w_0^{d_m}\right) = 0\right\}$$

in $P^n(\mathbf{C})$ *is hyperbolic.*

Proof. Consider a holomorphic map $f := (f_0 : f_1 : \cdots : f_n)$ of $\mathbf{C}$ into $V(\subset P^n(\mathbf{C}))$, where f_i are entire functions without common zeros. If $f_0 \equiv 0$, then $Q_{i_0}(0, f_1, \dots, f_n) \equiv 0$ for some i_0, whence f is a constant by (H1). Assume that $f_0 \not\equiv 0$. Then, $F(\varphi_1, \dots, \varphi_n) = 0$ for meromorphic functions $\varphi_i := Q_i(1, f_1, \dots, f_n)/f_0^{d_i}$. whence some φ_{i_0} is a constant and so f is a constant by (H1). This gives Theorem 3.1.

We give an example satisfying the assumptions of Theorem 3.1.

Proposition 3.2. *Set* $F(x, y) := x^p + y^p + x^ry^s + 1$ *for positive integers* p, r, s. *Assume that*

$$(1) \qquad p < t, \quad 6/p + 2/t < 1,$$

where $t := \min(r, s)$. *Then,* $F(x, y)$ *satisfies the assumptions* (i) *and* (ii) *of Theorem 3.1 for arbitrary positive integers* d_1 *and* d_2.

Proof. Obviously, (i) holds. To see (ii), take nonconstant meromorphic functions φ, ψ with $F(\varphi, \psi) = 0$. We write $\varphi = f_1/f_0, \psi = f_2/f_0$ with entire functions f_i such that f_1 and f_2 have no common zeros. Consider the holomorphic map $\Phi := (f_0^p : f_1^p : f_2^p) : \mathbf{C} \to P^2(\mathbf{C})$ and hyperplanes $H_j := \{w_{j-1} = 0\}$ for $j = 1, 2, 3$ and $H_4 := \{w_0 + w_1 + w_2 = 0\}$, which are in general position. Obviously, the pull-backs $\Phi^*(H_j)$ of H_j for

$j = 1, 2, 3$, considered as divisors, have no positive multiplicities smaller than p. Take a point z_0 in $f^{-1}(H_4)$. Since $f_0^p + f_1^p + f_2^p = -f_1^r f_2^s f_0^{p-(r+s)}$, if $f_0(z_0) \neq 0$, the multiplicity of $\Phi^*(H_4)$ at z_0 is at least t. Assume that $f_0(z_0) = 0$. Then, $f_1(z_0) \neq 0$ and $f_2(z_0) \neq 0$, because otherwise $\sum_{j=0}^{2} f_j(z_0)^p \neq 0$. This is impossible by the assumption $p < r + s$. In conclusion, $\Phi^*(H_4)$ has no positive multiplicities smaller than t. Then, there are constants c_0, c_1, c_2 with $(c_0, c_1, c_2) \neq (0, 0, 0)$ such that $c_0\varphi^p + c_1\psi^p + c_2 = 0$. Because, otherwise, the second main theorem for holomorphic curves in $P^n(\mathbf{C})$ gives $3(1-2/p)+(1-2/t) \leq 3$, which contradicts the assumption(cf., [5, Theorem 3.3.15]). If $c_2 = 0$, then φ and ψ are obviously constants. Otherwise, we have $c_0 f_0^p + c_1 f_1^p + c_2 f_2^p = 0$. Since $p \geq 4$ by the assumption, Φ is a constant. This gives Proposition 3.2.

By Theorem 3.1 and Proposition 3.2, we have the following:

Proposition 3.3. *Let $Q_i(w)$ be H-polynomials of degree d_i $(i = 1, 2)$ in $n+1$ variables $w = (w_0, w_1, \ldots, w_n)$ and p, r, s positive integers satisfying the condition* (1)*. Then, the zero locus of the polynomial*

$$R(w) := Q_1(w)^p w_0^{d-pd_1} + Q_2(w)^q w_0^{d-pd_2} + w_0^d - Q_1(w)^r Q_2(w)^s$$

is a hyperbolic hypersurface in $P^n(\mathbf{C})$ of degree $d := rd_1 + sd_2$.

This improves Masuda-Noguchi's Theorem as follows:

Theorem 3.4. *For each $n \geq 3$ we can take a positive integer $d(n)$ such that there are hyperbolic hypersurfaces of degree d for every $d \geq d(n)$ in $P^n(\mathbf{C})$. Here, for example, we can take*

$$(2) \quad d(n) := 9(2^n + 5 \times 3^{n-2}) + 2^n(5 \times 3^{n-2} - 1) + 5 \times 3^{n-2}(2^n - 1).$$

For the proof of Theorem 3.4, we give the following Lemma:

Lemma 3.5. *Let d_1 and d_2 be mutually prime positive integers. For arbitrarily given positive integer m_0, every integer d with*

$$d \geq m_0(d_1 + d_2) + d_1(d_2 - 1) + d_2(d_1 - 1)$$

can be written as $d = rd_1 + sd_2$ with $r, s \geq m_0$.

This is easily shown by the fact that, for each number ℓ with $0 \leq \ell < d_1$, we can find integers r, s with $|r| < d_2$, $|s| < d_1$ such that $\ell = rd_1 + sd_2$.

The proof of Theorem 3.4. To this end, for each $n(\geq 3)$ we set $d_1(n) := 2^n$ and $d_2(n) := 5 \times 3^{n-2}$. As is mentioned in the previous section, we can find H-polynomials Q_1 and Q_2 of degree $d_1(n)$ and $d_2(n)$

respectively. Define $d(n)$ by (2). By Lemma 3.5, we can write every $d \geq d(n)$ as $d = rd_1(n)+sd_2(n)$ with $r, s \geq m_0 := 9$, because $d_1(n)$ and $d_2(n)$ are mutually prime. For $p := 8$ and these r, s, which satisfy the condition (1), we apply Proposition 3.3 to find a homogeneous polynomial R of degree d such that $V := \{R = 0\}$ is a hyperbolic hypersurface in $P^n(\mathbf{C})$.

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A link between the asymptotic expansions of the Bergman kernel and the Szegö kernel

Kengo Hirachi

Introduction

Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$. Then the Bergman kernel K^{B} and the Szegö kernel K^{S} of Ω have singularities at the boundary diagonal. These singularities admit asymptotic expansions in powers and log of the defining function of Ω ([3], [2]) and, moreover, the coefficients of which can be expressed in terms of local invariants of the CR structure of the boundary $\partial\Omega$ as an application of the parabolic invariant theory developed in [4], [5], [1], [8], [6] and others. While these works provide a geometric algorithm of expressing the expansion of each kernel, it is not easy to read relations between them from this construction — for example, we can say very little about the relation between the log term coefficients of K^{B} and K^{S}, cf. §2.

In this note we present a method of relating these asymptotic expansions. Our strategy is to construct a meromorphic family of kernel functions K_s, $s \in \mathbb{C}$, such that K^{B} and K^{S} are realized as special values of K_s. In the case of the unit ball, $\{|z| < 1\}$, such a family is given by

$$K_s(z) = \pi^{-n}\Gamma(n-s)\,(1-|z|^2)^{s-n},$$

where $\Gamma(\alpha)$ is the gamma function, and K_{-1}, K_0 give K^{B}, K^{S}, respectively. Note that, for $s < 0$, K_s is characterized as the Bergman kernel for the weighted L^2 norm defined by the measure $(1-|z|^2)^{-s-1}/\Gamma(-s)dV$, see §1. For general strictly pseudoconvex domains, we begin by defining K_s for $s < 0$ as the weighted Bergman kernel, and then extend to $s \in \mathbb{C}$ by analytic continuation. Here we only consider the asymptotic expansion of K_s and define the analytic continuation as a meromorphic family of formal series, see §2. We then apply the invariant theory to express K_s in terms of geometric invariants of the boundary (Theorem 2). In these expansions, all K_s contain the same invariants up to universal

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constants depending polynomially on s. These formulae, in particular, give a relation between $K^{\mathrm{B}} = K_{-1}$ and $K^{\mathrm{S}} = K_0$.

Note that the kernel functions K_s for $s \in \mathbb{Z}$ have been introduced in Hirachi–Komatsu [7] and the present note is a continuation of that work. In [7], K_s are defined as the solutions of simple holonomic systems, which naturally arise from Kashiwara's microlocal analysis of the Bergman kernel [9]. While this point of view is not given explicitly in this note, this is also the main tool of the proofs of Theorems 1 and 2; the details will be given in my forthcoming paper.

§1. Weighted Bergman kernels

Let $\Omega \subset \mathbb{C}^n$ be a domain with C^∞ smooth boundary. Then there is a function $r \in C^\infty(\overline{\Omega})$, called a *defining function*, such that $\Omega = \{r > 0\}$ and $dr \neq 0$ on $\partial\Omega$. Fixing such an r, we define for $s < 0$ a weighted L^2 norm on Ω by

$$\|f\|_s^2 = \int_\Omega |f(z)|^2 \frac{r(z)^{-s-1}}{\Gamma(-s)} dV(z), \tag{1}$$

where $dV(z)$ is the standard Lebesgue measure on $\mathbb{C}^n$. Let

$$H_s(\Omega, r) := \{f \in \mathcal{O}(\Omega) : \|f\|_s < \infty\},$$

the Hilbert space of weighted L^2 holomorphic functions on Ω. If we take a complete orthonormal system $\{h_j\}_{j=0}^\infty$ of $H_s(\Omega, r)$, then the series

$$K_s[r](z, \overline{w}) := \sum_j h_j(z)\, \overline{h_j(w)}$$

converges for $(z, w) \in \Omega \times \Omega$ and define a function, which is shown to be independent of the choice of $\{h_j\}$. We call $K_s[r]$ the *weighted Bergman kernel.* Note that the Bergman kernel K^{B} is given by $K_{-1}[r]$, which is clearly independent of the choice of r.

In case $s = 0$, the right-hand side of (1) does not make sense because $\Gamma(-s)$ has simple poles at $s \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. However, we may define $\|\cdot\|_0$ by taking the limit

$$\lim_{s \to -0} \|f\|_s^2 = \int_{\partial\Omega} |f|^2 d\sigma(z), \quad f \in C^0(\overline{\Omega}),$$

where $d\sigma$ is the volume element on $\partial\Omega$ normalized by the condition

$$d\sigma \wedge dr = dV \quad \text{on } \partial\Omega.$$

Thus it is natural to define $H_0(\Omega, r) := \ker \overline{\partial}_b \subset L^2(\partial\Omega, d\sigma)$, where $\overline{\partial}_b$ is the tangential Cauchy–Riemann operator of $\partial\Omega$. Since each $f \in H_0(\Omega, r)$ admits an extension to $f \in \mathcal{O}(\Omega)$, we may also regard $H_0(\Omega, r) \subset \mathcal{O}(\Omega)$. The Szegö kernel is then defined by $K^{\mathrm{S}}[r](z, \overline{w}) := \sum_j h_j(z)\overline{h_j(w)}$, where $\{h_j\}_j$ is a complete orthonormal system of $H_0(\Omega, r)$.

Model case. In the case of the unit ball Ω_0, we may take $r(z) = 1 - |z|^2$. Then the monomials of z form a complete orthogonal system of $H_s(\Omega_0, r)$ (cf. [7]) and thus

$$K_s[r](z, \overline{w}) = \sum_\alpha \frac{z^\alpha \overline{w}^\alpha}{\|z^\alpha\|_s^2} = \frac{\Gamma(n-s)}{\pi^n}(1 - z \cdot \overline{w})^{s-n}.$$

The right-hand side is a meromorphic function of $s \in \mathbb{C}$ (with parameters $z, w \in \Omega$) and, thus $K_s[r]$ ($s < 0$) can be analytically continued to a meromorphic function of $s \in \mathbb{C}$, which we also denote by $K_s[r]$. Then, in particular, $K_0[r]$ gives the Szegö kernel $K^{\mathrm{S}}[r]$.

§2. Asymptotic expansions of the weighted Bergman kernels

In what follows, we assume that Ω is strictly pseudoconvex, and mainly consider the restriction to the diagonal of the kernel functions $K_s[r](z) := K_s[r](z, \overline{z})$.

It is known from the work of Fefferman [3] that the boundary singularity of the Bergman kernel $K^{\mathrm{B}}(z)$ takes the form $\varphi r^{-n-1} + \psi \log r$, where $\varphi, \psi \in C^\infty(\overline{\Omega})$. Based on his analysis, G. Komatsu has shown that the weighted Bergman kernels $K_s[r]$ admit similar expansions.

Theorem ([10]). *For $s < 0$, the weighted Bergman kernel $K_s[r]$ admits the following asymptotic expansion at the boundary:*

$$K_s[r] = \begin{cases} \varphi^{(s)}[r]\, r^{s-n} + \psi^{(s)}[r] \log r & \text{if } s \in \mathbb{Z}, \\ \varphi^{(s)}[r]\, r^{s-n} & \text{if } s \notin \mathbb{Z}, \end{cases} \tag{2}$$

where $\varphi^{(s)}[r], \psi^{(s)}[r] \in C^\infty(\overline{\Omega})$.

If we introduce the functions

$$\Phi_s[r] = \begin{cases} \Gamma(-s)\, r^s & \text{if } s \in \mathbb{C} \setminus \mathbb{N}_0, \\ \frac{(-1)^{s+1}}{s!}\, r^s \log r & \text{if } s \in \mathbb{N}_0, \end{cases}$$

then we may rewrite the expansions (2) in a unified form:

$$K_s[r](z) = \sum_{j=0}^\infty \varphi_j^{(s)}[r](z) \Phi_{s-n+j}[r](z), \quad \varphi_j^{(s)}[r] \in C^\infty(\overline{\Omega}). \tag{3}$$

Here the coefficients $\varphi_j^{(s)}[r]$ are not uniquely determined because r and z are not independent.

Our basic result that enables us to define the meromorphic family $K_s[r]$, $s \in \mathbb{C}$, is the following

Theorem 1. *The coefficients* $\varphi_j^{(s)}[r]$ *of* (3) *can be chosen so that* $\varphi_j^{(s)}[r] = \sum_{k=0}^{2j} a_{j,k}[r]\, s^k$ *holds for functions* $a_{j,k}[r] \in C^\infty(\overline{\Omega})$ *that are independent of* s.

Taking $\varphi_j^{(s)}[r]$ as in the theorem above and then using the relation $s\,\Phi_{s+j}[r] = -r\,\Phi_{s+j-1}[r] - j\,\Phi_{s+j}[r]$, we may rewrite (3) in the form

$$K_s[r] = \sum_{j=-\infty}^{\infty} a_j[r]\Phi_{s-n+j}[r], \tag{4}$$

where $a_j[r] \in C^\infty(\overline{\Omega})$ are independent of s and satisfies $a_j[r] = O(r^{-2j})$ for $j < 0$ (hence the boundary singularity of $a_j[r]\Phi_{s-n+j}[r]$ gets weaker as $|j| \to \infty$). Note that $a_j[r]$ modulo $O(r^\infty)$ is now uniquely determined by r, and moreover it is shown that map $r \mapsto a_j[r]$ is given by a partial differential operator.

Now we *define* $K_s[r]$ for $s \in \mathbb{C} \setminus (-\infty, 0)$ by the formula (4), which is regarded as formal series. Then we can show, in particular, that $K_0[r]$ gives the asymptotic expansion of the Szegö kernel $K^{\mathrm{S}}[r]$.

§3. Transformation law and an invariant expansion of $K_s[r]$

We next examine the transformation law of $a_j[r]$ under biholomorphic maps $F\colon \widetilde{\Omega} \to \Omega$. Recall [3] that F can be extended to a diffeomorphism up to the boundary. So, for a defining function r of Ω, we may give a defining function of $\widetilde{\Omega}$ by

$$\widetilde{r} := |\det F'|^{-2/(n+1)} r \circ F, \tag{5}$$

where $\det F'$ is the holomorphic Jacobian of F. Now from the definition of the norm $\|\cdot\|_s$, we see that the weighted Bergman kernel transforms according to

$$K_s[\widetilde{r}] = |\det F'|^{2(n-s)/(n+1)} K_s[r] \circ F. \tag{6}$$

Thus, substituting these transformation laws into (4), we get

$$a_j[\widetilde{r}] = |\det F'|^{2j/(n+1)} a_j[r] \circ F \tag{7}$$

by the uniqueness of the expansion (4).

Our next task is to construct functionals of r that transform like this under biholomorphic maps — and hopefully express $a_j[r]$ in terms of these functionals. Here we utilize the ambient metric construction of [4]. Associated to each r, we first define a Lorentz-Kähler metric $g = g[r]$ on a neighborhood of $\mathbb{C}^* \times \partial\Omega \subset \mathbb{C}^* \times \mathbb{C}^n$ by $g[r] = \sum_{j,k=0}^n g_{j\overline{k}}\, dz_j d\overline{z}_k$, where $g_{j\overline{k}} = \partial^2 r_\# / \partial z_j \partial \overline{z}_k$. Let $R = R[r]$ be the curvature of g and $R^{(p,q)} = \overline{\nabla}^{q-2}\nabla^{p-2}R$ be its iterated covariant derivatives. Then consider complete contractions of the form

$$W_\# = \operatorname{contr}\Big(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_m,q_m)}\Big),$$

with $\sum p_l = \sum q_l = m + w$. Such a contraction $W_\#$ assigns to each r a smooth function $W[r] := W_\#[r]|_{z_0=0}$ on $\overline{\Omega}$ near $\partial\Omega$. We call the functional $r \mapsto W[r]$ a *Weyl functional of weight* w. If W has weight w, then under (5), we have the desired transformation law

$$W[\widetilde{r}] = |\det F'|^{2w/(n+1)} W[r] \circ F.$$

It is a natural hope that all a_j can be expressed in terms of these Weyl functionals. However, at this stage, it is hard to deal with the case of arbitrary r. So we here choose a good class of defining functions in such a way that we can apply the invariant theory of [4], [1], [6]. To specify a class of defining functions, following [6], we consider the following complex Monge-Ampère equation

$$(-1)^n \det\left(\partial^2 U / \partial z^j \partial \overline{z}^k\right)_{0 \le j,k \le n} = |z_0|^{2n}$$

for a function $U(z_0, z)$ on $\mathbb{C}^* \times \overline{\Omega}$. This equation admits asymptotic solutions along $\mathbb{C}^* \times \partial\Omega$ of the form

$$U = r_\# + r_\# \sum_{k=1}^{\infty} \eta_k \cdot (r^{n+1} \log r_\#)^k,$$

where r is a C^∞ defining function of Ω, $r_\#(z_0, z) = |z_0|^2 r(z)$ and $\eta_k \in C^\infty(\overline{\Omega})$. For such a solution U, the smooth part $r_\# = |z_0|^2 r$ is uniquely determined. So, for each Ω, we may define $\mathcal{F}_\Omega$ to be the totality of r that arises as the smooth part of an asymptotic solution U. This class $\mathcal{F}_\Omega$ is shown to be preserved under the pull-back (5).

Now we use Weyl functionals to express $K_s[r]$ for $r \in \mathcal{F}_\Omega$. The invariant theory of [6] implies that each $a_j[r]$ admits an asymptotic expansion

$$a_j[r] = \sum_{k=0}^{\infty} W_{j,k}[r]\, r^k, \qquad r \in \mathcal{F}_\Omega, \tag{8}$$

where $W_{j,k}$ is a linear combination of Weyl functionals of weight $j+k$. Hence, using $r\Phi_{s-m}[r] = (m-s)\Phi_{s-m+1}[r]$ to absorb all explicit r in (8) into other $\Phi_{s-l}[r]$, we get

Theorem 2. *If $r \in \mathcal{F}_\Omega$, then $K_s[r]$ admits an expansion*

$$K_s[r] = \sum_{j=0}^{\infty} W_j^{(s)}[r]\, \Phi_{s-n+j}[r], \tag{9}$$

where each $W_j^{(s)}$ is a linear combination of Weyl functionals of weight j whose coefficients are polynomials in s of degree $\le 2j$.

The first three terms of the expansion are given by

$$\pi^n K_s[r] = \Phi_{s-n}[r] + \frac{1}{24}\|R\|^2_{z_0=1}\Phi_{s-n+2}[r] + O(r^{s-n-3}).$$

Here the second term $W^{(s)}_{s-n+1}$ vanishes. Thus we see in particular that the Bergman and the Szegö kernels have the same expansion in $\Phi_s[r]$ up to this order.

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On the non-existence of smooth Levi-flat hypersurfaces in $\mathbb{CP}_n$

Andrei Iordan

Abstract.

We prove that there exists no C^m Levi-flat real hypersurface in $\mathbb{CP}_n$ for $n \geqslant 2$ and $m \geqslant 4$. This is an improvement of the regularity in a theorem of Y.-T. Siu who proved this result for $m \geqslant 8$.

In [9] Y-T. Siu proved the following theorem:

Theorem 1 ([9]). *There exists no C^m Levi-flat real hypersurface in $\mathbb{CP}_n$ for $n \geqslant 2$ and $m \geqslant 8$.*

This theorem answers to a question raised by D. Cerveau [1]. The real analytic case of Theorem 1 was proved by A. Lins Neto [5] for $n \geqslant 3$ and by T. Ohsawa [7] for $n \geqslant 2$. The case $n \geqslant 3$ and $m \geqslant 3n/2 + 7$ of Theorem 1 was proved by Siu [8].

The proof of Theorem 1 is based on the following regularity result for the $\overline{\partial}$-operator:

Theorem 2 ([9]). *Let Ω be a domain with C^{m+1} Levi-flat boundary in $\mathbb{CP}_2$, $m \geqslant 3$. Let g be a C^{m+1} $\overline{\partial}$-closed $(0,1)$-form on Ω which is C^m up to the boundary of Ω. Then there exists u belonging to the Sobolev space $W^m(\Omega)$ such that $\overline{\partial} u = g$.*

A recent paper of G. M. Henkin and the author [4] study the regularity of the $\overline{\partial}$-operator on peudoconcave domains in $\mathbb{CP}_n$.

By using the results of [4] and Theorem 2 we prove in this note that there exists no C^m Levi-flat real hypersurface in $\mathbb{CP}_n$ for $n \geqslant 2$ and $m \geqslant 4$. The methods are the same as in [4].

Let Ω be a domain of $\mathbb{CP}_n$ and E a holomorphic hermitian vector bundle over Ω. We denote by $W^k_{(p,q)}(\Omega; E)$ the (p,q)-forms on Ω with coefficients in the Sobolev space $W^k(\Omega)$ and values in the bundle E endowed with the Sobolev norm $\|\cdot\|_k$ (or $\|\cdot\|_{k,\Omega}$), $A^\infty_{(p,q)}(\Omega; E)$ the set of $\overline{\partial}$-closed (p,q)-forms on Ω with values in E which have a C^∞ extension

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to $\overline{\Omega}$ and $AW^k_{(p,q)}(\Omega; E)$ the set of $\overline{\partial}$-closed (p,q)-forms contained in $W^k_{(p,q)}(\Omega; E)$.

Let $\delta(z)$ be the distance from $z \in \Omega$ to the boundary of Ω with respect to the Fubini-Study metric. A theorem of Takeuchi [10] shows that for every pseudoconvex domain Ω there exists a positive constant $\mathcal{K}_n \geqslant 1/3$ such that $i\partial\overline{\partial}(-\log\delta) \geqslant \mathcal{K}_n\omega$ where ω is the Kähler form of the Fubini-Study metric (see also [2], [6]). We denote by $L^2_{(p,q)}(\Omega; \delta^k; E)$ the set of E-valued (p,q)-forms f on Ω such that $\delta^k f$ is an L^2-form on Ω.

We say that a domain $\Omega \subsetneq \mathbb{CP}_n$ is pseudoconcave if $\mathbb{CP}_n \backslash \overline{\Omega}$ is pseudoconvex.

Let Ω_- be a pseudoconcave domain in $\mathbb{CP}_n$, k a positive integer and $f \in W^k_{(p,n-1)}(\Omega_-; \mathcal{O}(m))$ a $\overline{\partial}$-closed form. We set $\Omega_+ = \mathbb{CP}_n \backslash \overline{\Omega_-}$. We say that f verifies the moment condition of order k if there exists an extension $\widetilde{f} \in W^k_{(p,n-1)}(\mathbb{CP}_n; \mathcal{O}(m))$ of f such that $\overline{\partial}\widetilde{f} \in L^2_{(p,n)}(\Omega_+; \delta^{-k+1}; \mathcal{O}(m))$ and $\int_{\Omega_+} \overline{\partial}\widetilde{f} \wedge h = 0$ for every holomorphic form $h \in L^2_{(n-p,0)}(\Omega_+; \delta^{k-1}; \mathcal{O}(-m))$. Every form $f = \overline{\partial}u$ where $u \in W^{k+1}_{(p,n-2)}(\Omega_-; \mathcal{O}(m))$ verifies the moment condition of order k.

We recall here the following consequence of Theorem 7.1 and Theorem 8.7 of [4]:

Theorem 3 ([4]). *Let Ω_- be a pseudoconcave domain with Lipschitz boundary in $\mathbb{CP}_n$ and $k \geq 1$ an integer such that $2(k-1)\mathcal{K}_n - m + n + 1 > 0$. Then for every $\overline{\partial}$-closed form $f \in C^\infty_{(n,n-1)}(\overline{\Omega_-}; \mathcal{O}(m))$ verifying the moment condition of order k there exists $u \in W^k_{(n,n-2)}(\Omega_-; \mathcal{O}(m)) \cap C^\infty_{(n,n-2)}(\Omega_-; \mathcal{O}(m))$ such that $\overline{\partial}u = f$ and $\|u\|_k \leq C_k \|f\|_k$, where C_k is a constant independent of f.*

We use also the following approximation lemma (Lemma 8.3 of [4]):

Lemma 1 ([4]). *Let Ω be a relatively compact domain with Lipschitz boundary in a complex manifold, E a holomorphic bundle on X. Suppose that there exists a fundamental system of neighborhoods $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of Ω with the following property: for every $\overline{\partial}$-exact form $\Phi = \overline{\partial}\psi$ with $\psi \in A^\infty_{(p,q)}(\Omega_\varepsilon; E)$, there exists $0 < \varepsilon' < \varepsilon$ and $\varphi \in W^s_{(p,q)}(\Omega_{\varepsilon'}; E) \cap C^\infty_{(p,q)}(\Omega_{\varepsilon'}; E)$ such that $\overline{\partial}\varphi = \Phi$ and $\|\varphi\|_{s,\Omega_{\varepsilon'}} \leq C \|\Phi\|_{s,\Omega_{\varepsilon'}}$ with C independent of Φ and ε. Then, every $f \in AW^s_{(p,q)}(\Omega; E) \cap C^\infty_{(p,q)}(\Omega; E)$ belongs to the closure of $A^\infty_{(p,q)}(\Omega; E)$ in $W^s_{(p,q)}(\Omega; E)$.*

From Theorem 3 and Lemma 1 we obtain:

Proposition 1. *Let Ω_- be a pseudoconcave domain with Lipschitz boundary of $\mathbb{CP}_2$. Then $A^\infty(\Omega_-;\mathcal{O}(1))$ is dense in $AW^3(\Omega_-;\mathcal{O}(1))$.*

Proof. We identify the $\mathcal{O}(1)$-valued sections of $A^\infty(\Omega_-;\mathcal{O}(1))$ with the $\mathcal{O}(4)$-valued $(2,0)$-forms of $A^\infty_{(2,0)}(\Omega_-;\mathcal{O}(4))$. Since $\mathcal{K}_2 \geqslant 1/3$, it follows that $2(k-1)\mathcal{K}_2 - m + n + 1 > 0$ for $k = 3$ and $m = 4$. Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a fundamental neighborhood system $\{\Omega_\varepsilon\}_{\varepsilon>0}$ of $\overline{\Omega_-}$ such that Ω_ε is a pseudoconcave domain with Lipschitz boundary of $\mathbb{CP}_2$ for each $\varepsilon > 0$. Since every form $f = \overline{\partial} u$ where $u \in A^\infty_{(2,0)}(\Omega_\varepsilon;\mathcal{O}(4))$ verifies the moment condition of order 4, Proposition 1 follows from Theorem 3 and Lemma 1. Q.E.D.

Since

$$\dim A^\infty(\Omega_-;\mathcal{O}(1)) = \dim A^\infty_{(2,0)}(\Omega_-;\mathcal{O}(4)) = 3$$

(see Proposition 10.1 of [4]), from Proposition 1 we obtain:

Corollary 1. $\dim AW^3(\Omega_-;\mathcal{O}(1)) = 3$.

Theorem 4. *There exists no domain with C^k Levi-flat boundary in $\mathbb{CP}_n$ for $n \geqslant 2$ and $k \geqslant 4$.*

Proof. The proof is done by using Theorem 2 and an extension argument as in the proof of Proposition 4.3 of [4]. By using projections it is enough to prove the result for $n = 2$.

Let Ω be a domain with C^4 Levi-flat boundary in $\mathbb{CP}_2$, $a \in \Omega$ and $b \in \mathbb{CP}_2 \backslash \overline{\Omega}$. We denote by H the complex projectiv line through a and b and we choose homogeneous coordinates $z = (z_0; z_1; z_2)$ for a point $[z] \in \mathbb{CP}_2$ such that the complex projectiv line through a and b is given by $H = \{[z] \,|\, z_0 = 0\}$. Let Ω' be an open neighborhood of $\overline{\Omega}$ which does not contain the point b and $h \in H^{0,0}(H \cap \Omega';\mathcal{O}(1))$. By [3] there exists a Stein neighborhood V of $H \cap \Omega'$ and let $\widetilde{h} \in H^{0,0}(V;\mathcal{O}(1))$ an extension of h.

Let χ be a C^∞ function on $\mathbb{CP}_2$ with support contained in V such that $\chi \equiv 1$ near $H \cap \Omega$. By identifying the sections of $\mathcal{O}(1)$ with the 1-homogeneous functions in homogeneous coordinates, $\frac{\widetilde{h}\overline{\partial}\chi}{z_0}$ defines a form $g \in C^\infty_{(0,1)}\left(\overline{\Omega}\right)$. By Theorem 2 there exists $u \in W^3(\Omega)$ such that $\overline{\partial} u = g$. Then $\chi\widetilde{h} - z_0 u$ defines a section $f \in AW^3(\Omega;\mathcal{O}(1))$ such that $f = h$ on $H \cap \Omega$. This implies that $AW^3(\Omega;\mathcal{O}(1))$ is infinite dimensional and it contradicts the Corollary 1. Q.E.D.

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Recent development on Grauert domains

Su-Jen Kan

§1. Introduction

The purpose of this article is to give a short survey on the recent development of a canonical complex structure, the so called *adapted complex structure*, on the tangent bundle of a real-analytic Riemannian manifold.

It was observed by Grauert [G] that a real-analytic manifold X could be embedded in a complex manifold as a maximal totally real submanifold. One way to see this is to complexify the transition functions defining X. However, this complexification is not unique. In [G-S] and [L-S], Guillemin-Stenzel and independently Lempert-Szöke encompass certain conditions on the ambient complex structure to make the complexification canonical for a given real-analytic Riemannian manifold. In short, they were looking for a complex structure, on part of the cotangent bundle T^*X, compatible with the canonical symplectic structure on T^*X. Equivalently, it is to say that there is a unique complex structure, the *adapted complex structure*, on part of the tangent bundle of X making the leaves of the Riemannian foliation on TX into holomorphic curves. The set of tangent vectors of length less than r equipped with the adapted complex structure is called a *Grauert tube* T^rX. For each X, there corresponds a $r_{max}(X) \geq 0$ which is the maximal real number such that the adapted complex structure is defined on T^rX for all $r \leq r_{max}(X)$. Though each Grauert tube over the same Riemannian manifold are diffeomorphic to each other, it was proved in [K1] and [Sz1] that T^rX and T^sX are biholomorphically nonequivalent when $r \neq s$. A domain D in which the adapted complex structure is defined and $X \subset D \subset TX$, is called a *Grauert domain.* The largest one of such Grauert domains is called the *maximal Grauert domain* in TX. In general, the maximal Grauert domain is strictly larger than $T^{r_{max}}X$. They are the same when X is a symmetric space of rank-one. The domain of definition depends on the geometry of X. Lempert and Szöke have the following estimate on the existence of domain of definition.

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Theorem (Lempert-Szöke). *If the sectional curvatures of X are $\geq \lambda, \lambda < 0$ and the adapted complex structure exists on T^rX then $r < \frac{\pi}{2\sqrt{-\lambda}}$.*

§2. Rigidity of Grauert tubes

Since the adapted complex structure is constructed canonically associated to the Riemannian metric g of X, the differentials of the isometries of X are automorphisms of T^rX. Conversely, it is interesting to see whether all automorphisms of T^rX come from the differentials of the isometries of X or not. When the answer is affirmative, we say the Grauert tube is *rigid*.

With respect to the adapted complex structure, the length square function $\rho(x,v) = |v|^2, v \in T_xX$, is strictly plurisubharmonic. When the center X is compact, the Grauert tube T^rX is exhausted by ρ, hence is a Stein manifold with smooth strictly pseudoconvex boundary when the radius is less than the critical one. Applying the existence theorem of Cheng-Yau, there exists an invariant complete Kähler-Einstein metric g_{KE} of negative scalar curvature -1. Let ω_{KE}, which is a symplectic form on T^rX, denote the imaginary part of g_{KE}. Burns and Hind proved that (T^rX, ω_{KE}) is symplectomorphic to $(T^*X, d(pdq))$ via a symplectomorphism fixing X where pdq is the canonical Liouville 1-form on the cotangent bundle. Together with the fact that the automorphism group of T^rX is a compact Lie group, they (cf. [B], [B-H]) were able to prove the following rigidity result for Grauert tubes over compact real-analytic Riemannian manifolds.

Theorem (Burns-Hind). *Any Grauert tube of finite radius over a compact real-analytic Riemannian manifold is rigid.*

When X is non-compact nothing particular is known, not even to the general existence of a Grauert tube over X, i.e., the r_{max} could very well shrink to zero. When X is non-compact, most of the good properties in the compact cases were lacking since the length square function ρ is no longer an exhaustion. By now, the only two non-compact cases we are sure about the existence of Grauert tubes are those over co-compact real-analytic Riemannian manifolds, the Grauert tubes are simply the lifting of the Grauert tubes over their compact quotients, and Grauert tubes over real-analytic homogeneous Riemannian manifolds. In [K2], the author proved the following characterization on Grauert tubes.

Theorem (Kan 1). *If a Grauert tube T^rX is covered by the ball, then X is the real hyperbolic space.*

Using this and an extended version of Wong-Rosay theorem on the characterization of the unit ball, Kan and Ma (cf. [K-M 1,2] and [K3]) proved the rigidity for Grauert tubes over compact or non-compact locally symmetric spaces.

Later on, the author generalized the Wong-Rosay characterization to a general setting in any complex manifold and hence obtained:

Theorem (Kan 2). *Let T^rX be a Grauert tube over homogeneous Riemannian manifold of $r < r_{max}$. Then T^rX is either rigid or the ball.*

Here we need the condition $r < r_{max}$ since the proof heavily relies on the strictly pseudoconvexity of some good boundary points. We don't know whether it is possible to have more general rigidity other than this since the homogeneous spaces seem to be the best we could expect for Grauert tubes' construction to exist.

§3. Maximal Grauert domains

It is interesting to see whether the rigidity holds for $T^{r_{max}}X$ when X is not compact. As mentioned in the introduction, the maximal Grauert domain coincide with $T^{r_{max}}X$ when X is a symmetric space of rank-one. In [BHH], the authors considered the maximal Grauert domains over non-compact symmetric spaces. They showed that such maximal Grauert domains could be described algebraically which are correspondent to domains defined and studied by Akhiezer and Gindikin in [A-G]. They proved that

Theorem (Burns-Halverscheid-Hind 1).

(1) *The maximal Grauert domain over a non-compact symmetric space is either rigid or Hermitian symmetric.*
(2) *When X is a non-compact symmetric space of rank-one, $T^{r_{max}}X$ is never rigid.*

They also verified a conjecture of Akhiezer and Gindikin on the Steinness of such domains.

Theorem (Burns-Halverscheid-Hind 2). *The maximal Grauert domain over a non-compact symmetric space is Stein.*

By now, all examples we know are Stein. It is natural to ask whether all Grauert tubes or maximal Grauert domains are Stein. Recently Halverscheid and Iannuzzi [H-I] answer this question negatively. The example they consider is the 3-dimensional Heisenberg group. Their calculation works for generalized Heisenberg groups as well.

Theorem (Halverscheid-Iannuzzi). *The maximal Grauert domain over a generalized Heisenberg group is neither holomorphically separable nor holomorphically convex.*

§4. On the Kähler potential and CR invariants

Another characteristic feature of a Grauert tube over a compact Riemannian manifold is that it is exhausted by a non-negative strictly plurisubharmonic function whose square root satisfies the complex homogeneous Monge-Ampère equation away form the zero section. Emphasizing on this Monge-Ampère equation, some very nice results were obtained by Aguilar and by Stenzel.

In this section, we ask X to be compact. It is clear from the construction that a Grauert tube $T^rX, r < r_{max}$ is a Stein manifold with smooth strictly pseudoconvex boundary points. The existence of an invariant complete Kähler-Einstein metric of negative scalar curvature -1 was guaranteed. Since the construction of a Grauert tube is decided by the Monge-Ampère equation, it was expected that there might be a chance that this Kähler-Einstein metric is completely determined by the length square function ρ. R. Aguilar established a connection between potentials for Kähler-Einstein metrics in a neighborhood of X and the Riemannian density function of X. He proved that this occurs only when the density function of X depends solely on the geodesic distance function(such kind of manifold is called a *harmonic manifold*).

Theorem (Aguilar). *Suppose the Grauert tube T^rX admits a Kähler-Einstein metric with a Kähler potential that solely depends on ρ. Then X is a harmonic manifold.*

It is clear that the $(2n-1)$-dimensional strictly pseudoconvex boundary $\partial(T^rX)$ of the Grauert tube T^rX is a CR manifold when X is compact. The one-form $\theta = -Im\ \partial\rho$ has provided a pseudohermitian structure on it. There are two natural families of curves on $\partial(T^rX)$: the orbits of the geodesic flows coming from the Riemannian metric of X and chains, which are CR-invariants used to characterize CR manifolds.

In [St], Stenzel asked the question that whether the above two kinds of curves are related. He studied this pseudohermitian structure via the Fefferman metric and then related the pseudohermitian invariants of $\partial(T^rX)$ to the invariants of the ambient Kähler metric and eventually to the original metric of X.

Theorem (Stenzel).

(1) *Suppose there exists a $\delta > 0$ such that the orbits of the geodesic*

flows are chains on $\partial(T^r X)$ for all $r < \delta$. Then X is an Riemannian Einstein manifold.

(2) *If X is a harmonic manifold, then the orbits of the geodesic flows are chains on $\partial(T^r X)$, for all $r < r_{max}$.*

§5. Unbounded Grauert tubes

When $r = \infty$, i.e., when the whole tangent bundle TX is a Grauert tube of infinite radius, the situation is completely different from the cases of finite radii. In this case, we call TX an *unbounded Grauert tube.*

One trivial example is by taking $X = S^2$ with the standard metric. The adapted complex structure is defined on the whole tangent bundle, which is biholomorphic to the complex quadric $\mathcal{Q} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 | z_1^2 + z_2^2 + z_3^2 = 1\}$. The unbounded Grauert tube TS^2 is clearly not rigid.

One interesting question is to ask whether unbounded Grauert tubes over compact Riemannian manifolds have algebraic embeddings in $\mathbb{C}^N$ similar to the above round sphere case. Verifying the existence of a pair of real-valued exhaustion functions with the growth properties related to Demailly's conjecture on the characterization of affine algebraic manifolds. Aguilar and Burns proved the following

Theorem (Aguilar-Burns 1).
Suppose $\Omega = TX$ is an unbounded Grauert tube over a compact manifold X. Then Ω is an affine algebraic manifold.

They also classify all possible unbounded Grauert tubes TX when X is of dimension 2.

Theorem (Aguilar-Burns 2).
Suppose Ω is an unbounded Grauert tube over a compact manifold X^2. Then Ω is biholomorphic to one of $\mathbb{C}^* \times \mathbb{C}^*$, $(\mathbb{C}^* \times \mathbb{C}^*)/Z_2$, $\mathcal{Q}$ or $\mathcal{Q}/Z_2$.

§6. Other applications

There are also some interesting applications to this adapted complex structure done by R. Szöke in [Sz2] and [Sz3]. In [Sz2], Szöke tried to link the adapted complex structure over compact rank-one symmetric spaces to a complex structure J_S defined on the punctured tangent bundle. The latter is preserved by the normalized geodesic flow which makes it possible to quantize the energy function over the symplectic manifold

$\overset{\circ}{T}X$. He showed that the limit of the push forward of the adapted complex structure under an appropriate family of diffeomorphism exists and agrees with J_S.

In [Sz3], Szöke extended the method to treat all compact symmetric spaces. He proved that after appropriate rescalings, the bundle of (1,0) tangent vectors with respect to the adapted complex structure on TX has a specific limit bundle.

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Analytic polyhedra with non-compact automorphism group

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Abstract.

The main theme of this article concerns the characterization problem of analytic polyhedra in $\mathbf{C}^n$ with non-compact automorphism group. In particular, we include a proof that every bounded convex analytic polyhedron in $\mathbf{C}^n$ is biholomorphic to the product of a Kobayashi hyperbolic convex cone and a bounded convex domain. Several related recent developments are also introduced.

§1. Introduction

The study of the automorphism groups of domains in $\mathbf{C}^n$ is one of the traditional themes in the research of analytic functions in several complex variables. By an automorphism we mean a biholomorphic self-mapping of the given domain. They form naturally a topological group, endowed with the law of composition and the compact-open topology.

This paper concerns the important special collection of domains that are called the analytic polyhedra. An *analytic polyhedron* is a bounded domain Ω in $\mathbf{C}^n$ which admits holomorphic functions $f_1, \ldots, f_N$ defined on an open neighborhood U of the closure of Ω such that Ω is defined by the set of inequalities

$$|f_1(z)| < 1, \ldots, |f_N(z)| < 1.$$

The main interest of this article is in the characterization problem of analytic polyhedra which possess non-compact automorphism groups. Notice that this line of research is resonant with the widely known theorems of Wong [13], Rosay [12], Bedford and Pinchuk [1], Greene and Krantz [6], Kim [7], Fu and Wong [5] and others. Here, we present an account of recent developments on the characterization problem of analytic polyhedra with non-compact automorphism groups.

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§2. The Case of Convex Polyhedral Domains

Note that the boundary of an analytic polyhedron is Levi flat wherever the boundary is smooth. Thus, the class of analytic polyhedron is a subset of the collection of polyhedral domains defined as follows.

We call a bounded domain D in $\mathbf{C}^n$ *polyhedral*, if it admits the real valued smooth functions $\rho_1, \ldots, \rho_N$ defined in an open neighborhood U of the closure of D satisfying:

(1) D is defined by the inequalities $\rho_1(z) < 0, \ldots, \rho_N(z) < 0$.
(2) The boundary of D is defined by the relations $\rho_{i_1} = \ldots = \rho_{i_k} = 0$ for a non-empty collection of indices $\{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$.
(3) Each surface defined by $\rho_j = 0$ in U is C^∞ smooth Levi flat, for $j = 1, \ldots, k$.

Notice that the analytic polyhedra are polyhedral domains. Even if the choices for the defining system $\rho_1, \ldots, \rho_N$ are not in general unique for a polyhedral domain, they are essentially unique in almost all practical situations.

The typical generic subclass is also commonly considered; we call a polyhedral domain *normal*, if the only singularities in the boundary are produced by a complex normal crossing singularities. Now we introduce the following theorem, followed by a simpler and descriptive proof.

Theorem 2.1 (Kim [7]). *Let D be a convex normal polyhedral domain in $\mathbf{C}^n$. If the automorphism group $Aut(D)$ is non-compact, then D is biholomorphic to the product of the unit open disc and a convex domain in $\mathbf{C}^{n-1}$.*

Corollary 2.2. *A convex normal polyhedral domain in $\mathbf{C}^2$ possesses a non-compact automorphism group if, and only if, it is biholomorphic to the bidisc.*

Proof. We present here a proof of Theorem 2.1, which is simpler and in fact more general in its implication than the one originally presented in [7]. Since the automorphism group is non-compact, we have a sequence $\varphi_j \in \mathrm{Aut}\,(D)$, a point $q \in D$ and a boundary point $p \in \partial D$ such that

$$\lim_{j \to \infty} \varphi_j(q) = p.$$

Now, let $\rho_1, \ldots, \rho_m$ be a minimal set of defining functions for D. Then without loss of generality we may assume that

$$\rho_1(p) = \ldots = \rho_k(p) = 0 \text{ and } \rho_{k+1}(p) < 0, \ldots, \rho_m(p) < 0,$$

and that the gradient vectors $\nabla\rho_1(p), \ldots, \nabla\rho_k(p)$ are linearly independent over $\mathbf{C}$. Thus in particular, we have $1 \le k \le n$. Now, consider

$$\Sigma_\ell = \{z \mid \rho_\ell(z) = 0\}$$

for each $\ell = 1, \ldots, m$. This is a Levi flat surface defined in an open neighborhood of $\overline{D}$, and hence is foliated by smooth complex analytic varieties of complex dimension $n - 1$. But then, due to convexity, the analytic varieties contained in Σ_ℓ are in fact a linear subvariety. (Convexity and the maximum principle imply that the variety, say $V \subset \Sigma_\ell$ is contained in the real affine linear subspace, say $\tilde{V}$ of $\mathbf{C}^n$ of real codimension one. Then, being a complex subvariety of $\tilde{V}$ of real codimension one, V itself is a linear subvariety, linearly biholomorphic to a domain in $\mathbf{C}^{n-1}$.) Now let V_ℓ be the maximal (with respect to the inclusion) varieties though p in Σ_ℓ for each $\ell = 1, \ldots, m$. Then the maximal analytic variety in ∂D passing through p is in fact

$$X = V_1 \cap \cdots \cap V_k.$$

The linear independency condition implies that $\dim_{\mathbf{C}} X = n - k$.

Now consider the sequence $q_j := \varphi_j(q)$, which we shall call *an automorphism orbit* of q, accumulating at p. Then we change coordinates linearly at q_j, by a linear affine biholomorphism $\Psi_j : \mathbf{C}^n \to \mathbf{C}^n$, so that the new coordinate system $\zeta := \Psi_j(z)$ satisfy:

- $\Psi_j(q_j) = 0$ for each $j = 1, 2, \ldots$.
- $d\Psi_j|_{q_j}(\nabla\rho_\ell(p)) = (0, \ldots, 0, 1, 0, \ldots, 0)$ (the ℓ^{th} component is 1) for $\ell = 1, \ldots, k$.
- $\Psi_j(X) = \{\zeta_1 = \ldots = \zeta_k = 0\} \cap \partial D$.

Then we consider the scaling map $L_j : \mathbf{C}^n \to \mathbf{C}^n$ defined by

$$L_j(\zeta_1, \ldots, \zeta_n) = \left(\frac{\zeta_1}{\lambda_1^{(j)}}, \ldots, \frac{\zeta_k}{\lambda_k^{(j)}}, \zeta_{k+1}, \ldots, \zeta_n \right)$$

where $\lambda_\ell^{(j)}$ is the distance from the origin to $\Psi_j(\Sigma_\ell)$. Then we consider the sequence

$$\omega_j := L_j \circ \Psi_j \circ \varphi_j : D \to \mathbf{C}^n$$

of holomorphic imbedding maps. First notice that

$$\omega_j(D) = L_j \circ \Psi_j(D)$$

since $\varphi_j(D) = D$. Then, the closure of $L_j \circ \Psi_j(X)$ forms a sequence that converges, since it is in fact a sequence of closed convex subsets of $\mathbf{C}^n$. Notice that each member of this sequence is the closure of a convex domain in a complex affine subspace of codimension k, the limit set, say $\check{X}$ is also the closure of the same type. Notice here that Ψ_j converges to a non-degenerate complex affine mapping of $\mathbf{C}^n$. Therefore, the definition of $L_j \circ \Psi_j$ implies that $\check{X}$ has a non-empty $n-k$ complex dimensional interior in $\{\zeta \in \mathbf{C}^n \mid \zeta_1 = \ldots = \zeta_k = 0\}$. We shall denote by $\widehat{X}$ this interior of $\check{X}$.

Finally, we let

$$\widehat{D} := \big\{(\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n \mid \\ \Re\zeta_1 < 1, \ldots, \Re\zeta_k < 1, \text{ and} \\ (0, \ldots, 0, \zeta_{k+1}, \ldots, \zeta_n) \in \widehat{X}\big\}.$$

Notice that $\widehat{D}$ is biholomorphic to $\Delta^k \times \widehat{X}$, where Δ^k denotes the k-dimensional polydisc.

Now, examining the construction so far, one can easily see that for each compact subset $K \subset\subset D$, there exists j_0 such that $\omega_j(K) \subset \widehat{D}$ for every $j > j_0$. Moreover, for any compact subset K' of $\widehat{D}$, one can see that there exists j_1 such that $K' \subset \omega_j(D)$ for every $j > j_1$. Moreover, observe that $\omega_j(q) = 0$ for every j, and that the origin 0 is an interior point of $\widehat{D}$. Altogether, Montel's theorem now implies that both ω_j and ω_j^{-1} form convergent normal families. Then, choosing a subsequence and applying Cartan's generalization of Schwarz's lemma, we can conclude that D is in fact biholomorphic to the domain $\widehat{D}$. This establishes the theorem as claimed. Q.E.D.

Notice that one of the key roles of convexity of the analytic polyhedron in consideration is that the analytic varieties in the boundary are necessarily affine linear subsets of $\mathbf{C}^n$. In fact, it is true that *the normality condition is* not *essential* in the preceding proof. Therefore, with a small modification of the preceding arguments regarding the scaling method part, we arrive at the following slightly more general result.

Theorem 2.3. *Let Ω be a convex analytic polyhedron in $\mathbf{C}^n$. Then, Ω is biholomorphic to the product of a Kobayashi hyperbolic convex cone and a bounded domain if, and only if, the automorphism group $Aut(\Omega)$ is non-compact.*

§3. Recent Developments and Concluding Remarks

In light of preceding arguments, the natural direction to study is obviously on the analytic polyhedra that are not necessarily convex.

In fact, the case of normal analytic polyhedra in complex dimension two admitting a non-compact automorphism group has been analyzed further. We introduce

Theorem 3.1 (Kim-Pagano [10], 2001). *If $\Omega \subset \mathbf{C}^2$ is a normal analytic polyhedron with a non-compact automorphism group, then the holomorphic universal covering space of Ω is biholomorphic to the bidisc.*

While this theorem clarifies the situation without the convexity assumption, one aspect in contrast to consider is that the holomorphic quotients of the bidisc admitting a non-compact automorphism group is usually quite special. It had been conjectured that the deck transformation group acts only on one component of the bidisc resulting that the polyhedron be biholomorphic to the product of the disc and a Riemann surface. This conjecture was well analyzed recently and answered affirmatively by the author in a collaboration with S.G. Krantz and A.F. Spiro.

Theorem 3.2 (Kim/Krantz/Spiro [9]). *Let $\Omega \subset \mathbf{C}^2$ be a normal analytic polyhedron with a non-compact automorphism orbit accumulating at a boundary point $p \in \partial\Omega$. Let V_p denote the maximal analytic variety at p in $\partial\Omega$. Then, Ω is biholomorphic to the product of V_p and the unit open disc in $\mathbf{C}$.*

Since the case of normal analytic polyhedra in $\mathbf{C}^2$ with a noncompact automorphism group has received such a comprehensive result, the direction to progress seems pointing to the general analytic polyhedra without normality assumption.

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Problems related to hyperbolicity of almost complex structures

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The contents of my talk at this conference are in two papers [4] and [5]. So the emphasis here is on what I was unable to deliver at the conference for lack of time.

§1. Generic almost complex structures and hyperbolicity

Let (M, J) be an almost complex manifold. Because of paucity of local holomorphic functions in general, there is no complex function theory on (M, J). However, there is an abundant supply of holomorphic mappings from a disk of $\mathbf{C}$ into (M, J) [6], and we can define the intrinsic pseudo-distance d_M and hyperbolicity for an almost complex manifold M exactly in the same way as in the complex manifold case.

It is obvious that if M is hyperbolic, every holomorphic map $f\colon \mathbf{C} \to M$ is constant. Conversely, if M is compact and if there exist no non-constant holomorphic maps from $\mathbf{C}$ into M, then M is hyperbolic. In order to state the theorem a little more precisely, let z denote the natural coordinate system in $\mathbf{C}$, and take a length function E on M. We call a non-constant holomorphic map $f\colon \mathbf{C} \to M$ a **complex line** if

$$f^* E^2 \le C dz d\bar{z}$$

for some constant C. If $f(\mathbf{C})$ is contained in a compact subset of M, then this condition is independent of the choice of E. Let S be a subset (usually a domain) in M. We say that a complex line $f\colon \mathbf{C} \to M$ is a **limit complex line coming from** S if on each disk $D_R = \{|z| < R\}$ of radius R the mapping $f|_{D_R}$ is the limit of a sequence of holomorphic mappings of D_R into S. In this case, we have $f(\mathbf{C}) \subset \bar{S}$. Trivially, every complex line in M is a limit complex line coming from M.

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The proof for the following Brody's hyperbolicity criterion is exactly the same as in the complex case ([3; pp.100-103]).

(1.1) **Theorem.** *If a compact almost complex manifold M is not hyperbolic, then there is a complex line $f\colon \mathbf{C} \to M$.*

The following almost complex version of (3.6.8) in [3; p.106] holds.

(1.2) **Theorem.** *Let Z be an almost complex manifold, and Y a compact almost complex submanifold of Z. If Y is hyperbolic, there is a relatively compact neighborhood U of Y which is hyperbolically imbedded in Z.*

(1.3) **Corollary.** *Let $\pi\colon Z \longrightarrow X$ be an almost complex fiber space with compact fiber. If the fiber $\pi^{-1}(p_0)$ at a point $p_0 \in X$ is hyperbolic, then in a small neighborhood of p_0 every fiber is hyperbolic.*

Remark. The infinitesimal form F_M of the pseudo-distance d_M can be defined as in the complex case. As we remarked in [3; p.101], for the proofs of the results above we use only the most basic properties of F_X that are obvious from the definition. We need not know whether F_X is upper semi-continuous and d_M is the integrated form of F_X, although this is also an interesting question.

In view of (1.3) it seems to be reasonable to conjecture that if (M, J_0) is a compact hyperbolic almost complex manifold, all nearby almost complex structures J are hyperbolic. (By "nearby" we mean the first and second partial derivatives of J are close to those of J_0). Unlike the moduli space of complex structures on a compact manifold, the set of almost complex structures (modulo diffeomorphisms) is huge and has no nice structures. So, (1.3) by itself does not prove the conjecture.

If (M, J_0, g_0) is an almost Hermitian manifold with its holomorphic sectional curvature bounded by a negative constant, then for J sufficiently close to J_0 and for the Hermitian metric g defined by

$$g(u,v) = \frac{1}{2}(g_0(u,v) + g_0(Ju, Jv)),$$

the holomorphic sectional curvature remains bounded by a negative constant. On the other hand, as we have shown in [4], an almost Hermitian manifold with its holomorphic sectional curvature bounded by a negative constant is hyperbolic. So this is also another supporting evidence for the conjecture above.

A related question is hyperbolicity of a generic almost complex structure. Let (M, J_0) be a compact non-hyperbolic almost complex manifold. In view of (1.1) it seems that an arbitrarily small, but suitable deformation of J_0 would result in a hyperbolic almost complex structure.

§2. Automorphisms of almost complex manifolds

Generalizing the old theorem of Bochner for compact complex manifolds, Boothby, Wang and I proved in [1] that the automorphism group $\mathrm{Aut}(M, J)$ of a compact almost complex manifold (M, J) is a Lie group with Lie algebra $\mathbf{aut}(M, J)$ consisting of infinitesimal automorphisms of (M, J). The condition that a (real) vector field u is an infinitesimal automorphism of (M, J) is given by

$$(2.1) \qquad L_u(Jv) = J(L_u v) \qquad \text{for all vector fields} \quad v,$$

where L_u denotes the Lie differentiation with respect to u. Since $L_u v = [u, v]$, the condition above may be written as

$$(2.2) \qquad [u, Jv] = J[u, v] \qquad \text{for all vector fields} \quad v.$$

In the complex case, the automorphism group is a complex Lie group. This is because if $u \in \mathbf{aut}(M, J)$, then $Ju \in \mathbf{aut}(M, J)$. However, this is not the case for almost complex manifolds.

The integrability condition for J is given by vanishing of the Nijenhuis tensor N defined by

$$N(u, v) = [Ju, Jv] - J[Ju, v] + J(J[u, v] - [u, Jv]).$$

So, if $u, Ju \in \mathbf{aut}(M, J)$, then $N(u, v) = 0$ for all v. It is now clear that we cannot expect to have, in general, a complex Lie group acting on an almost complex manifold.

Now, if (M, J) is a compact hyperbolic almost complex manifold, $\mathrm{Aut}(M, J)$ is compact since it preserves the intrinsic distance d_M. We know that for a compact hyperbolic complex manifold (M, J), the group $\mathrm{Aut}(M, J)$ is finite. The reason is that if $\dim \mathrm{Aut}(M, J) > 0$, then $\mathrm{Aut}(M, J)$ has a complex one-parameter subgroup and the action of this one-parameter subgroup gives rise to nonconstant holomorphic maps from $\mathbf{C}$ into M, in violation of the hyperbolicity. Clearly, this argument cannot be used in the almost complex case.

However, we can circumvent this obstacle by using a slightly modified argument. If $u \in \mathbf{aut}(M, J)$, then by (2.2) we have

$$[u, Ju] = J[u, u] = 0.$$

Hence, the one-parameter groups e^{su} and e^{tJu} commute. Given a point $p_0 \in M$, the map $f\colon \mathbf{C} \to M$ defined by

$$f(s + ti) = e^{su + tJu}(p_0), \qquad s + ti \in \mathbf{C}$$

is holomorphic. For a suitable choice of p_0 this map is nonconstant, which proves the following theorem.

(2.3) **Theorem**. *The automorphism group of a compact hyperbolic almost complex manifold is finite.*

Let X and Y be compact almost complex manifolds, $\mathrm{Hol}(X, Y)$ be the family of holomorphic maps from X into Y, and $\mathrm{Sur}(X, Y)$ the family of surjective holomorphic maps from X to Y. If Y is hyperbolic, then $\mathrm{Hol}(X, Y)$ and $\mathrm{Sur}(X, Y)$ are compact. If, moreover, X and Y are complex manifolds, then $\mathrm{Sur}(X, Y)$ is finite. This has been proved under various additional assumptions and finally by Noguchi [7] in the most general form, see also [3; Chapter 6, §6]. The natural question is whether this holds also in the almost complex case.

At the moment, for a complete generalization there are too many obstacles. However, in some special cases it should be possible to find arguments avoiding the use of complex structures.

Consider, for example, Urata's theorem [9] which says that the family of surjective holomorphic maps *with connected fibers* from a compact complex manifold X to a compact hyperbolic complex manifold Y is finite. The simplified proof of this theorem by Simha [8] depends on the following two facts: (i) finiteness of $\mathrm{Aut}(Y)$ and (ii) constancy of a bounded holomorphic function on a compact complex space. The latter fact is used to show that a holomorphic map from a closed complex subspace of X into a coordinate neighborhood in Y is constant.

Simha's proof (which does not make us of the complex analytic structure of $\mathrm{Hol}(X, Y)$) seems to be adaptable to the almost complex case. As we have shown in (2.3) above, we have (i) in the almost complex case as well. As for (ii), from the elliptic differential equation satisfied by a holomorphic map between almost complex manifolds (see (2.2) in [1]), it is not hard to see that a holomorphic map from a compact almost complex manifold V into a coordinate neighborhood in Y is constant. However, we need to know this when V is a fiber of a surjective holomorphic map from X to Y, which may have singularities. In other words, we have to consider almost complex spaces (with singularities) whatever their definition may be.

If a holomorphic map $f: X \to Y$ from an almost complex manifold X to a hyperbolic almost complex manifold Y is finite-to-one, then X is also hyperbolic. This is a result in metric space topology, see (1.3.14) of [3; p.13]. If we can prove something like the Stein factorization theorem for almost complex manifolds, then we would be one step closer to dropping the assumption of connected fibers from Urata's theorem.

§3. Local hyperbolicity

One of the sufficient conditions for an almost complex manifold to be (complete) hyperbolic (in the sense that its intrinsic pseudo-distance is a (complete) distance) is that it admits a (complete) Hermitian metric with holomorphic sectional curvature bounded above by a negative constant, (see [4]).

As an application, we proved that every point of an almost complex manifold has a hyperbolic neighborhood. (In real dimension 4, the existence of a complete hyperbolic neighborhood was established by Debalme and Ivashkovich [2] by a completely different method.) In [4] I claimed that it has a *complete* hyperbolic neighborhood. However, at this conference it was pointed out by Forstnerič that the neighborhood I had constructed might not be complete. (The almost Hermitian metric constructed in [5] is a little simpler although it does not essentially differ from the one in [4].)

So the problem of constructing a *complete* hyperbolic neighborhood is still open.

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Ideals of multipliers

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Ideals of multipliers were introduced in [8] to find conditions on domains in complex manifolds under which subellipticity of the $\bar{\partial}$-Neumann problem holds. Similar ideals were used to study subellipticity on of $\Box_b$ on CR manifolds (see [9]). In [10] such ideals are used to study the situation when subellipticity breaks down but regularity still holds. Ideals of holomorphic multipliers in a somewhat different context have been used by Nadel (see [15]) and by Siu (see [16]) to prove global theorems in algebraic geometry. Here we will be concerned with the ideals that arise in the study of local regularity. We will briefly explain the use of subelliptic estimates then we define local and microlocal multipliers and show how to use them to derive subelliptic estimates. We also discuss the use of subelliptic multipliers when subellipticity fails. Finally we show how subelliptic multipliers give rise to invariants of complex analytic varieties.

§1. Definitions

A **CR manifold** is a compact C^∞ manifold M of dimension $2n+1$ endowed with an **integrable CR structure** which consists of a subbundle $T^{1,0}(M)$ of the complexified tangent bundle $\mathbb{C}T(M)$ satisfying the following. The complex fiber dimension of $T^{1,0}(M)$ is n,

$$T^{1,0}(M) \cap \overline{T^{1,0}(M)} = \{0\},$$

and if L and L' are local sections of $T^{1,0}(M)$ then $[L, L'] = LL' - L'L$ is also a local section of $T^{1,0}(M)$.
Let $\mathcal{A}_b^{p,q}$ denote the (p,q)-forms on M, let

$$\bar{\partial}_b : \mathcal{A}_b^{p,q} \to \mathcal{A}_b^{p,q+1}$$

denote the corresponding exterior derivative, and let $\bar{\partial}_b^* : \mathcal{A}_b^{p,q} \to \mathcal{A}_b^{p,q-1}$ denote the L_2 adjoint of $\bar{\partial}_b$.

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We define the **complex energy form** Q_b on $\mathcal{A}_b^{p,q}$ by

$$Q_b(\varphi, \psi) = (\bar{\partial}_b\varphi, \bar{\partial}_b\psi) + (\bar{\partial}_b^*\varphi, \bar{\partial}_b^*\psi),$$

where (,) denotes the L_2 inner product on forms. We define the **complex laplacian**

$\Box_b : \mathcal{A}_b^{p,q} \to \mathcal{A}_b^{p,q}$ by $\Box_b = \bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b$. Let

$$\mathcal{H}_b^{p,q} = \{\varphi \in L_2^{p,q} \mid \Box_b\varphi = 0\}$$

Note that if $\alpha \perp \mathcal{H}_b^{p,q}$ and then $\Box_b\varphi = \alpha$ if and only if $Q_b(\varphi, \psi) = (\alpha, \psi)$, for all ψ.

If $u \in C_0^\infty(\mathbb{R}^m)$ and if $s \in \mathbb{R}$ we define $\|u\|_s$ the **Sobolev** s**-norm** of u by

$$\|u\|_s^2 = \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 dV$$

If $u \in C^\infty(M)$ we define $\|u\|_s$ by choosing a partition of unity $\{\zeta_\nu\}$ which is subordinate to a covering by coordinate charts and set $\|u\|_s^2 = \sum \|\zeta_\nu u\|^2$.

§2. Subelliptic estimates

If $P \in M$ we say that a **subelliptic estimate** for (p, q)-forms holds at P if there exists a neighborhood U of P and constants C and ε such that

$$(\bullet_q) \qquad \|\varphi\|_\varepsilon^2 \leq C(Q_b(\varphi, \varphi) + \|\varphi\|^2),$$

for all $\varphi \in \mathcal{A}_b^{p,q}$ with support in U.

The above estimate has the following consequences (see [12]).

1. If $\Box_b\varphi = \alpha$ and if α is C^∞ on U then φ is C^∞ on U.

2. $\mathcal{H}_b^{p,q} \subset C^\infty(M)$.

3. If α is a (p, q)-form which is C^∞ on U and if ψ is a $(p, q-1)$-form orthogonal to $\mathcal{H}_b^{p,q-1}$ such that $\bar{\partial}_b\psi = \alpha$ then ψ is C^∞ on U.

4. Let $S_b : L_2^{p,q-1} \to \mathcal{N}^{p,q-1}(\bar{\partial}_b)$, where $\mathcal{N}^{p,q-1}(\bar{\partial}_b)$ denotes the null space of $\bar{\partial}_b$ and S_b the orthogonal projection. If $\theta \in L_2^{p,q-1}$ with θ in C^∞ on U then $S_b\theta$ is C^∞ on U.

Duality

Let $\{L_1, \dots, L_n\}$ be an orthonormal basis for $(1,0)$ vector fields on a neighborhood $U \subset M$ of P and let $\{L_1, \dots, L_n, \bar{L}_1, \dots, \bar{L}_n, T\}$ be a basis for the complex vector fields on U with $\bar{T} = -T$. Let $\{\omega_1, \dots, \omega_n\}$ be the dual basis of $(1,0)$ forms. Then if $\varphi \in \mathcal{A}_b^{0,q}$ with support in U we have $\varphi = \sum \varphi_I \bar{\omega}^I$ where I runs through the strictly in creasing q-tuples of integers between 1 and n and where $\bar{\omega}^I = \bar{\omega}_{i_1} \wedge \cdots \wedge \bar{\omega}_{i_q}$. We define $F^q\varphi \in \mathcal{A}^{0,n-q}$ by

$$F^q\varphi = \sum \epsilon^I_{I'} \bar{\varphi}_I \bar{\omega}^{I'},$$

where I' denotes the increasing $(n-q)$-tuple consisting of integers between 1 and n which are not in I, and $\epsilon^I_{I'}$ is defined by

$$\epsilon^I_{I'} \bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_n = \bar{\omega}^I \wedge \bar{\omega}^{I'}.$$

Then we have:

$$\bar{\partial}_b F^q\varphi = F^{q-1}\bar{\partial}_b^*\varphi + \sum a_{IJ}\bar{\varphi}_I\bar{\omega}^J$$

and

$$\bar{\partial}_b^* F^q\varphi = F^{q+1}\bar{\partial}_b\varphi + \sum b_{IJ}\bar{\varphi}_I\bar{\omega}^J.$$

Hence,

$$Q_b(\varphi,\varphi) = Q_b(F^q\varphi, F^q\varphi) + 0(\|\varphi\|^2).$$

Therefore, since $\|\varphi\|_\varepsilon = \|F^q\varphi\|_\varepsilon$, we conclude that $(\bullet_q)$ holds if and only if $(\bullet_{n-q})$ holds.

Microlocalization

Let $\{x_1, \dots, x_{2n}, t\}$ be real coordinates on U with origin at P such that

$$\frac{\partial}{\partial x_j} = \Re(L_j|_P), \ \frac{\partial}{\partial x_{j+n}} = \Im(L_j|_P),$$

and $\frac{\partial}{\partial t} = \sqrt{-1}T$. Let $\{\xi_1, \dots, \xi_{2n+1}\}$ denote the dual coordinates. If $u \in C_0^\infty(U)$ we have the microlocal decomposition $u = u^+ + u^- + u^0$, where $\mathcal{F}u^+$ and $\mathcal{F}u^-$ have supports in conical neighborhoods of $(0, \dots, 0, 1)$ and $(0, \dots, 0, -1)$, respectively and $\mathcal{F}u^0$ is supported in the union of the unit ball and the complement of the above conical neighborhoods (here $\mathcal{F}$ denotes the Fourier transform).

Let $U' \supset \bar{U}$ be a small neighborhood and let $\zeta \in C_0^\infty(U')$ with $\zeta = 1$ on U. Then we have

$$(\bullet_q^0) \qquad \|\zeta\varphi^0\|_1^2 \le C(Q_b(\zeta\varphi^0, \zeta\varphi^0) + \|\varphi\|^2),$$

for all $\varphi \in C_0^\infty(U)$. Thus to prove $\bullet_q$ it suffices to establish the corresponding estimates $(\bullet_q^+)$ and $(\bullet_q^-)$ for $\|\zeta\varphi^+\|_\varepsilon^2$ and for $\|\zeta\varphi^-\|_\varepsilon^2$, respectively.

Let $\Omega \subset X$ be a domain in a complex manifold X which has a smooth boundary M and such that $\bar{\Omega}$ is compact. We then say that the $\bar{\partial}$-Neumann problem for (p,q)-forms at $P \in M$ is subelliptic if there exists a neighborhood U of P and constants ε and C such that

$$(\bullet\bullet_q) \qquad |||\varphi|||_\varepsilon^2 \leq C(Q(\varphi,\varphi) + |||\varphi|||^2),$$

for all $\varphi \in Dom(\bar{\partial}^*) \cap \mathcal{A}^{p,q}$ with support in $U \cap \bar{\Omega}$. Here $\mathcal{A}^{p,q}$ denotes the space of (p,q)-forms in $C^\infty(\bar{\Omega})$,

$$Q(\varphi,\varphi) = ((\bar{\partial}\varphi, \bar{\partial}\varphi)) + ((\bar{\partial}^*\varphi, \bar{\partial}^*\varphi)),$$

and $|||\ |||$, $((\ ))$, $|||\ |||_\varepsilon$ denote the L_2 norm, the L_2 inner product, and the Sobolev ε-norm on $\bar{\Omega}$, respectively. The estimate $(\bullet\bullet_q)$ has the following consequences (see [12]).

1. If $\Box\varphi = \alpha$ and if α is C^∞ on $U \cap \bar{\Omega}$ then φ is C^∞ on $U \cap \bar{\Omega}$. Here $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ with domain consisting of $\{\varphi \in Dom(\bar{\partial}) \cap Dom(\bar{\partial}^*) \mid \bar{\partial}\varphi \in Dom(\bar{\partial}^*)$ and $\bar{\partial}^*\varphi \in Dom(\bar{\partial})\}$.

2.$\mathcal{H}^{p,q} \subset C^\infty(\bar{\Omega})$, where $\mathcal{H}^{p,q} = \{\varphi \mid \Box\varphi = 0\}$, is finite dimensional.

3. If α is a (p,q)-form which is C^∞ on $U \cap \bar{\Omega}$ and if ψ is a $(p,q-1)$-form orthogonal to $\mathcal{N}^{p,q-1}(\bar{\partial})$, where $\mathcal{N}^{p,q-1}(\bar{\partial})$ denotes the null space of $\bar{\partial}$, such that $\bar{\partial}_b\psi = \alpha$ then ψ is C^∞ on $U \cap \bar{\Omega}$.

4. If $B : L_2^{p,q-1}(\Omega) \to \mathcal{N}^{p,q-1}(\bar{\partial})$ is the orthogonal projection and if $\theta \in L_2^{p,q-1}(\Omega)$ with θ in C^∞ on U then $B\theta$ is C^∞ on $U \cap \bar{\Omega}$.

Denote by M the boundary of Ω and suppose that in a neighborhood of M there exists a real valued function r such that $r < 0$ in Ω which on M satisfies $r = 0$ and $dr \neq 0$. Let $\{z_1, \ldots, z_{n+1}\}$ be local holomorphic coordinates with origin at $P \in M$ such that $r_{z_i}(P) = 0$ for $i = 1, \ldots, n$ and $r_{z_{n+1}}(P) = 1$. Let

$$L_i = \frac{\partial}{\partial z_i} - r_{z_i}\frac{\partial}{\partial z_{n+1}}$$

and

$$T = r_{\bar{z}_{n+1}}\frac{\partial}{\partial z_{n+1}} - r_{z_{n+1}}\frac{\partial}{\partial \bar{z}_{n+1}}.$$

Theorem 2.1. *The $\bar{\partial}$-Neumann problem on Ω is subelliptic for (p,q)-forms at P, that is $(\bullet\bullet_q)$ holds, if and only if $(\bullet_q^+)$ holds on M.*

§3. Local and microlocal multipliers

Definition 3.1. *If $P \in M$ a* **subelliptic multiplier** *for (p,q)-forms at P is a germ of a C^∞ function f such that there exists a neighborhood U of P and positive constants ε and C so that*

$$(*_q) \qquad \|f\varphi\|_\varepsilon^2 \leq C(Q_b(\varphi,\varphi) + \|\varphi\|^2),$$

for all $\varphi \in \mathcal{A}_b^{p,q}$ with support in U. Note that this estimate is independent of p.

Let $\mathcal{I}_q$ denote the set of subelliptic multipliers. $\mathcal{I}_q$ satisfies the following.

1. $\mathcal{I}_q$ is an ideal.

2. ${}^{\mathbb{R}}\sqrt{\mathcal{I}_q} \subset \mathcal{I}_q$, here ${}^{\mathbb{R}}\sqrt{\mathcal{I}_q}$ denotes the real radical of $\mathcal{I}_q$ consisting of all germs g such that there exist $m \in \mathbb{Z}^+$ and $f \in \mathcal{I}_q$ with $|g|^m \leq |f|$.

Analogously we define $\mathcal{I}_q^+$ and $\mathcal{I}_q^-$ by the estimates

$$(*_q^+) \qquad \|f\zeta\varphi^+\|_\varepsilon^2 \leq C(Q_b(\zeta\varphi^+,\zeta\varphi^+) + \|\zeta'\varphi^+\|^2)$$

and

$$(*_q^-) \qquad \|f\zeta\varphi^-\|_\varepsilon^2 \leq C(Q_b(\zeta\varphi^-,\zeta\varphi^-) + \|\zeta'\varphi^-\|^2).$$

Then $\mathcal{I}_q = \mathcal{I}_q^+ \cap \mathcal{I}_q^-$ and we have that: $(\bullet_q^+)$ holds if and only if $1 \in \mathcal{I}_q^+$, $(\bullet_q^-)$ holds if and only if $1 \in \mathcal{I}_q^-$, $(\bullet_q)$ holds if and only if $1 \in \mathcal{I}_q$, and $(\bullet\bullet_q)$ holds if and only if $1 \in \mathcal{I}_q^+$.

These ideals satisfy the following duality property

$$\mathcal{I}_q^+ = \mathcal{I}_{n-q}^-.$$

This follows since $\|\varphi^+\| = \|(\bar{\varphi})^-\|$ and

$$\|f\zeta\varphi^+\|_\varepsilon = \|f\zeta(F_q\varphi)^-\|_\varepsilon + O(\|\varphi\|)$$

and

$$Q_b(\zeta\varphi^+,\zeta\varphi^+) = Q_b(\zeta(F_q\varphi)^-,\zeta(F_q\varphi)^-) + O(\|\varphi\|^2).$$

Pseudoconvexity

We define the **Levi form** in an open set $U \subset M$ to be the hermitian form $\mathcal{L}_P$ on $T_P^{1,0}$, for each $P \in U$ defined as follows. Let γ be a real one form in U such that $\gamma \neq 0$ and $\gamma(L) = 0$ for all $L \in T^{1,0}$. Then we set $\mathcal{L}(L,L') = \sqrt{-1} < d\gamma, L \wedge \bar{L}' >$. Then M is **pseudoconvex** if it can be covered by open sets on which $\mathcal{L}$ is positive semi-definite. In terms of the above basis we have $\mathcal{L}(L_i, L_j) = c_{ij}$ and

$$[L_i, \bar{L}_j] = c_{ij}T \quad \mod (L_1, \dots, L_n, \bar{L}_1, \dots, \bar{L}_n).$$

If M is pseudoconvex in a neighborhood of P we will construct a sequence of ideals

$$\mathcal{I}^+_{q,k} \subset \mathcal{I}^+_{q,k+1} \subset \mathcal{I}^+_q.$$

We define the quadratic form c_{IJ}, with q-tuples I and J, by

$$c_{IJ} = \sum_{i,j,K} \epsilon_I^{iK} \epsilon_J^{jK} c_{ij},$$

where K runs over all ordered $(q-1)$-tuples. Each of the coefficients ϵ_I^{iK} is either $0, 1$, or -1 defined as follows. First, if $i \notin K$ we denote by $\langle iK \rangle$ the ordered q-tuple containing i and the elements of K. Then we define

$$\epsilon_I^{iK} = \begin{cases} 0 & \text{if } i \in K \\ 0 & \text{if } \langle iK \rangle \neq I \\ \operatorname{sgn} \left\langle {iK \atop I} \right\rangle & \text{if } \langle iK \rangle = I, \end{cases}$$

where $\operatorname{sgn} \left\langle {iK \atop I} \right\rangle$ denotes the sign of the permutation $\{i, K\} \to I$. We observe the following.

A. If $(c_{ij}) \geq 0$ then $(c_{IJ}) \geq 0$.

B. If $(c_{ij}) \geq 0$ then $\sqrt[\mathbb{R}]{(\det c_{IJ})}$ equals the real radical of the ideal generated by the $(n-q+1) \times (n-q+1)$ subdeterminants of (c_{ij}).

Integration by parts gives.

$$\sum (c_{IJ} T \varphi_I, \varphi_J) + \sum \|\bar{L}_i \varphi_I\|^2 = Q_b(\varphi, \varphi) + \textit{ error}.$$

Substituting $F^q\varphi$ for φ and conjugating we get

$$-\sum (c_{I'J'} T \varphi_I, \varphi_J) + \sum \|L_i \varphi_I\|^2 = Q_b(\varphi, \varphi) + \textit{ error}.$$

Substituting $\zeta\varphi^+$ and $\zeta\varphi^-$ for φ in the first and second equation, respectively; we obtain

$$\|(\det c_{IJ})\zeta\varphi^+\|^2_{\frac{1}{2}} \leq C(Q_b(\zeta\varphi^+, \zeta\varphi^+) + \|\varphi\|^2)$$

and

$$\|(\det c_{I'J'})\zeta\varphi^-\|^2_{\frac{1}{2}} \leq C(Q_b(\zeta\varphi^-, \zeta\varphi^-) + \|\varphi\|^2).$$

Hence the $(n-q+1)\times(n-q+1)$ subdeterminants of (c_{ij}) are in $\mathcal{I}_q^+$ and the $(q+1)\times(q+1)$ subdeterminants of (c_{ij}) are in $\mathcal{I}_q^-$.

Given germs of C^∞ functions $f_1,\dots,f_n$ we define $n\times 2n$ matrix $M(f_1,\dots,f_n)$ by

$$M(f_1,\dots,f_n)=\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n}\\ c_{21} & c_{22} & \dots & c_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ c_{n1} & c_{n2} & \dots & c_{nn}\\ L_1f_1 & L_2f_1 & \dots & L_nf_1\\ L_1f_2 & L_2f_2 & \dots & L_nf_2\\ \vdots & \vdots & \ddots & \vdots\\ L_1f_n & L_2f_n & \dots & L_nf_n \end{pmatrix}.$$

Denote by $(Det^jM(f_1,\dots,f_n))$ the ideal generated by the $j\times j$ subdeterminants of $M(f_1,\dots,f_n)$.

Theorem 3.2. *If the $f_1,\dots,f_n$ are in $\mathcal{I}_q^+$ then $Det^{n-q+1}M(f_1,\dots,f_n)\subset\mathcal{I}_q^+$ and if the $f_1,\dots,f_n$ are in $\mathcal{I}_q^-$ then $Det^{q+1}M(f_1,\dots,f_n)\subset\mathcal{I}_q^-$.*

We define $\mathcal{I}_{q,k}^+$ by induction on k:

$$\mathcal{I}_{q,1}^+=\sqrt[\mathbb{R}]{(Det^{n-q+1}M(0))}$$

and

$$\mathcal{I}_{q,k+1}^+=\sqrt[\mathbb{R}]{(\mathcal{I}_{q,k}^+,\ \mathcal{D}^{n-q+1}(\mathcal{I}_{q,k}^+))},$$

where $\mathcal{D}^{n-q+1}(\mathcal{I}_{q,k}^+)$ is the set of all $(n-q+1)\times(n-q+1)$ subdeterminants of $M(f_1,\dots,f_n)$ for all n-tuples $(f_1,\dots,f_n)$ in $\mathcal{I}_{q,k}^+$. Similarly we define $\mathcal{I}_{q,k}^-$ by:

$$\mathcal{I}_{q,1}^-=\sqrt[\mathbb{R}]{(Det^{q+1}M(0))}$$

and

$$\mathcal{I}_{q,k+1}^-=\sqrt[\mathbb{R}]{(\mathcal{I}_{q,k}^-,\ \mathcal{D}^{q+1}(\mathcal{I}_{q,k}^-))}.$$

We then have:

$\mathcal{I}_{q,k}^+\subset\mathcal{I}_{q,k+1}^+\subset\mathcal{I}_q^+$,

$\mathcal{I}_{q,k}^+\subset\mathcal{I}_{q+1,k}^+$, and

$\mathcal{I}_{q,k}^+=\mathcal{I}_{n-q,k}^-$.

Hence if we set

$$\mathcal{I}_{q,k} = \mathcal{I}^+_{q,k} \cap \mathcal{I}^-_{q,k} = \mathcal{I}^+_{\min\{q,n-q\},k} \subset \mathcal{I}_q.$$

we conclude that if for some k

$$(**_q) \qquad 1 \in \mathcal{I}_{q,k}$$

then the subelliptic estimate $(\bullet_q)$ holds. The condition $(**_q)$ is called **finite ideal q-type**.

The conjecture is that $(**_q)$ is a necessary condition for the subelliptic estimate $(\bullet_q)$. Generalizing the work of Greiner (see [7]) this can be established when (c_{IJ}) and $(c_{I'J'})$ are diagonalizable on U. This diagonalizability condition implies that it is not necessary to use radicals in deriving $1 \in \mathcal{I}^+_{q,k}$. More generally Catlin (see [1]) has shown that subellipticity for the $\bar{\partial}$-Neumann problem is equivalent to the condition of **D'Angelo finite q-type**. The passage from the $\bar{\partial}$-Neumann problem to CR manifolds is routine. Thus the problem is to prove that finite ideal q-type is equivalent to finite D'Angelo type (see [4]). It is easy to prove that finite D'Angelo q-type implies finite ideal q-type, so the problem is to prove the converse. In case the CR manifold is real analytic this follows by use of methods developed by Diederich and Fornaess (see [5]).

§4. When subellipticity fails

The Fedii example in $\mathbb{R}^2$ is

$$Eu = -\frac{\partial^2 u}{\partial x^2} - a(x)\frac{\partial^2 u}{\partial t^2} = f,$$

where $a(x) > 0$ when $x \neq 0$ (see [F]). This equation is always hypoelliptic, it is elliptic if and only if $a(0) > 0$ and it is subelliptic if and only if $a(x) > c|x|^m$. The best way to see this is to note that a is a subelliptic multiplier in the sense that:

$$\|au\|_1^2 \leq C((\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}) + (a\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t})) = C(Eu, u).$$

In the Kusuoka and Stroock example (see [13]) in $\mathbb{R}^3$

$$E = -\frac{\partial^2}{\partial x^2} - a(x)\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2}$$

where $a(x) > 0$ when $x \neq 0$, E is hypoelliptic if and only if

$$\lim_{x \to 0} x \log a(x) = 0.$$

Generalization of the Fedii example on $\mathbb{R}^n_x \times \mathbb{R}^m_t$, $E = E_1 + c(x,t)E_2$, where

$$E_1 = -\sum a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$E_2 = -\sum b_{ij}(x,t)\frac{\partial^2}{\partial t_i \partial t_j},$$

$(a_{ij}) \geq 0$, $(b_{ij}) \geq 0$, and the E_1 and E_2 are uniformly subelliptic on $\mathbb{R}^n_x$ and on $\mathbb{R}^m_t$, respectively. Then E is hypoelliptic whenever there exists a manifold $S \subset \mathbb{R}^n \times \mathbb{R}^m$ which is transversal to $\mathbb{R}^n_x$ and $c(x,t) > 0$ whenever $(x,t) \notin S$.

The analogous statement for $\Box_b$ for (p,q)-forms would be that $\Box_b$ is hypoelliptic if there exists $f \in \mathcal{I}_q$ and a manifold $S \subset M$ of holomorphic dimension $\leq \min\{n-q-1,\ q-1\}$ such that $f \neq 0$ outside of S.

Christ (see [3]) has shown this does not hold in general but it does hold in case $M \subset \mathbb{C}^{n+1}$ given by a defining function r with special symmetries (see [K4]), such as: $r = \Re(z_{n+1}) - F(\sum |z_i|^2)$.

To find estimates for the $\bar\partial$-Neumann problem for pseudoconvex domains in $\mathbb{C}^2$, Christ has used the method of superlogarithmic estimates (see [2]), developed by Morimoto (see [14]). Christ's results can easily be generalized to the study of $\Box_b$ on (p,q)-forms on pseudo convex CR manifolds when the quadratic forms c_{IJ} and $c_{I'J'}$ are diagonalizable. More generally the result (pr0ven in [10]) is:

Theorem 4.1. *$\Box_b$ is hypoelliptic if there exists $f \in \mathcal{I}_q$ and a manifold $S \subset M$ of holomorphic dimension $\leq \min\{n-q-1,\ q-1\}$ such that $f \neq 0$ outside of S and*

$$\lim_{x \to S} d(x,S)\log|f(x)| = 0,$$

where $d(x,S)$ denotes the distance from x to S.

To prove this theorem in general we need the following localization lemma.

Lemma 4.2. *If M is pseudoconvex if $P \in S \subset M$ with S a submanifold of holomorphic dimension $\leq \min\{n-q-1,\ q-1\}$, Then there exists a neighborhood U of P such that if*
$S_a = \{Q \in U \mid dist(Q,S) \leq a$ then there exists $C > 0$ such that

$$\|\varphi\|^2_{S_a} \leq C(a^2 Q_b(\varphi,\varphi) + \|\varphi\|^2_{M-S_a} + \|\varphi\|^2_{-1}),$$

for all $\varphi \in \mathcal{A}^{p,q}_b$ with support in U. Here $\|..\|_X$ denotes the L_2-norm over X.

§5. Multipliers associated with singularities

Let $\{h_1, \ldots, h_m\}$ be holomorphic functions defined in a neighborhood of the origin in $\mathbb{C}^n$, with $h_j(0) = 0$. Let $M \subset \mathbb{C}^{n+1}$ be a pseudoconvex CR manifold which near the origin is defined by

$$\Re(z_{n+1}) = \sum |h_j(z_1, \ldots, z_n)|^2.$$

If $\mathcal{G}$ is an ideal of germs of holomorphic functions in $\mathbb{C}^n$ at the origin we define $\mathbf{B}(\mathcal{G})$ to be the set of all $n \times p$ matrices with $p \geq n$

$$B(g_1, \ldots, g_p) = \begin{pmatrix} g_{1z_1} & \cdots & g_{1z_n} \\ \vdots & \ddots & \vdots \\ g_{pz_1} & \cdots & g_{pz_n} \end{pmatrix},$$

for all n-tuples in $\mathcal{G}$. Let $\mathcal{D}^j(\mathbf{B}(\mathcal{G}))$ denote the ideal generated by the set of all $j \times j$ subdeterminants of $B(g_1, \ldots, g_p)$ for all $B(g_1, \ldots, g_p) \in \mathbf{B}(\mathcal{G})$.

Set

$$J_1^q(\mathcal{G}) = \sqrt{\mathcal{D}^{n-q}(\mathbf{B}(\mathcal{G}))}\ .$$

Inductively we define

$$J_{k+1}^q(\mathcal{G}) = J_1^q(\mathcal{G}, J_1^q(\mathcal{G}), \ldots, J_k^q(\mathcal{G})).$$

Let $\mathcal{H} = (h_1, \ldots, h_m)$, the ideal generated by $h_1, \ldots, h_m$. The following result shows how the ideals $J_k^q(\mathcal{H})$ determine subellipticity on M.

Proposition 5.1. $1 \in J_k^q(\mathcal{I}_{q,k}^+)$ *if and only if* $1 \in \mathcal{I}_{q,k}^+$.

Denoting by $V(\mathcal{H})$ the variety of $\mathcal{H}$, we have

$$\dim V(\mathcal{H}) = q \iff \begin{cases} 1 \in J_k^{q+1}(\mathcal{H}) & \text{for some } k \\ 1 \notin J_k^q(\mathcal{H}) & \text{for all } k. \end{cases}$$

Suppose $\dim V(\mathcal{H}) = q$ let k_0 be the least k for which $1 \in J_k^{q+1}(\mathcal{H})$. Note that $k_0 = 1$ if and only if 0 is not a singular point and that $V(\mathcal{H}, J_1^{q+1}(\mathcal{H}))$ is the singular variety of $V(\mathcal{H})$. If $q_1 = \dim V(\mathcal{H}, J_1^{q+1}(\mathcal{H}))$ we let k_1 be the least k for which $1 \in J_k^{q_1+1}(\mathcal{H}, J_1^{q+1}(\mathcal{H}))$. We continue defining $k_2, k_3, \ldots$ and these numbers are invariants of the singularity.

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The Bergman kernel of Hartogs domains and transformation laws for Sobolev-Bergman kernels

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Introduction

If we consider the Bergman kernel of strictly pseudoconvex domains, we can discuss a scalar invariant theory associated with CR geometry of the boundaries. This is Fefferman's program proposed in [3] and then developed in [6], [10], [1], [11], [8] and others. What will happen if the Bergman kernel is replaced by reproducing kernels associated with spaces of holomorphic functions contained in L^2 Sobolev spaces? Let us restrict ourselves to the case where the Sobolev order is a half integer $s/2$ ($s \in \mathbb{Z}$). The case $s = 0$ corresponds to the Bergman kernel. The case $s = 1$ corresponds to the Szegö kernel, and the invariant theory is essentially the same as that of the Bergman kernel ([10], [11]). The situation changes with the signature of this s. More precisely, it is at first necessary that the inner product of the Hilbert space which admits the reproducing kernel must satisfy a transformation law under biholomorphic mappings. Existence of such an inner product is obvious when $s \leq 0$ ($s \in \mathbb{R}$), whereas it is unknown for $s > 0$ except for $s = 1$. Next, boundary invariants will be contained in the singularity of the reproducing kernel, and if the singularity is of the same type as that of the Bergman kernel ([3], [2]) then in particular $s \geq 0$ is necessary ([9]). This fact suggests that the type of the singularities of the reproducing kernels for $s < 0$ are different from that of the Bergman kernel. Is it possible to avoid considering such new singularities? In what follows, we shall give an almost affirmative answer by considering Hartogs domains and Hirachi's formulation in [8] of a biholomorphic transformation law for local defining functions of strictly pseudoconvex domains.

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§1. Hartogs domains and biholomorphic transformation laws

Points of $\mathbb{C}^{n+t} = \mathbb{C}^n \times \mathbb{C}^t$ $(n \geq 2)$ will be denoted by (z, z_0), (w, w_0), etc. Recall that a domain $\Omega \subset \mathbb{C}^n$ is said to have C^∞ boundary if there exists a real valued function $\rho \in C^\infty(\overline{\Omega}) = C^\infty(\mathbb{C}^n)|_{\overline{\Omega}}$ such that

$$\Omega = \{z \mid \rho(z) > 0\}, \qquad d\rho \neq 0 \text{ on } \partial\Omega;$$

we then write $\rho \in C^\infty_{\text{def}}(\overline{\Omega})$. Given such a *defining function* $\rho \in C^\infty_{\text{def}}(\overline{\Omega})$, the *Hartogs domain* $D = D^t_\rho \subset \mathbb{C}^{n+t}$ associated with it is defined by

$$D := \{(z, z_0) \mid \lambda(z, z_0) > 0\}, \qquad \lambda(z, z_0) := \rho(z) - |z_0|^2;$$

thus $\lambda \in C^\infty_{\text{def}}(\overline{D})$ which depends on $t \in \mathbb{N}$ and $\rho \in C^\infty_{\text{def}}(\overline{\Omega})$.

Remark 1. D is defined even when $\rho \notin C^\infty(\overline{\Omega})$, but if $\partial\Omega \in C^\infty$ then $\partial D \in C^\infty$ because $\partial\lambda = \partial\rho - \overline{z}_0 \cdot dz_0$. If in addition $\partial\Omega$ is strictly pseudoconvex, so is ∂D on $z_0 = 0$. If furthermore $-\rho$ is strictly plurisubharmonic, so is $-\lambda$ and thus ∂D is everywhere strictly pseudoconvex.

In what follows, we assume $\rho \in C^\infty_{\text{def}}(\overline{\Omega})$ and consider for simplicity only *strictly pseudoconvex domains* Ω. For subscripts $i = 1, 2$, we use the following notation:

$$\rho_i \in C^\infty_{\text{def}}(\overline{\Omega}_i), \quad \lambda_i = \rho_i - |z_0|^2, \quad D_i = D^t_{\rho_i} \subset \mathbb{C}^{n+t}.$$

By elementary operations on determinants, we have:

Fact 1. *The Levi determinants (i.e. the complex Monge-Ampère operators) on Ω and D satisfy*

$$J_\Omega[\rho] := (-1)^n \det \begin{pmatrix} \rho & \rho_{\overline{k}} \\ \rho_j & \rho_{j\overline{k}} \end{pmatrix} = J_D[\lambda],$$

where the subscripts j, $\overline{k}$ stand for differentiation with respect to z_j, $\overline{z}_k$.

Recall by Fefferman [4] that if $\Phi : \Omega_1 \to \Omega_2$ is biholomorphic then $J_{\Omega_1}[u_1] = J_{\Omega_2}[u_2] \circ \Phi$ with $u_1 := |\det \Phi'|^{-2/(n+1)} (u_2 \circ \Phi)$ for functions u_2 in Ω_2, where Φ' denotes the holomorphic Jacobian matrix of Φ.

Lemma 1. *Given a biholomorphic map $\Phi : \Omega_1 \to \Omega_2$, let*

$$\Psi : (z, z_0) \mapsto (\Phi(z), m(z)\, z_0), \quad m(z) := [\det \Phi'(z)]^{1/(n+1)}.$$

Then $\Psi : D_1 \to D_2$ is a biholomorphic lift, provided

$$\rho_1 = |\det \Phi'|^{-2/(n+1)} (\rho_2 \circ \Phi). \tag{1.1}$$

Incidentally,

$$\text{(1.2)}\qquad \det \Psi'(z,z_0) = [\det \Phi'(z)]^{w(-t)/(n+1)}, \quad w(-t) := n+1+t.$$

Proof. It follows from (1.1) that $\lambda_2(\Psi(z,z_0)) = |m(z)|^2\,\lambda_1(z,z_0)$ so that $\Psi(D_1) \subset D_2$, and similarly $\Psi^{-1}(D_2) \subset D_1$. Now (1.2) is easy.

Remark 2. The lift Ψ is motivated by that of Fefferman [4], [5]:

$$\Phi_\# : (z_{\mathrm{F}}, z) \mapsto \big(m(z)^{-1} z_{\mathrm{F}},\, \Phi(z)\big), \quad z_{\mathrm{F}} \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}.$$

The map $\Phi_\# : \mathbb{C}^* \times \Omega_1 \to \mathbb{C}^* \times \Omega_2$ is biholomorphic. The multiplicative factor of the variable z_{F} in $\Phi_\#$ is the inverse of that in Ψ. Thus it is natural to consider the Lorentz-Kähler potential $|z_{\mathrm{F}}|^2\,\rho(z)$ upstairs (cf. [4], [5]), whereas we consider $\lambda(z,z_0) = \rho(z) - |z_0|^2$ in Lemma 1.

Following Hirachi [8], we lift the Levi determinant $J_\Omega[\,\cdot\,]$ on Ω to Fefferman's $\mathbb{C}^*$ bundle in [4], [5]. That is, we set, for functions $U = U(z_{\mathrm{F}}, z)$ in $\mathbb{C}^* \times \Omega$,

$$J_{\Omega,\#}[U] := (-1)^n \det \begin{pmatrix} U_{\mathrm{F}\overline{\mathrm{F}}} & U_{\mathrm{F}\bar{k}} \\ U_{j\overline{\mathrm{F}}} & U_{j\bar{k}} \end{pmatrix},$$

where the subscripts F, $\overline{\mathrm{F}}$ stand for differentiation with respect to z_{F}, $\bar{z}_{\mathrm{F}}$. Then, as in the proof of Fact 1, we have:

Fact 2. *Let $\Lambda(z_{\mathrm{F}}, z, z_0) := U(z_{\mathrm{F}}, z) - |z_{\mathrm{F}}|^2\,|z_0|^2$ in $\mathbb{C}^* \times D$ for functions $U = U(z_{\mathrm{F}}, z)$ in $\mathbb{C}^* \times \Omega$. Then*

$$J_{D,\#}[\Lambda] := (-1)^{n+t} \det \begin{pmatrix} \Lambda_{\mathrm{F}\overline{\mathrm{F}}} & \Lambda_{\mathrm{F}\bar{k}} & \Lambda_{\mathrm{F}\bar{0}} \\ \Lambda_{j\overline{\mathrm{F}}} & \Lambda_{j\bar{k}} & \Lambda_{j\bar{0}} \\ \Lambda_{0\overline{\mathrm{F}}} & \Lambda_{0\bar{k}} & \Lambda_{0\bar{0}} \end{pmatrix} = |z_{\mathrm{F}}|^{2t}\, J_{\Omega,\#}[U].$$

Remark 3. Roughly speaking, there does not exist any natural family, in the context of local biholomorphic invariant theory, of C^∞ (local) defining functions which satisfy the transform law (1.1) (cf. Theorem 2 of [9] for a precise statement). That is, (1.1) necessarily contains an error (cf. [5], [6], [7], [1], [11]). According to Hirachi's theory in [8], this difficulty in Fefferman's program for the invariant theory of the Bergman kernel can be avoided by considering asymptotic solutions of the complex Monge-Ampère equation upstairs

$$J_{\Omega,\#}[U] = |z_{\mathrm{F}}|^{2n} \;\;\&\;\; U > 0 \;\text{ in }\; \mathbb{C}^* \times \Omega, \qquad U = 0 \;\text{ on }\; \mathbb{C}^* \times \partial\Omega.$$

More precisely, asymptotic solutions are of the form

$$U = \rho_\# + \rho_\# \sum_{k=1}^{\infty} \eta_{k,\Omega} \left(\rho^{n+1} \log \rho_\#\right)^k, \qquad \eta_{k,\Omega} \in C^\infty(\overline{\Omega}),$$

where $\rho_\#$ takes the form $\rho_\#(z_{\mathrm{F}}, z) = |z_{\mathrm{F}}|^2 \rho(z)$ with special $\rho \in C^\infty_{\mathrm{def}}(\overline{\Omega})$. (This ρ involves an *ambiguity parameter* but transforms by (1.1), because the class of these ρ's are so chosen and an action is defined on the ambiguity parameter. See [8] for the detail.) On the other hand, $\Lambda := U - |z_{\mathrm{F}}|^2 |z_0|^2$ in Fact 2 formally satisfies

$$J_{D,\#}[\Lambda] = |z_{\mathrm{F}}|^{2n+2t} \;\;\&\;\; \Lambda > 0 \text{ in } \mathbb{C}^* \times D, \qquad \Lambda = 0 \text{ on } \mathbb{C}^* \times \partial D.$$

It might be interesting to study the role of Λ in the framework of Hirachi's theory [8].

§2. Sobolev-Bergman kernels of Ω in terms of the Bergman kernel of D

We denote the Bergman kernel of a Hartogs domain $D = D^t_\rho \subset \mathbb{C}^{n+t}$ by

$$K^{\mathrm{B}}_D((z, z_0), (w, w_0)) \qquad \big((z, z_0), (w, w_0) \in D\big),$$

and the restriction to the diagonal by $K^{\mathrm{B}}_D((z, z_0)) = K^{\mathrm{B}}_D((z, z_0), (z, z_0))$.

Lemma 2. *If $\Phi : \Omega_1 \to \Omega_2$ is biholomorphic, then under the condition* (1.1) *in Lemma 1,*

$$K^{\mathrm{B}}_{D_1}((z,0)) = K^{\mathrm{B}}_{D_2}((\Phi(z),0)) \,|\det \Phi'(z)|^{2w(-t)/(n+1)}. \tag{2.1}$$

More precisely, for the lift $\Psi : D_1 \to D_2$ in Lemma 1,

$$K^{\mathrm{B}}_{D_1}((z,z_0)) = K^{\mathrm{B}}_{D_2}(\Psi(z,z_0)) \,|\det \Phi'(z)|^{2w(-t)/(n+1)}. \tag{2.2}$$

Proof. It follows from the transformation law in general for the Bergman kernel that if $\Psi : D_1 \to D_2$ is biholomorphic then

$$K^{\mathrm{B}}_{D_1}((z,z_0)) = K^{\mathrm{B}}_{D_2}(\Psi(z,z_0)) \,|\det \Psi'(z,z_0)|^2.$$

Thus (2.2) follows from (1.2). Setting $z_0 = 0$ in (2.2), we get (2.1).

This lemma makes sense when it is combined with the following elementary observation by Ligocka in [12]. Recall by definition that the Bergman kernel of D is the reproducing kernel associated with the

Hilbert space $H^{\mathrm{B}}(D) = L^2(D) \cap \mathcal{O}(D)$, where $\mathcal{O}(D)$ denotes the totality of holomorphic functions in D. Let us set, for $k \in \mathbb{N}_0$,

$$(2.3) \qquad g_k(z) = g_k[\rho](z) := c_k(t)\, \rho(z)^{t+k}, \qquad c_k(t) := \frac{1}{t+k} \frac{\pi^t}{\Gamma(t)}$$

and consider the Hilbert space $H^{\mathrm{B}}(\Omega, g_k) := L^2(\Omega, g_k) \cap \mathcal{O}(\Omega)$ with respect to the measure having each $g_k(z)$ as the weight function. Denoting the reproducing kernel by $K^{\mathrm{B}}_{g_k}(z,w)$, we set $K^{\mathrm{B}}_{g_k}(z) = K^{\mathrm{B}}_{g_k}(z,z)$. It will be sometimes clearer if we factor out the positive constant $c_k(t)$ and consider the following (then $\ell = t+k$ is not necessary):

$$H^{\mathrm{B}}(\Omega, \rho^\ell) = L^2(\Omega, \rho^\ell) \cap \mathcal{O}(\Omega), \quad K^{\mathrm{B}}_{\rho^\ell}(z) = K^{\mathrm{B}}_{\rho^\ell}(z,z) \quad (\ell \in \mathbb{N}_0).$$

$K^{\mathrm{B}}_{\rho^\ell}$ is called the *Sobolev-Bergman kernel* of Sobolev order $-\ell/2$ in [9]. Then, it is shown in Ligocka [12] that

$$(2.4) \qquad K^{\mathrm{B}}_D((z,z_0),(w,w_0)) = \sum_{k \in \mathbb{N}_0} K^{\mathrm{B}}_{g_k}(z,w) \sum_{|\alpha|=k} z_0^\alpha\, \overline{w}_0^\alpha.$$

Theorem. *Given a biholomorphic map $\Phi : \Omega_1 \to \Omega_2$, consider the Hartogs domains $D_i = D^t_{\rho_i} \subset \mathbb{C}^{n+t}$ $(i = 1,2)$ defined by $\rho_i \in C^\infty_{\mathrm{def}}(\overline{\Omega}_i)$ satisfying the condition* (1.1) *in Lemma 1. Then the reproducing kernel $K^{\mathrm{B}}_{g_k[\rho]}(z)$ associated with the Hilbert space $H^{\mathrm{B}}(\Omega, g_k[\rho])$ defined via the function $g_k = g_k[\rho]$ in* (2.3) *satisfies the following transformation law*

$$(2.5) \qquad K^{\mathrm{B}}_{g_k[\rho_1]}(z) = K^{\mathrm{B}}_{g_k[\rho_2]}(\Phi(z))\, |\det \Phi'(z)|^{2w(-t-k)/(n+1)}.$$

That is, $K^{\mathrm{B}}_{\rho_1^{t+k}}(z) = K^{\mathrm{B}}_{\rho_2^{t+k}}(\Phi(z))\, |\det \Phi'(z)|^{2w(-t-k)/(n+1)}$.

Proof. If we set $z_0 = 0$ or $w_0 = 0$ in (2.4), then all terms in the right vanish except for $\alpha = 0$ (i.e. $k = 0$), so that $K^{\mathrm{B}}_D((z,0)) = K^{\mathrm{B}}_{g_0}(z)$. Thus (2.5) for $k = 0$ follows from (2.1) in Lemma 2. The result (2.5) for general $k \in \mathbb{N}_0$ also follows similarly by using (2.2) and Lemma 1.

Remark 4. Taking $k = 0$, we may write $K^{\mathrm{B}}_{\rho^t}(z) = c_0(t)\, K^{\mathrm{B}}_D((z,0))$ with $c_0(t) = \pi^t/\Gamma(t+1)$. Varying the dimension t, we get Sobolev-Bergman kernels of any negative half-integral order $-t/2$ $(t \in \mathbb{N})$. On the other hand, if we take $t = 1$, then we have $g_k(z) = c_k(1)\, \rho(z)^{k+1}$ with $c_k(1) = \pi/(k+1)$ and

$$K^{\mathrm{B}}_D((z,z_0)) = \sum_{k=0}^{\infty} K^{\mathrm{B}}_{g_k}(z)\, |z_0|^{2k} \qquad ((z,z_0) \in D \subset \mathbb{C}^{n+1}).$$

Varying this time the power k of $|z_0|^2$, we again get Sobolev-Bergman kernels of any negative half-integral order $-(k+1)/2$ $(k \in \mathbb{N}_0)$.

Remark 5. The singularities of these Sobolev-Bergman kernels of Ω are computable from that of the Bergman kernel K_D^{B} of the Hartogs domain $D = D_\rho^t$, but there remains a problem of localizing the singularity of K_D^{B}. The author expects that the argument here will be used rather as a heuristics of formulating a local or microlocal version.

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An approach to the Cartan geometry II : CR manifolds

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Introduction

One of the prominent features in the post-Oka development of the several complex variables is the extensive use of the Cauchy-Riemann partial differential equations. We also note the development of the CR geometry induced on the boundary. This geometry is introduced by E. Cartan [3] in low dimensional cases. The general case is developed by N. Tanaka [9], S.-S. Chern-J. Moser [4], S. Webster [10], and D. Burns. Jr.-S. Shnider [1]. This geometry will be the vehicle to set the Cauchy-Riemann equation geometrically.

The CR geometry is a special case of the Cartan geometry, which is regarded as a deformation of the Klein's classical geometry. Namely, for each classical geometry given as a homogenous space G/H we have the Cartan geometries modeled after G/H. For example, Riemann geometry is modeled after the euclidean geometry, which is the quotient of the group of euclidean motions by the orthogonal group. On a space X we have a Cartan geometry modeled after G/H when we have (1) a principal H-bundle E formed by frames, i.e. ways to identify up to equivalence (infinitesimally up to certain order) its neighborhood with open sets in G/H. (2) A Cartan connection on E valued in the Lie algebra of G.

CR geometry may be regarded as the case of Cartan geometry when the homogenous space is the unit ball in complex euclidean space acted by the group of holomorphic automorphisms. We constructed CR geometry in [6] from the above view point. However, we did not construct the frame bundle directly. We first construct the bundle of the frames of the first (infinitesimal) order and then we prolong it to the frame bundle. In this paper, we construct CR geometry by defining frames directly. We also write down the normal CR Cartan connections and discuss its global aspect.

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§1. The Homogenous CR manifolds

We fix a non-degenerate hermitian $n \times n$ matrix

$$(1) \qquad (\underline{h}_{\alpha\bar{\beta}}), \quad \alpha, \beta = 1, ..., n.$$

We consider, as our model, the CR-structure on the hypersurface $\mathcal{M}$ in $\mathbf{C}^{n+1} = \{(z^1, ..., z^n, w)\}$, given by

$$(2) \qquad \Im w = \frac{1}{2}\langle z, z\rangle, \quad \langle z, z\rangle = \underline{h}_{\alpha\bar{\beta}} z^\alpha \overline{z^\beta}.$$

A) We embed $\mathbf{C}^{n+1}$ in the complex projective space $\mathbf{CP}^{n+1}$ sending $(z^1,..., z^n, w)$ to the point with the homogenous coordinate $[1, z^1,..., z^n, w]$. The subgroup $\mathcal{G}$ of the projective group which preserves the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ acts transitively on the closure. Thus $\overline{\mathcal{M}}$ is the homogenous space on which we model our CR geometry.

B) We find that $\mathcal{G}$ decomposes to the product of the translation group and the isotropy group. Namely,

$$(3) \qquad \mathcal{G} = \mathcal{L} \cdot \mathcal{H},$$

(4)

$$\mathcal{L} = \{\ l(z, x) = \begin{pmatrix} 1 & 0 & 0 \\ z & I & 0 \\ w & iz^* & 1 \end{pmatrix} : z = (z^1, ..., z^n)^{\mathrm{tr}},\ w = x + \frac{i}{2}\langle z, z\rangle\ \}$$

where $(z^*)_\alpha = \underline{h}_{\alpha\bar{\beta}} \overline{z^\beta}$.

$\mathcal{H} = H/\mathrm{center}$, where H is the group of $(n+2) \times (n+2)$ matrixes:

$$(5) \qquad h = h(a, u, \beta, s) = \begin{pmatrix} a & \nu^* & b \\ 0 & u & \beta \\ 0 & 0 & 1/\bar{a} \end{pmatrix}, \quad \text{where}$$

a is a non-zero complex number, u a complex $n \times n$-matrix, β is a column complex n-vector β , and s is a real number satisfying:

$$(6) \qquad u^* u = I, \quad \frac{a}{\bar{a}} \det u = 1, \quad \nu = i\bar{a} u^* \beta, \quad \frac{b}{a} = s - \frac{i}{2}\langle \beta, \beta\rangle,$$

$(u^*)^\alpha_\beta = \underline{h}^{\alpha\bar{\gamma}} \underline{h}_{\beta\bar{\sigma}} \overline{u^\sigma_\gamma}$, and I is the identity $n \times n$-matrix. The center is the finite group

$$(7) \qquad \{h(e^{im'}, e^{im'} I, 0, 0)\ :\ m' = \frac{m}{n+2} 2\pi, \quad m = 0, 1, ..., n+1\}.$$

C) The Lie algebra $\mathbf{g}$ of $\mathcal{G}$ has the grading:

(8) $$\mathbf{g} = \mathbf{g}_{(-2)} + \mathbf{g}_{(-1)} + \mathbf{g}_{(0)} + \mathbf{g}_{(1)} + \mathbf{g}_{(2)}, \quad \text{where}$$

(9.1) $$\mathbf{g}_{(-2)} = \{\ \{\dot{x}\}_{(-2)} = (\frac{d(l(0, s\dot{x}))}{ds})_{s=0}\ :\ \dot{x} \in \mathbf{R}\},$$

(9.2) $$\mathbf{g}_{(-1)} = \{\ \{\dot{z}\}_{(-1)} = (\frac{d(l(s\dot{z}, 0))}{ds})_{s=0}\ :\ \dot{z} \in \mathbf{C}^n\},$$

(9.3) $$\mathbf{g}_{(0)} = \mathbf{R}\pi + \mathbf{R}\mu + \{\mathbf{su}(n)\}, \quad \text{where for } \dot{u} \in \mathbf{su}(n)$$

(9.4) $$\{\dot{u}\} = (\frac{dh(1, e^{s\dot{u}}, 0, 0)}{ds})_{s=0}, \quad \pi = (\frac{dh(e^s, I, 0, 0)}{ds})_{s=0},$$
$$\mu = (\frac{dh(e^{is}, e^{-\frac{2}{n}is} I, 0, 0)}{s})_{s=0},$$

(9.5) $$\mathbf{g}_{(1)} = \{\ \{\dot{\beta}\}_{(1)} = (\frac{dh(1, I, s\dot{\beta}, 0)}{ds})_{s=0}\ :\ \dot{\beta} \in \mathbf{R}^m\},$$

(9.6) $$\mathbf{g}_{(2)} = \{\ \{\dot{b}\}_{(2)} = (\frac{dh(1, I, 0, s\dot{b}))}{ds})_{s=0}\ :\ \dot{b} \in \mathbf{R}\},$$

(9.7) $$\dot{u} \in \mathbf{su}(n) \text{ if and only if } \underline{h}_{\sigma\bar{\gamma}}\dot{u}^{\sigma}_{\alpha} + \underline{h}_{\alpha\bar{\sigma}}\overline{\dot{u}^{\sigma}_{\gamma}} = 0.$$

(10) $$\mathbf{h} = \mathbf{g}_{(0)} + \mathbf{g}_{(1)} + \mathbf{g}_{(2)} \text{ is the Lie algebra of } H.$$

For $\dot{g} \in \mathbf{g}$ we set

(11)
$$\dot{g} = \{\dot{g}_{[-2]}\}_{(-2)} + \{\dot{g}_{[-1]}\}_{(-1)} + \dot{g}_{\pi}\pi + \dot{g}_{\mu}\mu + \{\dot{g}_{\mathbf{su}}\} + \{\dot{g}_{[1]}\}_{(1)} + \{\dot{g}_{[2]}\}_{(2)}.$$

D) In terms of the decomposition (3) the action of $g \in \mathcal{G}$ on $(z', w') \in \mathcal{M}$ is given by

(12)
$$T_{l(z,x)}(z', w') = (z' + z, w' + w + i\langle z', z\rangle), \quad \text{where } \langle z', z\rangle = \underline{h}_{\alpha\bar{\beta}}(z')^{\alpha}\overline{z^{\beta}}.$$

(13)
$$T_h(z', w') = (\frac{1}{a\lambda}(uz' + w'\beta), \frac{1}{\lambda}\frac{1}{|a|^2}w'), \quad \text{where } \lambda = 1 - i\langle uz', \beta\rangle + \frac{b}{a}w'.$$

E) The $\partial_\flat$-operators of the CR structure on $\mathcal{M}$ is generated by

$$P^\alpha = \frac{\partial}{\partial \overline{z^\alpha}} - i z_*^\alpha \frac{\partial}{\partial \bar{w}}, \quad z_*^\alpha = \underline{h}_{\beta\bar{\alpha}} z^\beta. \tag{14}$$

We have

$$[P^\alpha, \overline{P^\beta}] = i\underline{h}_{\beta\bar{\alpha}} \frac{\partial}{\partial \theta_{\mathcal{M}}}, \quad \frac{\partial}{\partial \theta_{\mathcal{M}}} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}. \tag{15}$$

F) The Maurer-Cartan form ω_G has the expression:

$$\omega_G = Ad(h^{-1})(\{\theta_{\mathcal{M}}\}_{(-2)} + \{dz\}_{(-1)}) + \omega_H, \tag{16}$$

where $\omega_H = h^{-1}dh$ is the Maurer-Cartan form of H and

$$\theta_{\mathcal{M}} = dx + \frac{i}{2}\langle z, dz\rangle - \frac{i}{2}\langle dz, z\rangle. \tag{17}$$

It then follows by calculation that using the terminology in (11)

$$(\omega_G)_{[-2]} = |a|^2 \theta_{\mathcal{M}}, \quad (\omega_G)_{[-1]} = a u^\star (dz - \bar{a}\beta\theta_{\mathcal{M}}). \tag{18}$$

Note that for matrix valued 1-forms α and β

$$[\alpha, \beta] = \alpha \wedge \beta + \beta \wedge \alpha. \tag{19}$$

We then find that the structure equation : $d\omega_G + [\omega_G, \omega_G]/2 = 0$ is rewritten in terms of the grading (8) as

$$d(\omega_G)_{[-2]} - i\langle(\omega_G)_{[-1]}, (\omega_G)_{[-1]}\rangle - 2(\omega_G)_\pi \wedge (\omega_G)_{[-2]} = 0. \tag{20.1}$$

$$\begin{aligned} d(\omega_G)_{[-1]} + \{(\omega_G)_{\mathbf{su}} - ((\omega_G)_\pi + \frac{n+2}{n} i(\omega_G)_\mu)I\} \wedge (\omega_G)_{[-1]} \\ + (\omega_G)_{[1]} \wedge (\omega_G)_{[-2]} = 0, \end{aligned} \tag{20.2}$$

$$d(\omega_G)_\pi - \Im\langle(\omega_G)_{[-1]}, (\omega_G)_{[1]}\rangle + (\omega_G)_{[2]} \wedge (\omega_G)_{[-2]} = 0, \tag{20.3}$$

$$d(\omega_G)_\mu + \Re\langle(\omega_G)_{[-1]}, (\omega_G)_{[1]}\rangle = 0, \tag{20.4}$$

$$\begin{aligned} d(\omega_G)_{\mathbf{su}} + (\omega_G)_{\mathbf{su}} \wedge (\omega_G)_{\mathbf{su}} + i(\omega_G)_{[1]} \wedge (\omega_G)^*_{[-1]} \\ - i(\omega_G)_{[-1]} \wedge (\omega_G)^*_{[1]} + \frac{2}{n} i\Re\langle(\omega_G)_{[-1]}, (\omega_G)_{[1]}\rangle = 0, \end{aligned} \tag{20.5}$$

(20.6)
$$d(\omega_G)_{[1]}+((\omega_G)_{\mathbf{su}}+((\omega_G)_\pi-\frac{n+2}{n}i(\omega_G)_\mu)I)\wedge w^*_{[1]}+(\omega_G)_{[-1]}\wedge(\omega_G)_{[2]}=0$$

(20.7)
$$d(\omega_G)_{[2]}+i\langle(\omega_G)_{[1]},(\omega_G)_{[1]}\rangle+2(\omega_G)_\pi\wedge(\omega_G)_{[2]}=0.$$

G) Note by calculation that for $g=l(z_0,w_0)h$

(21)
$$\begin{aligned}\overline{P^\alpha}T^\gamma_g(0)&=\frac{1}{a}u^\gamma_\alpha, & \frac{\partial}{\partial\theta_{\mathcal{M}}}T^\alpha_g(0)&=\frac{1}{a}\beta^\alpha,\\ \overline{P^\alpha}T^0_g(0)&=\frac{i}{a}\underline{h}_{\gamma\bar\sigma}\,u^\gamma_\alpha\,\overline{z^\sigma_0}, & \frac{\partial}{\partial\theta_{\mathcal{M}}}T^0_g(0)&=\frac{1}{|a|^2}+\frac{i}{a}\langle\beta,z_0\rangle.\end{aligned}$$

(22)
$$\frac{\partial}{\partial\theta_{\mathcal{M}}}\overline{P^\gamma}T^\alpha_g(0)=-\frac{b}{a}\frac{1}{a}u^\alpha_\gamma+i\underline{h}_{\sigma\bar\nu}u^\sigma_\gamma\frac{1}{a}\beta^\alpha\overline{\beta^\nu}.$$

H) We find by calculation that, setting

(23)
$$\mathrm{Ad}(h^{-1})(\{\dot g\}_{(l)})=A(h,\dot g,l),\quad\text{we have}$$

(24.1)
$$\begin{aligned}&A(h,\dot x,-2)_{[-2]}=|a|^2\dot x,\quad A(h,\dot x,-2)_{[-1]}=-|a|^2\dot x u^*\beta,\\ &A(h,\dot x,-2)_\pi+iA(h,\dot x,-2)_\mu=-a\bar b\dot x,\\ &A(h,\dot x,-2)_{[\mathbf{su}]}=i|a|^2\dot x(u^*\beta)\otimes(\beta^*u)+\frac{2i}{n}A(h,\dot x,\mu)I\\ &A(h,\dot x,-2)_{[1]}=-\bar a b\dot x u^*\beta,\quad A(h,\dot x,-2)_{[2]}=-|b|^2\dot x,\end{aligned}$$

(24.2)
$$\begin{aligned}&A(h,\dot z,-1)_{[-2]}=0,\quad A(h,\dot z,(-1))_{[-1]}=au^*\dot z,\\ &A(h,\dot z,-1)_\pi+iA(h,\dot z,-1)_\mu=ia\langle\dot z,\beta\rangle,\\ &A(h,\dot z,-1)_{[\mathbf{su}]}=-ia(u^*\dot z)\otimes(\beta^*u)-i\bar a(u^*\beta)\otimes(\dot z^*u)+\frac{2i}{n}A(h,\dot z,\mu)I,\\ &A(h,\dot z,-1)_{[1]}=bu^*\dot z-i\bar a\langle\beta,\dot z\rangle u^*\beta,\quad A(h,\dot z,-1)_{[2]}=2\Re ib\langle\dot z,\beta\rangle,\end{aligned}$$

(24.3)
$$\mathrm{Ad}(h^{-1})\pi=\pi+\{u^*\beta\}_{(1)}+\{2\Re\frac{b}{a}\}_{(2)},$$

(24.4)
$$\mathrm{Ad}(h^{-1})\mu=\mu-\{\frac{n+2}{n}iu^*\beta\}_{(1)}+\{\frac{n+2}{n}\langle\beta,\beta\rangle\}_{(2)},$$

$$(24.5)\quad \mathrm{Ad}(h^{-1})\{\sigma\} = \{u^*\sigma u\} + \{u^*\sigma\beta\}_{(1)} + \{i\langle\sigma\beta,\beta\rangle\}_{(2)}\ (\sigma \in \mathbf{su}(n)),$$

$$(24.6)\qquad \mathrm{Ad}(h^{-1})\{\gamma\}_{(1)} = \{\frac{1}{\bar{a}}u^*\gamma\}_{(1)} + \{2\Re\frac{i}{a}\langle\gamma,\beta\rangle\}_{(2)}\quad (\gamma \in \mathbf{C}^n),$$

$$(24.7)\qquad \mathrm{Ad}(h^{-1})\{s\}_{(2)} = \{\frac{s}{|a|^2}\}_{(2)}.$$

§2. CR coframes of infinitesimal order 1

A) Let M be a CR manifold with non-degenerate Levi-form, given by a subbundle $T_\flat''M$ of $\partial_\flat$ differential operators. We may identify M with a hypersurface in $\mathbf{C}^{n+1}$ passing the origin p_0 defined by an equation:

$$(1)\qquad r = 0.$$

We regard p_0 as the reference point and interested in the local aspect near p_0. Hence we may shrink M if necessary. We consider a chart $\{(z^1, ..., z^n, w)\}$ of $\mathbf{C}^{n+1}$. By a holomorphic linear change of chart we may assume

$$(2)\qquad \frac{\partial r}{\partial w} - \frac{\partial r}{\partial \bar{w}} \neq 0 \text{ at } p_0, \quad \frac{\partial r}{\partial z^\alpha} = O(1).$$

We set $r_\alpha = \partial/\partial z^\alpha$, $r_{\bar{\alpha}} = \partial/\partial\overline{z^\alpha}$, etc. Our model is the case

$$(3)\qquad r = r_{\mathcal{M}} = \frac{1}{i}(w - \bar{w}) - \langle z, z\rangle.$$

B) The space $T_\flat''M$ of the $\overline{\partial}_\flat$ differential operators of M is generated by

$$(4)\qquad Q^\alpha = \frac{\partial}{\partial\overline{z^\alpha}} - \frac{r_{\bar{\alpha}}}{r_{\bar{w}}}\frac{\partial}{\partial\bar{w}}. \quad \text{Set}$$

$$(5)\qquad \frac{\partial}{\partial\theta_M} = \frac{2}{r_w - r_{\bar{w}}}\left(r_w\frac{\partial}{\partial\bar{w}} - r_{\bar{w}}\frac{\partial}{\partial w} \right).$$

$\partial/\partial\theta_M$ is tangential to M. $Q^\alpha, \overline{Q^\alpha}, \partial/\partial\theta_M$ form a base of the complex tangent space $\mathbf{C}TM$.

C) For a differential form λ on $\mathbf{C}^{n+1}$ we also use the same letter to denote its restriction to M. $\overline{\partial}_\flat$ operators and their bar generate the subbundle of complex tangent space $\mathbf{C}TM$ defined by

$$(6)\qquad \theta_M = 0, \quad \text{where } \theta_M = \frac{1}{2}\left(dw + d\bar{w} + \frac{r_\beta}{r_w}dz^\beta + \frac{r_{\bar{\beta}}}{r_{\bar{w}}}\overline{dz^\beta} \right).$$

$dz^\alpha, \overline{dz^\alpha}, \theta_M$ form a base of $\mathbf{C}T^*M$ dual to the above mentioned base of $\mathbf{C}TM$. $T_\flat''M$ is given by he equation:

$$dz^\alpha = 0, \quad \theta_M = 0. \tag{7}$$

Since $T''M$ is closed under bracket, we see by the expression of Q^α in (2)

$$[Q^\alpha, Q^\beta] = 0. \tag{8.1}$$

Because of the Definition of the Levi-form we may set

$$[Q^\alpha, \overline{Q^\beta}] \equiv ic^{\alpha\bar\beta}\frac{\partial}{\partial\theta_M} \qquad (\text{mod } Q^\gamma, \overline{Q^\gamma}). \tag{8.2}$$

In view of (15) §1 and (3) we may assume that

$$c^{\alpha\bar\beta}(p_0) = \underline{h}_{\beta\bar\alpha}. \tag{8.3}$$

Because of the above mentioned duality, when l is a function on M,

$$dl = (\overline{Q^\alpha}l)dz^\alpha + (Q^\alpha l)d\bar z^\alpha + \frac{\partial l}{\partial\theta_M}\theta_M. \tag{9}$$

D) Consider a manifold N and a map $f\ : N \to M$. Since f is also a map into $\mathbf{C}^{n+1}$ we have in terms of the standard chart $(z^1, ..., z^n, w)$ the expression $f = (f^1, ..., f^n, f^0)$. Note that for any vector field X on N and a function l on M we have $X(l\circ f) = \langle dl, dfX\rangle \circ f$. Therefore by (9)

(10.1)

$$X(l\circ f) = (Xf^\alpha)\ (\overline{Q^\alpha}l)\circ f + (X\overline{f^\alpha})\ (Q^\alpha l)\circ f + (R_X f)\ \frac{\partial l}{\partial\theta_M}\circ f, \quad \text{where}$$

$$R_X f = \frac{1}{2}(Xf^0 + X\overline{f^0} + \frac{r_\alpha}{r_w}\circ f\ Xf^\alpha + \frac{r_{\bar\alpha}}{r_{\bar w}}\circ f\ X\overline{f^\alpha}). \tag{10.2}$$

Since dfX is tangential to M,

$$r_w\circ fXf^0 + r_{\bar w}\circ fX\overline{f^0} + r_\alpha\circ fXf^\alpha + r_{\bar\alpha}\circ fX\overline{f^\alpha} = 0. \tag{10.3}$$

Therefore we also have the expressions:

$$\begin{aligned} R_X f =& \frac{1}{2}(\frac{r_{\bar w} - r_w}{r_{\bar w}r_w}\circ f)\ \{(r_w\circ f)Xf^0 + (r_\alpha\circ f)Xf^\alpha\} \\ =& \frac{1}{2}(\frac{r_w - r_{\bar w}}{r_{\bar w}r_w}\circ f)\ \{(r_{\bar w}\circ f)X\overline{f^0} + (r_{\bar\alpha}\circ f)X\overline{f^\alpha}\}. \end{aligned} \tag{10.4}$$

E) Let $f : \mathcal{M} \to M$ be a map sending the origin 0 to p_0. Then by (6)

$$f^*\theta_M = \frac{1}{2}(df^0 + d\overline{f^0} + \frac{r_\beta}{r_w} \circ f df^\beta + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f d\overline{f^\beta}). \tag{11}$$

Apply (9) to the case $N = M = \mathcal{M}$ and $l = f^0$ as well as $l = f^\beta, f^{\bar\beta}$. We then find

$$\begin{aligned} f^*\theta_M =& C_f\ \theta_{\mathcal{M}} + C^0_{\alpha f}\ dz^\alpha_{\mathcal{M}} + C^0_{\bar\alpha f}\ d\overline{z^\alpha}_{\mathcal{M}}, \quad \text{where} \\ C_f =& \frac{1}{2}(\frac{\partial f^0}{\partial \theta_{\mathcal{M}}} + \frac{\partial \overline{f^0}}{\partial \theta_{\mathcal{M}}} + \frac{r_\beta}{r_w} \circ f\ \frac{\partial f^\beta}{\partial \theta_{\mathcal{M}}} + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f\ \frac{\partial \overline{f^\beta}}{\partial \theta_{\mathcal{M}}}), \\ C^0_{\alpha f} =& \frac{1}{2}(\overline{P^\alpha} f^0 + \overline{P^\alpha} \overline{f^0} + \frac{r_\beta}{r_w} \circ f\ \overline{P^\alpha} f^\beta + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f\ \overline{P^\alpha} \overline{f^\beta}), \\ C^0_{\bar\alpha f} =& \frac{1}{2}(P^\alpha f^0 + P^\alpha \overline{f^0} + \frac{r_\beta}{r_w} \circ f\ P^\alpha f^\beta + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f\ P^\alpha \overline{f^\beta}). \end{aligned} \tag{12.1}$$

Similarly, we find

$$\begin{aligned} f^*dz^\gamma =& C^\gamma_{0f}\ \theta_{\mathcal{M}} + C^\gamma_{\alpha f}\ dz^\alpha_{\mathcal{M}} + C^\gamma_{\bar\alpha f}\ d\overline{z^\alpha}_{\mathcal{M}}, \\ C^\gamma_{0f} =& \frac{\partial f^\gamma}{\partial \theta_{\mathcal{M}}}, \quad C^\gamma_{\alpha f} = \overline{P^\alpha} f^\gamma, \quad C^\gamma_{\bar\alpha f} = P^\alpha f^\gamma. \end{aligned} \tag{12.2}$$

Since $r \circ f = 0$, we also have

$$r_w \circ f\ \frac{\partial f^0}{\partial \theta_{\mathcal{M}}} + r_{\bar w} \circ f\ \frac{\partial \overline{f^0}}{\partial \theta_{\mathcal{M}}} + r_\beta \circ f\ \frac{\partial f^\beta}{\partial \theta_{\mathcal{M}}} + r_{\bar\beta} \circ f\ \frac{\partial \overline{f^\beta}}{\partial \theta_{\mathcal{M}}} = 0. \tag{13.1}$$

$$r_w \circ f\ P^\alpha f^0 + r_{\bar w} \circ f\ P^\alpha \overline{f^0} + r_\beta \circ f\ P^\alpha f^\beta + r_{\bar\beta} \circ f\ P^\alpha \overline{f^\beta} = 0. \tag{13.2}$$

Set

$$W = \frac{\partial f^0}{\partial \theta_{\mathcal{M}}} + \frac{r_\alpha}{r_w} \circ f \frac{\partial f^\alpha}{\partial \theta_{\mathcal{M}}}. \tag{14.1}$$

By the Definition of C_f in (12.1) and (13.1) we find that

$$W + \overline{W} = 2C_f, \quad r_w \circ f\ W + r_{\bar w} \circ f\ \overline{W} = 0. \tag{14.2}$$

Hence $(r_{\bar w} - r_w) \circ f\ W = 2(r_{\bar w} \circ f) C_f$. Therefore

$$C_f = \frac{r_{\bar w} - r_w}{2 r_{\bar w}\ r_w} \circ f (r_w \circ f\ \frac{\partial f^0}{\partial \theta_{\mathcal{M}}} + r_\beta \circ f\ \frac{\partial f^\beta}{\partial \theta_{\mathcal{M}}}). \tag{15}$$

F) We define the CR attaching maps of M as the maps which preserve infinitesimally the defining equation (7) of our CR structure. Namely,

(16) **Definition.** *$f : \mathcal{M} \to M$ is called a CR attaching map of order m when f is a diffeomorphism near 0 and*

$$C^0_{\alpha f} = O(m), \quad C^\gamma_{\bar\alpha f} = O(m), \quad C_f(0) > 0. \tag{16.1}$$

(17) **Proposition.** *Let $f : \mathcal{M} \to M$ be a CR attaching map of order m. Then*

$$P^\alpha f^j = O(m) \quad \text{for } j = 0, 1, ..., n;\ \alpha = 1, ..., n. \tag{17.1}$$

Conversely $f : \mathcal{M} \to M$ satisfying (17.1) is a CR attaching map of order m, provided C_f given by (15) is positive at the origin. We also have

$$r_w \circ f\ \overline{P^\alpha} f^0 + r_\beta \circ f\ \overline{P^\alpha} f^\beta = O(l). \tag{17.2}$$

Proof. Set for an arbitrary $f : \mathcal{M} \to M$

$$W^1_\alpha = P^\alpha f^0 + \frac{r_\beta}{r_w} \circ f\ P^\alpha f^\beta, \quad W^2_\alpha = P^\alpha \overline{f^0} + \frac{r_{\bar\beta}}{r_{\bar w}} \circ f\ P^\alpha \overline{f^\beta}. \tag{18.1}$$

We see by (13.2) and (12.1) that

$$r_w \circ f\ W^1_\alpha + r_{\bar w} \circ f\ W^2_\alpha = 0, \quad W^1_\alpha + W^2_\alpha = C^0_{\bar\alpha f}. \tag{18.2}$$

In the case f is a CR attaching map of order m, we have $W^1_\alpha = O(m)$, $W^2_\alpha = O(m)$. Therefore (17.2) holds. Since $P^\alpha f^\gamma = O(m)$ by (16.1) and $W^1_\alpha = O(m)$, (17.1) also holds. The converse holds, because (17.1) implies $W^1_\alpha = O(m)$ and by the 1st formula in (18.2) we have $W^2_\alpha = O(m)$. Q.E.D.

(19) **Proposition.** *For any $p \in M \subset \mathbf{C}^{n+1}$ there is an attaching map of order 3.*

Proof. We may assume that p is the origin. In view of the theorem of Chern and Moser we may assume that M is given by the equation: $r = 0$, where

$$(20.1) \qquad r = \frac{1}{i}(w - \bar{w}) - \langle z, z \rangle - F(w, x^0), \quad x^0 = \frac{1}{2}(w + \bar{w})$$

where $F \equiv 0$ (mod $(z,\ \bar{z})^4$). Then the map

$$(20.2) \qquad f : \mathcal{M} \ni (z, w) \to (z, w + iF(z, \frac{1}{2}(w + \bar{w}))$$

is a CR attaching map of order 3, because

$$(20.3) \qquad f^0 \equiv w \quad (\text{mod } (z,\ \bar{z})^4), \quad f^\alpha \equiv z^\alpha \quad (\text{mod } (z,\ \bar{z})^4).$$

Q.E.D.

G) Let N be a manifold. We denote by $J_0^l(\mathcal{M}, N)$ the space of l-jets at the reference point 0 of maps of $\mathcal{M}$ into N.

(21) **Definition.** *$J \in J_0^l(\mathcal{M}, M)$ is called a CR l-jet when there is a CR attaching map f of order l representing J. Denote by $J_0^l(M)_{CR}$ the space of CR l-jets.*

Since $P^\alpha, \overline{P^\alpha}, \partial/\partial\theta_{\mathcal{M}}$ form a base of $\mathbf{C}T\mathcal{M}$, $J_0^1(\mathcal{M}, \mathbf{C}^{n+1})$ has the standard chart $(.., p^{(0)j}, ..., p_\alpha^{(1)j}, ..., p_{\bar{\alpha}}^{(1)j}, ..., p_0^{(1)j}, ...)$, where $j = 0, 1, ..., n$. Namely, for $J \in J_0^1(\mathcal{M}, \mathbf{C}^{n+1})$ represented by a map $f : \mathcal{M} \to \mathbf{C}^{n+1}$

$$(22) \qquad \begin{aligned} p^{(0)j}(J) &= f^j(0), \quad p_\alpha^{(1)j}(J) = \overline{P^\alpha} f^j(0), \\ p_{\bar{\alpha}}^{(1)j}(J) &= P^\alpha f^j(0), \quad p_0^{(1)j}(J) = \frac{\partial f^j}{\partial \theta_{\mathcal{M}}}(0). \end{aligned}$$

$J^1(\mathcal{M}, M) \subset J^1(\mathcal{M}, \mathbf{C}^{n+1})$ is the submanifold defined by

$$(23.1) \qquad p^{(0)} = (p^{(0)1}, ..., p^{(0)n}, p^{(0)0}) \in M,$$

$$(23.2) \qquad \Re(r_w(r^{(0)})p_0^{(1)0} + r_\gamma(p^{(0)})p_0^{(1)\gamma}) = 0,$$

$$(23.3) \quad r_w(p^{(0)})p_\alpha^{(1)0} + r_{\bar{w}}(p^{(0)})\overline{p_{\bar{\alpha}}^{(1)0}} + r_\gamma(p^{(0)})p_\alpha^{(1)\gamma} + r_{\bar{\gamma}}(p^{(0)})\overline{p_{\bar{\alpha}}^{(1)\gamma}} = 0.$$

Note that the map

(24)

$$J \in J_0^1(\mathcal{M},\mathbf{C}^{n+1}) \to (p^{(0)}(J),...,p_{\bar\alpha}^{(1)j}(J),...,\Re(r_w(p^{(0)}(J))p_0^{(1)0}(J)$$
$$+r_{\bar\alpha}(p^{(0)}(J))p_\alpha^{(1)0}(J)),...,r_w(p^{(0)}(J))p_\alpha^{(1)0}(J)+r_\beta(p^{(0)}(J))p_\alpha^{(1)\beta}(J),...)$$
$$\in M\times\mathbf{C}^{n(n+1)}\times\mathbf{R}\times\mathbf{C}^n$$

is of maximal rank. Note also that $C_{\alpha f}^0(0)=0$ is a consequence of $p_\alpha^{(1)j}=0$ and (23.2). In view of (17), it then follows that

(25) **Proposition.** *$J_0^1(M)_{CR}$ is the subspace of $J_0^1(\mathcal{M},M)$ defined by the equations:*

(25.1) $$p_{\bar\alpha}^{(1)j}=0,\quad C^{(1)}>0,$$

where $C^{(1)}$ is defined by

(26) $$C^{(1)}=\frac{r_{\bar w}-r_w}{2r_{\bar w}r_w}(p^{(0)})\{r_w(p^{(0)})p_0^{(1)0}+r_\gamma(p^{(0)})p_0^{(1)\gamma}\}.$$

(27) **Proposition.** *For any $p\in M$, complex numbers C_j^γ ($\gamma=1,...,n;j=0,1...,n$), and $C>0$ there is unique $J\in J_0^1(M)_{CR}$ such that*

(28) $$p^{(0)}(J)=p,\quad p_j^{(1)\gamma}(J)=C_j^\gamma,\quad p_\alpha^{(1)0}(J)=-\frac{r_\gamma}{r_w}(p^{(0)})C_\alpha^\gamma,$$
$$p_0^{(1)0}(J)=\frac{2r_{\bar w}}{r_{\bar w}-r_w}(p(0))\,C-\frac{r_\gamma}{r_w}(p(0))\,p_0^{(1)\gamma}(J).$$

We thus have a chart $(x,...,C_j^\gamma,...,C)$ of $J_0^1(M)_{CR}$, called standard.

H) Because of the duality we have for an attaching map f of order 1 at $x\in M$

(29) $$(f_*\overline{P^\alpha})_x=C_{\alpha f}^\gamma(0)(\overline{Q^\gamma})_x,$$
$$(f_*\frac{\partial}{\partial\theta}_{\mathcal{M}})_x=C_{0f}^\gamma(0)(\overline{Q^\gamma})_x+\overline{C_{0f}^\gamma}(0)(Q^\gamma)_x+C_f(0)(\frac{\partial}{\partial\theta}_M)_x.$$

We call $(f_*\overline{P^\alpha})_x,(f_*\frac{\partial}{\partial\theta}_{\mathcal{M}})_x$ the CR frame of order 1 associated to a CR 1-jet $J=j_0^1f$. The space of CR frame of order 1 is diffeomorphic to $J_0^1(M)_{CR}$. The CR coframe $...,\omega_J^j,...$ of order 1 associated to CR 1-jet

J at $x \in M$ is defined as the dual to a CR frame of order 1 associated to J. We then find

$$(30) \quad \omega_J^\alpha = (C^{-1})_\gamma^\alpha(J)((dz_M^\gamma)_x - \frac{C_0^\gamma(J)}{C(J)}(\theta_M)_x), \quad \omega_J^0 = \frac{1}{C(J)}(\theta_M)_x.$$

where $((C^{-1})_\gamma^\alpha(J))$ is the inverse matrix of the matrix $(C_\alpha^\gamma(J))$.

We may regard ω_J^j as a 1-form Ω^j on $J_0^1(M)_{CR}$. Hence using the standard chart

$$(31) \qquad \Omega^\alpha = (C^{-1})_\gamma^\alpha(dz_M^\gamma - \frac{C_0^\gamma}{C}\theta_M), \quad \Omega^0 = \frac{1}{C}\theta_M.$$

Remark. In the case $M = \mathcal{M}$ we see by (17)-(18) §1 that $\Omega^\alpha = (\omega_G)_{[-1]}^\alpha$, $\Omega^0 = (\omega_G)_{([-2]}$.

I) Note that the isotropy group $\mathcal{H}$ at 0 acts on $\mathcal{M}$ as a CR isomorphism group. Hence, when f is a CR attching map of oder l and $h \in H$, $f \circ T_h$ (cf. (13) §1) is a CR attaching map of order l. Therefore we have the action of h on $J_0^1(M)_{CR}$, which we denote by R_h. We then find by (21) §1 and calculation that for $J \in J_0^1(M)_{CR}$

$$(32) \quad C_\alpha^\gamma(R_hJ) = C_\sigma^\gamma(J)\frac{1}{a}u_\alpha^\sigma, \quad C_0^\gamma(R_hJ) = C_0^\gamma(J)\frac{1}{|a|^2} + C_\sigma^\gamma(J)\frac{1}{a}\beta^\sigma,$$

$$(33) \qquad C(R_hJ) = C(J)\frac{1}{|a|^2}.$$

§3. CR coframe of infinitesimal order 2

A) Let $f : \mathcal{M} \to M \subset \mathbf{C}^{n+1}$ be a CR attaching map of order m. Then

$$(1) \qquad f^*\theta_M = C_f\theta_{\mathcal{M}} + O(m). \quad \text{Hence}$$

$$(2) \qquad \begin{aligned} f^*d\theta_M =& C_f\, d\theta_{\mathcal{M}} + dC_f \wedge \theta_{\mathcal{M}} + O(m-1) \\ =& iC_f \, < dz_{\mathcal{M}}, dz_{\mathcal{M}} > + dC_f \wedge \theta_{\mathcal{M}} + O(m-1). \end{aligned}$$

Since $f^*dz^\alpha = C_{\alpha f}^\gamma \, dz_{\mathcal{M}}^\gamma + C_{0f}^\alpha \, \theta_{\mathcal{M}} + O(m)$, we find that

$$(3) \qquad dz_{\mathcal{M}}^\gamma = C_\alpha^{\gamma f}\{f^*dz^\alpha - C_{0f}^\alpha \, \theta_{\mathcal{M}}\} + O(m),$$

where $(C^{\alpha f}_{\gamma})$ is the inverse matrix of $(C^{\alpha}_{\gamma f})$. Therefore

$$\begin{aligned}f^*d\theta_M = iC_f\underline{h}_{\gamma\bar{\sigma}}C^{\gamma f}_{\alpha}\overline{C^{\sigma f}_{\beta}}\{f^*dz^{\alpha}\wedge f^*\overline{dz^{\beta}} + C^{\alpha}_{0f}f^*\overline{dz^{\beta}} \\ - \overline{C^{\beta}_{0f}}f^*dz^{\alpha})\wedge\theta_{\mathcal{M}}\} + dC_f\wedge\theta_{\mathcal{M}} + O(m-1).\end{aligned} \tag{4}$$

For a function l on M we have by taking d of (9) §2

$$\frac{\partial l}{\partial\theta_M}d\theta_M = -d(\overline{Q^{\alpha}}l)\wedge dz^{\alpha} - d(Q^{\alpha}l)\wedge\overline{dz^{\alpha}} - d\frac{\partial l}{\partial\theta_M}\wedge\theta_M. \tag{5}$$

Applying (9) §2 again when l is replaced $Q^{\alpha}l$, etc. we find that

$$\begin{aligned}\frac{\partial l}{\partial\theta_M}d\theta_M =&[Q^{\beta},\overline{Q^{\alpha}}]l\; dz^{\alpha}\wedge\overline{dz^{\beta}} \\ &-\{[\overline{Q^{\alpha}},\frac{\partial}{\partial\theta_M}]l\; dz^{\alpha} + [Q^{\alpha},\frac{\partial}{\partial\theta_M}]l\;\overline{dz^{\alpha}}\}\wedge\theta_M.\end{aligned} \tag{6}$$

Applying the above in the case $l=(w+\bar{w})/2$, we find by (8.2) §2 that

$$d\theta_M = ic^{\beta\bar{\alpha}}dz^{\alpha}\wedge\overline{dz^{\beta}} + (\overline{c^{\alpha}}dz^{\alpha} + c^{\alpha}\overline{dz^{\alpha}})\wedge\theta_M, \tag{7}$$

where

$$c^{\beta\bar{\alpha}} = \frac{1}{2i}[Q^{\beta},\overline{Q^{\alpha}}](w+\bar{w}),\quad c^{\alpha} = \frac{1}{2}[\frac{\partial}{\partial\theta_M},Q^{\alpha}](w+\bar{w}). \tag{8}$$

Hence

$$\begin{aligned}f^*d\theta_M =&ic^{\beta\bar{\alpha}}\circ f\; f^*dz^{\alpha}\wedge f^*\overline{dz^{\beta}} \\ &+ (\overline{c^{\alpha}}\circ f\; f^*dz^{\alpha} + c^{\alpha}\circ f\; f^*\overline{dz^{\alpha}})\wedge f^*\theta_M.\end{aligned} \tag{9}$$

Comparing the above with (4), we find that

$$c^{\beta\bar{\alpha}}\circ f = C_f\underline{h}_{\gamma\bar{\sigma}}C^{\gamma f}_{\alpha}\overline{C^{\sigma f}_{\beta}} + O(m-1), \tag{10}$$

$$C_f c^{\alpha}\circ f = ic^{\alpha\bar{\beta}}\circ f\; C^{\beta}_{0f} + \overline{C^{\beta f}_{\alpha}}P^{\beta}C_f + O(m-1). \tag{11}$$

B) Denote by $J^2_0(M)$ the space of 2-jets of maps f of neighborhoods of 0 in $\mathcal{M}$ into M. When $\tilde{J}=j^2_0(f)$, we set

(12)

$$p^{(2)j}_{\alpha\beta}(\tilde{J})=\overline{P^{\alpha}}\;\overline{P^{\beta}}f^j(0),\;\; p^{(2)j}_{\bar{\alpha}\beta}(\tilde{J})=P^{\alpha}\overline{P^{\beta}}f^j(0),\;\; p^{(2)j}_{\alpha 0}(\tilde{J})=\overline{P^{\alpha}}\frac{\partial}{\partial\theta_{\mathcal{M}}}f^j(0),$$

$$p^{(2)j}_{00}(\tilde{J})=\frac{\partial^2}{\partial\theta^2_{\mathcal{M}}}f(0),\quad C^{(2)}_{\bar{\alpha}}(\tilde{J})=P^{\alpha}C_f(0),\quad C^{(2)}_{\alpha}(\tilde{J})=\overline{P^{\alpha}}C_f(0).$$

Denote by $J_0^2(M)_{CR}$ the space of 2-jets of CR attaching map to M of order 2. We set

$$E_1 = \rho_1^2(J_0^2(M)_{CR}) \subset J_0^1(M)_{CR}. \tag{13}$$

Let $(c_{\beta\bar{\alpha}})$ be the inverse matrix of $(c^{\beta\bar{\alpha}})$. We have by (10)-(11)

(14) **Proposition.** *For* $J = \rho_1^2(J^2) \in E_1$ *with* $J^2 \in J_0^2(M)_{CR}$

$$\frac{1}{C(J)} p_\gamma^{(1)\alpha}(J) \underline{h}^{\gamma\bar{\sigma}} \overline{p_\sigma^{(1)\beta}(J)} = c_{\beta\bar{\alpha}}(p^{(0)}(J)), \tag{15}$$

$$C_\beta^{(2)}(J^2) = p_\beta^{(1)\sigma}(J)\{ic^{\alpha\bar{\sigma}}(p^{(0)}(J))\overline{p_0^{(1)\alpha}(J)} + \overline{c^\sigma}(p^{(0)}(J))C(J)\}. \tag{16}$$

The action of H on $J_0^1(M)_{CR}$ (cf. (35)-(36) §2) preserves E_1. We find by (36) §2 that H acts transitively on the subspace of $J_0^1(M)_{CR}$ defined by the equation: In terms of the standard chart $(x, ..., C_j^\alpha, ..., C)$ of $J^1(M)_{CR}$

$$C_\gamma^\alpha\ \underline{h}^{\gamma\bar{\sigma}} \overline{C_\sigma^\beta} = C\ c_{\beta\bar{\alpha}}(x). \tag{17}$$

In view of (16) we conclude that

(18) **Proposition.** E_1 *is the subspace of* $J_0^1(M)_{CR}$ *defined by the equation* (17).

C) We also find that the subgroup H_1 of H which acts as the identity transformation is given by

$$a = 1, \quad u = I, \quad \beta = 0. \tag{19}$$

Hence H_1 is a 1 dimensional subgroup parametrized by

$$s = \Re\frac{b}{a}. \tag{20}$$

Therefore E_1 is a principal bundle with the structure group H/H_1.

We wish to define the CR frame bundle E by the following diagram:

$$\begin{array}{ccc} J^1(M)_{CR} & \leftarrow & \tilde{J}^2 \\ \uparrow & & \downarrow \\ E_1 & \leftarrow & E \end{array} \tag{21}$$

where $\tilde{J}^2_{CR}$ is a suitable subspace of $J^2(M)_{CR}$. (22) §1 and (20) suggest that we use as the above downward arrow the map

$$\tilde{\rho}\ :\ J^2 \to p^{(2)}_{\sharp}(J^2) = -\frac{1}{n}\Re(C^{-1})^{\gamma}_{\alpha}(J)p^{(2)\alpha}_{0_\gamma}(J^2). \tag{22}$$

D) We justify the above choice.

Since $p^{(2)}_{\sharp}$ may be regarded as a small deformation of $\Re(b/a)$ by (22) §1, $\tilde{\rho}$ is a projection. It remains to show that H acts on E making E a principal H-bundle. We define $\tilde{J}^2_{CR}$ as the space of 2-jets representable by a CR attaching map of order 3. We need to show that $p^{(2)}_{\sharp}(R_h\tilde{J})$ is a function of $p^{(2)}_{\sharp}(\tilde{J})$ and of h , provided $\tilde{J} \in \tilde{J}^2(M)_{CR}$.

We find by (16) §2 that for $f\ :\ \mathcal{M} \to M$

$$\begin{aligned}\frac{\partial}{\partial\theta_{\mathcal{M}}}\overline{P^{\gamma}}(f^{\alpha}\circ T_h)(x) = \{&\frac{\partial}{\partial\theta_{\mathcal{M}}}(\overline{P^{\gamma}}T^{\sigma}_h)(x)\}(\overline{P^{\sigma}}f^{\alpha})(T_hx)\\ &+(\overline{P^{\gamma}}T^{\sigma}_h)(x)\frac{\partial}{\partial\theta_{\mathcal{M}}}\{(\overline{P^{\sigma}}f^{\alpha})\circ T_h\}(x).\end{aligned} \tag{23}$$

We apply (16) §2 to $\frac{\partial}{\partial\theta_{\mathcal{M}}}\{(\overline{P^{\sigma}}f^{\alpha})\circ T_h\}(x)$ in the case $N = M = \mathcal{M}$ and (X, l, f) is $(\partial/\partial\theta_{\mathcal{M}}, \overline{P^{\sigma}}f^{\alpha}, T_h)$. We then find by (21)-(22) §2

(24)

$$\begin{aligned}p^{(2)\alpha}_{0\gamma}(R_h\tilde{J}) =&\frac{1}{a}u^{\sigma}_{\gamma}\{\frac{1}{|a|^2}p^{(2)\alpha}_{0\sigma}(\tilde{J}) + \frac{1}{a}\beta^{\mu}p^{(2)\alpha}_{\sigma\mu}(\tilde{J}) + i\underline{h}_{\sigma\bar{\mu}}\frac{1}{\bar{a}}\overline{\beta^{\mu}}C^{\alpha}_0(J)\}\\ &+\{-\frac{b}{a}\frac{1}{a}u^{\sigma}_{\gamma} + i\underline{h}_{\mu\bar{\nu}}u^{\mu}_{\gamma}\frac{1}{a}\beta^{\sigma}\overline{\beta^{\nu}}\}C^{\alpha}_{\sigma}(J).\end{aligned}$$

Therefore it is enough to show that $p^{(2)\alpha}_{\sigma\gamma}(\tilde{J})$ is a function on E_1, provided $\tilde{J}$ is represented by an attaching map of order 3.

By (10) we have for a CR attaching map f of order 3

$$C_f\ c_{\phi\bar{\alpha}}\circ f = P^{\nu}\overline{f^{\phi}}\ \underline{h}^{\gamma\bar{\nu}}\ \overline{P^{\gamma}}f^{\alpha} + O(2). \tag{25}$$

Applying $\overline{P^{\sigma}}$, we find that

(26)

$$\begin{aligned}&(\overline{P^{\sigma}}C_f)c_{\phi\bar{\alpha}}\circ f + C_f\overline{P^{\sigma}}(c_{\phi\bar{\alpha}}\circ f)\\ &\quad= (\overline{P^{\sigma}}\ \overline{P^{\gamma}}f^{\alpha})\underline{h}^{\gamma\bar{\nu}}(P^{\nu}\overline{f^{\phi}}) + (\overline{P^{\gamma}}f^{\alpha})\underline{h}^{\gamma\bar{\nu}}(\overline{P^{\sigma}}P^{\nu}\overline{f^{\phi}}) + O(1).\end{aligned}$$

Hence we see by (16)

(27)

$$p^{(2)\alpha}_{\sigma\gamma}(\tilde{J})\underline{h}^{\gamma\bar{\nu}}\overline{C^{\phi}_{\nu}}(J) = C^{(2)}_{\sigma}(\tilde{J})c_{\phi\bar{\alpha}}(x)+C(J)C^{\nu}_{\sigma}(J)\overline{Q^{\nu}}c_{\phi\bar{\alpha}}(x)+iC^{\alpha}_{\sigma}(J)\overline{C^{\phi}_0}(J).$$

In view of (16) we now conclude that $p_{\gamma\sigma}^{(2)\alpha}$ is a function on E_1 and consequently H acts on the space E.

We write down the formula for the operation of H on $p_\sharp^{(2)}$. Since $C_\gamma^\alpha(R_h J) = C_\sigma^\alpha(J)u_\gamma^\sigma/a$, we find

$$
\begin{aligned}
(28)\qquad & (C^{-1})_\alpha^\gamma(R_h J)p_{0\gamma}^{(2)\alpha}(R_h\tilde{J}) = (C^{-1})_\alpha^\gamma(J)\{\frac{1}{|a|^2}p_{0\gamma}^{(2)\alpha}(\tilde{J}) \\
& +\frac{1}{a}\beta^\sigma p_{\gamma\sigma}^{(2)\alpha}(\tilde{J}) + i\underline{h}_{\gamma\bar{\sigma}}\frac{1}{\bar{a}}\overline{\beta^\sigma}C_0^\alpha(J)\} - n\frac{b}{a} + i < \beta, \beta > .
\end{aligned}
$$

Since $C(C^{-1})_\mu^\nu = \underline{h}^{\nu\bar{\gamma}}\overline{C_\gamma^\sigma}c^{\sigma\bar{\mu}}$ we have on the other hand
(29)

$$
p_{\gamma\sigma}^{(2)\alpha}(\tilde{J})(C^{-1})_\alpha^\gamma(J) = \frac{n}{C(J)}C_\sigma^{(2)}(\tilde{J})+C_\sigma^\gamma(J)\{c^{\tau\bar{\alpha}}\overline{Q^\gamma}c_{\tau\bar{\alpha}}(x)+i\frac{\overline{C_0^\alpha}(J)}{C(J)}c^{\alpha\bar{\gamma}}\}.
$$

We then find after some cancellation
(30)

$$
p_\sharp^{(2)}(R_h\tilde{J}) = \frac{1}{|a|^2}p_\sharp^{(2)}(\tilde{J})+\Re\frac{b}{a}-\Re\frac{1}{a}\beta^\alpha\{\frac{C_\alpha^{(2)}(\tilde{J})}{C(J)}+\frac{1}{n}C_\alpha^\gamma(J)(\overline{Q^\gamma}c_{\sigma\bar{\mu}}(x))c^{\sigma\bar{\mu}}(x)\}.
$$

Therefore

$$
(30)\quad \underline{p_\sharp^{(2)}} = \frac{1}{|a|^2}p_\sharp^{(2)} + \Re\frac{b}{a} - \Re\frac{1}{a}\beta^\alpha C_\alpha^\gamma\{\overline{c^\gamma} + ic^{\sigma\bar{\gamma}}\frac{\overline{C_0^\sigma}}{C} + \frac{1}{n}c^{\sigma\bar{\mu}}(\overline{Q^\gamma}c_{\sigma\bar{\mu}})\}.
$$

§4. The normal CR Cartan Connections

Let $\omega : TE \to \mathbf{g}$ be a Cartan connection on the CR frame bundle E.

A) ω is called a CR Cartan connection (cf. (31) §2) when

$$
(1)\qquad \omega_{[-1]}^\alpha = \Omega^\alpha, \quad \omega_{[-2]} = \Omega^0.
$$

Let $U = \{(x)\} = \{(z, x^0)\}$ be a chart open set of M. In terms of a local trivialization $U \times H$ of E we have an expression :

$$
(2)\qquad \omega = Ad(h^{-1})w + \omega_H,
$$

where w is a $\mathbf{g}$-valued 1-form on U and ω_H is the Maurer-Cartan form of H regarded as a $\mathbf{h}$-valued 1-form. Its curvature form has the expression:

$$
(3)\qquad K = d\Omega + \frac{1}{2}[\Omega, \Omega] = Ad(h^{-1})k, \quad \text{where} \quad k = dw + \frac{1}{2}[w, w].
$$

B) A local trivialization of E over U is given, using a section $J : U \to E$, by

$$U \times H \ni (x,h) \to R_h J(x) \in E. \tag{4}$$

We find by (31) §2 that ω in (2) is a CR Cartan connection when
(5)

$$w^{\alpha}_{[-1]}(x) = (C^{-1})^{\alpha}_{\gamma}(x)(dz^{\gamma} - \frac{C^{\gamma}_0(x)}{C(x)}\theta_M), \quad w_{[-2]} = \frac{1}{C(x)}\theta_M, \quad \text{where}$$

$$J(x) = (..., C^{\alpha}_j(x), ..., C(x), p^{(2)}_{\sharp}(x)) \tag{6}$$

is the standard chart expression of $J(x)$. We see by the above that we have to determine $w_\pi, w_\mu, w_{\mathbf{su}}, w_{[1]}, w_{[2]}$ (cf (11) §1) to determine a CR Cartan connection. We put curvature conditions so that we have CR Cartan connections unique up to isomorphism.

C) As we obtained (20) §1 we find that k in (3) has the expression:

$$k_{[-2]} = dw_{[-2]} - i\langle w_{[-1]}, w_{[-1]}\rangle - 2w_\pi \wedge w_{[-2]}. \tag{7.1}$$

$$\begin{aligned} k_{[-1]} = dw_{[-1]} &+ \{w_{\mathbf{su}} - (w_\pi + \frac{n+2}{n} i w_\mu)I\} \wedge w_{[-1]} \\ &+ w_{[1]} \wedge w_{[-2]}, \end{aligned} \tag{7.2}$$

$$k_\pi = dw_\pi - \Im\langle w_{[-1]}, w_{[1]}\rangle + w_{[2]} \wedge w_{[-2]}, \tag{7.3}$$

$$k_\mu = dw_\mu + \Re\langle w_{[-1]}, w_{[1]}\rangle, \tag{7.4}$$

$$\begin{aligned} k_{\mathbf{su}} = dw_{\mathbf{su}} &+ w_{\mathbf{su}} \wedge w_{\mathbf{su}} + i w_{[1]} \wedge w^*_{[-1]} \\ &- i w_{[-1]} \wedge w^*_{[1]} + \frac{2}{n} i \Re\langle w_{[-1]}, w_{[1]}\rangle, \end{aligned} \tag{7.5}$$

$$k_{[1]} = dw_{[1]} + i(w_{\mathbf{su}} + (w_\pi - \frac{n+2}{n} i w_\mu)I) \wedge w^*_{[1]} + w_{[-1]} \wedge w_{[2]}, \tag{7.6}$$

$$k_{[2]} = dw_{[2]} + i < w_{[1]}, w_{[1]} > +2w_\pi \wedge w_{[2]}. \tag{7.7}$$

D) In order to carry out the program mentioned at the end of B), we set

$$\underline{C} = \text{the matrix } (C^{\alpha}_{\beta}(x)), \quad \hat{C} = (..., C^{\alpha}_0(x), ...). \tag{8}$$

We also omit x in $C(x)$, etc. We see by (7) §3, (10) §3, and (5) that

(9) $$d\theta_M = iC\langle w_{[-1]}, w_{[-1]}\rangle - 2\Re(ic^{\gamma\bar{\alpha}}C_0^{\alpha} - Cc^{\gamma})\overline{dz^{\gamma}} \wedge w_{[-2]}.$$

We then find that

(10) $$dw_{[-2]} - i\langle w_{[-1]}, w_{[-1]}\rangle - q^0 \wedge w_{[-2]} = 0, \quad dw_{[-1]} + q\wedge w_{[-1]} + q_{[1]}\wedge w_{[-2]} = 0,$$

(11) $$\begin{aligned} q^0 &= \Re(-\frac{i}{C}c^{\gamma\bar{\alpha}}C_0^{\alpha} + c^{\gamma})\overline{dz^{\gamma}} - \frac{1}{2}d\log C, \\ q &= \underline{C}^{-1}d\underline{C} - i\underline{C}^{-1}\hat{C}\otimes w^*_{[-1]}, \\ q_{[1]} &= \underline{C}^{-1}\hat{C}2\Re(-\frac{i}{C}c^{\beta\bar{\alpha}}C_0^{\alpha} + c^{\beta})\overline{dz^{\beta}} + Cd\frac{\underline{C}^{-1}\hat{C}}{C}. \end{aligned}$$

(12) **Lemma.** *We can find a unique set of a complex valued 1-form b^0, an $\mathbf{su}(n)$-valued 1-form $b_{\mathbf{su}}$, a $\mathbf{C}^n$-valued 1-form $b_{[1]}$, such that*

(13.1) $$b^0,\ b_{\mathbf{su}},\ b_{[1]} \equiv 0 \pmod{w_{[-1]}, \overline{w_{[-1]}}},$$

(13.2) $$\begin{aligned} &dw_{[-2]} - i\langle w_{[-1]}, w_{[-1]}\rangle - 2\Re b^0 \wedge w_{[-2]} = 0, \\ &dw_{[-1]} + (b_{\mathbf{su}} - b^0 I)\wedge w_{[-1]} + b_{[1]}\wedge w_{[-2]} = 0. \end{aligned}$$

Proof. By using the type with respect to $w_{[-1]}, \overline{w_{[-1]}}$, we check the uniqueness. To show the existence, note by (10) that $d\langle w_{[-1]}, w_{[-1]}\rangle - q^0 \wedge \langle w_{[-1]}, w_{[-1]}\rangle \equiv 0 \pmod{w_{[-2]}}$. We then find

(14.1) $$(dw^{\alpha}_{[-1]})^{(2,0)} - \underline{h}^{\alpha\bar{\beta}}\underline{h}_{\sigma\bar{\gamma}}w^{\sigma}_{[-1]} \wedge \overline{q^{\gamma}_{\beta\bar{\mu}}w^{\mu}_{[-1]}} - (q^0)^{(1,0)} \wedge w^{\alpha}_{[-1]} = 0,$$

where $(q^{\gamma}_{\beta})^{(0,1)} = q^{\gamma}_{\beta\bar{\mu}}\overline{w^{\mu}_{[-1]}}$. On the other hand we see by (10) that

(14.2) $$(dw^{\alpha}_{[-1]})^{(1,1)} + q^{\alpha}_{\sigma\bar{\mu}}\,\overline{w^{\mu}_{[-1]}} \wedge w^{\sigma}_{[-1]} = 0.$$

Therefore we find that (13) is valid when we set

(15) $$\begin{aligned} (b_{\mathbf{u}})^{\alpha}_{\gamma} &= q^{\alpha}_{\gamma\bar{\sigma}}\overline{w^{\sigma}_{[-1]}} - \underline{h}^{\alpha\bar{\beta}}\underline{h}_{\gamma\bar{\nu}}\overline{q^{\nu}_{\beta\bar{\sigma}}}w^{\sigma}_{[-1]}, \\ (b_{\mathbf{su}})^{\alpha}_{\beta} &= (b_{\mathbf{u}})^{\alpha}_{\beta} - (b_{\mathbf{u}})^{\gamma}_{\gamma}\delta^{\alpha}_{\beta}, \quad b^0 = (q^0)^{(1,0)} + (b_{\mathbf{u}})^{\gamma}_{\gamma}, \\ b_{[1]} &\equiv q_{[1]} - \frac{1}{C}\underline{C}^{-1}\frac{\partial\underline{C}}{\partial\theta_M}w_{[-1]} \pmod{w_{[-2]}}. \end{aligned}$$

Q.E.D.

E) For a differential form α we set

(16) $\alpha = \alpha^{+} + \alpha^{(0)} \wedge w_{[-2]}$, where $\alpha^{+}, \alpha^{(0)}$ do not contain $w_{[-2]}$.

By the Lemma we find the followings:

(17) **Proposition.** *$k_{[-2]} = 0$ if and only if $w_{\pi}^{+} = \Re\, b^0$.*

(18) **Proposition.** *Assume that $k_{[-2]} = 0$. Then $k_{[-1]} = 0$ if and only if*

$$w_{\mathbf{su}}^{+} = b_{\mathbf{su}}, \quad w_{\mu}^{+} = \frac{n}{n+2}\Im\, b^0, \quad w_{[1]}^{+} = b_{[1]} + (b_{\mathbf{su}}^{(0)} - (b^0)^{(0)} I) w_{[-1]}.$$

From now on we consider only CR Cartan connections satisfying the conditions in (17) and (18). We next examine conditions $k_{\pi} = 0$, $k_{\mu} = 0$.

By taking the exterior derivative of the first equality in (13.2), we find that

$$(d\Re\, b^0 - \Im\, \langle w_{[-1]}, b_{[1]}\rangle) \wedge w_{[-2]} = 0. \tag{19}$$

Therefore, we have the expression:

$$d\Re\, b^0 - \Im\, \langle w_{[-1]}, b_{[1]}\rangle + b_{[2]} \wedge w_{[-2]} = 0, \quad b_{[2]} = b_{[2]}^{+}. \tag{20}$$

Hence we find that

$$\begin{aligned} k_{\pi} =& \Im\langle w_{[-1]}, b_{[1]} - w_{[1]}\rangle + w_{\pi}^{(0)} i \langle w_{[-1]}, w_{[-1]}\rangle \\ &+ \ (dw_{\pi}^{(0)} + w_{[2]} - b_{[2]} + 2w_{\pi}^{(0)}\Re b^0) \wedge w_{[-2]}. \end{aligned} \tag{21}$$

(22) **Proposition.** *Assume that $k_{[-2]} = k_{[-1]} = 0$. Then $k_{\pi} = 0$ if and only if*

$$w_{[2]}^{+} = b_{[2]} - (dw_{\pi}^{(0)})^{+} - 2w_{\pi}^{(0)}\Re b^0 - \Im\langle w_{[-1]}, w_{[1]}^{(0)}\rangle.$$

We find

(23)

$$\begin{aligned} k_{\mu} = & \ \frac{n}{n+2} d(\Im b^0) + \Re\langle w_{[-1]}, b_{[1]} + (w_{\mathbf{su}}^{(0)} - \frac{n+2}{n} i w_{\mu}^{(0)} I) w_{[-1]}\rangle \\ &+ i w_{\mu}^{(0)} \langle w_{[-1]}, w_{[-1]}\rangle + (dw_{\mu}^{(0)} + \Re\langle w_{[-1]}, w_{[1]}^{(0)}\rangle + w_{\mu}^{(0)} 2\Re b^0) \wedge w^{[-2]}. \end{aligned}$$

By taking the exterior derivative of the 2nd formula in (13.2), we find by (20) that
(24)
$$\{(db_{\mathbf{su}} - id\Im b^0 I)^+ - \Im\langle w_{[-1]}, b^+_{[1]}\rangle I + b^+_{\mathbf{su}} \wedge b^+_{\mathbf{su}} + ib^+_{[1]} \otimes w^*_{[-1]}\} \wedge w_{[-1]} = 0.$$

Then it follows that

(25) $$\frac{n}{n+2}(d\Im b^0)^{(0,2)} - \frac{1}{2}\langle b^{(0,1)}_{[1]}, w_{[-1]}\rangle = 0.$$

Therefore we find the following: Set
(26)
$$(d(\Im b^0))^{(1,1)} = (d\Im b^0)_{\alpha\bar\beta} w^\alpha_{[-1]} \wedge \overline{w\beta_{[-1]}}, \quad \Re b^0 = (\Re b^0)_\alpha w^\alpha_{[-1]} + (\Re b^0)_{\bar\beta}\overline{w^\beta_{[-1]}},$$
$$(d\Im b^0)^{(0)} = \tilde b^0_\alpha w^\alpha_{[-1]} + \tilde b^0_{\bar\alpha}\overline{w^\alpha_{[-1]}}, \quad b^\alpha_{[1]} = b^\alpha_{[1]\gamma} w^\gamma_{[-1]} + b^\alpha_{[1]\bar\beta}\overline{w^\beta_{[-1]}}.$$

(27) **Proposition.** *Assume that* $k_{[-2]} = k_{[-1]} = k_\pi = 0$. *Then* $k_\mu = 0$ *if and only if*

$$w^{(0)}_\mu = \frac{1}{2}\frac{n+2}{n(n+1)}(\Im b_{[1]})^\alpha_\alpha + \frac{i}{2(n+1)}\underline{h}^{\alpha\bar\beta}(d\Im b^0)_{\alpha\bar\beta},$$
$$(w^{(0)}_{\mathbf{su}})^\alpha_\beta = \frac{n}{n+2}\underline{h}^{\alpha\bar\gamma}(d\Im b^0)_{\beta\bar\gamma} - \frac{1}{n+2}\underline{h}^{\kappa\bar\gamma}(d\Im b^0)_{\kappa\bar\gamma}\delta^\alpha_\beta + \frac{1}{2}\underline{h}^{\alpha\bar\kappa}\underline{h}_{\beta\bar\gamma}\overline{b^\gamma_{[1]\kappa}}$$
$$- \frac{1}{2}b^\alpha_{[1]]\beta} + \frac{1}{n}(\Im b_{[1]})^\gamma_\gamma \delta^\alpha_\beta,$$
with $dw^{(0)}_\mu = \tilde w_{\mu\alpha} w^\alpha_{[-1]} + \tilde w_{\mu\bar\alpha}\overline{w^\alpha_{[-1]}}$,
$$w^{(0)\alpha}_{[1]} = -2\underline{h}^{\alpha\bar\beta}\{\tilde w_{\mu\bar\beta} + w^{(0)}_\mu(\Re b^0)_{\bar\beta} + \frac{1}{2}\tilde b^0_{\bar\beta}\}.$$

Finally we put the condition:

(28) $$\operatorname{tr} k_{[2]} = 0,$$

where for a 2-form ϕ

(29) $$\operatorname{tr}\phi = \underline{h}^{\alpha\bar\beta}\phi_{\alpha\bar\beta}, \quad \phi^{(1,1)} = \phi_{\alpha\bar\beta} w^\alpha_{[-1]} \wedge \overline{w^\beta_{[-1]}}.$$

(30) **Proposition.** *tr* $k_{[2]} = 0$ *if and only if*

$$w^{(0)}_{[2]} = \frac{1}{n}\{itr\, dw^+_{[2]} - tr\,\langle w_{[1]}, w_{[1]}\rangle + 2itr\,(w^+_\pi \wedge w^+_{[2]})\}.$$

F) Note by (23)-(24) §1 that

$$k_{[-2]} = k_{[-1]} = k_\pi = k_\mu = \mathrm{tr}\, k_{[2]} = 0 \text{ if and only if} \qquad K_{[-2]} = K_{[-1]} = K_\pi = K_\mu = \mathrm{tr}\, K_{[2]} = 0. \tag{31}$$

(32) **Definition.** *A CR Cartan connection is called normal when its curvature satisfies the above conditions.*

Clearly the normality condition is a globally defined condition. We also see

(33) **Proposition.** *When we fix a chart (z, x^0) and a local cross-section (4), for arbitrary choice of $w_\pi^{(0)}$ there is a unique normal CR Cartan connection. The isomorphism class of the normal CR Cartan connections is unique.*

G) We next discuss the global aspect of the normal CR Cartan connetions.

Fix a chart $x = (z, x^0)$. Beside the local cross-section $J(x)$ given in (4)-(6) consider a new cross-section

$$\underline{J}(x) = R_{h(x)} J(x) \quad \text{for a } H\text{-valued function } h(x). \tag{34}$$

$\underline{J}(x)$ induces a chart $(x, \underline{h})$, which is related to the original chart (x, h) by

$$h = h(x)\underline{h}. \tag{35}$$

A Cartan connection (2) has the two expressions:

$$\omega = \mathrm{Ad}(h^{-1})w(x) + h^{-1}dh = \mathrm{Ad}(\underline{h}^{-1})\underline{w}(x) + \underline{h}^{-1}d\underline{h}. \tag{36}$$

Therefore

$$\underline{w}(x) = \mathrm{Ad}(h(x)^{-1})(w(x) + h(x)^{-1}dh(x)). \tag{37}$$

From now we omit (x) for simplicity. By the above and by (23) §1 we find that

$$\underline{w}_{[-2]} = |a|^2 w_{[-2]}. \tag{38.1}$$

$$\underline{w}^{\alpha}_{[-1]} = a(u^{-1})^{\alpha}_{\gamma} w^{\gamma}_{[-1]} - |a|^2 (u^{-1}\beta)^{\alpha} w_{[-2]}. \tag{38.2}$$

$$\underline{w}_{\pi} = w_{\pi} + \Re ia\langle w_{[-1]}, \beta\rangle - |a|^2 s\, w_{[-2]} + d\log|a|. \tag{39}$$

For a 1-form ϕ set

$$\phi = \phi_{\alpha} w^{\alpha}_{[-1]} + \phi_{\bar{\alpha}} \overline{w^{\alpha}_{[-1]}} + \phi^{(0)} w_{[-2]} = \tilde{\phi}_{\alpha} \underline{w}^{\alpha}_{[-1]} + \tilde{w}_{\bar{\alpha}} \overline{\underline{w}^{\alpha}_{[-1]}} + \tilde{\phi}^{(0)} \underline{w}_{[-2]}. \tag{40}$$

Then

$$\tilde{\phi}_{\alpha} = \phi_{\gamma} \frac{1}{a} u^{\gamma}_{\alpha}, \quad \tilde{\phi}^{(0)} = \frac{1}{|a|^2} \phi^{(0)} + \phi_{\alpha} \frac{1}{a} \beta^{\alpha} + \phi_{\bar{\alpha}} \frac{1}{\bar{a}} \overline{\beta^{\alpha}}. \tag{41}$$

Setting $\underline{w}^{(0)}_{\pi} = \tilde{\underline{w}}^{(0)}_{\pi}$ for simplicity, we then find

$$\begin{aligned} \underline{w}^{(0)}_{\pi} = {} & \frac{1}{|a|^2} w^{(0)}_{\pi} - s + 2\Re w_{\pi\alpha} \frac{1}{a} \beta^{\alpha} + 2\Re \frac{1}{a} \beta^{\alpha} (d\log|a|)_{\alpha} \\ & + \frac{1}{|a|^2} (d\log|a|)^{(0)}. \end{aligned} \tag{42}$$

On the other hand we see by (30) §3, (11), and (17) that

$$\underline{p}^{(2)}_{\sharp} = \frac{1}{|a|^2} p^{(2)}_{\sharp} + s - 2\Re \frac{1}{a} \beta^{\alpha} \{ w_{\pi\alpha} + \frac{1}{2} (d\log C)_{\alpha} + \frac{1}{n} c^{\gamma\bar{\sigma}} \overline{Q^{\alpha}} c_{\gamma\bar{\sigma}} \}. \tag{43}$$

Therefore

$$\underline{w}^{(0)}_{\pi} + \underline{p}^{(2)}_{\sharp} = \frac{1}{|a|^2} (w^{(0)}_{\pi} + p^{(2)}_{\sharp}) + R, \quad \text{where} \tag{44}$$

(45)

$$R = 2\Re \frac{1}{a} \beta^{\alpha} \{ (d\log|a|)_{\alpha} - \frac{1}{2} (d\log C)_{\alpha} - \frac{1}{n} C^{\gamma}_{\alpha} c^{\nu\bar{\sigma}} \overline{Q^{\gamma}} c_{\nu\bar{\sigma}} \} + \frac{1}{|a|^2} (d\log|a|)^{(0)}.$$

Note that we have the standard chart $(\underline{C}, \underline{C}^{\alpha}_0, \underline{C}^{\alpha}_{\gamma})$ induced by the local cross-section $\underline{J}$. We find by (32)-(33) §2 and (30) §3 that

$$\underline{C} = \frac{1}{|a|^2} C, \quad \underline{C}^{\alpha}_0 = \frac{1}{|a|^2} C^{\alpha}_0 + \frac{1}{a} \beta^{\gamma} C^{\alpha}_{\gamma}, \quad \underline{C}^{\alpha}_{\gamma} = \frac{1}{a} C^{\alpha}_{\nu} u^{\nu}_{\gamma}. \tag{46}$$

Set

$$U = (d\log C)^{(0)} - C^{\alpha}_0 c^{\gamma\bar{\sigma}} \overline{Q^{\alpha}} c_{\gamma\bar{\sigma}}. \tag{47}$$

Then we see by calculation that

$$\underline{U} = \frac{1}{|a|^2}U - R. \tag{48}$$

Therefore the condition: $w^0_\pi + p^{(2)}_\sharp + U = 0$ is a globally defined condition. We conclude

(49) **Proposition.** *When we choose*

$$w^{(0)}_\pi = -p^{(2)}_\sharp - U, \tag{50}$$

the normal CR Cartan connection is globally defined.

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The $\bar{\partial}$ equation in N variables, as N varies

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§1. Introduction

In this work we shall be concerned with solving the $\bar{\partial}$ equation in N dimensional balls, and the emphasis will be on understanding how the control that we have on the sup norm of the solution depends on the number of variables. The primary motivation for this line of research comes from the infinite dimensional theory of the $\bar{\partial}$ equation. Indeed, if it turns out that solutions of the N dimensional $\bar{\partial}$ equation can be estimated independently of N, one should expect that by passing to some limit a solution of the infinite dimensional $\bar{\partial}$ equation will be obtained as well. More on this later. However, our topic of the day is also related, perhaps only in spirit, to other areas of mathematics and beyond, where one studies systems with a large number N of degrees of freedom and investigates how properties of the system change as $N \to \infty$. One example would be statistical physics, another algorithmic complexity.

In the next section of the present work we first review the relevant estimates for the $\bar{\partial}$ equation available in the literature. None of them is known to be optimal; on the other hand they all involve N exponentially. In fact, exponential dependence on the dimension seems to be the rule in analysis and geometry, even beyond the theory of the $\bar{\partial}$ equation. This will be discussed at some length in section 2. Nevertheless we shall find one instance (Theorems 2.1 and 2.2) when the exponentially diverging estimates can be converted into dimension free estimates. As a consequence we obtain that on the level of $(0,1)$ forms the equation $\bar{\partial}u = f$ is solvable in pseudoconvex open subsets of the Banach space l^1 of summable sequences. This was already proved in [L1,2] for local resp. global solvability. Our treatment here does overlap with that of [L1], but is simpler. In addition, it gives a stronger result: in Theorem 4.2 the regularity assumption on f is weaker than Hölder continuity, while [L1] dealt with Lipschitz continuous f. This stronger result is sharp in that

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in l^1 mere continuity of f is not sufficient for the solvability of $\bar\partial u = f$, see [L1, Theorem 9.1].

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§2. The estimates

2.1. Rather than studying the $\bar\partial$ equation just in Euclidean balls, we fix $p \in [1, \infty)$ and consider

$$B_{N,p}(R) = B_N(R) = \{z \in \mathbb{C}^N : \|z\|_p < R\}, \text{ where}$$
$$\|z\|_p = \|z\| = \left(\sum_{\nu=1}^{N} |z_\nu|^p\right)^{1/p}, \qquad z = (z_\nu).$$

Given a $k \in [0, \infty)$, $r \in (0, 1]$ and a closed form $f \in C^k_{0,1}(B_N(1))$, we want to solve the equation

$$\bar\partial u = f|B_N(r) \tag{2.1}$$

with estimate

$$|u|_{C^0(B_N(r))} \leq c_N |f|_{C^k(B_N(1))}, \tag{2.2}$$

where c_N is independent of f, but may depend on p, k, r—that we think of as fixed—, and of course on N. The norm on the left hand side of (2.2) is $\sup_{B_N(r)} |u|$. The more general C^k norms on the right must be defined with a little care, since various seemingly natural choices behave somewhat differently as $N \to \infty$. The correct definition is gotten by using the Banach space structure of $(\mathbb{C}^N, \|\ \|_p)$ only, ignoring coordinates. Thus, when $(X, \|\ \|)$ is any Banach space and $\Omega \subset X$ is open, for $0 < k < 1$ and $u : \Omega \to \mathbb{C}$ one writes

$$|u|_{C^k(\Omega)} = \sup_\Omega |u| + \sup_{z \neq \zeta \in \Omega} \frac{|u(z) - u(\zeta)|}{\|z - \zeta\|^k}.$$

For $k \geq 1$, $|u|_{C^k(\Omega)}$ is defined inductively: one thinks of du as a function on $\Omega \times B$, $B \subset X$ the unit ball, and sets $|u|_{C^k(\Omega)} = \sup_\Omega |u| + |du|_{C^{k-1}(\Omega \times B)}$. Similarly, a 1-form f on Ω is a function on $\Omega \times B$, and the $C^k(\Omega \times B)$ norm of this function is what is meant by $|f|_{C^k(\Omega)}$.

Back to (2.1), (2.2), the question is how c_N depends on N—the hope being that it does not. There are various ways to solve (2.1) with estimates: the Hilbert space methods of Hörmander or, in case of smooth boundary, of Kohn; and integral formulas. Integral formulas of Grauert-Lieb, Henkin, Øvrelid, and others directly estimate $|u|_{C^0(B_N(1))}$, especially in the strongly pseudoconvex case $p = 2$, while Hörmander and Kohn only estimate the $L^2(B_N(1))$ norm of a solution, which then has to be converted into sup norm on smaller balls $B_N(r)$, $r < 1$. When one works one's way through the constants that occur, all the above methods give $c_N \approx \gamma^N$ with $\gamma = \gamma(p, k, r) > 1$ for $r < 1$. (For infinite dimensional applications it suffices to consider arbitrarily small but fixed $r > 0$. However, it is of some interest to see what happens to $\gamma(p, k, r)$ as $r \to 1$. The Hilbert space methods yield $\gamma(p, k, r)$ that blows up as $(1 - r)^{-1}$, while integral formulas, at least some of the time, yield $\gamma(p, k, r)$ that is uniformly bounded. For example one can take $\gamma(p, k, r) = 2$ when $p = 1$ or 2.)

2.2. Now an exponentially diverging c_N is not what we were after, but it is noteworthy that three different methods and their variants all produce such constants in (2.2). In fact, looking even beyond the theory of the $\bar\partial$ equation it seems that the natural place for the number of variables is in the exponent. A host of examples suggests the following general if vague principle: *In geometrical and analytical results the number of dimensions appears in the exponent, as c^N (or not at all, if $c = 1$).*

Here are some instances of this principle.

1° Scaling of volume in N dimensions, probably the source of all other examples: if $D \subset \mathbb{R}^N$ and $\lambda > 0$ then $\mathrm{Vol}(\lambda D) = \lambda^N \mathrm{Vol}\ D$.

2° The singularity of the harmonic Green function in N dimensions

$$G(x, y) \sim \mathrm{const}|x - y|^{2-N}, \qquad x \to y.$$

3° Weyl's law for the number $s(x)$ of eigenvalues $< x$ of the Laplacian on a compact N-dimensional Riemannian manifold: $s(x) \sim \mathrm{const}\ x^{N/2}$, $x \to \infty$.

4° With $L \to X$ a holomorphic line bundle over a compact base, the Euler characteristic $\chi(L^{\otimes m})$ is a polynomial in m of degree $\le N = \dim X$.

5° Sobolev's embedding theorem $W^{m,p}(\mathbb{R}^N) \subset C(\mathbb{R}^N)$, provided $m > N/p$. Here it takes a little arguing to get N in the exponent. For instance, when $p = 2$, the Sobolev space $W^{m,p}$ for the critical value $m = N/2$ consists of those $f \in L^2(\mathbb{R}^N)$ whose Fourier transform $\hat f$ satisfies

$$\int |\hat f(\xi)|^2 (1 + |\xi|^2)^{N/2} d\xi < \infty :$$

N indeed appears exponentially.

There are many more examples, but counterexamples as well. One counterexample we have just glossed over occurs in 2° above. Indeed, the constant there also depends on N: its expression contains $\Gamma(N/2-1)$, in addition to N in an exponent. This in itself is nothing to seriously worry about, though. The occurence of $\Gamma(N/2-1)$ has to do with the particular normalization of the translation invariant measure one uses in $\mathbb{R}^N$, so that a different normalization would lead to const $\equiv 1$. This little manipulation, however, exposes the fact that the ratio of the volumes of the unit ball and the unit cube in $\mathbb{R}^N$ also contains N inside the Γ function, an exception to the principle formulated above that should be taken more seriously.

To sum up: even if the dimension does not always appear in the exponent, it seems to do so extensively. This phenomenon definitely deserves some explanation. It indicates that dimensional dependence is subject to generals laws that should be uncovered and analyzed. The analysis in the present paper is of this kind, in the context of the $\bar\partial$ equation. We shall show that in one instance it is possible to start with exponentially diverging c_N in (2.1), (2.2), and convert this into a dimension independent estimate by means of some rather soft analysis.

2.3. The main result is

Theorem 2.1. *Let $p=1$. Given $k>0$ there is a number a such that for any N and any closed $f\in C^k_{0,1}(B_N(1))$ equation (2.1) has a solution u satisfying*

$$|u|_{C^0(B_N(r))}\le a|f|_{C^k(B_N(1))}, \tag{2.3}$$

provided $r=10^{-3}$.

Once (2.3) is known, it is routine to improve it to a similar estimate of $|u|_{C^k(B_N(r))}$, or even $|u|_{C^{k+1}(B_N(r))}$ when $k\notin\mathbb{N}$, at the price of scaling a and r by a dimension independent factor. In some ways Theorem 2.1 is sharp. It would not hold when $k=0$, nor would it hold for all $k>0$ if $p>1$ (the proof of [L1, Theorem 9.1] shows both). On the other hand, it might very well be true for arbitrary p and $k+1>\lceil p\rceil$ (= the least integer $\ge p$).

However, there is a norm better suited to the problem than Hölder norms C^k, which we now proceed to define. Let $D\subset\mathbb{C}^N$ be a bounded domain, with $(x,y)\in\mathbb{C}^N\times\mathbb{C}^N$ associate the map

$$\varphi_{xy}:\overline{B_1(1)}\ni s\mapsto x+sy\in\mathbb{C}^N, \tag{2.4}$$

and let $\Omega = \{(x,y) : \varphi_{xy}(\overline{B_1(1)}) \subset D\}$. Given $f \in C^0_{0,1}(D)$, for each $(x,y) \in \Omega$ try to solve the equation $\bar\partial v_{xy} = \varphi^*_{xy} f$. If this can be done with $v_{xy} \in C^1(B_1(1))$ depending continuously on x, y, put

$$[f]_D = |f|_{C^0(D)} + \inf_{\{v_{xy}\}} \sup\{\|y\|^{-1}|v_{xy}|_{C^1(B_1(1))} : (x,y) \in \Omega,\ y \neq 0\},$$

the inf taken over all families $\{v_{xy}\}$ as above. Otherwise define $[f]_D = \infty$. This norm transforms simply under affine maps $\alpha : \mathbb{C}^n \to \mathbb{C}^N$ of form $\alpha(x) = Ax + b$, A linear and injective:

$$[\alpha^* f]_{\alpha^{-1}D} \leq \|A\| [f]_D. \tag{2.5}$$

Here $\|A\|$ is the operator norm of A induced by the l^1 norms on $\mathbb{C}^n$, $\mathbb{C}^N$. To verify (2.5) note that $\alpha \circ \varphi_{xy} = \varphi_{\alpha(x),Ay}$. Hence with the family v_{xy} in the definition of $[f]_D$, $w_{xy} = v_{\alpha(x),Ay}$ will be a corresponding family for $\alpha^* f$. Since

$$\|y\|^{-1}|w_{xy}|_{C^1(B_1(1))} \leq \|A\|\ \|Ay\|^{-1}|v_{\alpha(x),Ay}|_{C^1(B_1(1))},$$

and moreover

$$|\alpha^* f|_{C^0(\alpha^{-1}D)} \leq \|A\|\ |f|_{C^0(D)},$$

(2.5) follows. In particular, (2.5) applied with homotheties α shows that $[f]$ is homogeneous of order 1, i.e. (diam $D)[f]_D$ is scale invariant. The significance of this norm is that $[\bar\partial u]_D < \infty$ implies u is (locally) C^1, as one easily shows using one variable Cauchy representations for the holomorphic function $\varphi^*_{xy} u - v_{xy}$.

If f is a Hölder continuous form then

$$[f]_{B_N(R)} \leq \text{const}|f|_{C^k(B_N(R))}, \qquad k > 0,$$

with dimension independent constant, since an admissible v_{xy} can be gotten by taking the Cauchy transform of (a C^k extension of) $\varphi^*_{xy} f$. Therefore Theorem 2.1 follows from

Theorem 2.2. *Let $p = 1$. There is a constant a such that for all closed $f \in C^0_{0,1}(B_N(1))$ (2.1) has a solution u with*

$$|u|_{C^0(B_N(r))} \leq a[f]_{B_N(1)}, \quad r = 10^{-3}. \tag{2.6}$$

Moreover, u can be chosen to depend linearly on f.

As explained above, u will be C^1 when the right hand side of (2.6) is finite. Conversely, if $\bar\partial u = f$ has a solution $u \in C^1(B_N(1))$ then $[f]_{B_N(1)} < \infty$: indeed, one can take $v_{xy} = \varphi^*_{xy} u - u(x)$.

Since from this point on only C^0 norms will matter, we shall abbreviate $|\ |_{C^0(D)} = |\ |_D$. We shall also drop the superscript from $C^0(D)$, $C^0_{0,1}(D)$. Finally, we shall write $B_N = B_N(1)$.

§3. Proofs

3.1. To prove Theorem 2.2 we shall start with the exponentially diverging estimate (2.2), where $c_N = \gamma^N$, and, as promised, we shall convert it into a dimension independent estimate. While standard by now, the proof of (2.2) is not easy: whether derived by Hilbert space techniques or by Cauchy–Fantappiè formulas, it requires serious analysis. In comparison, conversion to a dimension independent estimate will be smooth sailing, involving some combinatorics and some routine analysis of the one dimensional $\bar\partial$ operator. The only nonstandard analytical component concerns a certain property of holomorphic functions in $B_N(R)$, to which we now turn.

For the rest of the paper, $p = 1$. Let $\#z$ denote the number of nonzero coordinates of $z \in \mathbb{C}^N$.

Theorem 3.1. *Suppose $h \in \mathcal{O}(B_N(R))$ satisfies $|h(z)| \le q^{\#z}$ with some $q > 1$. Then*

$$(3.1) \qquad |h(z)| \le \frac{R}{R - eq\|z\|}, \qquad \text{if} \quad eq\|z\| < R.$$

It is here that an estimate, exponential in dimension, is turned into a dimension independent one. Indeed, the assumption means that $\sup_P |h| \le q^{\dim P}$ for each coordinate plane P; and one concludes that near 0 $h(z)$ can be bounded irrespective of the dimension of the coordinate plane in which z sits.

Proof. We shall assume $R = 1$; the general case will then follow by a substitution $z = Rz'$. Expand h in a homogeneous series $\sum h_m$, where

$$h_m(z) = \int_0^1 h(e^{2\pi i t} z) e^{-2\pi i m t}\, dt, \qquad m = 0, 1, \ldots.$$

Clearly $|h_m(z)| \le q^{\#z}$ if $z \in B_N$. With each h_m associate the symmetric m-linear form

$$(3.2) \qquad P_m(z^1, \ldots, z^m) = \frac{1}{2^m m!} \sum_{\epsilon_j = \pm 1} \epsilon_1 \ldots \epsilon_m h_m\Big(\sum_{j=1}^m \epsilon_j z^j\Big),$$

then $h_m(z) = P_m(z, \dots, z)$. If each z^j is a possibly rotated basis vector of form $(0, \dots, e^{i\theta}, \dots, 0)$ and $z = \sum \epsilon_j z^j$, then $\|z\|$, $\#z \le m$. Hence $|h_m(z)| \le q^m m^m$ and (3.2) implies

$$|P_m(z^1, \dots, z^m)| \le q^m m^m / m! \le e^m q^m.$$

The same must hold if each z^j is in the convex hull of rotated basis vectors, i.e. whenever $\|z^j\| \le 1$. (It is here that $p = 1$ is essential.) This in turn implies $|h_m(z)| \le e^m q^m \|z\|^m$, and (3.1) follows.

The theorem would be outright false if $p > 1$, as $h(z) = \sum z_\nu$ shows.

3.2. The point of departure in the proof of Theorem 2.2 is the estimate from [L1, Corollary 3.2], a simple consequence of Hörmander's L^2 estimate [Ho, Theorem 4.4.2].

Proposition 3.2. *If $f \in C_{0,1}(B_n(R))$ is closed, $\bar\partial u = f$ has a solution $u \in C(B_n(R))$ that satisfies*

$$\begin{aligned} |u(z)| &\le 2(1 + 2\sqrt{n}) R \left(\frac{R}{R - \|z\|} \right)^n |f|_{B_n(R)} \\ &\le 3R \left(\frac{2R}{R - \|z\|} \right)^n |f|_{B_n(R)}. \end{aligned}$$

In particular u can be chosen to be the solution with minimal $L^2(B_n(R))$ norm, in which case it will depend linearly on f.

First we shall improve this to an estimate that is still exponential but in $\#z$ rather than in n:

Proposition 3.3. *If $f \in C_{0,1}(B_N(R))$ is closed, the equation $\bar\partial u = f$ has a solution $u \in C(B_N(R))$ that satisfies*

$$|u(z)| \le 3R \left(\frac{5R}{R - \|z\|} \right)^{\#z} |f|_{B_N(R)}. \tag{3.3}$$

Again, u can be chosen to depend linearly on f.

Proof. We shall take $R = 1$. For a subset $\mathcal{P} \subset \{1, \dots, N\}$ let $P = \{z \in \mathbb{C}^N : z_\nu = 0 \text{ if } \nu \notin \mathcal{P}\}$ denote the corresponding coordinate plane and $B_{\mathcal{P}} = B_N \cap P$; and similarly with $\mathcal{Q}$, Q. Let π_ν denote the projection of $\mathbb{C}^N$ on the ν'th coordinate hyperplane, so that $\prod_{\nu \notin \mathcal{P}} \pi_\nu$ is projection on P.

By Proposition 3.2 for each $\mathcal{P}$ there is a $u_{\mathcal{P}} \in C(B_{\mathcal{P}})$ solving $\bar{\partial} u_{\mathcal{P}} = f|B_{\mathcal{P}}$ such that

$$|u_{\mathcal{P}}(z)| \leq 3 \left(\frac{2}{1 - \|z\|} \right)^{|\mathcal{P}|} |f|_{B_N}. \tag{3.4}$$

If there were a $u \in C(B_N)$ with $u|B_{\mathcal{P}} = u_{\mathcal{P}}$ for all $\mathcal{P}$, this u would satisfy (3.3). While there is no reason for such a u to exist, there is a simple way to produce u for which $u|B_{\mathcal{P}} \approx u_{\mathcal{P}}$.

Quite generally, suppose we are given a system of l-forms $u_{\mathcal{P}} \in C_l(B_{\mathcal{P}})$, $\mathcal{P} \subset \{1, \dots, N\}$. Define

$$u = \sum_{\mathcal{P}} \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*)(\prod_{\nu \notin \mathcal{P}} \pi_\nu^*) u_{\mathcal{P}} \in C_l(B_N). \tag{3.5}$$

We shall need the following properties of this operation.

1° If $u_{\mathcal{P}} = v|B_{\mathcal{P}}$ with some $v \in C_l(B_N)$ then $u = v$.

2° The operation (3.5) commutes with $\bar{\partial}$.

3° If $\bar{\partial} u_{\mathcal{P}} = f|B_{\mathcal{P}}$ with some $f \in C_{l+1}(B_N)$ then $\bar{\partial} u = f$.

4° If $\mathcal{Q} \subset \{1, \dots, N\}$ then

$$u|B_{\mathcal{Q}} = \sum_{\mathcal{P} \subset \mathcal{Q}} \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*)(\prod_{\nu \notin \mathcal{P}} \pi_\nu^*) u_{\mathcal{P}}|B_{\mathcal{Q}}.$$

To verify 1° replace $u_{\mathcal{P}}$ by v in (3.5) and note that on $C_l(B_N)$

$$\sum_{\mathcal{P}} \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*) \prod_{\nu \notin \mathcal{P}} \pi_\nu^* = \prod_{\nu=1}^{N} (1 - \pi_\nu^* + \pi_\nu^*) = 1.$$

2° is obvious and 3° follows from 1° and 2°. Finally, observe that $\pi_\mu^* \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*) = 0$ when $\mu \in \mathcal{P}$ so that

$$(\prod_{\mu \notin \mathcal{Q}} \pi_\mu^*) u = \sum_{\mathcal{P} \subset \mathcal{Q}} \prod_{\nu \in \mathcal{P}} (1 - \pi_\nu^*)(\prod_{\nu \notin \mathcal{P}} \pi_\nu^*) u_{\mathcal{P}},$$

which is equivalent to 4°.

Now apply (3.5) with our $u_{\mathcal{P}}$ initially constructed. By 3° $\bar{\partial} u = f$. Also, if $z \in B_N$ and $\mathcal{Q} = \{\nu : z_\nu \neq 0\}$ then one can estimate $u(z)$ using 4°, collecting the contributions of $\mathcal{P}$ of fixed cardinality i, and applying

(3.4):

$$
\begin{aligned}
|u(z)| &\le \left(\sum_{\mathcal{P}\subset\mathcal{Q}}\prod_{\nu\in\mathcal{P}}(1-\pi_\nu^*)\Big(\prod_{\nu\notin\mathcal{P}}\pi_\nu^*\Big)|u_{\mathcal{P}}|\right)(z)\\
&\le \sum_{i=0}^{|\mathcal{Q}|}\binom{|\mathcal{Q}|}{i}2^i\cdot 3\left(\frac{2}{1-\|z\|}\right)^i|f|_{B_N} = 3\left(1+\frac{4}{1-\|z\|}\right)^{|\mathcal{Q}|}|f|_{B_N}\\
&\le 3\left(\frac{5}{1-\|z\|}\right)^{\#z}|f|_{B_N},
\end{aligned}
$$

as claimed.

If f of Proposition 3.3 vanishes on a hyperplane, one can choose u that also vanishes there:

Proposition 3.4. *Let $0\le\rho<R$ and suppose a closed $g\in C_{0,1}(B_N(R))$ vanishes when restricted to the hyperplane $z_N=\rho$. Then the equation $\bar\partial w=g$ has a solution $w\in C(B_N(R))$ that vanishes on the hyperplane and satisfies*

$$
|w(z)|\le 4R\left(\frac{5R^2}{(R-\rho)(R-\|z\|)}\right)^{\#z+1}|g|_{B_N(R)}. \tag{3.6}
$$

Proof. Again we take $R=1$. Define

$$
\pi(z)=\frac{z'}{1-z_N}\in\mathbb{C}^{N-1},\qquad z=(z',z_N)\in B_N,
$$

and check that $\|\pi(z)\|\le\|z\|$. If $\epsilon: B_{N-1}\to B_N$ denotes the embedding $\epsilon(z')=((1-\rho)z',\rho)$ then $\pi\circ\epsilon=\mathrm{id}$. By Proposition 3.3 there is a $v\in C(B_N)$ that satisfies $\bar\partial v=g$ and

$$
|v(z)|\le 3\left(\frac{5}{1-\|z|}\right)^{\#z}|g|_{B_N}.
$$

Now $\bar\partial\epsilon^*v=\epsilon^*g=0$ so that $w=v-\pi^*\epsilon^*v$ also solves $\bar\partial w=g$. In addition, w vanishes on the hyperplane $z_N=\rho$. Since

$$
\|\epsilon\pi(z)\|=(1-\rho)\|\pi(z)\|+\rho\le(1-\rho)\|z\|+\rho
$$

and $\#\epsilon\pi(z) \leq \#z + 1$, one can estimate $w(z) = v(z) - v(\epsilon\pi(z))$:

$$|w(z)| \leq 3\left(\left(\frac{5}{1-\|z\|}\right)^{\#z} + \left(\frac{5}{(1-\rho)(1-\|z\|)}\right)^{\#z+1}\right)|g|_{B_N}$$
$$\leq 4\left(\frac{5}{(1-\rho)(1-\|z\|)}\right)^{\#z+1}|g|_{B_N}.$$

3.3. Propositions 3.2, 3.3, and 3.4 would hold for all $p \geq 1$, with modified constants. For the proof of the next, key proposition $p = 1$ is essential.

Proposition 3.5. *If $f \in C_{0,1}(B_N)$ is closed and $Z \in B_N(1/6)$, the equation $\bar\partial U = f|B_N(1/6)$ has a solution $U \in C(B_N(1/6))$ that satisfies*

$$|U(z)| \leq c\|z - Z\| q^{\#z}[f]_{B_N}, \qquad \|z\| < 1/6. \tag{3.7}$$

One can take $q = 16$, $c = 10^5$.

Proof. The claim is true when $N = 0$; we shall prove it for general N by induction. Assume it true with N replaced by $N-1$, and also assume without loss of generality that $Z_N = \rho \geq 0$. We shall borrow π, ϵ from the previous proof.

The inductive hypothesis applied with $f' = \epsilon^* f$ gives a $U' \in B_{N-1}(1/6)$ that satisfies $\bar\partial U' = f'|B_{N-1}(1/6)$ and

$$|U'(z')| \leq c\|z' - \pi(Z)\| q^{\#z'}[f']_{B_{N-1}}, \qquad \|z'\| < 1/6. \tag{3.8}$$

Set $g = f - \pi^* f'$, and apply Proposition 3.4 with $R = 5/6$, to obtain a solution of $\bar\partial w = g|B_N(5/6)$ that satisfies $w(\cdot, Z_N) = 0$ and

$$|w(z)| \leq 65 \cdot q^{\#z}|g|_{B_N(5/6)}, \qquad \|z\| < 1/2, \tag{3.9}$$

with $q = 16$. If

$$U = \pi^* U' + w \tag{3.10}$$

then $\bar\partial U = \pi^* f' + f - \pi^* f' = f$. It remains to estimate U in terms of $[f]_{B_N}$.

By (2.5) $[f']_{B_{N-1}} = [\epsilon^* f]_{B_{N-1}} \leq (1 - Z_N)[f]_{B_N}$. Also

$$\begin{aligned}\|\pi(z) - \pi(\zeta)\| &= \left\|\frac{z' - \zeta'}{1 - \zeta_N} + \frac{(z_N - \zeta_N)z'}{(1-\zeta_N)(1-z_N)}\right\| \\ &\leq \frac{\|z' - \zeta'\| + \|z\|\,|z_N - \zeta_N|}{|1 - \zeta_N|}.\end{aligned} \tag{3.11}$$

Hence (3.8) implies

$$|U'(\pi(z))| \le c(\|z' - Z'\| + |z_N - Z_N|/6)q^{\#z}[f]_{B_N}, \tag{3.12}$$

for $\|z\| < 1/6$. Next $|d\pi|_{B_N(5/6)} \le 6$ by (3.11), whence $|\pi^* f'|_{B_N(5/6)} \le 6|f'|_{B_{N-1}} \le 6[f]_{B_N}$ and $|g|_{B_N(5/6)} \le 7[f]_{B_N}$. Thus by (3.9)

$$|w(z)| \le 460 q^{\#z}[f]_{B_N}, \qquad \|z\| < 1/2. \tag{3.13}$$

This can be refined as follows. If $\|z\| < 1/6$, consider the map

$$\varphi = \varphi_{xy} : \overline{B}_1 \ni s \mapsto ((1 - s/4)\pi(z), s/4) \in B_N(1/2).$$

Then $\bar\partial \varphi^* w = \varphi^* g = \varphi^* f$, since φ maps into a fiber of π. By the definition of $[f]_{B_N}$, there is a $v = v_{xy}$ such that $\bar\partial v = \varphi^* f$ and $|v|_{C^1(B_1)} \le [f]_{B_N}$. Thus $h = \varphi^* w - v$ is holomorphic. Since the hyperbolic distance between $4z_N, 4Z_N \in B_1(2/3) \subset B_1$ is $\le 8|z_N - Z_N|$, Schwarz's lemma implies

$$\begin{aligned} |h(4z_N) - h(4Z_N)| &\le 8|z_N - Z_N|\ |h|_{B_1} \\ &\le 8|z_N - Z_N|(|\varphi^* w|_{B_1} + |v|_{B_1}). \end{aligned}$$

Now $v(4z_N) - v(4Z_N)$ can also be estimated in terms of $z_N - Z_N$, therefore

$$\begin{aligned} w(z) &= w(\varphi(4z_N)) - w(\varphi(4Z_N)) \\ &= h(4z_N) - h(4Z_N) + v(4z_N) - v(4Z_N) \end{aligned}$$

too. All added up one obtains for $\|z\| < 1/6$

$$\begin{aligned} |w(z)| &\le |z_N - Z_N|(8|\varphi^* w|_{B_1} + 12|v|_{C^1(B_1)}) \\ &\le 7 \cdot 10^4 |z_N - Z_N| q^{\#z}[f]_{B_N} \end{aligned}$$

by (3.13), taking into account that $\#\varphi(s) \le \#z + 1$. Thus by (3.10), (3.12)

$$|U(Z)| \le c\left\{\|z' - Z'\| + \left(\frac{1}{6} + \frac{7}{c}10^4\right)|z_N - Z_N|\right\} q^{\#z}[f]_{B_N},$$

and (3.7) follows, provided $c \ge 10^5$.

3.4. Theorem 2.2 is now easily proved.

We shall verify that u given in Proposition 3.3, with $R = 1$, satisfies (2.6). Take an arbitrary $Z \in B_N(r)$ and construct U as in Proposition 3.5. Then with $h = u - U \in \mathcal{O}(B_N(1/6))$ and $z \in B_N(1/6)$ we have

$$|h(z)| \leq (3 \cdot 6^{\#z} + c \cdot 16^{\#z})[f]_{B_N} \leq 2c \cdot 16^{\#z}[f]_{B_N}$$

by (3.3), (3.7). Hence from Theorem 3.1, applied with $R = 1/6$

$$|u(Z)| = |h(Z)| \leq \frac{2c}{1 - 96e\|Z\|}[f]_{B_N} \leq 4c[f]_{B_N}, \qquad \|Z\| < r,$$

and (2.6) holds with $a = 4 \cdot 10^5$.

3.5. Above we have not insisted on sharp constants, and indeed it is possible to obtain somewhat stronger results. First off, if integral formulas are used rather than L^2 estimates, it is possible to show that in Proposition 3.2 the base of the exponential can be taken to be 2. With a little more care in subsequent estimates in Proposition 3.5 one could replace 1/6 by an arbitrary $\rho < 1$ and q by an arbitrary number > 5. As a consequence, r of Theorem 2.2 can be anything $< 1/(5e)$. It would be of interest to know whether one can take r arbitrarily close to 1, or perhaps even equal to 1. I don't believe this is possible, even if $[f]_{B_N}$ is replaced by $|f|_{C^k(B_N)}$, as long as k is fixed. If I am right, phase transition would occur in the Cauchy–Riemann equations: there would be a critical radius $r_0 \in (0,1)$ such that for closed $f \in C^k_{0,1}(B_N)$ the equation $\bar\partial u = f|B_N(r)$ can be solved with dimension independent bounds on u if $r < r_0$, but not if $r > r_0$. In the latter regime $|u|_{B_N(r)}$ would diverge exponentially.

§4. Infinite dimensions

Now we shall see what Theorem 2.2 implies about the $\bar\partial$ equation in infinite dimensions. Let Γ be an arbitrary set and

$$l^1(\Gamma) = \{z : \Gamma \to \mathbb{C} \mid \sum_{\gamma\in\Gamma} |z(\gamma)| = \|z\| < \infty\}.$$

Given an open $D \subset l^1(\Gamma)$ and $f \in C_{0,1}(D)$ closed we ask if there is a $u \in C^1(D)$ that solves $\bar\partial u = f$. (For basics of $\bar\partial$ in Banach spaces see [L1,2].) In [L1] we showed how to pass from finite dimensional estimates for $\bar\partial$ to infinite dimensional results. This can be done with the improved estimates of Theorem 2.2, and we obtain the following result. If $x, y \in l^1(\Gamma)$, $s \in \bar B_1$, define $\varphi_{xy}(s) = x + sy$ as in (2.4), and $\Omega = \{(x,y) : \varphi_{xy}(\bar B_1) \subset D\}$.

Theorem 4.1. *Suppose each $(\xi,\eta)\in\Omega$ has a neighborhood Ω_0 such that if $(x,y)\in\Omega_0$, the equation $\bar\partial v_{xy}=\varphi^*_{xy}f$ can be solved with $v_{xy}\in C^1(B_1)$ depending continuously on x,y. Then in a neighborhood of an arbitrary $z\in D$ the equation $\bar\partial u=f$ is solvable with u a C^1 function.*

Global solvability can also be obtained:

Theorem 4.2. *If Γ is countable and D is pseudoconvex then $\bar\partial u=f$ has a solution $u\in C^1_{\mathrm{loc}}(D)$ if and only if the hypothesis of Theorem 4.1 is satisfied.*

Thus solvability or nonsolvability of $\bar\partial u=f$ depends only on solvability on one dimensional slices.

Theorem 4.2 follows from Theorem 4.1 and the main result of [L2]. Indeed, if f satisfies the hypothesis then D can be covered by open sets V so that some $u_V\in C^1(V)$ solves $\bar\partial u_V=f|V$. By [L2, Theorem 0.1] the holomorphic cocycle (u_V-u_W) is exact, hence of form (h_V-h_W) with $h_V\in\mathcal{O}(V)$. It follows that $u(z)=u_V(z)-h_V(z)$ if $z\in V$ defines the required solution $u\in C^1_{\mathrm{loc}}(D)$.

Very little is known about solving the $\bar\partial$ equation in Banach spaces other than l^1, or for forms of higher degree. Patyi in [P] gives an example of a Banach space in which $\bar\partial u=f$ is not solvable for some closed C^∞ form f. It would be of great interest to explore the solvability of the $\bar\partial$ equation in classical Banach spaces such as l^p, $L^p[0,1]$, $C[0,1]$.

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Levi form of logarithmic distance to complex submanifolds and its application to developability

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§1. Introduction

Let M be a complex manifold of codimension q defined in an open subset U of $\mathbb{C}^n$ and let $\delta_M(P)$ be the Euclidean distance from $P \in U$ to M. Then it is well-known that the function $\varphi := -\log \delta_M$ is, near M, weakly q-convex i.e., the Levi form $L(\varphi)$ of φ has $n-q+1$ nonnegative eigenvalues. Moreover, $L(\varphi)$ is positive semi-definite in the tangential direction of dimension $n-q$ to M (cf. [M2]).

The purpose of the present article is to calculate the Levi form $L(\varphi)$ explicitly near M and to give a necessary and sufficient condition for defining functions of M that $L(\varphi)$ degenerates in the tangential direction (§2, Theorem 1). Such calculation was first done by Matsumoto-Ohsawa [M-O] to study Levi flat hypersurfaces in complex tori of dimension two. As its application, by combining it with the theorem of Fischer-Wu [F-W], developability of a complex submanifold M $(\subset \mathbb{C}^n)$ is characterized by the Levi form of $-\log \delta_M$ if $\dim M = 1$, 2 or $n-1$ (§3, Theorem 2).

§2. Levi form of logarithmic distance

Let r, q and n be integers with $r+q=n$, $r \geq 1$ and $q \geq 1$, and let M be a complex submanifold of dimension r in $\mathbb{C}^n$ defined by

$$M = \{(t, f(t)) \mid t = (t_1, \dots, t_r) \in V\}$$

for open $V \subset \mathbb{C}^r$ and holomorphic $f = (f_1, \dots, f_q) : V \longrightarrow \mathbb{C}^q$. Let $(z, w) = (z_1, \dots, z_r; w_1, \dots, w_q)$ be a (given) coordinate system of $\mathbb{C}^n = \mathbb{C}^r \times \mathbb{C}^q$. By a translation and a unitary transformation of (z, w) if necessary we may assume that $0 = (0, \dots, 0) \in V$ and

$$(1) \qquad f_\mu(0) = 0, \quad \frac{\partial f_\mu}{\partial t_i}(0) = 0$$

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for $1 \leq i \leq r$ and $1 \leq \mu \leq q$. We denote by $\delta_M(z,w)$ the Euclidean distance from $(z,w) \in \mathbb{C}^n$ to M and put $\varphi(z,w) := -\log \delta_M(z,w)$.

We define the (r,r)-matrices $\Phi(w)$ and $F_\mu(t)$, $1 \leq \mu \leq q$, by

$$\Phi(w) := \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(0,w)\right)_{1 \leq i,j \leq r}, \quad F_\mu(t) := \left(\frac{\partial^2 f_\mu}{\partial t_i \partial t_j}(t)\right)_{1 \leq i,j \leq r}$$

and put

$$\mathcal{F}(w) := \sum_{\mu=1}^{q} \overline{F_\mu(0)} w_\mu.$$

$F_\mu(t)$ and $\mathcal{F}(w)$ are symmetric and $\Phi(w)$ is Hermitian.

Then we obtain the following (see [M-O], Lemma for $q = r = 1$).

Theorem 1. *There exists $\varepsilon > 0$ such that*

$$\Phi(w) = \frac{1}{2||w||^2} \overline{\mathcal{F}(w)} \mathcal{F}(w) [E - \overline{\mathcal{F}(w)} \mathcal{F}(w)]^{-1}$$

for $0 < ||w|| < \varepsilon$, where $||w||^2 := \sum_{\mu=1}^{q} |w_\mu|^2$ and E denotes the identity matrix. In particular, two matrices $\Phi(w)$ and $\mathcal{F}(w)$ have the same rank for each w with $0 < ||w|| < \varepsilon$.

Proof. If we put

$$\alpha(z,w,t) := \sum_{i=1}^{r} |z_i - t_i|^2 + \sum_{\mu=1}^{q} |w_\mu - f_\mu(t)|^2 \tag{2}$$

for $(z,w) \in \mathbb{C}^r \times \mathbb{C}^q$ and $t \in V$, then

$$\frac{\partial \alpha}{\partial t_i} = \overline{t_i - z_i} + \sum_{\mu=1}^{q} \frac{\partial f_\mu}{\partial t_i} \{\overline{f_\mu(t) - w_\mu}\} \tag{3}$$

for $1 \leq i \leq r$. By the implicit function theorem we can find C^ω-functions $t_k = t_k(z,w)$, $1 \leq k \leq r$, defined near $(0,0) \in \mathbb{C}^r \times \mathbb{C}^q$ such that

$$\frac{\partial \alpha}{\partial t_i}(z,w,t(z,w)) = 0, \quad \frac{\partial \alpha}{\partial \bar{t}_i}(z,w,t(z,w)) = 0 \tag{4}$$

for $1 \leq i \leq r$ (cf. [M1]). Then by (1) we have $t_k(0,w) = 0$ for $1 \leq k \leq r$.

If we put $\beta(z,w) := \alpha(z,w,t(z,w))$ then $\beta(z,w) = \delta_M(z,w)^2$ near $(0,0) \in \mathbb{C}^r \times \mathbb{C}^q$. By applying (4) and (2) we have

$$\frac{\partial \beta}{\partial z_i} = \frac{\partial \alpha}{\partial z_i} = \overline{z_i - t_i}, \quad \frac{\partial^2 \beta}{\partial z_i \partial \bar{z}_j} = \delta_{ij} - \frac{\partial \bar{t}_i}{\partial \bar{z}_j} \tag{5}$$

for $1 \le i, j \le r$. By differentiating (4) we have

$$(6) \qquad \begin{cases} \dfrac{\partial^2 \alpha}{\partial t_i \partial z_j} + \displaystyle\sum_{k=1}^{r} \left(\dfrac{\partial^2 \alpha}{\partial t_i \partial t_k} \dfrac{\partial t_k}{\partial z_j} + \dfrac{\partial^2 \alpha}{\partial t_i \partial \bar{t}_k} \dfrac{\partial \bar{t}_k}{\partial z_j} \right) = 0 \\ \dfrac{\partial^2 \alpha}{\partial \bar{t}_i \partial z_j} + \displaystyle\sum_{k=1}^{r} \left(\dfrac{\partial^2 \alpha}{\partial \bar{t}_i \partial t_k} \dfrac{\partial t_k}{\partial z_j} + \dfrac{\partial^2 \alpha}{\partial \bar{t}_i \partial \bar{t}_k} \dfrac{\partial \bar{t}_k}{\partial z_j} \right) = 0 \end{cases}$$

and by differentiating (3) we have

$$\frac{\partial^2 \alpha}{\partial t_i \partial z_j} = 0, \qquad \frac{\partial^2 \alpha}{\partial \bar{t}_i \partial z_j} = -\delta_{ij},$$

$$\frac{\partial^2 \alpha}{\partial t_i \partial t_j} = \sum_{\mu=1}^{q} \frac{\partial^2 f_\mu}{\partial t_i \partial t_j} \{\overline{f_\mu(t) - w_\mu}\}, \qquad \frac{\partial^2 \alpha}{\partial t_i \partial \bar{t}_j} = \delta_{ij} + \sum_{\mu=1}^{q} \frac{\partial f_\mu}{\partial t_i} \frac{\partial \bar{f}_\mu}{\partial \bar{t}_j}.$$

Now if $(z, w) = (0, w)$ then $t(0, w) = 0$ and by (1) we have

$$(7) \qquad \frac{\partial^2 \alpha}{\partial t_i \partial t_j}(0, w, 0) = -\sum_{\mu=1}^{q} \frac{\partial^2 f_\mu}{\partial t_i \partial t_j}(0) \bar{w}_\mu, \qquad \frac{\partial^2 \alpha}{\partial t_i \partial \bar{t}_j}(0, w, 0) = \delta_{ij}.$$

If we put

$$(8) \qquad \mathcal{F}(w)_{ij} := \sum_{\mu=1}^{q} \frac{\partial^2 \bar{f}_\mu}{\partial \bar{t}_i \partial \bar{t}_j}(0) w_\mu$$

then $\mathcal{F}(w)_{ij}$ is the (i, j)-component of the symmetric matrix $\mathcal{F}(w)$. By substituting (7) and (8) for (6) we have

$$(9) \qquad \begin{cases} \dfrac{\partial \bar{t}_i}{\partial z_j}(0, w) = \displaystyle\sum_{k=1}^{r} \overline{\mathcal{F}(w)}_{ik} \dfrac{\partial t_k}{\partial z_j}(0, w) \\ \dfrac{\partial t_i}{\partial z_j}(0, w) - \delta_{ij} = \displaystyle\sum_{k=1}^{r} \mathcal{F}(w)_{ik} \dfrac{\partial \bar{t}_k}{\partial z_j}(0, w) \end{cases}$$

and hence

$$\frac{\partial t_i}{\partial z_j}(0, w) - \delta_{ij} = \sum_{k=1}^{r} \mathcal{F}(w)_{ik} \sum_{l=1}^{r} \overline{\mathcal{F}(w)}_{kl} \frac{\partial t_l}{\partial z_j}(0, w).$$

Since $\mathcal{F}(0)$ is the zero matrix, we thus obtain

$$(\partial t_i / \partial z_j(0, w))_{1 \le i, j \le r} = [E - \mathcal{F}(w) \overline{\mathcal{F}(w)}]^{-1}$$

for sufficiently small w and therefore by (5) we have

$$\begin{aligned}(\partial^2\beta/\partial z_i\partial\bar{z}_j(0,w))_{1\le i,j\le r} &= E - [E - \overline{\mathcal{F}(w)}\mathcal{F}(w)]^{-1} \\ &= -\overline{\mathcal{F}(w)}\mathcal{F}(w)[E - \overline{\mathcal{F}(w)}\mathcal{F}(w)]^{-1}.\end{aligned}$$

On the other hand, $\beta = {\delta_M}^2$ and

$$\frac{\partial^2(-\log\delta_M)}{\partial z_i\partial\bar{z}_j} = \frac{1}{2}\left(-\frac{1}{\beta}\frac{\partial^2\beta}{\partial z_i\partial\bar{z}_j} + \frac{1}{\beta^2}\frac{\partial\beta}{\partial z_i}\frac{\partial\beta}{\partial\bar{z}_j}\right).$$

Moreover by (2) and (5) we have $\beta(0,w) = ||w||^2$ and $\partial\beta/\partial z_i(0,w) = 0$ for $1 \le i \le r$. This proves the theorem. Q.E.D.

Remark. The complex Hessian matrix of $\varphi(z,w) := -\log\delta_M(z,w)$ at $(z,w) = (0,w)$, $0 < ||w|| < \varepsilon$, is written as

$$\begin{pmatrix} (\partial^2\varphi/\partial z_i\partial\bar{z}_j) & (\partial^2\varphi/\partial z_i\partial\bar{w}_\nu) \\ (\partial^2\varphi/\partial w_\mu\partial\bar{z}_j) & (\partial^2\varphi/\partial w_\mu\partial\bar{w}_\nu)\end{pmatrix}(0,w) = \begin{pmatrix}\Phi(w) & O \\ O & \Psi(w)\end{pmatrix},$$

where $\Phi(w)$ is the (r,r)-matrix defined as above and $\Psi(w)$ is the (q,q)-matrix defined by $\Psi(w) := (\partial^2(-\log||w||)/\partial w_\mu\partial\bar{w}_\nu)_{1\le\mu,\nu\le q}$.

§3. Developability of complex submanifolds

Let $M = \{(t, f(t)) \mid t \in V\}$ $(\subset \mathbb{C}^n)$ be as in §2. If we put $J(t) := (F_1(t), \dots, F_q(t))$ then ${}^tJ(t)$ is the Jacobian matrix of the Gauss map

$$t \longmapsto \left(\frac{\partial f_1}{\partial t_1}, \dots, \frac{\partial f_1}{\partial t_r}, \dots, \frac{\partial f_q}{\partial t_1}, \dots, \frac{\partial f_q}{\partial t_r}\right).$$

By Fischer-Wu [F-W] (cf. [F-P]), the complex submanifold M of dimension r is developable almost everywhere (i.e., at each point $(t, f(t))$ where $\operatorname{rank} J(t)$ is maximal) if and only if $\operatorname{rank} J(t) < r$ for all t.

As an application of Theorem 1, we can obtain the following.

Theorem 2. *In the case* $\dim M = 1$, 2 *or* $n-1$, *M is developable almost everywhere if and only if the Levi form of* $-\log\delta_M$ *degenerates in the tangential direction at each point near M.*

For the proof we use the following.

Lemma. *Let $A_1, \dots, A_q$ be complex symmetric matrices of degree r and let $w = (w_1, \dots, w_q) \in \mathbb{C}^q$. Then*

(i) $\max_{w\in\mathbb{C}^q} \operatorname{rank}\sum_{\mu=1}^q A_\mu w_\mu \le \operatorname{rank}(A_1, \dots, A_q)$.

(ii) *The equality holds if $r = 1, 2$ or if $q = 1$.*

(iii) *The equality does not hold in general if $r \ge 3$ and $q \ge 2$.*

Proof. (i) is trivial and (ii) is also trivial if $r = 1$ or $q = 1$. (In these cases the matrices $A_1, \dots, A_q$ need not be symmetric.)

If $(2,2)$-matrices $A_1, \dots, A_q$ are symmetric and $\det(\sum_{\mu=1}^{q} A_\mu w_\mu) \equiv 0$ then $\det(A_{\mu_1} w_{\mu_1} + A_{\mu_2} w_{\mu_2}) \equiv 0$ for any pair (μ_1, μ_2) with $1 \leq \mu_1 < \mu_2 \leq q$, and the coefficients of the polynomial of degree 2 with respect to (w_{μ_1}, w_{μ_2}) are all zero. From this it is easy to see that $\operatorname{rank}(A_{\mu_1}, A_{\mu_2}) \leq 1$ for all (μ_1, μ_2) and hence $\operatorname{rank}(A_1, \dots, A_q) \leq 1$, which proves (ii).

(iii) follows from the next example. Q.E.D.

Example. Consider the real symmetric matrices

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\operatorname{rank}(A_1, A_2) = 3$, although $\det(A_1 w_1 + A_2 w_2) \equiv 0$. Therefore, if $M \subset \mathbb{C}^5 = \mathbb{C}^3 \times \mathbb{C}^2$ is the complex submanifold defined by

$$M = \{(z, w) \in \mathbb{C}^5 \mid w_1 = z_1 z_2, w_2 = z_1 z_2 + z_1 z_3\}$$

then $-\log \delta_M$ degenerates in the tangential direction at $(0, w)$ for all w near $0 \in \mathbb{C}^2$, but M is not developable at the origin $(0,0) \in M$.

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Numerical characterisations of hyperquadrics

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Abstract.

Smooth quadric hypersuraces in $\mathbb{P}^{n+1}(\mathbb{C})$ are numerically characterised as the smooth Fano n-folds of length n, *i.e.*, a smooth Fano n-fold X is isomorphic to a hyperquadric if and only if the minimum of the intersection number $(C, -K_X)$ is n, where C runs through the rational curves on X.

Introduction

This article is a supplement to the author's joint paper [2], where we characterised projective n-space as a unique smooth Fano n-fold of length $n+1$, the largest value possible. The purpose of this article is to characterise smooth hyperquadrics as Fano manifolds of the the second largest length n.

Given a Fano manifold X [resp. a pair (X, x_0) of a Fano manifold X and a closed point x_0 on it], we define the (global) *length* $l(X)$ of X [resp. the *local lenghth* $l(X, x_0)$ of (X, x_0)] to be the positive integer

$$\min_{C \subset X} \{(C, -K_X)\},$$

where C runs through the set of the rational curves contained in X [resp. the set of the rational curves such that $x_0 \in C \subset X$].

The local lenght $l(X, x_0)$ is a lower semiconitinuous function in x_0 and the global lenghth $l(X)$ is by definition equal to $\inf_{x_0 \in X} l(X, x_0)$. For a given closed point $x_0 \in X$, it is known that $l(X, x_0) \leq \dim X + 1$, the equality holding if and only if X is projective space [2].

In terms of the notions above, our main result is the following

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Theorem 0.1. *Let X be a smooth Fano variety of dimension $n \geq 3$ defined over an algebraically closed field k of characteristic zero. Then the following three conditions are equivalent:*

(1) *X is isomorphic to a smooth hyperquadric $Q_n \subset \mathbb{P}^{n+1}$.*
(2) *The global length $l(X)$ is n.*
(3) *$\rho(X) = 1$ and $l(X, x_0) = n$ for a sufficiently general point $x_0 \in X$, where $\rho(X)$ stands for the Picard number.*[1]

This simple numerical result involves the preceding characterisations due to Brieskorn [1], Kobayashi-Ochiai [6], and Cho-Sato [3][4] as immediate corollaries. Namely

Theorem 0.2. *For a smooth X Fano n-fold ($n \geq 3$) over $\mathbb{C}$, the three conditions in* (0.1) *are also equivalent to the following four:*

(4) *There is a homotopy equivalence between X and Q_n such that the inducedcohomology isomorphism $\mathrm{H}^2(Q_n, \mathbb{Z}) \to \mathrm{H}^2(X, \mathbb{Z})$ identifies the anticanonical classes.*
(5) *The anticanonical class $c_1(X)$ is divisible by n in* $\mathrm{Pic}(X)$.
(6) *The tangent bundle Θ_X is not ample, but $\wedge^2\Theta_X$ is ample.*[2]
(7) *There is a surjective morphism $Q_m \to X$, $m \geq n$, and $X \not\simeq \mathbb{P}^n$.*

Let us briefly outline our strategy to the proof of Theorem 0.1, the essential part of which is the implication (3) $\Rightarrow$ (1) proved in §3.

Assume that a smooth Fano n-fold X satisfies the condition (3). Because smooth Fano 3-folds with Picard number one are completely classified by Iskovskih [5], we may assume that $n \geq 4$ (this assumption is of course of purely technical nature). Pick up two general points $x_+, x_- \in X$. We consider an (arbitrary) irreducible component $W\langle x_+, x_-\rangle$ of the closed subset

$$\{C \subset X \mid C \text{ is a connected union of rational curves, } C \supset \{x_+, x_-\}, (C, -K_X) = 2n\}$$

of the Chow scheme $\mathrm{Chow}(X)$.

Under our hypothesis, it is easy to show that $\dim W\langle x_+, x_-\rangle = n-1$. Each closed point $w \in W\langle x_+, x_-\rangle$ represents either an irreducible rational curve $C \subset X$ or a connected union of two irreducible rational curves $L_+ \cup L_- \subset X$ with $L_\pm \ni x_\pm$, $L_\pm \not\ni x_\mp$.

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[1]The condition on the Picard number is essential; see Remark 4.2 below.
[2]A differential-geometric analogue of this condition (positivity of the holomorphic bisectional curvature with one-dimensional degeneracy) is given in [8].

Let $V\langle x_+, x_-\rangle \subset W\langle x_+, x_-\rangle \times X$ be the associated incidence variety with natural surjective projection $\mathrm{pr}_X\colon V\langle x_+, x_-\rangle \to X$. Let $\overline{V}\langle x_+, x_-\rangle$ denote the normalisation of $V\langle x_+, x_-\rangle$ and $\overline{\mathrm{pr}}_X\colon \overline{V}\langle x_+, x_-\rangle \to X$ the induced projection. The inverse image of $x_\pm$ via this projection determines a distinguished section $\sigma_\pm \subset \overline{V}\langle x_+, x_-\rangle$ over the normalisation $\overline{W}\langle x_+, x_-\rangle$ of $W\langle x_+, x_-\rangle$.

Given a smooth curve T and a morphism $f\colon T \to \overline{W}\langle x_+, x_-\rangle$, the fibre product $T \times_{\overline{W}\langle x_+, x_-\rangle} \overline{V}\langle x_1, x_2\rangle$ is a very special conic bundle over T, the properties of which are studied in §2. With the aid of the results obtained in §2, we show that $\overline{\mathrm{pr}}_X$ lifts to an isomophism between $\overline{V}\langle x_+, x_-\rangle$ and the two-point blowup $\mathrm{Bl}_{\{x_+, x_-\}}X$ of X, inducing isomorphisms

$$\overline{W}\langle x_+, x_-\rangle \simeq \sigma_\pm \simeq E_\pm \simeq \mathbb{P}^{n-1},$$

where $E_\pm \subset \mathrm{Bl}_{\{x_+, x_-\}}X$ is the exceptional divisor over $x_\pm \in X$. The pullback $\tilde{H}_0 = \mathrm{pr}^*_{\overline{W}}L$ of the hyperplane divisor $L \subset \overline{W}\langle x_+, x_-\rangle \simeq \mathbb{P}^{n-1}$ is a semiample divisor on $\overline{V}\langle x_+, x_-\rangle \simeq \mathrm{Bl}_{\{x_1, x_2\}}X$. Then we show that $\tilde{H}_0$ contracts to an ample divisor H_0 on X and that the complete linear system $|H_0|$ defines an isomorphism from X to a hyperquadric in $\mathbb{P}^{n+1}$.

The parameter space $W\langle x_+, x_-\rangle$ eventually turns out to be the dual projective space of the complete liner system $|\mu^* H_0 - E_+ - E_-| \simeq \mathbb{P}^{n-1}$ on $\mathrm{Bl}_{\{x_+, x_-\}}X$, which is viewed as the sublinear system $|H_0(-x_+ - x_-)| \subset |H_0| \simeq \mathbb{P}^{n+1}$ on X. To be more explicit, for each $n-1$-dimensional linear subspace Λ of

$$\mathrm{H}^0(X, \mathcal{I}_{x_+}\mathcal{I}_{x_-}(H_0)) \subset \mathrm{H}^0(Q_n, \mathcal{O}(1)),$$

we associate $[C] \in W\langle x_+, x_-\rangle$, where C is the plane conic cut out of Q_n by the $n-1$ hyperplanes $\in \Lambda$ through x_+, x_-.

Convention: In what follows, every scheme is defined over the complex number field. Schemes are often identified with the set of their complex points, regarded as analytic spaces with Euclidean topology.

For mathematical notation, we basically follow the convention in [2], to which we refer the reader for technical details as well.

§1. Review of basic facts

In this section, we review several elementary facts and some basic results of [2] concerning unsplitting family of rational curves.

Given a projective variety X, the *Chow scheme* $\mathrm{Chow}(X)$ and the *Hilbert scheme* $\mathrm{Hilb}(X)$ are defined as the parameter spaces of effective cycles and closed subschemes, respectively. They are known to exsist as disjoint union of projective schemes. An effective cycle (or a closed subscheme) $\Gamma \subset X$ will be denoted by $[\Gamma]$ when viewed as a point in $\mathrm{Chow}(X)$ (or of $\mathrm{Hilb}(X)$).

For two projecive varieties X, Y, the morphisms from Y to X form a locally closed (and hence quasiprojective) subset $\mathrm{Hom}(Y, X)$ of $\mathrm{Hilb}(Y \times X)$. When X is smooth and Y is a curve, we have the local dimension estimate

$$\chi(Y, f^*\Theta_X) \leq \dim_{[f]} \mathrm{Hom}(Y, X) \leq \dim \mathrm{H}^0(Y, f^*\Theta_X)$$

at a given closed point $[f]$. The second inequality becomes equality if and only if $\mathrm{Hom}(Y, X)$ is smooth at $[f]$.

The following is an immediate consequence of well known Sard's theorem.

Proposition 1.1. *Let X be a projective variety, M a smooth scheme of finite type* [resp. *locally of finite type*] *and let $h\colon M \to \mathrm{Hom}(\mathbb{P}^1, X)$ a morphism. Assume that the naturally induced morphism $\Phi_h\colon M \times \mathbb{P}^1 \to X$ is dominant (i.e. the image contains a nonempty open subset of X). Choose a general* [resp. *sufficiently general*] *nonsingular closed point $x_0 \in X_{\mathrm{reg}} = X \setminus \mathrm{Sing}(X)$ and take an arbitrary closed point $y \in M$. Then the natural $\mathbb{C}$-linear differential map*

$$\Theta_{M,y} \oplus \Theta_{\mathbb{P}^1,p} \to \Theta_{X,x_0}$$

is surjective at any closed point $(y, p) \in \Phi^{-1}(x_0)$. Specifically when h is a locally closed embedding of M into $\mathrm{Hom}(\mathbb{P}^1, X)$ with $y = [f] \in \mathrm{Hom}(\mathbb{P}^1, X)$, $f(\mathbb{P}^1) \subset X_{\mathrm{reg}}$, the natural evaluation map gives a surjection from $\Theta_{M,[f]} \subset \mathrm{H}^0(\mathbb{P}^1, f^*\Theta_X)$ onto Θ_{X,x_0} (under the condition that $f(\mathbb{P}^1)$ passes through the (sufficiently) general closed point $x_0 \in X$).

Let $\mathcal{U} \subset \mathrm{Chow}(X)$ be a locally closed subset. The *incidence variety* attached to $\mathcal{U}$ is the closed subset $\mathcal{G} \subset \mathcal{U} \times X$ defined by

$$\mathcal{G} = \{([Y], x) |\ [Y] \in \mathcal{U}, x \in Y \subset X\}.$$

We let $\mathrm{pr}_{\mathcal{U}}$ and pr_X denote the natural projections from the incidence variety to $\mathcal{U}$ and to X, respectively.

Corollary 1.2. *Let $A \subset X$ be an arbitrary finite set of closed points on a smooth projecive variety X. Let $\mathcal{U}\langle A\rangle$ be the locally closed* *subset* $\subset \mathrm{Chow}(X)$ *of finite type consisting of irreducible, reduced, smooth rational curves which contain* A*, and let* $\mathcal{G}\langle A\rangle \subset \mathcal{U}\langle A\rangle \times X$ *denote the associated incidence variety. If the projection* $\mathrm{pr}_X : \mathcal{G}\langle A\rangle \to X$ *is dominant and* $x_0 \in X$ *is a general closed point, then, for each element* $[C]$ *of the closed subset*

$$\mathcal{U}\langle A, x_0\rangle = \{[C] \in \mathcal{U}\langle A\rangle |\ C \ni x_0\},$$

the sheaf $\Theta_X \otimes_{\mathcal{O}_X} \mathcal{O}_C(-A)$ *is generated by global sections.*[3] *In particular,* $\mathcal{U}\langle A, x_0\rangle$ *is smooth, with Zariski tangent space* $\mathrm{H}^0(C, \mathcal{N}_{X/C}(-A - x_0))$ *at* $[C]$.

Proof. Since $\mathcal{U}\langle A\rangle$ consists of smooth rational curves on X, it is thought of as a locally closed subscheme of $\mathrm{Hilb}(X)$ in an obvious way, with $\mathcal{G}\langle A\rangle$ being the associated universal family. Its Zariski tangent space at $[C]$ is naturally identified with $\mathrm{H}^0(C, \mathcal{N}_{C/X}(-A))$. By assumption, the universal family $\mathcal{G}\langle A\rangle$ dominates X so that the differential $\Theta_{\mathcal{G}\langle A\rangle} \to \mathrm{pr}_X^*\Theta_X$ is onto at anly point $p \in \mathcal{G}\langle A\rangle$ over the general point $x_0 \in X$. This differential naturally induces homomorphisms

$$\Theta_{\mathcal{G}\langle A\rangle/\mathcal{U}\langle A\rangle}|_{\{[C]\}\times C} \to \Theta_C,$$
$$\mathrm{pr}_{\mathcal{U}\langle A\rangle}^*\Theta_{\mathcal{U}\langle A\rangle}|_{\{[C]\}\times C} \to \mathcal{N}_{X/C}.$$

The second homomorphism is generically surjective whenever $C \ni x_0$. In particular, $\mathrm{H}^0(C, \mathcal{N}_{X/C}(-A))$ generically generates $\mathcal{N}_{X/C}(-A)$, meaning that

$$\mathcal{N}_{X/C}(-A) \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}(d_i), \quad d_i \geq 0.$$

We have therefore

$$\mathrm{H}^1(C, \mathcal{N}_{X/C}(-A)) = \mathrm{H}^1(C, \mathcal{N}_{X/C}(-A - x_0)) = 0$$

and hence $\mathcal{U}\langle A\rangle$ and $\mathcal{U}\langle A, x\rangle$ are both smooth at $[C] \in \mathcal{U}\langle A, x_0\rangle$.

So far, we have been dealing with general families of rational curves. From now on, we will exclusively treat rational curves of low degree.

Let X be a smooth, projective, uniruled variety with an ample divisor H and $x_0 \in X$ a closed point. Define the *minimum degree* $\mathrm{Mindeg}(X, x_0, H)$ *of the rational curves through* x_0 to be the minimum of the intersection numbers (C, H), C running through the irreducible

[3] Here A is viewed as an effective divisor on the nonsingular curve C.

rational curves containing x_0 in X. (Of course $\mathrm{Mindeg}(X, x_0, H) = l(X, x_0)$ when X is Fano and $H = -K_X$, the case we are interested in.) If a rational curve $C \ni x_0$ satisfies $(C, H) = \mathrm{Mindeg}(X, x_0, H)$, call C a *rational curve of minimum degree through* x_0. The rational curves of minimum degree through the base point x_0 form a closed (and hence projective) subscheme of finite type of $\mathrm{Chow}(X)$, and so does its arbitrary irreducible component $S\langle x_0\rangle \subset \mathrm{Chow}(X)$. The associated incidence variety

$$F\langle x_0\rangle = \{([C], x) |\ [C] \in S\langle x_0\rangle, x \in C \subset X\}$$

is naturally a closed subscheme of $S\langle x_0\rangle \times X$ with two projections pr_S and pr_X to $S\langle x_0\rangle$ and X.

The family $\mathrm{pr}_S\colon F\langle x_0\rangle \to S\langle x_0\rangle$ (of rational curves of minimum degree through x_0) is an *unsplitting family* of rational curves, *i.e.*, every closed fibre $C = F_s$ is a reduced, irreducible rational curve on X.

Proposition 1.3. *In the above notation, assume that the base point x_0 is general in X and that* $\dim S\langle x_0\rangle \geq 2$. *Let Y be the image* $\mathrm{pr}_X(F\langle x_0\rangle)$. *Let $\overline{S}\langle x_0\rangle$, $\overline{F}\langle x_0\rangle$ and $\overline{Y}$ be the normalisations of $S\langle x_0\rangle$, $F\langle x_0\rangle$ and Y, and denote by $\overline{\mathrm{pr}}_{\overline{S}}\colon \overline{F}\langle x_0\rangle \to \overline{S}\langle x_0\rangle$ and $\overline{\mathrm{pr}}_{\overline{Y}}\colon \overline{F}\langle x_0\rangle \to \overline{Y}$ the naturally induced morphism. Then we have*

(1) *If $[L] \in S\langle x_0\rangle$ is a general member, then the rational curve $L \subset X$ is smooth $\mathbb{P}^1$ and its normal bundle $\mathcal{N}_{L/X}$ in X is isomorphic to $\mathcal{O}(1)^{\oplus r} \oplus \mathcal{O}^{\oplus n-r-1}$, where* $2 \leq r = \dim Y - 1 = (C, -K_X) - 2 \leq n - 1$.
(2) *Only finitely many members L of $S\langle x_0\rangle$ can have singulrities at the base point x_0.*
(3) *Only finitely many members L of $S\langle x_0\rangle$ can have cuspidal singularities and no member has a cuspidal singularity at the base point x_0.*
(4) *The first projection $\overline{\mathrm{pr}}_{\overline{S}}\colon \overline{F}\langle x_0\rangle \to \overline{S}\langle x_0\rangle$ is a $\mathbb{P}^1$-bundle.*
(5) *The scheme theoretic inverse image $\overline{\mathrm{pr}}_Y^*(x_0) \subset \overline{F}\langle x_0\rangle$ of the base point x_0 via the second projection $\overline{\mathrm{pr}}_Y : \overline{F}\langle x_0\rangle$ is a disjoint union of a specified section σ_0 and a (zero-dimensional) closed subscheme τ away from σ_0. In particular, locally around the Cartier divisor σ_0, the projection $\overline{\mathrm{pr}}_{\overline{Y}}$ naturally lifts to a morphism $\tilde{\mathrm{pr}}_{\tilde{Y}} : \overline{F}\langle x_0\rangle \to \tilde{Y}$, where $\tilde{Y}$ is the normalisation of the one-point blowup $\mathrm{Bl}_{x_0}Y$ of Y at x_0.*
(6) *The second projection $\overline{\mathrm{pr}}_{\overline{Y}}$ is unramified over $\overline{Y} \setminus \bigl(\mathrm{Sing}(\overline{Y}) \cup \{\overline{y}_0\}\bigr)$, where $\overline{y}_0 = \overline{\mathrm{pr}}_{\overline{Y}}(\sigma_0) \in \overline{Y}$ is a point over $x_0 \in Y$. In particular, the induced morphism from a small open neighbourhood*

of σ_0 in $\overline{F}\langle x_0\rangle$ to a neighbourhood of the exceptional divisor E_{x_0} in $\tilde{Y}$ is unramified in codimension one.

Proof. The statements (1) through (5) are proved in [2, §§2 – 3]. In order to prove (6), we blowup the zero-dimensional subscheme τ and eliminate the indeterminacy to get a morphism $\mathrm{Bl}_\tau\overline{F}\langle x_0\rangle \to \tilde{Y}$. (Note that $\mathrm{Bl}_\tau\overline{F}\langle x_0\rangle$ is normal with only A-type rational double points as singularities by [2, 4.2, Step 2].) It is easy to show that the strict transform of a general member L of $S\langle x_0\rangle$ in $\tilde{Y}$ is a smooth rational curve lying on the nonsingular locus of $\tilde{Y}$ and has trivial normal bundle. Then we can photocopy the proof of [2, Theorem 4.2].

§2. Conic bundles

While in [2] we relied on special properties of $\mathbb{P}^1$-bundles over curves, the key ingredient in the present paper is the theory of two-dimensional conic bundles, *i.e.*, one-parameter families of plane conics. To be more precise, a flat projective family $\pi\colon \mathcal{C} \to T$ over a smooth curve T is said to be a (two-dimensional) *conic bundle* if

(1) a general fibre of π is a smooth $\mathbb{P}^1$, and
(2) there exists an étale open covering[4] $\{p_\alpha\colon U_\alpha \to T\}$ of T and a family of vector bundles $\mathcal{E}_\alpha$ of rank three on U_α such that $\mathcal{C}_\alpha = U_\alpha \times_T \mathcal{C}$ is isomorphic to a hypersurface $\in |2\mathbf{L}_{\mathcal{E}_\alpha}|$ in the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E}_\alpha)$ with tautological line bundle $\mathbf{L}_{\mathcal{E}_\alpha}$.

A singular fibre of a conic bundle is either a union of two lines meeting at a single point or a double line (a non-reduced fibre). The singular loci of the fibres $\mathcal{C}_t$ form a closed subset $\mathrm{Cr}(\mathcal{C}) \subset \mathcal{C}$, called the *critical locus.*

Let $\hat{\pi}\colon \hat{\mathcal{C}} \to T$ be a projective morphism from an irreducible (possibly singular) surface onto a smooth curve. Let $\mathcal{C}$ denote the normalisation of $\hat{\mathcal{C}}$, and $\pi\colon \mathcal{C} \to T$ the morphism naturally induced by $\hat{\pi}$.

Lemma 2.1. *In the above notation, assume that*

(a) *for each cosed point $t \in T$, the effective Cartier divisor $\hat{\mathcal{C}}_t = \hat{\pi}^*(t)$ is reduced and contains at most two irreducible components, and that*
(b) *a general fibre $\hat{\mathcal{C}}_t$ is smooth $\mathbb{P}^1$.*

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[4]alternatively, an analytic open covering

Then

(1) *the fibration* $\pi\colon \mathcal{C} \to T$ *is a conic bundle without non-reduced fibres, and*

(2) $\mathcal{C}$ *has at worst* A-*type Du Val points as singularities. Any singular point of* $\mathcal{C}$ *is contained in the unique intersection point of the two components of some reducible fibre of* π.

Proof. Pick up an arbitrary closed point $t \in T$. Since the base T is smooth and the reduced closed fibre $\hat{\mathcal{C}}_t$ is smooth outside a finite set $\Sigma_t \subset \hat{\mathcal{C}}_t$, we see that $\hat{\mathcal{C}}$ is smooth along $\hat{\mathcal{C}}_t \setminus \Sigma_t$. Therefore $\mathcal{C}$ and $\hat{\mathcal{C}}$ are isomorphic in codimension one, so that the closed fibre $\mathcal{C}_t \subset \mathcal{C}$ is also reduced having at most two irreducible components.

Take the minimal resolution $\mu\colon \tilde{\mathcal{C}} \to \mathcal{C}$. The smooth surface $\tilde{\mathcal{C}}$ is flat over T, and we have $\tilde{\mathcal{C}}_t K_{\tilde{\mathcal{C}}} = -2$. Furthermore, $\tilde{\mathcal{C}}$ is obtained as a blowup of a suitable $\mathbb{P}^1$-bundle over T. Each irreducible component E of $\tilde{\mathcal{C}}_t$ is thus smooth $\mathbb{P}^1$ with nonpositive self intersection, and $E^2 = 0$ if and only if $E = \tilde{\mathcal{C}}_t$. By the adjunction formula, $EK_{\tilde{\mathcal{C}}} = -2 - E^2 \geq -1$ unless $E = \tilde{\mathcal{C}}_t$. If E is contracted to a point on $\mathcal{C}$, then $EK_{\tilde{\mathcal{C}}} \geq 0$ because our resolution is minimal.

We have two cases:

Case 1. $\mathcal{C}_t$ *is irreducible.* In this case, we have a unique component $\tilde{\mathcal{C}}_t^0$ of $\tilde{\mathcal{C}}_t$ which surjects onto $\mathcal{C}_t$. Any other component is contracted to a point and has non-negative intersection with $K_{\tilde{\mathcal{C}}}$, while $\tilde{\mathcal{C}}_t K_{\tilde{\mathcal{C}}} = -2$. This implies that $\tilde{\mathcal{C}}_t^0 K_{\tilde{\mathcal{C}}} \leq -2$, so that $\tilde{\mathcal{C}}_t^0 = \tilde{\mathcal{C}}_t$ or, equivalently, $\mathcal{C}_t$ is a smooth fibre.

Case 2. $\mathcal{C}_t$ *is the union of two irreducible components* $\mathcal{C}_{t\pm}$. In this case, there are at most two irreducible components with $EK_{\tilde{\mathcal{C}}} = -1$ and all the other compoents have nonnegative intersection with $K_{\tilde{\mathcal{C}}}$, while the sum of the intersection numbers is -2. This means that the two strict transforms $\tilde{\mathcal{C}}_{t\pm}$ of $\mathcal{C}_{t\pm}$ are (-1)-curves and the other components are (-2)-curves. If we write

$$\tilde{\mathcal{C}}_t = \tilde{\mathcal{C}}_{t+} + \tilde{\mathcal{C}}_{t-} + \sum_i a_i E_i,$$

then

$$1 = -(\tilde{\mathcal{C}}_{t+})^2 = \tilde{\mathcal{C}}_{t+}\tilde{\mathcal{C}}_{t-} + \sum_i a_i \tilde{\mathcal{C}}_{t+} E_i,$$

meaning that $\tilde{\mathcal{C}}_{t+}$ meets with a single reduced irreducible component E_+. If E_+ is $\tilde{\mathcal{C}}_{t-}$, then, by symmetry, $E_- = \tilde{\mathcal{C}}_{t+}$ is the unique component which meets $\tilde{C}_{t-}$, so that $C_t = \tilde{C}_t = \tilde{C}_{t+} + \tilde{C}_{t-}$. If E_+ is one of the (-2)-curves, then the blowdown of $\tilde{C}_{t+}$ affects the single component E_+ to produce a new (-1)-curve, and we get a similar situation, $\tilde{C}_{t+}$ being replaced with the image of E_+. Reiterating the same process, we arrive at the situation where $E_+ = \tilde{C}_{t-}$. Thus $\tilde{C}_t$ is a single chain

$$\tilde{C}_{t+} + E_1 + \cdots + E_m + \tilde{C}_{t-},$$

of which the two ends are the (-1)-curves. Since the intermediary curves form a chain of (-2)-curves, we can contract the chain to an A_m-singularity. After contracting all such chains on $\tilde{C}$, we get a normal surface C^*. By construction, the resolution $\mu\colon \tilde{C} \to C$ factors through C^*, which is finite over C. Hence, by Zariski's Main Theorem, $C^* = C$.

The relative anticanonical divisor $-K_{C/T}$ gives a closed embedding of C into the projective bundle $\mathbb{P}(\mathrm{pr}_{T*}\mathcal{O}_C(-K_{C/T}))$, defining a standard conic bundle structure on C.

When it has an A_m-singularity (a smooth point is considered as an A_0-singularity) on a reducible fibre C_t, the normal surface C is locally defined by the equation $\xi_1\xi_2 = \tau^{m+1}$ in $T \times \mathbb{P}^2$, where τ is a local parameter of T and ξ_0, ξ_1, ξ_2 are homogeneous coordinates of $\mathbb{P}^2$.

Proposition (2.1) determines the rational Néron-Severi group of the conic bundle C. In fact we have the following

Corollary 2.2. *Let the notation and assumptions be as in* (2.1). *Let C° denote the non-critical locus $C\setminus\mathrm{Cr}(C)$. Then there exists a section $\sigma\colon T \to C^\circ \subset C$ of the projection π. The surface C is $\mathbb{Q}$-factorial, i.e., every Weil divisor is Cartier if multiplied by a suitable positive integer. The $\mathbb{Q}$-Néron-Severi group $\mathrm{NS}(C)_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} \mathrm{NS}(C)$ is a $\mathbb{Q}$ vector space freely generated by σ, $\mathfrak{f} = [C_t]$ and δ_i, $i = 1, \ldots, r$, where $\delta_i = [C_{t_i+}] - [C_{t_i-}]$ and the $C_{t_i} = C_{t_i+} + C_{t_i-}$, $i = 1, \ldots, r$ are the decomposition of the singular fibres such that $\sigma C_{t_i+} = 1$. If C has an A_{m_i}-singularity at $C_{t_i+} \cap C_{t_i-}$, we have the following intersection table:*

$$\begin{aligned} &\mathfrak{f}^2 = \mathfrak{f}\delta_i = \delta_i\delta_j = 0, \quad i \neq j, \\ &\delta_i^2 = -\frac{4}{m_i + 1}, \\ &\sigma\mathfrak{f} = \sigma\delta_i = 1. \end{aligned}$$

Proof. Let $\mu\colon \tilde{C} \to C$ be the minimal resolution and E_{ik} a (-2)-curve over the singualar point on C_{t_i}. Denoting $\tilde{C}_{t_i+}$ denote the strict transform of $\mathcal{C}_{t_i+}$, we can write $\mu^*\mathcal{C}_{t_i+} = \tilde{\mathcal{C}}_{t_i+} + \sum_k a_k E_{ik}$, while $E_{ik}\mu^*\mathcal{C}_{t_i+} = 0, k = 1, \ldots, m_i$. This determines the coefficients a_k, yielding

$$\mu^*\mathcal{C}_{t_i+} = \tilde{\mathcal{C}}_{t_i+} + \frac{1}{m_i+1}\sum_k (m_i + 1 - k)E_{ik}.$$

Then the above intersection table follows from simple computation.

Definition 2.3. Let $\pi\colon \mathcal{C} \to T$ be a normal conic bundle over a smooth projective curve T and B a nef and big Cartier divisor on $\mathcal{C}$. The fibre space $\pi\colon \mathcal{C} \to T$ (or the total space $\mathcal{C}$, by abuse of terminology) is said to be an *B-symmetric* conic bundle if $B\mathcal{C}_{t+} = B\mathcal{C}_{t-}$ whenever a closed fibre $\mathcal{C}_t$ is a union of two components $\mathcal{C}_{t+}$, $\mathcal{C}_{t-}$.

Assume that $\pi\colon \mathcal{C} \to T$ has two distinct sections σ_+, σ_-. The triple $(\mathcal{C}; \sigma_+, \sigma_-)$ is said to be *strongly B-symmetric* if the following four conditions are satisfied:

(a) $\mathcal{C}$ is B-symmetric;
(b) σ_+ and σ_- are mutually disjoint divisors contained in the non-critical locus $\mathcal{C}^\circ = \mathcal{C} \setminus \mathrm{Cr}(\mathcal{C})$;
(c) $B\sigma_+ = B\sigma_-$;
(d) For any reducible fibre $\mathcal{C}_t = \mathcal{C}_{t+} + \mathcal{C}_{t-}$, we have

$$\sigma_+\mathcal{C}_{t+} = \sigma_-\mathcal{C}_{t-} = 1,$$
$$\sigma_+\mathcal{C}_{t-} = \sigma_-\mathcal{C}_{t+} = 0,$$

(possibly after suitable reindexing of the irreducible components $\mathcal{C}_{t\pm}$).

Proposition 2.4. *Let $\pi\colon \mathcal{C} \to T$ be a normal conic bundle over a smooth projective curve with a nef big divisor B and two sections σ_+, σ_-. Assume that $(\mathcal{C}; \sigma_+, \sigma_-)$ is strongly B-symmetric and let s denote the number of the singular fibres. Let $\mu\colon \tilde{\mathcal{C}} \to \mathcal{C}$ be the minimal resolution. Then we have:*

(1) $B \approx d(\sigma_+ + \sigma_-) + a\mathfrak{f}$, $d \in \mathbb{N}$, $a \in \mathbb{Q}$.
(2) $\sigma_- \approx \sigma_+ + \sum \frac{m_i+1}{2}\delta_i$.
(3) *Let $\sigma \subset \mathcal{C}$ be a section of π and $\tilde{\sigma} \subset \tilde{\mathcal{C}}$ its strict transform. Let*

$$\tilde{\mathcal{C}}_{t_i} = \sum_{k=0}^{m_i+1} E_{ik} = \tilde{\mathcal{C}}_{t_i-} + E_{i1} + \cdots + E_{im_i} + \tilde{\mathcal{C}}_{t_i+}$$

be the irreducible decomposition of a singular fibre of $\tilde{\pi}\colon \tilde{\mathcal{C}} \to T$ over t_i and let $E_{i\kappa_i}$ be the unique component which meets $\tilde{\sigma}$.

Then

$$\tilde{\sigma} = \mu^*\sigma - \sum_i \Big(\sum_{k=1}^{\kappa_i} \frac{k(m_i + 1 - \kappa_i)}{m_i + 1} E_{ik} + \sum_{k=\kappa_i+1}^{m_i} \frac{(m_i + 1 - k)\kappa_i}{m_i + 1} E_{ik}\Big).$$

(4) $\sigma_+^2 = \sigma_-^2 = -\frac{e}{2} \leq 0$, *where*

$$e = \sum_i (m_i + 1),$$

the sum being taken over the the reducible fibres C_{t_i}, *on which* C *has singularities of type* A_{m_i} (*of course we define* $m_i = 0$ *if* C *is nonsingular near* C_{t_i}).

(5) *If its strict transform* $\tilde{\sigma} \subset \tilde{C}$ *has negative self intersection, then a section* $\sigma \subset C$ *coincides with one of the two specified sections* $\sigma_\pm$. *In particular,* σ *is one of the* $\sigma_\pm$ *once a section* $\sigma \subset C$ *satisfies* $\sigma^2 < 0$. *If* $\sigma \neq \sigma_\pm$ *and its strict transform* $\tilde{\sigma}$ *satisfies* $\tilde{\sigma}^2 = 0$, *then* σ *is disjoint with* $\sigma_\pm$. *If, furthermore,* $\sigma^2 = 0$, *then it is away from* $\mathrm{Sing}(C)$.

(6) *If there are two sections* $\sigma_1, \sigma_2 \neq \sigma_\pm \subset C$ *such that the strict transforms* $\tilde{\sigma}_1, \tilde{\sigma}_2$ *are mutually disjoint in* $\tilde{C}$, *then* $\sigma_1 \cup \sigma_2$ *is away from* $\sigma_+ \cup \sigma_-$.

Proof. The first three statements are direct consequences of the intersection table in (2.2) and we leave the proof to the reader.

Take the minimal resolution $\tilde{C}$ of C. Let $\tilde{\sigma}_\pm \subset \tilde{C}$ denote the strict transforms of the sections $\sigma_\pm$. Starting from $\tilde{C}$, we can find a series of blowdowns

$$\tilde{C} = C_0 \to C_1 \to \cdots \to C_e$$

to reach a $\mathbb{P}^1$-bundle C_e. The number of the blowdowns is computed by

$$e = \sum_{t \in T} \big((\text{the number of the components of } \tilde{C}_t) - 1\big) = \sum_{i=1}^{s} (m_i + 1).$$

We denote by $\sigma_{\pm k}$ the image of $\tilde{\sigma}_\pm = \sigma_{\pm 0}$ in C_k.

The choice of blowdowns is not unique. Our choice is inductively made in such a way that at each step the (-1)-curve to be contracted must intersect σ_{+k} (or, equivalently, that the (-1)-curve does not touch σ_{-k}). In such a (unique) choice of blowdowns, we can easily see that the two divisors $\sigma_{\pm e}$ are still disjoint on the $\mathbb{P}^1$-bundle C_e. A $\mathbb{P}^1$-bundle with two disjoint sections is canonically a projective bundle $\mathbb{P}(\mathcal{L}_+ \oplus \mathcal{L}_-)$, the two direct summand corresponding to the two sections. Thus we have $\sigma_{+e}^2 = -\sigma_{-e}^2 = \deg \mathcal{L}_+\mathcal{L}_-^{-1}$. By construction, $\sigma_{-e}^2 = \tilde{\sigma}_-^2 = \sigma_-^2 = \sigma_+^2$, while $\sigma_{+e}^2 \geq \sigma_{+0} = \tilde{\sigma}_+^2 = \tilde{\sigma_+}^2 = \sigma_-^2$, the equality holding if and only if $e = 0$, $\mathcal{C}_e = \tilde{\mathcal{C}} = \mathcal{C}$. Therefore $\sigma_+^2 + e = -\sigma_-^2 = -\sigma_+^2$, whence follows (4).

We trace back the blowdown procedure by starting from the $\mathbb{P}^1$-bundle $\mathcal{C}_e$ with two disjoint sections $\sigma_{\pm e}$ and by successively blowing up points on the strict transforms σ_{+k} on $\mathcal{C}_k$, eventually to reach $\tilde{C} = \mathcal{C}_0$.

Let $\sigma \subset \mathcal{C}$ be a section different from $\sigma_\pm$. Its strict transform $\tilde{\sigma}$ in $\tilde{\mathcal{C}}$ is mapped to a section σ_e on the $\mathbb{P}^1$-bundle $\mathcal{C}_e$. Putting $a = \sigma_e\sigma_{-e} \geq 0$, we have $\sigma_e\sigma_{+e} = e + a \geq e$, $\sigma_e^2 = e + 2a$. Let $\mathcal{C}_{t_ie} \simeq \mathbb{P}^1$ be the strict transform in $\mathcal{C}_e$ of the singular fibre $\mathcal{C}_{t_i} \subset \mathcal{C}$. Let κ_i denote the local intersection number $(\sigma_e, \sigma_{+e})_{\mathrm{loc}}$ at the single point $\mathcal{C}_{t_ie} \cap \sigma_{+e}$, with the obvious inequality $\sum_{i=1}^s \kappa_i \leq \sigma_e\sigma_{+e} = e + a$. By the description of the blowing up $\mathcal{C}_0 \to \mathcal{C}_e$, the selfintersection $\tilde{\sigma}$ is computed by $e + 2a - \sum_i \kappa_i \geq a \geq 0$, the equalities are attained if and only if $a = 0, \sum \kappa_i = e + a = e$, meaning that $\tilde{\sigma}$ is disjoint with $\tilde{\sigma}_\pm$ in this case. These facts in mind, we readily deduce (5) and (6) from the easy inequality $\sigma^2 \geq \tilde{\sigma}^2$, the equality holding if and only if σ does not pass through the singular points.

If none of the two sections $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ coincides with $\tilde{\sigma}_\pm$, the both divisors are necessarily nef with non-negative selfintersection. When one of them has positive self-intersection, they must intersect by Hodge index theorem. If both have self intersection zero, then they cannot meet $\tilde{\sigma}_\pm$ by (6) (recall that $\sigma_\pm$ is not affected by the resolution).

Corollary 2.5. *Let X be a projective variety with an ample divisor H and let $\pi: \mathcal{C} \to T$ be a normal conic bundle over a smooth curve. Let $f: \mathcal{C} \to X$ be a morphism with two-dimensional image such that its restriction to each fibre $\mathcal{C}_t$ is finite. Assume that π admits two sections $\sigma_\pm$ such that*

(1) *$f(\sigma_\pm)$ is a single point $x_\pm \in X$, $x_+ \neq x_-$,*
(2) *$(\mathcal{C}_t, f^*H) = 2\,\mathrm{mindeg}(X, x_1; H) = 2\mathrm{mindeg}(X, x_2; H)$ for each closed fibre $\mathcal{C}_t$ of π, and that*
(3) *no irreducible component of a singlular fibre $\mathcal{C}_{t_i}$ of π simultaneously meets both σ_+ and σ_-.*

Then $\mathcal{C}$ is a strongly f^*H-symmetric conic bundle and f is finite over $X \setminus \{x_+, x_-\}$.

Proof. The first statement follows from the condition (3) plus the equalities $\sigma_\pm f^*H = 0$ and $f(\mathcal{C}_{t+})H = f(\mathcal{C}_{t-})H = \mathrm{mindeg})(X, x_i; H)$ for a reducible fibre $\mathcal{C}_t = \mathcal{C}_{t+} \cup \mathcal{C}_{t-}$. In order to prove the second statement, assume that there is a curve $\sigma \subset \mathcal{C}$ which is contracted to a point by f. By considering a suitable base change if necessary, we may assume that σ is a section without loss of generality. Then, by the equality $\sigma f^*H = 0$ and the Hodge index theorem, we infer that $\sigma^2 < 0$, contradicting (2.4).

Proposition 2.6. *Let $\pi\colon \mathcal{C} \to T$ be a two-dimensional normal conic bundle and $f\colon \mathcal{C} \to X$ a morphism with two-dimensional image as in* (2.5). *Assume that*

(1) *$\mathcal{C}$ is f^*H-symmetric, that*
(2) *There are two sections $\sigma_\pm$ such that $f(\sigma_\pm)$ is a single point $x_\pm \in X$, $x_+ \neq x_-$, and that*
(3) *there is a third section $\sigma \subset \mathcal{C}$ such that $f(\mathcal{C}_t)$ has a cuspidal singularity at $f(\sigma \cap \mathcal{C}_t)$ for each irreducible fibre $\mathcal{C}_t$.*

Then σ is away from one of the $\sigma_\pm$.

Proof. Let $\mathcal{I}_\sigma \subset \mathcal{O}_\mathcal{C}$ denote the ideal sheaf of the closed subscheme $\sigma \subset \mathcal{C}$. Let $R \subset \mathbb{C}(T)\mathcal{O}_\mathcal{C} \subset \mathbb{C}(\mathcal{C})$ be the $\mathbb{C}(T)$-subalgebra generated by 1, $\mathcal{I}_\sigma^2$, $\mathcal{I}_\sigma^3$. We define the $\mathcal{O}_T$-subalgebra $\mathcal{O}_\mathcal{G} \subset \mathcal{O}_\mathcal{C}$ by $\mathcal{O}_\mathcal{G} = R \cap \mathcal{O}_\mathcal{C}$. $\mathcal{O}_\mathcal{G}$ determines a family $\dot{\pi}\colon \mathcal{G} \to T$ of singular rational curves, which factors $f\colon \mathcal{C} \to X$ into the natural projection $\mathcal{C} \to \mathcal{G}$ and $g\colon \mathcal{G} \to X$. Let $\mathrm{Pic}(\mathcal{G}/T) = \coprod_d \mathrm{Pic}^d(\mathcal{G}/T)$ be the relative Picard group scheme, $\mathrm{Pic}^d(\mathcal{G}/T)$ consisting of the equivalence classes of line bundles of degree d on each fibre.

If a closed fibre $\mathcal{G}_t$ is an irreducible cuspidal curve, then $\mathrm{Pic}^0(\mathcal{G}/T)_t = \mathrm{Pic}^0(\mathcal{G}_t)$ is naturally isomorphic to $\mathbb{G}_a \simeq \mathbb{A}^1$. The line bundle $\mathcal{O}_\mathcal{G}(g^*H)$ determines a global section of $\mathrm{Pic}^d(\mathcal{G}/T) \to T$, and, at a generic point $t \in T$, there is a unique section λ such that $\lambda^{\otimes d} \sim \mathcal{O}(g^*H)$, determining a unique rational (and hence holomorphic) section $\sigma^*\colon T \to \mathcal{C}$ such that $\sigma^*(t) \in \mathcal{C}_t \setminus \sigma(t) \simeq \mathcal{G}_t \setminus \sigma(t)$ and that $\mathcal{O}(\sigma^*(t)) \sim \lambda(t)$ for general $t \in T$.

Take the minimal resolution $\tilde{\mathcal{C}} \to \mathcal{C}$ and let $\tilde{\sigma}, \tilde{\sigma}^*$, etc. be the strict transforms in $\tilde{\mathcal{C}}$ of σ, σ^*, etc. $\subset \mathcal{C}$. Let us check that $\tilde{\sigma}^* \subset \tilde{\mathcal{C}}$ is away from $\tilde{\sigma}$.

By construction, $\tilde{\sigma}^*$ does not meet $\tilde{\sigma}$ outside the singular fibres.

The local structure of $\tilde{\sigma}^*$ around a singular fibre $\tilde{\mathcal{C}}_{t_i}$ is also very simple. Let $\tilde{\mathcal{C}}_{t_i} = \tilde{\mathcal{C}}_{t_i-} + E_{i1} + \cdots + E_{im_i} + \tilde{\mathcal{C}}_{t+}$ be the irreducible decomposition of the singular fibre, a chain of smooth $\mathbb{P}^1$'s. For the strict transform $\tilde{\sigma} \subset \tilde{\mathcal{C}}$ of $\sigma \subset \mathcal{C}$, let $E_{i\kappa_i}$ denote the unique component which meets $\tilde{\sigma}$ (we set $E_{i0} = \tilde{\mathcal{C}}_{t_i-}, E_{i,m_i+1} = \tilde{\mathcal{C}}_{t+}$, by convention). As we have seen in (2.4.3), there is a unique solution $(y_{ik}) \in \mathbb{Q}^e$ (actually

$\in \mathbb{Z}^e)$, $e = \sum_i (m_i + 1)$, which satisfies the linear equations

$$(\tilde{f}^* H + \sum_{i,k} y_{ik} E_{i+1}) E_{ik} = d\delta_{k\kappa_i}.$$

Noting that there are two (-1)-curves as two ends of the chain $\tilde{C}_{t_i}$, we can blow down $\tilde{\mathcal{C}}$ to a smooth $\mathbb{P}^1$-bundle $\mathcal{C}^\dagger$ in such a way that all the components of $\tilde{C}_{t_i} \setminus E_{i\kappa_i}$ are contracted to points. Then the divisor $\tilde{f}^* H + \sum_{i,k} y_{ik} E_{i+1}$ is a pull-back of a divisor $H^\dagger$ on $\mathcal{C}^\dagger$. Let $\sigma^\dagger$ denote the image of $\tilde{\sigma}$ on $\mathcal{C}^\dagger$. Starting from $\mathcal{C}^\dagger$ and $\sigma^\dagger$, we can easily construct a family of cupidal plane cubics $\mathcal{G}^\dagger \to T$ which coincides with $\mathcal{G} \to T$ over a general point t. The divisor $H^\dagger$ is a global section of $\mathrm{Pic}^d(\mathcal{G}^\dagger/T)$ and we find a unique section

$$\sigma^{*\dagger} \subset \mathrm{Pic}^1(\mathcal{G}^\dagger/T) \simeq \mathcal{G}^\dagger \setminus \sigma^\dagger$$

such that $\sigma^{\dagger \otimes d} \sim H^\dagger$ on $\mathcal{G}^\dagger_t$. The section $\tilde{\sigma}^*$ on $\tilde{\mathcal{C}}$ is then the strict transform of $\sigma^{*\dagger} \subset \mathcal{C}^\dagger$, and in particular is off $\tilde{\sigma}$, the strict transform of $\sigma^\dagger$. If $\tilde{\sigma}^*$ is one of the $\tilde{\sigma}_\pm$, say $\tilde{\sigma}_+$, then its image σ does not meet σ_+ on $\mathcal{C}$ (because the resolution $\tilde{\mathcal{C}} \to \mathcal{C}$ does not affect $\sigma_\pm$). If $\tilde{\sigma}^* \subset \tilde{\mathcal{C}}$ is not one of the $\tilde{\sigma}_\pm$, then, by (2.4.6), σ does not intersect $\sigma_\pm$.

§3. Fano n-manifolds with Picard number one and local length n

In this section, we prove the essential part of Theorem 0.1, the implication (3) $\Rightarrow$ (1). Recall that Theorem 0.1 is known for Fano 3-folds.

Throughout the section, we assume:

(a) X is a Fano manifold of dimension $n \geq 4$ with Picard number one.
(b) The two closed points $x_+, x_- \in X$ are general.
(c) $l(X, x_\pm; -K_X) = n$.

Consider an irreducible component W of the closed subset $\mathcal{W} \subset \mathrm{Chow}(X)$ which consists of the connected rational curves C with $(C, -K_X) = 2n$. Let $\mathrm{pr}_W \colon V \to W$ be the associated incidence variety. Let $D \subset W$ denote the *descriminant locus*, the locus consisting of the reducible rational curves and the non-reduced curves. The induced subfamily of curves over D is denoted by V_D.

The symbol $W\langle x_+, x_- \rangle$ [resp. $D\langle x_+, x_- \rangle$] stands for the closed subset of the curves $\in W$ [resp. $\in D$] passing through the two points

x_+, x_-. The associated incidence varieties are denoted by $V\langle x_+, x_-\rangle$ and $V_D\langle x_+, x_-\rangle$.

By our construction, the following assertion is immediate.

Proposition 3.1. *The fibre of* pr_W *over a point* $w \in D\rangle x_+, x_- \subset W$ *is either a connected union of two irreducible, reduced rational curves or a non-reduced rational curve of generic multiplicity two. Given a smooth curve* T *and a non-constant morphism* $T \to W\langle x_+, x_-\rangle \subset W$ *of which the image is not contained in* $D\langle x_+, x_-\rangle$, *the normalisation of the fibre product* $T \times_W V$ *is a symmetric conic bundle over* T.

Proposition 3.2. *In the above notation, we have*

(1) $\dim D\langle x_+, x_-\rangle = n-2$ *and the image of the projection* $\mathrm{pr}_X\colon V_D\langle x_+, x_-\rangle \to X$ *is a divisor* Y *on* X.

(2) *The divisor* Y *a union of two divisors* Y_+, Y_- *such that* x_i *is contained in* Y_j *if and only if* $i = j$ $(i, j = +, -)$.

(3) *An arbitrary element* $[C] \in D\langle x_+, x_-\rangle$ *is a reduced reducible curve* $L_+ \cup L_-$ *with* $L_\pm \ni x_\pm$, $L_\pm \subset Y_\pm \subset X$.

(4) $\dim\big(\mathrm{Sing}(Y_+) \cap Y_-\big) \le n-3$ *and a general member* $L_+ \cup L_-$ *does not pass through this set.*

(5) *If* $L_+ \cup L_-$ *is a general member of* $D\langle x_+, x_-\rangle$, *then* $L_\pm$ *is smooth with normal bundle* $\simeq \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$.

(6) *A general member* $L_+ \cup L_-$ *of* $D\langle x_+, x_-\rangle$ *deforms to an irreducible rational curve* C *such that* $C \supset \{x_+, x_-\}$. *More precisely, there is a commutative diagram*

$$\begin{array}{ccc}
\mathcal{C}_0 & \longrightarrow & L_+ \cup L_- \\
\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & V\langle x_+, x_-\rangle \\
\downarrow & & \downarrow \\
\Delta & \longrightarrow & W\langle x_+, x_-\rangle
\end{array}$$

where

$$\mathcal{C} = \{((x:y:z), t); xy = t\} \subset \mathbb{P}^2 \times \Delta$$

is a nonsingular conic bundle over a small disk Δ *with reducible central fibre* $\mathcal{C}_0$.

Proof. The rational curves L with $L(-K_X) = n$ form a family F parametrised by a variety S of dimension $\ge 2n-3$. By our assumption (c), the closed subfamily $F\langle x_\pm\rangle \to S\langle x_\pm\rangle$ consisting of the members through a general base point $x_\pm$ is a non-empty unsplitting family parametrised by $S\langle x_\pm\rangle$. Then we apply (1.) to a general member L of $S\langle x_\pm\rangle$, to deduce that

a) the parameter space $S\langle x_\pm\rangle$ has dimension $n-2$, that
b) the projection $\mathrm{pr}_X\colon F\langle x_\pm\rangle \to X$ is finite over $X\setminus\{x_\pm\}$ and that
c) $Y_\pm = \mathrm{pr}_X(F\langle x_\pm\rangle)$ is a divisor.

By our genericity condition, $x_\pm \in X\setminus Y_\mp$, and so $S\langle x_+\rangle \cap S\langle x_-\rangle = \emptyset$. In particular, any member of $D\langle x_+, x_-\rangle$ is a reduced, reducible curve $L_+ \cup L_-$ such that $L_\pm \ni x_\pm$.

If $[L_-] \in S\langle x_-\rangle$, then $L_- \not\subset Y_+$ and $L_- \cap Y_+ \neq \emptyset \subset X$ because $\rho(X) = 1$, meaning that we can find a curve $[L_+] \in S\langle x_+\rangle$ so that L_+ meets L_-, *i.e.*, $[L_+ \cup L_+] \in D\langle x_+, x_-\rangle$. Furthermore, since $L_- \cap Y_+$ is a finite set, we have only finitely many choices of such L_+ (because $F\langle x_+\rangle$ is an unsplitting family of rational curves). Put in another way, the projection $D\langle x_+, x_-\rangle \to S\langle x_-\rangle$, $[L_+ \cup L_-] \mapsto [L_-]$ is surjective and finite (and so is the other projection $D\langle x_+, x_-\rangle \to S\langle x_+\rangle$ by symmetry).

In particular, $\dim D\langle x_+, x_-\rangle = \dim S\langle x_+\rangle = n-2$. If $[L_+ \cup L_-]$ is a general point in $D\langle x_+, x_-\rangle$, then so is $[L_\pm]$ in $S\langle x_\pm\rangle$ and

$$\Theta_X|_{L_\pm} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}.$$

It is clear that $Y = \mathrm{pr}_X(F_D\langle x_+, x_-\rangle$ is the union of the divisors $Y_\pm = \mathrm{pr}_X(F\langle x_\pm\rangle)$.

The members L_- that meet $\mathrm{Sing}(Y_+)$ form a closed subset of $S\langle x_-\rangle$. Suppose that this subset is the whole space $S\langle x_-\rangle$. Then it follows that there is an $n-2$-dimensional irreducible component Σ of $\mathrm{Sing}(Y_+)$ such that every member L_- of $S\langle x_-\rangle$ passes through Σ. If follows that, for a general closed point $x_0 \in \Sigma$, there is a member L_- of $S\langle x_0, x_-\rangle$. On the other hand, $\dim F\langle x_0\rangle$ is $n-1$ near L_-, so that $\mathrm{pr}_X(F\langle x_0\rangle)$ is a divisor on X. If we replace x_- by another general point $\notin \mathrm{pr}_X(F\langle x_0\rangle)$, we cannot find L_- which connects x_0 and x_-, meaning that $\mathrm{pr}_X(F\langle x_-\rangle) \cap \Sigma \neq \Sigma$ for a generic choice of x_-.

We have so far checked the statements (1) – (5). In order to prove (6), consider a nonsingular conic bundle

$$\mathcal{C} = \{((x:y:z),t); xy = t\} \subset \mathbb{P}^2 \times \Delta$$

over a small disk Δ with reducible central fibre $\mathcal{C}_0$. Choose a general member $L_+ \cup L_-$ of D and fix a birational map $f\colon \mathcal{C}_0 \to L_+ \cup L_- \subset X$. The graph Γ_f of this map is a locally complete intersection in $\mathcal{C} \times X$ with normal bundle $\mathcal{N} \simeq f^*\Theta_X \oplus \mathcal{O}$. Hence $\dim \mathrm{H}^0(\Gamma_f, \mathcal{N}) = 3n+1$,

$\mathrm{H}^1(\Gamma_f, \mathcal{N}) = 0$. On the other hand, the deformation of $f: \mathcal{C}_0 \to X$ has dimension $3n$ by the splitting type of $\Theta_X|_{L_i}$. This shows that, locally around $D\langle x_+, x_-\rangle$, we have the dimension estimate $\dim W \geq \dim D + 1$.

Recall that $W\langle x_+, x_-\rangle$ [resp. $D\langle x_+, x_-\rangle$] is naturally identified with the inverse image of (x_+, x_-) via the natural projection $V^{(2)} \to X \times X$ [resp. $V_D^{(2)} \to X \times X$]. Then elementary dimension count gives the following equalities:

$$\dim W\langle x_+, x_-\rangle = \dim W + 2 - 2n,$$
$$\dim D\langle x_+, x_-\rangle = \dim D + 2 - 2n,$$

so that $\dim W\langle x_+, x_-\rangle \geq \dim D\langle x_+, x_-\rangle + 1$. In other words, there is an irreducible member $C \supset \{x_+, x_-\}$ which is a deformation of $L_+ \cup L_-$.

The above deformation argument also shows that W is smooth at a general point of D, and so is $W\langle x_+, x_-\rangle$ at a general point of $D\langle x_+, x_-\rangle$. It is easy to show that $\overline{V}\langle x_+, x_-\rangle$ is smooth along a general member $L_+ \cup L_-$ of $D\langle x_+, x_-\rangle$. (Analytically-locally, it looks like $\mathcal{C} \times D\langle x_+, x_-\rangle$).

We list below a few corollaries of Proposition 3.2.

Corollary 3.3. *Let T be a smooth curve and $f: T \to W\langle x_+, x_-\rangle$ a non-constant morphism with image not contained in $D\langle x_+, x_-\rangle$. Then the normalisation of the fibre product $T \times_{W\langle x_+, x_-\rangle} V\langle x_+, x_-\rangle$ is a strongly $\mathrm{pr}_X^* H$-symmetric conic bundle over T.*

Proof. By construction.

Corollary 3.4. *Let V_t be an arbitrary irreducible fibre of the family $V\langle x_+, x_-\rangle \to W\langle x_+, x_-\rangle$. Then $C = \mathrm{pr}_X(V_t)$ is not contained in the divisor $Y \subset X$. Given any non-empty irreducible closed subset $R \subset W\langle x_+, x_-\rangle \backslash D\langle x_+, x_-\rangle$ of dimension $\leq n-2$ and the associated subfamily $V_R\langle x_+, x_-\rangle \to R$, the image $\mathrm{pr}_X(V_R\langle x_+, x_-\rangle) \subset X$ can neither contain any irreducible component of $Y_\pm$ nor be contained in $Y_\pm$.*

Proof. If C is contained in Y, then C must be contained in one of Y_+, Y_- because C is irreducible. Then C cannot pass one of the $x_\pm$, which is absurd. In particular, the irreducible constructible set $\mathrm{pr}_X(V_R\langle x_+, x_-\rangle)$ of dimension $\leq n-1$ cannot be contained in any of the irreducible component of the divisor $Y_\pm$ and and so cannot contain any component of $Y_\pm$.

Consider the fibre product $V^{(3)} = V \times_W V \times_W V$ with the natural projection $\mathrm{pr}_X^{(3)}: V^{(3)} \to X \times X \times X$.

Corollary 3.5. *In the above notation, we have*

(1) $\dim W = 3n-3$ *(near* $W\langle x_+, x_-\rangle$*). and the projection* $\mathrm{pr}_X^{(3)}\colon V^{(3)} \to X \times X \times X$ *is dominant.*

(2) *A general element* $[C] \in W$ *is irreducible and if* $f\colon \mathbb{P}^1 \to C \subset X$ *is the normalisation,* $f^*\Theta_X \simeq \mathcal{O}(2)^{\oplus n}$.

Proof. Everything is considered around $W\langle x_+, x_-\rangle$.

By the condition $C(-K_X) = 2n$, $[C] \in W$, we have the inequality $\dim W \geq 3n-3$, so that $\dim V^{(3)} \geq 3n$. Hence (1) follows if we check that the inverse image $(\mathrm{pr}_X^{(3)})^{(-1)}(x_+, x_-, x_0)$ of a general point (x_+, x_-, x_0) of $\mathrm{pr}_X^{(3)}(V^{(3)})$ is finite. The inverse image of $\{(x_+, x_-)\} \times X$ is naturally identified with $V\langle x_+, x_-\rangle$, family of rational curves passing through x_+, x_-, parametrised by the closed subset $W\langle x_+, x_-\rangle \subset W$. By (2.5), the projection $V\langle x_+, x_-\rangle \to X$ is finite over $X \setminus \{x_+, x_-\}$, which in particular means that the inverse image of x_0 is finite.

Take a general nonsingular point $[C] \in W\langle x_+, x_-\rangle$ and let $\overline{V}\langle x_+, x_-\rangle$ denote the normalisation of $V\langle x_+, x_-\rangle$. $\overline{V}\langle x_+, x_-\rangle$ is locally a $\mathbb{P}^1$ bundle over a small smooth neighbourhood of $[C]$. Since the projection $V\langle x_+, x_-\rangle \to X$ is dominant, the natural map $\Theta_{\overline{V}\langle x_+,x_-\rangle} \to \overline{\mathrm{pr}}_X^*\Theta_X$ is surjective at a general point of $\overline{C}$. This means that $\mathrm{H}^0(\overline{C}, f^*\Theta_X(-x_+ - x_-))$ and $\Theta_{\overline{C}} \simeq \mathcal{O}(2)$ generates a subsheaf of rank n of $f^*\Theta_X$. It follows that the direct sum decomposition $f^*\Theta_X \simeq \oplus\mathcal{O}(d_i)$ satisfies $d_i \geq 2$, while $\sum d_i = C(-K_X) = 2n$, whence follows (2).

Let $\overline{V}\langle x_+, x_-\rangle$ and $\overline{W}\langle x_+, x_-\rangle$ denote the normalisation of $V\langle x_+, x_-\rangle$ and $W\langle x_+, x_-\rangle$, and let

$$\overline{\mathrm{pr}}_{\overline{W}}\colon \overline{V}\langle x_+, x_-\rangle \to \overline{W}\langle x_+, x_-\rangle,\ \overline{\mathrm{pr}}_X\colon \overline{V}\langle x_+, x_-\rangle \to X$$

be the natural projections. Let $\overline{D}\langle x_+, x_-\rangle \subset \overline{W}\langle x_+, x_-\rangle$ denote the inverse image of $D\langle x_+, x_-\rangle$. It is known that $\overline{V}\langle x_+, x_-\rangle$ is a $\mathbb{P}^1$-bundle if restricted over the open subset

$$\overline{W}^\circ\langle x_+, x_-\rangle = \overline{W}\langle x_+, x_-\rangle \setminus \overline{D}\langle x_+, x_-\rangle.$$

Let $R \subset \overline{V}^\circ\langle x_+, x_-\rangle$ be the ramification locus of $\overline{\mathrm{pr}}_X|_{\overline{V}^\circ\langle x_+,x_-\rangle}$.

The following statement follows from standard deformation theory:

Proposition 3.6. *In the notation above,* R *is the union of* σ_+, σ_- *and* $\overline{\mathrm{pr}}_{\overline{W}}^{-1}(B)$, *where* $B \subset \overline{W}^\circ\langle x_+, x_-\rangle$ *is the closed subset*

$$\{s \in \overline{W}^\circ\langle x_+, x_-\rangle;\ \overline{\mathrm{pr}}_X^*\Theta_X|_{\overline{V}_s} \not\simeq \mathcal{O}(2)^{\oplus n}\}.$$

Combined with (3.4) and (3.5), this means

Corollary 3.7. *The closure of $\overline{\mathrm{pr}}_X(R) \subset X$ does not contain $Y_\pm = \mathrm{pr}_X(F\langle x_\pm\rangle)$. In particular, if $[L_+ \cup L_-]$ is a general member of $D\langle x_+, x_-\rangle$, any closed subset Γ of the inverse image $\overline{\mathrm{pr}}_X^{-1}(L_+) \subset \overline{V}\langle x_+, x_-\rangle$ is not contained in the closure of R as long as Γ surjects onto L_1.*

Let $\Gamma \subset \overline{V}\langle x_+, x_-\rangle$ be an irreducible curve which surjects onto $L_+ \subset X$, where $[L_+ \cup L_-]$ is a general point of $D\langle x_+, x_-\rangle$. There are three cases:

A. Γ is contained in a fibre of $\overline{\mathrm{pr}}_{\overline{W}}$ (in this case, Γ is simply the normalisation of the first irreducible component L_+ of the fibre $L_+ \cup L_- \subset V\langle x_+, x_-\rangle$).

B. $\overline{\mathrm{pr}}_{\overline{W}}(\Gamma)$ is a curve on $\overline{D}\langle x_+, x_-\rangle$.

C. $\overline{\mathrm{pr}}_{\overline{W}}(\Gamma)$ is a curve not contained in $\overline{D}\langle x_+, x_-\rangle$.

Lemma 3.8. *In Case* **B**, Γ *does not intersect* σ_+, σ_-.

Proof. It is trivial that $\Gamma \not\ni \sigma_-$ because $L_+ \not\ni x_-$. The fibre space $\overline{V}_D\langle x_+, x_-\rangle \to \overline{D}\langle x_+, x_-\rangle$ is a union of two irreducible components $F^*\langle x_+\rangle$ and $F^*\langle x_-\rangle$, and Γ must be a curve on $F^*\langle x_+\rangle$. $F^*\langle x_+\rangle \to \overline{D}\langle x_+, x_-\rangle$ is the base change of the fibre space $F\langle x_+\rangle \to S\langle x_+\rangle$ given by the finite morphism $D\langle x_+, x_-\rangle \to S\langle x_+\rangle$. In particular, $\Gamma \subset F^*\langle x_+\rangle$ comes from a curve $\Gamma_0 \subset \overline{F}\langle x_+\rangle$.

$\overline{F}\langle x_+\rangle$ is a $\mathbb{P}^1$-bundle over $\overline{S}\langle x_+\rangle$ and surjects onto the divisor Y_+. Furthermore $\overline{\mathrm{pr}}_X(\Gamma_0) = L_+$ passes through a general point of Y_+. This means that the differential homomorphism $\Theta_{\overline{F}\langle x_+\rangle} \to \overline{\mathrm{pr}}_X^*\Theta_X$ is of rank $n-1$ at a general point of Γ_0.

Put $\Delta = \overline{\mathrm{pr}}_{\overline{S}}(\Gamma_0) \subset \overline{S}\langle x_+\rangle$. Given a finite morphism $\tilde{\Delta} \to \Delta$, let $\mathcal{F} \to \tilde{\Delta}$ denote the induced $\mathbb{P}^1$-bundle over $\tilde{\Delta}$ with the natural finite-to-one morphisms $h\colon \mathcal{F} \to \overline{F}\langle x_+\rangle$ and generically finite-to-one morphism $f = \overline{\mathrm{pr}}_X h\colon \mathcal{F} \to X$. $\mathcal{F}$ carries the specified section $\tilde{\sigma}_+ = h^{-1}(\sigma_+)$. If $\tilde{\Delta}$ is suitably chosen, the inverse image of Γ_0 is a union of sections σ_i. By construction,

$$\Theta_{\mathcal{F}}|_{\sigma_i} \subset h^*\Theta_{\overline{F}\langle x_+}|_{\sigma_i} \subset f^*\Theta_X|_{\sigma_i},$$

inducing an injection $\mathcal{N}_{\sigma_i/\mathcal{F}} \hookrightarrow f^*\mathcal{N}_{L_+/X}$.

Recalling the isomorphism $\mathcal{N}_{L_+/X} \simeq \mathcal{O}(1)^{\oplus n-2} \oplus \mathcal{O}$, we infer that the self intersection number σ_i^2 of the effective divisor $\sigma_i \subset \mathcal{F}$ is bounded from above by a_i, where a_i is the mapping degree of the surjection $\sigma_i \to L_+$.

Let H be an ample divisor on X and set $d = L_+H$. Then, for each fibre $\mathcal{F}_s$, $s \in \tilde{\Delta}$, we have three equalities $F_s f^*H = d$, while $\tilde{\sigma}_+ f^*H = 0$, $\sigma_i f^*H = a_i d$. Since the Néron-Severi group of $\mathcal{F}$ is generated by $\tilde{\sigma}_+$ and the fibre $\mathcal{F}_s$, the first two equalities yield the numerical equivalence $f^*H \approx d(\tilde{\sigma}_+ + e\mathcal{F}_s)$, where $e = -\tilde{\sigma}_+^2 > 0$. Similarly, if we put $\sigma_i \approx \tilde{\sigma}_+ + a'_i\mathcal{F}_s$ for a suitable integer a'_i, the third equality gives

$$a_i d = \sigma_i f^*H = (\tilde{\sigma}_+ + a'_i\mathcal{F}_s)f^*H = a'_i\mathcal{F}_s f^*H = a'_i d,$$

so that $a'_i = a_i$. Then the inequality $\sigma_i^2 \leq a_i$ shown above is rewritten into

$$a_i \geq \sigma_i^2 = (\tilde{\sigma}_+ + a_i F_s)^2 = -e + 2a_i,$$

or, equivalently, $a_i \leq e$, and we get the inequality $\sigma_i\tilde{\sigma}_+ = -e + a_i \leq 0$. By our assumption $\sigma_i \neq \tilde{\sigma}_+$, this means that σ_i does not meet $\tilde{\sigma}_+$ for every i. In other words, Γ is off σ_+.

Lemma 3.9. *In Case* **C**, Γ *does not intersect* σ_+, σ_-.

Proof. In this case, $\hat{\Delta} = \overline{\mathrm{pr}}_{\overline{W}}(\Gamma)$ is not contained in $B \subset W\langle x_+, x_-\rangle$, where $R = \overline{\mathrm{pr}}_{\overline{S}}^{-1}(B)$ is the ramification locus of $\overline{\mathrm{pr}}_X$. By taking a suitable covering $\tilde{\Delta} \to \hat{\Delta}$, we get a conic bundle $\mathcal{C} = \tilde{\Delta} \times_{\overline{W}\langle x_+,x_-\rangle} \overline{V}\langle x_+, x_-\rangle \to \tilde{\Delta}$, on which the inverse image of Γ is a union of sections Γ_i. Let $\tilde{\mathcal{C}} \to \mathcal{C}$ be the minimal resolution, $\tilde{\Gamma}_i$ and $\tilde{\sigma}_\pm$ being the strict transforms of Γ_i and $\sigma_\pm$. (The $\tilde{\sigma}_\pm$ are also the total transforms because $\sigma_\pm$ lies on the nonsingular locus of $\mathcal{C}$.) In this situation, what we have to show is that the divisor $\tilde{\Gamma}_i$ on $\tilde{\mathcal{C}}$ is away from the specified sections $\tilde{\sigma}_\pm$. By (2.4.5) and (2.4.6), this will follow from the inequality $\tilde{\Gamma}_i^2 \leq 0$.

In order to establish this inequality, we start with the following observation.

Let $G \supset \Gamma$ be an irreducible component of the closed subset $\overline{\mathrm{pr}}_X^{-1}(Y_+) \subset \overline{V}\langle x_+, x_-\rangle$. G is a divisor on $\overline{V}\langle x_+, x_-\rangle$ which surjects onto $\overline{W}\langle x_+, x_-\rangle$. In particular, G is a multi-section of the (generically) conic fibration $\overline{V}\langle x_+, x_-\rangle \to \overline{W}\langle x_+, x_1\rangle$. At a general closed point of Γ (which is also a general closed point of G), we have local isomorphisms

$$\Theta_{\overline{V}\langle x_+,x_-\rangle} \simeq \overline{\mathrm{pr}}_X^*\Theta_X$$
$$\overline{\mathrm{pr}}_{\overline{W}}^*\Theta_{\overline{W}\langle x_+,x_-\rangle} \simeq \Theta_G \simeq \overline{\mathrm{pr}}_X^*\Theta_{Y_+},$$

implying that the composite of natural homomorphisms

$$\begin{aligned}\Theta_{\overline{V}\langle x_+,x_-\rangle/\overline{W}\langle x_+,x_-\rangle}|_\Gamma &\to \Theta_{\overline{V}\langle x_+,x_-\rangle}|_\Gamma \to (\overline{\mathrm{pr}}_X|_\Gamma)^*(\Theta_X|_{L_+}) \\ &\to (\overline{\mathrm{pr}}_X|_\Gamma)^*\big((\Theta_X|_{L_+}/(\Omega^1_{Y_+}|_{L_+})^*\big)/(\text{torsion}) \simeq \mathcal{O}_\Gamma\end{aligned}$$

is non-zero (and hence injective) at a general point of Γ. (Here we used the fact that $\Theta_X|_{L_+} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-2} \oplus \mathcal{O}$, and its ample part of rank $n-1$ exactly corresponds to Θ_{Y_+} at a general point of L_+.)

Going back to the conic bundle $\tilde{\mathcal{C}} \to \tilde{\Delta}$ with the section $\tilde{\Gamma}_i$, this observation tells us that, at a general point of Γ_i, the natural homomorphism $\mathcal{N}_{\tilde{\Gamma}_i,\mathcal{C}} \simeq \Theta_{\tilde{\mathcal{C}}/\tilde{\Delta}}|_{\Gamma_i} \to \mathcal{O}_{\tilde{\Gamma}_i}$ is non-zero. This shows that $\deg \mathcal{N}_{\tilde{\Gamma}_i/\tilde{\mathcal{C}}} = \tilde{\Gamma}_i^2 \leq 0$.

Lemma 3.10. *No member C of $W\langle x_+, x_-\rangle$ has a cuspidal singularity at x_+ or x_-.*

Proof. Let $W\langle\mathrm{cusp}\rangle \subset W$ denote the closure of the locus of irreducible cuspidal curves. Let $V\langle\mathrm{cusp}\rangle$ be the associated family and $\Sigma \subset V\langle\mathrm{cusp}\rangle$ the locus of the cuspidal points of the fibres. What we are going to show is that the natural projection $\Sigma \times_{W\langle\mathrm{cusp}\rangle} V\langle\mathrm{cusp}\rangle \to X \times X$ is not surjective, meaning that there is no member of W which has a cusp at x_+ and passes through x_- when (x_+, x_-) are general.

For simplicity of the notation, we put

$$\begin{aligned} Z &= V\langle\mathrm{cusp}\rangle \times_{W\langle\mathrm{cusp}\rangle} V\langle\mathrm{cusp}\rangle, \\ \Sigma_1 &= \Sigma \times_{W\langle\mathrm{cusp}\rangle} V\langle\mathrm{cusp}\rangle \\ \Sigma_2 &= V\langle\mathrm{cusp}\rangle \times_{W\langle\mathrm{cusp}\rangle} \Sigma. \end{aligned}$$

Suppose that Σ_1 and Σ_2 dominate $X \times X$ via the natural projection $Z \to X \times X$. Let $Z \xrightarrow{g} Y \xrightarrow{h} X \times X$ be the Stein factorisation: namely, h is finite and the fibre Z_y of g over a general point $y \in Y$ is an irreducible variety. Our hypothesis amounts to the condition $\dim Z_y \cap \Sigma_i = a \geq 0$, so that $\dim Z_y = a + 1 \geq 1$. Hence we can find an irreducible curve $f\colon T \to Z_y \subset Z = V\langle\mathrm{cusp}\rangle \times_{W\langle\mathrm{cusp}\rangle} V\langle\mathrm{cusp}\rangle$ such that

($\clubsuit$) *$f(T)$ is not contained in $\Sigma_1 \cup \Sigma_2$ but connects these two divisors.*

Let T' be the image of $f(T)$ in $W\langle\mathrm{cusp}\rangle$. Every member C of T' contains $\{x_+, x_-\}$, where (x_+, x_-) is the image of $y \in Y$ in $X \times X$. The condition ($\clubsuit$) above says that a general member of T' has no cusp at $x_\pm$ but some member does; thus $T' \subset W\langle\mathrm{cusp}\rangle$ defines a nontrivial one-parameter family of cuspidal curves passing through x_+, x_-. However, (2.6) asserts that the cupidal locus cannot pass through one of the $x_\pm$, which contradicts our construction.

Corollary 3.11. *Let $\mathfrak{M}_{x_+} \subset \mathcal{O}_X$ be the maximal ideal which defines x_+. Then*

(1) *$\mathfrak{M}_{x_+}\mathcal{O}_{\overline{V}\langle x_+,x_-\rangle} = \mathfrak{J}(-\sigma_+)$, where $\mathfrak{J} \subset \mathcal{O}_{\overline{V}\langle x_+,x_-\rangle}$ is an ideal*

sheaf of a closed subscheme away from σ_+*.*

(2) *If* C *is a general member of* $W\langle x_+, x_-\rangle$*, then*

$$\overline{\mathrm{pr}}_X^{-1} C = \sigma_+ + \sigma_- + \overline{V}_{[C]} + B + E$$

where B *is a union of finitely many curves away from* $\sigma_+ \cup \sigma_-$ *and* E *is a finite set* $\subset \overline{V}\langle x_+, x_-\rangle \setminus (\sigma_+ \cup \sigma_-)$ *such that* $\overline{\mathrm{pr}}_X(E) \subset \{x_+, x_-\}$*.*

Proof. (1) The statement is a direct consequence of (3.10). In particular, on an open neighbourhood of the Cartier divisor σ_+, the projection $\overline{\mathrm{pr}}_X : \overline{V}\langle x_+, x_-\rangle \to X$ lifts to a morphism $\tilde{\mathrm{pr}}_X$ to $\mathrm{Bl}_{x_+}(X)$. The scheme theoretic inverse image of C in $\mathrm{Bl}_{x_+} X$ is $\mathcal{I}_{\tilde{C}}(-E_+)$, where $\tilde{C}$ the strict transform and E_+ the exceptional divisor.

(2) Since $\overline{\mathrm{pr}}_X$ is finite over $X \setminus \{x_+, x_-\}$ and $C \subset X$ is a locally complete intersection of codimension $n-1$, it is clear that there is a decomposition of the above type and we have only to show that B is away from $\sigma_+ \cup \sigma_-$. If it meets $\sigma_+ \cup \sigma_-$ for general C, then the same should hold for any specialisation of C, which is not the case for $C = L_+ + L_-$ by (3.8) and (3.9).

Corollary 3.12. *Take a small open analytic neighbourhood* U^* *of* x_+ *in* X*. Then* $\overline{\mathrm{pr}}_X^{-1}(U^*) \subset \overline{V}\langle x_+, x_-\rangle$ *is a disjoint union of a small open neighbourhood* U *of* σ_+ *and an extra open subset* U'*.*

The X*-projection* $\overline{\mathrm{pr}}_X$ *induces proper bimeromorphic morphisms between a small analytic neighbourhood* $U \to U^*$ *finite over* $U^* \setminus \{x_+\}$ *and* $U \to \tilde{U}^*$*, where* $\tilde{U}^* \subset \mathrm{Bl}_{x_+}(X)$ *is the inverse image of* U^**. In particular, the ramification locus of* $\overline{\mathrm{pr}}_X$ *has codimension* ≥ 2 *on* $\overline{V}\langle x_1, x_2\rangle \setminus (\sigma_+ \cup \sigma_-)$*.*

Proof. Let C be a general member of $W\langle x_+, x_-\rangle$ and $x \in C$ a closed point sufficiently close to x_+ but not equal to $x_\pm$. Then (3.11) asserts that $\overline{\mathrm{pr}}_X^{-1}(x) \cap U = \{([C], x)\}$, a single point. Hence $\overline{\mathrm{pr}}_X$ is a bimeromorphism on U, finite over $U^* \setminus \sigma_+$.

If ramification locus of $\overline{\mathrm{pr}}_X$ contains an $(n-1)$-dimensional irreducible component $R_0 \neq \sigma_\pm$, it must be a pull-back $\overline{\mathrm{pr}}_{\overline{W}}^* B_0$ of a divisor B_0 on $\overline{W}\langle x_+, x_-\rangle$. However, this contradicts the fact that $\overline{\mathrm{pr}}_X$ is unramified in codimension one on $U \setminus \sigma_+$.

Corollary 3.13. (1) $\overline{\mathrm{pr}}_X$ *is birational.*
(2) $\overline{V}\langle x_+, x_-\rangle \setminus (\sigma_+ \cup \sigma_-) \simeq X \setminus \{x_+, x_-\}$*.*
(3) $\overline{W}\langle x_+, x_-\rangle$ *is nonsingular.*

(4) *There are isomorphisms*

$$\overline{V}\langle x_+, x_-\rangle \simeq V\langle x_+, x_-\rangle \simeq \mathrm{Bl}_{\{x_+,x_-\}}(X)$$
$$\overline{W}\langle x_+, x_-\rangle \simeq W\langle x_+, x_-\rangle \simeq E_\pm \simeq \mathbb{P}^{n-1}.$$

Proof. Because the Fano manifold X is smooth and simply connected, a generically finite morphism $f\colon Y \to X$ with branch locus of dimension $\leq n-2$ is necessarily birational, whence (1) follows. By Grothendieck's version of Zariski's Main Theorem, the inverse image of x_+ in $\overline{V}\langle x_+, x_-\rangle$ is connected, so that $\mathfrak{M}_{x_\pm}\mathcal{O}_{\overline{V}\langle x_+,x_-\rangle} = \mathcal{O}(-\sigma_\pm)$. This shows that $\overline{F}\langle x_+, x_-\rangle \to X \setminus \{x_+, x_-\}$ is a well defined, proper, finite, birational morphism, and hence an isomorphism.

Since $\overline{\mathrm{pr}}_{\overline{W}}|_{\overline{V}\langle x_+,x_-\rangle\setminus(\sigma_+\cup\sigma_-)}$ has reduced fibres and hence admits an analytic local section over any closed point $w \in \overline{W}\langle x_+, x_-\rangle$, the smoothness of the total space implies that of the base space, *i.e.*, the assertion (3).

The fibre space $V\langle x_+, x_-\rangle$ has fibres smooth near the section $\sigma_\pm$, and hence the smoothness of the base $\overline{W}\langle x_+, x_-\rangle$ is inherited by the total space near $\sigma_\pm$, thereby showing the global smoothness of $\overline{F}\langle x_+, x_-\rangle$. Once the smoothness is established, the purity of ramification loci tells us that the naturally induced morphism $\overline{\mathrm{pr}}_{\tilde{X}}\colon \overline{V}\langle x_+, x_-\rangle \to \mathrm{Bl}_{\{x_+,x_-\}}(X)$, which has ramification of codimension ≥ 2, is an isomorphism, thereby inducing $\sigma_\pm \simeq E_\pm$.

We now arrive at the conclusion:

Corollary 3.14. *The pullback* $\mathrm{pr}_W^* L$ *of the hyperplane divisor* L *on* $W\langle x_+, x_-\rangle \simeq \mathbb{P}^{n-1}$ *is linearly equivalent to* $\mathrm{pr}_X^* H_0 - \sigma_+ - \sigma_-$*, where* H_0 *is an ample divisor on* X *with* $H_0^n = 2$*. The linear system* $|H_0|$ *is free from base points, defining an isomorphism* $X \to Q_n \subset \mathbb{P}^{n+1}$*.*

Proof. Since $\mathrm{pr}_W^* L$ cuts out a hyperplane from the section $\sigma_\pm = \tilde{\mathrm{pr}}_X^* E_\pm$, it is linearly equivalent to $\mathrm{pr}_X^* H_0 - \sigma_+ - \sigma_-$. It follows that $H_0^n = L^n + 2 = 2$. Noting that $|L|$ is free from base point, we see that $|H_0|$ has no base point outside $\{x_+, x_-\}$. On the other hand, since $\mathrm{Pic}(X) \simeq \mathbb{Z}$ is discrete, the linear system $|H_0|$ does not depend on the choice of the general base points $x_\pm \in X$, meaning that it is free from base points and has dimension $\dim |L| + 2 = n + 2$.

The semiample divisor H_0 is ample or, equivalently, $(\Gamma, H_0) > 0$ for every irreducible curve $\Gamma \subset X$. Indeed, for every irreducible curve $\Gamma \not\subset \sigma_+ \cup \sigma_-$, we have $(\Gamma, \tilde{\mathrm{pr}}_X^* H_0) \geq (\Gamma, \mathrm{pr}_W^* L)$, the equality holding if and only if Γ is away from $\sigma \cup \sigma_2$. By construction $(\Gamma, \mathrm{pr}_W^* L) \geq 0$, the equality holding if and only if Γ is an irreducible component of the fibre of pr_W. Hence $(\Gamma, \tilde{\mathrm{pr}}_X^* H_0) \geq (\Gamma, \mathrm{pr}_W^* L) \geq 0$ and at least one of the inequalities is strict.

Thus $|H_0|$ defines a finite morphims $X \to \mathbb{P}^{n+1}$ onto a non-degenerate hypersurface of degree ≤ 2, which is necessarily an isomorphism onto a hyperquadric.

§4. Proof of main theorems and concluding remarks

Let us complete the proof of (0.1) and (0.2).

In Theorem 0.1, the implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial, and it suffices to show that $\rho(X) = 1$ when the global length $l(X) = n$.

Lemma 4.1. *Let X be a Fano n-fold of dimension $n \geq 3$. If $l(X) = n$, then the Picard number $\rho(X)$ is one.*

Proof. Suppose $\rho(X) \geq 2$. Fix an extremal ray, and we have a non-trivial extremal contraction $\pi\colon X \to Y$ (see, for instance, [7, §3]). The fibre of a closed point $y \in Y$ is uniruled.

When π is birational, take the exceptional locus E of π. Let $C \subset E$ be a rational curve which is contracted to a point in Y and suppose that $(C, -K_X)$ attains the minimum among such curves. Then any deformation of the normalisation morphism $f\colon \mathbb{P}^1 \to C$ belongs to $\mathrm{Hom}(\mathbb{P}^1, E)$, and thanks to [2, Theorem 2.8] we have

$$(C,-K_X)+n \leq \dim_{[f]}\mathrm{Hom}(\mathbb{P}^1,X) = \dim_{[f]}\mathrm{Hom}(\mathbb{P}^1,E) \leq 2\dim E+1 \leq 2n-1,$$

contradicting the inequality $(C, -K_X) \geq n$.

In case X is a fibre space over Y, take a rational curve C contained in a smooth fibre X_y, and assume that $(C, -K_X)$ attains the minimum among such. Then we have $\dim X_y + 1 \geq (C, -K_{X_y}) = (C, -K_X) \geq n$, so that $\dim X_y = n-1$ and $X_y \simeq \mathbb{P}^{n-1}$. Choose another extremal ray inducing a sectond morphism $\varphi\colon X \to Z$. By what we have seen before, φ defines another fibre space structure on X.

A fibre X_y of π is $\mathbb{P}^{n-1} \subset X$ which is non-trivially mapped to Z, a projective variety. The pullback of an ample divisor H on Z is non-trivial on $X_y \simeq \mathbb{P}^{n-1}$ and hence ample, so that H^{n-1} cannot be numerically trivial on Z. In particular $\dim Z \geq n-1$, and a general fibre X_z of φ must be $\mathbb{P}^1$ with $(X_z, -K_X) = 2 < n$, another contradiction.

Remark 4.2. In Theorem 0.1, we cannot drop the condition $\rho(X) = 1$ in (3). For instance, let A be a smooth hypersurface of degree $d \leq n$

of the linear subspace $\mathbb{P}^{n-1} = H = \{x_n = 0\} \subset \mathbb{P}^n$ and let $\mu\colon X \to \mathbb{P}^n$ be the blowup along A. X is a smooth Fano manifold with $\rho(X) = 2$, $-K_X = (n+1)\mu^*H - E$, where E stands for the exceptional divisor. If $x_0 \in X \setminus H$, then the local length $l(X, x_0)$ is n, which is attained by the strict transforms of the lines connecting x_0 and A.[5] In this case, any curve C with $(C, -K_X) = -2n$, $C \ni x_+, x_-$ is a disjoint union of two components provided $x_\pm$ are general.

In Theorem 0.2, the following implication relations are trivial:

— (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2),
— (1) $\Rightarrow$ (6),
— (1) $\Rightarrow$ (7),

while the equivalence between (1)(2)(3) were established by (4.1). Thus it suffices to check the implications (6) $\Rightarrow$ (2) and (7) $\Rightarrow$ (3) to complete the proof of (0.2).

The implication (6) $\Rightarrow$ (3) follows from

Lemma 4.3. *Let X be a smooth Fano n-fold, $n \geq 3$. If $\wedge^2\Theta_X$ is ample, then $l(X) \geq n$.*

Proof. Let C be an arbitrary rational curve on X and let $\nu\colon \mathbb{P}^1 \to C \subset X$ denote a birational map induced by the normalisation of C. Put $\nu^*\Theta_X \simeq \bigoplus_{i=1}^n \mathcal{O}(d_i)$, $d_1 \leq d_2 \leq \cdots \leq d_2$. Then the condition on $\wedge^2\Theta_X$ implies that $2d_2 \geq d_2 + d_1 \geq 1$. If $d_1 \geq 1$, then $(C, -K_X) = \sum_i d_i \geq n$. If $d_1 = 0$, then $d_1 + 0 \geq 1$, while $d_n \geq 2$ thanks to the inclusion $\Theta_{\mathbb{P}^1} \simeq \mathcal{O}(2) \subset \nu^*\Theta_X$. Suppose that $d_1 < 0$. Then $d_2 \geq -d_1 + 1$ so that

$$(C, -K_X) = \sum_{i=1}^n d_i = d_1 + \sum_{i=2}^n d_i \geq d_1 + (n-1)(-d_1+1) = n-1+(n-2)(-d_1).$$

Since $n \geq 3$, we have $(C, -K_X) \geq n$ whenever $d_1 < 0$.

Finally we have

Lemma 4.4. *Assume that $n \geq 3$. Let $f\colon Q_m \to X$ be a surjective morphism from an smooth hyperquadric in $\mathbb{P}^{m+1}$ to a smooth projective vaiety of dimension n. Then X is a Fano n-fold with Picard number one and the local length satisfies $l(X, x_0) \geq n$ if x_0 is away from the branch locus of f.*

Proof. Because $\rho(Q_m) = 1$, $m \geq n \geq 3$, the pullback f^*H of the hyperplane bundle H on X is ample, implying $m = n$ and the equality

[5] The global length $l(X)$ is of course 1 attained by the fibres of the $\mathbb{P}^1$-bundle $E \to A$.

$\rho(X) = 1$ as well. Because $\mathcal{O}(-f^*K_X)$ contains the ample line bundle $\mathcal{O}(-f^*K_{Q_n})$, X with Picard number one must be Fano.

Let $C \subset X$ be a rational curve passing through x_0 and $\nu\colon \mathbb{P}^1 \to C \subset X$ the normalisation morphism. Then $\nu^*\Theta_X \simeq \sum_i \mathcal{O}(d_i)$, $d_1 \leq d_2 \leq \cdots \leq d_n$, $d_n \geq 2$. Hence $(C, K_X) \geq n$ follows if we show that $d_1 \geq 0$, $d_2 \geq 1$.

Let $\Gamma \subset Q_n$ be an irreducible curve which surjects onto C, with normalisation $\nu'\colon \tilde{\Gamma} \to \Gamma \subset Q_n$. Consider the commutative diagram

$$\begin{array}{ccccc}
\tilde{\Gamma} & \xrightarrow[\nu']{} & \Gamma & \longrightarrow & Q_n \\
\downarrow \tilde{f}_\Gamma & & \downarrow f_\Gamma & & \downarrow f \\
\mathbb{P}^1 & \xrightarrow[\nu]{} & C & \longrightarrow & X.
\end{array}$$

Noticing that the ramification locus of f does not contain Γ (which meets $f^{-1}(x_0)$), we have a natural inclusion

$$\nu'^*\Theta_{Q_n} \subset \nu'^*f^*\Theta_X = \tilde{f}_\Gamma^*\Big(\bigoplus_{i=1}^n \mathcal{O}(d_i)\Big).$$

Then the semipositivity of Θ_{Q_n} gives $d_1 \geq 0$, while the ampleness of $\wedge^2\Theta_{Q_n}$ yields $2d_2 \geq d_1 + d_2 > 0$.

Remark 4.5. In the proof of Theorem 0.1, we used the condition that X is nonsingular in order to establish the dimension estimates for $S\langle x_\pm\rangle$, $W\langle x_+, x_-\rangle$ and the birationality (generic one-to-one property) of pr_X. If we relax the smoothness condition into normality, we obtain the following

THEOREM. *Let X be a normal, projective, $\mathbb{Q}$-factorial, $\mathbb{Q}$-Fano n-fold with Picard number one defined over the complex numbers. Let x_0 be a sufficiently general closed point of X and assume that any rational curve passing through x_0 deforms in $n-2$ independent parameters. Then X is a finite quotient of a normal hyperquadric $\subset \mathbb{P}^{n+1}$ (possibly with irreducible singular locus of dimension $\leq n-2$) by a finite group action without divisorial fixed point set. In particular, X is isomorphic to a normal hyperpquadric if and only if the open subset $X \setminus \mathrm{Sing}(X)$ is simply connected.*

The proof of Theorem 0.1 carries over into this situation without essential change. The variety $\overline{V}\langle x_+, x_-\rangle$ is now a two-point blowup of a normal hyperquadric, while pr_X is unramified over $X \setminus \big(\{x_+, x_-\} \cup \mathrm{Sing}(X)\big)$.

Remark 4.6. The author does not know if Theorem 0.1 stays true in positive characteristics. Almost all of our arguments work well regardless of the characteristic. The exceptions are those related to Sard's theorem, which, unfortunately, permeate throughout the paper. The most serious question to be checked is the separability of the projection $\mathrm{pr}_X : V\langle x_+, x_-\rangle \to X$.

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Meromorphic mappings and deficiencies

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Abstract.

In this note, we shall discuss elimination theorems of defects of hypersurfaces or rational moving targets for a meromorphic mapping or a holomorphic curve into $\mathbf{P}^n(\mathbf{C})$ by its small deformation.

§1. Introduction.

Value distribution theory is to study how intersects the image of a mapping to divisors in a target space. Liouville theorem asserts that the image of a meromorphic function is dense in the projective space $\mathbf{P}^1(\mathbf{C})$, and also Picard theorem asserts that the image covers all points on $\mathbf{P}^1(\mathbf{C})$ except for at most two points. Nevanlinna theory is a quantitative refinement of Picard theorem. Nevanlinna deficiency $\delta_f(a)$ express that $\delta_f(a) = 1$ if the image $f(\mathbf{C})$ omits a-point and $\delta_f(a) > 0$ if f covers a point a relatively few times. For a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, Nevanlinna's defect relations or Crofton's formulae assert that Nevanlinna defects or Valiron defects of a mapping are very few.

We shall now discuss on defects for a family of mappings, that is, elimination theorems of defects of hyperplanes, hypersurfaces or rational moving targets for a meromorphic mapping or a holomorphic curve into $\mathbf{P}^n(\mathbf{C})$ by its small deformation. Here a samll deformation $\tilde{f}$ of f means that the difference of order functions of $\tilde{f}$ and f is relatively small.

§2. Preliminaries.

Let $z = (z_1, ..., z_m)$ be the natural coordinate system in $\mathbf{C}^m$. Set

$$\langle z, \xi \rangle = \sum_{j=1}^{m} z_j \xi_j \text{ for } \xi = (\xi_1, ..., \xi_m), \|z\|^2 = \langle z, \bar{z} \rangle, \ B(r) = \Big\{ z \Big| \|z\| < r \Big\},$$

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$\partial B(r) = \left\{ z \Big| \|z\| = r \right\}$, $\psi = dd^c \log \|z\|^2$ and $\sigma = d^c \log \|z\|^2 \wedge \psi^{m-1}$, where $d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$ and $\psi^k = \psi \wedge \cdots \wedge \psi$ (k-times).

Let f be a nonconstant meromorphic mapping f of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and $\mathcal{L} = [\mathbf{H}^d]$ be the line bundle over $\mathbf{P}^n(\mathbf{C})$ which is determined by d-th tensor power of the hyperplane bundle $[\mathbf{H}]$. A hypersurface D of degree d in $\mathbf{P}^n(\mathbf{C})$ is given by the divisor of a holomorphic section $s \in H^0(\mathbf{P}^n(\mathbf{C}), \mathcal{O}(\mathcal{L}))$ which is determined by a homogeneous polynomial $P(w)$ of degree d. A metric $a = \{a_\alpha\}$ on the line bundle $\mathcal{L}$ is given by $a_\alpha = (\sum_{j=0}^n |w_j/w_\alpha|^2)^d$ in a neighborhood $U_\alpha = \{w \in \mathbf{P}^n(\mathbf{C}) | w_\alpha \neq 0\}$.

The Nevanlinna's order function $T_f(r, \mathcal{L})$ of f for the line bundle $\mathcal{L}$ is given by:

$$T_f(r, \mathcal{L}) := \int_{r_0}^{r} \frac{dt}{t} \int_{B(t)} f^* \omega \wedge \psi^{m-1},$$

where $\omega = \{\omega_\alpha\} = dd^c \log(\sum_{j=0}^n |w_j/w_\alpha|^2)^d$ in U_α. We say that f is transcendental if $\lim\limits_{r\to+\infty} \dfrac{T_f(r, \mathcal{L})}{\log r} = +\infty$. The norm of a section s is given by

$$\|s\|^2 := \frac{|s_\alpha|^2}{a_\alpha} = \frac{|P(w)|^2}{(\sum_{j=0}^n |w_j|^2)^d}.$$

The proximity function $m_f(r, D)$ of D is defined by

$$m_f(r, D) := \int_{\partial B} \log \frac{1}{\|s_f\|} \sigma = \int_{\partial B} \log \frac{\|f\|^d}{|P(f)|} \sigma.$$

The Nevanlinna deficiency $\delta_f(D)$ and the Valiron deficiency $\Delta_f(D)$ of D for f is defined by

$$\delta_f(D) := \liminf_{r\to\infty} \frac{m_f(r, D)}{T_f(r, \mathcal{L})} \text{ and } \Delta_f(D) := \limsup_{r\to\infty} \frac{m_f(r, D)}{T_f(r, \mathcal{L})}.$$

Using Stok's theorem, the Nevanlinna's order function $T_f(r) := T_f(r, [\mathbf{H}])$ of f for the hyperplane bundle $[\mathbf{H}]$ is written as:

$$T_f(r) = \int_{\partial B(r)} \log\Big(\sum_{j=0}^n |f_j|^2\Big)^{1/2} \sigma + O(1) = \int_{\partial B(r)} \log \sum_{j=0}^n |f_j| \sigma + O(1).$$

Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, and ϕ be a meromorphic mapping of $\mathbf{C}^m$ into the dual projective space $\mathbf{P}^n(\mathbf{C})^*$ which is called a moving target for f. Then the proximity function $m_f(r, \phi)$ of a moving target ϕ into $\mathbf{P}^n(\mathbf{C})^*$ is given by:

$$m_f(r, \phi) := \int_{\partial B} \log \frac{\|f\| \|\phi\|}{|\langle f, \phi \rangle|} \sigma.$$

The Nevanlinna deficiency $\delta_f(\phi)$ and the Valiron deficiency $\Delta_f(\phi)$ of a moving target ϕ for f are defined similarly. (See [5])

Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Then f has a reduced representation $(f_0 : \dots : f_n)$, and we write $f = (f_0, \dots, f_n)$ the same letter as the mapping f. Denote $D^\alpha f = (D^\alpha f_0, \dots, D^\alpha f_n)$ for a multi-index α, where $D^\alpha f_j = \partial^{|\alpha|} f_j / \partial z_1^{\alpha_1} \cdots \partial z_m^{\alpha_m}, \alpha = (\alpha_1, \dots, \alpha_m)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$.

Fujimoto [2] defined the generalized Wronskian of f by

$$W_{\alpha^0, \dots, \alpha^n}(f) = \det(D^{\alpha^k} f : 0 \leq k \leq n),$$

for $n+1$ multi-indices $\alpha^k = (\alpha_1^k, \dots, \alpha_m^k)$, $(0 \leq k \leq n)$.

§ 2-2. Some Results

Molzon-Shiffman-Sibony [6] defined the projective logarithmic capacity $C(E)$ of a set E on $\mathbf{P}^n(\mathbf{C})$, and they gave a criterion of positivity of projective logarithmic capacity for a subset of $\mathbf{P}^n(\mathbf{C})$

Proposition 1 ([3]). *Let f be a nonconstant meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Then, for $H \in \mathbf{P}^n(\mathbf{C})^*$,*

$$\lim_{r \to +\infty} \frac{m_f(r, H)}{T_f(r)} = 0,$$

outside a set $E \subset \mathbf{P}^n(\mathbf{C})^*$ of projective logarithmic capacity zero.

Proposition 2 ([3]).

$$\mathcal{A} := \left\{ (1, a_1, \dots, a_n, a_1^2, a_1 a_2, \dots, a_1^{i_1} \cdots a_n^{i_n}, \dots, \prod_{k=1}^{n} a_k^d) \mid a_j \in \mathbf{C} \right\}$$

is of positive projective logarithmic capacity.

§3. Elimination of defects of meromorphic mappings.

For a meromorphic mapping f of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, we can eliminate all defects by a small deformation of f.

Theorem 1. *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a given transcendental meromorphic mapping, and d is a positive integer. Then there exists a regular matrix $L = (l_{ij})_{0 \leq i,j \leq n}$ of the form $l_{i,j} = c_{ij} g_j + d_{ij}$, $(c_{ij}, d_{ij} \in \mathbf{C} : 0 \leq i, j \leq n)$ such that $\det L \neq 0$ and $\tilde{f} = L \cdot f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ is a meromorphic mapping without Nevanlinna defects of hypersurfaces of degree at most d, and satisfies $|T_f(r) - T_{\tilde{f}}(r)| = O(\log r)$ $(r \to \infty)$, where g_j $(j = 1, \dots, n)$ are some monomials on $\mathbf{C}^m$.*

Theorem 2. *Let $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be a given transcendental holomorphic curve. Then there exists a regular matrix $L = (l_{ij})_{0 \leq i,j \leq n}$ of the form $l_{i,j} = c_{ij} g_j + d_{ij}$, $(c_{ij}, d_{ij} \in \mathbf{C} : 0 \leq i, j \leq n)$ such that* $\det L \neq 0$ *and $\tilde{f} = L \cdot f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ is a holomorphic curve without Nevanlinna defects of rational moving targets and satisfies $|T_f(r) - T_{\tilde{f}}(r)| = o(T_f(r))(r \to \infty)$, where g_j $(j = 1, ..., n)$ are some transcendental entire functions on $\mathbf{C}$ satisfying $T_{g_j}(r) = o(T_{g_{j+1}}(r))$, $(j = 1, ..., n-1)$ and $T_{g_n}(r) = o(T_f(r))$ $(r \to \infty)$ which are constructed by using Edrei-Fuchs' theorem* [1].

Note that we cannot replace all transcendental entire functions g_j by rational functions.

Remark 1. In Theorem 1 and 2, mappings f may be linearly degenarate or of infinite order, and also if f is of finite order we can replace "Nevanlinna deficiency" by "Valiron deficiency" in the conclusion.

Remark 2. I first proved Theorem 1 for a meromorphic mapping $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ and hyperplanes [3], and also for a holomorphic curve $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ and hypersurfaces [4]. The case where $m > 1$ in Theorem 1 is not yet published. Theorem 2 is found in [5].

We now give a very short sketch of the proof of Theorem 1 for $m \geq 1$. We need following lemmas.

Lemma 1. *There are monomials $g_1, ..., g_n$ in $\mathbf{C}^m$ such that any n derivatives in $\{D^\alpha g := (D^\alpha g_1, ..., D^\alpha g_n) | |\alpha| \leq n+1\}$ are linearly independent over the field $\mathcal{M}$ of meromorphic functions on $\mathbf{C}^m$, where $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbf{Z}_{\geq 0}$ is a multi-index and $D^\alpha g_k = \partial^{|\alpha|} g_k / \partial {z_1}^{\alpha_1} \cdots \partial {z_m}^{\alpha_m}$.*

Lemma 2. *Let $h = (h_0 : h_1 : \cdots : h_n)$ be a reduced representation of a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and $g_1, ..., g_n$ linearly independent monomials as in Lemma 1. Then there exists $(\tilde{a}_1, ..., \tilde{a}_n)$ such that*

$$f := (h_0 : h_1 + \tilde{a}_1 g_1 h_0 : h_2 + \tilde{a}_2 g_2 h_0 : \cdots : h_n + \tilde{a}_n g_n h_0)$$

is a reduced representation of a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$.

Sketch of the proof of Theorem 1:
There is a regular linear change L_1 of $\mathbf{P}^n(\mathbf{C})$ such that $h := L_1 \cdot f \equiv (h_0 : \cdots : h_n) : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ is a reduced representation of the meromorphic mapping h which satisfies

$$m_h(r, H_j) = o(T_h(r)) \quad (r \to +\infty), \quad (j = 0, 1, .., n),$$

where $H_j = \{(w_0 : \cdots : w_n) | w_j = 0\}$.

Consider the Veronese mapping v_d given by monomials of degree d. We first deform a meromorphic mapping h to $\tilde{h} := (h_0 : h_1 + \tilde{a}_1 g_1 h_0 : h_2 + \tilde{a}_2 g_2 h_0 : \cdots : h_n + \tilde{a}_n g_n h_0)$ by using $g_1, ..., g_n$ as in Lemma 1, and compose it to the Veronese mapping v_d. We write the composed mapping as $\tilde{f} = v_d \circ \tilde{h} = (\tilde{f}_0, ..., \tilde{f}_s)$.

We next choose a sequence of integers $\{m_{j,i}\}$ with large gaps such that $m_{j,i}^{(s+1)^2} < m_{j,i+1}$ for (j=1,...,n; i=1,...,m). We consider monomials $g_j = g_{j,1}(z_1) \cdots g_{j,m}(z_m)$, where $g_{j,i}(z_i) = z_i^{m_{j,i}}$ (j=1,...,n; i=1,...,m). Then we can prove Lemma 1 and Lemma 2. In the proof of Theorem 1, the key point is an auxiliary mapping F which is constructed by using the generalized Wronskian of $\tilde{f}_0, ..., \tilde{f}_s$. By using Proposition 1 and 2, we can choose complex numbers $\tilde{a}_1, ..., \tilde{a}_n$ in Lemma 2 such that F is nonconstant and $\Delta_F(H_{\mathbf{a}}) = 0$ for some suitable vector $\mathbf{a} \in \mathbf{C}^{s+1} \setminus \{0\}$ constructed by using $\tilde{a}_1, ..., \tilde{a}_n$. Another part of the proof is essentially similar to the method of [3]. Detail is omitted here.

§4. A space of meromorphic mappings.

We shall introduce a distance on the space $\mathcal{F}$ of meromorphic mappings into $\mathbf{P}^n(\mathbf{C})$. Let $f = (f_0 : ... : f_n)$ and $g = (g_0 : ... : g_n)$ be reduced representations of meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Then we define the distance $d(f, g) := d_1(f, g) + d_2(f, g)$, where

$$d_1(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \int_n^{n+1} dt \int_{\partial B(t)} \inf_{\theta} \left\| \frac{f(z)}{\|f(z)\|} - e^{i\theta} \frac{g(z)}{\|g(z)\|} \right\| \sigma \leq 1,$$

which is a distance and it can not distinguish mappings which are rational or transcendental, and

$$d_2(f, g) := \liminf_{\alpha \to +1} \limsup_{r \to \infty} \left\{ \left| \frac{T_f(r)}{(\log r)^{\alpha} + T_f(r)} - \frac{T_g(r)}{(\log r)^{\alpha} + T_g(r)} \right| \right\},$$

which is a pseudodistance and it distinguishs mappings which are rational or transcendental.

In our case, a small deformation $\tilde{f}$ is represented as a form $\tilde{f} = (h_0, h_1 + a_1 g_1 h_0 :, ..., : h_n + a_n g_n h_0)$. Also, we can choose $(a_1, ..., a_n)$ such that $\|\mathbf{a}\| := |a_1| + \cdots + |a_n|$ is as small as possible. So, we can choose $\hat{f} := L_1^{-1} \cdot \tilde{f}$ which is also a small deformation without Nevanlinna defects such that $d(\hat{f}, f)$ is as small as possible. Hence we see meromorphic mappings without Nevanlinna defects are dense in the subset $\mathcal{F}_T \subset \mathcal{F}$ of transcendental meromorphic mappings on this distance.

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Intersection multiplicities of holomorphic and algebraic curves with divisors

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Abstract.

Here we discuss the intersection multiplicities of holomorphic and algebraic curves with divisors on an algebraic variety as an analogue to the abc-Conjecture. We announce some new results.

§1. Introduction

We discuss the truncation of counting functions in the second main theorem (S.M.T.) for holomorphic curves and algebraic ones, that is, the bound of intersection multiplicities of holomorphic curves with divisors in transcendental and algebraic cases; the problem has a strong analogue with the abc-Conjecture of Masser-Oesterlé.

abc-Conjecture. *Let $a, b, c \in \mathbf{Z}$ be coprime and satisfy*

$$a + b + c = 0.$$

Then for an arbitrary $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that

$$\max\{|a|, |b|, |c|\} \leqq C(\epsilon) \left(\prod_{\substack{p>1 \text{ prime} \\ p|abc}} p \right)^{1+\epsilon}. \tag{1.1}$$

It is the key point of the conjecture that the multiplicities of primes of a, b and c in the right hand side of (1.1) are counted only by "$1 + \epsilon$". The analogue for holomorphic curves $f : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ is known:

Theorem 1.2. (Nevanlinna-Cartan-Nochka '83) *Let m be the dimension of the linear span of $f(\mathbf{C}) \subset \mathbf{P}^n(\mathbf{C})$. Let $H_i, 1 \leqq i \leqq q$, be*

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hyperplanes in general position. Then,

$$(q-2n+m-1)T_f(r) \leqq \sum_{i=1}^{q} N_m(r, f^*H_i) + S_f(r),$$
$$S_f(r) = O(\log r) + \delta \log T_f(r)||_{E(\delta)}, \quad \text{meas}\, E(\delta) < \infty.$$

Here we use the notation:

$$T_f(r) = T_f(r; O(1)) = \int_1^r \frac{dt}{t} \int_{|z|<t} f^* c_1(O(1)),$$
$$N_k(r, f^*D) = \int_1^r \frac{\sum_{|z|<t} \min\{k, \text{ord}_z f^*D\}}{t} dt,$$

and "$||_{E(\delta)}$" means that the estimate holds for $r \notin E(\delta)$ as $r \to \infty$.

H. Fujimoto '82 obtained similar estimates for associated curves in Ahlfors-Weyl-Stoll theory.

§2. Lemma on logarithmic differential

Let φ be a logarithmic differential on V with C^∞-coefficients. For a meromorphic mapping $g : \mathbf{C}^m \to V$ set

$$g^*\varphi = \sum_{i=1}^{m} \xi_i dz_i.$$

Let γ denote the rotationary invariant probability measure on the sphere $\{\|z\| = r\}$.

Lemma 2.1. (Nog. '77–) *Let the notation be as above. Then we have*

$$m(r, \xi_i) := \int_{\|z\|=r} \log^+ \xi_i(z)\gamma(z) = S_f(r).$$

Vitter '77 proved this for $f : \mathbf{C}^m \to \mathbf{P}^1(\mathbf{C})$ and $\varphi = dw/w$.

Theorem 2.2. (Griffiths, Carlson, King, Sakai, Shiffman, Stoll, Kodaira, ..., Nog.) *Let $f : \mathbf{C}^n \to V$ be a differentiably non-degenerate meromorphic mapping to a complex projective algebraic manifold V. Let $L \to V$ be a line bundle with complete linear system $|L|$. Let $D = \sum D_i \in |L|$ be a divisor only with simple normal crossings. Then,*

$$T_f(r, L) + T_f(r, K_V) \leqq \sum_i N_1(r, f^*D_i) + S_f(r).$$

In stead of the curvature method employed by Griffiths et al, we may apply Lemma 2.1 directly to prove Theorem 2.2, where a positivity assumption for L is not needed.

Fundamental conjecture for holomorphic curves. *Let $L \to V$ be as above and* $\dim V = n$. *Let $D = \sum D_i \in |L|$. Let $f : \mathbf{C} \to V$ be a Zariski non-degenerate holomorphic curve. Then there exists a number $k = k(D, n)$ such that*

$$T_f(r, L) + T_f(r, K_V) \leqq \sum N_k(r, f^*D_i) + S_f(r).$$

If D has only simple normal crossings, then $k = n$

§3. Holomorphic mappings

We call a complex Lie group M a *semi-torus* if it admits an exact sequence:

$$0 \to (\mathbf{C}^*)^t \to M \to M_0 \to 0,$$

where M_0 is a complex torus. Using the compactification $(\mathbf{P}^1(\mathbf{C}))^t \supset (\mathbf{C})^t$, we take a compactification $\bar{M}$ of M, and set $\partial M = \bar{M} \setminus M$. Let $\bar{D}$ be a reduced divisor on $\bar{M}$ such that no irreducible component of $\bar{D}$ is contained in ∂M. If every l irreducible components of $\bar{D} + \partial M$ has the intersection of pure codimension l, $\bar{D} + \partial M$ is said to be *in general position.*

Theorem 3.1. ([NWY00, NWY02]) *Let M be a semi-torus, and let $L \to \bar{M}$ be a line bundle. Let $\bar{D} \in |L|$ be a divisor without support in ∂M. If M is not compact, we assume that $\bar{D} + \partial M$ is in general position. Set $D = \bar{D} \cap M$. Let $f : \mathbf{C} \to M$ be a holomorphic curve which is Zariski-nondegenerate as a curve in $\bar{M}$. Then we have*

$$T_f(r; L) = N_{k_0}(r; f^*D) + S_f(r).$$

Here, if the order ρ_f of f is finite, $k_0 = k_0(D, \rho_f)$; otherwise, $k_0 = k_0(D, f)$.

Remark. (i) The assumption for $\bar{D} + \partial M$ being in general position is necessary, by examples.

(ii) By examples of singular D on abelian M, k_0 must depend on D. In the case of abelian varieties, Yamanoi lately obtained

$$T_f(r; L) = N_1(r; f^*D) + o(T_f(r; c_1(D))).$$

There is an application of Theorem 3.1 to the Kobayashi hyperbolicity of a covering over abelian varieties, ramified over D.

Using the method of the proof of Theorem 3.1, we have

Theorem 3.2. *Let $f : \mathbf{C}^n \to M$ be a differentiably non-degenerate holomorphic mapping to a semi-torus M of dimension n. Let $L \to \bar{M}$ be a line bundle and $\bar{D} \in |L|$. Set $D = \bar{D} \cap M$. Then,*

$$T_f(r, L) \leqq N_n(r, f^*D) + S_f(r).$$

Taking this opportunity, we would like to make some minor corrections to [N98] and [NWY02].

Remark. (i) The statements given in [N98], Proposition (1.8) (ii), and in [SY96], Theorem (2.2), are too strong to claim; there were gaps in both proofs. The proof given in [N98] implies only that

$$\dim \mathrm{St}(X_k(f)) > 0.$$

But this gives no effect to the other part of the paper, in particular to the proof of the Main Theorem of [M98], for it was not used. If the holomorphic curve is of finite order, then Proposition (1.8) (ii) in [N98] will hold, but in general there is a counter-example (cf. [NW02c]).

(ii) Because of the correction above, in [NWY02] p. 146, 4th–2nd lines from below, the space W_k is to be defined as the π_2-projection of the Zariski closure of $J_k(f)(\mathbf{C})$ as in [NWY00].

§4. Function fields

Nevanlinna theory is an approximation theory of complex numbers by meromorphic functions, as the Diophantine approximation is the approximation of algebraic numbers by rationals or algebraic numbers of a fixed number field (the inverse of Vojta's observation). Over algebraic function fields, one may think the approximation of rational functions by rational functions.

Let R be an algebraic curve of genus g, and let L be a line bundle on R with degree $\deg L$.

Theorem 4.1. *Let $H_j, 1 \leqq j \leqq q$, be linear forms in general position on $\mathbf{P}^n(\mathbf{C})$ with coefficients in $H^0(R, L)$. For an arbitrarily given $\epsilon > 0$, set*

$$k(\epsilon) = (n+1)\left\{\left(\left[\max\left\{g + \frac{2n}{\epsilon}, 2g-2\right\}\right] + 2\right)\deg L - g + 1\right\} - 1.$$

Then for an arbitrary $x : R \to \mathbf{P}^n(\mathbf{C})$, we have

$$(q - 2n - \epsilon)\mathrm{ht}(x) \leqq \sum_{j=1}^{q} N_{k(\epsilon)}(H_j(x)) + C(\epsilon, q).$$

Here $\mathrm{ht}(x) = \deg x^*O(1)$ and $N_{k(\epsilon)}(\cdot)$ is the counting function of zeros of a linear form with truncation level $k(\epsilon)$.

In the proof we use the results of Nochka '83, the method of Steinmetz '85-Shirosaki '91, J. Wang '00, the author '97, and the Riemann-Roch.

Motivated by the similar problem modeled after the "abc-Conjecture" over abelian varieties and here over function fields, A. Buium ([Bu98]) proved

Theorem 4.2. *Let A be an abelian variety, and let D be an effective divisor on A such that D is hyperbolic. Then there is a constant $N(R, A, D)$ such that for an arbitrary morphism $f : R \to A$ with $f(R) \not\subset D$,*

$$\mathrm{ord}_x f^*D \leqq N(R, A, D), \qquad x \in R.$$

A. Buium used a method based on Kolchin's differential algebra. He conjectured there should be a proof by standard algebraic geometry, and the theorem should be valid for ample D. It is indeed true in more general form:

Theorem 4.3. ([NW0x]) *There is a function*

$$N : \mathbf{N} \times \mathbf{N} \times \mathbf{N} \to \mathbf{N}$$

such that the following statement holds: Let C be a smooth compact curve of genus g, let A be an abelian variety of dimension n, let D be an ample effective divisor on A with $d = c_1(D)^n$, and let $f : C \to A$ be a morphism with $f(C) \not\subset D$. Then

$$\mathrm{ord}_x f^*D \leqq N(g, n, d), \qquad x \in C.$$

As an application we have a finiteness

Theorem 4.4. ([NW0x]) *Let C' be an affine algebraic curve, let A be an abelian variety and let D be an ample effective divisor on A. Then, either $\exists f : C' \to D$, non-constant, or there are only finitely many non-constant morphisms from C' to $A \setminus D$.*

Remark. If there is a nonconstant $f : C' \to D$, there may be infinitely many non-constant $g : C' \to A \setminus D$, by example.

N.B. The following list of references is not intended to be complete, but sufficient to trace up the necessary papers by referring their references.

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Generalization of a precise L^2 division theorem

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§ Introduction

The purpose of this article is to generalize the following.

Theorem 1 (cf. [O-3]). *Let D be a bounded pseudoconvex domain in $\mathbf{C}^n$ and let $z = (z_1, \ldots, z_n)$ be the coordinate of $\mathbf{C}^n$. Then there exists a constant C depending only on the diameter of D such that, for any plurisubharmonic function φ on D and for any holomorphic function f on D satisfying*

$$\int_D |f(z)|^2 e^{-\varphi(z)} |z|^{-2n}\, d\lambda < \infty \tag{1}$$

there exists a vector valued holomorphic function $g = (g_1, \ldots, g_n)$ on D satisfying

$$f(z) = \sum_{i=1}^{n} z_i g_i(z) \tag{2}$$

with

$$\int_D |g(z)|^2 e^{-\varphi(z)} |z|^{-2n+2}\, d\lambda \leq C \int_D |f(z)|^2 e^{-\varphi(z)} |z|^{-2n}\, d\lambda. \tag{3}$$

Here $d\lambda$ denotes the Lebesgue measure.

We generalize this in order to establish an understanding that the measure $e^{-\varphi}|z|^{-2n}\, d\lambda$ in (1) consists of three parts, i.e. $e^{-\varphi(z)}$ for any plurisubharmonic function φ, $|z|^{-2}$ as the quotient fiber metric associated to the morphism $g \mapsto \sum z_i g_i$, and $|z|^{-2n+2}\, d\lambda$ as the residue of a volume form on $(D \setminus \{0\}) \times \mathbf{P}^{n-1}$ with respect to the embedding of $D \setminus \{0\}$ by $z \mapsto (z, [z])$, where $[z] = (z_1 : \cdots : z_n)$.

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In our generalized circumstance there will be given a complex manifold M and a surjective morphism $\gamma : E \to Q$, where E and Q are holomorphic vector bundles over M.

It was first asked by H. Skoda [S-2] to find an L^2 surjectivity condition for the morphism induced from γ. More precisely speaking, by specifying a C^∞ volume form dV_M on M, a C^∞ fiber metric h_E of E and the fiber metric h_Q of Q induced from h_E via γ, a surjectivity criterion was looked for with respect to the induced morphism

$$\gamma_* : A^2(M,E) \longrightarrow A^2(M,Q)$$

where $A^2(M,\,\cdot\,)$ $(= A^2(M,\,\cdot\,,dV_M))$ denotes the space of L^2 holomorphic sections and $\gamma_*(g) := \gamma \circ g$.

Here we shall relax the L^2 condition by considering another volume form dV'_M on M and ask for a surjectivity condition for the induced operator

$$\gamma_* : A^2(M,E,dV_M) \longrightarrow A^2(M,Q,dV'_M)$$

where γ_* is only defined as a map from a linear subspace of $A^2(M,E,dV_M)$.

To state our main result, let us introduce some notation.

Let $Q^\vee$, $E^\vee$ denote the duals of Q, E, let $\gamma^\vee : Q^\vee \to E^\vee$ be the dual of γ, and let

$$P(Q^\vee) = \coprod_{x\in M} P(Q_x^\vee), \quad P(E^\vee) = \coprod_{x\in M} P(E_x^\vee),$$

where $P(Q_x^\vee) = \{\mathbf{C}v \mid v \in Q_x^\vee \setminus \{0\}\}$ and $P(E_x^\vee) = \{\mathbf{C}w \mid w \in E_x^\vee \setminus \{0\}\}$. We shall indentify $P(Q^\vee)$ as a complex submainfold of $P(E^\vee)$ via $\gamma^\vee$.

Let us define a line bundle $L(E^\vee)$ over $P(E^\vee)$ by

$$L(E^\vee) = \coprod_{\xi\in P(E^\vee)} L(E^\vee)_\xi$$

where $L(E^\vee)_\xi = \xi$. Then $L(E^\vee)^\vee$ is, as a holomorphic line bundle over $P(E^\vee)$, naturally indentified with

$$\coprod_{x,\xi} E_x/\operatorname{Ker}\xi \quad (x \in M, \xi \in P(E_x^\vee))$$

where $\operatorname{Ker}\xi := \operatorname{Ker}\alpha$ for any $\alpha \in E_x^\vee$ with $\xi = \mathbf{C}\alpha$. The line bundle $(\gamma^\vee)^* L(E^\vee)^\vee$ over $P(Q^\vee)$ will be naturally indentified with

$$\coprod_{x,\xi} Q_x/\operatorname{Ker}\xi \quad (x \in M, \xi \in P(Q_x^\vee))$$

and denoted simply by $L(E^\vee)^\vee | P(Q^\vee)$.

Let $\sigma : P(E^\vee)^\sim \to P(E^\vee)$ be the monoidal transform of $P(E^\vee)$ along $P(Q^\vee)$. For simplicity we put

$$\Sigma = \sigma^{-1}(P(Q^\vee)).$$

Let $p = \operatorname{rank} E$ and $q = \operatorname{rank} Q$. Then the canonical bundles $K_{P(E^\vee)^\sim}$ and $K_{P(E^\vee)}$ are related by a canonical isomorphism

$$K_{P(E^\vee)^\sim} \simeq \sigma^* K_{P(E^\vee)} \otimes [\Sigma]^{p-q-1}.$$

Here Σ denotes the line bundle associated to the divisor Σ. Hence a volume form $dV_{P(E^\vee)^\sim}$ on $P(E^\vee)^\sim$ is induced from dV_M, h_E and a fiber metric of $[\Sigma]$. There is a canonical fiber metric of $[\Sigma]$ induced from h_E, but we shall not stick to it for the sake of generality.

For any Hermitian line bundle L, its curvature form is denoted by Θ_L. For simplicity, the curvature form of the volume form, as a fiber metric of the anticanonical bundle $K_\bullet^\vee$, is denoted by $\mathrm{Ric}_\bullet$.

In this situation, a generalization of Theorem 1 is

Theorem 2. *Suppose that the following are satisfied.*

1. *There exists a closed subset $A \subset M$ such that*

(1.a) *$M \setminus A$ is a Stein mainfold*

and

(1.b) *For any point $x \in A$ and for any neighborhood $U \ni x$, all the L^2 holomorphic finctions on $U \setminus A$ extend holomorphically to U.*

2. *$[\Sigma]$ admits a fiber metric such that*

(2.a) *There exists a bounded canonical section, say s, of $[\Sigma]$.*

(2.b) *There exists a constant R_1 such that $dV_M \leq R_1(\varpi \circ \sigma)_* dV_{P(E^\vee)^\sim}$, where ϖ denotes the projection from $P(E^\vee)$ to M.*

(2.c) *There exists a positive number ε_0 such that*

$$\sqrt{-1}(\sigma^*\Theta_{L(E^\vee)^\vee} + \sigma^* \mathrm{Ric}_{P(E^\vee)} - (p-q+\varepsilon)\Theta_{[\Sigma]}) \geq 0 \quad \text{for all } \varepsilon \in [0, \varepsilon_0].$$

Then the operator $\gamma_* : A^2(M, E, dV_M) \to A^2(M, Q, dV'_M)$ admits a bounded right inverse if there exists a constant R_2 such that

$$R_2 dV'_M \geq (\pi \circ \sigma)_* dV_\Sigma.$$

Here π denotes the projection from $P(Q^\vee)$ to M and dV_Σ denotes the volume form on Σ induced from $dV_{P(E^\vee)^\sim}$ and the fiber metric of $[\Sigma]$.

Corollary 3. *Let D be a pseudoconvex domain in $\mathbf{C}^n$, let $h_1, \dots, h_p$ be bounded holomorphic functions on D, whose first order derivatives are also bounded, let φ be a plurisubharmonic function on D and let f be a holomorphic function on D satisfying*

$$\|f\|^2 := \int_D |f|^2 e^{-\varphi} |h|^{-2} \bigwedge^n \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log|h|^2) < \infty$$

where $h = (h_1, \dots, h_p)$. Then there exist holomorphic functions $g_1, \dots, g_p$ on D such that $f = \sum_{i=1}^p g_i h_i$ and

$$\int_D |g|^2 e^{-\varphi} d\lambda \le C \|f\|^2.$$

Here C is a constant depending only on h. Moreover, if the Ricci curvature of $\bigwedge^n \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log|h|^2)$ is semipositive, then there exist holomorphic functions $l_1, \dots, l_p$ on D such that $f = \sum_{i=1}^p l_i h_i$ and

$$\int_D |l|^2 e^{-\varphi} \bigwedge^n \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log|h|^2) \le C' \|f\|^2$$

where C' is a constant depending only on h.

Obviously the latter part of Corollary 3 contains Theorem 1.

Corollary 4. *Let D, h and φ be as above. Then, for any holomorphic function f on D satisfying*

$$\int_D |f|^2 e^{-\varphi} |h|^{-2k-2} |dh|^{2k} d\lambda$$

where $k = \inf(n, p-1)$, there exist holomorphic functions $g_1, \dots, g_p$ such that $f = \sum_{i=1}^p g_i h_i$ and

$$\int_D |g|^2 e^{-\varphi} d\lambda \le C'' \int_D |f|^2 e^{-\varphi} |h|^{-2k-2} |dh|^{2k} d\lambda$$

where C'' is a constant depending only on h.

The paper is organized as follows. In Section 1 we briefly review the L^2 extension theorem for the reader's convenience. Theorem 2 will be proved in Section 2. In Section 3, we shall recall Skoda's L^2 division theorem and its consequence which is weaker than Theorem 1. We dare to do this because we want to show by a counterexample that a naïve improvement of Skoda's theorem, from which Theorem 1 would follow immediately, is false. This may well mean that our formulation of a generalized L^2 division theorem gives a new insight into the division properties of holomorphic functions.

§1. Preliminaries – L^2 extension theorem

Let N be a complex mainfold of dimension m and let $F \to N$ be a holomorphic line bundle with a C^∞ fiber metric h_F. (The symbols M, n, E, h_E are reserved for the division theory.)

Let $S \subset N$ be a closed complex submainfold of codimension one, and let $[S]$ be the holomorphic line bundle defined by a system of transition functions $e_{\alpha\beta} = s_\alpha / s_\beta$, where s_α are local defining functions of S associated to some open covering of N. Any holomorphic section s of $[S]$ is called a canonical section if $S = s^{-1}(0)$ and $ds|S$ is nowhere zero. Once for all we fix a C^∞ fiber metric b of $[S]$ and a canonical section $s = \{s_\alpha\}$ with $s_\alpha = e_{\alpha\beta} s_\beta$.

Given any C^∞ volume form dV_N on N, a volume form $dV_{N,b}$ on S is induced from dV_N, s and b via the canonical isomorphism

$$(K_M \otimes [S])|S \simeq K_S$$

which is given by

$$\frac{\omega \wedge ds_\alpha}{s_\alpha} \longmapsto \omega|S.$$

One may write on S

$$dV_{N,b} = \frac{dV_N}{\sqrt{-1} b_\alpha ds_\alpha \wedge d\bar{s}_\alpha}.$$

Here the fiber metric b is represented by a system of positive C^∞ functions b_α satisfying $b_\alpha = |e_{\beta\alpha}|^2 b_\beta$. More explicitly writing, let x be any point of S and let $(z_1, \dots, z_n)$ be a holomorphic local coordinate around x such that $z_n = s_\alpha$ for some α around x, and such that

$$dV_N = \sqrt{-1}^n dz_1 \wedge d\bar{z}_1 \wedge \dots dz_n \wedge d\bar{z}_n$$

holds at x. Then, identifying $(z_1, \dots z_{n-1})$ with a local coordinate of S around x, we have

$$dV_{N,b} = \sqrt{-1}^{n-1} b_\alpha^{-1} dz_1 \wedge d\bar{z}_1 \wedge \dots dz_{n-1} \wedge d\bar{z}_{n-1}$$

at x.

Besides the induced volume form $dV_{N,b}$, there is a volume form associated to the function $\log |s|^2$, which turned out to be more natural in the L^2 extension theory. In general, given any continuous function $\psi : N \to \mathbf{R} \cup \{-\infty\}$ such that $\psi - \log |s|^2$ is bounded near every point of S, we define a positive Radon measure $dV_N[\psi]$ on S by

$$\int_S f dV_N[\psi] = \overline{\lim_{t\to\infty}} \frac{1}{\pi} \int_{\psi^{-1}((-t-1,-t))} f e^{-\psi} dV_N.$$

Here f runs through compactly supported nonnegative continuous fuction on N.

However it is easy to see that

$$(\dagger) \qquad dV_N[\log|s|^2] = \frac{dV_N}{\sqrt{-1}b_\alpha ds_\alpha \wedge d\overline{s}_\alpha} = dV_{N,b},$$

whose verification is left to the reader.

Let $A^2(N, F, h_F, dV_N)$ (resp. $A^2(S, F, h_F, dV_N[\log|s|^2])$) be the Hilbert space of L^2 holomorphic sections of F over N (resp. over S) with respect to (h_F, dV_N) (resp. w.r.t. $(h_F, dV_N[\log|s|^2])$).

Theorem 1.1. *Let N, dV_N, F, h_F, S, b and s be as above, and assume that the following are satisfied.*

(1.1) *N contains a Stein open subset N' such that*
  (1.1.a) *N' intersects with every connected component of S and*
  (1.1.b) *For any point $x \in N \setminus N'$ and for any neighborhood $U \ni x$, all the L^2 holomorphic functions on $U \cap N'$ extend holomorphically to U.*

(1.2) $\sup_N |s| < \infty$.

(1.3) *There exists a positive number ε_0 such that*

$$\sqrt{-1}(\Theta_F + \mathrm{Ric}_N - (1+\varepsilon)\Theta_{[S]}) \geq 0 \quad \textit{for all } \varepsilon \in [0, \varepsilon_0].$$

Then there exists a bounded linear operator I from $A^2(S,F,h_F,dV_N[\log[s]^2])$ to $A^2(N,F,h_F,dV_N)$ such that $I(f)|S = f$ for any $f \in A^2(S,F,h_F,dV_N[\log[s]^2])$. Here the norm of I is bounded by a constant dependly only on $\sup_N |s|$ and ε_0.

This result is essentially contained in [O-2, Theorem 4]. Nevertheless we want to prove it here because the curvature assumption (1.3) is somewhat weaker than that of [O-2].

Let us recall first a basic L^2 existence theorem for the $\overline{\partial}$-equation whose proof is contained in [O-2].

Theorem 1.2. *Let (N, g) be a complete Kähler manifold of dimension m, let η be a bounded positive C^∞ function on N and let c be a positive continuous function on $(0, \infty)$ such that $c(\eta)$ is bounded. Let*

(F, h_F) be a Hermitian holomorphic line bundle over N whose curvature form Θ_F satisfies

$$\kappa := \sqrt{-1}(\eta\Theta_F - \partial\overline{\partial}\eta - c(\eta)^{-1}\partial\eta \wedge \overline{\partial}\eta) \geq 0.$$

Then, for any positive integer q and for any $\overline{\partial}$-closed locally square integrable F-valued (m, q) form u on N satisfying $((\kappa\Lambda_g)^{-1}u, u) < \infty$, there exists a square integrable F-valued $(m, q-1)$ form v such that

$$\overline{\partial}(\sqrt{\eta + c(\eta)}v) = u \quad \text{and} \quad \|v\|^2 \leq ((\kappa\Lambda_g)^{-1}u, u).$$

Here Λ_g denotes the adjoint of $u \mapsto$ (the fundamental form of g) $\wedge\, u$.

The proof of Theorem 1.2 is a straightforward application of Hahn-Banach's theorem. (We note that the boundedness assumption on η and $c(\eta)$ was missing in [O-2]. See alse [O-1].)

Proof of Theorem 1.1. By (1.1) it suffices to prove that, for any relatively compact Stein open subset $\Omega \subset N$ with C^2 strongly pseudoconvex boundary, there exists a bounded linear operator

$$I_\Omega : A^2(S, F, h_F, dV_N[\log|s|^2]) \longrightarrow A^2(\Omega, F, h_F, dV_N)$$

such that $I_\Omega(f)|S \cap \Omega = f|S \cap \Omega$ for any $f \in A^2(S, F, h_F, dV_N[\log|s|^2])$ and that $\|I_\Omega\|$ is bounded by a constant that depends only on $\sup_N |s|^2$ and ε_0.

Once for all we fix such Ω and f. Then, by extending f to a neighborhood of $\overline{\Omega \cap S}$ as a holomorphic section of F, say $\tilde{f}$, we consider a C^∞ extension of f to $\overline{\Omega}$ of the form

$$\tilde{f}_t = \chi(\log|s|^2 + t + 2)\tilde{f} \quad (t \gg 1)$$

where χ is a C^∞ function $\mathbf{R}$ satisfying $\chi(x) = 1$ for $x < 1$ and $\chi(x) = 0$ for $x > 2$.

By solving the equation $\overline{\partial}v_t = \overline{\partial}\tilde{f}_t/s$ on Ω with an L^2 norm estimate and by taking a weak limit of $\tilde{f}_t - sv_t$ on Ω, we shall obtain a holomorphic extension of f with a required L^2 norm bound.

For that we regard $\overline{\partial}\tilde{f}_t/s$ as a $K_N^\vee \otimes F \otimes [S]^\vee$-valued $(m, 1)$ form on Ω, and apply Theorem 1.2 for any complete Kähler metric on Ω. Note that Ω carries a complete Kähler metric because Ω is Stein (cf. [G]). Multiplying s by a constant if necessary, we may assume that $\sup_N \log|s| < -1$. Then we put $\Psi = \log|s|^2$, $\Phi = \log(|s|^2 + e^{-t})$ and

$$\eta = \frac{1}{\min(\varepsilon_0, 1)} + \log(|s|^2 + e^{-t}) + \log(-\log(|s|^2 + e^{-t})).$$

By a straightforward computation we obtain

$$\partial\bar{\partial}\Phi = e^{-\Phi}|s|^2\partial\bar{\partial}\Psi + e^{-2\Phi-t}|s|^2\partial|s|^2 \wedge \bar{\partial}|s|^2 \quad \text{on } \Omega \setminus S$$

and

$$-\partial\bar{\partial}\eta = \Big(1 - \frac{1}{\Phi}\Big)^2 \partial\bar{\partial}\Phi + \Phi^{-2}\partial\Phi \wedge \bar{\partial}\Phi.$$

Let us choose t_0 so that $\Phi < -2$ if $t > t_0$. Then, for all $t > t_0$ we have

$$\begin{aligned}
&\sqrt{-1}(\Phi^{-2}\partial\Phi \wedge \bar{\partial}\Phi - \eta^{-3}\partial\eta \wedge \bar{\partial}\eta) \\
&= \sqrt{-1}\Big(\Phi^{-2}\partial\Phi \wedge \bar{\partial}\Phi - \frac{1}{(\Phi + \log(-\Phi))^3}\Big(1 - \frac{1}{\Phi}\Big)^2 \partial\Phi \wedge \bar{\partial}\Phi\Big) \\
&\geq \sqrt{-1}(\Phi^{-2} - \Phi^{-3})\partial\Phi \wedge \bar{\partial}\Phi \geq \frac{\sqrt{-1}}{8}\partial\Phi \wedge \bar{\partial}\Phi.
\end{aligned}$$

Therefore if we put

$$\kappa = \sqrt{-1}(\eta\Theta_{F\otimes K_N^\vee \otimes [S]^\vee} - \partial\bar{\partial}\eta - \eta^{-3}\partial\eta \wedge \bar{\partial}\eta)$$

and $\varepsilon_1 = \min(\varepsilon_0, 1)$, on $\Omega \setminus S$ we have

$$\begin{aligned}
\kappa &\geq \frac{1}{\varepsilon_1}\Theta_{F\otimes K_N^\vee \otimes [S]^\vee} + \Big(1 - \frac{1}{\Phi}\Big)^2 \partial\bar{\partial}\Phi + \frac{\sqrt{-1}}{8}\partial\Phi \wedge \bar{\partial}\Phi \\
&\geq \frac{1}{\varepsilon_1}(\Theta_{F\otimes K_N^\vee \otimes [S]^\vee} + \varepsilon_1 e^{-\Phi}|s|^2\partial\bar{\partial}\Psi) + \frac{\sqrt{-1}}{8}\partial\Phi \wedge \bar{\partial}\Phi \\
&\geq \frac{1}{\varepsilon_1}(\Theta_F + \mathrm{Ric}_N - (1 + \varepsilon_1 e^{-\Phi}|s|^2)\Theta_{[S]}) + \frac{\sqrt{-1}}{8}\partial\Phi \wedge \bar{\partial}\Phi.
\end{aligned}$$

Since $e^{-\Phi}|s|^2 < 1$, the first term in the last inequality is semipositive by assumption. Therefore we obtain

$$\kappa \geq \frac{\sqrt{-1}}{8}\partial\Phi \wedge \bar{\partial}\Phi \quad \text{on } \Omega.$$

Hence, for any Hermitian metric g on Ω we obtain

$$\Big((\kappa\Lambda_g)^{-1}\Big(\frac{\bar{\partial}\tilde{f}_t}{s}\Big), \frac{\bar{\partial}\tilde{f}_t}{s}\Big) \leq C_0\|f\|^2, \quad \text{for } t \gg 1.$$

Here the L^2 norm $\|f\|$ of f is with respect to h_F and $dV_N[\log|s|^2]$, the inner product on the left hand side is with respect to h_F, dV_N and g, and C_0 depends only on $\sup|\chi'|$.

Therefore, choosing g to be a complete Kähler metric on Ω, we may apply Theorem 1.2 and obtain a square integrable $F\otimes K_N^\vee\otimes[S]^\vee$-valued $(m,0)$ form w satisfying

$$\overline{\partial}(\sqrt{\eta+\eta^3}w)=u$$

and

$$\|w\|^2\le C_0\|f\|^2.$$

Clearly $\sup_N|s\sqrt{\eta+\eta^3}|\le C_1$, where C_1 depends only in $\sup_N|s|$ and ε_0.

Therefore $\sqrt{\eta+\eta^3}w$ $(=\sqrt{\eta_t+\eta_t^3}w_t)$ is a wanted solution to the $\overline{\partial}$-equation $\overline{\partial}v_t=\overline{\partial}\tilde{f}_t/s$.

§2. Proof of Theorem 2

Let the notation be as in Theorem 2 and let ϖ be the projection from $P(E^\vee)$ to M. Then we have a canonical commutative diagram

$$\begin{array}{ccccc} L(E^\vee)^\vee & \longleftarrow & \varpi^*E & \longrightarrow & E \\ & \searrow & \downarrow & & \downarrow \\ & & P(E^\vee) & \longrightarrow & M \end{array}$$

to which an isomorphism

$$\begin{aligned} A^2(M,E,dV_M) &\xrightarrow{\sim} A^2(P(E^\vee),L(E^\vee)^\vee) \\ &\qquad (=A^2(P(E^\vee),L(E^\vee)^\vee,\varpi^*dV_M\wedge dV_{FS})) \end{aligned}$$

is associated, which is an isometry up to multiplication by the volume of $\mathbf{P}^{p-1}$. Here dV_{FS} denotes the Fubini-Study volume form on the fibers of $P(E^\vee)$. Identifying $L(E^\vee)^\vee|P(E^\vee)$ with $L(Q^\vee)^\vee$ as in the introduction we have a commutative diagram

$$\begin{array}{ccc} A^2(M,E,dV_M) & \xrightarrow{\sim} & A^2(P(E^\vee),L(E^\vee)^\vee) \\ \downarrow{\scriptstyle\gamma_*} & & \downarrow{\scriptstyle\rho} \\ A^2(M,Q,dV_M') & \xrightarrow{\sim} & A^2(P(Q^\vee),L(Q^\vee)^\vee) \end{array}$$

where ρ denotes the natural restriction operator.

Now suppose that (1.a)–(2.c) and $R_2dV_M'\ge(\pi\circ\sigma)_*(dV_\Sigma/|ds|^2)$ are satisfied. Then, to prove the existence of the right inverse of γ_*, it suffices to prove that the restriction operator

$$\tilde{\rho}:A^2(P(E^\vee)^\sim,\sigma^*L(E^\vee)^\vee)\longrightarrow A^2(\Sigma,\sigma^*L(E^\vee)^\vee,dV_\Sigma/|ds|^2)$$

admits a bounded right inverse. For that we shall verify the conditions (1.1)–(1.3) of Theorem 1.1 for $N = P(E^\vee)^\sim$ and $S = \Sigma$.

(1.1): Since $M \setminus A$ is Stein and $\varpi^{-1}(M \setminus A)$ is a $\mathbf{P}^{p-1}$-bundle over $M \setminus A$, $\varpi^{-1}(M \setminus A)$ admits a positive line bundle, and therefore so is $\sigma^{-1}(\varpi^{-1}(M \setminus A))$, too. Hence $\sigma^{-1}(\varpi^{-1}(M \setminus A))$ contains as ample effective divisor Z which intersects with every component of Σ transversally. One may then put $N' = Z^c$.

(1.2) follows from (2.a). (1.3) follows from (2.c) because $\mathrm{Ric}_{P(E^\vee)^\sim} = \sigma^* \mathrm{Ric}_{P(E^\vee)} - (p-q-1)\Theta_{[\Sigma]}$ by the definition of the volume form $dV_{P(E^\vee)^\sim}$.

Hence, by Theorem 1.1, the restriction operator from $A^2(P(E^\vee)^\sim, \sigma^* L(E^\vee)^\vee)$ to $A^2(\Sigma, \sigma^* L(E^\vee)^\vee, dV_{P(E^\vee)^\sim}[\log |s|^2])$ admits a bounded right inverse. This completes the proof of Theorem 2 because $dV_{P(E^\vee)^\sim}[\log |s|^2] = dV_\Sigma$ by (†). □

To deduce Corollary 3 from Theorem 2, we put $M = D \setminus h^{-1}(0)$, $E = M \times \mathbf{C}^p$, $Q = M \times \mathbf{C}$ and $\gamma(z, \zeta) = \sum \zeta_i h_i(z)$. Then we may put $A = h_i^{-1}(0)$ for any nonzero h_i. As for the fiber metric of $[\Sigma]$, we may take $|\zeta|^{-2} \sum_{i \neq j} |\zeta_i h_j - \zeta_j h_i|^2$ as the squared length of the canonical section $s = \left\{h_j \frac{\zeta_i}{\zeta_j} - h_i\right\}_{i \neq j}$ where the local expression $h_j \frac{\zeta_i}{\zeta_j} - h_i$ is effective on the completement of the proper transform of the set $\{h_j \zeta_i - h_i \zeta_j = 0\}$ in $\{\zeta_j \neq 0\}$. Clearly $|s|$ is bounded on M, so what remains is to verify (2.c) and the estimates for the volume forms.

For that we notice that

$$dV_\Sigma = \frac{|\zeta|^2 dV_{P(E^\vee)^\sim}}{\sqrt{-1}\left(\sum_{i \neq j} |\zeta_i h_j - \zeta_j h_i|^2\right) d\left(h_l - \frac{\zeta_l}{\zeta_k} h_k\right) \wedge d\left(\overline{h}_l - \frac{\overline{\zeta}_l}{\overline{\zeta}_k} \overline{h}_k\right)}$$

where

$$dV_{P(E^\vee)^\sim} = \frac{|\zeta|^{2p-4}}{\left(\sum_{i \neq j} |\zeta_i h_j - \zeta_j h_i|^2\right)^{p-2}} \bigwedge^{n+p-1} \sigma^*(\sqrt{-1}\partial\overline{\partial}(|z|^2 + \log |\zeta|^2)).$$

From this expression of $dV_{P(E^\vee)^\sim}$ it is easy to see that the curvature condition (2.c) holds true.

To see that the required estimates for $dV_{P(E^\vee)^\sim}$ and dV_Σ hold, we consider an embedding

$$\begin{array}{ccc} D \times \mathbf{P}^{p-1} & \hookrightarrow & D \times \mathbf{C}^p \times \mathbf{P}^{p-1} \\ \Cup & & \Cup \\ (z, \zeta) & \longmapsto & (z, h(z), \zeta) \end{array}$$

and the associated commutative diagram between the blow ups

$$\begin{array}{cccc} \iota: & (D \times \mathbf{P}^{p-1})^{\sim} & \hookrightarrow & D \times (\mathbf{C}^p \times \mathbf{P}^{p-1})^{\sim} \\ & \downarrow \sigma_1 & & \downarrow \sigma_2 \\ & D \times \mathbf{P}^{p-1} & \hookrightarrow & D \times \mathbf{C}^p \times \mathbf{P}^{p-1}. \end{array}$$

Since $\sup_D |dh| < \infty$ by assumption, there exists a constant C such that

$$(*) \quad C^{-1} dV_{P(E^\vee)^\sim} < \iota^* \left\{ \frac{|\zeta|^{2p-4}}{\left(\sum_{i \neq j} |\zeta_i w_j - \zeta_j w_i|^2 \right)^{p-2}} \bigwedge^{n+p-1} \sigma_2^* (\sqrt{-1}\partial\bar{\partial}(|z|^2 + |w|^2 + \log |\zeta|^2)) \right\} < C dV_{P(E^\vee)^\sim}$$

where w denotes the coordinate of $\mathbf{C}^p$.

In particular, $dV_{P(E^\vee)^\sim}$ dominates the pull back of a bounded $(n + p - 1, n + p - 1)$ form on $D \times (\mathbf{C}^p \times \mathbf{P}^{p-1})^{\sim}$, so that

$$\text{const.}(\varpi \circ \sigma)_* dV_{P(E^\vee)^\sim} \geq \bigwedge^n \sqrt{-1}\partial\bar{\partial}|z|^2.$$

$(*)$ also shows that dV_Σ is quasi-equivalent to the pull back of $\bigwedge^{n+p-2} \omega$ for some smooth positive $(1, 1)$ form, say ω, on the exceptional set of σ_2.

Clearly

$$\sigma_{2*}\omega \leq \text{const.} \sqrt{-1}\partial\bar{\partial}(|z|^2 + |w|^2 + \log |\zeta|^2)$$

in the sense of current, so that

$$\begin{aligned} (\varpi \circ \sigma)_* dV_\Sigma &\leq \text{const.} \bigwedge^n \sqrt{-1}\partial\bar{\partial}(|z|^2 + |h(z)|^2 + \log |h(z)|^2) \\ &\leq \text{const.} \bigwedge^n \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log |h(z)|^2). \end{aligned}$$

The first part of Corollary 3 follows from this by regarding $e^{-\varphi}$ as an increasing limit of smooth fiber metrices of E whose curvature forms are semipositive. To obtain the latter part we have only to set $dV_M = \bigwedge^n \sqrt{-1}\partial\bar{\partial}(|z|^2 + \log |h|^2)$. □

Corollary 4 follows imediately from Corollary 3.

§3. A note on Skoda's division theorem

It might be worthwhile to compare our results with the following which are due to Skoda [S-2] (see also [D]).

Theorem 3.1. *Let M be a complex manifold of dimension n admitting a Kähler metric and a plurisubharmonic exhaustion function of class C^2, let E be a holomorphic Hermitian vector bundle of rank p over M whose curvature form is semipositive in the sense of Griffiths, and let $\gamma : E \to Q$ be a surjective morphism to a holomorphic vector bundle Q of rank q. Then, for any holomorphic Hermitian line bundle L whose curvature form satisfies*

$$(S) \qquad \sqrt{-1}(\Theta_L - \Theta_{\det E} - k\Theta_{\det Q}) \geq 0$$

for some $k > \inf(n, p-q)$, the induced linear map

$$\gamma_* : A^2(M, E \otimes K_M \otimes L) \longrightarrow A^2(M, Q \otimes K_M \otimes L)$$

is surjective.

Corollary 3.2. *Let D be a pseudoconvex domain in $\mathbf{C}^n$, let $h_1, \ldots, h_p$ be holomorphic functions on D, and let $k = \inf(n, p-1)$. Then, for any positive number ε, there exists a constant C_ε such that, for any plurisubharmonic function φ on D and for any holomorphic function f on D satisfying*

$$\int_D |f|^2 e^{-\varphi} |h|^{-2k-2-\varepsilon} d\lambda < \infty$$

there exist holomorphic functions $g_1, \ldots g_p$ such that $f = \sum_{i=1}^p g_i h_i$ and

$$\int_D |g|^2 e^{-\varphi} |h|^{-2k-\varepsilon} d\lambda \leq C_\varepsilon \int_D |f|^2 e^{-\varphi} |h|^{-2k-2-\varepsilon} d\lambda.$$

There are two points to be noted here. One point is that Corollary 3.2 is not contained in Corollary 3 because we had to assume the boundedness of h and its first derivative. The other point is that one cannot drop the above ε by weaking the inequality $k > \inf(n, p-q)$ in the hypothesis to $k \geq \inf(n, p-q)$, as the following counterexample shows.

Let $\mathcal{O}(k)$ denote the holomorphic line bundle of degree k over $\mathbf{P}^1$ ($\mathcal{O} := \mathcal{O}(0)$).

Define a morphism $\iota : \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1)$ by $\iota(z, \zeta) = (z, (z\zeta, (z+1)\zeta))$, and let $0 \to \mathcal{O} \xrightarrow{\iota} \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O}(2) \to 0$ be the associated exact sequence. Tensoring $\mathcal{O}(-1)$ to this we have

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Letting $M = \mathbf{P}^1$, $E = \mathcal{O} \oplus \mathcal{O}$, $Q = \mathcal{O}(1)$, $L = \mathcal{O}(1)$ and $k = \inf(n, p - q) = 1$, we have

$$\deg L = \deg(\det E) - k \deg(\det Q) = 1 - 0 - 1 = 0.$$

Hence (S) is satisfied, but

$$A^2(M, K_M \otimes E \otimes L) = H^0(\mathbf{P}^1, \mathcal{O}(-1) \oplus \mathcal{O}(-1)) = \{0\}$$

and

$$A^2(M, K_M \otimes Q \otimes L) = H^0(\mathbf{P}^1, \mathcal{O}) \neq \{0\}.$$

Therefore γ_* is not surjective!

Open Question. *Establish a general L^2 division theory that unifies Theorem 2 and Theorem 3.1.*

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Amoebas, convexity and the volume of integer polytopes

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Abstract.

To any given Laurent polynomial f on $\mathbf{C}_*^n$ we associate two natural convex functions M_f and N_f on $\mathbf{R}^n$. We compute the Hessian of M_f and obtain an explicit formula for the volume of the Newton polytope Δ_f. We also establish asymptotic formulas relating our convex functions to coherent triangulations of Δ_f and to the secondary polytope.

§1.

Let $A \subset \mathbf{Z}^n$ be a finite set and consider a general Laurent polynomial $f(z) = \sum_{\alpha\in A} a_\alpha\, z^\alpha$, with complex coefficients and $z \in \mathbf{C}_*^n$. The Newton polytope Δ_f is defined as the convex hull of A (in $\mathbf{R}^n \supset \mathbf{Z}^n$), or more accurately, as the convex hull of those α for which $a_\alpha \neq 0$. The amoeba $\mathbf{A}_f$ is defined to be the image of the zero set of f under the mapping $\mathrm{Log} : \mathbf{C}_*^n \to \mathbf{R}^n$ given by $(z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$. In the sequel we use the notation $|z_j| = t_j$ and $\log|z_j| = x_j$.

We are going to deal with the two functions

$$M_f(x) = \log\Big(\sum_{\alpha\in A} |a_\alpha|\, e^{\langle\alpha,x\rangle}\Big)$$

and

$$N_f(x) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log\big|f(e^{x+i\theta})\big|\, d\theta_1 \wedge \cdots \wedge d\theta_n.$$

They are both convex functions in $\mathbf{R}^n$ with the property that their gradient mappings map $\mathbf{R}^n$ to the Newton polytope Δ_f. More precisely, the mapping $\mathrm{grad}\, M_f$ is a diffeomorphism $\mathbf{R}^n \to \mathrm{int}\,\Delta_f$, whereas

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$\operatorname{grad} N_f$ maps $\mathbf{R}^n$ onto the closed polytope Δ_f with each connected component of $\mathbf{R}^n \setminus \mathbf{A}_f$ being sent to one of the integer vectors $\Delta_f \cap \mathbf{Z}^n$, called the order of that connected component. (See [5] for more on this.)

Introducing the corresponding Monge–Ampère measures

$$\operatorname{Hess} M_f = \operatorname{Jac} \operatorname{grad} M_f \quad \text{and} \quad \operatorname{Hess} N_f = \operatorname{Jac} \operatorname{grad} N_f,$$

we conclude from general facts on convex functions, see [6], that these are both positive measures with total masses equal to $\operatorname{Vol} \Delta_f$.

Let us order the set A as $\{\alpha^0, \alpha^1, \ldots, \alpha^N\}$, and consider, for any increasing multi-index $J = \{j_0, \ldots, j_n\} \in \{0, 1, \ldots, N\}^{1+n}$, the square matrix A_J having the $(1+n)$-vectors $(1, \alpha^{j_k})$ as its columns. Observe that $|\det(A_J)|$ equals $n!$ times the volume of the simplex σ_J with vertices in $\alpha^{j_0}, \ldots, \alpha^{j_n}$. We begin with an explicit computation.

Proposition 1.1 *The push-forward of the measure* $\operatorname{Hess} M_f$ *under the mapping* $\operatorname{Exp}\colon \mathbf{R}^n \to \mathbf{R}^n_+$ *defined by* $(x_1, \ldots, x_n) \mapsto (e^{x_1}, \ldots, e^{x_n})$, *is given by Lebesgue measure times a rational function* h_f / F^{1+n}, *with the polynomial* h_f *explicitly given by*

$$h_f(t) = {\sum_{|J|=1+n}}' \det{}^2(A_J)\, |a_{\alpha^{j_0}}|\, t^{\alpha^{j_0}} \cdots |a_{\alpha^{j_n}}|\, t^{\alpha^{j_n}}.$$

Here the summation is over all increasing multi-indices J, and F is obtained from f by replacing each coefficient a_α by $|a_\alpha|$.

Proof: The gradient of M_f equals the moment map (cf. [4], p.198)

$$\operatorname{grad} M_f(x) = \frac{\sum_{\alpha \in A} \alpha\, |a_\alpha|\, e^{\langle \alpha, x \rangle}}{\sum_{\alpha \in A} |a_\alpha|\, e^{\langle \alpha, x \rangle}} = \frac{\sum_{\alpha \in A} \alpha\, |a_\alpha|\, t^\alpha}{\sum_{\alpha \in A} |a_\alpha|\, t^\alpha},$$

which means that $\operatorname{Hess} M_f(x) = \det\bigl(\partial^2 M_f(x)/\partial x_j\, \partial x_k\bigr)$ is equal to

$$\left| \frac{\sum_{\alpha \in A} \alpha_j \alpha_k\, |a_\alpha|\, t^\alpha}{\sum_{\alpha \in A} |a_\alpha|\, t^\alpha} - \frac{\bigl(\sum_{\alpha \in A} \alpha_j\, |a_\alpha|\, t^\alpha\bigr)\bigl(\sum_{\alpha \in A} \alpha_k\, |a_\alpha|\, t^\alpha\bigr)}{\bigl(\sum_{\alpha \in A} |a_\alpha|\, t^\alpha\bigr)^2} \right|,$$

and if we introduce the abbreviation $c_\alpha = |a_\alpha|\, t^\alpha$ we may re-write the above $n \times n$-determinant as the following $(1+n) \times (1+n)$-determinant:

$$\frac{1}{\bigl(\sum c_\alpha\bigr)^{1+n}} \begin{vmatrix} \sum c_\alpha & \sum \alpha_1 c_\alpha & \cdots & \sum \alpha_n c_\alpha \\ \sum \alpha_1 c_\alpha & \sum \alpha_1 \alpha_1 c_\alpha & \cdots & \sum \alpha_1 \alpha_n c_\alpha \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \sum \alpha_n c_\alpha & \sum \alpha_n \alpha_1 c_\alpha & \cdots & \sum \alpha_n \alpha_n c_\alpha \end{vmatrix}. \qquad (*)$$

Now we consider the $(1+n)\times(1+N)$-matrix

$$B = \begin{pmatrix} \sqrt{c_{\alpha^0}} & \sqrt{c_{\alpha^1}} & \dots & \sqrt{c_{\alpha^N}} \\ \alpha_1^0\sqrt{c_{\alpha^0}} & \alpha_1^1\sqrt{c_{\alpha^1}} & \dots & \alpha_1^N\sqrt{c_{\alpha^N}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \alpha_n^0\sqrt{c_{\alpha^0}} & \alpha_n^1\sqrt{c_{\alpha^1}} & \dots & \alpha_n^N\sqrt{c_{\alpha^N}} \end{pmatrix},$$

and make two observations. First, the determinant $(*)$ is equal to $\det(B\,B^{\mathrm{tr}})/F(t)^{1+n}$. Second, the polynomial h_f is equal to the sum of the squares of all the maximal minors of B. The desired identity $\operatorname{Hess} M_f = h_f/F^{1+n}$ therefore follows from the Cauchy–Binet formula, see [3], which says that the determinant of the product $B\,B^{\mathrm{tr}}$ is indeed equal to the sum of the squares of the minors of B.

We remark that h_f is the non-homogeneous toric Jacobian of the extended gradient $(F, t_1\partial_1 F, \dots, t_n\partial_n F)$, see [2] and Proposition 1.2 in [1], where a similar computation was carried out. Combining our Proposition 1.1 with the fact that the total mass of $\operatorname{Hess} M_f$ is equal to $\operatorname{Vol}\Delta_f$, we obtain the following explicit, elementary, and apparently new formula for the volume of the Newton polytope.

Theorem 1.2 *The volume of the Newton polytope Δ_f can be computed by means of the closed formula*

$$\operatorname{Vol}\Delta_f = \int_{\mathbf{R}_+^n} \frac{h_f(t)}{(F(t))^{1+n}}\,\frac{dt_1\wedge\dots\wedge dt_n}{t_1\cdots t_n}. \qquad (**)$$

We knew a priori that this integral should converge, since the measure $\operatorname{Hess} M_f$ has a finite mass, but the convergence now also follows from the obvious fact that the Newton polytope of h_f is contained in the interior of $(1+n)\,\Delta_f$.

Regarding the function N_f, we recall the following result from [5]. Remember that a polyhedral subdivision is a generalized triangulation whose elements are polyhedra (but not necessarily simplices).

Theorem 1.3 *The piecewise linear convex function* $\max_\alpha(c_\alpha + \langle\alpha, x\rangle)$, *where* $c_\alpha + \langle\alpha, x\rangle = N_f(x)$ *in the component of* $\mathbf{R}^n\setminus\mathbf{A}_f$ *of order* α, *defines a polyhedral subdivision of* $\mathbf{R}^n$ *whose* $(n-1)$*-skeleton is contained in* $\mathbf{A}_f$, *while its Legendre transform similarly defines a dual polyhedral subdivision* $\mathbf{T}_f$ *of* Δ_f. *A vector* α *is a vertex in* $\mathbf{T}_f$ *if and only if* $\mathbf{R}^n\setminus\mathbf{A}_f$ *has a component of order* α.

§**2.**

In this section we shall study the asymptotic behaviour of Theorems 1.2 and 1.3 as the coefficients a_α tend to infinity. More precisely, we will set $a_\alpha = \lambda^{s_\alpha}$ for some fixed vector $(s_\alpha) \in \mathbf{R}^A$ and $\mathbf{R} \ni \lambda \to \infty$. We recall from [4] that the so-called secondary polytope $\Sigma_A \subset \mathbf{Z}^A$ has the property that its vertices are in bijective correspondence with the coherent triangulations of Δ_f, and that a triangulation is coherent if it can be defined by a convex (or concave) piecewise linear function (as in Theorem 1.3).

For any vertex v of Σ_A, the normal cone N_v, which consists of all vectors $(s_\alpha) \in \mathbf{R}^A$ such that $(s, v) = \max_{w \in \Sigma_A}(s, w)$, has a non-empty interior. Any vector (s_α) from $\operatorname{int} N_v$, that is, such that $(s, v) > (s, w)$ for all $w \in \Sigma_A$ with $v \neq w$, can be used to produce the associated coherent triangulation $\mathbf{T}_v$ of Δ_f in the following way. Let g_s be the piecewise linear concave function on Δ_f whose graph equals the upper boundary of the convex hull of the union of half lines $\{(\alpha, y);\ \alpha \in A, y \leq s_\alpha\}$. Then $\mathbf{T}_v$ is obtained by projecting the linear pieces of the graph of g_s down to Δ_f. Notice that $-g_s$ is the Legendre transform of the piecwise linear convex function $\max_\alpha(s_\alpha + \langle \alpha, x \rangle)$ on $\mathbf{R}^n$.

The polynomial h_f, and hence the whole volume formula in Theorem 1.2, contains one term for each subsimplex σ_J with vertices in A. Asymptotically, it is only the terms corresponding to the disjoint simplices of a coherent triangulation that survive, as shown by the following theorem.

Theorem 2.1 *Let v be a vertex of the secondary polytope Σ_A, and take a vector $(s_\alpha) \in \mathbf{R}^A$ in the interior of the normal cone N_v. Set the coefficients a_α of f equal to λ^{s_α}. Then the term $I_J(\lambda)$ in* (**) *corresponding to the multi-index J satisfies*

$$\lim_{\lambda \to \infty} I_J(\lambda) = \begin{cases} \operatorname{Vol} \sigma_J, & \textit{if } \sigma_J \in \mathbf{T}_v\,, \\ 0, & \textit{otherwise}\,. \end{cases}$$

Proof: Recalling the formula for h_f, we see that

$$I_J(\lambda) = \int_{\mathbf{R}^n_+} \frac{\det^2(A_J)\, \lambda^{s_{\alpha^{j_0}}}\, t^{\alpha^{j_0}} \cdots \lambda^{s_{\alpha^{j_n}}}\, t^{\alpha^{j_n}}}{(\lambda^{s_{\alpha^0}}\, t^{\alpha^0} + \lambda^{s_{\alpha^1}}\, t^{\alpha^1} + \ldots \lambda^{s_{\alpha^N}}\, t^{\alpha^N})^{1+n}} \frac{dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n}.$$

If we perform the monomial substituion $u_k = \lambda^{s_{\alpha^{j_k}}} t^{\alpha^{j_k}} / \lambda^{s_{\alpha^{j_0}}} t^{\alpha^{j_0}}$, for $k = 1, \ldots, n$, we arrive at

$$I_J(\lambda) = \int_{\mathbf{R}_+^n} \frac{|\det(A_J)|\, du_1 \wedge \cdots \wedge du_n}{(1 + u_1 + \cdots + u_n + \delta(\lambda))^{1+n}},$$

where $\delta(\lambda)$ is a finite sum of fractional monomials $\lambda^{r_0} u_1^{r_1} \cdots u_n^{r_n}$, with $r \in \mathbf{Q}^{1+n}$ and $r_0 \neq 0$. Now, it is not hard to verify that the simplex σ_J belongs to the triangulation $\mathbf{T}_v$ precisely if all the exponents r_0 are negative. In this case the term $\delta(\lambda)$ tends to zero, and since the integral of $du_1 \wedge \cdots \wedge du_n/(1+u_1+\cdots+u_n)^{1+n}$ over the positive orthant is equal to $1/n!$, we conclude that $I_J(\lambda) \to |\det(A_J)|/n!$ as claimed. Otherwise, the denominator in the integrand goes to infinity, and the integral $I_J(\lambda)$ tends to zero.

The proof of the next result is essentially parallel to that of Theorem 9 in [7] and will be omitted.

Theorem 2.2 *Let v be a vertex of the secondary polytope Σ_A, and take a vector (s_α) as in Theorem 2.1. Set the coefficients a_α of f equal to λ^{s_α} and denote the new polynomial by f^λ. For large values of the parameter λ the polyhedral subdivision $\mathbf{T}_{f^\lambda}$ from Theorem 1.3 will then coincide with the coherent triangulation $\mathbf{T}_v$.*

We end with a closer look at a one-dimensional case.

Example 2.3 Consider a one-variable polynomial of the form $f(t) = 1 + a_1 t + \cdots + a_{n-1} t^{n-1} + t^n$. For each $m = 0, 1, \ldots, 2n-2$ the so-called Ostrogradski method for finding the rational part of a primitive function can be realized with the explicit formula

$$\int \frac{t^m\, dt}{f(t)^2} = -\frac{P_m(t)}{f(t)} + \int \frac{Q_m(t)\, dt}{f(t)},$$

where the P_m and Q_m are polynomials of degrees $n-1$ and $n-2$ respectively. To be specific, one has $P_m(t) = \sum_{k=0}^{n-1} A_{m,k} t^k$ and $Q_m(t) = P_m'(t) + \sum_{\ell=0}^{n-2} B_{m,\ell} t^\ell$, with the $(2n-1)\times(2n-1)$-matrix $(B_{m,\ell}, A_{m,k})$ being the inverse of the standard Sylvester matrix (see [4], p.405) whose determinant equals the discriminant D_n of f. Now, if we collect terms in h_f and write $t^{-1} h_f(t) = \sum_{m=0}^{2n-2} C_m t^m$, then it holds that $\sum_m A_{m,k} C_m = (n-k) a_k$ and $\sum_m B_{m,\ell} C_m = -(\ell+1)(n-\ell-1) a_{\ell+1}$. (Here $a_0 = a_n = 1$.) This implies in particular that if we replace the individual terms

$$\int_0^\infty \frac{(j_1 - j_0)^2\, a_{j_0} a_{j_1}\, t^{j_0 + j_1 - 1}\, dt}{f(t)^2}$$

in formula $(**)$ by their principal parts

$$-\frac{(j_1-j_0)^2\, a_{j_0}a_{j_1}\, P_{j_0+j_1-1}(t)}{f(t)}\Bigg|_0^\infty = (j_1-j_0)^2\, a_{j_0}a_{j_1}\, A_{j_0+j_1-1,0}$$

then they still sum to $\operatorname{Vol}\Delta_f = n$. In other words, the individual terms of $(**)$, which are not themselves rational functions of the coefficients a_j, can be replaced by rational expressions so that the volume formula still holds true. Since these expressions all have the discriminant D_n as their denominator, this means we have in a canonical way associated polynomials (the numerators) with all subsimplices $[j_0, j_1]$ so that their sum is equal to nD_n. In fact, the linear form on the vector space $\langle 1, t, \ldots, t^{2n-2}\rangle$ given by

$$t^m \mapsto P_m(0) \quad (= A_{m,0})$$

coincides with the toric residue associated to the mapping (f, tf').

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On the decomposition of holomorphic functions by integrals and the local CR extension theorem

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Among the classical and far reaching applications of integrals to the decomposition of holomorphic functions is P. Cousin's use of the Cauchy Integral Formula to obtain the most basic version which underlies the solution of what is now known as the additive Cousin problem. Concretely, if L is an (oriented) line segment in the complex plane $\mathbb{C}$, and if f is holomorphic in a neighborhood of L, then

$$F^{\pm}(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)d\zeta}{\zeta - z}, \quad z \notin L$$

defines F^+ on the left side of L (resp. F^- on the right), both F^+ and F^- extend holomorphically across L, and

$$f(z) = F^+(z) - F^-(z) \text{ on } L.$$

In 1942, K. Oka [O] used a version of this principle with a Bergman-Weil type integral formula for polyhedra in his solution of the Levi problem.

Another well known application of this principle arises in the classical proof of the Hartogs extension theorem by means of the Bochner-Martinelli formula, discovered independently by E. Martinelli and S. Bochner in the early 1940s. Suitably modified, this principle allows also a simple natural proof of the corresponding *global* CR extension theorem of Severi and Fichera[1].

In this note I shall discuss an application of this principle to a proof of a version of the *local* CR extension theorem, valid under minimal regularity hypotheses. The main step reduces the question of CR extension to the classical problem of extension of holomorphic functions. While versions of this reduction have been known for a long time (see,

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[1] See [R1] for details.

for example, [AH]), the techniques used require stronger differentiability hypotheses, and typically involve some loss of regularity. In contrast, the Bochner-Martinelli kernel provides a simple mechanism to carry out the reduction in an optimal way. More precisely, I shall prove the following result.

Theorem 1. *Let S be a closed C^1 hypersurface in an open set $\Omega \subset \mathbb{C}^n$. Let $U \subset\subset \Omega$ be a neighborhood of $p \in S$ with $U \setminus S = U^+ \cup U^-$, where U^+ and U^- are disjoint and connected. Suppose U^- is not a domain of holomorphy at p, i.e., every $f \in \mathcal{O}(U^-)$ extends holomorphically to p. Then there exists a neighborhood W of p, such that every CR function $f \in C(U \cap S)$ extends holomorphically to $U^+ \cap W$.*

The hypothesis at the point p is satisfied, for example, in the classical situation where U^+ is strictly pseudoconvex at p. More generally, if S is of class C^2, it holds whenever the Levi form of S, viewed as part of the boundary of U^+, has at least one positive eigenvalue. Of course, the hypothesis on the given function f has to be interpreted in the weak sense, i.e., $\int_S f\bar{\partial}\varphi = 0$ for all $C^\infty_{(n,n-2)}$ forms with compact support in U.

Interest in this phenomenon was rekindled by the recent discovery of a long forgotten 1936 paper by Hellmuth Kneser [K], in which this theorem was proved for strictly pseudoconvex boundary points in $\mathbb{C}^2$, fully 20 years *before* Hans Lewy's famous 1956 theorem [H], which for a long time had been viewed as the first result of this sort[2].

Let us briefly recall some basic results about the Bochner-Martinelli kernel

$$K_{BM} = \frac{(n-1)!}{(2\pi i)^n} \frac{\sum_{j=1}^n (\overline{\zeta_j - z_j}) d\zeta_j \wedge (\Lambda_{k \neq j} d\overline{\zeta_k} \wedge d\zeta_k)}{|\zeta - z|^{2n}}.$$

(Complete proofs may be found, for example, in [R1].)

K_{BM} is real analytic in z. So, if S is an oriented C^1 hypersurface and $f \in C(S)$ has compact support, the Bochner-Martinelli transform

$$T_S f(z) = \int_S f(\zeta) K_{BM}(\zeta, z)$$

defines a real analytic function on $\mathbb{C}^n \setminus S$. With $U \setminus S = U^+ \cup U^-$ as in the theorem, one may consider the restrictions $T_S^+ f = T_S f \mid U^+$ and $T_S^- f = T_S f \mid U^-$. If f is Hölder continuous of some positive order,

[2]The reader may find a comprehensive account of the history of the local and global CR extension phenomena in the author's article [R2].

then $T_S^+ f$ and $T_S^- f$ extend continuously to S, and if the orientation of S agrees with the one it carries as part of the boundary of U^+, one has the "jump formula"

$$(1) \qquad f(z) = T_S^+ f(z) - T_S^- f(z) \text{ for } z \in U \cap S.$$

More generally, this formula remains valid for continuous f, whenever one can prove the continuous extension from at least one of the sides (continuity from the other side then follows as well).

Since K_{BM} is not holomorphic if $n > 1$, $T_S f$ will not be holomorphic on $\mathbb{C}^n \setminus S$ in general. Instead, one has

$$(2) \qquad \overline{\partial_z} K_{BM} = -\overline{\partial_\zeta} K_1 \text{ on } \mathbb{C}^n \times \mathbb{C}^n \setminus \{\zeta = z\},$$

where K_1 is an explicit double form of type $(0,1)$ in z and $(n, n-2)$ in ζ. A simple application of Stokes' theorem then implies that if S is compact without boundary, say if $S = bD$ for $D \subset\subset \mathbb{C}^n$, and if $f \in \mathcal{O}(S)$, then $T_S f$ is holomorphic on $\mathbb{C}^n \setminus S$. In fact, only the weaker hypothesis $\overline{\partial_b} f = 0$ on S is needed for this conclusion.

When S has nonempty boundary, $T_S f$ is no longer holomorphic in general. Instead, one has the following weaker result.

Lemma 2. *Suppose S is a C^1 hypersurface in $\mathbb{C}^n$, and $f \in C(S)$ has compact support in S. Let U be an open set, such that f is weakly CR on $S \cap U$. Then $\overline{\partial}(T_S f)$ extends to a C^∞ $(0,1)$ form on U.*

The important fact is that application of $\overline{\partial}$ eliminates the discontinuity of $T_S f$ across S.

Proof. $T_S f$ is clearly C^∞ outside S. We need to show that $T_S f$ extends C^∞ to any point $p \in S \cap U$. Fix p, and choose a neighborhood $V(p) \subset\subset U$ and $\chi \in C_0^\infty(U)$ with $\chi \equiv 1$ on V. Then

$$T_S f = \int_S f \chi K_{BM} + \int_S f(1-\chi) K_{BM},$$

where the 2nd integral is clearly C^∞ on V (indicated by +..... in the following). On $V \setminus S$ one therefore has

$$\overline{\partial_z} T_S f(z) = \int_S f(\zeta)\chi(\zeta)\overline{\partial_z} K_{BM}(\zeta, z) + \ldots = (\text{by}(2)) - \int_S f \chi \overline{\partial_\zeta} K_1 + \ldots =$$

$$= - \int_S f \; \overline{\partial_\zeta}(\chi K_1) + \int_S f \; (\overline{\partial_\zeta} \chi) K_1 + \ldots .$$

In the last equation, the first integral is 0 by the hypothesis on f [3], and hence extends trivially across $S \cap V$, while the 2nd integral is C^∞ on V since $\overline{\partial_\zeta \chi} \equiv 0$ on V.

Corollary 3. *If U is Stein, there exists $u \in C^\infty(U)$, such that $H^\pm = T_S^\pm f - u$ is holomorphic on U^+ (resp. U^-).*

Proof. Let u be any solution of $\overline{\partial} u = \overline{\partial} T_S f$ on U. Note that if U is convex with smooth boundary (for example a ball), such solutions can be found by means of elementary integral formulas.

The proof of the theorem is now very easy. Without loss of generality we may assume that U is a ball centered at p, and that f has compact support in S. Consider $T_S f$, and choose u as in the Corollary. By the hypothesis on p, the function H^- extends holomorphically across p, say to a neighborhood W of p (which depends only on the complex geometry of S near p, i.e., W can actually be chosen independently of H^- and f). Since u is C^∞ on U, $T_S^- f = H^- + u$ extends continuously (in fact C^∞) across $S \cap W$. Hence $T_S^+ f$, and then $H^+ = T_S^+ f - u$, also extends continuously from $U^+ \cap W$ to $S \cap W$, and the jump formula (1) holds on $S \cap W$. It follows that

$$f(z) = (T_S^+ f(z) - u) - (T_S^- f(z) - u) = H^+(z) - H^-(z) \text{ for } z \in W \cap S,$$

and thus $H^+ - H^-$ yields the desired holomorphic extension of f to $U^+ \cap W$.

Remark. The proof shows that the extension $H^+ - H^-$ of f is continuous on $W \cap \overline{U^+}$. In case $f \in C^1(S)$, one easily shows that the extension is in $C^1(W \cap \overline{U^+})$, and that analogous results hold when S and f are differentiable of higher order. The proofs follow by the same techniques used in the corresponding regularity results for the global CR extension theorem (see [R1] for example).

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The monodromy covering of the versal deformation of cyclic quotient surface singularities

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Abstract.

We give a short survey on some new (and old) results on deformations of cyclic quotient surface singularities which are mainly contained in the doctoral thesis of STEPHAN BROHME.

§1. Introduction

By studying the special case of cyclic quotient surface singularities several general aspects of deformation theory of complex–analytic singularities have been detected, e. g. the existence of many components of the base space of the versal deformation (which we also call the *versal base space* for short) and their monodromy coverings and the existence of embedded components. Even more: the (necessarily) smooth reduced components, the deformations thereon including the discriminant and the adjacencies and the monodromy coverings can be explicitly described and are very well understood (c. f. [4], [5], [6], [7], [8], [11], [16]; see also [2]).

The versal base space itself has - in the first interesting case of embedding dimension $e = 5$ - quite simple equations ([13], [14]). Later, ARNDT [1] calculated those equations for embedding dimension 6 and gave a "quasi–algorithmic" structure theorem for the general case (see also [9] and [10] for another approach for the much wider classes of *rational surface singularities with reduced fundamental cycle* and *sandwiched singularities*). In his dissertation, BROHME [6] proposes an explicit algorithm to produce equations in the cyclic case which are closer related to the continued fractions than those given in [9] and proves that his algorithm really leads to correct equations up to embedding dimension 8. It should also be mentioned that MIYAJIMA [12] has done some calculations on the versal deformation space by means of the deformation theory of CR–structures.

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However, all these sets of equations are extremely complicated (therefore, they are not reproduced here due to lack of space). In particular, it is almost impossible to draw any geometric conclusions from them. Despite the beautiful "picture method" of DE JONG and VAN STRATEN, there was in my opinion a "satisfactory" construction of the versal deformation - in the case of cyclic quotients - in terms of combinatorics, i. e. in terms of the *continued fraction* associated to such a singularity still missing. In order to remedy this unpleasant situation, I sketched in August 1996 an explicit construction of (a finite covering of) the *reduced* versal deformation space (the main idea is already contained in [15]). In the following I shall state the result after some preparatory notions and remarks; a proof is contained in [6]. Due to explicit computer algebra calculations via *Singular* in small embedding dimensions with the help of Brohme's equations, I am convinced that also the embedded components can successfully be "attached" to this construction.

This work would not have been possible without the pioneering work of JAN CHRISTOPHERSEN, JAN STEVENS and KURT BEHNKE on the component structure of the deformation space of the cyclic quotients.

§2. Some notions

Recall that a quotient surface singularity is given by natural numbers n, q with $1 \leq q < n$ and $\gcd(n, q) = 1$ which determine the singularity $X_{n,q}$ as the quotient of $\mathbb{C}^2$ by the linear action of the group $C_{n,q} \subset \mathrm{GL}(2, \mathbb{C})$ generated by the diagonal matrix $\mathrm{diag}(\zeta_n, \zeta_n^q)$ where ζ_n denotes a primitive n-th root of unity. It is well-known that all quotients of $\mathbb{C}^2$ by a finite cyclic group are of this form (up to analytic isomorphism), and $X_{n,q} \cong X_{n,q'}$ if and only if $q = q'$ or $qq' \equiv 1$. Moreover, the embedding dimension $e = e_{n,q} = \mathrm{emb}\, X_{n,q}$ is equal to

$$e_{n,q} = 3 + \sum_{k=1}^{\ell} (b_k - 2)$$

with the coefficients b_k of the Hirzebruch–Jung continued fraction expansion

$$\frac{n}{q} = b_1 - \underline{1 \rfloor b_2} - \cdots - \underline{1 \rfloor b_\ell}\,, \quad b_k \geq 2$$

or, resp., $e_{n,q} = r + 2$, $r = r_{n,q}$ the codimension of $X_{n,q}$, where

$$\frac{n}{n-q} = a_1 - \underline{1 \rfloor a_2} - \cdots - \underline{1 \rfloor a_r}\,, \quad a_j \geq 2\,.$$

(Note that we changed our notations of [14] in accordance with the work of JAN CHRISTOPHERSEN [7] and JAN STEVENS [16]). In other words, the system $(a_1, \ldots, a_r)$ of exponents (as well as the system of selfintersection numbers $b_1, \ldots, b_\ell$) is an analytic invariant of the singularity up to reversal of the order.

The $r(r+1)/2$ equations for the singularity $X_{n,q}$ can be written down with these exponents in *quasideterminantal* form (see e. g. [15]). For our construction, the r *leading* equations

$$x_0 x_2 - x_1^{a_1} = 0, \quad x_1 x_3 - x_2^{a_2} = 0, \ldots, x_{r-1} x_{r+1} - x_r^{a_r} = 0$$

are of special importance as well as the *last* one:

$$x_0 x_{r+1} - x_1^{a_1-1} x_2^{a_2-2} \cdot \ldots \cdot x_{r-1}^{a_{r-1}-2} x_r^{a_r-1} = 0 .$$

It follows from the work of Christophersen and Stevens that for fixed $r \geq 2$ there exist only finitely many so-called *r–chains* (representing zero) $(k_1, \ldots, k_r) \in \mathbb{N}_+^r$ such that the reduced components of the versal deformation space of a cyclic quotient surface singularity $X_{n,q}$ of codimension r are in $1:1$ correspondence to those r–chains $\underline{k} = (k_1, \ldots, k_r)$ satisfying $\underline{k} \leq \underline{a} := (a_1, \ldots, a_r)$, i. e.

$$k_j \leq a_j , \quad j = 1, \ldots, r .$$

Before we proceed further we recall the definition of r–chains $\underline{k}$ by Christophersen. Define $\alpha_0 = 0$, $\alpha_1 = 1$ and inductively $\alpha_{j+1} = k_j \alpha_j - \alpha_{j-1}$, $j = 1, \ldots, r$. Then $\underline{k}$ is an r–chain if $\alpha_j \geq 1$, $j = 1, \ldots, r$, and $\alpha_{r+1} = 0$. This is equivalent to saying that the continued fraction

$$k_1 - \underline{1 \rfloor} \overline{k_2} - \cdots - \underline{1 \rfloor} \overline{k_r}$$

is well defined and has the value 0. Let us list here all these chains for the cases $r = 2, 3, 4$ together with their corresponding α–series which are also necessary for understanding the construction.

r	$\underline{k} = (k_1, \ldots, k_r)$	$\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$
2	(1, 1)	(1, 1)
3	(1, 2, 1)	(1, 1, 1)
	(2, 1, 2)	(1, 2, 1)
4	(1, 2, 2, 1)	(1, 1, 1, 1)
	(1, 3, 1, 2)	(1, 1, 2, 1)
	(2, 1, 3, 1)	(1, 2, 1, 1)
	(2, 2, 1, 3)	(1, 2, 3, 1)
	(3, 1, 2, 2)	(1, 3, 2, 1)

We will write K_r for the set of r–chains; its cardinality is the famous CATALAN number

$$\frac{1}{r}\binom{2\,(r-1)}{r-1}\,.$$

To each $\underline{k} \in K_r$ we can associate a certain "cross and circle" diagram $\nabla_{\underline{k}}$ which can be used as a *format* for a cyclic quotient surface singularity $X_{n,q}$ of codimension r and exponents $\underline{a} = (a_1, \ldots, a_r)$: if $\underline{k} \leq \underline{a}$ holds, then $\nabla_{\underline{k}}$ determines in a completely algorithmic manner a system of equations $P_{ij}^{\nabla_{\underline{k}}}$, $0 \leq i, j \leq r+1$, $i+1 < j$ with $P_{i-1,i+1}^{\nabla_{\underline{k}}}$, $i = 1, \ldots, r$, always being the leading equations as above (for more details, see [5] and [15]). The *last* equation is of the form $P_{0,r+2}^{\nabla_{\underline{k}}} = x_0\, x_r - x_1^{\alpha_1\,(a_1-k_1)} \cdot \ldots \cdot x_r^{\alpha_r\,(a_r-k_r)}$. Moreover, for different $\underline{k}$, the last equations are also different. The quasideterminantal format belongs to the r–chain $(1,\, 2,\, 2,\, \ldots, 2,\, 1)$.

§3. The construction

We first describe the main features of our construction in the special case of cyclic *double points* and then in general. The situation for the A_{n-1}–singularity is extremely simple. It can be described by the linear action on $\mathbb{C}^2$ of the subgroup $C_{n,n-1} \subset \mathrm{SL}\,(2,\, \mathbb{C})$ which is generated by the diagonal matrix $\mathrm{diag}\,(\zeta_n,\, \zeta_n^{-1})$. Since $u^n,\, uv,\, v^n$ are generating polynomials of the invariant ring $\mathbb{C}\,[u,\, v]^{C_{n,n-1}}$, the singularity is given by the equation

$$x_0\, x_2 = x_1^n \quad \text{in} \quad \mathbb{C}^3\,.$$

To find a nice family $\mathcal{Y} \to T$ we replace the polynomial on the righthand side with a *generic* product of linear factors:

$$(*) \qquad x_0\, x_2 = (x_1 + t_1) \cdot \ldots \cdot (x_1 + t_n)\,.$$

Interpreting this equation as giving a hypersurface $\mathcal{Y} \subset \mathbb{C}^3 \times \mathbb{C}^n$, the projection to the second factor $T = \mathbb{C}^n$ yields an n–parameter deformation $\mathcal{Y} \to T$ of $X_{n,\,n-1}$ on which the symmetric group $\mathfrak{S}_n$ on n symbols acts. Dividing out the action of $\mathfrak{S}_n$, we get the deformation

$$(**)\ x_0\, x_2 = x_1^n + s_1\, x_1^{n-1} + \cdots + s_n\,, \quad (s_1, \ldots, s_n) \in S = \mathbb{C}^n\,,$$

where $s_j = s_j\,(t_1, \ldots, t_n)$ denotes the j^{th} elementary symmetric function in the elements $t_1, \ldots, t_n$, e. g. $s_1 = t_1 + \cdots + t_n, \ldots, s_n = t_1 \cdot \ldots \cdot t_n$. It is well–known that restriction of $(*)$ to the hyperplane $H = \{t_1 + \cdots + t_n = 0\}$ gives the (minimal) versal deformation $x_0\, x_2 = x_1^n + s_2\, x_1^{n-2} + \cdots + s_n$, and it is easily checked that the stabilizer subgroup of $\mathfrak{S}_n$ on H is isomorphic to $\mathfrak{S}_{n-1}$, the Weyl group of A_{n-1}–type playing here the role of the monodromy group.

The lesson to be learned by this example is <u>not to try</u> to construct a *minimal* family from the beginning. In fact, our base space $T_{n,q}$ for general cyclic quotients $X_{n,q}$ will be too large; but it is canonically a product of vector spaces, and minimizing the family means just to restrict to hyperplanes as above in some or all of these vector spaces.

We now explain our <u>Ansatz</u>. We make the leading equations completely generic in a fully symmetric way by taking the risk to not getting the minimal family (former attempts sacrificed the symmetry because of minimality and got lost in a not manageable mess of unnecessary conditions). To be more precise, we start with equations of type

$$
\begin{aligned}
x_0\,(x_2 + t_2^{(r)}) &= (x_1 + t_1^{(1)}) \cdot (x_1 + t_1^{(2)}) \cdot \ldots \cdot (x_1 + t_1^{(a_1)}) =: X_1^{(a_1)}, \\
(x_1 + t_1^{(\ell)})\,(x_3 + t_3^{(r)}) &= (x_2 + t_2^{(1)}) \cdot \ldots \cdot (x_2 + t_2^{(a_2)}) = X_2^{(a_2)}, \\
&\;\;\vdots \\
(x_{r-2} + t_{r-2}^{(\ell)})\,(x_r + t_r^{(r)}) &= X_{r-1}^{(a_{r-1})}, \\
(x_{r-1} + t_{r-1}^{(\ell)})\,x_{r+1} &= X_r^{(a_r)}\,.
\end{aligned}
$$

Here, of course, the upper indices (r) and (ℓ) are standing for "right" and "left" (not to be confused with the numbers r and ℓ). For inductive reasons one even should x_0 and x_{r+1} replace by $x_0 + t_0^{(\ell)}$ and $x_{r+1} + t_{r+1}^{(r)}$, resp. In order to minimalize we have later to put again $t_0^{(\ell)} = t_{r+1}^{(r)} = 0$ and $\sum_{k=1}^{a_j} t_j^{(k)} = 0\,,\ j = 1, \ldots, r$. Concerning the Weyl group or monodromy group, we introduce $W := W_1 \times \cdots \times W_r$, where $W_j \cong \mathfrak{S}_{a_j}$ denotes the symmetric group on a_j elements acting on the variables $t_j^{(1)}, \ldots, t_j^{(a_j)}$ by permutation (and on the others including $t_j^{(r)}$, $t_j^{(\ell)}$, if existing, trivially).

Our goal is to construct a W–invariant deformation of $X_{n,q}$ over a subspace of the vector space of all t–parameters. In order to do so, we follow formally for all r–chains $\underline{k} \le \underline{a}$ the "pattern" of the format $\nabla_{\underline{k}}$. This leads to *meromorphic* equations. More precisely, it will turn out that to each $\underline{k}$ there correspond further r–tuples $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r)$, $\underline{\rho} = (\rho_1, \ldots, \rho_r)$ with $\lambda_r = 0$, $\rho_1 = 0$ independently of $\underline{a}$ such that the last equation becomes

$$
(x_0 + t_0^{(\ell)})\,(x_{r+1} + t_{r+1}^{(r)}) = \frac{X_1^{(a_1)\alpha_1}}{(x_1 + t_1^{(\ell)})^{\lambda_1}} \cdot \frac{X_2^{(a_2)\alpha_2}}{(x_2 + t_2^{(\ell)})^{\lambda_2}\,(x_2 + t_2^{(r)})^{\rho_2}} \cdot
$$
$$
\cdots \frac{X_{r-1}^{(a_{r-1})\alpha_{r-1}}}{(x_{r-1} + t_{r-1}^{(\ell)})^{\lambda_{r-1}}\,(x_{r-1} + t_{r-1}^{(r)})^{\rho_{r-1}}} \cdot \frac{X_r^{(a_r)\alpha_j}}{(x_r + t_r^{(r)})^{\rho_r}}\,.
$$

We now put

$$t_1^{(\ell)} = t_1^{(1)} = \cdots = t_1^{(\lambda_1)} \quad \text{and} \quad t_r^{(r)} = t_r^{(1)} = \cdots = t_r^{(\rho_r)}$$

or correspondingly with all other combinations of equations we get by the action of $W_1 \times \cdots \times W_r$ on the righthand side. In the middle terms $2 \leq j \leq r-1$, we set $t_j^{(r)} = t_j^{(\ell)} = t_j^{(1)} = \cdots = t_j^{(\lambda_j+\rho_j)}$ or etc. for $\alpha_j = 1$; for $\alpha_j > 1$, we can choose $t_j^{(r)}$ and $t_j^{(\ell)}$ independently as before.

It is easily seen that all equations of type $\nabla_{\underline{k}}$, not only the last one, are then in fact *holomorphic* on the corresponding linear subspaces since the exponents λ_j and ρ_j satisfy sufficiently good properties.

§4. The main result

By the construction of the preceding section, we can attach to any cyclic quotient surface singularities $X = X_{n,q}$ a (reduced) subspace

$$T = T_{n,q} \subset \mathbb{C}^N, \quad N = N_{n,q},$$

consisting of a huge bunch of linear subspaces on which a subgroup $W = W_{n,q}$ of the symmetric group $\mathfrak{S}_N$ acts in a canonical way such that the following is satisfied (for details, see [6]).

i) On each component T' of T there lives a canonical deformation $\mathcal{Y}'$ of X;
ii) for two such components T', T'' these deformations $\mathcal{Y}'$, $\mathcal{Y}''$ coincide on the intersection $T' \cap T''$ thus defining a deformation

$$\mathcal{Y} = \mathcal{Y}_{n,q} = \bigcup \mathcal{Y}' \longrightarrow T,$$

iii) W acts equivariantly in a canonical way on $\mathcal{Y} \to T$;
iv) if $W' = W'_{n,q}$ denotes the stabilizer subgroup of W on a component T' of T, then W' acts as a reflection group such that

$$S' = T'/W'$$

is a smooth component of $S := T/W$, and W' acts also on $\mathcal{Y}' = \mathcal{Y}|T' \to T'$ equivariantly, inducing a deformation $\mathcal{X}' = \mathcal{Y}'/W' \to T'/W' = S'$;
v) each component S' is a component of the (reduced) base space of $X_{n,q}$, and all of these appear precisely once such that $\mathcal{X} \to S := T/W$ is the (reduced) versal deformation of $X_{n,q}$.

Remarks. 1. If the exponents a_j are big enough, the base space of the versal deformation of $X_{n,q}$ is *stable*, i. e. a product of a fixed space, depending only on the embedding dimension, with a smooth factor, as is well–known by the work of THEO DE JONG and DUCO VAN STRATEN [10] (the conditions $a_j \geq r-1$, $j = 1, \ldots, r$, should suffice). Hence, in these cases we have the maximal number of irreducible components.
2. On each component $\mathcal{Y}'$, the quotient mapping $\mathcal{Y}' \to \mathcal{Y}'/W' = \mathcal{X}'$ is the monodromy covering of $\mathcal{X}'$ in the sense of BEHNKE and CHRISTOPHERSEN [4]. Hence one may call the family $\mathcal{Y} \to T$ the *monodromy covering* of the versal deformation $\mathcal{X} \to S$ with *monodromy group* W. It is quite unclear to which extend the existence of such a family is a special feature of the cyclic quotient singularities only.
3. The highly symmetric "Ansatz" which is leading to our family is also interesting and promising with respect to other aspects of (cyclic) quotient surface singularities. It should, e. g. help to put the *toric structures* on the components together in an intelligent manner.

§5. Embedded components

With his equations, Brohme was able to carry out some calculations with *Singular*; e. g. for $e = 7$ and the (generic) exponents $(4, 4, 4, 4, 4)$, there are 11 extra *embedded* components in addition to the 14 reduced ones, 8 of them "supported" on the Artin component, 3 on other components of highest dimension. For smaller exponents there are in general fewer embedded components. It turns out that the result has a combinatorial description, too. One has to regard the following 5–chains:

$$\begin{array}{l}(2,\,2,\,2,\,2,\,2),\\ (1,\,3,\,2,\,2,\,2), (3,\,1,\,3,\,2,\,2), (2,\,3,\,1,\,3,\,2), (2,\,2,\,3,\,1,\,3), (2,\,2,\,2,\,3,\,1),\\ (3,\,2,\,2,\,2,\,2), (2,\,3,\,2,\,2,\,2), (2,\,2,\,3,\,2,\,2), (2,\,2,\,2,\,3,\,2), (2,\,2,\,2,\,2,\,3).\end{array}$$

Then embedded components correspond to chains which are smaller than the sequence of the a_j and are supported (on the monodromy covering) on easily describable *linear* subspaces of nonembedded components.

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Moduli as algebraic spaces

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Abstract.

We give a general criterion for the existence of a coarse moduli space as an algebraic space.

§1. Introduction

Given the class of inhomogeneously polarized, projective manifolds over $\mathbb{C}$, whose Hilbert polynomial is fixed, Matsusaka's big theorem ensures the boundedness of the corresponding moduli functor so that a coarse moduli space arises from an open subscheme of a certain Hilbert scheme. Mumford proved in [7, p. 217 ff.] that a coarse moduli space for non-uniruled manifolds exists and carries the structure of an algebraic space over $\mathbb{C}$ under the additional assumption that the automorphism groups of all objects are finite. Viehweg extended in [16, Theorem 9.16] the proof that the quotient of a scheme by an equivalence relation is an algebraic space, to those equivalence relations, whose equivalence classes are equidimensional. In this note, we show that in moduli theoretic situations the equidimensionality follows automatically.

§2. Fibered groupoids

In this section, we provide the formal framework. We denote by $p : \mathbf{F} \to \mathbf{A}$ a fibered groupoid in the sense of Grothendieck (categorie cofibrè en groupoides, [1, 9]). In our case $\mathbf{A}$ shall denote either the category $\mathbf{A}_c$ of complex spaces or the category $\mathbf{A}_s$ of schemes over $\mathbb{C}$ (separated and of finite type). By definition $p : \mathbf{F} \to \mathbf{A}$ is characterized by the following properties.

(i) For any morphism $g : R \to S$ in $\mathbf{A}$ and any $a \in Obj(\mathbf{A})$ over S there is a $g' : b \to a$ in $\mathbf{F}$ with $p(g') = g$. The object b is sometimes denoted by g^*a or $a \times_S R$.

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(ii) Let $\alpha : a \to c$ and $\beta : b \to c$ be morphisms in $\mathbf{F}$ such that there exists a morphism $\varphi : p(a) \to p(b)$ in $\mathbf{A}$ with $p(\beta) \circ \varphi = p(\alpha)$. Then there exists a unique morphism $\gamma : a \to b$ over φ such that $\beta \circ \gamma = \alpha$.

Property (ii) justifies the notation in (i).

Let S be in $Obj(A)$, then $\mathbf{F}(S)$ is by definition the category whose objects are objects in $\mathbf{F}$, which are mapped to S under p with morphisms over id_S. Passing to direct limits, one can assign to any fibered groupoid over the category of complex spaces a fibered groupoid over the category of complex space germs.

For any fibered groupoid there is an induced moduli functor $\mathfrak{M} : \mathbf{A} \to (Sets)$ where $\mathfrak{M}(S) = \overline{F(S)}$ is the set of isomorphism classes from $F(S)$, and where morphisms are defined in the obvious way. It is necessary in most cases to assign a sheafified moduli functor $\mathcal{M}$ to $\mathfrak{M}$, where the topology is the classical or étale topology depending on the choice of $\mathbf{A}$).

For the category of complex spaces $\mathbf{A}_c$ a coarse moduli space M is a morphism of functors

$$\Phi : \mathcal{M} \to M$$

with the following property

(i) for any complex space N and any morphism of functors $F : \mathcal{M} \to N$ there exists a unique morphism $f : M \to N$ such that $f \circ \Phi = F$.

(ii) the map $\mathcal{M}(Spec(\mathbb{C})) \to M(Spec(\mathbb{C}))$ is bijective.

(Here the complex space M is identified with $Hom(-, M)$).

If $\mathbf{A} = \mathbf{A}_s$, a coarse moduli space will be an *algebraic space* over $\mathbb{C}$: First to any scheme X the functor $Hom(-, X)$ from the category of affine schemes to the category of sets is assigned, inducing a sheaf of sets with respect to the étale topology. The latter is by definition a $\mathbb{C}$-space. An equivalence relation in the latter category is defined for all affine U and defines a quotient presheaf, and the corresponding sheaf is finally an algebraic space.

Let $a_0 \in \mathbf{F}(Spec(\mathbb{C}))$ be given. Then $\mathbf{F}_{a_0}$ denotes the induced fibered groupoid over the category of spaces with base point or space germs, whose objects are morphisms $a_0 \to a$ over $0 \to S$ implying a relationship to deformation theory: The usual deformation functor D_{a_0} from the category of complex space germs to the category of sets is equal to $D_{a_0}(S) = \overline{\mathbf{F}_{a_0}(S)}$, where the latter denotes the set of isomorphy classes of objects from $\mathbf{F}_{a_0}(S)$. Such deformation functors satisfy the axioms of Schlessinger [10] automatically, if the following condition holds (cf. [14, Lemma 2.6,2.7]):

(D) For all $a_0 \in \mathbf{F}(Spec(\mathbb{C}))$ there exists a semi-universal object in D_{a_0} over the category of complex spaces germs.

Let $\mathbf{A} = \mathbf{A}_c$, and let $a, b \in \mathbf{F}(S)$ for some complex space S. The functor $Isom_S(a,b)$: (Complex spaces $/S$) $\to$ (Sets) assigns to any complex space $R \to S$ the set of isomorphisms $a \times_S R \to b \times_S R$, and for morphisms in the category of complex spaces over S this functor is defined in an obvious way. We shall assume below that any such $Isom_S(a,b)$ is representable by a complex space over S. At the same time, we consider the functor $Isom_S(a,b)$ for a fibered groupoid over schemes. Because of the base change property, any such fibered groupoid defines a fibered groupoid over the category of $\mathbb{C}$-spaces. If $Isom_S(a,b)$ is representable by a $\mathbb{C}$-scheme (for any a, b), then the induced $\mathbb{C}$-space represents the induced functor for $\mathbb{C}$-spaces.

Now we come to the typical situation of a fibered groupoid $p : \mathbf{F} \to \mathbf{A}_s$, which is the restriction of a fibered groupoid $p_c : \mathbf{F}_c \to \mathbf{A}_c$ with $\mathbf{F}$ being a subcategory of $\mathbf{F}_c$. Under the assumption that for all Artinian schemes S_0 the category $\mathbf{F}(S_0)$ is a full subcategory of $\mathbf{F}_c(S_0)$ we call p_c a *complexification* of p. Let S be a $\mathbb{C}$-scheme and $a, b \in Obj(\mathbf{F}(S))$ and let $I = Isom_S(a,b) \to S$ represent the isomorphism functor. If follows easily that the induced morphism of corresponding complex spaces provides a representation of the isom-functor for $\mathbf{F}_c$. We state the following condition:

(Pr) For any a, b in $\mathbf{F}$ with $p(a) = p(b) = S$ the functor $Isom_S(a,b)$ is representable by a scheme $I = Isom_S(a,b) \to S$ proper over S.

Remark 1. *Let H be a scheme and $a \in \mathbf{F}(H)$, then $\psi : Isom_{H\times H}(a \times H, H \times a) \to H \times H$ defines an equivalence relation on H.*

We mention two technical conditions, which will be satisfied in our applications.

(R1) The morphism $p_2 = pr_2 \circ \psi : Isom_{H\times H}(a \times H, H \times a) \to H$ is smooth.

(R2) For $h \in H$ the morphism $\psi_h : p_2^{-1}(h) \to H \times \{h\}$ induced by ψ is smooth over its image.

Theorem 1. *Let $p : \mathbf{F} \to \mathbf{A}_s$ be a fibered groupoid with complexification $p_c : \mathbf{F}_c \to \mathbf{A}_c$, where semi-universal objects exist for the induced deformation functors. Suppose that the above condition (Pr) holds. Assume that for any $a_0 \in \mathbf{F}(Spec(\mathbb{C}))$ there exists an object a over a scheme H which induces a complete deformation of a_0 such that also (R1) and (R2) hold for a. Then there exists a coarse moduli space in the category of algebraic spaces over $\mathbb{C}$.*

We first show that p_c possesses a coarse moduli space M_c (in the category of complex spaces). Observe that any object a_0 from $\mathbf{F}$ over $Spec(\mathbb{C})$ is the restriction of some b from $\mathbf{F}$ such that the restriction of b to some classical open set induces a semi-universal deformation. This fact, together with the representability of the deformation functor is sufficient for the proof from [12, 13], (and [14] for the nonreduced case). Let $c \in \mathbf{F}(S)$. We write $Aut_S(c) = Isom_S(c,c)$. Let S be connected, and denote by $Aut^0_S(c)$ the connected component, which contains the identity section.

Proposition 1. *The fibers of $Aut^0_S(c) \to S$ are complex Lie groups of constant dimension.*

Corollary 1. *Under the above conditions any semi-universal deformation of $a_0 \in \mathbf{F}(Spec(\mathbb{C}))$ is universal.*

The proposition and the corollary follow like in [13] and [14, Theorem 5.1] from the existence of a semi-universal deformation and the properness assumption.

Now the theorem is a consequence of [16, Thm. 9.1] (cf. [7, App. 5A.]).

§3. Applications to polarized varieties

Let (X, λ_X) be a projective manifold equipped with an inhomogeneous polarization λ_X i.e. an ample divisor up to numerical equivalence, we write λ_X as $c_{1,\mathbb{R}}(L)$ for some ample line bundle L. Let $h \in \mathbb{Q}[T]$ with $h(\mathbb{Z}) \subset \mathbb{Z}$, and consider those (X, λ_X) with $\chi(X, \mathcal{O}_X(L^k)) = h(k)$. Families of polarized projective manifolds with fixed h define a groupoid $p : \mathbf{F} \to \mathbf{A}_s$ with complexification. Let $\mathcal{M}_h : \mathbf{A}_s \to$ (Sets) be the induced sheafified moduli functor. Let $m > 0$ satisfy the statement of Matsusaka's theorem [5]: For all such X and L (with given dimension, and fixed Hilbert polynomial) the m-th powers L^m are very ample, and $H^j(X, \mathcal{O}_X(L^m)) = 0$ for all $j > 0$. In particular, the linear system of all sections of L^m provides an embedding of X into $\mathbb{P}_N$, where $N = h(m)-1$.

Denote by $\mathcal{H} \subset \mathrm{Hilb}^{h(m\cdot t)}_{\mathbb{P}_N}$ the Zariski open subspace of all smooth $X \subset \mathbb{P}_N$ with Hilbert polynomial $h(m\cdot t)$ such that $\mathcal{O}_X(1)$ is divisible by m in $\mathrm{Pic}(X)$. (Assume that $\mathcal{H}$ is connected). The induced family $\mathcal{X} \hookrightarrow \mathbb{P}_N \times \mathcal{H} \to \mathcal{H}$ over $\mathcal{H}$ gives rise to the set-theoretic moduli space M_h, which always carries a natural topology induced by the classical topology on $\mathcal{H}$. Let $\psi : Isom_{\mathcal{H}\times\mathcal{H}}(\mathcal{X} \times \mathcal{H}, \mathcal{H} \times \mathcal{X}) \to \mathcal{H} \times \mathcal{H}$ be the canonical map. The necessary Hausdorff condition for M_h is the properness of $\mathrm{Im}(\psi) \to \mathcal{H}\times\mathcal{H}$ with respect to the classical topology. A slightly stronger condition is the properness of ψ.

Theorem 2. *Suppose that $\psi : Isom_{\mathcal{H}\times\mathcal{H}}(\mathcal{X}\times\mathcal{H}, \mathcal{H}\times\mathcal{X}) \to \mathcal{H}\times\mathcal{H}$ is proper. Then there exists a coarse moduli space, which is an algebraic space over $\mathbb{C}$.*

We verify the assumptions of Theorem 1: The properness of the isomorphism functor for any two given families of polarized projective manifolds can be proved easily using Hilbert schemes, and properties (R1) and (R2) follow like in [7] and [16]: (R1) follows from the Hilbert scheme construction, and (R2) is essentially Proposition 1 (cf. also [14]).

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Prolongation of holomorphic vector fields on a tube domain and its applications

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Introduction

In general, in the study of the holomorphic equivalence problem for complex manifolds, that is to say, the problem of investigating what happens when two complex manifolds are biholomorphically equivalent, it is one of standard ways to direct our attention to biholomorphic invariant objects. As a typical and good example of such objects, we have holomorphic automorphism groups. In fact, when Poincaré showed that a ball and a polydisk in $\mathbf{C}^2$ are not biholomorphically equivalent, he looked at their holomorphic automorphism groups, and showed that the dimensions do not coincide. One of the foundations of observations like this is the pioneer result of H. Cartan that the holomorphic automorphism group of a complex bounded domain has the structure of a Lie group.

Now, when a holomorphic automorphism group has the structure of a Lie group, what advantage do we have? It seems that one advantage is that conjugacy theorems in Lie group theory can be applied. The conjugacy theorems are very powerful tools, and if they can be applied well, splendid achievements are produced. But, in order to apply the conjugacy theorems, we need to know a lot about the Lie group structure of a holomorphic automorphism group. So, since Lie algebra provides much useful information about Lie group, we are led to turning our eyes to the Lie algebra of complete holomorphic vector fields corresponding to the Lie algebra of a holomorphic automorphism group. Then, in the process of investigating such Lie algebras, we often come up against the problem of completeness, or the fundamental problem of judging whether a vector field is complete or not. In general, a judgement on the completeness of a vector field is very difficult to deal with. Actually, given a vector field, the problem of whether its integral curve is lengthened to

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infinity or not has complicated aspects as the problem of solutions of autonomous systems in the theory of nonlinear oscillations. But, in some geometric setting, there is a nice algebraic criterion on the completeness of a vector field. In this article, we discuss such a criterion in the case of holomorphic vector fields on a tube domain. Our objects of consideration are polynomial vector fields on a tube domain T_Ω. We give a method of determining higher degree complete polynomial vector fields on T_Ω from the data on lower degree complete polynomial vector fields on T_Ω, which we call prolongation. Furthermore, we give its applications to the holomorphic equivalence problem for tube domains.

§1. Basic concepts and results on tube domains

We first recall some notation and terminology. An automorphism of a complex manifold M means a biholomorphic mapping of M onto itself. The group of all automorphisms of M is denoted by $\mathrm{Aut}(M)$. We denote by $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$ the subgroup of $\mathrm{Aut}(\mathbf{C}^n)$ consisting of all transformations of the form

$$\mathbf{C}^n \ni z \longmapsto Az + \beta \in \mathbf{C}^n,$$

where $A \in GL(n, \mathbf{R})$ and $\beta \in \mathbf{C}^n$. Two complex manifolds are said to be holomorphically equivalent if there is a biholomorphic mapping between them. For a Lie group G, we denote by G° the identity component of G and by $\mathrm{Lie}\, G$ the Lie algebra of G. If $E = \{\cdots\}$ is a subset of a vector space V over a field $\mathbf{F}$, the linear subspace of V spanned by E is denoted by $E_{\mathbf{F}} = \{\cdots\}_{\mathbf{F}}$.

We now recall basic concepts and results on tube domains. A tube domain T_Ω in $\mathbf{C}^n$ is a domain in $\mathbf{C}^n$ given by $T_\Omega = \mathbf{R}^n + \sqrt{-1}\Omega$, where Ω is a domain in $\mathbf{R}^n$ and is called the base of T_Ω. Clearly, each element $\xi \in \mathbf{R}^n$ gives rise to an automorphism $\sigma_\xi \in \mathrm{Aut}(T_\Omega)$ defined by

$$\sigma_\xi(z) = z + \xi \quad \text{for } z \in T_\Omega.$$

Write $\Sigma = \mathbf{R}^n$. The additive group Σ acts as a group of automorphisms on T_Ω by

$$\xi \cdot z = \sigma_\xi(z) \quad \text{for } \xi \in \Sigma \text{ and } z \in T_\Omega.$$

The subgroup of $\mathrm{Aut}(T_\Omega)$ induced by Σ is denoted by Σ_{T_Ω}. Note that if $\varphi \in GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$, then $\varphi(T_\Omega)$ is a tube domain in $\mathbf{C}^n$, and we have $\varphi \Sigma_{T_\Omega} \varphi^{-1} = \Sigma_{T_\Xi}$, where $T_\Xi = \varphi(T_\Omega)$.

Consider a biholomorphic mapping $\varphi\colon T_{\Omega_1} \to T_{\Omega_2}$ between two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbf{C}^n$. Then, by what we have noted above and [3, Section 1, Proposition], φ is given by an element of $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$

if and only if φ is equivariant with respect to the Σ-actions. Biholomorphic mappings between tube domains equivariant with respect to the Σ-actions may be considered as natural isomorphisms in the category of tube domains. In view of this observation, we say that two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbf{C}^n$ are affinely equivalent if there is a biholomorphic mapping between them given by an element of $GL(n, \mathbf{R}) \ltimes \mathbf{C}^n$.

If the convex hull of the base Ω of a tube domain T_Ω in $\mathbf{C}^n$ contains no complete straight lines, then T_Ω is holomorphically equivalent to a bounded domain in $\mathbf{C}^n$ and, by a well-known theorem of H. Cartan, the group $\mathrm{Aut}(T_\Omega)$ of all automorphisms of T_Ω forms a Lie group with respect to the compact-open topology. The Lie algebra $\mathfrak{g}(T_\Omega)$ of the Lie group $\mathrm{Aut}(T_\Omega)$ can be identified canonically with the finite-dimensional real Lie algebra consisting of all complete holomorphic vector fields on T_Ω. Throughout this article, we are concerned with tube domains whose bases have the convex hulls containing no complete straight lines.

Let $z_1, \cdots, z_n$ be the complex coordinate functions of $\mathbf{C}^n$ and, for $j = 1, \cdots, n$, we write $\partial_j = \partial/\partial z_j$. Let D be a domain in $\mathbf{C}^n$. Then every holomorphic vector field Z on D can be written in the form

$$Z = \sum_{j=1}^{n} f_j(z)\partial_j,$$

where $f_1(z), \cdots, f_n(z)$ are holomorphic functions on D. The vector field Z is called a polynomial vector field if $f_1(z), \cdots, f_n(z)$ are polynomials in $z_1, \cdots, z_n$. The maximum value of the degrees of the polynomials $f_1(z), \cdots, f_n(z)$ is called the degree of Z. The following result is fundamental in our study.

Structure Theorem ([3, Section 2, Theorem]). *To each tube domain T_Ω in $\mathbf{C}^n$ whose base Ω has the convex hull containing no complete straight lines, there is associated a tube domain $T_{\tilde{\Omega}}$ which is affinely equivalent to T_Ω such that $\mathfrak{g}(T_{\tilde{\Omega}})$ has the direct sum decomposition*

$$\mathfrak{g}(T_{\tilde{\Omega}}) = \mathfrak{p} + \mathfrak{e}$$

for which

$$\mathfrak{p} = \{X \in \mathfrak{g}(T_{\tilde{\Omega}}) \,|\, X \textit{ is a polynomial vector field}\},$$

$$\mathfrak{e} = \sum_{i=1}^{r} \{E_i^+, E_i^-\}_{\mathbf{R}},$$

$$E_i^{\pm} = e^{\pm z_i}\left(\partial_i \pm \sum_{j=r+1}^{n} \sqrt{-1}a_i^j\partial_j\right), \quad i = 1, \cdots, r,$$

where r is an integer between 0 and n and a_i^j, $i = 1, \cdots, r$, $j = r + 1, \cdots, n$, are real constants.

The integer r is called the exponential rank of the tube domain T_Ω, and is denoted by $e(T_\Omega)$. This is well-defined, because it is readily verified that if two tube domains T_{Ω_1} and T_{Ω_2} are affinely equivalent, then we have $e(T_{\Omega_1}) = e(T_{\Omega_2})$. When a tube domain T_Ω satisfies $e(T_\Omega) = 0$, we call T_Ω a tube domain with polynomial infinitesimal automorphisms.

Our main theme in this article is a study of tube domains with polynomial infinitesimal automorphisms. This is motivated by the holomorphic equivalence problem for tube domains, which we will explain below.

In terms of the notion of the affine equivalence of tube domains, the holomorphic equivalence problem for tube domains may be formulated as the problem of studying the connection between the two equivalences - the holomorphic equivalence and the affine equivalence - of tube domains. It is clear that if two tube domains in $\mathbf{C}^n$ are affinely equivalent, then they are holomorphically equivalent. What we have to ask is whether the converse assertion holds or not:

Problem. If two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbf{C}^n$ are holomorphically equivalent, then are they affinely equivalent?

When Ω_1 and Ω_2 are convex cones in $\mathbf{R}^n$, an affirmative answer is given (see Matsushima [1]). On the other hand, when Ω_1 and Ω_2 are arbitrary domains in $\mathbf{R}^n$ whose convex hulls contain no complete straight lines, there is a simple counter example. In fact, consider the upper half plane

$$T_{(0,\infty)} = \{x + \sqrt{-1}y \in \mathbf{C} \mid x \in \mathbf{R},\, y > 0\}$$

and the strip

$$T_{(0,\pi)} = \{x + \sqrt{-1}y \in \mathbf{C} \mid x \in \mathbf{R},\, 0 < y < \pi\}$$

in the complex plane. Then the tube domains $T_{(0,\infty)}$ and $T_{(0,\pi)}$ in $\mathbf{C}$ are holomorphically equivalent, but not affinely equivalent. We can clarify what causes a phenomenon like this by making use of the Structure Theorem stated above.

Let T_{Ω_1} and T_{Ω_2} be tube domains in $\mathbf{C}^n$ whose bases Ω_1 and Ω_2 have the convex hulls containing no complete straight lines. Since the exponential rank of a tube domain is an affine invariant, it is natural to reformulate the holomorphic equivalence problem for tube domains as follows:

Problem (*). If $e(T_{\Omega_1}) = e(T_{\Omega_2})$ and if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then are T_{Ω_1} and T_{Ω_2} affinely equivalent?

The counter example shown above corresponds to the case where $e(T_{\Omega_1}) \neq e(T_{\Omega_2})$, because $e(T_{(0,\infty)}) = 0$ and $e(T_{(0,\pi)}) = 1$. On the other hand, when Ω_1 and Ω_2 are bounded domains in $\mathbf{R}^n$, it is shown ([5]) that if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then we have $e(T_{\Omega_1}) = e(T_{\Omega_2})$, and T_{Ω_1} and T_{Ω_2} are affinely equivalent.

Specifying Problem (*), we consider the following problem which has fundamental importance:

Problem (**). If $e(T_{\Omega_1}) = e(T_{\Omega_2}) = 0$ and if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then are T_{Ω_1} and T_{Ω_2} affinely equivalent?

When Ω_1 and Ω_2 are convex cones in $\mathbf{R}^n$, we have $e(T_{\Omega_1}) = e(T_{\Omega_2}) = 0$ (see [1]), and an affirmative answer to Problem (**) is given, as stated above. For an attempt to solve Problem (**) in the case where T_{Ω_1} and T_{Ω_2} are arbitrary tube domains with polynomial infinitesimal automorphisms, we need a further study of the structure of $\mathfrak{g}(T_\Omega)$. The Prolongation Theorem given in the next section enables us to make a more detailed analysis of the structure of $\mathfrak{g}(T_\Omega)$ and, applying this, together with the classification result in [6] and so on, we can give an affirmative answer to Problem (**) in various cases [4], [8], [9].

§2. Prolongation of complete polynomial vector fields on a tube domain and tube domains with polynomial infinitesimal automorphisms

Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines. For a polynomial vector field Z on T_Ω of degree 2, we write

$$Z = \sum_{k=0}^{2} \left(X^{(k)} + \sqrt{-1} Y^{(k)} \right),$$

where $X^{(k)}, Y^{(k)}$ are polynomial vector fields whose components with respect to $\partial_1, \cdots, \partial_n$ are homogeneous polynomials in $z_1, \cdots, z_n$ with real coefficients of degree k, and set

$$\begin{aligned} Z_{[b]} &= X^{(2)} + \sqrt{-1} Y^{(1)}, \\ Z_{[a]} &= X^{(1)} + \sqrt{-1} Y^{(0)}, \\ Z_{[s]} &= X^{(0)}. \end{aligned}$$

Note that $Z = Z_{[s]} + Z_{[a]} + Z_{[b]} + \sqrt{-1}Y^{(2)}$. Our criterion on the completeness of Z is given in the following theorem.

Prolongation Theorem ([7, Section 2, Prolongation Theorem]). *Let Z be a polynomial vector field on T_Ω of degree 2. Then Z is complete on T_Ω if and only if one has $Y^{(2)} = 0$, and the vector fields $[\partial_i, Z]$, $i = 1, \cdots, n$, and $Z_{[a]}$ are all complete on T_Ω. Consequently, if Z is complete on T_Ω, then $Z_{[b]}$ is complete on T_Ω. Also, if $Z = Z_{[b]}$ and if the vector fields $[\partial_i, Z]$, $i = 1, \cdots, n$, are all complete on T_Ω, then Z is complete on T_Ω.*

The proof of this theorem is based on the fact that every infinitesimal isometry on a complete Riemannian manifold is complete. It follows from this fact that Z is complete on T_Ω if and only if the coefficient functions of Z satisfy the system of certain linear partial differential equations, and it is represented as the condition stated in the Prolongation Theorem.

Now, when we are discussing tube domains T_Ω with polynomial infinitesimal automorphisms, it is one of the key points that a polynomial gives the Taylor expansion around the origin of the function it represents. In what follows, we give some fundamental results on $\mathfrak{g}(T_\Omega)$ obtained by combining the Prolongation Theorem above with this fact.

2.1. General observations on an isotropy subalgebra of $\mathfrak{g}(T_\Omega)$

Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω has the convex hull containing no complete straight lines. We may assume without loss of generality that T_Ω contains the origin of $\mathbf{C}^n$. Every element Z of $\mathfrak{g}(T_\Omega)$ has the Taylor expansion around the origin given as

$$Z = \sum_{k=0}^{\infty} Z^{((k))},$$

where $Z^{((k))}$ is a polynomial vector field whose components with respect to $\partial_1, \cdots, \partial_n$ are homogeneous polynomials in $z_1, \cdots, z_n$ of degree k. We write

$$Z^{((1))} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} c_{ji}(Z) z_i \right) \partial_j,$$

where $c_{ji}(Z)$, $j, i = 1, \cdots, n$, are complex constants. Let $\mathfrak{k}$ denote the isotropy subalgebra of $\mathfrak{g}(T_\Omega)$ at the origin. Then $\mathfrak{k}$ consists of those elements Z of $\mathfrak{g}(T_\Omega)$ which satisfy $Z^{((0))} = 0$. An application of H.

Cartan's uniqueness theorem [2, Chapter 5, Proposition 1] yields the following result.

Lemma 1. *If Z is an element of $\mathfrak{k}$ and if $Z^{((1))} = 0$, then $Z = 0$.*

This result implies that the linear representation of $\mathfrak{k}$ given by

$$\mathfrak{k} \ni Z \longmapsto (c_{ji}(Z)) \in \mathfrak{gl}(n, \mathbf{C})$$

is faithful, where $\mathfrak{gl}(n, \mathbf{C})$ denotes the set of complex n by n matrices viewed as the Lie algebra of $GL(n, \mathbf{C})$. We recall here that T_Ω has the Bergman metric $ds^2_{T_\Omega}$. Using the invariance of $ds^2_{T_\Omega}$ under the action of Σ_{T_Ω}, after a suitable real linear change of coordinates we may assume that the holomorphic vector fields $\partial_1, \cdots, \partial_n$ form an orthonormal basis at the origin with respect to $ds^2_{T_\Omega}$. Then the matrix $(c_{ji}(Z))$ is a skew-Hermitian matrix for every element Z of $\mathfrak{k}$. Indeed, this follows from the fact that every automorphism of T_Ω is an isometry with respect to $ds^2_{T_\Omega}$.

2.2. Consequences of the Prolongation Theorem

Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines, and suppose further that $e(T_\Omega) = 0$, or $\mathfrak{g}(T_\Omega)$ consists of all polynomial vector fields which are complete on T_Ω. Then every element Z of $\mathfrak{g}(T_\Omega)$ can be written in the form

$$Z = \sum_{k=0}^{\infty} Z^{(k)}, \tag{$\sharp$}$$

where $Z^{(k)}$ is a polynomial vector field whose components with respect to $\partial_1, \cdots, \partial_n$ are homogeneous polynomials in $z_1, \cdots, z_n$ of degree k. Note that, in ($\sharp$), only finitely many $Z^{(k)}$'s are not equal to zero. We may assume without loss of generality that T_Ω contains the origin, and that $\partial_1, \cdots, \partial_n$ form an orthonormal basis at the origin with respect to the Bergman metric $ds^2_{T_\Omega}$. Then ($\sharp$) gives the Taylor expansion of Z around the origin. For $k = 0, 1, 2, \cdots$, we write

$$Z^{(k)} = X^{(k)} + \sqrt{-1}Y^{(k)},$$

where $X^{(k)}, Y^{(k)}$ are polynomial vector fields whose components are homogeneous polynomials with real coefficients of degree k. We define real vector subspaces $\mathfrak{q}, \mathfrak{s}, \mathfrak{a}_*, \mathfrak{b}$ of $\mathfrak{g}(T_\Omega)$ by

$$\mathfrak{q} = \left\{ Z \in \mathfrak{g}(T_\Omega) \;\middle|\; Z = \sum_{k=0}^{2} Z^{(k)} = \sum_{k=0}^{2} \left(X^{(k)} + \sqrt{-1} Y^{(k)} \right) \right\},$$
$$\mathfrak{s} = \{\partial_1, \cdots, \partial_n\}_{\mathbf{R}},$$
$$\mathfrak{a}_* = \left\{ Z \in \mathfrak{g}(T_\Omega) \;\middle|\; Z = X^{(1)} + \sqrt{-1} Y^{(0)} \right\},$$
$$\mathfrak{b} = \left\{ Z \in \mathfrak{g}(T_\Omega) \;\middle|\; Z = X^{(2)} + \sqrt{-1} Y^{(1)} \right\}.$$

The Prolongation Theorem shows that $\mathfrak{q}$ has the direct sum decomposition

$$\mathfrak{q} = \mathfrak{s} + \mathfrak{a}_* + \mathfrak{b}.$$

Note that $\mathfrak{b}$ is contained in the isotropy subalgebra $\mathfrak{k}$ of $\mathfrak{g}(T_\Omega)$ at the origin. The following result on $\mathfrak{b}$ is useful for a further study of the structure of $\mathfrak{g}(T_\Omega)$.

Lemma 2 ([7, Section 4, Lemma 4.2]). *Let $Z = X^{(2)} + \sqrt{-1} Y^{(1)}$ be an element of $\mathfrak{b}$ and write*

$$Y^{(1)} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} b_{ji}(Z) z_i \right) \partial_j,$$

where $b_{ji}(Z)$, $j, i = 1, \cdots, n$, are real constants. Then the following hold.

i) *$X^{(2)} = 0$ if and only if $Y^{(1)} = 0$.*

ii) *The real n by n matrix $(b_{ji}(Z))$ is symmetric for every element Z of $\mathfrak{b}$.*

As a consequence of ii) of Lemma 2, it should be observed that, when $\mathfrak{b}$ is an abelian subalgebra of $\mathfrak{g}(T_\Omega)$, the matrices $(b_{ji}(Z))$, $Z \in \mathfrak{b}$, are simultaneously diagonalizable by a suitable orthogonal change of coordinates.

§3. An application of Lie group theory to the holomorphic equivalence problem for tube domains

The following result plays an important role in the study of the equivalence of Siegel domains.

Conjugacy Theorem (cf. Matsushima [1]). *Any two maximal triangular subalgebras of a real Lie algebra are conjugate to each other under an inner automorphism.*

As a consequence of this result, we obtain a useful observation on an application of Lie group theory to the holomorphic equivalence problem for tube domains. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two real Lie algebras. Consider a subalgebra $\mathfrak{s}_1$ of $\mathfrak{g}_1$ sucu that $ad\,X$ is nilpotent on $\mathfrak{g}_1$ for every $X \in \mathfrak{s}_1$. Then, in view of Engel's theorem, there exists a maximal triangular subalgebra $\mathfrak{t}_1$ of $\mathfrak{g}_1$ containing $\mathfrak{s}_1$. Similarly, consider a subalgebra $\mathfrak{s}_2$ of $\mathfrak{g}_2$ sucu that $ad\,X$ is nilpotent on $\mathfrak{g}_2$ for every $X \in \mathfrak{s}_2$, and let $\mathfrak{t}_2$ be a maximal triangular subalgebra of $\mathfrak{g}_2$ containing $\mathfrak{s}_2$. Note that $\mathfrak{s}_1$ is contained in the nilradical $\mathfrak{n}_1$ of $\mathfrak{t}_1$, while $\mathfrak{s}_2$ is contained in the nilradical $\mathfrak{n}_2$ of $\mathfrak{t}_2$. Suppose now that there is a Lie algebra isomorphism $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Since $\Phi(\mathfrak{t}_1)$ is a maximal triangular subalgebra of $\mathfrak{g}_2$, it follows from the Conjugacy Theorem that there exists an inner automorphism σ of $\mathfrak{g}_2$ such that $\sigma(\Phi(\mathfrak{t}_1)) = \mathfrak{t}_2$. Since $\sigma(\Phi(\mathfrak{n}_1)) = \mathfrak{n}_2$, we see that $\sigma(\Phi(\mathfrak{s}_1))$ and $\mathfrak{s}_2$ are subalgebras of $\mathfrak{n}_2$.

To apply the above observation to our study, let T_{Ω_1} and T_{Ω_2} be two tube domains in $\mathbf{C}^n$ with polynomial infinitesimal automorphisms, and set $\mathfrak{g}_1 = \mathfrak{g}(T_{\Omega_1})$ and $\mathfrak{g}_2 = \mathfrak{g}(T_{\Omega_2})$. Since $\mathfrak{g}(T_{\Omega_1})$ consists of polynomial vector fields, it follows that $ad\,X$ is nilpotent on $\mathfrak{g}(T_{\Omega_1})$ for every element X of the subalgebra $\mathrm{Lie}\,\Sigma_{T_{\Omega_1}}$ of $\mathfrak{g}(T_{\Omega_1})$ corresponding to $\Sigma_{T_{\Omega_1}}$. Therefore we can set $\mathfrak{s}_1 = \mathrm{Lie}\,\Sigma_{T_{\Omega_1}}$. Similarly, we can set $\mathfrak{s}_2 = \mathrm{Lie}\,\Sigma_{T_{\Omega_2}}$. Suppose now that T_{Ω_1} and T_{Ω_2} are holomorphically equivalent. The above observation shows that we can find a solvable subalgebra $\mathfrak{t}_1$ of $\mathfrak{g}_1$ containing $\mathfrak{s}_1$, a solvable subalgebra $\mathfrak{t}_2$ of $\mathfrak{g}_2$ containing $\mathfrak{s}_2$, and a biholomorphic mapping $\psi : T_{\Omega_1} \to T_{\Omega_2}$ between T_{Ω_1} and T_{Ω_2} such that $\Psi(\mathfrak{t}_1) = \mathfrak{t}_2$, where Ψ is a Lie algebra isomorphism of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ given as the differential of the Lie group isomorphism $\mathrm{Aut}(T_{\Omega_1}) \ni g \mapsto \psi \circ g \circ \psi^{-1} \in \mathrm{Aut}(T_{\Omega_2})$. Note that both $\Psi(\mathfrak{s}_1)$ and $\mathfrak{s}_2$ are n-dimensional abelian subalgebras of the nilradical of the solvable Lie algebra $\mathfrak{t}_2$, and that if $\Psi(\mathfrak{s}_1)$ and $\mathfrak{s}_2$ are conjugate under an inner automorphism of $\mathfrak{g}_2$, then we can conclude by [3, Section 1, Proposition] that T_{Ω_1} and T_{Ω_2} are affinely equivalent. Thus the problem reduces to the investigation of certain solvable Lie algebras, and, as one direction to complete the story of our study, it seems to be important to study tube domains with solvable groups of automorphisms.

§4. A class of tube domains with solvable groups of automorphisms

Among tube domains with polynomial infinitesimal automorphisms, tube domains T_Ω whose bases Ω are convex cones are characteristic in the point that they have the property that if $\mathrm{Aut}(T_\Omega)$ is solvable, then the identity component of $\mathrm{Aut}(T_\Omega)$ necessarily consists of affine transformations. On the other hand, when Ω is an arbitrary convex domain in $\mathbf{R}^n$ containing no complete straight lines, there is a tube domain T_Ω in $\mathbf{C}^n$ such that $\mathrm{Aut}(T_\Omega)$ is solvable, but contains nonaffine automorphism, as the following theorem shows.

Theorem 1. *Let T_Ω be a tube domain in $\mathbf{C}^n$ whose base Ω is a convex domain in $\mathbf{R}^n$ containing no complete straight lines and let $n \geq 2$. Assume that:*

i) *T_Ω is a tube domain with polynomial infinitesimal automorphisms;*
ii) *$\mathrm{Aut}(T_\Omega)$ is a solvable Lie group;*
iii) *T_Ω contains the origin of $\mathbf{C}^n$ and the orbit of $G(T_\Omega)$ through the origin has dimension $n+1$, where $G(T_\Omega) = \mathrm{Aut}(T_\Omega)^\circ$.*

Then, in the notation of Subsection 2.2, $\mathfrak{g}(T_\Omega)$ coincides with $\mathfrak{q}$. Moreover, according to the cases of a) $\mathfrak{b} \neq \{0\}$ *and* b) $\mathfrak{b} = \{0\}$, *the following hold.*

a) *One has $n \geq 3$ and, after a real linear change of coordinates in $\mathbf{C}^n$, $\mathfrak{a}_*$, $\mathfrak{b}$ and the nilradical $\mathfrak{n}$ of $\mathfrak{g}(T_\Omega)$ are given by*

$$\begin{aligned}\mathfrak{a}_* &= \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}} + \mathfrak{k} \cap \mathfrak{a}_* \quad \textit{(direct sum)},\\ \mathfrak{b} &= \{\sqrt{-1}z_1\partial_1 + z_1^2\partial_2\}_{\mathbf{R}},\\ \mathfrak{n} &= \mathfrak{s} + \{\sqrt{-1}\partial_1 + 2z_1\partial_2\}_{\mathbf{R}}.\end{aligned}$$

Also, any n-dimensional abelian subalgebra of $\mathfrak{n}$ is conjugate to $\mathfrak{s}$ by an inner automorphism of $\mathfrak{g}(T_\Omega)$.

b) *The nilradical $\mathfrak{n}$ of $\mathfrak{g}(T_\Omega)$ has dimension less than or equal to $n+1$. Also, any n-dimensional abelian subalgebra of $\mathfrak{n}$ coincides with $\mathfrak{s}$.*

Combining this structure theorem with the observation given in Section 3, we can give an answer to the holomorphic equivalence problem for a class of tube domains with solvable groups of automorphisms.

Theorem 2. *Let T_{Ω_1} and T_{Ω_2} be two tube domains in $\mathbf{C}^n$ whose bases Ω_1 and Ω_2 are convex domains in $\mathbf{R}^n$ containing no complete straight lines and let $n \geq 2$. Assume that:*

i) *T_{Ω_1} and T_{Ω_2} are tube domains with polynomial infinitesimal automorphisms;*
ii) *$\mathrm{Aut}(T_{\Omega_1})$ is a solvable Lie group;*
iii) *There exists a point z_0 of T_{Ω_1} such that the orbit of $G(T_{\Omega_1})$ through z_0 has dimension $n+1$.*

Under these assumptions, if T_{Ω_1} and T_{Ω_2} are holomorphically equivalent, then they are affinely equivalent.

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Hypersurfaces and uniqueness of holomorphic mappings

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Abstract.

— It is possible to determine meromorphic functions on $\mathbb{C}$ by inverse images of some sets since R. Nevanlinna. However, analogous problems to holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ are complicated. In this paper some results for such problems are given. —

§1. Inroduction

Let $\mathcal{F}$ be a family of nonconstant holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ and $S_1, \cdots, S_q$ hypersurfaces of $\mathbb{P}^n(\mathbb{C})$. Then, what S_j have the property that $f^*S_j = g^*S_j$ $(1 \leq j \leq q)$ imply $f = g$ for $f, g \in \mathcal{F}$? Here, we consider S_j as divisors and f^*S_j are pull-backs. Also, we say that a hypersurface S has the uniquness property for $\mathcal{F}$ if $f^*S = g^*S$ implies $f = g$ for $f, g \in \mathcal{F}$.

The origin of this problem is Nevanlinna's unicity theorems:

Theorem N.1 ([N]). *Let a_j $(1 \leq j \leq 5)$ be distinct points in $\overline{\mathbb{C}}$. If nonconstant meromorphic functions f and g satisfy*

$$f^{-1}(a_j) = g^{-1}(a_j) \quad (1 \leq j \leq 5),$$

then $f = g$.

Theorem N.2 ([N]). *Let $a_1, \cdots, a_4$ be distinct points in $\overline{\mathbb{C}}$ such that the nonharmonic ratio is not -1 in each permutaion. If nonconstant meromorphic functions f and g satisfy*

$$f^{-1}(a_j) = g^{-1}(a_j) \quad (\textit{counting multiplicity}) \quad (1 \leq j \leq 4),$$

then $f = g$.

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§2. Uniqueness range sets

A uniqueness range set for entire (meromorphic) functions which has abbreviation URSE(URSM) is a discrete subset $S \subset \overline{\mathbb{C}}$ which has the property that entire (meromorphic) functions f and g such that $f^*S = g^*S$ are identical. For example, the zero set of $e^z + 1$ is not a URSE, but the zero set of $e^z + z$ is a URSE.

Theorem Y.1 ([Y2]). *Let p and d be relatively prime integers such that $d > 2p+4, p \geq 1$ and a, b nonzero complex constant such that $P(w) := w^d + aw^{d-p} + b = 0$ has no multiple root. Then, the zero set S of $P(w)$ is a URSE.*

The smallest d which satisfies the condition is 7 ($p = 1$). Therefore, there is a URSE with seven elements.

Also, Fujimoto showed a clss of URSM and one of URSE in [F3].

§3. Hypersurfaces with the uniqueness property

Now we consider hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, and $w_0, \cdots, w_n$ represent homogeneous coordinates of the space. Let $v_j = (a_{j0}, \cdots, a_{jn})$ $(0 \leq j \leq n+1)$ be vectors in general position. We consider the hypersurface S defined by

$$\sum_{j=0}^{n+1} \left(\sum_{k=0}^{n} a_{jk} w_k \right)^d = 0.$$

We denote by A_j the $(n+1) \times (n+1)$ matrix which is obtained by omitting the row v_j from $(n+2) \times (n+1)$ matrix $\begin{pmatrix} v_0 \\ \vdots \\ v_{n+1} \end{pmatrix}$, and assume that

$$\left(\frac{\det A_j}{\det A_k} \right)^d \neq \left(\frac{\det A_\mu}{\det A_\nu} \right)^d$$

for $0 \leq j, k, \mu, \nu \leq n+1$ such that $j \neq k, \mu \neq \nu, (j,k) \neq (\mu, \nu)$.

Theorem S.2([S]). *Assume $d \geq (2n+1)^2$. Then the hypersurface S has the uniqueness property for the family of linearly non-degenerate holomorphic mappings.*

Example. Let $v_0 = (1, 0, \cdots, 0), \cdots, v_n = (0, \cdots, 0, 1)$, $v_{n+1} = (a_0, \cdots, a_n)$, where $a_0 \cdots a_n \neq 0$. Then $\det A_j = (-1)^{n-j} a_j$, $\det A_{n+1}$

$= 1$. If we assume that

$$(-1)^{k-j}\frac{a_j}{a_k} \neq (-1)^{\nu-\mu}\frac{a_\mu}{a_\nu} \quad \text{for} \quad j \neq k, \mu \neq \nu, (j,k) \neq (\mu,\nu),$$

then the assumption of the theorem is satisfied, where $a_{n+1} = -1$. Now our hypersurface is defined by

$$w_0{}^d + \cdots + w_n{}^d + (a_0 w_0 + \cdots + a_n w_n)^d = 0.$$

Moreover, if $a_0\eta_0 + \cdots + a_n\eta_n \neq 1$ for any $(d-1)$-st roots η_j of $-a_j$, the hypersurface is non-singular.

§4. Some hypersurfaces case

Now the problem of uniqueness by inverse images of some hypersurfaces are treated.

Let n and m be positive integers and put $w = \exp(2\pi i/n)$, $u = \exp(2\pi i/m)$.

Theorem Y.2 ([Y1]). *Let $S_1 = \{a+b, a+bw, \cdots, a+bw^{n-1}\}$ and $S_2 = \{c\}$ with $n > 4, b \neq 0, c \neq a, (c-a)^{2n} \neq b^{2n}$. If $f^*S_j = g^*S_j$ $(j = 1, 2)$ for nonconstant entire functions f and g, then $f = g$.*

Theorem Y.3 ([Y1]). *Let $S_1 = \{a_1+b_1, a_1+b_1w, \cdots, a_1+b_1w^{n-1}\}$ and $S_2 = \{a_2+b_2, a_2+b_2u, \cdots, a_2+b_2u^{m-1}\}$ with $n > 4, m > 4, b_1b_2 \neq 0, a_1 \neq a_2$. If $f^*S_j = g^*S_j$ $(j = 1, 2)$ for nonconstant entire functions f and g, then $f = g$.*

Let f and g be holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ and H_j $(1 \leq j \leq q)$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Assume that

$(*)$ $\quad f^{-1}(H_j) = g^{-1}(H_j)$ (counting multiplicity) $(1 \leq j \leq q)$.

Theorem F.1 ([F1]). *If f and g are linearly non-degenerate and $q \geq 3n+2$, then $f = g$.*

Theorem F.2 ([F2]). If f and g are algebraically non-degenerate and $q \geq 2n+3$, then $f = g$.

Take $(a_{jk})_{0 \leq j,k \leq n} \in GL(n+1, \mathbb{C})$. Let p_1 and p_2 be positive integers and p the least common multiple of them. Consider hypersurfaces

$$S_1 : w_0{}^{p_1} + \cdots + w_n{}^{p_1} = 0,$$

$$S_2 : \sum_{j=0}^{n} \left(\sum_{k=0}^{n} a_{jk} w_k \right)^{p_2} = 0.$$

As an anologue of Theorem Y.3 we have

Theorem SU([SU]). *Assume that* $p_1,\ p_2 \geq (2n+1)^2$ *and that* $(a_{jk})^{2p} \neq (a_{\mu\nu})^{2p}$ *for any* (j,k) *and* (μ,ν) *with* $(j,k) \neq (\mu,\nu)$.*If linearly non-degenerate holomorphic mappings* f *and* g *of* $\mathbb{C}$ *into* $\mathbb{P}^n(\mathbb{C})$ *satisfy* $f^*S_j = g^*S_j$ $(j=1,2)$, *then* $f=g$.

Under the same condition of Theorem F1 and Theorem F2, the following was concluded without the nondegeneracy of f and g but with the addtional conditions $f(\mathbb{C}) \not\subset H_j,\ g(\mathbb{C}) \not\subset H_j$:

Theorem F.4 ([F1]). *If* $q = 3n+1$, *then* $g = Lf$ *by some projective linear transformation* L.

For $n=2$ and any $q \geq 6$, however, Fujimoto gave an example of hyperplanes in general position $H_1, \cdots, H_q$ such that there exist distinct f and g which satisfy $(*)$ and $f(\mathbb{C}) \not\subset H_j,\ g(\mathbb{C}) \not\subset H_j$. Of course, f and g are linearly degenerate, and one is a projective linear transformation of the other.

Problem. Do there exist hypersurfaces $S_1, \cdots, S_q$ such that non-constant holomorphic mapping $f,\ g$ satisfying $f^*S_j = g^*S_j$ $(1 \leq j \leq q)$ are identical?

Next, we consider the case that the family $\mathcal{F}$ is the family of non-constant holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. We consider the case of $n=2$.

Take $v_j = (a_{j0}, a_{j1}, a_{j2}) \in \mathbb{C}^3 (1 \leq j \leq q)$. Assume the following conditions:

(1) $a_{jk} \neq 0 \quad (1 \leq j \leq q,\ 0 \leq k \leq 2)$;

(2) $v_1, \cdots, v_q$ are in general position;

(3) for distinct $1 \leq j_1, j_2, j_3, j_4 \leq q$ and $k = 0,1,2$,

$$\frac{a_{j_1 k}}{a_{j_2 k}} \neq \frac{\det\ ({}^t v_{j_1}, {}^t v_{j_3}, {}^t v_{j_4})}{\det\ ({}^t v_{j_2}, {}^t v_{j_3}, {}^t v_{j_4})};$$

(4) for distinct $1 \leq j_1, \ldots, j_6 \leq q$ and distinct $1 \leq k_1, \cdots, k_6 \leq q$, and for d-th roots of one $\omega_1, \cdots, \omega_6$, if

$$\det \begin{pmatrix} a_{j_1 0} & a_{j_1 1} & a_{j12} & \omega_1 a_{k_1 0} & \omega_1 a_{k_1 1} & \omega_1 a_{k_1 2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j_6 0} & a_{j_6 1} & a_{j62} & \omega_6 a_{k_6 0} & \omega_1 a_{k_1 1} & \omega_1 a_{k_1 2} \end{pmatrix} = 0,$$

then $j_1 = k_1, \cdots, j_6 = k_6, \omega_1 = \cdots = \omega_6$.

Moreover we assume $p \geq 4$, $q \geq 10$, $d \geq (2q-1)^2$ and consider the hypersurface

$$S \ : \ \sum_{j=1}^{q} (a_{j0}w_0{}^p + a_{j1}w_1{}^p + a_{j2}w_2{}^p)^d = 0.$$

Theorem S.3. *Let $f = (f_0 : f_1 : f_2)$ and g be nonconstant holomorphic mappings of $\mathbb{C}$ into $\mathbb{P}^2(\mathbb{C})$. If $f^*S = g^*S$, then $g = (f_0 : \omega_1 f_1 : \omega_2 f_2)$, where ω_1, ω_2 are d-th roots of one.*

Corollary S.4. *There exist hypersurfaces S_1 and S_2 with the proprety that nonconstant holomorphic mappings f and g of $\mathbb{C}$ into $\mathbb{P}^2(\mathbb{C})$ satisfying $f^*S_j = g^*S_j$ $(j = 1, 2)$ are identical.*

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Seshadri constants and a criterion for bigness of pseudo-effective line bundles

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Abstract.

We state some fundamental properties of the intersection theory for pseudo-effective line bundles defined by Tsuji, and some applications of that theory. As one of applications, we generalize Seshadri's criterion for ampleness of nef line bundles as a criterion for bigness of pseudo-effective line bundles.

§1. Intersection theory for pseudo-effective line bundles

We consider a smooth complex projective variety X of dimension n and a line bundle L on X. A line bundle L is *pseudo-effective* if it has a singular Hermitian metric h with positive curvature: $\Theta_h \geq 0$ in the sense of current, here $\Theta_h := (2\pi)^{-1}\sqrt{-1}\,\overline{\partial}\partial \log h$. This is equivalent to the original algebraic definition of the pseudo-effectivity, namely the first Chern class $c_1(L) \in \overline{N}_{eff}$, here $\overline{N}_{eff}$ is the closure of the convex cone in the (real) Néron-Severi group $NS_{\mathbf{R}}(X) \subset H^2(X, \mathbf{R})$ generated by first Chern classes of effective $\mathbf{R}$-line bundles. By definition, a singular Hermitian metric h on L is written as $h = e^{-\varphi}h_0$ for a smooth Hermitian metric h_0 on L and $\varphi \in L^1_{loc}(X)$. The *multiplier ideal sheaf*

$$\mathcal{I}(h) = \mathcal{I}(X, h)$$

of $h = e^{-\varphi}h_0$ is defined by the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is integrable. By a theorem of Nadel (cf. [D1, §5]), $\mathcal{I}(h)$ is coherent provided φ is quasi-plurisubharmonic, i.e., the current $\sqrt{-1}\partial\overline{\partial}\varphi$ is bounded from below by a smooth real $(1,1)$-form on X (in particular case $\Theta_h \geq 0$). We sometimes use a variant $\mathcal{I}_+(h) := \mathcal{I}_+(e^{-\varphi})$ $:= \lim_{\varepsilon \downarrow 0} \mathcal{I}(e^{-(1+\varepsilon)\varphi})$. In case φ is quasi-plurisubharmonic, we regard the (complete pluri-)polar set of φ as the singular locus of h:

$$\operatorname{Sing} h := \{x \in X;\ \varphi(x) = -\infty\}.$$

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For a subvariety Y in X (i.e., a closed integral subscheme of X), we say that the restriction $h|_Y$ is *well-defined*, if $Y \not\subset \operatorname{Sing} h$.

Definition 1.1 ([T, Definition 2.9]). Let h be a singular Hermitian metric on L with $\Theta_h \geq 0$. Let C be an integral (i.e., reduced and irreducible) curve in X such that $h|_C$ is well-defined. The *intersection number* $(L, h) \cdot C$ is the real number

$$(L, h) \cdot C := \limsup_{m \to \infty} m^{-1} h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m) \cdot \mathcal{O}_C).$$

The original definition [T, Definition 2.9] is given for a slightly wider class of curves. The following proposition gives alternative definitions.

Proposition 1.2. *Let $C \subset X$ be an integral curve such that $h|_C$ is well-defined. Then*

$$\begin{aligned}(L, h) \cdot C &= \lim_{m \to \infty} m^{-1} h^0(C, \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m) \cdot \mathcal{O}_C) \\ &= \lim_{m \to \infty} m^{-1} \deg_C \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m).\end{aligned}$$

More precisely we assert that limits exist and they coincide. As for the last term, we define $\deg_C \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m) := mL \cdot C + \deg_C \mathcal{I}(h^m)$, and "$\deg_C$" as follows: Let $\nu : C' \longrightarrow C \subset X$ be the normalization of C, and $\mathcal{I} \subset \mathcal{O}_X$ a coherent ideal sheaf such that $C \not\subset \operatorname{supp} \mathcal{O}_X/\mathcal{I}$. Then we set $\deg_C \mathcal{I} := \deg_{C'} \nu^{-1}\mathcal{I} \cdot \mathcal{O}_{C'}$ as the degree of the invertible sheaf $\nu^{-1}\mathcal{I} \cdot \mathcal{O}_{C'}$ on C'.

By Demailly [D2, 4.1.1], every pseudo-effective line bundle L has a unique (up to certain equivalence of singularities) class of singular Hermitian metrics $h_{\min}$ with minimal singularities such that $\Theta_{h_{\min}} \geq 0$. In case L is semi-ample, we have $\mathcal{I}(h_{\min}^m) = \mathcal{O}_X$ for every $m \in \mathbf{N}$, and hence $(L, h_{\min}) \cdot C = L \cdot C$. In case L is big and it admits the so-called Zariski decomposition $L = P + N$ (i.e., P and N are $\mathbf{Q}$-divisors such that P nef, N effective, and that the natural injection $H^0(X, [mP]) \longrightarrow H^0(X, mL)$ is bijective for all $m \in \mathbf{N}$), $(L, h_{\min}) \cdot C = P \cdot C$ holds for $C \not\subset \operatorname{SBs}|L| := \bigcap_{m \in \mathbf{N}} \operatorname{Bs}|mL|$ the stable base locus (cf. [Tk2, 2.11]).

In [Tk2], we introduce a variant of the above intersection numbers. It is defined by using a variant of multiplier ideals due to Ein and Kawamata (refer [DEL, 1.7], [L] for a systematic treatment). These are defined algebraically, and therefore they fit into algebraic methods. In the rest of this section, let us assume the Kodaira-Iitaka dimension $\kappa(X, L) \geq 0$, and denote $h_{\min}$ the singular Hermitian metric with minimal singularities on L. Let k be a positive integer such that $H^0(X, \mathcal{O}_X(kL)) \neq 0$. We take a log-resolution $\mu : X' \longrightarrow X$ of the linear system $|kL|$ such that $\mu^*|kL| = |V| + E$, where $|V|$ is a free linear system, E the fixed part, and $E + \mathrm{Exc}(\mu)$ has simple normal crossing support. Given such a log-resolution plus a rational number $c > 0$, we define $\mathcal{J}(c \cdot |kL|) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [cE])$, this is independent of the choice of μ. In case L is big, there exists a large $k_0 \in \mathbf{N}$ such that the family of ideals $\{\mathcal{J}(\frac{c}{k} \cdot |kL|)\}_{k > k_0}$ has a unique maximal element. We denote the maximal element by

$$\mathcal{J}(c \cdot ||L||) = \mathcal{J}(X, c \cdot ||L||)$$

and call it the *asymptotic multiplier ideal sheaf* associated to c and $|L|$. Even in case L is not big, but still $\kappa(X, L) \geq 0$, one can modify the above definition and can define $\mathcal{J}(c \cdot ||L||)$. Let us recall the following fundamental properties.

Lemma 1.3. (1) $\mathcal{J}(||L||) \subset \mathcal{I}(h_{\min})$.
(2) $\mathcal{I}_+(h_{\min}) \subset \mathcal{J}(||L||) \subset \mathcal{I}(h_{\min})$, *provided L is big.*
(3) *The following natural inclusions are isomorphisms for every m:*

$$H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{J}(||mL||)) \longrightarrow H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m_{\min})) \longrightarrow H^0(X, \mathcal{O}_X(mL)).$$

Those above mentioned multiplier ideals $\mathcal{J}(||mL||), \mathcal{I}(h^m_{\min})$ are used as an important tool in their proofs of the invariance of plurigenera due to Siu [S] and Kawamata [K]. Using asymptotic multiplier ideals, we introduce a variant of intersection numbers.

Definition-Proposition 1.4 ([Tk2]). *Let $C \subset X$ be an integral curve such that $C \not\subset \mathrm{SBs}\,|L|$. Then*
(1) *the following limit exists:*

$$||L; C|| := \lim_{m \to \infty} m^{-1} \deg_C \mathcal{O}_C(mL) \otimes \mathcal{J}(||mL||).$$

(2) $0 \leq ||L; C|| \leq (L, h_{\min}) \cdot C \leq L \cdot C$.
(3) $||L; C|| = (L, h_{\min}) \cdot C$, *provided L is big.*

The middle and the last inequalities in (2) can be strict.

§2. Seshadri-type criterion

We generalize the following Seshadri's criterion.

Definition-Theorem 2.1 (cf. [H, I §7]). *Assume L is nef.* Seshadri's constant *of L at $x \in X$ is defined by*

$$\varepsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all integral curve C passing through x, and $\mathrm{mult}_x C$ *is the multiplicity of C at x.*

(1) Seshadri's criterion: *L is ample if and only if the global Seshadri's constant $\varepsilon(L) := \inf_{x \in X} \varepsilon(L, x)$ is positive.*

(2) (cf. [D1, 7.9]) *For every positive d-dimensional subvariety Y passing through x, $L^d \cdot Y \geq \varepsilon(L, x)^d \mathrm{mult}_x Y$ holds.*

It is suggestive to think of the Seshadri's constant $\varepsilon(L, x)$ as measuring how positive L is locally near x. By means of our intersection theory, we introduce an invariant to measure a local positivity of pseudo-effective line bundles.

Definition 2.2. Let h be a singular Hermitian metric on L with $\Theta_h \geq 0$. *Seshadri's constant of* (L, h) at $x \in X$ is defined by

$$\varepsilon((L, h), x) := \inf_{C \ni x} \frac{(L, h) \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all integral curve C passing through x and $h|_C$ is well-defined.

One can also define this type of invariant in terms of $||L; C||$. On the other hand, we have a global invariant.

Definition 2.3. The *volume* of L is the real number

$$v(L) = v(X, L) := \limsup_{m \to \infty} \frac{n!}{m^n} h^0(X, \mathcal{O}_X(mL)).$$

By definition, a line bundle is *big* if its volume is positive. In case L is nef, $v(L) = L^n$ holds. In case L is big, it has a singular Hermitian metric h with $\Theta_h \geq 0$ such that $\varepsilon((L, h), x) > 0$ for every point x outside some divisor (this is an easy consequence of Kodaira's lemma). Conversely, as in Seshadri's criterion, our criterion guarantees a positivity of the global volume by means of local positivities.

Theorem 2.4. *A line bundle L is big if and only if it has a singular Hermitian metric h with $\Theta_h \geq 0$ such that $\varepsilon((L, h), x) > 0$ for some point $x \in X - \mathrm{Sing}\, h$. Moreover in that case*

$$v(L) \geq \varepsilon((L, h), x)^n.$$

We also have an analogous statement of Theorem 2.1(2). Theorem 2.4 is proved by using the following approximation result of Seshadri's constants, which is an analogue of the approximation theorem of volumes due to Fujita. In this sense Theorem 2.5 is a local version of Fujita [F] (see also [DEL, 3.2]): Theorem 2.6 below.

Theorem 2.5. *Let h be a singular Hermitian metric on L with $\Theta_h \geq 0$. Assume L is big, and write $L \equiv A + E$ for an ample $\mathbf{Q}$-divisor A and an effective $\mathbf{Q}$-divisor E (Kodaira's lemma). Let δ be a positive number. Then there exist a birational modification $\mu_\delta : X_\delta \longrightarrow X$ and a decomposition $\mu_\delta^* L \equiv A_\delta + E_\delta$ with an ample $\mathbf{Q}$-divisor A_δ and an effective $\mathbf{Q}$-divisor E_δ such that μ_δ is isomophic over $X - (\mathrm{Sing}\, h \cup E)$, $E_\delta \subset \mu_\delta^{-1}(\mathrm{Sing}\, h \cup E)$, and such that*

$$\varepsilon(A_\delta, \mu_\delta^{-1}(x)) \geq (1-\delta)\varepsilon((L,h),x)$$

for every $x \in X - (\mathrm{Sing}\, h \cup E)$. Moreover $A_\delta^n \geq v(L) - \delta$ holds as below.

Theorem 2.6 ([F]). *Let L be a big line bundle on X. Given any $\varepsilon > 0$, there exists a birational modification $\mu_\varepsilon : X'_\varepsilon \longrightarrow X$ and a decomposition $\mu_\varepsilon^* L \equiv A_\varepsilon + E_\varepsilon$, where E_ε is en effective $\mathbf{Q}$-divisor and A_ε is an ample $\mathbf{Q}$-divisor with $A_\varepsilon^n > v(L) - \varepsilon$.*

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Subadjunction theorem

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Abstract.

We give a subadjunction theorem which relates the multi-adjoint linear system of the ambient space and the linear system of the restricted bundle on a subvariety.

§1. Introduction

Let M be a complex manifold and L be a line bundle on M and S be a submanifold of M. It is a basic question whether the restriction map

$$H^0(M, \mathcal{O}_M(L)) \to H^0(S, \mathcal{O}_S(L))$$

is surjective.

In this paper we shall consider this question for multi-adjoint type line bundles under certain geometric conditions.

Let us state our result precisely. Let M be a complex manifold of dimension n and let S be a closed complex submanifold of M. Then we consider a class of continuous function $\Psi : M \longrightarrow [-\infty, 0)$ such that

1. $\Psi^{-1}(-\infty) \supset S$,
2. if S is k-dimensional around a point x, there exists a local coorinate $(z_1, \dots, z_n)$ on a neighbourhood of x such that $z_{k+1} = \cdots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} \mid \Psi(z) - (n-k)\log \sum_{j=k+1}^{n} \mid z_j \mid^2 \mid < \infty.$$

The set of such functions Ψ will be denoted by $\sharp(S)$.

For each $\Psi \in \sharp(S)$, one can associate a positive measure $dV_M[\Psi]$ on S as the minimum element of the partially ordered set of positive

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measueres $d\mu$ satisfying

$$\int_{S_k} f d\mu \geq \limsup_{t\to\infty} \frac{2(n-k)}{\sigma_{2n-2k-1}} \int_M f \cdot e^{-\Psi} \cdot \chi_{R(\Psi,t)} dV_M$$

for any nonnegative continuous function f with $\operatorname{supp} f \subset\subset M$. Here S_k denotes the k-dimensional component of S, σ_m denotes the volume of the unit sphere in $\mathbf{R}^{m+1}$, and $\chi_{R(\Psi,t)}$ denotes the characteristic funciton of the set

$$R(\Psi, t) = \{x \in M \mid -t-1 < \Psi(x) < -t\}.$$

Theorem 1.1. *Let M be a projective manifold with a continuous volume form dV_M, let L be a holomorphic line bundle over M with a C^∞-hermitian metric h_L, let S be a compact complex submanifold of M, let $\Psi : M \longrightarrow [-\infty, 0)$ be a continuous function and let K_M be the canonical bundle of M.*

1. $\Psi \in \sharp(S) \cap C^\infty(M \backslash S)$,
2. $\Theta_{h\cdot e^{-(1+\epsilon)\Psi}} \geq 0$ *for every* $\epsilon \in [0, \delta]$ *for some* $\delta > 0$,
3. *there is a positive line bundle on M.*

Then every element of $H^0(S, \mathcal{O}_S(m(K_M + L)))$ extends to an element of $H^0(M, \mathcal{O}_M(m(K_M + L)))$.

One may think that the assumption on the existence of the function Ψ is somewhat technical or restrictive. But as one see in the last section, this is not the case. In fact one may construct such a function by using an effective $\mathbf{Q}$-divisor on M.

The results in this paper may be considered as a generalization of [6] to the case of nontrivial normal bundles. We also note that there exists another type of subadjunction theorem due to Y. Kawamata ([2]). This is a reserch announcement. The detailed proof will be published elsewhere.

§2. Setch of the proof of Theorem 1.1

Here we shall give a sketch of the proof of Theorem 1.1. Let M, S, L be as in Theorem 1.1. Let h_S be a canonical AZD ([8]) of $K_M + L \mid_S$. Let A be a sufficiently ample line bundle on M. Let us define the singular hemitian metric on $m(K_M + L) \mid_S$ by

$$h_{m,S} := K(A + m(K_M + L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)^{-1}$$

Then as in [8], we see that

$$h_S := \liminf_{m\to\infty} \sqrt[m]{h_{m,S}}$$

holds. Hence $\{\sqrt[m]{h_{m,S}}\}$ is considered to be an algebraic approximation of h_S. We consider the Bergman kernel

$$K(S, A + m(K_M + L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S) = \sum_i \mid \sigma_i^{(m)} \mid^2,$$

where $\{\sigma_i^{(m)}\}$ is a complete orthonormal basis of $A^2(S, A+m(K_M+L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)$. We note that (cf. [3, p.46, Proposition 1.4.16])

$$K(S, A + m(K_M + L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)(x)$$

$$= \sup\{\mid\sigma\mid^2 (x) \mid \sigma \in A^2(S, A+m(K_M+L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S), \parallel\sigma\parallel = 1\}$$

holds for every $x \in S$. We note that there exists a positive constant C_0 independent of m such that

$$h_{m,S} \leq C_0 \cdot h_A \cdot h_S^m$$

holds for every $m \geq 1$ as in [8]. Let h_M be a canoncal AZD of $K_X + L$ and let ν denote the numerical Kodaira dimension of $(K_M + L, h_M)$, i.e.,

$$\nu := \lim_{m\to\infty} \frac{\log \dim H^0(M, \mathcal{O}_M(A + m(K_M + L)) \otimes \mathcal{I}(h_M^m))}{\log m}.$$

For simplicity we shall consider the case that ν is equal to the numerical Kodaira dimension of $K_M + L$. Otherwise the proof should be modified a little bit.

Inductively on m, we extend each

$$\sigma \in A^2(S, A + m(K_M + L) \mid_S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)$$

to a section

$$\tilde{\sigma} \in A^2(M, A + m(K_M + L), dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}, dV)$$

with the estimate

$$\parallel \tilde{\sigma} \parallel \leq C \cdot m^{-\nu} \parallel \sigma \parallel$$

where $\parallel \ \parallel$'s denote the L^2-norms respectively, C is a positive constant indpendent of m and we have defined

$$\tilde{K}_m(x) := \sup\{\mid \tilde{\sigma} \mid^2 (x) \mid \parallel \tilde{\sigma} \mid_S \parallel = 1, \parallel \tilde{\sigma} \parallel \leq C \cdot m^{-\nu}\}$$

and set

$$\tilde{h}_m = \frac{1}{\tilde{K}_m}.$$

If we take C sufficiently large, then $\tilde{h}_m$ is well defined for every $m \geq 0$. By easy inductive estimates, we see that

$$\tilde{h}_\infty := \liminf_{m\to\infty} \sqrt[m]{\tilde{h}_m}$$

exists and gives an extension of h_S. Then by [4] for every $m \geq 1$, we may extend every element of $A^2(m(K_M + L) \mid_S, dV_M^{-1} \cdot h_L \cdot h_S^{m-1}, dV_M[\Psi])$ to $A^2(m(K_M + L), dV_M^{-1} \cdot h_L \cdot \tilde{h}_\infty, dV_M)$. This completes the proof of Theorem 1.1.

§3. Generalization of Theorem 1.1

Let M be a smooth projective variety and let (L, h_L) be a singlar hermitian line bundle on M such that $\Theta_{h_L} \geq 0$ on M. Let dV be a C^∞-volume form on M. Let $\sigma \in \Gamma(\bar{M}, \mathcal{O}_{\bar{M}}(m_0 L) \otimes \mathcal{I}(h))$ be a global section. Let α be a positive rational number ≤ 1 and let S be an irreducible subvariety of M such that $(M, \alpha(\sigma))$ is logcanonical but not KLT(Kawamata log-terminal) on the generic point of S and $(M, (\alpha - \epsilon)(\sigma))$ is KLT on the generic point of S for every $0 < \epsilon << 1$. We set

$$\Psi = \alpha \log h_L(\sigma, \sigma).$$

We shall assume that S is not contained in the singular locus of h, where the singular locus of h means the set of points where h is $+\infty$.

For the moment we shall consider the case that S is smooth (when S is not smooth, we just need to take an embedded resolution of S). In this case Ψ may not belong to $\sharp(S)$, since Ψ may not have the prescribed singularity along S as in the definition of $\sharp(S)$. Then as in Section 2.1, we may define a (possibly singular measure) $dV[\Psi]$ on S. This can be viewed as follows. Let $f : N \longrightarrow M$ be a logresolution of $(X, \alpha(\sigma))$. Then as before we may define the singular volume form $f^*dV[f^*\Psi]$ on the divisorial component of $f^{-1}(S)$. The singular volume form $dV[\Psi]$ is defined as the fibre integral of $f^*dV[f^*\Psi]$.

The proof [4] and hence proof of Theorem 1.1 also works in this case except a minor difference. The difference is that $dV_M[\Psi]$ (which is defined similarly as above) may have singularities along some Zariski closed subset of S. Let $d\mu_S$ be a C^∞-volume form on S and let φ be the function on S defined by

$$\varphi := \log \frac{dV[\Psi]}{d\mu_S}.$$

Theorem 3.1. *Let M,S,Ψ be as above. Suppose that S is smooth. Then every element of $A^2(S, \mathcal{O}_S(m(K_M + dL)), e^{-(m-1)\varphi} \cdot dV^{-m} \cdot h_L^m, dV[\Psi])$ extends to an element of*

$$H^0(M, \mathcal{O}_M(m(K_M + dL))).$$

As we mentioned as above the smoothness assumption on S is just to make the statement simpler.

As an example of an application, we have :

Corollary 3.1 ([6]). *Let $\pi : X \longrightarrow \Delta$ be a semistable degeneration of projective variety over the unit disk. Let $X_0 = \pi^{-1}(0) = \sum_i D_i$ be the irreducible decomposition. Then we have that*

$$\sum_i P_m(D_i) \leq P_m(X_t)$$

holds where t is any regular value of π and P_m denotes the m-th plurigenus.

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Fixed points of polynomial automorphisms of $\mathbf{C}^n$

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Abstract.

We study the fixed point indices of some polynomial automorphisms of $\mathbf{C}^n$. In particular, it is shown that, for a composition of generalized Hénon maps, the sum of the fixed point indices vanishes. A consequence is that a generic polynomial automorphism of $\mathbf{C}^2$ has a saddle fixed point.

§1. Statement of the results

A bijective map F of the space of n complex variables $\mathbf{C}^n$ onto itself defined by polynomials $f_1(x), \dots, f_n(x)$, $x = (x_1, \dots, x_n)$, is said to be a polynomial automorphism of $\mathbf{C}^n$. The set $\mathrm{Aut}\,(\mathbf{C}^n)$ of all polynomial automorphisms of $\mathbf{C}^n$ forms a group under composition. Two maps $F_1, F_2 \in \mathrm{Aut}\,(\mathbf{C}^n)$ are conjugate if there exists a map $G \in \mathrm{Aut}\,(\mathbf{C}^n)$ such that $F_2 = G^{-1} \circ F_1 \circ G$.

For a fixed point of a holomorphic map of $\mathbf{C}^n$ to itself, holomorphic Lefschetz index can be defined (see §2, also Griffiths-Harris [2]). We will study the indices for the fixed points of polynomial automorphisms, since they are important invariants under conjugation.

For the case of two variables, Friedland-Milnor [1] showed that any map in $\mathrm{Aut}\,(\mathbf{C}^2)$ is conjugate to either (1) an affine map, (2) an elementary map or (3) a composition $F_m \circ \cdots \circ F_1$ of generalized Hénon maps

$$F_\mu(x, y) = (y, p_\mu(y) - \delta_\mu x), \quad \mu = 1, \dots, m,$$

where $p_\mu(y)$ are polynomials of degree ≥ 2 and $\delta_\mu \neq 0$.

We denote by H_0 the set consisting of compositions of generalized Hénon maps, and by H the set of all maps conjugate to one of the maps in H_0.

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Let $\mathrm{Fix}\,(F)$ denote the set of all fixed points of F. It was shown in [1] that, if $F \in H_0$ and $\deg F = k$, then F has k fixed points counting multiplicity. i.e., $\sum_{a\in \mathrm{Fix}(F)} \mathrm{Mult}\,(F, a) = k$.

Now we have

Theorem 1. *If $F \in H$, then we have*

$$\sum_{a\in \mathrm{Fix}(F)} \mathrm{Ind}\,(F, a) = 0.$$

We note that the formula fails in general for maps $\notin H$. A proof of this formula for a generalized Hénon map is given in [3]. A similar result for holomorphic maps on projective spaces is given in [4].

Corollary 1. *Let $F \in H$ and suppose that F has only simple fixed points a_j $(j = 1, \cdots, k)$. Let $\lambda_{j,1}, \lambda_{j,2}$ denote the eigenvalues of $F'(a_j)$. Then we have*

$$\sum_{j=1}^{k} \left(\frac{1}{1-\lambda_{j,1}} + \frac{1}{1-\lambda_{j,2}} \right) = k,$$

Corollary 2. *Let $F \in H$ and $\delta = \det F'$. Suppose that $|\delta| \neq 1$ or $\delta = 1$. Then* (1) *F has either a saddle fixed point or a multiple fixed point, and* (2) *F has infinitely many periodic points that are either saddle or multiple.*

The condition on δ cannot be dropped as the following example shows.

Example Let F be a Hénon map defined by

$$F(x, y) = (y, y^2 + c - \delta x).$$

Then F has at least one saddle fixed point if and only if $(\delta, c) \notin \Delta \cup \Gamma$, where $\Delta = \{(\delta + 1)^2 - 4c = 0\}$ and

$$\Gamma = \left\{ |\delta| = 1,\ \frac{c}{\delta} \text{ is real and} \sqrt{2(1 + \mathrm{Re}\,\delta)} - 1 \leq \frac{c}{\delta} < \frac{1 + \mathrm{Re}\,\delta}{2} \right\}.$$

We can generalize the index formula to maps of certain class of polynomial automorphisms of $\mathbf{C}^n$:

Theorem 2. *Let $F = F_m \circ \cdots \circ F_1$ be the composition of shift-like maps $F_\mu : \mathbf{C}^n \to \mathbf{C}^n$ $(\mu = 1, \ldots, m)$ defined by*

$$F_\mu(x_1, \ldots, x_n) = (x_2, \ldots, x_n, a_\mu x_1 + p_\mu(x_2, \ldots, x_n)),$$

where p_μ *are polynomials in* $n-1$ *variables. Suppose that there exist* ν $(2 \leq \nu \leq n)$ *such that*

$$P_\mu(x_2, \ldots, x_n) = c_\mu x_\nu^{k_\mu} + (\text{lower order terms}), \ c_\mu \neq 0.$$

Then we have $\sum_{a \in \mathrm{Fix}(F)} \mathrm{Ind}\,(F, a) = 0$.

We remark that, for general (compositions of) shift-like maps, the set $\mathrm{Fix}\,(F)$ may be non-isolated. Even if $\mathrm{Fix}\,(F)$ is isolated, the index formula does not necessarily hold.

Example Consider the map $F : \mathbf{C}^3 \to \mathbf{C}^3$ defined by

$$F(x, y, z) = (y, z, \delta x + (y - z)^2).$$

If $\delta \neq 1$, then $\mathrm{Fix}\,(F) = \{0\}$ and $\mathrm{Ind}\,(F, 0) = 1/(1-\delta)$. If $\delta = 1$, then $\mathrm{Fix}\,(F) = \{x = y = z\}$.

§2. Multiplicity and Index

Let $G : \mathbf{C}^n \to \mathbf{C}^n$ be a holomorphic map and suppose that a is an isolated zero of G. Then there exist neighborhoods U of a and V of 0 such that $G^{-1}(0) \cap U = \{a\}$ and that $G|U : U \to V$ is a branched cover. We define the zero multiplicity $\mathrm{mult}\,(G, a)$ of G at a to be the sheet number of this map $G|U$. We call that a is a simple zero of G if $\mathrm{mult}\,(G, a) = 1$, or in other words, if $\det G'(a) \neq 0$.

If a is a simple zero, we define the zero index by $\mathrm{ind}\,(G, a) = 1/\det G'(a)$. For the general case $\mathrm{ind}\,(G, a)$ is defined as follows: We set $\omega = dx_1 \wedge \cdots \wedge dx_n$ and

$$\eta = \frac{c_n}{\|x\|^{2n}} \sum_{i=1}^{n} (-1)^{i-1} \overline{x}_i d\overline{x}_1 \wedge \cdots \widehat{d\overline{x}_i} \cdots \wedge d\overline{x}_n$$

Where $c_n = \sqrt{-1}^{n^2} (n-1)!/(2\pi)^n$. We define

$$\mathrm{ind}\,(G, a) = \int_{\partial B} (G^* \eta) \wedge \omega$$

where B denotes a ball with center a of sufficiently small radius so that a is the only zero of G in B.

We will apply the following lemma in the proof of Theorem 2.

Lemma 3. *Let* $G(x) = (g_1(x), \ldots, g_n(x))$ *be a polynomial map of* $\mathbf{C}^n$ *to* $\mathbf{C}^n$. *Suppose that* g_ν *is of the form*

$$g_\nu(x) = c_\nu x_{\sigma(\nu)}^{k_\nu} + (\text{lower order terms}), \quad k_\nu \geq 2,\ c_\nu \neq 0, \quad (\nu = 1, \ldots, n).$$

where σ *is a permutation of* $\{1, \ldots, n\}$. *then* $\sum_{a \in G^{-1}(0)} \mathrm{ind}\,(G, a) = 0$.

To see this, we note that

$$\sum_{a \in G^{-1}(0)} \operatorname{ind}(G, a) = \int_{\partial B} (G^*\eta) \wedge \omega,$$

where B is a sufficiently large ball in $\mathbf{C}^n$. By estimating the integral, we conclude the lemma.

Now let $F : \mathbf{C}^n \to \mathbf{C}^n$ be a holomorphic map and suppose that a is an isolated fixed point of F. This is equivalent to say that a is an isolated zero of the map $Id - F$. We define the fixed point multiplicity and the fixed point index by

$$\operatorname{Mult}(F, a) = \operatorname{mult}(Id - F, a), \quad \operatorname{Ind}(F, a) = \operatorname{ind}(Id - F, a).$$

§3. Outline of the proof

3.1 To prove Theorem 2, let us first introduce the concept of vectorial shift-like map. We denote the points in $\mathbf{C}^{mn}$ as (m, n)-matrices and also as a row of column vectors: $\widehat{\xi} = (\xi_{ij}) = (\xi_1, \dots, \xi_n)$. A map $\Phi \in \operatorname{Aut}(\mathbf{C}^{mn})$ is said to be a vectorial shift-like map if it is of the form

$$\Phi(\xi_1, \dots, \xi_n) = (\xi_2, \dots, \xi_n, A\xi_1 + Q(\xi_2, \dots, \xi_n))$$

where $A \in GL(m, \mathbf{C})$ and Q is a column vector of polynomials in $m(n-1)$ variables ξ_{ij} $(1 \le i \le m; 2 \le j \le n)$.

The fixed points of Φ are of the form $\hat{b} = (b, \dots, b)$, where $b \in \mathbf{C}^m$ are the roots of the equation $A\xi + Q(\xi, \dots, \xi) = \xi$. We define a linear map $L : (\xi_1, \dots, \xi_n) \mapsto (\eta_1, \dots, \eta_n)$ by

$$\eta_\nu = \xi_\nu - \xi_{\nu+1} \ (\nu = 1, \dots, n-1) \quad \text{and} \quad \eta_n = \xi_n.$$

Then $(Id - \Phi) \circ L^{-1}$ takes the form $(\eta_1, \dots, \eta_n) \mapsto (\eta_1, \dots, \eta_{n-1}, \eta_n - A(\eta_1 + \dots + \eta_n) - Q(\eta_2 + \dots + \eta_n, \dots, \eta_n))$. The sum of the zero point indices of this map is equal to that of the map $\eta \mapsto \eta - A\eta - Q(\eta, \dots, \eta)$. If this satisfies the condition of Lemma 3, then $\sum_{\hat{b} \in \operatorname{Fix}(\Phi)} \operatorname{Ind}(\Phi, \hat{b}) = 0$.

3.2 Let $F_\mu : \mathbf{C}^n \to \mathbf{C}^n$ be holomorphic maps $(\mu = 1, \dots, m)$, and let $F = F_m \circ \dots \circ F_1$ be their composition. To study the fixed points of F, we consider the map $\hat{F} : \mathbf{C}^{mn} \to \mathbf{C}^{mn}$ defined as follows. We denote the points in $\mathbf{C}^{mn}$ by a (m, n)-matrix and also as a column of row vectors :

$\hat{x} = (x_{ij}) = {}^t(x_1, \dots, x_m)$. We define $\hat{F}$ by

$$\hat{F}(\hat{x}) = \hat{F}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} F_m(x_m) \\ F_1(x_1) \\ \vdots \\ F_{m-1}(x_{m-1}) \end{pmatrix}.$$

There is a one-to-one correspondence between the sets $\mathrm{Fix}\,(F)$ and $\mathrm{Fix}\,(\hat{F})$. In fact, if a is in $\mathrm{Fix}\,(F)$, then the point $\hat{a} = {}^t(a_1, \dots, a_m)$ with $a_1 = a, a_\mu = F_{\mu-1}(a_{\mu-1})$ $(\mu = 2, \dots, m)$ is in $\mathrm{Fix}\,(\hat{F})$. Conversely, if $\hat{a} = {}^t(a_1, \dots, a_m)$ is in $\mathrm{Fix}\,(\hat{F})$, then a_1 is in $\mathrm{Fix}\,(F)$.

Further we can prove that, if $a \in \mathrm{Fix}\,(F)$ and $\hat{a} \in \mathrm{Fix}\,(\hat{F})$ are corresponding fixed points, then

$$\mathrm{Mult}\,(F, a) = \mathrm{Mult}\,(\hat{F}, \hat{a}), \quad \text{and} \quad \mathrm{Ind}\,(F, a) = \mathrm{Ind}\,(\hat{F}, \hat{a}).$$

3.3 Now we apply the above obserbations to a composition $F = F_m \circ \cdots \circ F_1$ of shift-like maps F_μ. Then $\hat{F}(\hat{x})$ takes the form

$$\begin{pmatrix} x_{m2} & \cdots & x_{mn} & \delta_m x_{m1} + p_m(x_{m2}, \dots, x_{mn}) \\ x_{12} & \cdots & x_{1n} & \delta_1 x_{11} + p_1(x_{12}, \dots, x_{1n}) \\ \vdots & \ddots & \vdots & \vdots \\ x_{m-1,2} & \cdots & x_{m-1,n} & \delta_{m-1} x_{m-1,1} + p_{m-1}(x_{m-1,2}, \dots, x_{m-1,n}) \end{pmatrix}.$$

We can reduce $\hat{F}$ to a vectorial shift-like map by conjugation. To see this, consider the linear map $M : \mathbf{C}^{mn} \ni (x_{ij}) \mapsto (\xi_{ij}) \in \mathbf{C}^{mn}$ defined by $\xi_{ij} = x_{[i-j+1],j}$ where $[\ell]$ denotes the number such that $1 \le [\ell] \le m$ and $[\ell] \equiv \ell \mod m$. Then the conjugate $\Phi = M \circ \hat{F} \circ M^{-1}$ is a vectorial shift-like map $\Phi(\xi_1, \dots, \xi_n) = (\xi_2, \dots, \xi_n, A\xi_1 + Q(\xi_2, \dots, \xi_n))$, where

$$A\xi_1 + Q(\xi_2, \dots, \xi_n) = \begin{pmatrix} \delta_{[1-n]}\xi_{[1-n],1} + p_{[1-n]}(\xi_{[2-n],2}, \dots, \xi_{m,n}) \\ \delta_{[2-n]}\xi_{[2-n],1} + p_{[2-n]}(\xi_{[3-n],2}, \dots, \xi_{1,n}) \\ \vdots \\ \delta_{[m-n]}\xi_{[m-n],1} + p_{[m-n]}(\xi_{[1-n],2}, \dots, \xi_{m-1,n}) \end{pmatrix}.$$

The map $\eta \mapsto \eta - A\eta - Q(\eta, \dots, \eta)$ takes the form

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{pmatrix} \mapsto \begin{pmatrix} \eta_1 - \delta_{[1-n]}\eta_{[1-n]} - p_{[1-n]}(\eta_{[2-n]}, \dots, \eta_m) \\ \eta_2 - \delta_{[2-n]}\eta_{[2-n]} - p_{[2-n]}(\eta_{[3-n]}, \dots, \eta_1) \\ \vdots \\ \eta_m - \delta_{[m-n]}\eta_{[m-n]} - p_{[m-n]}(\eta_{[1-n]}, \dots, \eta_{m-1}) \end{pmatrix}.$$

Under the condition of Theorem 2, this map satisfies the condition of Lemma 3. Thus Theorem 2 is proved.

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On Nevanlinna theory for holomorphic curves in Abelian varieties

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Abstract.

We give some observations and results on Nevanlinna theory for holomorphic curves in algebraic varieties.

§1. Intersection theory and Nevanlinna theory

In this note, we consider Nevanlinna theory as non-compact, transcendental intersection theory. First we begin with an algebraic intersection theory. Let X be a smooth, projective algebraic variety and let $D \subset X$ be an effective reduced divisor. Let C be a smooth, projective curve and let S be a finite set of points on C, which will be fixed for the following discussion. Let $f : C \to X$ be an algebraic map such that $f(C) \not\subset \operatorname{supp} D$. Then we have

$$\sum_{x \in C \setminus S} \operatorname{ord}_x f^* D + \sum_{x \in S} \operatorname{ord}_x f^* D = \int_C f^*(c_1(D)). \tag{1}$$

The left hand side of (1) is a sum of local intersection numbers between $f(C)$ and D, while the right hand side is a cohomological invariant which only depend on f and $\mathcal{O}(D)$.

There is a kind of intersection theory for a holomorphic map $f : \mathbb{C} \to X$ which may be transcendental. This is called Nevanlinna's First Main Theorem. We want to count a intersection number between $f(\mathbb{C})$ and D. Since this number is infinite in general, we use an exhaustion $\mathbb{C} = \cup_{r>0}\{z \in \mathbb{C}; |z| < r\}$. We define the counting function as

$$N(r, f, D) = \int_1^r \left(\sum_{|z|<t} \operatorname{ord}_z f^* D \right) \frac{dt}{t}.$$

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As in the first term of the left hand side of (1), the counting function counts intersection numbers just on the non-compact part $\mathbb{C}$. Hence we need to count intersection number on the boundary of $\mathbb{C}$. This is the following proximity function which corresponds to the second term of the left hand side of (1). Let $L(D)$ be the associated line bundle for D. Let $||\cdot||$ be a Hermitian metric of $L(D)$ and let s_D be a section of $L(D)$ such that D is the zero divisor for s_D. Then we define the proximity function of D by

$$m(r,f,D) = \int_0^{2\pi} \log \frac{1}{||s_D \circ f(re^{i\theta})||} \frac{d\theta}{2\pi}.$$

We define an analogue of degree of f with respect to a line bundle L on X as

$$T(r,f,L) = \int_1^r \frac{dt}{t} \int_{\mathbb{C}(t)} f^* c_1(L) + O(1) \quad (r \to \infty),$$

which is called the order function. We define the height function of D by $T(r,f,D) = T(r,f,L(D)) + O(1)$. Then the First Main Theorem in Nevanlinna theory is

$$N(r,f,D) + m(r,f,D) = T(r,f,D) + O(1), \tag{2}$$

which is an analogue of (1). Here the left hand side depends on external geometry of $f(\mathbb{C})$ and D in X, while the right hand side only depend on $f(\mathbb{C})$ and a cohomology class of D.

§2. Conjectures

Our Problem is the following;

What happen if we don't count intersection multiplicity in (1) or (2) ?

Of course, we can't obtain an equality any more, but we hope that there is some inequality. We motivate this estimate by the following heuristic and optimal observation for an algebraic map $f: C \to X$. Let $\mathcal{M}_f$ be the connected component of the moduli space of f.
(i) For a generic $f_0 \in \mathcal{M}_f$, we have $\deg f_0^* D = \deg(f_0^* D)_{\mathrm{red}}$. This is because $f_0(C)$ and D would intersect transversely.
(ii) For an integer $k \geq 0$, put

$$\mathcal{M}_f^k = \{f \in \mathcal{M}_f; \deg f^* D - \deg(f^* D)_{\mathrm{red}} \geq k\}.$$

Then $\mathcal{M}_f^k$ is a Zariski closed subset of $\mathcal{M}_f$ and form a sequence

$$\mathcal{M}_f = \mathcal{M}_f^0 \supset \mathcal{M}_f^1 \supset \mathcal{M}_f^2 \supset \cdots .$$

(iii) We hope that $\operatorname{codim}(\mathcal{M}_f^{k+1}, \mathcal{M}_f^k) \geq 1$ in general.
(iv) Hence for $k = \dim \mathcal{M}_f + \epsilon$, we have "$\mathcal{M}_f^k = \emptyset$".
(v) We hope that $\dim \mathcal{M}_f = -\deg f^* K_X + \epsilon$ for the canonical line bundle K_X.
(vi) We have $\deg(f^*D)_{\mathrm{red}} = \deg_{C\setminus S}(f^*D)_{\mathrm{red}} + O(1)$ where $O(1)$ is a bounded term independent to f. This is because

$$\#S < \infty. \tag{3}$$

Hence we hope that the following conjecture is true (cf. [7]).

Conjecture 1. Let L be an ample line bundle on X and let $\epsilon > 0$. Then there exists a proper Zariski closed subset $\Lambda = \Lambda(X, D, L, \epsilon) \subsetneqq X$ such that

$$\deg f^* K_X(D) \leq \deg_{C\setminus S}(f^*D)_{\mathrm{red}} + \epsilon \deg f^* L + O_\epsilon(1)$$

for all algebraic map $f : C \to X$ with $f(C) \not\subset \Lambda$. Here $O_\epsilon(1)$ is a bounded term independent to f but dependent on ϵ and L.

For a closed subvariety Z of X with $\operatorname{codim}(Z, X) \geq 2$, we put

$$\mathcal{M}_f^k = \{f \in \mathcal{M}_f ; \deg f^* Z \geq k\},$$

and the same observation makes us to hope

Conjecture 2. There exists a proper Zariski closed subset $\Xi = \Xi(X, Z, L, \epsilon) \subsetneqq X$ such that

$$\deg f^* Z \leq -\deg f^* K_X + \epsilon \deg f^* L + O_\epsilon(1)$$

for all algebraic map $f : C \to X$ with $f(C) \not\subset \Xi$.

There are counterparts in Nevanlinna theory for the above conjectures (cf. [1]). Define the truncated counting function by

$$N^{(1)}(r, f, D) = \int_1^r \left(\sum_{|z|<t} \min(\operatorname{ord}_z f^* D, 1) \right) \frac{dt}{t}.$$

Conjecture 3. There exists a proper Zariski closed subset $\Lambda = \Lambda(X, D, L, \epsilon) \subsetneqq X$ such that

$$T(r, f, K_X(D)) \leq N^{(1)}(r, f, D) + \epsilon T(r, f, L) \ ||$$

for all holomorphic map $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset \Lambda$.

Conjecture 4. There exists a proper Zariski closed subset $\Xi = \Xi(X, Z, L, \epsilon) \subsetneqq X$ such that

$$N(r, f, Z) \leq -T(r, f, K_X) + \epsilon T(r, f, L) \ ||$$

for all holomorphic map $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset \Xi$.

Here the symbol $||$ means that the inequality holds for $r > 0$ outside a set of finite linear measure. In the above, conjectures 3 and 4 correspond to those of 1 and 2, respectively.

Remark. (1) The counterpart for inequality (3) in Nevanlinna theory is Nevanlinna's lemma on logarithmic derivatives for a meromorphic function φ, i.e., $m(r, \varphi'/\varphi, \infty) < O(\log(rT(r, \varphi, \infty))) \ ||$. To see this, we note that

$$\#S < \infty \Longleftrightarrow \sum_{x \in S} \mathrm{ord}_x(\partial\varphi/\varphi)^*(\infty) < O(1) \quad \text{for all } \varphi \in \mathbb{C}(C),$$

where ∂ is a vector field on C and $O(1)$ is a constant independent of φ.

(2) To be precise, we need the condition that D is simple normal crossing in the above conjectures (cf. [6]).

§3. The case for curves

When $\dim X = 1$, we have the natural morphism between logarithmic 1-forms $f^*\Omega^1_X(\log D) \to \Omega^1_C(\log(f^*D)_{\mathrm{red}})$ for algebraic map $f : C \to X$. Hence by taking degrees and using (3), we obtain Conjecture 1 in this case. For the holomorphic case $f : \mathbb{C} \to X$, the following result is classical (R. Nevanlinna, L. Ahlfors).

Theorem 1. *Suppose* $\dim X = 1$. *Then Conjecture 3 is true.*

Suppose $g(X) \geq 2$. Since we have $N^{(1)}(r, f, D) \leq T(r, f, D)$, Theorem 1 implies the inequality $T(r, f, K_X) \leq O(1)||$. But since K_X is ample, this inequality implies that f is constant. Hence we have

Corollary 1. *Suppose* $g(X) \geq 2$. *Then all holomorphic map* $f : \mathbb{C} \to X$ *is a constant map.*

The higher dimensional version of this corollary is the following conjecture (cf. [1]).

Conjecture 5. Let X be a projective variety of general type. Then there exists a proper Zariski closed subset $Y \subsetneqq X$ such that the image of all non-constant holomorphic map $f : \mathbb{C} \to X$ is contained in Y.

A remarkable fact is that Theorem 1 for $X = \mathbb{P}^1$ implies Corollary 1. Suppose $g(X) \geq 2$ and let $\pi : X \to \mathbb{P}^1$ be a ramified covering. Let $E' \subset X$ be the ramification divisor of π and put $D = \operatorname{supp} \pi_*(E')$, $E = \operatorname{supp} \pi^* D$. Then we have an equality $\pi^* K_{\mathbb{P}^1}(D) = K_X(E)$. Hence for a holomorphic map $f : \mathbb{C} \to X$, we apply Theorem 1 to $\pi \circ f$ and we have

$$\begin{aligned} T(r, f, K_X(E)) &= T(r, f, \pi^* K_{\mathbb{P}^1}(D)) = T(r, \pi \circ f, K_{\mathbb{P}^1}(D)) \\ &\leq N^{(1)}(r, \pi \circ f, D) = N^{(1)}(r, f, E) \leq T(r, f, E) \end{aligned}$$

modulo small term $\epsilon T(r, f, L) \,||$. Hence $T(r, f, K_X) \leq \epsilon T(r, f, L) \,||$ for all $\epsilon > 0$, which implies Corollary 1.

Remark. This argument is quite general. And it also works in the higher dimensional case: Conjecture 3 for X implies Conjecture 5 for X' which is a ramified covering of X.

§4. The case of Abelian varieties

In the higher dimensional case, the conjectures in section 2 seem to be difficult problem. But when X is an Abelian variety, we have interesting results. (cf. [2],[3],[4],[5],[8],[9])

Theorem 2. *Let X be an Abelian variety. Then Conjectures 3 and 4 are true.*

Remark This theorem holds without any restriction for the singularities of D.

As corollaries to this theorem, we have

Corollary 2. *Let X be a projective variety with irregularity condition* $\dim H^0(X, \Omega_X^1) \geq \dim X$. *Then Conjecture 5 is true for X.*

The case $\dim H^0(X, \Omega_X^1) > \dim X$ is famous Bloch-Ochiai's Theorem and our new part is the case $\dim H^0(X, \Omega_X^1) = \dim X$. To prove this case, we use the albanese map $X \to \operatorname{Alb}(X)$ which is a generically finite map, the argument for the remark in section 3 and the above Theorem 2.

The following Corollary is a unicity theorem for elliptic curves. Though there is a higher dimensional version for general Abelian varieties, we just present an one dimensional case for the sake of simplicity.

Corollary 3. *Let E_1, E_2 be elliptic curves and let $O_i \in E_i$ $(i = 1,2)$ be the points of identities. Let $f_i : \mathbb{C} \to E_i$ $(i = 1,2)$ be non-constant holomorphic maps such that* $\operatorname{supp} f_1^*(O_1) = \operatorname{supp} f_2^*(O_2)$. *Then there exists an isomorphism $\alpha : E_1 \to E_2$ such that $f_2 = \alpha \circ f_1$.*

The idea of the proof of this corollary is the following. Consider the holomorphic map $f_1 \times f_2 : \mathbb{C} \to E_1 \times E_2$ and suppose that the image $f_1 \times f_2(\mathbb{C})$ is Zariski dense in $E_1 \times E_2$. Then since $\operatorname{codim}(O_1 \times O_2, E_1 \times E_2) \geq 2$, Theorem 2 implies that $N^{(1)}(r, f_1 \times f_2, O_1 \times O_2)$ is very small term. On the other hand the assumption $\operatorname{supp} f_1^*(O_1) = \operatorname{supp} f_2^*(O_2)$ implies that $N^{(1)}(r, f_1 \times f_2, O_1 \times O_2) = N^{(1)}(r, f_1, O_1)$ but this right hand side is a big term. These give a contradiction, hence $f_1 \times f_2(\mathbb{C})$ is not Zariski dense in $E_1 \times E_2$. By Bloch-Ochiai's theorem, $f_1 \times f_2(\mathbb{C})$ is contained in some elliptic curve $F \subset E_1 \times E_2$ and this F gives the graph of α.

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Numerical characterization for affine varieties be a cone over nonsingular projective varieties

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Nikulin's $K3$ surfaces, adiabatic limit of equivariant analytic torsion, and the Borcherds Φ-function

Ken-Ichi Yoshikawa

Abstract.

In this note, we prove that the "adiabatic limit" of the equivariant analytic torsion of a Nikulin's $K3$ surface converges to the value of norm of the Borcherds Φ-function at its period point after a certain renormalization.

§0. Introduction

Let $\pi\colon M \to B$ be a submersion of compact Riemannian manifolds. Let g_M and g_B be Riemannian metrics on M and B, respectively. For $0 < \epsilon < \infty$, set $g_{M,\epsilon} := g_M + \epsilon^{-1}\pi^* g_B$. Let $T(g_M)$ be a geometric object depending on the metric g_M. The limit of $T(g_{M,\epsilon})$ as $\epsilon \to 0$ is called the adiabatic limit of T. The adiabatic limits of various geometric objects have been studied by many authors. In this note, we study a variant of this problem. (Although we will not discuss here, the work of Berthomieu-Bismut ([B-B]) seems to be very related to our subject.)

Let $\pi\colon X \to \mathbb{P}^1$ be an elliptic $K3$ surface. Let $\iota\colon X \to X$ be a holomorphic involution acting non-trivially on canonical forms on X. Let κ_X and $\kappa_{\mathbb{P}^1}$ be Kähler classes on X and $\mathbb{P}^1$, respectively. By Yau ([Ya]), the Kähler class $\kappa_{X,\epsilon} := \kappa_X + \epsilon^{-1}\pi^*\kappa_{\mathbb{P}^1}$ carries uniquely a Ricci-flat Kähler form ω_ϵ. We study the equivariant analytic torsion ([Bi]) of $(X, \iota, \omega_\epsilon)$ as $\epsilon \to 0$ in the case where (X, ι) is a class of $K3$ surfaces studied by Nikulin ([N]). As a result, we recover the Borcherds Φ-function of dimension 26 restricted to a certain locus of dimension 10.

Although we talked a little about the adiabatic limit of the invariant introduced in [Yo] at the conference, we will focus on that subject in this short note.

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§1. Nikulin's $K3$ surfaces

Let X be a $K3$ surface with canonical bundle K_X. Let $\eta_X \in H^0(X, K_X)$ be a nowhere vanishing holomorphic 2-form on X. Then $H^2(X, \mathbb{Z})$ equipped with the intersection pairing is isometric to the $K3$-lattice

$$\mathbb{L}_{K3} := U \oplus U \oplus U \oplus E_8 \oplus E_8, \tag{1.1}$$

where $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and E_8 is the negative definite lattice associated with the Cartin matrix of type E_8. An isometry $\phi\colon H^2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$ is called a marking of X, and the pair (X, ϕ) is called a marked $K3$ surface.

Set

$$\Omega := \{[x] \in \mathbb{P}(\mathbb{L}_{K3} \otimes \mathbb{C});\ \langle x, x\rangle = 0,\ \langle x, \bar{x}\rangle > 0\}. \tag{1.2}$$

For a marked $K3$ surface (X, ϕ), the point $[\phi(\eta_X)] \in \mathbb{P}(\mathbb{L}_{K3} \otimes \mathbb{C})$ is called the period of (X, ϕ). Then one can verify that $[\phi(\eta_X)] \in \Omega$.

Definition 1.1. Let $\iota\colon X \to X$ be a holomorphic involution acting non-trivially on $H^0(X, K_X)$, i.e., $\iota^*\eta_X = -\eta_X$. The pair (X, ι) is called a *Nikulin's $K3$ surface* if the ι^*-invariant part of $H^2(X, \mathbb{Z})$ is isometric to the lattice $\Lambda := U \oplus E_8(2)$. Here $E_8(2)$ denotes the lattice of rank 8 whose intersection form is twice of that on E_8.

Nikulin's $K3$ surfaces are constructed as follows:

Let $C_1, C_2 \subset \mathbb{P}^2$ be two smooth cubic curves in general position. Then C_1 meets C_2 transversally at 9 points; $C_1 \cap C_2 = \{p_1, p_2, \cdots, p_9\}$. Let $\mathbb{P}^2[9] \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ at these 9 points. Then $\mathbb{P}^2[9]$ is the blowing-up of the base points of the pencil spanned by C_1, C_2.

Fix homogeneous polynomials $f_1(z)$, $f_2(z)$ defining C_1, C_2, respectively. Then $\mathbb{P}^2[9]$ admits the elliptic fibration $\pi\colon \mathbb{P}^2[9] \to \mathbb{P}^1$ with fiber $\pi^{-1}(s:t) = \{[z] \in \mathbb{P}^2;\ sf_1(z) + tf_2(z) = 0\}$. Hence, $\mathbb{P}^2[9]$ is a rational elliptic surface.

Let $\widetilde{C}_1, \widetilde{C}_2 \subset \mathbb{P}^2[9]$ be the proper transform of C_1, C_2, respectively. Then the divisor $\widetilde{C}_1 + \widetilde{C}_2$ is the member of the double anti-canonical system $|-2K_{\mathbb{P}^2[9]}|$. Let $X_{C_1+C_2}$ be the double covering of $\mathbb{P}^2[9]$ with branch divisor $\widetilde{C}_1 + \widetilde{C}_2$. Let $\iota_{C_1+C_2}\colon X_{C_1+C_2} \to X_{C_1+C_2}$ be the non-trivial covering transformation. By the canonical bundle formula, $X_{C_1+C_2}$ is a $K3$ surface. By the rationality of $\mathbb{P}^2[9]$, $\iota_{C_1+C_2}$ acts non-trivially on $H^0(X_{C_1+C_2}, K_{X_{C_1+C_2}})$. Since the fixed point set of $\iota_{C_1+C_2}$ is identified with $C_1 + C_2$, it follows from Nikulin's classification of the fixed point set ([N, Th. 4.2.2]) that $(X_{C_1+C_2}, \iota_{C_1+C_2})$ is a Nikulin's $K3$ surface.

Let $\pi_{C_1+C_2}\colon X_{C_1+C_2} \to \mathbb{P}^1$ be the elliptic fibration associated to the linear system $|\widetilde{C}_1|$. Since the image of every member of $|\widetilde{C}_1|$ by $\iota_{C_1+C_2}$ is again a member of $|\widetilde{C}_1|$, there exists an involution $i_{\mathbb{P}^1}$ on $\mathbb{P}^1$ such that

$$\begin{array}{ccc} X_{C_1+C_2} & \xrightarrow{p} & \mathbb{P}^2[9] \\ {\scriptstyle \pi_{C_1+C_2}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 & \xrightarrow[q]{} & \mathbb{P}^1 \end{array} \tag{1.3}$$

is a commutative diagram, where $p\colon X_{C_1+C_2} \to \mathbb{P}^2[9] = X_{C_1+C_2}/\iota_{C_1+C_2}$ and $q\colon \mathbb{P}^1 \to \mathbb{P}^1 = \mathbb{P}^1/i_{\mathbb{P}^1}$ are the natural projections.

§2. The moduli space of Nikulin's $K3$ surfaces

Define an involution I_Λ on $\mathbb{L}_{K3}$ by

$$I_\Lambda(a,b,c,x,y) = (a,-b,-c,y,x) \qquad (a,b,c \in U,\ x,y \in E_8). \tag{2.1}$$

Then Λ is the invariant part of I_Λ. Let L be the anti-invariant part of I_Λ. Then L is the orthogonal complement of Λ in $\mathbb{L}_{K3}$, and

$$L = U \oplus U \oplus E_8(2). \tag{2.2}$$

Let (X,ι) be a Nikulin's $K3$ surface. Since the embedding $\Lambda \hookrightarrow \mathbb{L}_{K3}$ is unique up to an automorphism of $\mathbb{L}_{K3}$, there exists a marking ϕ of X such that $\phi \circ \iota^* \circ \phi^{-1} = I_\Lambda$. A marking with this property is called a marking of a Nikulin's $K3$ surface. By Definition 1.1, the period of a marked Nikulin's $K3$ surface lies in the following subset of Ω:

$$\Omega_\Lambda := \{[x] \in \mathbb{P}(L \otimes \mathbb{C});\ \langle x,x\rangle = 0,\ \langle x,\bar{x}\rangle > 0\}. \tag{2.3}$$

Then Ω_Λ consists of two connected components $\Omega_\Lambda^\pm$, each of which is isomorphic to a symmetric bounded domain of type IV of dimension 10. However, the period mapping omits the divisor $\mathcal{D}_\Lambda$ of Ω_Λ described as follows: For $l \in L$ with $l^2 := \langle l,l\rangle < 0$, set $H_l := \{[x] \in \Omega_\Lambda;\ \langle x,l\rangle = 0\}$. Let $\mathcal{D}_\Lambda$ be the discriminant locus of Ω_Λ:

$$\mathcal{D}_\Lambda := \bigcup_{d \in L,\, d^2=-2} H_d. \tag{2.4}$$

Let $O(L)$ be the isometry group of the lattice L. Then $O(L)$ acts naturally on Ω_Λ and preserves $\mathcal{D}_\Lambda$. In [Yo, Th. 1.8], we proved:

Theorem 2.1. *The coarse moduli space of Nikulin's $K3$ surfaces is isomorphic to the analytic space $\mathcal{M}_\Lambda^0 := (\Omega_\Lambda \setminus \mathcal{D}_\Lambda)/O(L)$ via the period mapping.*

§3. The restriction of the Borcherds Φ-function to Ω_Λ

In [Bo], Borcherds introduced a remarkable automorphic form on the 26-dimensional symmetric bounded domain of type IV associated with the even unimodular lattice $II_{2,26} := U \oplus U \oplus E_8 \oplus E_8 \oplus E_8$. His automorphic form is called the *Borcherds Φ-function* and is denoted by Φ. We refer to [Bo, Th. 10.1 and §10 Example 2] for more details about the Borcherds Φ-function.

Since $L \subset II_{2,26}$, one can restrict the Borcherds Φ-function to Ω_Λ. This automorphic form on Ω_Λ is denoted by Φ_Λ:

$$\Phi_\Lambda := \Phi|_{\Omega_\Lambda}. \tag{3.1}$$

Then we proved in [Yo, Lemma 8.5] that Φ_Λ is an automorphic form on Ω_Λ of weight 12 with zero divisor $\mathcal{D}_\Lambda$.

Fix a vector $\ell \in L \otimes \mathbb{R}$ such that $\ell^2 \geq 0$. The pointwise length of Φ_Λ is defined by

$$\|\Phi_\Lambda\|^2([z]) := \left(\frac{\langle z, \bar{z}\rangle_L}{|\langle z, \ell\rangle_L|^2}\right)^{12} |\Phi_\Lambda([z])|^2 \qquad ([z] \in \Omega_\Lambda). \tag{3.2}$$

Then $\|\Phi_\Lambda\|^2$ is an $O(L)$-invariant C^∞-function on Ω_Λ and is regarded as a function on $\mathcal{M}^0_\Lambda$.

§4. Equivariant analytic torsion of Nikulin's $K3$ surfaces

In [Bi], Bismut established the foundations of the theory of equivariant analytic torsion and equivariant Quillen metrics. Here, we recall his construction in the simplest case. We refer to [Bi] for more details about equivariant analytic torsion and equivariant Quillen metrics.

Let Y be a compact Kähler manifold. Let $\theta\colon Y \to Y$ be a holomorphic involution. Let $\mathbb{Z}_2 \subset \mathrm{Aut}(Y)$ be the subgroup generated by θ. Let γ_Y be a $\mathbb{Z}_2$-invariant Kähler metric on Y. Let $\square_q$ be the $\bar{\partial}$-Laplacian acting on $(0,q)$-forms on Y with respect to γ_Y. Let $\sigma(\square_q)$ be the spectrum of $\square_q$. For $\lambda \in \sigma(\square_q)$, let $E_q(\lambda)$ be the vector space of eigenforms of $\square_q$ with eigenvalue λ. Then $\mathbb{Z}_2$ preserves $E_q(\lambda)$.

For $g \in \mathbb{Z}_2$ and $s \in \mathbb{C}$, set $\zeta_q(g)(s) := \sum_{\lambda \in \sigma(\square_q)\setminus\{0\}} \mathrm{Tr}(g|_{E_q(\lambda)})\lambda^{-s}$. Classically, $\zeta_q(g)(s)$ converges absolutely when $\mathrm{Re}\, s > \dim Y$, admits a meromorphic continuation to $\mathbb{C}$, and is holomorphic at $s = 0$.

Definition 4.1. For $g \in \mathbb{Z}_2$, the *equivariant analytic torsion* of (Y, γ_Y) is defined by

$$\log \tau_{\mathbb{Z}_2}(Y, \gamma_Y)(g) := \sum_{q \geq 0} (-1)^{q+1} \zeta'_q(g)(0). \tag{4.1}$$

When $g = 1$, $\tau_{\mathbb{Z}_2}(Y, \gamma_Y)(1)$ coincides with the Ray-Singer analytic torsion of (Y, γ_Y) and is denoted by $\tau(Y, \gamma_Y)$.

§5. The adiabatic limit of $\tau_{\mathbb{Z}_2}$ for Nikulin's $K3$ surfaces

Let (X, ι) be a Nikulin's $K3$ surface. Let $C_1 + C_2$ be the set of fixed points of ι. Then C_1 and C_2 are mutually disjoint elliptic curves. Let $[(X, \iota)] \in \mathcal{M}^0_\Lambda$ be the $O(L)$-orbit of the period of (X, ι). By the $O(L)$-invariance of $\|\Phi_\Lambda\|$, the value $\|\Phi_\Lambda([X, \iota])\|$ makes sense.

Let $\pi\colon X \to \mathbb{P}^1$ be the elliptic fibration associated with the free linear system $|C_1|$. Then the image of an arbitrary fiber of π by ι is again a fiber of π, and ι induces an involution $i_{\mathbb{P}^1}$ on $\mathbb{P}^1$ verifying (1.3).

Let κ_X be an ι-invariant Kähler class on X. Let $\kappa_{\mathbb{P}^1}$ be a Kähler class on $\mathbb{P}^1$. For $0 < \epsilon < +\infty$, set

$$\kappa_\epsilon := \kappa_X + \epsilon^{-1}\pi^*\kappa_{\mathbb{P}^1}. \tag{5.1}$$

Then $\{\kappa_\epsilon\}_{0<\epsilon<+\infty}$ is a family of ι-invariant Kähler classes on X. Notice that the Kähler class on the fiber induced from κ_ϵ is independent of ϵ. By Calabi-Yau ([Ya]), there exists uniquely an ι-invariant Ricci-flat Kähler form ω_ϵ in κ_ϵ:

$$\mathrm{Ric}(\omega_\epsilon) \equiv 0, \quad \iota^*\omega_\epsilon = \omega_\epsilon, \quad [\omega_\epsilon] = \kappa_\epsilon \qquad (0 < \epsilon < +\infty). \tag{5.2}$$

Let $\mathrm{Vol}(X, \omega_\epsilon) := \int_X \omega_\epsilon^2/2!$ be the volume of (X, ω_ϵ). Let $F \in H_2(X, \mathbb{Z})$ be the class of fibers of $\pi\colon X \to \mathbb{P}^1$. Set $\mathrm{Vol}(F, \kappa|_F) := \int_F \kappa|_F$ and $\mathrm{Vol}(\mathbb{P}^1, \kappa_{\mathbb{P}^1}) := \int_{\mathbb{P}^1} \kappa_{\mathbb{P}^1}$. By (5.1) and the projection formula, we get

$$\mathrm{Vol}(X, \omega_\epsilon) = \mathrm{Vol}(X, \kappa) + \epsilon^{-1}\mathrm{Vol}(F, \kappa|_F)\mathrm{Vol}(\mathbb{P}^1, \kappa_{\mathbb{P}^1}). \tag{5.3}$$

The following is the main result of this note:

Theorem 5.1. *There exists a constant $C \neq 0$ depending only on the lattice Λ such that*

$$\lim_{\epsilon \to 0} \tau_{\mathbb{Z}_2}(X, \omega_\epsilon)(\iota) \cdot \mathrm{Vol}(X, \omega_\epsilon) = C\,\|\Phi_\Lambda([(X, \iota)])\|^{-\frac{1}{6}}. \tag{5.4}$$

Proof. For $\tau \in \mathbb{H}$, let $\Delta(\tau) = e^{2\pi i\tau}\prod_{n>0}(1 - e^{2\pi i n\tau})^{24}$ be the Jacobi-Δ function. Set $\|\Delta(\tau)\|^2 := (\mathrm{Im}\,\tau)^{12}|\Delta(\tau)|^2$, which is a $SL_2(\mathbb{Z})$-invariant function on $\mathbb{H}$. Let $[C_i] \in \mathbb{H}/SL_2(\mathbb{Z})$ be the period of the elliptic curve C_i. By the $SL_2(\mathbb{Z})$-invariance of $\|\Delta(\tau)\|$, the value $\|\Delta([C_i])\|$ is independent of the choice of a representative of $[C_i]$ in $\mathbb{H}$.

By [Yo, Th. 5.2 and Th. 8.7], there exists a constant $C_\Lambda \neq 0$ depending only on the lattice Λ such that

$$\begin{aligned}
(5.5) \qquad & \tau_{\mathbb{Z}_2}(X,\omega_\epsilon)(\iota)\cdot \mathrm{Vol}(X,\omega_\epsilon)\prod_{i=1}^{2}\tau(C_i,\omega_\epsilon|_{C_i})\cdot \mathrm{Vol}(F,\kappa_\epsilon|_F) \\
&= C_\Lambda\,\|\Phi_\Lambda([(X,\iota)])\|^{-\frac{1}{6}}\cdot\prod_{i=1}^{2}\|\Delta([C_i])\|^{-\frac{1}{6}}.
\end{aligned}$$

By [G-W, Th. 5.6], the family of Kähler forms $\{\omega_\epsilon|_{C_i}\}_{0<\epsilon<1}$ converges in arbitrary C^k-topology to the *flat* Kähler form ω_{C_i} on C_i with Kähler class $\kappa|_{C_i}$. Hence, we deduce from the anomaly formula for Quillen metrics that

$$(5.6) \qquad \lim_{\epsilon\to 0}\tau(C_i,\omega_\epsilon|_{C_i}) = \tau(C_i,\omega_{C_i}),\quad \mathrm{Vol}(F,\kappa_\epsilon|_F) = \mathrm{Vol}(C_i,\omega_{C_i}).$$

Since ω_{C_i} is flat, Kronecker's limit formula yields that

$$(5.7) \qquad \tau(C_i,\omega_{C_i})\cdot\mathrm{Vol}(C_i,\omega_{C_i}) = \|2^{12}\Delta([C_i])\|^{-\frac{1}{6}}.$$

The result follows from (5.5), (5.6), (5.7). Q.E.D.

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Nikulin's $K3$ surface and adiabatic limit

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