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Invariants of combinatorial line arrangements and Rybnikov's example

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Abstract.

Following the general strategy proposed by G.Rybnikov, we present a proof of his well-known result, that is, the existence of two arrangements of lines having the same combinatorial type, but non-isomorphic fundamental groups. To do so, the Alexander Invariant and certain invariants of combinatorial line arrangements are presented and developed for combinatorics with only double and triple points. This is part of a more general project to better understand the relationship between topology and combinatorics of line arrangements.

One of the main subjects in the theory of hyperplane arrangements is the relationship between combinatorics and topological properties. To be precise, one has to make the following distinction: for a given hyperplane arrangement $\mathcal{H} \subset \mathbb{P}^n$, one can study the topological type of the pair $(\mathbb{P}^n, \mathcal{H})$ or the topological type of the complement $\mathbb{P}^n \setminus \mathcal{H}$. For the first concept we will use the term *relative topology* of $\mathcal{H}$, whereas for the second one we will simply say *topology* of $\mathcal{H}$. It is clear that if two hyperplane arrangements have the same relative topology, then they have the same topology, but the converse is not known. For $n = 2$, topology, relative topology and combinatorics are also related via graph manifolds with the boundary of a compact regular neighbourhood of $\mathcal{H}$, see [14, 24].

In a well-known and very cited unpublished paper [26], G. Rybnikov found an example of two line arrangements L_1 and L_2 in the complex

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projective plane $\mathbb{P}^2$ having the same combinatorics but different topology. A better understanding of this paper has been the aim of several works since then ([7, 20, 8, 22]).

The most common way to prove that two topologies of line arrangements are different is to check that the fundamental groups of their complements are not isomorphic. This is usually not done directly, but by calculating invariants of the fundamental group, mostly *borrowed* from invariants of links, such as Alexander polynomials ([25] and references there for links, [16] for algebraic curves), character (or characteristic) varieties ([13] for links, [17] for algebraic curves), Alexander invariants and Chen groups ([12, 19, 28] for links, [15, 11, 6, 20, 8, 23] for line arrangements) just to mention a few (both invariants and publications). In [26], Rybnikov uses central extensions of Chen groups in order to study the relative topology of line arrangements (and the fundamental groups of their complements); in this work, we use truncations of the Alexander Invariant by the $\mathfrak{m}$-adic filtration, where $\mathfrak{m}$ is the augmentation ideal; such truncations were studied by L.Traldi in [27] for links.

Recently, the authors of this work have provided an example of two line arrangements with different relative topologies (see [2]). The contribution of [2] is that it refers to real arrangements, that is, arrangements that admit real equations for each line (note that Rybnikov's example does not admit real equations).

The proof proposed by Rybnikov has two steps. Let $G_i := \pi_1(\mathbb{P}^2 \setminus \bigcup L_i)$, $i = 1, 2$.

(R1) Recall that the homology of the complement of a hyperplane arrangement depends only on combinatorics. This way, one can identify the abelianization of G_1 and G_2 with an Abelian group H combinatorially determined. Rybnikov proves that no isomorphisms exist between G_1 and G_2 that induce the identity on H. In particular, this result proves that both arrangements have different relative topologies. The reason can be outlined as follows: any automorphism of the combinatorics of Rybnikov's arrangement can be obtained from a diffeomorphism of $\mathbb{P}^2$, thus inducing an automorphism of fundamental groups. Since any homeomorphism of pairs $(\mathbb{P}^2, \bigcup L_i)$ defines an automorphism of the combinatorics of $\bigcup L_i$, after composition one can assume that any homeomorphism of pairs induces the identity on H. The strategy rests on the study of the first terms of the Lower Central Series (LCS), which coincide with the first terms of the series producing Chen groups. Since L_1 and L_2 are constructed using the MacLane arrangement L_ω

(see Example 1.7), it is enough to study, by some combinatorial arguments, the LCS of L_ω with an extra structure (referred to as an ordered arrangement). Although this part is explained in [26, Section 3], computations are hard to verify.

(R2) The second step is essentially combinatorial. The main point is to truncate the LCS of G_i such that the quotient K depends only on the combinatorics. Rybnikov proposes to prove that an automorphism of K induces the identity on H (up to sign and automorphisms of the combinatorics). This proof is only outlined in [26, Proposition 4.2]. It is worth pointing out that such a result cannot be expected for any arrangement. Also [26, Proposition 4.3] needs some explanation of its own. The main difference between relative topology and topology of the complement in terms of isomorphisms of the fundamental group is that homeomorphisms of pairs induce isomorphisms that send meridians to meridians, whereas homeomorphisms of the complement can induce any kind of isomorphism, and even if we know that the isomorphism induces the identity on homology, this is not enough to claim that meridians are sent to meridians.

The aim of our work is to follow the idea behind Rybnikov's work and, using slightly different techniques, provide detailed proofs of his result. This is part of a more general project by the authors that aims to better understand the relationship between topology and combinatorics of line arrangements.

The following is a more detailed description of the layout of this paper. In Section 1, the more relevant definitions are set, as well as a description of Rybnikov's and MacLane's combinatorics. Sections 2 and 3 provide a proof of Step (R1). In order to do so, we propose a new approach related to Derived Series, which is also useful in the study of Characteristic Varieties and the Alexander Invariant. The Alexander Invariant of a group G, with a fixed isomorphism $G/G' \approx \mathbb{Z}^r$, is the quotient G'/G'' considered as a module over the ring $\Lambda := \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$, which is the group algebra of $\mathbb{Z}^r$. Using the truncated modules $\Lambda/\mathfrak{m}^j$, the problem is reduced to solving a system of linear equations. Note that other ideal could be used instead of $\mathfrak{m}$. Section 4 is devoted to the study of combinatorial properties of a line arrangement which ensure that any automorphism of the fundamental group of the complement *essentially* induces the identity on homology (that is, the analogous of [26, Proposition 4.2]). This is an interesting question that can be applied to general line arrangements. For the sake of simplicity, we only present our progress on line arrangements with double and triple points. This provides a proof for the second step (R2).

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§1. Settings and Definitions

In this section, some standard facts about line combinatorics and ordered line combinatorics will be described. Special attention will be given to MacLane and Rybnikov's line combinatorics.

Definition 1.1. A *combinatorial type* (or simply a *(line) combinatorics*) is a couple $\mathscr{C} := (\mathcal{L}, \mathcal{P})$, where $\mathcal{L}$ is a finite set and $\mathcal{P} \subset \mathcal{P}(\mathcal{L})$, satisfying that:

(1) For all $P \in \mathcal{P}$, $\#\mathcal{P} \geq 2$;
(2) For any $\ell_1, \ell_2 \in \mathcal{L}$, $\ell_1 \neq \ell_2$, $\exists! P \in \mathcal{P}$ such that $\ell_1, \ell_2 \in P$.

An *ordered combinatorial type* $\mathscr{C}^{\text{ord}}$ is a combinatorial type where $\mathcal{L}$ is an ordered set.

Notation 1.2. Given a combinatorial type $\mathscr{C}$, the multiplicity m_P of $P \in \mathcal{P}$ is the number of elements $L \in \mathcal{L}$ such that $P \in L$; note that $m_P \geq 2$. The *multiplicity of* a combinatorial type is the number $1 - \#\mathcal{L} + \sum_{P \in \mathcal{P}} (m_P - 1)$.

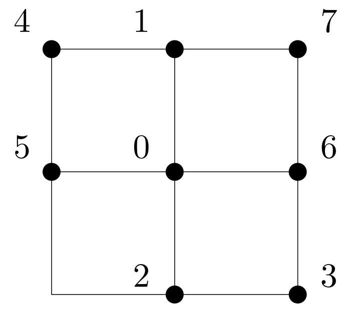

Fig. 1. Ordered MacLane *lines* in $\mathbb{F}_3^2$

Example 1.3 (MacLane's combinatorics)**.** Let us consider the 2-dimensional vector space on the field $\mathbb{F}_3$ of three elements. Such a plane contains 9 points and 12 lines, 4 of which pass through the origin. Consider $\mathcal{L} = \mathbb{F}_3^2 \setminus \{(0,0)\}$ and $\mathcal{P}$, the set of lines in $\mathbb{F}_3^2$ (as a subset of

$\mathcal{P}(\mathcal{L})$). This provides a combinatorial type structure $\mathscr{C}_{\mathrm{ML}}$ that we will refer to as *MacLane's combinatorial type.* Figure 1 represents an ordered MacLane's combinatorial type.

Definition 1.4. Let $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ be a combinatorial type. We say a complex line arrangement $\mathcal{H} := \ell_0 \cup \ell_1 \cup ... \cup \ell_r \subset \mathbb{P}^2$ is a *realization* of $\mathscr{C}$ if and only if there are bijections $\psi_1 : \mathcal{L} \to \{\ell_0, \ell_1, ..., \ell_r\}$ and $\psi_2 : \mathcal{P} \to \mathrm{Sing}(\mathcal{H})$ such that $\forall \ell \in \mathcal{H}, P \in \mathcal{P}$, one has $P \in \ell \Leftrightarrow \psi_1(\ell) \in \psi_2(P)$. If $\mathscr{C}^{\mathrm{ord}}$ is an ordered combinatorial type and the irreducible components of $\mathcal{H}$ are also ordered, we say $\mathcal{H}$ is an ordered realization if ψ_1 respects orders.

Notation 1.5. The space of all complex realizations of a line combinatorics $\mathscr{C}$ is denoted by $\Sigma(\mathscr{C})$. This is a quasiprojective subvariety of $\mathbb{P}^{\frac{r(r+3)}{2}}$, where $r := \#\mathscr{C}$. If $\mathscr{C}^{\mathrm{ord}}$ is ordered, we denote by $\Sigma^{\mathrm{ord}}(\mathscr{C}) \subset (\check{\mathbb{P}}^2)^r$ the space of all ordered complex realizations of $\mathscr{C}^{\mathrm{ord}}$.

There is a natural action of $\mathrm{PGL}(3; \mathbb{C})$ on such spaces. This justifies the following definition.

Definition 1.6. The *moduli space* of a combinatorics $\mathscr{C}$ is the quotient $\mathscr{M}(\mathscr{C}) := \Sigma(\mathscr{C}) / \mathrm{PGL}(3; \mathbb{C})$. The *ordered moduli space* $\mathscr{M}^{\mathrm{ord}}(\mathscr{C})$ of an ordered combinatorics $\mathscr{C}^{\mathrm{ord}}$ is defined accordingly.

Example 1.7. Let us consider the MacLane line combinatorics $\mathscr{C}_{\mathrm{ML}}$. It is well known that such combinatorics has no real realization and that $\#\mathscr{M}(\mathscr{C}_{\mathrm{ML}}) = 1$, however $\#\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}}) = 2$. The following are representatives for $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}})$:

$$\begin{gathered} \ell_0 = \{x = 0\} \quad \ell_1 = \{y = 0\} \quad \ell_2 = \{x = y\} \quad \ell_3 = \{z = 0\} \\ \ell_4 = \{x = z\} \quad \ell_5^{\pm} = \{z + \omega y = 0\} \quad \ell_6^{\pm} = \{z + \omega y = (\omega + 1)x\} \\ \ell_7^{\pm} = \{(\omega + 1)y + z = x\} \end{gathered} \tag{1}$$

where $\omega = e^{2\pi i/3}$.

We will refer to such ordered realizations as

$$L_\omega := \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5^+, \ell_6^+, \ell_7^+\}$$

and

$$L_{\bar{\omega}} := \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5^-, \ell_6^-, \ell_7^-\}.$$

Remark 1.8. Given a line combinatorics $\mathscr{C} = (\mathcal{L}, \mathcal{P})$, the automorphism group $\mathrm{Aut}(\mathscr{C})$ is the subgroup of the permutation group of $\mathcal{L}$ preserving $\mathcal{P}$. Let us consider an ordered line combinatorics $\mathscr{C}^{\mathrm{ord}}$. It is easily seen that $\mathrm{Aut}(\mathscr{C}^{\mathrm{ord}})$ acts on both $\Sigma^{\mathrm{ord}}(\mathscr{C}^{\mathrm{ord}})$ and $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}^{\mathrm{ord}})$. Note also that $\mathscr{M}(\mathscr{C}^{\mathrm{ord}}) \cong \mathscr{M}^{\mathrm{ord}}(\mathscr{C}^{\mathrm{ord}}) / \mathrm{Aut}(\mathscr{C}^{\mathrm{ord}})$.

Example 1.9. The action of $\mathrm{Aut}(\mathscr{C}_{\mathrm{ML}}) \cong \mathrm{PGL}(2, \mathbb{F}_3)$ on the moduli spaces is as follows: matrices of determinant $+1$ (resp. -1) fix (resp. exchange) the two elements of $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}})$. Of course complex conjugation also acts on $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}})$ exchanging the two elements. From the topological point of view one has that:

- There exists a homeomorphism $(\mathbb{P}^2, \bigcup L_\omega) \to (\mathbb{P}^2, \bigcup L_{\bar\omega})$ preserving orientations on both $\mathbb{P}^2$ and the lines. Such a homeomorphism does not respect the ordering.
- There exists a homeomorphism $(\mathbb{P}^2, \bigcup L_\omega) \to (\mathbb{P}^2, \bigcup L_{\bar\omega})$ preserving orientations on $\mathbb{P}^2$, but not on the lines. Such a homeomorphism respects the ordering.

Also note that the subgroup of automorphisms that preserve the set $L_0 := \{\ell_0, \ell_1, \ell_2\}$ is isomorphic to Σ_3, since the vectors $(1,0)$, $(1,1)$ and $(1,2)$ generate $\mathbb{F}_3^2$. We will denote by L_+ and L_- the sets of 5 lines such that $L_\omega = L_0 \cup L_+$ and $L_{\bar\omega} = L_0 \cup L_-$. Since any transposition of $\{0,1,2\}$ in $\mathscr{C}_{\mathrm{ML}}$ produces a determinant -1 matrix in $\mathrm{PGL}(2, \mathbb{F}_3)$, one concludes from the previous paragraph that any transposition of $\{0,1,2\}$ induces a homeomorphism $(\mathbb{P}^2, \bigcup L_\omega) \to (\mathbb{P}^2, \bigcup L_{\bar\omega})$ that exchanges L_ω and $L_{\bar\omega}$ as representatives of elements of $\mathscr{M}^{\mathrm{ord}}(\mathscr{C}_{\mathrm{ML}})$ and globally fixes L_0.

Example 1.10 (Rybnikov's combinatorics)**.** Let L_ω and $L_{\bar\omega}$ be ordered MacLane realizations as above, where $L_0 := \{\ell_0, \ell_1, \ell_2\}$. Let us consider a projective transformation ρ_ω (resp. $\rho_{\bar\omega}$) fixing the initial ordered set L_0 (that is, $\rho(\ell_i) = \ell_i$ $i = 0,1,2$) and such that $\rho_\omega L_\omega$ (resp. $\rho_{\bar\omega} L_{\bar\omega}$) and L_ω intersect each other only in double points outside the three common lines. Note that $\rho_\omega, \rho_{\bar\omega}$ can be chosen with real coefficients.

Let us consider the following ordered arrangements of thirteen lines: $R_{\alpha,\beta} = L_\alpha \cup \rho_\gamma L_\beta$, where $\alpha, \beta \in \{\omega, \bar\omega\}$ and $\gamma = \beta$ (resp $\bar\beta$) if $\alpha = \omega$ (resp. $\bar\omega$). They produce the following combinatorics $\mathscr{C}_{\mathrm{RYB}} := (\mathcal{R}, \mathcal{P})$ given by:

(2)
$$\mathcal{R} := \{\ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8, \ell_9, \ell_{10}, \ell_{11}, \ell_{12}\}$$
$$\mathcal{P}_2 := \left\{ \begin{array}{llll} \{\ell_2, \ell_3\}, & \{\ell_0, \ell_7\}, & \{\ell_1, \ell_6\}, & \{\ell_4, \ell_5\}, \\ \{\ell_2, \ell_8\}, & \{\ell_0, \ell_{12}\}, & \{\ell_1, \ell_{11}\}, & \{\ell_9, \ell_{10}\}, \\ \{\ell_i, \ell_j\} & 3 \le i \le 7, & 8 \le j \le 12 & \end{array} \right\}$$
$$\mathcal{P}_3 := \left\{ \begin{array}{llll} \{\ell_0, \ell_1, \ell_2\}, & \{\ell_3, \ell_6, \ell_7\}, & \{\ell_0, \ell_5, \ell_6\}, & \{\ell_1, \ell_4, \ell_7\}, \\ \{\ell_1, \ell_3, \ell_5\}, & \{\ell_2, \ell_4, \ell_6\}, & \{\ell_2, \ell_5, \ell_7\}, & \{\ell_0, \ell_3, \ell_4\}, \\ \{\ell_8, \ell_{11}, \ell_{12}\} & \{\ell_0, \ell_{10}, \ell_{11}\}, & \{\ell_1, \ell_9, \ell_{12}\}, & \{\ell_1, \ell_8, \ell_{10}\}, \\ \{\ell_2, \ell_9, \ell_{11}\}, & \{\ell_2, \ell_{10}, \ell_{12}\}, & \{\ell_0, \ell_8, \ell_9\} & \end{array} \right\}$$
$$\mathcal{P} := \mathcal{P}_2 \cup \mathcal{P}_3$$

Proposition 1.11. *The following combinatorial properties hold:*

(1) *The different arrangements $R_{\alpha,\beta}$ have the same combinatorial type $\mathscr{C}_{\mathrm{RYB}}$.*

(2) *The set of lines L_0 has the following distinctive combinatorial property: every line in L_0 contains exactly 5 triple points of the arrangement; the remaining lines only contain 3 triple points.*

(3) *For the other* 10 *lines we consider the equivalence relation generated by the relation of* sharing a triple point*. There are two equivalence classes which correspond to L_ε and $\rho L_{\varepsilon'}$, $\varepsilon, \varepsilon' = \pm$.*

By the previous remarks one can group the set $\mathcal{R}$ together in three subsets. One is associated with the set of lines L_0 (referred to as $\mathcal{R}_0$), and the other two are combinatorially indistinguishable sets ($\mathcal{R}_1$ and $\mathcal{R}_2$) such that $\mathcal{R}_0 \cup \mathcal{R}_1$ and $\mathcal{R}_0 \cup \mathcal{R}_2$ are MacLane's combinatorial types. Note that any automorphism of $\mathscr{C}_{\mathrm{RYB}}$ must preserve $\mathcal{R}_0$ and either preserve or exchange $\mathcal{R}_1$ and $\mathcal{R}_2$. Therefore, $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}}) \cong \Sigma_3 \times \mathbb{Z}/2\mathbb{Z}$. The following results are immediate consequences of the aforementioned remarks.

Proposition 1.12. *The following are (or induce) homeomorphisms between the pairs $(\mathbb{P}^2, \bigcup R_{\bar\omega,\bar\omega})$ and $(\mathbb{P}^2, \bigcup R_{\omega,\omega})$ (resp. $(\mathbb{P}^2, \bigcup R_{\omega,\bar\omega})$ and $(\mathbb{P}^2, \bigcup R_{\bar\omega,\omega})$) preserving the orientation of $\mathbb{P}^2$:*

(a) *Complex conjugation, which reverses orientations of the lines.*

(b) *A transposition in $\mathcal{R}_0$, which preserves orientations of the lines.*

We will refer to $R_{\omega,\omega}$ and $R_{\bar\omega,\bar\omega}$ (resp. $R_{\omega,\bar\omega}$ and $R_{\bar\omega,\omega}$) as a type $+$ (resp. type $-$) arrangements.

Proposition 1.13. *Any homeomorphism of pairs between a type $+$ and a type $-$ arrangement should lead (maybe after composing with complex conjugation) to an orientation-preserving homeomorphism of pairs between a type $+$ and a type $-$ arrangement.*

If such a homeomorphism existed, there should be an orientation-preserving homeomorphism of ordered MacLane arrangements of type L_ω and $L_{\bar\omega}$.

The purpose of the next section will be to prove that there is no orientation-preserving homeomorphism of ordered MacLane arrangements of type L_ω and $L_{\bar\omega}$.

§2. The truncated Alexander Invariant

Even though the Alexander Invariant can be developed for general projective plane curves, we will concentrate on the case of line arrangements. Let $\bigcup L \subset \mathbb{P}^2$ be a projective line arrangement where $L = \{\ell_0, \ell_1, ..., \ell_r\}$. Let us denote its complement $X := \mathbb{P}^2 \setminus \bigcup L$ and G its fundamental group. The derived series associated with this group is recursively defined as follows: $G^{(0)} := G$, $G^{(n)} := (G^{(n-1)})' = [G^{(n-1)}, G^{(n-1)}]$, $n \geq 1$, where G' is the derived subgroup of G, i.e. the subgroup generated by $[a,b] := aba^{-1}b^{-1}$, $a, b \in G$. Note that the consecutive quotients are Abelian. This property also holds for the lower central series defined as $\gamma_1(G) := G$, $\gamma_n(G) := [\gamma_{n-1}(G), G]$, $n \geq 1$. It is clear that $G^{(0)} = \gamma_1(G)$ and $G^{(1)} = \gamma_2(G)$.

Since $H_1(X) = G/G'$, one can consider the inclusion $G' \hookrightarrow G$ as representing the universal Abelian cover $\tilde{X}$ of X, where $\pi_1(\tilde{X}) = G'$, and therefore $H_1(\tilde{X}) = G'/G''$.

The group of transformations $H_1(X) = G/G' = \mathbb{Z}^r$ of the cover acts on G'. This results in an action by conjugation on $G'/G'' = H_1(\tilde{X})$, $G'' = G^{(2)}$:

$$\begin{array}{ccll} G/G' \times G'/G'' & \to & G'/G'' & \\ (g, [a,b]) & \mapsto & g * [a,b] \mod G'' & = [g,[a,b]] + [a,b], \end{array}$$

where $a * b := aba^{-1}$. This action is well defined since $g \in G'$ implies $g*[a,b] \equiv [a,b] \bmod G''$. Additive notation will be used for the operation in G'/G''.

This action endows the Abelian group G'/G'' with a G/G'-module structure, that is, a module on the group ring $\Lambda := \mathbb{Z}[G/G']$. If $x \in G$, then t_x denotes its class in Λ. For $i = 1, \ldots, r$, we choose $x_i \in G$ a meridian of ℓ_i in G; the class $t_i := t_{x_i} \in \Lambda$ does not depend on the particular choice of the meridian in ℓ_i. Note that $t_1, \ldots, t_r$ is a basis of $G/G' \cong \mathbb{Z}^r$ and therefore one can identify

$$\Lambda := \mathbb{Z}[G/G'] = \mathbb{Z}[t_1^{\pm 1}, ..., t_r^{\pm 1}]. \tag{3}$$

This module is denoted by M_L and is referred to as the *Alexander Invariant* of L. Since we are interested in oriented topological properties of $(\mathbb{P}^2, \bigcup L)$, the coordinates $t_1, \ldots, t_r$ are well defined.

Remark 2.1. The module structure of M_L is in general complicated. One of its invariants is the zero set of the Fitting ideals of the complexified Alexander Invariant of L, that is, $M_L^{\mathbb{C}} := M_L \otimes (\Lambda \otimes \mathbb{C})$. This sequence of invariants is called the sequence of characteristic varieties of L introduced by A. Libgober [17]. These are subvarieties of the torus $(\mathbb{C}^*)^r$; in fact, irreducible components of characteristic varieties are translated subtori [1].

Our approach in studying the structure of the Λ-module M_L is via the associated graded module by the augmentation ideal $\mathfrak{m} := (t_1 -$

$1, ..., t_r - 1)$. In order to do so, and to be able to do calculations, we need some formulæ on this module relating operations in G'/G''. For the sake of completeness, these formulæ are listed below. However, since they are straightforward consequences of the definitions, their proof will be omitted. The symbol " $\stackrel{2}{=}$ " means that the equality is considered in G'/G'':

Properties 2.2.

(1) $[x,p] \stackrel{2}{=} (t_x - 1)p \quad \forall p \in G'$,

(2) $[x_1 \cdot ... \cdot x_n, y_1 \cdot ... \cdot y_m] \stackrel{2}{=} \sum_{i=1}^{n} \sum_{j=1}^{m} T_{ij}[x_i, y_j]$, where $T_{ij} = \prod_{k=1}^{i-1} t_{x_k} \cdot \prod_{l=1}^{j-1} t_{y_l}$.

(3) $[p_1 \cdot ... \cdot p_n, x] \stackrel{2}{=} -(t_x - 1)(p_1 + ... + p_n) \quad \forall p_i \in G'$,

(4) $[p_x x, p_y y] \stackrel{2}{=} [x,y] + (t_x - 1)p_y - (t_y - 1)p_x \quad \forall p_x, p_y \in G'$.

(5) $[x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}, y_1^{\beta_1} \cdot \ldots \cdot y_m^{\beta_m}] \stackrel{2}{=} \sum_{i=1}^{n} \sum_{j=1}^{m} T_{ij}\Big([x_i, y_j] + \delta(i,j)\Big)$, where

$$\delta(i,j) = (t_{y_j} - 1)[\alpha_i, x_i] - (t_{x_i} - 1)[\beta_j, y_j].$$

(6) *Jacobi relations:*

$$J(x,y,z) := (t_x - 1)[y,z] + (t_y - 1)[z,x] + (t_z - 1)[x,y] \stackrel{2}{=} 0.$$

Let us recall a well-known result on presentations of fundamental groups of line arrangements based on the celebrated Zariski-Van Kampen method for computing the fundamental group of the complement of an algebraic curve. Let us recall briefly a description of this method applied to $X := \mathbb{P}^2 \setminus \bigcup L$. For a more detailed exposition see [9, p.121] or [21].

Let $P_0 \in \ell_0 \setminus (\ell_1 \cup ... \cup \ell_r)$ and consider the pencil of lines in $\mathbb{P}^2$ based on P_0. This defines a locally trivial fibration outside a finite number of points $\Delta := \{a_0, a_1, ..., a_s\} \subset \mathbb{P}^1$, that is, a fibration $X \setminus \pi^{-1}(\Delta) \xrightarrow{\pi|} \mathbb{P}^1 \setminus \Delta$, where $a_0 = \pi(\ell_0)$ and $\pi^{-1}(a_j) = H_j \cap X$ such that H_j is a line passing through P_0 and a singular point of $\bigcup L$. Let $* \in \mathbb{P}^1 \setminus \Delta$ be a base point and choose $\{\gamma_1, ..., \gamma_s\}$ a set of meridians on $\pi_1(\mathbb{P}^1 \setminus \Delta; *)$ such that γ_j is a meridian of a_j and $\gamma_1 \cdot \ldots \cdot \gamma_s$ is the inverse of a meridian of a_0. Let $y_* \in \pi^{-1}(*)$ ($|y_*|$ *big enough*) and consider $x_i \in \pi_1(\pi^{-1}(*); y_*) =: \mathbb{F}$, a meridian of ℓ_i. The group $\pi_1(\mathbb{P}^1 \setminus \Delta; *)$ of the base acts on the group $\mathbb{F}$ of the fiber in such a way that $\gamma_j(x_i)$ is a conjugate of x_i. This action comes from a morphism $\pi_1(\mathbb{P}^1 \setminus \Delta; *) \to \mathbb{B}_r$ and the Artin action of $\mathbb{B}_r$ on the free group $\mathbb{F}$ with the list of generators $\bar{x} := (x_1, \ldots, x_r)$.

A straightforward consequence of the Zariski-Van Kampen method ([30, 29], and [31, Chapter VIII]) is that

$$\langle \bar{x};\ x_i = x_i^{\gamma_j},\ i = 1, ..., r,\ j = 1, ..., s \rangle$$

is a presentation of G.

Moreover, one can describe the action of each γ_j in more detail as follows. Let D_j be a small enough disk around a_j, and g_j a path from $*$ to $p_j \in \partial D_j$ such that $\gamma_j = g_j \cdot \partial D_j \cdot g_j^{-1}$. Let us work on $\pi_1(\pi^{-1}(p_j), y_*)$.

The action of ∂D_j on $\pi_1(\pi^{-1}(p_j), y_*)$ can be described in a suitable set of free generators $\bar{y}$ of $\pi_1(\pi^{-1}(p_j), y_*)$ as follows. Let $P := \ell_{i_1} \cap ... \cap \ell_{i_p}$ be a singular point of $\bigcup L$ of multiplicity p on H_j and let $y_{i_1}, ..., y_{i_p}$ meridians of the lines such that $Y_P := y_{i_1} \cdots y_{i_p}$ is homotopic to a meridian of P on H_j. In that case $y_{i_k}^{\partial D_j} = y_{i_k}^{Y_P}$. If $H_j \cap \ell_i$ is not a singular point $\bigcup L$ then $y_i^{\partial D_j} = y_i$.

The path g_j induces a natural isomorphism $\beta_j : \pi_1(\pi^{-1}(p_j), y_*) \to \mathbb{F}$ of Artin type between $\bar{y}$ and $\bar{x}$; β_j is induced by a pure braid associated to g_j and we will identify these groups via β_j. Let us denote by $\mathbb{F}_P$ the subgroup of $\mathbb{F}$ generated by $y_{i_1}, ..., y_{i_p}$. Since each y_{i_k} is a conjugate of x_{i_k}, one obtains the following.

Proposition 2.3. *The group G admits a presentation of the form $\langle \bar{x}; \bar{W} \rangle$, where $\bar{W} := \{W_1(\bar{x}), ..., W_m(\bar{x})\}$ $m \geq 0$, and $W_i(\bar{x}) \in \mathbb{F}'$, $\forall i = 1, ..., m$.*

Moreover, $\bar{W}$ consists of words of type

$$[y_{i_k}, Y_P] \in [\mathbb{F}_P, Y_P],\ k = 1, \ldots, p-1. \tag{4}$$

for every $P \in \ell_{i_1} \cap ... \cap \ell_{i_p}$ ordinary multiple point of $\bigcup L$ of multiplicity p not belonging to ℓ_0.

Remark 2.4. The difficult part of actually finding a presentation is the computation of the pure braids mentioned above. Effective methods have been constructed in several works [3, 7, 5].

Remark 2.5. The relations $[y_{i_k}, Y_P]$ can also be written in the form $[x_{i_k}, X_{P,k}]$, where $X_{P,k}$ is a product of conjugates of $x_{i_1}, \ldots, x_{i_p}$. Moreover, we may use other relations to *simplify* the elements $X_{P,k}$.

Definition 2.6. Any presentation $\langle \bar{x}; \bar{W}(\bar{x}) \rangle$ of G as in Proposition 2.3 (or Remark 2.5) will be called a *Zariski presentation* of G. The free group $\mathbb{F} := \langle \bar{x} \rangle$ will be referred to as the *free group associated with* the given presentation.

Notation 2.7. Most of the following construction could be done on a broader variety of groups such as 2-formal, or 2-free groups, but since we want to apply this theory to a particular problem, we will only deal with Zariski presentations of groups of line arrangements.

For technical reasons it is important to consider the Alexander Invariant corresponding to the free group associated with a given presentation. Such a module will be denoted by $\tilde{M}_L$. The following is a standard presentation of the modules $\tilde{M}_L$ and $\tilde{M}_L$ in terms of a Zariski presentation of G.

The following is a presentation of the Alexander Invariant from a given Zariski presentation. For another presentation of the Alexander Invariant from the braid monodromy see [8, Theorem 5.3].

Proposition 2.8. *Let $\langle \bar{x}; \bar{W} \rangle$ be a Zariski presentation of G and let $\mathbb{F} := \langle \bar{x} \rangle$ be its associated free group, then the module $\tilde{M}_L$ admits a presentation $\tilde{\Gamma}/\mathcal{J}$, where*

$$\tilde{\Gamma} := \bigoplus_{1 \leq i < j \leq r} [x_i, x_j]\Lambda$$

and $\mathcal{J}$ is the submodule of $\tilde{\Gamma}$ generated by the Jacobi relations (Property 2.2(6))

$$J(i,j,k) := (t_i - 1)x_{jk} + (t_j - 1)x_{ki} + (t_k - 1)x_{ij}.$$

Moreover, the module M_L can be obtained as a quotient of $\tilde{M}_L$ as $\tilde{\Gamma}/(\mathcal{J} + \mathcal{W})$, where $\mathcal{W}$ is the submodule of $\tilde{\Gamma}$ generated by the relations $\bar{W}$.

Proof. First, the Reidemeister-Schreier method on $\mathbb{F}' \hookrightarrow \mathbb{F}$ can be used to obtain a system of generators and a generating system of relations of $\mathbb{F}'$. Let $\bar{\imath} = (i_1, ..., i_r) \in \mathbb{Z}^r$ and $\ell(\bar{\imath}) = \max\{k \mid i_k \neq 0\}$. Note that $x_{\bar{\imath}} := x_1^{i_1} \cdot \ldots \cdot x_r^{i_r}$ is a Reidemeister-Schreier system of representatives of $\mathbb{F}'/\mathbb{F}''$ (from now on, and to avoid ambiguities, we will write $\mathbb{F}'/\mathbb{F}''$ when referring to the group structure and $\tilde{M}_L$ when referring to the module structure). Hence the family

$$x[\alpha]_{\bar{\imath}} := x_{\bar{\imath}} x_\alpha x_{\bar{\imath}+e_\alpha}^{-1}, \quad \alpha = 1, ..., \ell(\bar{\imath}) - 1$$

(where e_α is such that $x_\alpha = x_{e_\alpha}$) represents a free system of generators of $\mathbb{F}'$. Our purpose now is to use the module structure in order to obtain a finite set of generators of $\mathbb{F}'/\mathbb{F}''$ as the module $\tilde{M}_L$. Let $\bar{\imath}_\alpha = (i_1, ..., i_{\alpha-1}, 0, ..., 0)$, where $\alpha = 1, ..., r$ and $\bar{\imath}_1 = (0, ..., 0)$. Note that:

a) $x[\alpha]_{\bar\imath} = [x_{\bar\imath}, x_\alpha][x_\alpha, x_{\bar\imath_\alpha}]$. Therefore, using Property 2.2(2), one has

$$x[\alpha]_{\bar\imath} \stackrel{2}{=} [x_{\bar\imath}, x_\alpha] - [x_{\bar\imath_\alpha}, x_\alpha] \stackrel{2}{=} \sum_{k=\alpha+1}^{r} T^{\bar\imath_k}[x_k^{i_k}, x_\alpha], \tag{5}$$

where $T^{\bar\imath} := t_1^{i_1} \cdots t_r^{i_r}$.

b) $[x_k^{i_k}, x_\alpha] \stackrel{2}{=} \dfrac{t_k^{i_k} - 1}{t_k - 1}[x_k, x_\alpha]$.

Hence, the module $\tilde M_L$ is generated by the elements $x_{ij} := [x_i, x_j]$, where $1 \leq i < j \leq r$. Let us define the following sets of elements in $\tilde M_L$:

$$\Gamma_1 := \{T^{\bar\imath} x_{jk} \mid \max\{i \mid i \text{ is a coordinate of } \bar\imath\} \leq j\}.$$

and

$$\Gamma_2 := \{T^{\bar\imath} x_{jk} \mid \max\{i \mid i \text{ is a coordinate of } \bar\imath\} > j\}.$$

Note that $\mathbb{F}'/\mathbb{F}''$ is generated by Γ_1. Moreover the elements in Γ_1 are independent, since

$$T^{\bar\imath} x_{jk} \stackrel{2}{=} x_{\bar\imath} x_j x_k x_j^{-1} x_k^{-1} x_{\bar\imath}^{-1} = (x_{\bar\imath} x_k x_j x_k^{-1} x_j^{-1} x_{\bar\imath}^{-1})^{-1}$$

$$= x[j]^{-1}_{\bar\imath + e_k} \stackrel{2}{=} -x[j]_{\bar\imath + e_k}.$$

Therefore the relations in the module $\tilde M_L$ come from rewriting the elements in Γ_2 in terms of the base Γ_1. In fact, it is enough to consider the elements in Γ_2 of the form $t_i x_{jk}$ where $i > j < k$. One has the following two situations:

(1) If $j < i \leq k$ then

$$t_i x_{jk} \stackrel{2}{=} x_i * x_{jk} = x_i x_j x_k x_j^{-1} x_k^{-1} x_i^{-1}$$

$$= (x_i x_j x_i^{-1} x_j^{-1})(x_j x_i x_k x_j^{-1} x_k^{-1} x_i^{-1}) =$$

$$= x[j]_{e_i} x[j]^{-1}_{e_i + e_k} \stackrel{2}{=} x[j]_{e_i} - x[j]_{e_i + e_k}.$$

Finally, applying (5),

$$t_i x_{jk} \stackrel{2}{=} x_{ij} - x_{ij} - t_i x_{kj} \stackrel{2}{=} t_i x_{jk}.$$

(2) If $j < k < i$, then

$$t_i x_{jk} \stackrel{2}{=} x_i * x_{jk} = x_i x_j x_k x_j^{-1} x_k^{-1} x_i^{-1} =$$

$$= (x_i x_j x_i^{-1} x_j^{-1})(x_j x_i x_k x_i^{-1} x_k^{-1} x_j^{-1})(x_j x_k x_i x_j^{-1} x_i^{-1} x_k^{-1})(x_k x_i x_k^{-1} x_i^{-1}) =$$

$$= x[j]_{e_i} x[k]_{e_j+e_i} x[j]^{-1}_{e_k+e_i} x[k]^{-1}_{e_i} \stackrel{2}{=} x[j]_{e_i} + x[k]_{e_j+e_i} - x[j]_{e_k+e_i} - x[k]_{e_i}.$$

Applying (5) one has

$$t_i x_{jk} \stackrel{2}{=} x_{ij} + t_j x_{ik} - x_{kj} - t_k x_{ij} - x_{ik} \stackrel{2}{=} x_{jk} - (t_k - 1)x_{ij} - (t_i - 1)x_{ki}.$$

which produces the Jacobi relation $J(j,k,i) = 0$.

The second statement follows from the abelianization of

$$\mathbb{W} \hookrightarrow \mathbb{F}' \twoheadrightarrow G',$$

where $\mathbb{W}$ is the normal subgroup of $\mathbb{F}$ generated by $\bar{W}$. Note that $\mathcal{W} = \mathbb{W}/(\mathbb{W} \cap \mathbb{F}')$ and hence $\mathcal{W}$ is generated by the projection of the system $\bar{W}$ in $\mathcal{W}$. Q.E.D.

Remark 2.9. Note that the expression (4) and Property 2.2(5) provide a method to rewrite the relations $\bar{W}$ as elements of $\tilde{\Gamma}$.

Example 2.10. As an example of how to obtain $\mathcal{W}$ note that, if the lines ℓ_i and ℓ_j in L intersect in a double point, then there is a relation in $\bar{W}$ of type $[x_i^{\alpha_i}, x_j^{\alpha_j}]$, where $\alpha_i, \alpha_j \in G$. Using Property 2.2(5), this relation can be written in $\tilde{M}_L$ as $x_{i,j} + (t_j - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_j, x_j] \in \mathcal{W}$.

Analogously, if the lines ℓ_i, ℓ_j and ℓ_k in L intersect at a triple point, one obtains relations in G of type $[x_i^{\alpha_i}, x_j^{\alpha_j} x_k^{\alpha_k}]$, where $\alpha_i, \alpha_j, \alpha_k \in G$, which can be rewritten in $\mathcal{W}$ as

$$x_{i,j} + (t_j - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_j, x_j] + t_j\big(x_{i,k} + (t_k - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_k, x_k]\big).$$

Let $\mathfrak{m}$ be the augmentation ideal in Λ associated with the origin, that is, the kernel of homomorphism of Λ-modules, $\varepsilon : \Lambda \to \mathbb{Z}$, $\varepsilon(t_i) := 1$, where $\mathbb{Z}$ has the trivial module structure.

One can consider the filtration on M_L associated with $\mathfrak{m}$, that is, $F^i M_L := \mathfrak{m}^i M_L$. The associated graded module $\operatorname{gr} M_L := \oplus_{i=0}^{\infty} \operatorname{gr}^i M_L$, where $\operatorname{gr}^i M_L := F^i M_L / F^{i+1} M_L$ is a graded module over $\operatorname{gr}_{\mathfrak{m}} \Lambda := \oplus_{i=0}^{\infty} F^i \Lambda / F^{i+1} \Lambda$.

Consider the rings $\Lambda_j := \Lambda/\mathfrak{m}^j$, obtained by taking the quotient of Λ by successive powers of the ideal $\mathfrak{m}$. This allows one to define truncations of the Alexander Invariant.

Definition 2.11. The Λ_j-module, $M_L^j := M_L \otimes_\Lambda \Lambda_j$ will be called the *j-th truncated Alexander Invariant of L*. The induced filtration is finite and will be denoted in the same way.

Example 2.12. From Example 2.10, it is easily seen that the relations in M_L^2 coming from double and triple points are as follows.

(1) If ℓ_i and ℓ_j intersect at a double point one has:

$$x_{ij} + (t_j - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_j, x_j] = 0, \tag{6}$$

$$(t_k - 1)x_{ij} = 0. \tag{7}$$

(2) If ℓ_i, ℓ_j and ℓ_k intersect at a triple point one has:

$$\begin{aligned} x_{i,j} + t_j x_{i,k} + (t_j - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_j, x_j] \\ +(t_k - 1)[\alpha_i, x_i] - (t_i - 1)[\alpha_k, x_k] = 0, \end{aligned} \tag{8}$$

$$(t_m - 1)x_{i,j} + (t_m - 1)x_{i,k} = 0. \tag{9}$$

For any $k \in \mathbb{N}$, there is a natural morphism $\varphi_k : G' \twoheadrightarrow M_L^k$. We will sometimes refer to $\varphi_k(g)$ as $g \pmod{\mathfrak{m}^k}$ and equalities in M_L^k will be denoted by $p_1 \overset{k}{\equiv} p_2$.

Remark 2.13. A Zariski presentation on G induces a (set-theoretical) section

$$s : M_L \to G',$$

$$s\left(\varepsilon(t_1 - 1)^{k_1} ... (t_r - 1)^{k_r} x_{ij}\right) := [x_1^{[k_1]}, x_2^{[k_2]}, ..., x_r^{[k_r]}, [x_i, x_j]]^\varepsilon,$$

defined inductively, where

$$[w_1, w_2, ..., w_n] := [w_1, [w_2, ..., w_n]], \quad [w_1^{[n]}, w_2] := [w_1^{[n-1]}, w_1, w_2].$$

This, accordingly, induces a section of φ_k on each M_L^k denoted by s_k.

Remark 2.14. From Property 2.2(1), we deduce that the kernel of φ_2 equals $\gamma_4(G)$. Moreover $\ker(G' \to M_L^1)$ equals $\gamma_3(G)$.

The previous construction can be summarized in the following.

Proposition 2.15. *Let $\psi(p_1, ..., p_m)$ be a word on the letters $\bar{p} := \{p_1, ..., p_m\}$. If $p_i, q_i \in G'$ and $p_i \overset{k}{\equiv} q_i$ $(i = 1, ..., m)$, then $[g, \psi(\bar{p})] \overset{k+1}{\equiv} [g, \psi(\bar{q})]$, $\forall g \in G$. In particular, if $p \in M_L^k$ then $[g, s_k(p)]$ is a well-defined element of M_L^{k+1}; if $g = x_i$ this element can be written $(t_i - 1)p \in M_L^{k+1}$.*

Remark 2.16. The ring Λ_k is not local, but note that an element $\lambda \in \Lambda_k$ is a unit if an only if $\varepsilon(\lambda) = \pm 1$. To see this note that $\Lambda_k = \mathbb{Z} \oplus \mathfrak{m}/\mathfrak{m}^k$ and the kernel of the evaluation map $\varepsilon : \mathbb{Z} \oplus \mathfrak{m}/\mathfrak{m}^k \to \mathbb{Z}$ is exactly $\mathfrak{m}/\mathfrak{m}^k$.

Notation 2.17. Note that everything in this section can also be reproduced by using the free group $\mathbb{F}$ associated with a Zariski presentation of G and will be denoted by adding a tilde. For instance, $\tilde{M}_L = \Lambda^{\binom{r}{2}}/\mathcal{J}$ is the Alexander Invariant associated with $\mathbb{F}_G$, and $F^i\tilde{M}_L$ is the filtration associated with $\mathfrak{m} \subset \Lambda$.

Note that any automorphism of G that sends x_i to $x_i\alpha_i$, (with $\alpha_i \in G'$) induces a filtered automorphism of M_L^k:

$$[x_i, x_j] \mapsto [x_i\alpha_i, x_j\alpha_j] \overset{2}{\equiv} [x_i, x_j] + t_j(t_i - 1)\alpha_j - t_i(t_j - 1)\alpha_i. \tag{10}$$

Note that this automorphism induces the identity on $\operatorname{gr} M_L^k$.

The following result is an immediate consequence of Proposition 2.15 and it explains why M_L^k is a more manageable object.

Corollary 2.18. *Under the above conditions the following formula holds in M_L^k:*

$$[x_i\alpha_i, x_j\alpha_j] \overset{k}{\equiv}$$

$$[x_i, x_j]+(t_i-1)\varphi_{k-1}(\alpha_j)-(t_j-1)\varphi_{k-1}(\alpha_i)+(t_j-1)(t_i-1)\varphi_{k-2}(\alpha_j-\alpha_i),$$

and hence the formula (10) *only depends on $\varphi_{k-1}(\alpha_i)$.*

Lemma 2.19. *Under the above conditions*

(1) *The Λ_1-module $\operatorname{gr}^0 M_L = \Lambda_1^{\binom{r}{2}}/\mathcal{W}$ is free of rank $g = \binom{r}{2} - v$, where v is the multiplicity of the combinatorial type of L (Notation* 1.2*).*

(2) *The Λ_2-module $\operatorname{gr}^1 M_L$ is combinatorial.*

The groups $\operatorname{gr}^k M_L \otimes \mathbb{Q}$ are combinatorial, see [23].

Proof. By Proposition 2.3 and the discussion previous to it, for any singular point $P = \ell_{i_1} \cap \ldots \cap \ell_{i_p}$ (of multiplicity $p \geq 2$) there are relations in G of type

(11)

$$R_P \equiv \begin{cases} [x_{i_1}^{\alpha_1^P}, x_{i_1}^{\alpha_1^P} \cdots x_{i_p}^{\alpha_{i_p}^P}] \overset{2}{\equiv} x_{i_1,i_2} + x_{i_1,i_3} + \ldots + x_{i_1,i_p} + \ldots \\ \hfill \text{terms in } F^1 M_L \\ [x_{i_2}^{\alpha_2^P}, x_{i_1}^{\alpha_1^P} \cdots x_{i_p}^{\alpha_{i_p}^P}] \overset{2}{\equiv} x_{i_2,i_1} + x_{i_2,i_3} + \ldots + x_{i_2,i_p} + \ldots \\ \hfill \text{terms in } F^1 M_L \\ \ldots \\ [x_{i_{p-1}}^{\alpha_{i_p}^P}, x_{i_1}^{\alpha_1^P} \cdots x_{i_p}^{\alpha_{i_p}^P}] \overset{2}{\equiv} x_{i_p,i_1} + x_{i_p,i_2} + \ldots + x_{i_{p-1},i_p} + \ldots \\ \hfill \text{terms in } F^1 M_L \end{cases}$$

Since $\mathcal{J} \subset F^1\tilde{M}_L$, one has that $\operatorname{gr}^0 \tilde{M}_L = \operatorname{gr}^0 \tilde{\Gamma} = \Lambda_1^{\binom{r}{2}}$ and hence, by Proposition 2.8, one has $\operatorname{gr}^0 M_L = \Lambda_1^{\binom{r}{2}}/\mathcal{W}^1$. Therefore (1) follows from the fact that the equations of R_P in (11) are independent, since each generator $x_{i_\bullet,i_p}$ appears only once.

To prove the second part, it is enough to see that $\operatorname{gr}^1 M_L$ must be generated (as an Abelian group) by the elements $(t_i-1)x_{j,k}$ and a generating system of relations is given by $J(i,j,k)\otimes\Lambda_2$ and $(\mathfrak{m}R_P)\otimes\Lambda_2$, for any $i,j,k\in\{1,...,r\}$ and $P\in\operatorname{Sing}(L)\setminus\ell_0$, which is a purely combinatorial system of relators. Q.E.D.

Notation 2.20. Since $\operatorname{gr}^0 M_L$ and $\operatorname{gr}^1 M_L$ only depend on the combinatorics, we will often use the notation $\operatorname{gr}^0 M_{\mathscr{C}}$ and $\operatorname{gr}^1 M_{\mathscr{C}}$ respectively to refer to such groups.

§3. Truncated Alexander Invariant and Homeomorphisms of Ordered Pairs

Let L_1 and L_2 be two ordered line arrangements sharing the ordered combinatorics $\mathscr{C}$. Consider two Zariski presentations $G_1=\langle\bar{x};\bar{W}^1(\bar{x})\rangle$ and $G_2=\langle\bar{x};\bar{W}^2(\bar{x})\rangle$ of the fundamental groups of X_{L_1} and X_{L_2}, where the subscripts of the generators $\bar{x}:=\{x_1,...,x_r\}$ respect the ordering of the irreducible components. The Abelian groups G_1/G_1' and G_2/G_2' can be canonically identified with $\operatorname{gr}^0 M_{\mathscr{C}}$ so that $x_i \pmod{G_1'} \equiv x_i \pmod{G_2'}$. Hence $\Lambda:=\Lambda_{L_1}=\Lambda_{L_2}$ We will study the existence of isomorphisms $h:G_1\to G_2$ such that $h_*:\operatorname{gr}^0 M_{\mathscr{C}}\to\operatorname{gr}^0 M_{\mathscr{C}}$ is the identity.

Definition 3.1. Let $\mathbb{F}_i$ be the free group associated with the Zariski presentation of G_i, $i=1,2$. A morphism $\tilde{h}:\mathbb{F}_1\to\mathbb{F}_2$ is called a *homologically trivial morphism* if $\tilde{h}_*:\mathbb{F}_1/\mathbb{F}_1'\to\mathbb{F}_2/\mathbb{F}_2'$ satisfies $\tilde{h}_*(x_i)=x_i$. A morphism $h:G_1\to G_2$ is called a *homologically trivial isomorphism* if it is induced by a homologically trivial morphism $\tilde{h}$, i.e., if $h_*:\operatorname{gr}^0 M_{\mathscr{C}}\to\operatorname{gr}^0 M_{\mathscr{C}}$ is the identity. Note that $\tilde{h}$ might not be unique.

Remarks 3.2.

(1) The above definition is mainly used for isomorphisms; in this setting, we consider that the trivial map is the identity and not the constant morphism. Other authors use *IA-automorphisms* [4] or homologically marked groups and homologically marked morphisms [22].

(2) In other words, a morphism $h:G_1\to G_2$ is homologically trivial if there exists $(\alpha_1,...,\alpha_r)\in(G_2')^r$ such that $h(x_i)=x_i\alpha_i$.

(3) Any homologically trivial isomorphism h induces a Λ-module morphism $h : M_1 := M_{L_1} \to M_{L_2} =: M_2$.
(4) Any homologically trivial isomorphism h respects the filtrations F and produces isomorphisms $\operatorname{gr}^i h : \operatorname{gr}^i M_1 \to \operatorname{gr}^i M_2$. By identifying $\operatorname{gr}^1 M_1 \equiv \operatorname{gr}^1 M_{\mathscr{C}} \equiv \operatorname{gr}^1 M_2$, $\operatorname{gr}^1 h$ is the *identity*.

In order to state some properties of homologically trivial isomorphisms, we need to introduce some notation. Note that the homologically trivial morphism $\tilde{h}$ also induces morphisms on the Alexander Invariants of the associated free groups $\tilde{M}_i$ $(i = 1, 2)$ and on their truncations $\tilde{M}_i^j$. Let us denote by $\tilde{h}^i : \tilde{M}_1^i \to \tilde{M}_2^i$ the induced homologically trivial morphisms of the truncated modules. A straightforward computation proves that

$$\tilde{h}(J(x_i, x_j, x_k)) = J(x_i, x_j, x_k) \in \mathbb{F}_2'/\mathbb{F}_2''. \tag{12}$$

Homologically trivial isomorphisms induce a particular kind of isomorphisms of the Λ-modules M_1, M_2 which are worth studying.

Remark 3.3. A direct attempt to prove that two modules are homologically trivial isomorphic is almost intractable. One would have to check if, for some choice $(\alpha_1, \ldots, \alpha_r) \in (G_2')^r \mod G_2''$, such a Λ-module isomorphism exists. The lack of linearity in this approach is the reason why we consider the truncated modules M_1^k, M_2^k.

Applying Corollary 2.18, we are faced with simply solving a linear system as follows. Let $h : G_1 \to G_2$ be a homologically trivial morphism, then there exists $(\alpha_1, \ldots, \alpha_r) \in (G_2')^r \mod G_2''$ such that $h(x_i) = x_i\alpha_i$. Therefore there exist Λ_k-morphisms $h^k : M_1^k \to M_2^k$ induced by h for any $k \in \mathbb{N}$. Note that

$$h^2(x_{i,j}) = x_{i,j} + \sum_{u,v} \alpha_{u,v}^j x_{i,u,v} - \sum_{u,v} \alpha_{u,v}^i x_{j,u,v}$$

where $x_{i,j} \overset{2}{\equiv} [x_i, x_j]$, $x_{i,u,v} \overset{2}{\equiv} (t_i - 1)x_{u,v}$, and

$$\alpha_w \overset{1}{\equiv} \sum_{u<v} \alpha_{u,v}^w x_{u,v}, \tag{13}$$

since

$$h^2(x_{i,j}) \overset{2}{\equiv} [h(x_i), h(x_j)] \overset{2}{\equiv} [x_i\alpha_i, x_j\alpha_j] \tag{14}$$

only depends on $\varphi_1(\alpha_i)$, the class of $\alpha_i \mod \mathfrak{m}$, by Proposition 2.18. In order to prove that h^2 is well defined, one must solve a linear system of equations on the variables $\alpha_{u,v}^i$ in the Abelian group M_2^k. If an integer solution exists, one can repeat the procedure on M_i^3, obtaining again a linear system of equations in the Abelian group M_2^3, and so on. In this work, we only need to consider h^2.

Let us consider an ordered line arrangement L with a fixed Zariski presentation $G = \langle \bar{x}; \bar{W} \rangle$. Let us denote $\mathscr{C}$ its ordered combinatorics.

In our particular case we can effectively compute the 2-nd truncated Alexander Invariant. The following result is an easy computation.

Lemma 3.4. *For any MacLane arrangement L, the Abelian group M_L^2 is free of rank* 29, *and its subgroup* $\operatorname{gr}^1 M_L^2 = \operatorname{gr}^1 M_{\mathscr{C}_{\mathrm{ML}}}$ *is free of rank* 21.

Theorem 3.5. *There is no homologically trivial isomorphism between $G_\omega := \pi_1(\mathbb{P}^2 \setminus \bigcup L_\omega)$ and $G_{\bar\omega} := \pi_1(\mathbb{P}^2 \setminus \bigcup L_{\bar\omega})$.*

Proof. Fix suitable Zariski presentations $G_\omega = \langle x_1, ..., x_7; W_1^\omega(\bar{x}), ..., W_{13}^\omega(\bar{x}) \rangle$ and $G_{\bar\omega} = \langle x_1, ..., x_7; W_1^{\bar\omega}(\bar{x}), ..., W_{13}^{\bar\omega}(\bar{x}) \rangle$ of the ordered arrangements L_ω and $L_{\bar\omega}$ (for instance, we have used the suitable Zariski presentations provided in [26] and other presentations obtained using the software in [5]). We identify the corresponding free groups $\mathbb{F}_\omega$ and $\mathbb{F}_{\bar\omega}$ with a free group $\mathbb{F}_7$. Recall that their combinatorial type has multiplicity 13, see Notation 1.2. We assume the relations to be ordered upon the following condition:

$$W_i^\omega(\bar{x}) \overset{1}{\equiv} W_i^{\bar\omega}(\bar{x}),$$

(in particular $W_i^\omega(\bar{x}) - W_i^{\bar\omega}(\bar{x}) \in F^1 \tilde{M}_{L_{\bar\omega}}$). Let us suppose that a homologically trivial homomorphism $h : G_\omega \to G_{\bar\omega}$ exists. Consider the corresponding elements $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_7$ that induce such a morphism (Remark 3.2(2)). Consider $M_\omega = M_{L_\omega}$ and $M_{\bar\omega} = M_{L_{\bar\omega}}$, the Alexander Invariants of L_ω and $L_{\bar\omega}$. Let $\Lambda := \mathbb{Z}[t_1^{\pm 1}, ..., t_7^{\pm 1}]$ be the ground ring of both Alexander Invariants, where $t_i \overset{1}{=} x_i$ as usual. This mapping induces a Λ_2-isomorphism $h^2 : M_\omega^2 \to M_{\bar\omega}^2$. By Corollary 2.18, h^2 only depends on the class $\alpha_i \mod \mathfrak{m}$. As in (13), one has

$$\alpha_k \overset{1}{\equiv} \sum_{1 \leq i < j \leq 7} \alpha_{ij}^k x_{ij}, \quad \alpha_{ij}^k \in \mathbb{Z}.$$

By (12) Jacobi relations play no role here.

Let us fix $i = 1, \ldots, 13$. Since $W_i^\omega(\bar{x}) \in \tilde{M}_\omega^2$ vanishes in M_ω^2, one deduces that $\tilde{h}^2(W_i^\omega(\bar{x})) \in \tilde{M}_{L_{\bar\omega}}^2$ should vanish in $M_{\bar\omega}^2$. Equivalently $\tilde{h}^2(W_i^\omega(\bar{x})) - W_i^{\bar\omega}(\bar{x}) \in F^1 \tilde{M}_{\bar\omega}^2$ should also vanish in $F^1 M_{\bar\omega}^2 = \operatorname{gr}^1 M_{\mathscr{C}_{\mathrm{ML}}}$. The vanishing of these terms, considered in the free abelian group $\operatorname{gr}^1 M_{\mathscr{C}_{\mathrm{ML}}}$, produces a system of linear equations in the variables

α_{ij}^k (actually, even though there are 147 variables, only 126 appear in the equations).

Solving a system with 137 equations and 126 variables is not an easy task, but any computer will help. Using Maple8, it takes 85 seconds of CPU time running on an Athlon at 1.4MHz and 256Kb RAM Memory to obtain the linear set of solutions. It is an affine variety of dimension 98 of the form $(\lambda_1, ..., \lambda_{98}, \kappa_1, ..., \kappa_{28})$ where

$$\kappa_i = q_i + \sum_{j=1}^{98} \varepsilon_j^i \lambda_j,$$

$\varepsilon_j^i \in \{0, \pm 1\}$, and $q_i \in \mathbb{Q}$. Since $\{q_i \mid i = 1, \ldots, 28\} = \{0, \pm 1, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{5}{3}\}$ one concludes that there is no integer solution[1]. Q.E.D.

Corollary 3.6. *There is no orientation preserving homeomorphism between the pairs of ordered arrangements* $(\mathbb{P}^2, \bigcup L_\omega)$ *and* $(\mathbb{P}^2, \bigcup L_{\bar{\omega}})$.

For Rybnikov's arrangements, one obtains similar results.

Lemma 3.7. *For any Rybnikov's arrangement R, the Abelian group M_R^2 is free of rank* 55, *and its subgroup* $\operatorname{gr}^1 M_R^2 = \operatorname{gr}^1 M_{\mathscr{C}_{\text{RYB}}}$ *is free of rank* 40.

Theorem 3.8. *There is no homologically trivial isomorphism between* $G_+ := \pi_1(\mathbb{P}^2 \setminus \bigcup R_{\omega,\omega})$ *and* $G_- := \pi_1(\mathbb{P}^2 \setminus \bigcup R_{\bar{\omega},\omega})$.

Proof. One way to prove this statement is to follow the computational strategy proposed for MacLane arrangements. First one needs Zariski presentations of $G_\pm$. This was done by means of the software in [5]. In this case the linear system obtained consists of 531 equations and 420 variables (again, out of the 792 variables α_{ij}^k, only 420 appear in the equations) and it took the same processor a total of 23,853 seconds of CPU time to compute the solutions. The space of solutions has dimension 252, that is, it can be written as $(\lambda_1, ..., \lambda_{252}, \kappa_1, ..., \kappa_{168})$, where

$$\kappa_i = q_i + \sum_{j=1}^{168} \varepsilon_j^i \lambda_j,$$

$\varepsilon_j^i \in \{0, \pm 1\}$, and $q_i \in \mathbb{Q}$. Since $\{q_i \mid i = 1, \ldots, 168\} = \{0, \pm 1, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{5}{3}\}$, one again concludes that there is no integer solution.

[1]The software is written for Maple8 and can be visited at the following public site `http://riemann.unizar.es/geotop/pub/`.

Another proof that doesn't depend as strongly on computations can be obtained from Theorem 3.5 as follows. Let us assume that a homologically trivial isomorphism exists between G_+ and G_-. Such an isomorphism induces an Λ_2-isomorphism between M^2_+ and M^2_-. Let $\widehat{\Lambda}_2 := \Lambda_2/\mathfrak{m}'$, where $\mathfrak{m}'$ is the ideal generated by $(t_8-1),...,(t_{12}-1)$, and let $\widehat{M}^2_\pm$ denote $M^2_\pm \otimes \widehat{\Lambda}_2$. Note that M^2_ω (resp. $M^2_{\bar\omega}$) can be considered as the $\widehat{\Lambda}_2$-module obtained from the inclusion of the complements $\mathbb{P}^2 \setminus \bigcup R_{\omega,\omega} \hookrightarrow \mathbb{P}^2 \setminus \bigcup L_\omega$ (resp. $\mathbb{P}^2 \setminus \bigcup R_{\bar\omega,\omega} \hookrightarrow \mathbb{P}^2 \setminus \bigcup L_{\bar\omega}$). Moreover, these inclusions define epimorphisms of $\widehat{\Lambda}_2$-modules $\pi : \widehat{M}^2_+ \twoheadrightarrow M^2_\omega$ and $\bar\pi : \widehat{M}^2_+ \twoheadrightarrow M^2_{\bar\omega}$. Proving the existence of a homologically trivial isomorphism $\tilde h^2$ that matches in the commutative diagram (15), and using Theorem 3.5 one obtains a contradiction.

$$
\begin{array}{ccc}
\widehat{M}^2_+ & \xrightarrow{\hat h^2} & \widehat{M}^2_- \\
\pi \downarrow & & \downarrow \bar\pi \\
M^2_\omega & \xrightarrow{\tilde h^2} & M^2_{\bar\omega}
\end{array}
\tag{15}
$$

Consider S_+ the $\widehat{\Lambda}_2$-submodule of $\widehat{M}^2_+$ generated by the elements $x_{i,j}$, $i,j \in \{1,...,7\}$ and consider the commutative diagram (16). Since $\pi(x_{i,j}) = x_{i,j}$ and $\bar h^2(x_{i,j}) \equiv x_{i,j} \mod \operatorname{gr}^1 M^2_{\bar\omega}$.

$$
\begin{array}{ccc}
\widehat{M}^2_+ & \xrightarrow{\hat h^2} & \widehat{M}^2_- \\
\pi \uparrow & & \downarrow \bar\pi \\
S_+ & \xrightarrow{\bar h^2} & M^2_{\bar\omega} \\
\pi_| \downarrow & & \\
M^2_\omega & &
\end{array}
\tag{16}
$$

Since M^2_ω and $M^2_{\bar\omega}$ are free Abelian of the same rank, (15) can be obtained from (16), by proving that $\pi_|$ is an isomorphism, which is the statement of Lemma 3.9. Q.E.D.

Lemma 3.9. *The epimorphism $\pi_|$ in* (16) *is injective*

Proof. We break the proof in several steps.

(O1) $(t_k - 1)x_{i,j} = 0$ in $\widehat{M}^2_+$ if $\{i,j\} \cap \{8,...,12\} \neq \emptyset$.

This can be proved case by case (all the equalities are considered in $\widehat{M}^2_+$):

- (a) If $i \in \{3,...,7\}$ and $j \in \{8,...,12\}$ (or vice versa, since $x_{i,j} = -x_{j,i}$): this is a consequence of Example 2.12(1) since the lines ℓ_i and ℓ_j intersect transversally.
- (b) If $i,j \in \{8,...,12\}$: this is a consequence of (a) and the Jacobi relations (Property 2.2(6)).

(c) If $i \in \{1,2\}, j \in \{8,...,12\}$: using the Jacobi relations (Property 2.2(6)) and (b) it is enough to check that $(t_k - 1)x_{i,j} = 0$, $i,k \in \{1,2\}$, $j \in \{8,...,12\}$. If ℓ_i and ℓ_j intersect at a double point, then (a) proves the result. Otherwise, there exists a line ℓ_m ($m \in \{7,...,12\}$) such that ℓ_i, ℓ_j and ℓ_m intersect at a triple point. By Example 2.12(2) one has $(t_k - 1)x_{i,j} + (t_k - 1)x_{m,j} = 0$, but $(t_k - 1)x_{m,j} = 0$ by (b), thus we are done.

(O2) $\operatorname{gr}^1(\widehat{M}^2_+) \subset S_+$. It is a direct consequence of (O1).

(O3) $\operatorname{gr}^0(\widehat{M}^2_+) = \dfrac{S_+}{\operatorname{gr}^1 \widehat{M}^2_+} \oplus \dfrac{\ker \pi + \operatorname{gr}^1 \widehat{M}^2_+}{\operatorname{gr}^1 \widehat{M}^2_+}$, i.e., $\dfrac{S_+}{\operatorname{gr}^1 \widehat{M}^2_+} \cong \operatorname{gr}^0 M^2_\omega$.

Since $\ker \pi$ is generated by $\langle x_{i,j} \rangle_{1 \leq i < j \leq 12, j > 7}$, it is clear that $\operatorname{gr}^0(\widehat{M}^2_+)$ decomposes in the required sum. It remains to prove that it is a direct sum.

One can consider $\operatorname{gr}^0(\widehat{M}^2_+)$ as a quotient $\dfrac{\langle x_{i,j} \rangle_{1 \leq i < j \leq 12}}{\mathcal{W}}$, hence it is enough to check that there is a system of generators $r_1, ..., r_n$ of $\mathcal{W}$ such that:

$$(*) \qquad \text{either} \quad r_i \in \frac{\ker \pi + \operatorname{gr}^1 \widehat{M}^2_+}{\operatorname{gr}^1 \widehat{M}^2_+}, \quad \text{or} \quad r_i \in \frac{S_+}{\operatorname{gr}^1 \widehat{M}^2_+}.$$

Note that a system of relators can be obtained combinatorially as $x_{i,j} = 0$ (if $\{\ell_i, \ell_j\}$ is a double point) or $x_{i,j} + x_{i,k} = 0$ (if $\{\ell_i, \ell_j, \ell_k\}$ is a triple point). Relations coming from double points satisfy (*). For the triple point relations note that any triple point $\{\ell_i, \ell_j, \ell_k\}$ such that $\{i,j\} \subset \{8,...,12\}$, verifies that $k \in \{8,...,12\}$; therefore, condition (*) is also satisfied.

(O4) $\operatorname{gr}^1(\widehat{M}^2_+) \cong \operatorname{gr}^1 M^2_\omega$.

By (O1), the Abelian group $\widehat{M}^2_+$ is generated by $x_{i,j}$, $i,j \in \{1,\ldots,12\}$, and $(t_k - 1)x_{i,j}$, $i,j,k \in \{1,\ldots,7\}$; the relators are obtained from the singular points (see Example 2.12) and the Jacobi relations $\mathcal{J}$.

By (O1) and the proof of Lemma 2.19(2), we find that $\operatorname{gr}^1(\widehat{M}^2_+)$ is generated by the elements $(t_k - 1)x_{i,j}$, $i,j,k \in \{1,\ldots,7\}$ and the relations are exactly those in $\mathcal{J}$ and the relations (7) and (9) in Example 2.12. The arguments used in (O3) also show that only double and triple points in $\mathscr{C}_{\mathrm{ML}}$ provide non-trivial relations and thus one obtains the same system of generators and relations of $\operatorname{gr}^1 M^2_\omega$.

Q.E.D.

Remark 3.10. The proof of Lemma 3.9 is combinatorial and depends strongly on the properties of $\mathscr{C}_{\text{RYB}}$. This lemma corresponds to a key statement of the proof of [26, Lemma 4.3] which is worth mentioning.

§4. Homologically Rigid Fundamental Groups

This last section will be devoted to proving that the fundamental groups of $R_{\omega,\omega}$ and $R_{\omega,\bar{\omega}}$ are not isomorphic.

Remark 4.1. Associated with a combinatorial type $\mathscr{C} := (\mathcal{L}, \mathcal{P})$, there is a family of groups, where

$$H_{\mathscr{C}} := \frac{\bigoplus_{\ell\in\mathcal{L}}\langle x_\ell\rangle\mathbb{Z}}{\langle\sum_{\ell\in\mathcal{L}} x_\ell\rangle\mathbb{Z}}$$

and $\operatorname{gr}^i M_{\mathscr{C}}$ is given by generators and relations as a quotient of $H_{\mathscr{C}}^{\otimes(i+1)}$, $i = 0, 1$ as described in Lemma 2.19. Note that, if $\mathscr{C}$ has a realization L, then one has identifications $H_{\mathscr{C}} \equiv H_1(\mathbb{P}^2\setminus\bigcup L; \mathbb{Z})$ and $\operatorname{gr}^i M_{\mathscr{C}} \equiv \operatorname{gr}^i M_L$.

Notation 4.2. There is a natural injective map $\Gamma(\mathscr{C}) \hookrightarrow \operatorname{Aut}(H_{\mathscr{C}})$ given by the permutation of the generators of $H_{\mathscr{C}}$; we identify $\Gamma(\mathscr{C})$ with its image in $\operatorname{Aut}(H_{\mathscr{C}})$. Another subgroup of $\operatorname{Aut}(H_{\mathscr{C}})$, denoted by $\operatorname{Aut}^1(H_{\mathscr{C}})$, is defined as those automorphisms of $H_{\mathscr{C}}$ that induce an automorphism of $\operatorname{gr}^1 M_{\mathscr{C}}$. It is easily seen that $\{\pm 1_{H_{\mathscr{C}}}\} \times \operatorname{Aut}(H_{\mathscr{C}}) \subset \operatorname{Aut}^1(H_{\mathscr{C}})$.

Definition 4.3. A line combinatorics $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ is called *homologically rigid* if $\operatorname{Aut}^1(H_{\mathscr{C}}) = \{\pm 1\} \times \Gamma(\mathscr{C})$.

The first goal of this section is to prove that Rybnikov's combinatorial type $\mathscr{C}_{\text{RYB}} := (\mathcal{R}, \mathcal{P})$ (described in (2)) is homologically rigid; we will follow the ordering (1). Results of this sort have been studied by M.Falk in [10, Corollary 3.24].

In order to do so, we are going to study $\operatorname{Aut}^1(H_{\mathscr{C}})$ for an ordered combinatorics $\mathscr{C} = (\mathcal{L}, \mathcal{P})$ having at most triple points. We will denote $\mathcal{P}_j := \{P \in \mathcal{P} \mid \#P = j\}$, $j = 2, 3$, and $\mathcal{L} := \{\ell_0, \ell_1, \ldots, \ell_r\}$. Let us first describe the groups $H_{\mathscr{C}}$ and $\operatorname{gr}^1 M_{\mathscr{C}}$:

$$H_{\mathscr{C}} := \frac{\langle x_0\rangle\mathbb{Z} \oplus \langle x_1\rangle\mathbb{Z} \oplus \cdots \oplus \langle x_r\rangle\mathbb{Z}}{\langle x_0 + x_1 + \ldots + x_r\rangle\mathbb{Z}}, \quad \operatorname{gr}^1 M_{\mathscr{C}} := \frac{\bigwedge^2 H_{\mathscr{C}}}{R_2 \oplus R_3}$$

where R_2 is the subgroup generated by $x_{i,j}$ ($\{\ell_i, \ell_j\} \in \mathcal{P}_2$) and R_3 is the subgroup generated by $x_{i,k} + x_{j,k}$ and $x_{i,j} + x_{i,k}$ ($\{\ell_i, \ell_j, \ell_k\} \in \mathcal{P}_3$).

Any isomorphism $\psi : H_{\mathscr{C}} \to H_{\mathscr{C}}$ induces a map $\wedge^2\psi : \bigwedge^2 H_{\mathscr{C}} \to \bigwedge^2 H_{\mathscr{C}}$. Let us represent $\psi : H_{\mathscr{C}} \to H_{\mathscr{C}}$ by means of a matrix $A^\psi :=$

$(a_i^j) \in \mathrm{Mat}(r+1,\mathbb{Z})$ such that $\psi(x_i) := \sum_{j=0}^{r} a_i^j x_j$ (note that such a matrix is not uniquely determined: each column is only well defined modulo the vector $\mathbb{1}_{r+1} := (1,\dots,1)$). The conditions required for this map to define a morphism on the quotient $\mathrm{gr}^1 M_{\mathscr{C}}$ are called *admissibility conditions* and can be expressed as follows:

(17)
$$\begin{vmatrix} a_u^i & a_v^i & 1 \\ a_u^j & a_v^j & 1 \\ a_u^k & a_v^k & 1 \end{vmatrix} = 0, \qquad \text{if } \{\ell_i,\ell_j,\ell_k\} \in \mathcal{P}_3, \{\ell_u,\ell_v\} \in \mathcal{P}_2$$

$$\begin{vmatrix} a_\bullet^i & a_u^i + a_v^i + a_w^i & 1 \\ a_\bullet^j & a_u^j + a_v^j + a_w^j & 1 \\ a_\bullet^k & a_u^k + a_v^k + a_w^k & 1 \end{vmatrix} = 0 \quad \text{if } \{\ell_i,\ell_j,\ell_k\}, \{\ell_u,\ell_v,\ell_w\} \in \mathcal{P}_3 \quad (\bullet = u,v,w)$$

(also note that such conditions are invariant on the coefficient vectors $(a_\bullet^0, a_\bullet^1, \dots, a_\bullet^{12})$ modulo $\mathbb{1}_{13}$). We summarize these facts.

Proposition 4.4. *Any morphism $\psi : H_{\mathscr{C}} \to H_{\mathscr{C}}$ whose associated matrix A^ψ satisfies the admissibility conditions* (17) *produces a well-defined morphism $\wedge^2\psi : M_{\mathscr{C}}^1 \to M_{\mathscr{C}}^1$.*

We are going to express the admissibility conditions (17) in a more useful way. Let $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}})$ and let A^ψ be a matrix representing ψ. Fix $P \in \mathcal{P}_3$ and consider the submatrix $A_P^\psi \in \mathrm{Mat}(3 \times 12, \mathbb{Z})$ of A^ψ which contains the rows associated with P. Let $\Sigma_k := \mathbb{Z}^{k+1}/\mathbb{1}_{k+1}$, $k \in \mathbb{N}$. We denote by $v_0(P), v_1(P)\dots, v_r(P) \in \Sigma_2$, the column vectors (mod $\mathbb{1}_3$) of A_P^ψ.

Lemma 4.5.

(1) *The vectors $v_0(P), v_1(P)\dots, v_r(P)$ span Σ_2.*
(2) $\sum_{j=0}^{r} v_j(P) = 0 \in \Sigma_2$.
(3) *For any $Q \in \mathcal{P}$ and for any $\ell_u \in Q$, the vectors $v_u(P)$ and $\sum_{\ell_i \in Q} v_i(P)$ are linearly dependent (i.e., span a sublattice of Σ_2 of rank less than two). In particular, if $\sum_{\ell_i \in Q} v_i(P) \neq 0$, then $\{v_i(P) \mid \ell_i \in Q\}$ spans a rank-one sublattice of Σ_2.*
(4) *There exists $Q \in \mathcal{P}_3$ such that $\{v_i(P) \mid \ell_i \in Q\}$ spans a rank-two sublattice of Σ_2 and $\sum_{\ell_i \in Q} v_i(P) = 0$.*

Proof.

(1) ψ is an automorphism.
(2) The sum of the columns of A^ψ is a multiple of $\mathbb{1}_{r+1}$.

(3) It is an immediate consequence of the admissibility conditions (17).
(4) If no such Q exists, then all the vectors $V_i(P)$ are linearly dependent, which contradicts (1). The last part is a consequence of (3).

Q.E.D.

Definition 4.6. Let $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ be a combinatorics; we say $\mathscr{C}' := (\mathcal{L}', \mathcal{P}')$ is a *subcombinatorics of* $\mathscr{C}$ if $\mathcal{L}' \subset \mathcal{L}$ and $\mathcal{P}' := \{P \cap \mathcal{L} \mid P \in \mathcal{P}, \#(P \cap \mathcal{L}) \geq 2\}$.

We define a subcombinatorics $\operatorname{Adm}_\psi(P) \subset \mathscr{C}$ as follows:

$$\mathcal{L}(\operatorname{Adm}_\psi(P)) := \{\ell_i \in \mathcal{R} \mid v_i(P) \neq 0\}.$$

Note that,

$$\ell_i \notin \mathcal{L}(\operatorname{Adm}_\psi(P)) \iff \text{the } i^{\text{th}} \text{ column of } A_P^\psi \text{ is a multiple of } \mathbb{1}_3. \tag{18}$$

This motivates the following definition.

Definition 4.7. A line combinatorics $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ with only double and triple points is called *3-admissible* if it is possible to assign a nonzero vector $v_i \in \mathbb{Z}^2$ to each $\ell_i \in \mathcal{L}$ such that:

(1) There exists $P \in \mathcal{P}_3$, such that $\{v_j \mid \ell_j \in P\}$ spans a rank-two sublattice.
(2) For every $P \in \mathcal{P}$ and for every $\ell_i \in P$, v_i and $\sum_{\ell_j \in P} v_j$ are linearly dependent.
(3) $\sum_{\ell_i \in \mathcal{L}} v_i = (0, 0)$.

Remarks 4.8. The conditions of Definition 4.7 can be made more precise.

(1) If $P = \{\ell_i, \ell_j\} \in \mathcal{P}_2$, then v_i and v_j are proportional, in notation, $v_i \| v_j$.
(2) If $P \in \mathcal{P}_3$ verifies condition (1) then $\sum_{\ell_i \in P} v_i = (0, 0)$.

Examples 4.9.

(1) With the above notation, $\operatorname{Adm}_\psi(P)$ is 3-admissible by Lemma 4.5.
(2) The combinatorics $\mathcal{M}_3$ of a triple point (that is, $\mathcal{L}_{\mathcal{M}_3} := \{0, 1, 2\}, \mathcal{P}_{\mathcal{M}_3} := \{\{0, 1, 2\}\}$) is 3-admissible, simply using $v_0 := (1, 0)$, $v_1 := (0, 1)$, $v_2 := (-1, -1)$.
(3) Let $\mathscr{C} := (\mathcal{L}, \mathcal{P})$ be a combinatorics such that
   - $\mathcal{L} := \mathcal{L}_0 \coprod \mathcal{L}_1 \coprod \mathcal{L}_2$;

- $\mathcal{L}_1$ and $\mathcal{L}_2$ define non-empty subcombinatorics in general position w.r.t. $\mathcal{L}_0$ (that is, $\ell_i \in \mathcal{L}_1$ and $\ell_j \in \mathcal{L}_2$ implies that $\{\ell_i, \ell_j\} \in \mathcal{P}$);
- at most one line of $\mathcal{L}_0$ *intersects* $\mathcal{L}_1 \cup \mathcal{L}_2$ in a non-multiple point of $\mathcal{L}_0$.

Then $\mathscr{C}$ is not 3-admissible. It is enough to see that if $\{v_i\}_{\ell_i \in \mathcal{L}}$ is a set of non-zero vectors satisfying conditions (2) and (3) of Definition 4.7, one has that $v_i \| v_j$.

- $\ell_i \in \mathcal{L}_1$ and $\ell_2 \in \mathcal{L}_2$; since $\{\ell_i, \ell_j\} \in \mathcal{P}$ then, by Remark 4.8(1), $v_i \| v_j$.
- $\ell_i, \ell_j \in \mathcal{L}_1$; considering any $\ell_k \in \mathcal{L}_2$ and using the previous case, one has $v_i \| v_k \| v_j$. The same argument works for $\ell_i, \ell_j \in \mathcal{L}_2$. In particular, $v_i \| v_j$ if $\ell_i, \ell_j \in \mathcal{L}_1 \cup \mathcal{L}_2$.
- $\ell_i \in \mathcal{L}_0$ and $P \in \mathcal{P}$ such that $P \cap \mathcal{L}_0 = \{\ell_i\}$. Since all the vectors associated with P but one are proportional, then this must also be the case for v_i.

All the vectors (but at most one) are proportional. To conclude we apply condition (3) of Definition 4.7.

(4) Ceva's line combinatorics is 3-admissible.

Proof. Ceva's line combinatorics is given by the following realization:

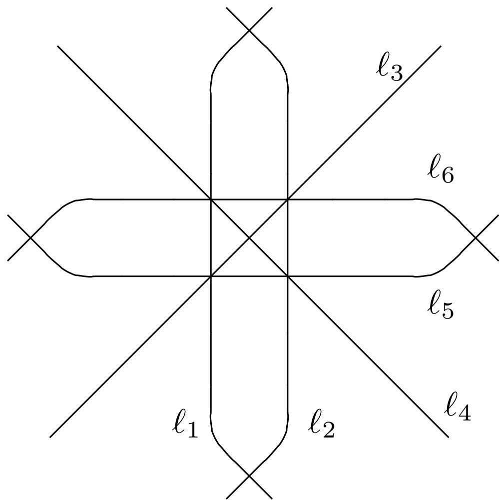

Fig. 2. Ceva's line combinatorics

$$\mathcal{L}_{\text{CEVA}} := \{1, 2, 3, 4, 5, 6\}$$

$$\mathcal{P}_{\text{CEVA}} := \{\{1,2\}, \{3,4\}, \{5,6\}, \{1,3,5\}, \{1,4,6\}, \{2,3,6\}, \{2,4,5\}\}.$$

For example, the following is a 3-admissible set of vectors for Ceva:

$$\{v_1 = v_2 = (1,0), v_3 = v_4 = (0,1), v_5 = v_6 = (-1,-1)\}$$

Q.E.D.

(5) MacLane's line combinatorics $\mathscr{C}_{\mathrm{ML}}$ is not 3-admissible.

Proof. We will use the combinatorics given in Figure 1. Let us assume that MacLane is 3-admissible, then one has a list of non-zero vectors $v_0, v_1, ..., v_7$ associated with each line. We will first see that v_0 and v_1 cannot be proportional. If they were ($v_0 \| v_1$), using $\{0,1,2\}$, $\{1,6\}$ and $\{2,3\}$ one would have that $v_0 \| v_2 \| v_6 \| v_3$ and finally, using $\{3,6,7\}$, one obtains $v_0 \| v_7$, and hence, by $\{1,4,7\}$ and $\{4,5\}$, all vectors are proportional, which contradicts condition (1) of Definition 4.7.

Therefore, v_0 and v_1 are linearly independent. After a change of basis, one can assume that $v_0 = (\alpha, 0)$, $v_1 = (0, \beta)$, $\alpha, \beta \in \mathbb{Z} \setminus \{0\}$, and therefore $v_2 = (-\alpha, -\beta)$. We will briefly describe the conditions and how they affect the vectors until a contradiction is reached:

$$\overset{\{0,7\}}{\underset{\{1,6\}}{\longrightarrow}} \begin{cases} v_7 = (\gamma, 0) \\ v_6 = (0, \delta) \end{cases} \overset{\{3,6,7\}}{\underset{\{0,5,6\}}{\longrightarrow}} \begin{cases} v_3 = (-\gamma, -\delta) \\ v_5 = (-\alpha, -\delta) \end{cases} \overset{\{1,3,5\}}{\longrightarrow} \begin{cases} \alpha = -\gamma \\ \beta = 2\delta. \end{cases}$$

Hence v_3 does not satisfy the condition (2) of Definition 4.7 with $v_2 = (\gamma, -2\delta)$ on the double point $\{2,3\}$. Q.E.D.

Definition 4.10. A combinatorics $\mathscr{C}$ with only double and triple points is *pointwise 3-admissible* if the only 3-admissible subcombinatorics of $\mathscr{C}$ are isomorphic to $\mathcal{M}_3$.

Remark 4.11. In fact, it can be proved that a combinatorics $\mathscr{C}$ is pointwise 3-admissible if and only if its first resonance variety does not contain non-local components ([18]).

Proposition 4.12. *If $\mathscr{C}$ is a pointwise 3-admissible combinatorics then any $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}})$ induces a permutation ψ_3 of $\mathcal{P}_3$.*

Proof. Let $\mathcal{L} := \{\ell_0, ..., \ell_r\}$ denote the set of lines of $\mathscr{C}$ and let $\mathcal{P}_3 \subset \mathcal{P}$ denote the set of triple points of $\mathscr{C}$. We will first prove that any isomorphism $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}})$ induces a map $\psi_3 : \mathcal{P}_3 \to \mathcal{P}_3$. Consider a triple point $P := \{\ell_i, \ell_j, \ell_k\} \in \mathcal{P}_3$; then $\mathrm{Adm}_\psi(P)$ is an admissible subcombinatorics of $\mathscr{C}$ (Example 4.9(1)) and defines a triple point. The map $\psi_3 : \mathcal{P}_3 \to \mathcal{P}_3$ given by $\psi_3(P) := \mathrm{Adm}_\psi(P)$ is defined.

We will next prove that such a map is indeed injective, and hence a permutation (in order for this to make sense, one has to assume that $\#\mathcal{P}_3 > 1$ and hence $r \geq 4$). Assume ψ_3 is not injective, and let $\psi_3(P_1) = \psi_3(P_2) = Q = \{\ell_u, \ell_v, \ell_w\}$. One has to consider two different cases depending on whether or not P_1 and P_2 share a line:

(1) If P_1, P_2 do not share a line, i.e, $P_1 := \{\ell_{i_1}, \ell_{j_1}, \ell_{k_1}\}$ and $P_2 := \{\ell_{i_2}, \ell_{j_2}, \ell_{k_2}\}$, where all the subscripts are pairwise different. By reordering the columns, let us write $Q = \{\ell_0, \ell_1, \ell_2\}$. Let $A^\psi_{P_1,P_2}$ be the submatrix of A^ψ corresponding to the rows $\{i_1, j_1, k_1, i_2, j_2, k_2\}$. Using (18):

$$A^\psi_{P_1,P_2} := \begin{pmatrix} a_0^{i_1} & a_1^{i_1} & a_2^{i_1} & a_3^{i_1} & \dots & a_r^{i_1} \\ a_0^{j_1} & a_1^{j_1} & a_2^{j_1} & a_3^{i_1} & \dots & a_r^{i_1} \\ a_0^{k_1} & a_1^{k_1} & a_2^{k_1} & a_3^{i_1} & \dots & a_r^{i_1} \\ a_0^{i_2} & a_1^{i_2} & a_2^{i_2} & a_3^{i_2} & \dots & a_r^{i_2} \\ a_0^{j_2} & a_1^{j_2} & a_2^{j_2} & a_3^{i_2} & \dots & a_r^{i_2} \\ a_0^{k_2} & a_1^{k_2} & a_2^{k_2} & a_3^{i_2} & \dots & a_r^{i_2} \end{pmatrix}, \tag{19}$$

where $a_0^{i_\bullet} + a_1^{i_\bullet} + a_2^{i_\bullet} = a_0^{j_\bullet} + a_1^{j_\bullet} + a_2^{j_\bullet} = a_0^{k_\bullet} + a_1^{k_\bullet} + a_2^{k_\bullet}$, $\bullet = 1, 2$. The sublattice K of Σ_5 generated by the columns (mod $\mathbb{1}_6$) should have maximal rank equal to 5. Note that rank (K) equals the rank of the matrix $\overline{A}^\psi_{P_1,P_2}$ obtained by subtracting the last row from the first ones, forgetting the last row and replacing the first column by the sum of the first three:

$$\overline{A}^\psi_{P_1,P_2} = \begin{pmatrix} b_0^{i_1} & b_1^{i_1} & b_2 & b_3 & \dots & b_r \\ b_0^{j_1} & b_1^{j_1} & b_2 & b_3 & \dots & b_r \\ b_0^{k_1} & b_1^{k_1} & b_2 & b_3 & \dots & b_r \\ b_0^{i_2} & b_1^{i_2} & 0 & 0 & \dots & 0 \\ b_0^{j_2} & b_1^{j_2} & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{20}$$

which does not have rank 5.

(2) If P_1, P_2 share a line, say $P_1 := \{i, j_1, k_1\}$ and $P_1 := \{i, j_2, k_2\}$. Then, analogously to the previous case, one obtains a similar matrix to (19) but where the rows i_1 and i_2 are identified, and we proceed in a similar way.

Q.E.D.

Definition 4.13. Three triple points $P, Q, R \in \mathcal{P}$ of a line combinatorics $(\mathcal{L}, \mathcal{P})$ are said to be *in a triangle* if $P \cap Q = \{\ell_1\}$, $P \cap R = \{\ell_2\}$ and $Q \cap R = \{\ell_3\}$ are pairwise different.

Proposition 4.14. *For any $\psi \in \operatorname{Aut}^1(H_{\mathscr{C}})$, $\mathscr{C}$ pointwise 3-admissible, ψ_3 satisfies the following Triangle Property: $\psi_3 : \mathcal{P}_3 \to \mathcal{P}_3$ preserves triangles, that is, if $P_1, P_2, P_3 \in \mathcal{P}_3$ are in a triangle, then $\psi_3(P_1), \psi_3(P_2), \psi_3(P_3)$ are also in a triangle.*

Proof. Let $P_1, P_2, P_3 \in \mathcal{P}_3$ be three triple points in a triangle, $P_1 := \{\ell_i, \ell_j, \ell_k\}$, $P_2 := \{\ell_k, \ell_l, \ell_m\}$, $P_3 := \{\ell_m, \ell_n, \ell_i\}$. Let us assume that $\psi_3(P_1), \psi_3(P_2), \psi_3(P_3)$ are not in a triangle. One has two possibilities, either two of them do not share a line or three of them share a line.

(1) Two of them, say $\psi_3(P_1), \psi_3(P_2)$ do not share a line. After reordering, we can suppose that $\psi_3(P_1) = \{\ell_0, \ell_1, \ell_2\}$ and $\psi_3(P_2) = \{\ell_3, \ell_4, \ell_5\}$. For $\psi_3(P_3)$ there are several possibilities but we may assume $\psi_3(P_3) \subset \{\ell_0, \ell_3, \ell_6, \ell_7, \ell_8\}$. Consider the submatrix $A^\psi_{P_1,P_2,P_3}$ of A^ψ given by the rows of P_1, P_2, P_3. Applying (18) successively to the rows defined by P_1, P_2, P_3 one has:

(21)
$$A^\psi_{P_1,P_2,P_3} := \begin{pmatrix} a_0^i & a_1^i & a_2^i & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0^j & a_1^j & a_2^j & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0 & a_1 & a_2 & a_3^l & a_4^l & a_5^l & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0 & a_1 & a_2 & a_3^m & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & \dots & a_r \\ a_0^n & a_1 & a_2 & a_3^n & a_4 & a_5 & a_6^n & a_7^n & a_8^n & a_9 & \dots & a_r \end{pmatrix};$$

moreover, $a_1^i = a_1$, $a_2^i = a_2$,

$$a_0^i + a_1 + a_2 = a_0^j + a_1^j + a_2^j = a_0 + a_1 + a_2 (\Rightarrow a_0^i = a_0),$$
$$a_3 + a_4 + a_5 = a_3^l + a_4^l + a_5^l = a_3^m + a_4 + a_5 (\Rightarrow a_3^m = a_3)$$

and

$$a_0 + a_3 + a_6 + a_7 + a_8 = a_0^n + a_3^n + a_6^n + a_7^n + a_8^n.$$

As in the proof of Proposition 4.12 we need rank $(\overline{A}^\psi_{P_1,P_2,P_3}) = 5$, where $\overline{A}^\psi_{P_1,P_2,P_3}$ is the matrix obtained from $A^\psi_{P_1,P_2,P_3}$ by subtracting the last row from the first ones and forgetting the last row. We obtain:

(22)
$$\bar{A}^\psi_{P_1,P_2,P_3} := \begin{pmatrix} b_0 & 0 & 0 & b_3 & 0 & 0 & b_6 & b_7 & b_8 & 0 & \dots & 0 \\ b_0^j & b_1^j & b_2^j & b_3 & 0 & 0 & b_6 & b_7 & b_8 & 0 & \dots & 0 \\ b_0 & 0 & 0 & b_3 & 0 & 0 & b_6 & b_7 & b_8 & 0 & \dots & 0 \\ b_0 & 0 & 0 & b_3^l & b_4^l & b_5^l & b_6 & b_7 & b_8 & 0 & \dots & 0 \\ b_0 & 0 & 0 & b_3 & 0 & 0 & b_6 & b_7 & b_8 & 0 & \dots & 0 \end{pmatrix}$$

which cannot have rank 5.

(2) If $\psi_3(P_1), \psi_3(P_2)$ and $\psi_3(P_3)$ have a common line, say ℓ_0, we follow the same strategy and obtain the desired result.

Q.E.D.

Our main goal is to check if ψ_3 is induced by an element of $\operatorname{Aut}(\mathscr{C})$. The next example shows that we need *enough* triangles.

Example 4.15. Note that Proposition 4.12 does not automatically ensure that in general an automorphism of the combinatorics is produced. For instance, consider the combinatorics $\mathscr{C}$ given by the lines $\{0, 1, \ldots, 6\}$, and the following triple points $\{0, 1, 2\}$, $\{2, 3, 4\}$, and $\{4, 5, 6\}$ (the remaining intersections are double points). It is easy to see that such a combinatorics is pointwise 3-admissible. Let $\psi : H_{\mathscr{C}} \to H_{\mathscr{C}}$ be given by the following matrix:

$$A^{\psi} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix},$$

satisfying the admissibility relations. This induces the following maps:

$$\begin{array}{ccc} \mathcal{P}_3 & \stackrel{\psi_3}{\to} & \mathcal{P}_3 \\ \{0,1,2\} & \mapsto & \{0,1,2\} \\ \{2,3,4\} & \mapsto & \{4,5,6\} \\ \{4,5,6\} & \mapsto & \{2,3,4\} \end{array} \qquad \begin{array}{ccc} \operatorname{gr}^1 M_{\mathscr{C}} & \stackrel{\wedge^2\psi}{\to} & \operatorname{gr}^1 M_{\mathscr{C}} \\ x_{0,1} & \mapsto & x_{0,1} \\ x_{2,3} & \mapsto & x_{4,5} \\ x_{4,5} & \mapsto & x_{2,3}, \end{array}$$

where $\operatorname{gr}^1 M_{\mathscr{C}} \cong \langle x_{0,1}\rangle\mathbb{Z} \oplus \langle x_{2,3}\rangle\mathbb{Z} \oplus \langle x_{4,5}\rangle\mathbb{Z}$.

However, the given permutation is not induced by an automorphism of the combinatorics, because the point $\{2, 3, 4\}$ (which is the only one that shares a line with the other two) is not fixed.

We want to apply the previous results to $\mathscr{C}_{\mathrm{RYB}}$. First we will check that $\mathscr{C}_{\mathrm{RYB}}$ is pointwise 3-admissible.

Lemma 4.16. *An admissible subcombinatorics of $\mathscr{C}_{\mathrm{RYB}}$ cannot have lines in both $\mathcal{R}_1 := \{3, 4, 5, 6, 7\}$ and $\mathcal{R}_2 := \{8, 9, 10, 11, 12\}$.*

Proof. Any subcombinatorics of $\mathscr{C}_{\mathrm{RYB}}$ having lines in both $\mathcal{R}_1$ and $\mathcal{R}_2$ verifies the conditions of Example 4.9(3). Q.E.D.

Lemma 4.17. *$\mathscr{C}_{\mathrm{ML}}$ is pointwise 3-admissible.*

Proof. It is not difficult to prove that any combinatorics (with only double and triple points) of less than 6 lines (other than $\mathcal{M}_3$) is not 3-admissible. In Example 4.9(5) it is shown for the whole combinatorics of eight lines. Let us check the remaining cases, that is, six and seven lines. Note that, up to a combinatorics automorphism, there is a unique way to remove one line. There are, however, two possible ways to remove two lines, depending on whether they intersect or not at a triple point. Therefore we only have to check the following cases:

(1) For 7 lines
$\{0,1,2,3,4,5,6\}$:
$\overset{\{3,6\}}{\to} v_3\|v_6 \overset{\{1,6\},\{2,3\}}{\to} v_3\|v_1\|v_2 \overset{\{1,4\}}{\to} v_3\|v_4 \overset{\{4,5\}}{\to} v_3\|v_5$.
(2) For 6 lines, removing two lines intersecting at a triple point
$\{0,2,3,4,5,6\}$: $\overset{\{3,5\}}{\to} v_5\|v_3 \overset{\{2,3\},\{3,6\},\{4,5\}}{\to} v_5\|v_2\|v_6\|v_4$.
(3) For 6 lines, removing two lines intersecting at a double point
$\{0,2,3,4,5,7\}$: $\overset{\{2,3\},\{2,4\}}{\to} v_2\|v_3\|v_4 \overset{\{3,7\},\{4,5\}}{\to} v_2\|v_5\|v_7$.

Q.E.D.

Proposition 4.18. $\mathscr{C}_{\mathrm{RYB}}$ *is pointwise 3-admissible.*

Proof. An immediate consequence of Lemmas 4.16 and 4.17.
Q.E.D.

Remarks 4.19. In order to prove that for any $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}_{\mathrm{RYB}}})$, ψ_3 comes from an element of $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}})$ we need to know more combinatorial properties of $\mathscr{C}_{\mathrm{RYB}}$ and $\mathscr{C}_{\mathrm{ML}}$.

(1) The triple point $\{0,1,2\} \in \mathcal{P}_3$ in $\mathscr{C}_{\mathrm{RYB}}$ is the only one that belongs to 36 triangles.
(2) Any triple point $P \in \mathcal{P}_3$ in $\mathscr{C}_{\mathrm{RYB}}$ except for $\{3,6,7\}$ and $\{8,11,12\}$ satisfies that $\{0,1,2\}$ and P are in a triangle.
(3) Any two triple points in $\mathscr{C}_{\mathrm{ML}}$ sharing a line belong to a triangle.
(4) For any three triple points P_1,P_2,P_3 in $\mathscr{C}_{\mathrm{ML}}$, there exists another triple point Q such that Q,P_i,P_j belong to a triangle ($i,j \in \{1,2,3\}$).

Proposition 4.20. *Let* $\psi \in \mathrm{Aut}^1(\mathscr{C}_{\mathrm{RYB}})$.

(1) $\psi_3(\{0,1,2\}) = \{0,1,2\}$
(2) ψ_3 *either preserves (resp. exchanges) the triple points of* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *in* $\mathscr{C}_{\mathrm{RYB}}$ *inducing an automorphism*
(3) *The action of* ψ_3 *on* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *comes from an automorphism (resp. isomorphism) of their combinatorics.*

Proof. Part (1) is true by Propositions 4.12-4.14 and Remark 4.19(1). By Remark 4.19(2), the points $\{3,6,7\}$ and $\{8,11,12\}$ are either preserved or exchanged. In order to prove (2) and (3), we may suppose that $\psi_3(\{3,6,7\}) = \{3,6,7\}$. Recall that the subcombinatorics defined by $\mathcal{R}_0 \cup \mathcal{R}_1$ is isomorphic to $\mathscr{C}_{\mathrm{ML}}$. Since triangles are preserved by ψ_3 (Proposition 4.14), according to Remark 4.19(3) the images of any two triple points in $\mathcal{R}_0 \cup \mathcal{R}_1$ sharing a line also share a line. This implies, using Remark 4.19(4) again, that the image of any three triple points on $\mathcal{R}_0 \cup \mathcal{R}_1$ sharing a line are also three points sharing a line. Since any line in $\mathcal{R}_1$ or $\mathcal{R}_2$ has at least three triple points, we conclude (2) and (3). Q.E.D.

Proposition 4.21. *Let $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}_{\mathrm{RYB}}})$; then $\psi_3 : \mathcal{P}_3 \to \mathcal{P}_3$ is induced by an automorphism of $\mathscr{C}_{\mathrm{RYB}}$.*

Proof. We will use Proposition 4.20 repeatedly. We can compose ψ with an element of $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}})$ in order to have ψ_3 preserve the triple points in $\mathcal{R}_i$. Recall that $\psi_3|_{\mathcal{R}_0 \cup \mathcal{R}_i}$ comes from an automorphism φ_i of $\mathscr{C}_{\mathrm{ML}}$ which respects $\{0,1,2\}$. Composing again with an element of $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}})$ we may suppose that φ_1 is the identity on $\{0,1,2\}$. It is enough to prove that it is also the case for φ_2. If it is not the case, we may assume (by conjugation with an element of $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}})$) that $\varphi_2(0) = 1$. There are two possibilities to be checked, depending on whether 9 and 10 are fixed or permuted. In both cases, the triple points $\{0,1,2\}$, $\{3,6,7\}$ and $\{8,11,12\}$ are fixed. Using the arguments in the proofs of Propositions 4.12 and 4.14, one can obtain the induced matrices A^ψ with all the admissibility relations, which, modulo $\mathbb{1}_{13}$ do not have a maximal rank in either case. Q.E.D.

Proposition 4.22. *$\mathscr{C}_{\mathrm{RYB}}$ is homologically rigid.*

Proof. Let $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}_{\mathrm{RYB}}})$; by Propositions 4.21 and 4.14, ψ_3 comes from an automorphism of the combinatorics. Composing with the inverse of such an automorphism, we may suppose that $\psi_3 = 1_{\mathcal{P}_3}$. It is hence enough to prove that any isomorphism $\psi \in \mathrm{Aut}^1(H_{\mathscr{C}_{\mathrm{RYB}}})$ that induces the identity on ψ_3 is just $\pm 1_{H_{\mathscr{C}_{\mathrm{RYB}}}}$. From the definition of $\mathrm{Adm}_\psi(P)$, $P \in \mathcal{P}_3$, we deduce the following. Let us fix the j^{th}-column; all the entries in this column corresponding to $P \in \mathcal{P}_3$ such that $j \notin P$ are equal. We deduce from this that we can choose A^ψ such that all the elements outside the diagonal are constant in their column. Adding multiples of $\mathbb{1}_{13}$, we obtain that A^ψ can be chosen to be diagonal. We also know that for each $P \in \mathcal{P}_3$, the diagonal terms corresponding to $\mathcal{P}$ are equal and since any two elements can be joined by a *chain* of triple points, we deduce that all the diagonal terms are equal. Since ψ is an automorphism, they are equal to ± 1 and then $\psi = \pm 1_{H_{\mathscr{C}_{\mathrm{RYB}}}}$. Q.E.D.

Therefore we can prove the main result.

Theorem 4.23. *The fundamental groups of the two complex realizations of Rybnikov's combinatorics are not isomorphic.*

Proof. Let G_+ and G_- be the fundamental groups of $R_{\omega,\omega}$ and $R_{\bar{\omega},\omega}$ respectively. Any isomorphism $\tilde{\psi} : G_+ \to G_-$ will produce an automorphism $\psi : H_{\mathscr{C}_{\mathrm{RYB}}} \equiv G_+/G'_+ \to G_-/G'_- \equiv H_{\mathscr{C}_{\mathrm{RYB}}}$, that is, we can consider $\psi \in \mathrm{Aut}^1(\mathscr{C}_{\mathrm{RYB}})$. By Theorem 4.22, ψ induces an automorphism of $\mathscr{C}_{\mathrm{RYB}}$. Since the identifications $H_{\mathscr{C}_{\mathrm{RYB}}} \equiv G_\pm/G'_\pm$ are made up to the action of $\mathrm{Aut}(\mathscr{C}_{\mathrm{RYB}})$, we may assume that ψ induces $\pm 1_{H_{\mathscr{C}_{\mathrm{RYB}}}}$. Moreover, eventually exchanging $R_{\omega,\omega}$ (resp. $R_{\bar{\omega},\omega}$) for $R_{\bar{\omega},\bar{\omega}}$ (resp. $R_{\omega,\bar{\omega}}$), see Example 1.10 by means of the automorphism given by complex conjugation, we may assume that $\psi = 1_{H_{\mathscr{C}_{\mathrm{RYB}}}}$ (Proposition 4.22). Therefore $\tilde{\psi}$ is a homologically trivial isomorphism between G_+ and G_-, something which is ruled out by Theorem 3.8. Q.E.D.

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On time averaged optimization of dynamic inequalities on a circle

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Abstract.

We analyze the maximum averaged profit for one parameter families of dynamic inequalities and profit densities on a circle. Generic singularities of the profit for stationary strategies are classified. They are shown to be stable.

§1. Introduction

A smooth function F on the tangent bundle TM of a smooth manifold M defines a *dynamic inequality*: a tangent vector $v \in TM$ is an *admissible* velocity of the inequality if $F(v) \leq 0$. We consider only inequalities (called inequalities with *locally bounded derivatives*) such that the set of admissible velocities over any base point of M is compact. We identify the space of inequalities with the space of functions F. In particular, a *family* of inequalities is a family of functions.

An *admissible motion* is an absolutely continuous mapping $t \mapsto x(t)$ of the time axis segment to the manifold M with the derivative $\dot{x}(t)$ belonging to the convex hull of the admissible velocities in the fiber over $x(t)$ (whenever the derivative exists).

Given a continuous *profit density* function $f : M \mapsto R$, an admissible motion $x, x = x(t)$, on the interval $[0, T]$, $T > 0$, provides the *profit*

$$P(T) = \int_0^T f(x(t))dt \quad \text{and the } \textit{averaged profit} \quad A(T) = P(T)/T.$$

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An important well known control problem is to find an admissible motion providing the maximum averaged profit on the *infinite horizon*, that is when $T \to \infty$, [1], [7], [14]. Such a motion is called *optimal.*

V.I.Arnold suggested a new approach to the problem based on the singularity theory methods. He proved that a constant map (=*stationary strategy*): $x(t) = x_0 \in M$ for any t, or periodic motions can be optimal [2] (see also [1], [3]). The case studied by Arnold is a reasonable model for cyclic process with a prescribed trajectory in multidimensional phase space. An example of this process is a motion along the closed route with the velocity depending on a chosen control.

In the present paper we follow this approach and analyze an analog of Arnold's model [2] defining admissible velocities by a dynamic inequality. We classify generic singularities of the maximum averaged profit provided by stationary strategies in one parameter families of dynamic inequalities and profit densities on the circle. We use Γ-*equivalence*: two germs of functions have the same singularity if their graphs are diffeomorphic via a *parameter diffeomorphism*, that is via a difeomorphism, which respects the natural projection to the parameter sending any fiber to a fiber. We prove also the stability of these singularities with respect to small perturbations: an object has stable singularity, if any sufficiently close object has equivalent singularity and the corresponding equivalence diffeomorphism can be taken close to the identity.

For multidimensional parameter or phase space the classification problem remains open.

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§2. Classification of singularities

In this section the main results are stated. We consider only one parameter families of inequalities and densities on the circle. The phase variable and the parameter are denoted by x and p, respectively. A *generic* or *typical* object is an object from an open dense subset of the space of objects endowed with smooth or sufficiently smooth fine topology.

2.1. Stationary domain and shadows

Clearly, a point of the phase space is a stationary strategy if and only if the zero level set of the dynamic inequality over this point contains both non-positive and non-negative velocities. Denote by P_- and P_+ the subsets of this level set which consists of non-positive and non-negative velocities, respectively. Hence, the set of all stationary strategies (=*stationary domain*) of an inequality is the intersection of the images (=*shadows*)$\pi(P_-), \pi(P_+)$ of these subsets parts the natural projection $\pi : (x, \dot{x}, p) \mapsto (x, p)$ along the velocity axis.

We denote stationary domain by S and its intersection with the fiber $p = p_0$ by S_{p_0}.

Using the results of [4], [10], [11], [12], [13] on generic singularities of restrictions of projections to submanifolds and submanifolds with boundary and taking into account that the sets P_- and P_+ have the same boundary, we prove the following

Theorem 2.1. *The germ of the stationary domain of a generic dynamic inequality at any boundary point is fiber diffeomorphic to the germ at the origin of one of the following (eight) sets*

(1) $\quad 1)\, x \geq 0;\ 2_\pm)\, p \geq \pm x^2;\ 3_\pm)\, p \geq \pm|x|;\ 4_\pm)\, x \geq \pm|p|;\ 5)\, x \geq p|p|$

Moreover the stationary domain of generic family is stable.

Remark 1. The theorem holds for a subset in the space of inequality families which is open in fine C^3-topology and dense in fine C^∞-topology.

Theorem 2.1 is proved in Subsection 3.1

2.2. Maximum profit for stationary strategies

The maximum averaged profit A_s for stationary strategies is a solution of the extremal problem

$$A_s(p) = \max_{x \in S_p} f(x, p) \tag{2}$$

over the set of all stationary strategies for parameter value p.

Theorem 2.2. *Any germ of the profit A_s for a generic pair of families of inequalities and profit densities is* Γ *- equivalent to the germ at the origin of one of the eight functions listed in the second column of Table 1.*

Remark 2. The third column of Table 1 contains more precise information on the equivalence used. Singularities 1–5 can be reduced

Table 1

No	Sing.	Eq.	Conditions
1	0	R^+	unique optimal strategy either of type 1) with $f_x \neq 0$ or an interior one of S
2	$\lvert p\rvert$	R^+	unique optimal strategy either of type 4_-) or the competition of two strategies with singularity 1
2_-	$-\lvert p\rvert$	R^+	unique optimal strategy of type 4_+)
3	$p\lvert p\rvert$	R^+	unique optimal strategy of type 1) with $f_x = 0$
4	$\sqrt{p}$	R^+	unique optimal strategy of type 2_+)
5	$0,\ p \geq 0$	R^+	unique optimal strategy of type 3_+)
6	$\begin{cases} 0, & p < 0 \\ 1+\sqrt{p}, & p \geq 0 \end{cases}$	Γ	competition of two strategies with singularity 1 and 4
7	$\begin{cases} 0, & p < 0 \\ 1, & p \geq 0 \end{cases}$	Γ	competition of two strategies with singularity 1 and 5

to normal form by a R^+-equivalence, which is a particular case of Γ-equivalence: the diffeomorphisms acts on each fiber just by a shift depending on a parameter [5].

The fourth column contains description of the type of strategy, and the type of singularity of the stationary domain from Theorem 2.1.

Theorem 2.2 is proved in Subsection 3.2.

§3. Proofs

Here Theorem 2.1 and Theorem 2.2 are proved sequentially.

3.1. Singularities of stationary domain

If a family of inequalities has no stationary strategies then this is also true for any family of inequalities sufficiently close to the given one in the fine C^0-topology.

Consider the case when the stationary domain is not empty. The zero level of a generic family of inequalities is non-critical. It is a smooth (hyper)surface. The restriction τ of the natural projection π along the velocity axis to this level is a proper map due to the imposed "locally bounded derivatives" condition.

J. Mather's theory for restrictions of projections [11] imply that for a generic family of inequalities the map τ is LR-stable . Moreover, it can have only Whitney fold and pleat singularities (as a map between two dimensional manifolds [10]). Hence, in a generic case the set C of critical values of the map τ is either empty or a smooth curve with cusps and transversal self-intersections.

LR-stability and transversality theorems imply that in a generic case the set C and its singularities take typical position with respect to the natural fibering over the parameter space. Consequently, sets C for a generic inequality and an inequality sufficiently close to it are parameter diffeomorphic (via a diffeomorphism close to the identity). For all the detailes which we omit here see [8], where a similar result is proven.

The stationary domain is the intersection of shadows of the sets P_- and P_+. Clearly, for a generic family of inequalities, the boundary of shadows belongs to the union of the set C and the intersection I of the zero section $\dot{x} = 0$ with zero level of the family of inequalities. Similar arguments show that in a generic case the union $C \cup I$ is also stable (with respect to parameter difeomorphisms). So, generic stationary domain and its local singularities are stable. Now we classify these singularities.

According to [10], a generic shadow of two dimensional manifold with boundary near any its boundary point in appropriate smooth local coordinates u, v takes the form of one of the three sets

$$a)\, u \geq 0 \quad \text{or} \quad b)\, v \leq |u|, \quad \text{or else} \quad c)\, v \geq u|u|$$

near the origin.

The first singularity occurs either at a Whitney fold critical point of the map τ outside the zero section $\dot{x} = 0$, or at a regular point of the map τ which belongs to the zero section.

For a generic family of inequalities, the second singularity is a transversal superposition of two singularities of the first type. Finally, the third singularity occurs at a Whitney fold critical point of the map τ which belongs to the zero section $\dot{x} = 0$.

Hence, the first and the third singularities are local (completely defined by the germ at a single point of the zero level of the family of dynamic inequalities), and the second singularity is defined by two germs.

To classify generic singularities of stationary domain one needs to study the singularities of the intersection of shadows of the subsets P_+ and P_-.

These intersections yield singularities a) - c)at the point which belongs to the interior of one of these shadows and to the boundary of the other. For a generic family of inequalities, we get a transversal superposition of these singularities when the point belongs to the boundaries of shadows but its critical inverse images in the zero level of the family are distinct. Due to the dimensions, only a superposition of two singularities of first type with transversal intersection of the boundaries is generic. In this case we obtain the normal form $v \geq |u|$ at the origin (up to a diffeomorphism).

Singularity of the type c) appears simultaneously on the boundaries of shadows of P_- and P_+. In a generic case, this is possible only when the intersection of these shadow are defined by germ of the zero level of the family of inequalities at the same point. Here the boundary of the intersection of shadows is determined by the set I, as it is easy to see. Hence, this gives normal form c) of the stationary domain (up to a diffeomorphism).

Taking into account all possible different generic position with respect to the natural fibering over the parameter of the singularities a) - c) we get exactly the list (1) of the theorem (up to parameter diffeomorphisms).

Theorem 2.1 is proved.

3.2. Singularities of maximum profit

Without loss of generality one can think that some stationary domain with typical singularities from the list (1) is fixed. A boundary point of the domain is called *singular* if at this point the singularity of the boundary is not 1) from this list. Due to Theorem 2.1 singular points of the boundary form a discrete set.

Lemma 3.1. *For a generic pair of one parameter families of inequalities and densities the derivative of the family of densities along the phase variable does not vanish at singular points of the boundary of the stationary domain.*

This lemma follows immediately from the stability of the stationary domain and Thom transversality theorem. It implies

Corollary 1. *For a generic pair of one-parametric families of inequalities and densities a singular point of the boundary of stationary domain does not provide maximum averaged profit for stationary strategies if at this point this domain has singularity* 2_-) *or* 3_-) *from the list (1).*

Corollary 2. *For a generic pair of one parameter families of inequalities and densities and a singular point* (x, p) *of the boundary of stationary domain the germ at the point* p *of the maximum averaged* *profit provided by stationary strategies which are sufficiently close to the strategy* (x, p) *is the germ at the origin of one of the four functions*

$$2_+)\,\sqrt{p};\quad 3_+)\,0,\,p \geq 0;\quad 4_\pm)\,\mp|p|;\quad 5)\,p|p| \tag{3}$$

up to R^+*-equivalence if at this point this domain has singularity* $2_+)$, $3_+)$, $4_\pm)$ *and* 5) *from the list (1), respectively.*

Thus to finish the proof of Theorem 2.2 one needs to study the singularities of the maximum averaged profit for stationary strategies which are provided either by

(a) an interior point of the stationary domain, or

(b) by a boundary point where the domain has singularity of type 1) from the list (1), or else

(c) by the competition of different stationary strategies.

Consider these three cases sequentially. For the first two cases let (x_0, p_0) be the unique stationary strategy providing the profit $A_s(p_0)$.

Case (a). For a generic family f of profit densities with one parameter at least one of the derivatives f_x, f_{xx} and f_{xxx} at any point is not zero due to Thom transversality theorem. So if the strategy (x_0, p_0) is an interior point of the stationary domain then at this point one has to have $f_x = 0$ and $f_{xx} < 0$. Due to continuity of f and closeness of S that implies that near the point p_0 that maximum is provided by the values of the family of densities on the set $f_x = 0$. Due to implicit function theorem near the point (x_0, p_0) this set is smoothly embedded curve $x = X(p)$ with some smooth map X, $X(p_0) = x_0$, due to $f_{xx}(x_0, p_0) < 0$. Hence the germ (A_s, p_0) is the germ of smooth function $f(X(p), p)$ at p_0. So it is R^+-equivalent to the germ of the zero function at the origin.

Case (b). Let the point (x_0, p_0) be the boundary point of the stationary domain with the singularity 1) from the list (1). Again due to Thom transversality theorem at least one of the derivatives f_x and f_{xx} does not vanish in a generic case.

When the derivative $f_x(x_0, p_0)$ is not zero then the germ (A_s, p_0) is the germ of the restriction of the family f to the boundary of the stationary domain near the point (x_0, p_0). Thus as above the germ (A_s, p_0) is the germ of a smooth function and it is R^+-equivalent to the germ of the zero function at the origin.

If the derivative $f_x(x_0, p_0)$ is zero then as above in the case of an interior point the derivative $f_{xx}(x_0, p_0)$ has to be negative. Due to Thom transversality theorem the differential of the restriction of the derivative f_x to the boundary do not vanish at the point (x_0, p_0) in a generic case. Consequently the profit A_s near the point p_0 is the maximum of the restrictions of the density family to the boundary of the stationary domain and to the part of the curve $f_x = 0$ of local maximums of the densities that belongs to the stationary domain.

At the point p_0 this boundary (the curve, respectively) is transversal to the natural fibering over the parameter due to the type 1) of singularity of the boundary (the inequality $f_{xx}(x_0, p_0) < 0$, respectively). Besides these boundary and curve do not tangent at this point because the differential of the restriction of the derivative f_x to the boundary do not vanish at this point. That implies that the profit A_s has singularity at the point p_0 provided by discontinuity of the second derivative, and the germ (A_s, p_0) is R^+-equivalent to the germ of function $p|p|$ at the origin.

Case (c). Due to multi jet transversality theorem in a generic case for a value p_0 of the parameter there can appear competition only of two stationary strategies. Moreover the germ of the problem at one of them s_1 has to provide the singularity 1 from Table 1. For the other strategy s_2 the value of the profit density is either equal or greater then one at the first. Otherwise there is no any competition. Consider these two subcases consequently.

In the first subcase the germ of the problem at the other strategy has to be also of type 1 from Table 1 due to multi jet transversality theorem. Moreover at the value p_0 the derivatives of best profits defined by the germs of the problem at the strategies s_1 and s_2 are different. Hence the competition gives singularity 2 from Table 1 up to R^+-equivalence.

In the second subcase the best averaged profit A_{s_2} for stationary strategies defined by the germ of the problem at the point s_2 can not provide the singularity 1 from Table 1, or the $4_\pm$) and 5) from the list (3). Otherwise there is no any competition of strategies s_1 and s_2 at the point p_0. Thus at the point p_0 the profit A_{s_2} can have up to R^+-equivalence only the singularity either 2_+) or 3_+) from the list (3). Consequently the maximum averaged profit for stationary strategies has at the point p_0 the singularities 6 and 7 from Table 1, respectively.

The stability of singularities of maximum averaged profit for stationary strategies with respect to small perturbations of generic problem follows from transversality theorems.

Finally, the stability of stationary domain up to small perturbation of generic inequality follows from the LR-stability of the map τ [11] and the stability of intersection of zero level of the inequality with the zero section of tangent bundle.

Remark 3. Besides the well-known singularities $|p|$, $\max\{0, 1 + \sqrt{p}\}$ of competition of strategies [2], [5], [8], [9], in the problem studied only one new generic singularity 7 from Table 1 appears. As we see above it is the result of the typical competition of the singularity 1 with the singularity 5 from this table.

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Thom polynomial computing strategies. A survey

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Abstract.

Thom polynomials compute the cohomology classes of degeneracy loci. In this paper we use a simple example to review the core ideas in different—mostly recently found—methods of computing Thom polynomials. Our goal is to show the underlying topology/geometry/algebra without involving combinatorics.

§1. Introduction

Global topology can force singularities to occur. That is, in a family of objects (where the 'object' can be a linear map, a map germ, a differential form, a diagram of maps, a variety, a stable bundle over a variety, etc) some has to be singular *because* of the topology of the family. This global aspect of singularities is encoded by their Thom polynomials.

Let G be a group acting on a vector space V, and let η be a G-invariant subvariety. Then the Poincaré dual of η in equivariant cohomology is called the Thom polynomial of η, denoted by $\mathrm{Tp}_\eta \in H^*_G(V) = H^*_G(\text{point}) = H^*(BG)$. Sometimes η is an open subset of a G-invariant subvariety. Then we define $\mathrm{Tp}_\eta := \mathrm{Tp}_{\bar{\eta}}$. Tracing back this definition one finds the following topological statement: *whenever a fiber bundle $E \to X$ with fiber V and structure group G is given, the cohomology class represented by the preimage S of the η-points under a generic section is equal to the Thom polynomial of the bundle.* That is, if V is the collection of 'objects', G is a natural equivalence on them, η is the collection of 'singular objects' then the mentioned sections are the 'families of objects' over the parameter space X, and S is the locus of points

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where the object is singular. Hence the knowledge of Thom polynomial tells us the (cohomology class of the) locus of the singular points.

The determination of concrete Thom polynomials is often difficult. What makes the case even worse is that Thom polynomial problems come in natural infinite series, and the combinatorial organization of calculating the infinitely many Thom polynomials at the same time often conceals the actual topological method used. The goal of this paper is to survey some Thom polynomial calculational methods without involving combinatorics. Hence we will deal with just one concrete (quite trivial) example, and show the calculation in five different ways.

Let $G = GL_3(\mathbf{C}) \times GL_3(\mathbf{C})$ act on the vector space of 3×3 matrices $V = \mathrm{Hom}(\mathbf{C}^3, \mathbf{C}^3)$, by $(A, B) \cdot M := BMA^{-1}$. Let Σ^2 denote the invariant set of matrices whose corank is 2, i.e. whose rank is 1. We will calculate the Thom polynomial of (the closure of) Σ^2. Hence $\mathrm{Tp} = \mathrm{Tp}_{\Sigma^2}$ is a degree 4 polynomial in $H^*(BGL_2 \times BGL_2) = \mathbf{Z}[A_1, A_2, A_3, B_1, B_2, B_3]$ (degree of the Chern class X_i is i), or what is the same, a degree 4 polynomial in $\mathbf{Z}[a_1, a_2, a_3, b_1, b_2, b_3]$ (degree of the Chern root x_i is 1), symmetric in a_1, a_2, a_3 and in b_1, b_2, b_3. Here $a_1 + a_2 + a_3 = A_1, a_1a_2 + a_1a_3 + a_2a_3 = A_2$, $a_1a_2a_3 = A_3$ and the same for the B's.

Theorem 1.1. *Tp_{Σ^2} is*

$$c_2^2 - c_1c_3, \tag{1}$$

where c_i is the i'th Taylor coefficient of

$$\frac{1 + B_1t + B_2t^2 + B_3t^3}{1 + A_1t + A_2t^2 + A_3t^3},$$

that is Tp_{Σ^2} =

$$B_2^2 - B_2A_1B_1 + B_2A_1^2 - 2B_2A_2 - A_1B_1A_2 + A_2^2 - B_1B_3+$$

$$+A_2B_1^2 + B_1A_3 + A_1B_3 - A_1A_3, \tag{2}$$

or in Chern roots, it is

$$(b_1b_2+b_1b_3+b_2b_3)^2-(b_1b_2+b_1b_3+b_2b_3)(a_1+a_2+a_3)(b_1+b_2+b_3)+\dots \tag{3}$$

In Sections 2-6 we will give 5 proofs. Before that we make two preliminary remarks. One is that the geometric counterpart of giving the Thom polynomial in Chern roots is that we restrict the group action to the maximal torus. Because of the splitting lemma, this does not mean any loss of information about Tp_η. The other remark is that when η happens to be smooth in V (e.g. it is a coordinate subspace), then Tp_η is the Euler class of the representation normal to η in 0. This follows from the definition. Some of the proofs below reduce the computation to this special case.

§2. The restriction equations

In this method, when computing the Thom polynomial of η, one needs to work with the *simpler* orbits (ones not contained in the closure of η). For such a ζ we pick a representative and find its stabilizer subgroup $G_\zeta \subset G$. This inclusion induces a map $BG_\zeta \to BG$ between the classifying spaces, and in turn a homomorphism $f_\zeta : H^*(BG) \to H^*(BG_\zeta)$.

Theorem 2.1. [14, Th. 2.4], [5, Th. 3.2] *Let ζ not be contained in the closure of η. Then the Thom polynomial of η vanishes at f_ζ. Moreover, if the representation satisfies a technical condition (see* [5, 3.4-3.5]*), then in the expected degree, only integer multiples of the Thom polynomial of η satisfy all these vanishing conditions.*

In our situation Σ^0 and Σ^1 play the role of ζ, with representatives the identity matrix and $\mathrm{diag}(1, 1, 0)$, respectively. Now G_{Σ^0} and G_{Σ^1} could be determined explicitly, but we will only compute their maximal tori—this is enough, since $H^*(BG)$ injects into $H^*(BT)$ in general. Thus we will take

$$G_{\Sigma^0} = \{(\mathrm{diag}(x, y, z), \mathrm{diag}(x, y, z)) : x, y, z \in \mathbf{C}^*\},$$

$$G_{\Sigma^1} = \{(\mathrm{diag}(x, y, u), \mathrm{diag}(x, y, v)) : x, y, u, v \in \mathbf{C}^*\}.$$

From these the induced map can be read, as follows:

$$f_{\Sigma^0} : \mathbf{Z}[A_1, A_2, A_3, B_1, B_2, B_3] \to \mathbf{Z}[x, y, z]$$

maps both A_i and B_i to the i'th elementary symmetric polynomial of x, y, z. The map

$$f_{\Sigma^1} : \mathbf{Z}[A_1, A_2, A_3, B_1, B_2, B_3] \to \mathbf{Z}[x, y, u, v]$$

maps A_i to the i'th elementary symmetric polynomial of x, y, u, while maps B_i to the i'th elementary symmetric polynomial of x, y, v.

We need the intersection of the kernels of these two homomorphisms. In fact, one factors through the other, so we only need $\ker f_{\Sigma^1}$, which turns out (Macaulay2) to be an ideal generated by polynomials in degrees

4, 5 and 6. The degree 4 generator, $A_2^2 - A_1A_3 - A_1A_2B_1 + A_3B_1 + \dots$ thus has to be $\pm$ the Thom polynomial of Σ^2. The sign can be determined by the so-called *principal equation* of [5, Th. 3.5], which states that the f_η image of the Thom polynomial of η is the equivariant Euler class of η. In our case

$$G_{\Sigma^2} = \{(\mathrm{diag}(x,u,v), \mathrm{diag}(x,w,z)) : x,u,v,w,z \in \mathbf{C}^*\},$$

and f_{Σ^2} is analogous to the above. The normal slice to Σ^2 at $\mathrm{diag}(1,0,0)$ is the space of matrices whose 1'st row and column is 0. Therefore the equivariant Euler class is $(w-u)(w-v)(z-u)(z-v)$. Computation shows that this is the image of the above polynomial at f_{Σ^2}, so the above polynomial is the sought Thom polynomial.

Remark 2.2. For a reference of this method as well as many applications see [5], [14], [10]. The restriction method is very effective if the representation has finitely many orbits. When dealing with natural infinite series, a connection with various resultant formulas can be established, see [3].

§3. Resolution and integral

In the following method it is assumed that η is a cone in V, and, instead of $\eta \subset V$, we consider the projectivization $\mathbf{P}\eta \subset \mathbf{P}V$. The starting point is looking for an equivariant resolution of $\mathbf{P}\eta$ considered as a map $\varphi : R \to \mathbf{P}V$.

Theorem 3.1. [6, Th. 3.1] *Let $\alpha_i \in H^*(BT)$ be the weights of the representation of G on V. Denote by q the polynomial*

$$\frac{\prod(x+\alpha_i) - \prod \alpha_i}{x}$$

in the equivariant cohomology ring

$$H_G^*(\mathbf{P}V) = \frac{H^*(BG)[x]}{\prod(x+\alpha_i)}.$$

Then the Thom polynomial of η is

$$\int_R \varphi^*(q).$$

In our case $\mathbf{P}\Sigma^2 = \mathbf{P}^2 \times \mathbf{P}^2$ is already smooth, hence $\varphi : R = \mathbf{P}^2 \times \mathbf{P}^2 \to \mathbf{P}^8$ is the Segre embedding. The ring $H_G^*(\mathbf{P}^2 \times \mathbf{P}^2)$ is $H^*(BG)[y,z]$

modulo the two relations $r_1 := \prod_{i=1}^3(y-a_i)$ and $r_2 := \prod_{i=1}^3(z+b_i)$. Since $\varphi^*(x) = y+z$ (x, y, z are the classes of hyperplane sections of $\mathbf{P}^8$ and the two copies of $\mathbf{P}^2$'s, respectively) we have that the Thom polynomial of Σ^2 is

$$\int_{\mathbf{P}^2\times\mathbf{P}^2} \frac{\prod_{i=1}^3\prod_{j=1}^3(y+z-a_i+b_j) - \prod_{i=1}^3\prod_{j=1}^3(-a_i+b_j)}{y+z}.$$

Integration means taking the top coefficient, i.e. the coefficient of yz. Hence the procedure is to consider the integrand above, use the relations r_1, r_2 to reduce its (y,z)-degree to $(1,1)$, and take the coefficient of yz. Note that taking the minimal degree representative in a factor ring is *automatically* done in computer algebra packages, which makes this method very easy to code.

Remark 3.2. For a reference of this method, see [6]. It is most effective if we can find a resolution with simple cohomology ring. In these cases the integration part is often encoded as an interpolation problem, so the combinatorics of divided differences enters the calculations.

§4. Resolution and integral via localization

The method presented in this section is not really a new method, it's rather an improvement of that of Section 3. The novelty is that we compute the integral $\int_R \varphi^*(q)$, which is the Thom polynomial, by localization techniques. This is a vital help when R is more complicated than a projective space or Grassmannian.

We will use the Atiyah-Bott localization formula [1], as follows. Let a torus T act on a manifold with fixed point set the disjoint union of some F_i's. Then the integral of an equivariant cohomology class $\alpha \in H_T^*(M)$ can be 'localized':

$$\int_M \alpha = \sum_i \int_{F_i} \frac{j_i^*\alpha}{e(\nu_i)},$$

where $j_i : F_i \subset M$ is the embedding and ν_i is its normal bundle. When the fixed point set is discrete we can integrate by just "counting":

$$\int_M \alpha = \sum_i \frac{j_i^*\alpha}{e(T_{F_i}M)}. \tag{4}$$

In our case R is $\mathbf{P}^2\times\mathbf{P}^2$, with 9 fixed points $P_{1,1} := ((1:0:0),(1:0:0))$, $P_{1,2} := ((1:0:0),(0:1:0))$, etc. It will be convenient to use a different form of $q \in H_G^*(\mathbf{P}^8)$, namely $q = \frac{-\prod\prod(-a_i+b_j)}{x}$ (recall that $H^*_G(\mathbf{P}V) = H^*(BG)[x]/\prod\prod(x - a_i + b_j))$. Then the term in (4) corresponding to e.g. $P_{1,1}$ is

$$L_{1,1} := \prod_{i=1}^{3}\prod_{j=1}^{3}(-a_i+b_j)\frac{1}{(-a_1+b_1)\cdot(a_2-a_1)(a_3-a_1)(b_2-b_1)(b_3-b_1)}.$$

The Thom polynomial is then the sum of 9 similar terms, or

$$\mathrm{Tp}_{\Sigma^2} = \frac{1}{4}\sum_{\sigma\in S_3\times S_3} L_{\sigma(1,1)}.$$

Remark 4.1. See [7] for a general reference. This method is most effective if we can simplify the resulting sum using algebra (e.g. Lagrange interpolation). In cases like ours, when the resolution is trivial, the localized integral formula coincides with the *Dusitermaat-Heckman* formula.

§5. Gröbner degeneration

The goal of this method is to "perturb" Σ^2 in $\mathrm{Hom}(\mathbf{C}^3, \mathbf{C}^3)$ without changing its Thom polynomial, and eventually degenerate it to another set, whose Thom polynomial is trivial to compute. The first obstacle is that Σ^2 can not be perturbed at all to another G-invariant subset. However, we can restrict the group action to the maximal torus T without losing any Thom polynomial information, and there are lots of T-invariant perturbations.

Let us consider the following example. The torus $GL_1(\mathbf{C})\times GL_1(\mathbf{C})$ acts on $\mathbf{C}^3 = \mathbf{C}\{x, y, z\}$ by $(\alpha, \beta).(x, y, z) = (\alpha^2 x, \beta^2 y, \alpha\beta z)$. Then the cone $xy - z^2$ is invariant. But so is $xy - t\cdot z^2$ for every $t \in \mathbf{R}$. In the $t = 0$ limit case we get $xy = 0$, which is the union of two planes: $x = 0$ with Thom polynomial $(2b)(a+b)$, and $y = 0$ with Thom polynomial $(2a)(a+b)$ (see the last paragraph of the Introduction). It is easy to believe that the perturbation did not change the Thom polynomial, hence the Thom polynomial of the cone is $(2a)(a + b) + (2b)(a + b)$.

What are the "legal" perturbation (where the Thom polynomial does not change), and how to imitate this process when the variety has higher codimension? The theory of Gröbner basis gives an answer (for a general reference for Gröbner basis theory see e.g. [4, Ch. 15]).

Let I be the ideal of the torus-invariant variety X. Fix a term-order, and consider $in(I)$, the ideal generated by the initial terms of polynomials in I. Then the variety (scheme) corresponding to $in(I)$ is a *flat* deformation of X, hence their Thom polynomials are the same. (Well, one has to be a little careful about the multiplicities of the irreducible components of $in(I)$.)

Note that if we have a Gröbner basis f_i of I, then the leading terms of the f_i's generate $in(I)$. In our case

$$I = I(\Sigma^2) = (a_{11}a_{22} - a_{12}a_{13}, a_{11}a_{23} - a_{13}a_{21}, \dots)$$

(the 9 2×2 minors) is given by a Gröbner basis with respect to e.g. the "graded reverse lexicographic" term order generated by $a_{11} > a_{12} > a_{13} > a_{21} > \dots$. Thus

$$in(I) = (a_{12}a_{13}, a_{13}a_{21}, \dots)$$

(the 'antidiagonals' of the 9 2×2 minors). A computer algebra package (e.g. Macaulay2 "primaryDecomposition in(I)") can be used to find the primary decomposition of $in(I)$ which is:

$$(a_{12}, a_{13}, a_{22}, a_{23}), \qquad (a_{12}, a_{13}, a_{23}, a_{31}),$$

$$(a_{13}, a_{21}, a_{23}, a_{31}), \qquad (a_{12}, a_{13}, a_{31}, a_{32}),$$

$$(a_{13}, a_{21}, a_{31}, a_{32}), \qquad (a_{21}, a_{22}, a_{31}, a_{32}).$$

They all describe linear spaces, whose Thom polynomials are obtained by the last remark of the Introduction, hence the Thom polynomial of Σ^2 is the sum of the following polynomials

$$(b_1-a_2)(b_1-a_3)(b_2-a_2)(b_2-a_3), \qquad (b_1-a_2)(b_1-a_3)(b_2-a_3)(b_3-a_1),$$

$$(b_1-a_3)(b_2-a_1)(b_2-a_3)(b_3-a_1), \qquad (b_1-a_2)(b_1-a_3)(b_3-a_1)(b_3-a_2),$$

$$(b_1-a_3)(b_2-a_1)(b_3-a_1)(b_3-a_2), \qquad (b_2-a_1)(b_2-a_2)(b_3-a_1)(b_3-a_2),$$

which turns out to be (3).

Remark 5.1. The theory behind this method is worked out in [11], see also [12]. An advantage is that the Thom polynomial is obtained as a sum with *positive* coefficients, which is sometimes important in enumerative geometry. When working with natural infinite series one meets subtle combinatorics (e.g. the "pipe dreams" of [12]).

§6. Porteous' method of embedded resolution

As a preparation we study Gysin maps associated with Grassmann bundles. Let $E^3 \to X$ be a bundle of rank 3 and $\pi : Gr_2(E^3) \to X$ its associated Grassmann-2 bundle (ie. we replace the fiber over $e \in E$ from E_e to $Gr_2(E_e)$.) The goal is to understand the Gysin map $\pi_!$ on some naturally defined cohomology classes of $Gr_2(E^3)$—namely the Chern monomials of the tautological 2-bundle S on $Gr_2(E^3)$. We claim that $\pi_!(c_{\lambda_1,\lambda_2})(-S) = c_{\lambda_1-1,\lambda_2-1}(-E)$. Here $c_{u,v}$ is the determinant of the matrix $\begin{pmatrix} c_u & c_{u+1} \\ c_{v-1} & c_v \end{pmatrix}$. Moreover, if F is any other bundle on X, and we denote its pullback to $Gr_2(E^3)$ also by F, then $\pi_!(c_{\lambda_1,\lambda_2})(F-S) = c_{\lambda_1,\lambda_2}(F-E)$, for a recent reference see [8, p.43].

With this knowledge we can calculate Tp_{Σ^2} as follows. Consider two 3-bundles E and F over X, and a generic homomorphism h between them. We want to resolve the closure of $\Sigma^2(h) \subset X$. Let $\pi : Gr_2(E) \to X$ be as above and consider the bundles S, E, F over $Gr_2(E)$. Let $\bar{h} : S \to F$ be the composition of the natural map $S \to E$ with the pullback of h. The 0-points of $\bar{h}$ can also be considered as $\Sigma^2(\bar{h})$. Now one fact [Port] is that the genericity of h implies that $\bar{h}$ is transversal to the 0-section of $\mathrm{Hom}(S, F)$, so we know the cohomology class $[\Sigma^2(\bar{h})] = e(\mathrm{Hom}(S, F))$. The other fact [Port] is that π restricted to $\Sigma^2(\bar{h})$ is a resolution of $\Sigma^2(h)$, thus $\pi_![\Sigma^2(\bar{h})] = [\Sigma^2(h)]$, what we want to compute. In the light of the above description of $\pi_!$ we only need to write $e(\mathrm{Hom}(S, F))$ as a linear combinations of $c_{\lambda_1,\lambda_2}(F-S)$'s. The Euler class $e(\mathrm{Hom}(S, F))$ is the product of differences of Chern roots of F and S, which is the same as $c_{3,3}(F-S)$. Hence $\pi_!(e(\mathrm{Hom}(S, F)) = c_{2,2}(F-E)$, which is (1), what we wanted to prove.

Remark 6.1. This method was historically the first, applied in many different situations, see [13], [15], [9] (singularities), works of Pragacz, Fulton, Harris-Tu, Buch and others (algebraic geometry, see [8] for references and e.g. [2] for a recent application). The effective usage of this method requires the handling of the combinatorics of Gysin homomorphisms, Schur and Schubert polynomials, Young tableaux, etc.

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The complex crystallographic groups and symmetries of J_{10}

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Abstract.

We show that finite order symmetries of the function singularity J_{10} give rise to some of complex crystallographic groups listed in [12]. The groups are extracted from the equivariant monodromy of the function. This is the first appearance of affine reflection groups in a singularity context.

A series of papers [8, 9, 10, 5] has related finite order symmetries of simple function singularities to certain finite unitary reflection groups of Shephard and Todd [13]. This paper makes the next step in the same direction: we study finite order symmetries of one of Arnold's parabolic singularities [1, 3, 4], J_{10}, and construct complex crystallographic groups from the relevant monodromy.

Our approach is similar to that introduced in [8]. First of all, the cyclic group action on the homology of a two-dimensional symmetric Milnor fibre splits the homology over $\mathbb{C}$ into character subspaces H_χ. In a number of cases, as classified in [11], this splits the two-dimensional kernel K of the intersection form between two subspaces H_χ corresponding to two distinct conjugate characters: $K = K_{\chi_1} \oplus K_{\chi_2}$. In the corresponding character subspaces in the cohomology, we consider the affine hyperplanes of all 2-cocycles taking a fixed non-zero value on a fixed non-trivial element of K_{χ_i}. The equivariant monodromy on such a hyperplane turns out to be a complex crystallographic group.

Altogether our construction yields seven different affine groups. A question which naturally arises from this paper is that of existence of any version of a crystallographic group discriminant which gives hypersurfaces isomorphic to the discriminants of the symmetric J_{10} functions, similar to the relation between the discriminants of the Shephard-Todd groups and of the symmetric ADE singularities observed in [8, 9, 10].

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The structure of the paper is as follows.

Section 1 introduces the affine complex reflection groups involved. Section 2 recalls, from [11], the list of symmetries of the J_{10} singularities which may lead to monodromy realisations of the groups. Here we also formulate our main result on relating the singularities and crystallographic groups. The result is then proved in Section 3. The proof is based on the consideration of the Dynkin diagrams of the symmetric singularities, which therefore may be treated as an analog of the affine diagrams of Weyl groups for the Shephard-Todd groups concerned. While the diagrams for the invariant functions have been obtained in [11], the diagram of the only equivariant function appearing is constructed in Section 4.

Finally, in Section 5, we give an example of a complex crystallographic group which is not contained in Popov's tables in [12]. This reopens Borel's problem of complete classification of complex crystallographic groups.

§1. The complex crystallographic groups

An *affine reflection* in $\mathbb{C}^n$ is an affine unitary transformation identical on a hyperplane, which is called the *mirror* of the reflection. A group generated by such reflections and having a compact fundamental domain is called *complex crystallographic*. Such groups were classified by V. L. Popov in [12].

Let $L \subset U_n$ be the linear part of a complex crystallographic group W, that is the image of W under the natural map $W \to U_n$. Of course, L must be a Shephard-Todd group. We denote by T the maximal translation subgroup of W. Then W is an extension of L by T:

$$0 \to T \to W \to L \to \{id\}$$

is an exact sequence. Unlike the real case, W may not be the semi-direct product of its linear and translation parts. However, all the groups arising in this paper from our singularity constructions are such products.

We shall now list the groups involved. Mirrors of L will be identified by their normals which we shall call *roots*.

The linear parts of the groups we will need are the Shephard-Todd groups $L = G(6,1,2)$, $G_3(6)$, G_5, G_8, G_{26}, G_{31}, G_{32}. Dynkin diagrams of these groups are given in Figure 1. The vertex set of a diagram there represents a set of generating reflections. Each vertex is a unit root and is marked with the order of the reflection, order 2 traditionally omitted. An edge $a \to b$ is equipped with the hermitian product $\langle a, b\rangle$. As usual, $\omega = e^{2\pi i/3}$. The edge orientation is omitted if the product is real, and there is no edge at all if the roots are orthogonal. We borrow the G_{31} diagram from [10] (cf. [6]). All the other diagrams were constructed using the roots from Table 2 of [12] (see also [6]). The rank of the group G_{31} is 4. The rank of any other group is equal to the number of vertices in its diagram.

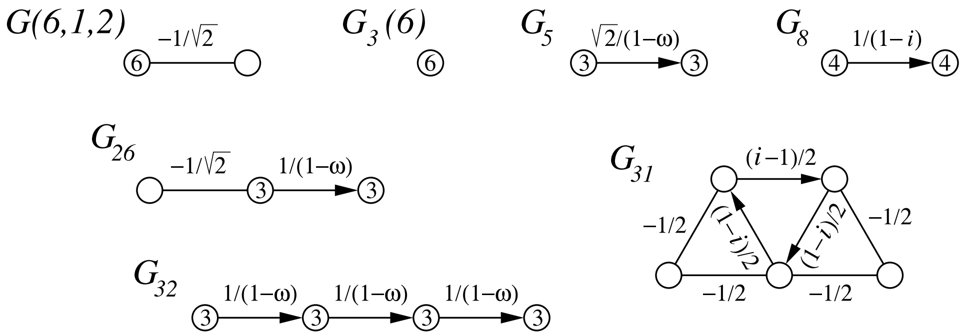

Figure 1. Dynkin diagrams of the Shephard-Todd groups.
All the roots are unit.

The crystallographic group with $L = G_{26}$ arising in our situation will be shown to be Popov's $[K_{26}]_2$. Its translation subgroup T is the lattice spanned by the L-orbit of a non-zero root of any of its order 2 reflections.

The other groups will be Popov's $[G(6,1,2)]$, $[K_3(6)]$, $[K_5]$, $[K_8]$, $[K_{31}]$ and $[K_{32}]$ respectively. For each of them, the lattice T is the span of the L-orbit of any non-zero root of L. For all of these groups, except for $[K_5]$, this leaves no ambiguity in the choice since $L \neq G(6,1,2)$ is transitive on the set of its mirrors, while both obvious possibilities for $G(6,1,2)$ give the same lattice. As for $[K_5]$, when the mirror set consists of two G_5-orbits (the lattice choice between which clearly leads to the same crystallographic group), we will be a bit more specific about the preferable orbit later.

All our crystallographic groups have the conjugate versions, with i and ω replaced by their conjugates. However, the conjugations yield the same groups.

§2. Automorphisms of J_{10}

Now we introduce the singularities we will be dealing with.

Let f be a holomorphic function-germ on $(\mathbb{C}^n, 0)$. Consider a diffeomorphism-germ g of $(\mathbb{C}^n, 0)$ sending the hypersurface $f = 0$ into itself. It multiplies f by a function c not vanishing at the origin. In what follows g will have a finite order, so c is just a constant, a root of unity.

Consider the space $\mathcal{O}(g,c)$ of all holomorphic function-germs on $(\mathbb{C}^n,0)$ multiplied by c under the action of g. The group $\mathcal{R}_g$ of biholomorphism-germs of $(\mathbb{C}^n,0)$ commuting with g acts on $\mathcal{O}(g,c)$. The corresponding equivalence is a geometric equivalence in the sense of Damon [7]. Therefore, the base of an $\mathcal{R}_g$-miniversal deformation of f in $\mathcal{O}(g,c)$ is smooth and such a deformation can be constructed in the standard way [7, 4].

Definition 2.1. An automorphism g of a hypersurface $f = 0$ is called *smoothable* if an $\mathcal{R}_g$-versal deformation of function f contains members with smooth zero sets.

In [11] the list of all smoothable quasihomogeneous automorphisms of all the members of the function family

$$J_{10} : x^3 + ax^2y^2 + xy^4\,, \qquad a \neq 4\,,$$

was obtained. Moreover, a further selection of cases with a potential to yield complex crystallographic groups was carried out in [11].

The selection was based on the construction of an affine reflection group from a semi-definite hermitian form with a one-dimensional kernel, which we briefly mentioned in the introduction. For this, we lift a smoothable automorphism of a J_{10} curve to a smoothable automorphism g of its one-variable stabilisation. As a result, the second homology of the symmetric Milnor fibre in $\mathbb{C}^3$ splits into a direct sum $\oplus_\chi H_\chi$ of character subspaces, so that g acts on an individual summand as a multiplication by the root of unity χ, $\chi^{\mathrm{order}(g)} = 1$.

We want to split the 2-dimensional kernel of the J_{10} intersection form between two different H_χ. On the other hand, since we are going to extract a crystallographic group from the monodromy and since such a group has at least two generators, we need the discriminant of an $\mathcal{R}_g$-miniversal deformation of our function to be at least of multiplicity 2. Smoothable automorphisms of J_{10} satisfying these two requirements were called *interesting* in [11].

In fact, an automorphism g is used just to split the homology and does not affect any monodromy on the summands H_χ obtained. Therefore, we should not distinguish between automorphisms producing same splittings. In particular, we should not distinguish between automorphisms generating the same cyclic group. As it was shown in [11], there are just 8 different interesting symmetries of the J_{10} functions modulo such identifications. We recall them in Table 1. Notice that none of the cases contains the modulus.

The table contains seven invariant and one equivariant ($J_{10}/\mathbb{Z}_4$) singularities. In the table, the *versal monomials* are those whose addition with arbitrary complex coefficients to f gives an $\mathcal{R}_g$-miniversal deformation. In all the cases, their number is equal to the dimension of an H_χ on which the intersection form degenerates. The *affine groups* are those we are going to construct on such character subspaces. The *notation* of the symmetric singularities in the last column is in the spirit of that in [9]. Participation of a Weyl group in the notation (including the Weyl G_2) indicates that the discriminant of a symmetric singularity is that of the Weyl group, and hence the monodromy groups on the H_χ and the corresponding crystallographic group are in fact representations of the relevant generalised braid group, with certain powers of the generators set to be the identities.

Table 1. Symmetric J_{10} singularities

f	$g : x, y, z \mapsto$	$\lvert g\rvert$	versal monomials	kernel χ	affine group	notation
$x^3 + y^6 + z^2$	$\omega x, \omega y, z$	3	$1, y^3, xy^2$	$\omega, \overline{\omega}$	$[G(6,1,2)]$	$J_{10}\vert\mathbb{Z}_3$
	$x, -\overline{\omega}y, z$	6	$1, x$	$-\omega, -\overline{\omega}$	$[K_3(6)]$	$A_2^{(6)}$
	$\omega x, -\omega y, z$	6	$1, xy^2$	$-\omega, -\overline{\omega}$	$[K_3(6)]$	$G_2^{(6)}$
	$\omega x, -y, z$	6	$1, y^2, y^4$	$-\omega, -\overline{\omega}$	$[K_5]$	$B_3^{(3,3)}$
	$x, \omega y, z$	3	$1, x, y^3, xy^3$	$\omega, \overline{\omega}$	$[K_{26}]_2$	$F_4^{(3)}$
	$\omega x, y, z$	3	$1, y, y^2, y^3, y^4$	$\omega, \overline{\omega}$	$[K_{32}]$	$A_5^{(3)}$
$x^3 + xy^4 + z^2$	x, iy, z	4	$1, x, y^4$	$i, -i$	$[K_8]$	$C_3^{(4)}$
	$-x, -y, iz$	4	y, y^3, y^5, x, xy^2	$i, -i$	$[K_{31}]$	$J_{10}/\mathbb{Z}_4$

Theorem 2.1. *Consider an automorphism g of a J_{10} function singularity from the table. Let H_χ be its character subspace in the second homology of a g-symmetric Milnor fibre, on which the intersection form has a non-trivial kernel. Let Γ be a hyperplane in the space dual to H_χ formed by all the*

cohomology classes taking a fixed non-zero value on a fixed element of the kernel. Let π_1 be the fundamental group of the complement to the discriminant in the base of an $\mathcal{R}_g$-versal deformation of the function. Then the monodromy group induced by π_1 on Γ is the complex crystallographic group of the table.

The proof of the theorem is given in the next section. It will use the Dynkin diagrams of the singularities for the subspaces H_χ of the table. The diagrams are given in Figure 2. Their elements represent both the degenerate intersection forms on the H_χ and the relations for the corresponding Picard-Lefschetz operators. Namely:

(1) the vertex set is a distinguished set of vanishing χ-cycles, that is of elements of H_χ which are symmetric analogues of Morse vanishing cycles (see [8, 9, 10] for details, cf. [2]);
(2) beside each vertex the self-intersection number of the χ-cycle is given;
(3) non-orthogonal χ-cycles are joined by an oriented edge labelled with the intersection number similar to how this was done for the group diagrams;
(4) however, the edge orientation is omitted in all the tree diagrams since the χ-cycles are defined up to multiplication by powers of χ and up to a choice of their own orientation (for the same reason each tree diagram serves both conjugate values of χ);
(5) inside each vertex the order of the corresponding Picard-Lefschetz operator is written (order 2 omitted);
(6) the multiplicity of an edge between vertices a and b illustrates the length of the braiding relation between the Picard-Lefschetz operators:
   - commutativity if there is no edge;
   - $h_a h_b h_a = h_b h_a h_b$ if the edge is simple;
   - $(h_a h_b)^2 = (h_b h_a)^2$ for a double edge;
   - $(h_a h_b)^3 = (h_b h_a)^3$ if the edge is triple.

In relation to Theorem 2.1, the Picard-Lefschetz operators will yield the generating affine reflections of the crystallographic groups.

The "skeleton" of a diagram with a Weyl group in the notation is the Dynkin diagram of the Weyl group. This reflects the fact that the discriminant of the symmetric singularity is isomorphic to that of the Weyl group.

For all the invariant cases, the diagrams of Figure 2 were constructed in [11] following the methods of [8, 9, 10] in a very straightforward way. The $J_{10}/\mathbb{Z}_4$ diagram will be obtained in Section 4.

Remark 2.1. The only difference between two character subspaces H_χ for each of the singularities with tree diagrams comes out in the actual Picard-Lefschetz operators. Each of them is a transformation

$$h_a : c \mapsto c - (1-\lambda)\langle c, a\rangle a / \langle a, a\rangle \tag{1}$$

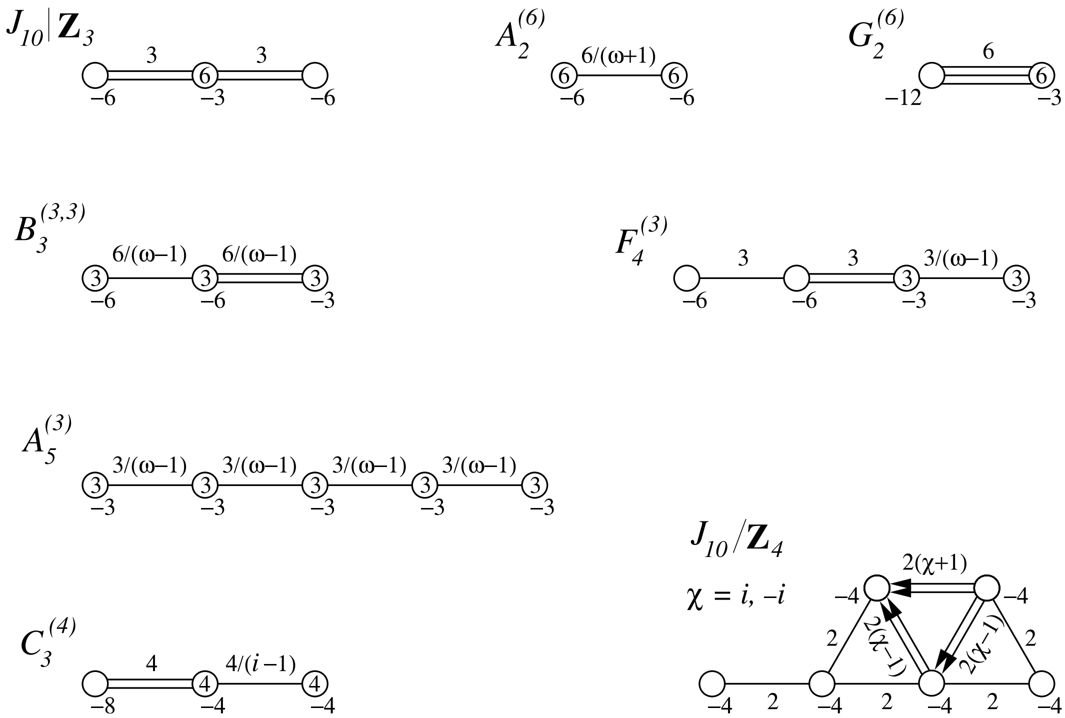

Figure 2. Dynkin diagrams of symmetric J_{10} singularities.

defined not only by its root a, but also by the eigenvalue $\lambda \neq 1$. For the operators of order greater than two, $\lambda = -\chi$ for $J_{10}|\mathbb{Z}_3$ and $B_3^{(3,3)}$, and $\lambda = \chi$ otherwise. Since the characters come in conjugate pairs, this means that the realisations of the crystallographic groups come in conjugate pairs too.

Definition 2.2. If the hermitian form $\langle \cdot, \cdot \rangle$ has a one-dimensional kernel, a transformation (1) will be called a *pseudo-reflection* provided a is not in the kernel.

Remark 2.2. The diagrams $G_2^{(6)}$ and $B_3^{(3,3)}$ are the results of folding of the diagrams $J_{10}|\mathbb{Z}_3$ and $A_5^{(3)}$ in two, similar to how the B_k diagram can be obtained from that of A_{2k-1} (see [2], cf. [8, 9, 10]). This corresponds to the symmetry groups of the second pair of singularities being index two subgroups in the symmetry groups of the first pair.

§3. Proof of Theorem 2.1

We start with some general considerations and then apply their results in a case-by-case study.

3.1. Hermitian forms of corank 1

Let $e_0, e_1, \dots, e_k$ be coordinate vectors in $\mathbb{C}^{k+1}$. Consider a semi-definite hermitian form $\widetilde{q}$ on $\mathbb{C}^{k+1}$ with a one-dimensional kernel K spanned by $\widetilde{\mathbf{a}} = e_0 + a_1 e_1 + \cdots + a_k e_k$, where not all of the constants a_j are zero. In the basis $\{e_j\}$, the form is given by the matrix

$$\widetilde{Q} = (\widetilde{q}_{ij}) = (\widetilde{q}(e_i, e_j)) = \begin{pmatrix} \widetilde{q}_{00} & \overline{Q}_0^T \\ & \\ Q_0 & Q \end{pmatrix}$$

where Q is a $k \times k$ non-degenerate Hermitian matrix, $\widetilde{q}_{00} = \widetilde{q}(e_0, e_0)$ and Q_0 is the column of products of the $e_{i>0}$ with e_0. From the form of the kernel we see that

$$Q_0 = -Q\,\overline{\mathbf{a}} \qquad \text{and} \qquad \widetilde{q}_{00} = \mathbf{a}^T Q\,\overline{\mathbf{a}}$$

where $\mathbf{a} = (a_1, \dots, a_k)^T$ is the *truncated kernel vector*. Matrix Q is actually the matrix of the non-degenerate hermitian form q induced by $\widetilde{q}$ on $\mathbb{C}^{k+1}/K \simeq \mathbb{C}^k$ and written in the basis formed by the projections of the $e_{j>0}$.

Assume that for each of the basic vectors e_j we have a pseudo-reflection on $\mathbb{C}^{k+1}$ with the eigenvalue $\lambda_j \neq 1$:

$$\widetilde{h}_j : c \mapsto c - (1 - \lambda_j)\frac{\widetilde{q}(c, e_j)}{\widetilde{q}(e_j, e_j)} e_j\,.$$

The matrix of the transformation $\widetilde{h}_0$ in our basis is

$$\begin{pmatrix} \lambda_0 & -\beta_0 Q_0^T \\ 0 & I_k \end{pmatrix}, \qquad \text{where} \quad \beta_0 = (1 - \lambda_0)/\widetilde{q}_{00}\,.$$

The matrices of the other $\widetilde{h}_j$ are similarly constructed from the columns of $\widetilde{Q}$ and differ from I_{k+1} in the jth rows only. The pseudo-reflections $\widetilde{h}_{j>0}$ project to $\mathbb{C}^{k+1}/K$ to the reflections h_j preserving the form q there.

Since all the $\widetilde{h}_j$ fix K, in the dual space $\mathbb{C}^{k+1,*}$ of linear functionals on $\mathbb{C}^{k+1}$, the dual operators $\widetilde{h}_j^*$ send each hypersurface formed by all the functionals taking a fixed value on $\widetilde{\mathbf{a}} \in K$ into itself.

Take one of such hyperplanes,

$$\Gamma = \{\alpha_0 + \alpha_1 a_1 + \cdots + \alpha_k a_k = b\} \subset \mathbb{C}^{k+1,*}\,, \qquad b \neq 0,$$

where the α_j are the coordinates dual to those we had on $\mathbb{C}^{k+1}$. Let h_0^* be the restriction of $\widetilde{h}_0^*$ to Γ. Then, in the coordinates $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)^T$ on Γ we have

$$h_0^*(\boldsymbol{\alpha}) = \begin{pmatrix} -\beta_0 Q_0 & I_k \end{pmatrix} \begin{pmatrix} b - \mathbf{a}^T \boldsymbol{\alpha} \\ \boldsymbol{\alpha} \end{pmatrix} = A_0^T \boldsymbol{\alpha} - \beta_0 b\, Q_0$$

where $A_0^T = I_k + \beta_0 Q_0\, \mathbf{a}^T$.

All the $h^*_{j>0}$ are homogeneous in $\boldsymbol{\alpha}$ and their matrices A_j^T are obtained by deleting the first row and first column of the $\widetilde{h}_j$ matrices and then taking the transposes. The deletion here means passing to the matrices of the h_j, and since these reflections preserve the form q, we have

$$A_j^T Q \overline{A}_j = Q \qquad \Longrightarrow \qquad \overline{Q}^{\,-1} = (A_j^T)^T \overline{Q}^{\,-1} (\overline{A_j^T})\,.$$

Thus, the reflections $h^*_{j>0}$ preserve the hermitian form $\boldsymbol{\alpha}^T \overline{Q}^{-1} \overline{\boldsymbol{\alpha}}$ on Γ. It is easily checked that the same is true for the linear part A_0^T of h_0^* and that, moreover, the translation vector of h_0^* is a λ_0-eigenvector of A_0^T. Therefore, h_0^* is indeed an affine reflection.

Let us now pass to the coordinates $\boldsymbol{\alpha}' = Q^{-1}\boldsymbol{\alpha}$ on Γ. Then

$$\boldsymbol{\alpha}^T \overline{Q}^{-1} \overline{\boldsymbol{\alpha}} = \boldsymbol{\alpha}'^T \overline{Q} \overline{\boldsymbol{\alpha}}' \qquad \text{and} \qquad \begin{aligned} h_0^*(\boldsymbol{\alpha}') &= Q^{-1}(A_0^T Q \boldsymbol{\alpha}' - \beta_0 b Q_0) \\ &= \overline{A}_0^{\,-1} \boldsymbol{\alpha}' + \beta_0 b \overline{\mathbf{a}}\,. \end{aligned}$$

Similarly, the matrices of all the other reflections become the $\overline{A}_j^{\,-1}$ now. Thus, in the coordinates $\boldsymbol{\alpha}'$, we have ended up with reflections (one of them, h_0^*, affine) preserving the hermitian form with the matrix $\overline{Q}$.

Conclusion. Omit the leftmost vertex from each singularity diagram of Figure 2. It is easy to see that the subdiagrams obtained produce on the H_χ involved the monodromy groups coinciding with the linear parts L of the crystallographic groups of Section 1. Indeed orienting all non-oriented edges from the left to the right, changing the sign of the hermitian form (this move does not affect any reflections) and dividing the roots by positive numbers to make them unit we immediately get from our subdiagrams to the diagrams of Figure 1.

The only point remaining now for a verification of Theorem 2.1 is to check that the truncated kernel vector in each case is normal to a relevant mirror of L (as in the discussion of the lattices in Section 1). We carry this out in the next subsection.

In the $[K_{31}]$ case the rank 4 group G_{31} is generated by 5 reflections, but this makes no difference in the approach.

Remark 3.1. In terms of the singularities, the vertex omission mentioned above corresponds to the adjacencies of the symmetric J_{10} functions to the symmetric ADE singularities of [8, 9, 10].

3.2. The case-by-case analysis

For all the singularities, we assume that the vertices of the diagrams of Figure 2 are ordered from the left to the right starting with 0 (for $J_{10}/\mathbb{Z}_4$ the 4-valent vertex will be number 5). The components of the truncated kernel vector $\mathbf{a}$ are ordered respectively. The markings of all non-oriented edges are understood as the intersection numbers $\widetilde{q}_{j,j+1} = \langle e_j, e_{j+1}\rangle$. The $A_{j>0}$ are the matrices of the reflections h_j on $\mathbb{C}^k$ corresponding to its basic vectors e_j. Their determinants are assumed to be -1, ω, i, $-\omega$.

The one-dimensional cases $A_2^{(6)}$ and $G_2^{(6)}$ are trivial.

Elementary calculations for the other tree diagrams give the following.

$$\begin{array}{rcclcl}
J_{10}|\mathbb{Z}_3: & \mathbf{a} & = & (2,1) & = & A_1^3 e_2 \\
B_3^{(3,3)}: & \mathbf{a} & = & (\overline{\omega}-1,-2\overline{\omega}) & = & \overline{\omega} A_1^{-1} A_2 e_1 \\
C_3^{(4)}: & \mathbf{a} & = & (2,-1-i) & = & (1+i) A_1^{-1} A_2 e_1 \\
F_4^{(3)}: & \mathbf{a} & = & (2,3,\overline{\omega}-1) & = & A_1 A_2 A_3^{-1} A_2 e_1 \\
A_5^{(3)}: & \mathbf{a} & = & (\overline{\omega}-1,-2\overline{\omega},\overline{\omega}-\omega,\omega) & = & \overline{\omega} A_1^{-1} A_2 A_3 A_4^{-1} A_3 A_2 e_1
\end{array}$$

Notice that the symmetry of the G_5 diagram is destroyed in the $B_3^{(3,3)}$ case as it was promised in Section 1 and assumed by the diagram of Figure 2: the translation lattice of this realisation of $[K_5]$ is spanned by the G_5-orbit of a multiple of e_1, not of e_2.

For $J_{10}/\mathbb{Z}_4$, $\chi = i$, we get $\mathbf{a} = (2,3,2(1-i),1-i)$. According to [13, 6], for the $e_{j>0}$ in $\mathbb{C}^4$ with the diagonal hermitian form $-\sum_{s=1}^4 |z_s|^2$ we can take

$$e_1 = (2,0,0,0)\,, \qquad e_2 = (-1,-1,-1,-1)\,, \qquad e_3 = (0,1+i,0,1+i)\,,$$

$$e_4 = (0,-1-i,1+i,0)\,, \qquad e_5 = (-1,i,-1,i)\,.$$

This gives $\mathbf{a} = (1,-1,-1,1)$ which is also a root of G_{31} [13, 6]. Passing to $\chi = -i$ conjugates $\mathbf{a}$ and the e_j settings, but gives the same affine group since the mirror set of G_{31} is sent by the conjugation into itself.

This finishes the proof of Theorem 2.1.

§4. The $J_{10}/\mathbb{Z}_4$ Dynkin diagrams

We shall now construct Dynkin diagrams for the $J_{10}/\mathbb{Z}_4$ singularity starting with the two-variable case and then passing to three variables. This involves two sets of parallel objects differing just by the absence or presence of the square of an extra variable z. In order not to repeat the definitions and settings twice, all the notations for the 3-variable case will be the same as for two variables, but with the tilde on the top. This will be slightly inconsistent with the notations used in the previous sections, but will not be confusing.

4.1. The plane curve

All through this subsection $g = -id$ will be the central symmetry of $\mathbb{C}^2$, and we shall be working with g-equivariant holomorphic functions on the plane, that is series containing monomials of odd degrees only: $f(-x,-y) = -f(x,y)$.

Starting with a g-equivariant function-germ f with an isolated singularity at the origin, we slightly deform it in a generic (but still equivariant) way to a function f_* with a smooth zero set. Localising this set in an appropriate ball as it is routinely done in singularity theory, we obtain a curve V_*, a symmetric Milnor fibre of the germ f. For a generic line in the function space, to define vanishing cycles on V_*, one naturally takes the family of levels $f_* + \alpha\ell = 0$, where ℓ is a generic linear function on the plane and α a complex parameter. Morse 1-cycles in this family vanish in symmetric pairs, $e_1 = ge_0$. Such a pair defines vanishing χ-cycles in the character spaces of g in $H_1(V_*) = H_{\chi=1} \oplus H_{\chi=-1}$:

$$(2) \qquad e_0 + e_1 \in H_{\chi=1} \qquad \text{and} \qquad e_0 - e_1 \in H_{\chi=-1}\,.$$

Now take a system of paths on $\mathbb{C}_\alpha$ starting at the origin and leading to the critical values of α. Assume they have no mutual- and self-intersections, that is the system is *distinguished*. Then the corresponding distinguished systems of vanishing χ-cycles generate the H_χ [10]. However, these cycles are no longer independent: we get too many critical values of α.

For the $J_{10}/\mathbb{Z}_4$ function, it is convenient to start with a sabirification, that is a perturbation with all critical points real and all saddles on the zero level, rather than with a complete smoothing of the zero set. So we take the one-parameter family

$$f_\alpha = x(x + y^2 + y - 1)(x - y^2 + y + 1) + \alpha y\,.$$

The zero levels for two values of α, zero and sufficiently small positive α_*, are shown in Figure 3. The point α_* will now be our base point. The level $f_{\alpha_*} = 0$ will be denoted V_*, and all the cycles will be constructed in $H_1(V_*)$.

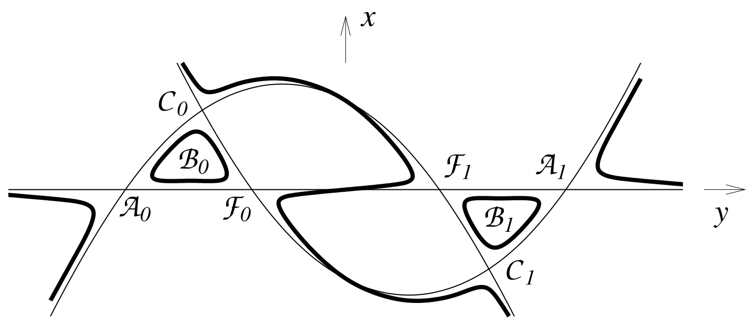

Figure 3. The curve $f_0 = 0$ (thin) and its smoothing $V_* = \{f_{\alpha_*} = 0\}$.

There are four distinct critical values of α: zero (triple), one positive (greater than α_*) and a pair of conjugates with the real part negative (see Figure 4a). The cycles vanishing on V_* along the real paths shown in Figure 4a may be traced in Figure 3. These are the $\mathcal{A}_j$, $\mathcal{C}_j$ and $\mathcal{F}_j$ vanishing at the relevant nodes of the curve $f_0 = 0$, and the ovals $\mathcal{B}_j$. The cycles vanishing along the two remaining paths will be denoted respectively $\mathcal{D}_j$ and $\mathcal{E}_j$, $j = 0, 1$. We assume that the orientations in the pairs are such that the symmetry g interchanges the cycles without affecting the orientation.

Routine calculations of the intersections yield that, within the remaining flexibility in choosing the orientations, the Dynkin diagram for the twelve 1-cycles is the one shown on the left in Figure 5.

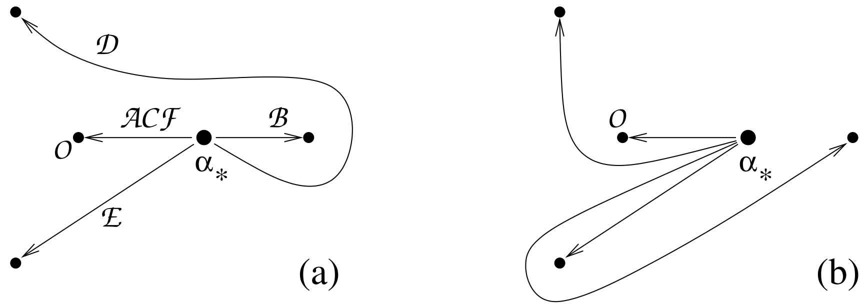

Figure 4. Distinguished path systems in $\mathbb{C}_\alpha$ leading to the critical values of the parameter α.

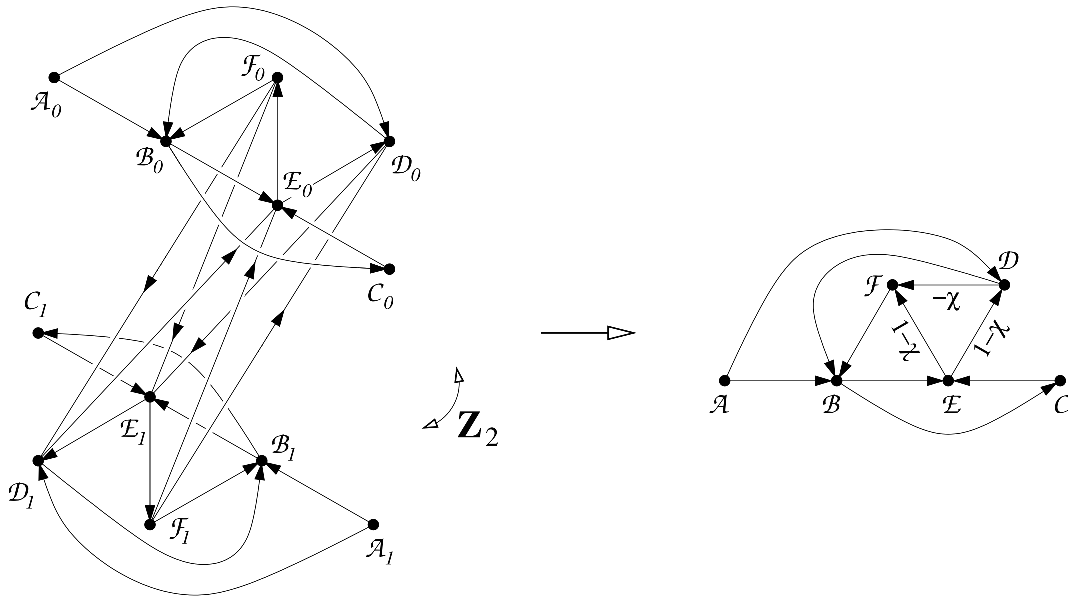

Figure 5. Folding the curve diagram to the intersection diagrams for the character subspaces $H_{\chi=\pm1}$. On the left: an edge $a \to b$ means $\langle a, b\rangle = 1$. On the right: the label on an edge $a \to b$ is $\langle a, b\rangle/2$, with the marking 1 omitted. All the self-intersections are 0.

Passing to the intersections of the χ-cycles as in (2), we fold the 12-vertex diagram in two and obtain the diagram on the right in Figure 5.

To simplify the last diagram, we change the paths in $\mathbb{C}_\alpha$ as shown in Figure 4b. The cycles $\mathcal{B}$ and $\mathcal{D}$ are then transformed by the relevant Picard-Lefschetz operators

$$h_{\mathcal{X}} : \mathcal{Y} \mapsto \mathcal{Y} - \langle \mathcal{Y}, \mathcal{X}\rangle \mathcal{X}/2\,.$$

We also reorient the cycle $\mathcal{C}$, and multiply $\mathcal{F}$ by $-\chi$. The moves provide the diagram of Figure 6 in which the modified cycles are primed.

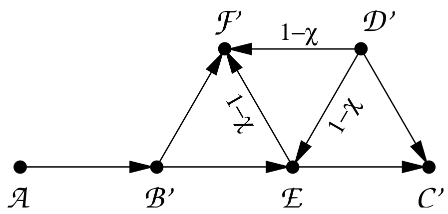

Figure 6. The Dynkin diagram for the $J_{10}/\mathbb{Z}_4$ curve corresponding to the path system of Figure 4b, $\chi = \pm 1$. The conventions are as is in Figure 5 right.

Remark 4.1. Figure 6 suggests that $\mathcal{E} - \mathcal{D}' = \mathcal{F}'$. We shall see why this is indeed the case in the next subsection.

4.2. The surface

This time we have the transformation $\widetilde{g}(x, y, z) = (-x, -y, iz)$. We restrict our attention to the functions $f(x, y) + z^2$ multiplied by $\widetilde{g}$ by -1. Morse 2-cycles of such functions vanish in symmetric pairs again, and we shall order and orient them so that

$$\widetilde{g} : \widetilde{e}_0 \mapsto \widetilde{e}_1 \mapsto -\widetilde{e}_0 \,.$$

Hence we have $H_2(\widetilde{V}_*, \mathbb{C}) = H_{\widetilde{\chi}=i} \oplus H_{\widetilde{\chi}=-1}$ with the summands spanned respectively by the $\widetilde{\chi}$-cycles

$$\widetilde{e}_0 - i\widetilde{e}_1 \qquad \text{and} \qquad \widetilde{e}_0 + i\widetilde{e}_1 \,. \tag{3}$$

For the $J_{10}/\mathbb{Z}_4$ singularity $x^3 + xy^4 + z^2$, we shall now construct the vanishing cycles on $\widetilde{V}_* = \{f_{\alpha_*} + z^2 = 0\}$ from those we obtained on V_* in the previous subsection. For this, we first recall an interpretation of the suspension of the real 1-cycle e on $x^2 + y^2 - 1 = 0$ to the real 2-cycle $\widetilde{e}$ on $x^2 + y^2 - 1 + z^2 = 0$. For this, one considers the family of levels $\phi(x, y) = \beta$ of the function $\phi = x^2 + y^2 - 1$ whose only critical value is -1. Changing β from 0 to -1, we contract the cycle e to a point and, thus, get a thimble $\tau(e)$ on the surface $\{x^2 + y^2 - 1 = \beta\} \subset \mathbb{C}^3_{xy\beta}$. Setting now $\beta = -z^2$ and taking the inverse image of $\tau(e)$ in the xyz-space we get there the 2-cycle $\widetilde{e}$.

Consider now the 2-parameter family of levels $f_0(x, y) + \alpha y = \beta$ (f_0 as in the previous subsection). Let α be close to one of its critical values α' of Subsection 4.1. Consider a Morse 1-cycle c which has nearly vanished on the level $f_0(x, y) + \alpha y = 0$. It is the boundary of the thimble $\tau(c) \subset \{f_0(x, y) + \alpha y = \beta\} \subset \mathbb{C}^3_{x,y,\beta}$ that contracts c to the nearby critical value β' of f_α along the straight path γ from 0 to β' in $\mathbb{C}_\beta$. Let us now move α along the path in $\mathbb{C}_\alpha$ from α' to α_*. This deforms γ to a path in $\mathbb{C}_\beta$ from the origin to the relevant critical value of f_{α_*}. Respectively the thimble $\tau(c)$ becomes the thimble that contracts the cycle c (now brought to V_*) along the new path. Setting $\beta = -z^2$ doubles the new thimble and makes it into a 2-cycle in $\widetilde{V}_*$.

Applying this procedure to each of our twelve 1-cycles, we obtain twelve 2-cycles in $\widetilde{V}_{\alpha_*}$ defined by their equators in V_{α_*} and by paths in $\mathbb{C}_\beta$ leading from the origin to the critical values of f_{α_*}. To make the calculations easier, we better have the paths without mutual- and self-intersections. However, two pairs of the paths must share the same final points as there are just ten critical values. And of course the paths corresponding to a pair of g-symmetric 1-cycles should be centrally symmetric in $\mathbb{C}_\beta$.

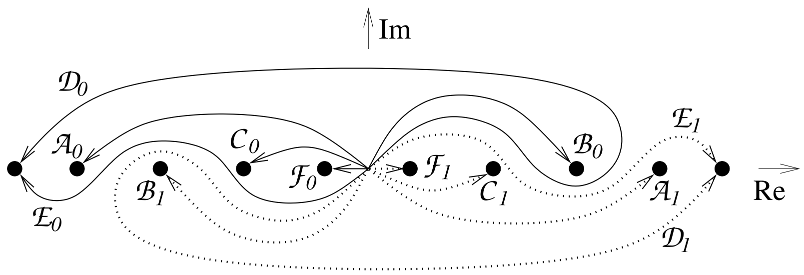

Figure 7. A path system in $\mathbb{C}_\beta$ contracting the cycles on V_* to the critical points of the function f_{α_*}.

A path system in $\mathbb{C}_\beta$ corresponding to the path system of Figure 4a and satisfying all these conditions is shown in Figure 7. To orient the resulting 2-cycles, we first orient the inverse images in $\mathbb{C}_z$ of the paths in $\mathbb{C}_\beta$. The z-paths corresponding to the $\mathcal{X}_0$ will be oriented at the origin by the tangent vectors with the positive real parts, and those for the $\mathcal{X}_1$ by the vectors with the positive imaginary parts. We orient the 2-cycle $\widetilde{\mathcal{X}}_j$ along its equator $\mathcal{X}_j$ by the orientation of $\mathcal{X}_j$ followed by the chosen orientation $z_{\mathcal{X}_j}$ of the z-path. Then

$$\langle \widetilde{a}, \widetilde{b} \rangle = -\langle a, b \rangle \cdot sgn(z_a, z_b)$$

if the two cycles meet only at the equators.

The result of the construction is the Dynkin diagram on the left in Figure 8.

The only intersection numbers in Figure 8 which still need explanation are the $\langle \widetilde{D}_j, \widetilde{E}_j \rangle$ since the cycles meet not just at the equators but at the poles too. To obtain these intersections, we notice that the path system of Figure 7 demonstrates that, in terms of the monodromy operators, the 1-cycles of the 2-variable case satisfy the relations

$$\mathcal{E}_j = h_{\mathcal{F}_j} h_{\mathcal{C}_j} h_{\mathcal{A}_j} h_{\mathcal{B}_j}(\mathcal{D}_j) = \mathcal{D}_j - \mathcal{B}_j + \mathcal{C}_j - \mathcal{F}_j\,. \tag{4}$$

The fact that the sign of $\mathcal{E}_j$ here is plus rather than minus can be easily checked by comparing appropriate intersection numbers. Relations (4), in particular,

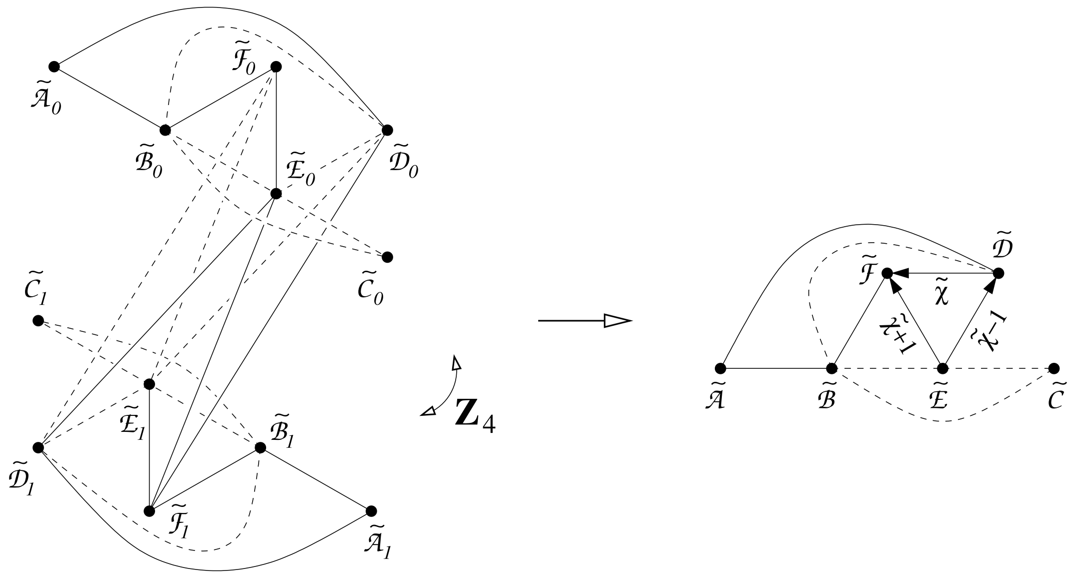

Figure 8. Folding the surface diagram to the intersection diagrams for the character subspaces $H_{\widetilde{\chi}=\pm i}$. On the left: each cycle has the self-intersection -2, a simple (dashed) edge denotes the intersection number 1 (respectively -1). On the right: the self-intersections are -4, the label on an edge is half the intersection number, marking 1 is omitted and marking -1 is presented by a dashed edge.

imply the relation of Remark 4.1. Moreover, from the same figure, we see that the same relations, but with the tildes added everywhere, hold for the 2-cycles we are considering now (the sign choice on the left can be verified like before). This gives us the numbers $\langle \widetilde{D}_j, \widetilde{E}_j \rangle$.

To get the Dynkin diagrams for the homology $H_{\widetilde{\chi}=\pm i}$, we follow the settings of (3) and fold the 12-vertex diagram in two to the diagram on the right in Figure 8, $\widetilde{\chi} = \pm i$.

Switching to the path system of Figure 4b, that is applying appropriate Picard-Lefschetz operators

$$\widetilde{h}_{\widetilde{\mathcal{X}}} : \widetilde{\mathcal{Y}} \mapsto \widetilde{\mathcal{Y}} + \langle \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}} \rangle \widetilde{\mathcal{X}}/2 ,$$

and introducing $\widetilde{\mathcal{F}}' = -\widetilde{\chi}\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{C}}' = -\widetilde{\mathcal{C}}$ afterwards, we end up with the diagram of Figure 9. Bearing in mind the notational difference, we see that the result is exactly the $J_{10}/\mathbb{Z}_4$ diagram of Figure 2, $\widetilde{\chi} = \chi = \pm i$. Relations (4) yield $\widetilde{\mathcal{E}} - \widetilde{\mathcal{D}} = \widetilde{\mathcal{F}}'$.

This finishes the construction.

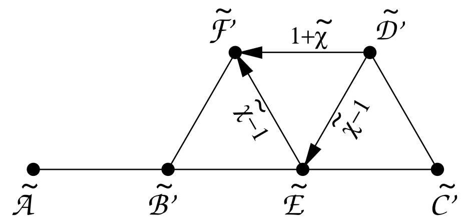

Figure 9. The Dynkin diagram for the $J_{10}/\mathbb{Z}_4$ surface corresponding to the path system of Figure 4b, $\widetilde{\chi} = \pm i$. The conventions are as in Figure 8 right.

Remark 4.2. As usual, one can order the paths of Figure 4 anticlockwise in the order they leave the base point. This provides the Dynkin diagrams of Figures 6 and 9 with a standard ordering of the vertices: $\mathcal{A}\mathcal{C}'\mathcal{F}'\mathcal{D}'\mathcal{B}'\mathcal{E}$ and the same with the tildes.

§5. An extra complex crystallographic group

This group came to our attention when the order of the Picard-Lefschetz operator corresponding to the central vertex of the $J_{10}|\mathbb{Z}_3$ diagram in Figure 2 was mistakenly taken to be 3 in [11] and the constructions of Section 3 were applied to that diagram. The rank 2 group obtained turned out to be complex crystallographic, but not contained in Popov's classification tables in [12]. We describe it now. In the spirit of Popov's notations, the group will be denoted $[G(6,2,2)]^*$.

We start with the Shephard-Todd group $G(3,1,2)$. It acts on $\mathbb{C}^2$, equipped with the hermitian form $|z_1|^2 + |z_2|^2$, by multiplying either coordinate by ω and by swapping z_1 and z_2. Therefore, it is generated by the order 3 reflection r_1 defined by the root u_1 and by the order two reflection r_2 corresponding to the root $u_2 - u_1$ (the u_j are the unit coordinate vectors in $\mathbb{C}^2$).

The group $[G(6,2,2)]^*$ is the result of the addition to $G(3,1,2)$ of the affine reflection

$$r_0 : (z_1, z_2) \mapsto (-z_2, -z_1) + (1,1)\,.$$

The reflection has root $u_1 + u_2$. Therefore, the linear part of the new group is $G(6,2,2)$ [13, 6, 10]. We shall see that the group itself is not a semi-direct product of its linear part and the translation lattice T.

Let us find the maximal translation subgroup T of $[G(6,2,2)]^*$. For this, it will be more convenient to use the transformation

$$R_0 = r_2 r_0 : (z_1, z_2) \mapsto (-z_1, -z_2) + (1,1)\,.$$

instead of r_0. Since R_0 is of order 2, any element of $[G(6,2,2)]^*$ is of the form

$$\phi = a_s R_0 a_{s-1} R_0 \dots R_0 a_2 R_0 a_1\,, \qquad a_1, \dots, a_s \in G(3,1,2)\,,$$

where a_s and a_1 may be the identity. The linear part of ϕ is $(-1)^{s-1}a_s \dots a_1$. Since $-id \notin G(3,1,2)$, for ϕ to be a translation we need $s = 2k+1$ and $a_{2k+1} = (a_{2k} \dots a_1)^{-1}$. Setting $b_j = a_j a_{j-1} \dots a_1$ so that $a_j = b_j b_{j-1}^{-1}$, we get

$$\phi = (b_{2k+1}^{-1} R_0 b_{2k+1} b_{2k}^{-1} R_0 b_{2k})(b_{2k-1}^{-1} R_0 b_{2k-1} b_{2k-2}^{-1} R_0 b_{2k-2}) \\ \dots (b_2^{-1} R_0 b_2 b_1^{-1} R_0 b_1)\,.$$

Hence the lattice T is spanned by the translations of the form $b_2^{-1} R_0 b_2 b_1^{-1} R_0 b_1$, $b_1, b_2 \in G(3,1,2)$. These are translations by the vectors

$$b_2^{-1} R_0 b_2 b_1^{-1} R_0 b_1(0) = b_2^{-1}(t) - b_1^{-1}(t)\,, \qquad t = u_1 + u_2\,.$$

Since the $G(3,1,2)$-orbit of the vector $t = (1,1)$ consists of the nine vectors whose coordinates are 1, ω and $\overline{\omega}$, this gives

$$T = (1-\omega)\mathbb{Z}[u_1, \omega u_1, u_2, \omega u_2]\,. \tag{5}$$

Let us check that $[G(6,2,2)]^*$ has a compact fundamental domain. First of all we notice that the semi-direct product W of $G(3,1,2)$ with the lattice T of (5) is a realisation of the crystallographic group $[G(3,1,2)]_1$ of [12]. On the other hand, let W' be a similar realisation of $[G(3,1,2)]_1$, but with the finer lattice $\mathbb{Z}[u_1, \omega u_1, u_2, \omega u_2]$. Denote by W'' the group generated by W' and $-id$. Since

$$W \subset [G(6,2,2)]^* \subset W''$$

and the two groups on the sides have compact fundamental domains, the same holds for the group in the middle.

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tt^* geometry and mixed Hodge structures

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tt^* geometry is a generalization of variation of Hodge structures (section 2). Also the nilpotent orbits of Schmid and the relation to polarized mixed Hodge structures generalize; part of this is still a conjecture (section 3). tt^* geometry turns up in the unfoldings of holomorphic functions with isolated singularities (section 4).

This short paper is an introduction and a survey. It gives definitions, results, conjectures and references, but no proofs. It follows closely a talk which was given at the conference on *Singularity theory and its applications* (MSJ-IRI2003) in Sapporo, Japan, on 16-25 September 2003.

§1. Motivation and history

An isolated hypersurface singularity comes equipped with a polarized mixed Hodge structure (PMHS) on the middle cohomology of a Milnor fiber [St]. If one considers a semiuniversal unfolding with base space M of such a singularity, one obtains a variation of PMHS's on a subspace of M, the μ-constant stratum. But in fact, the variation of PMHS's extends to a variation of a more general structure on the whole base space.

This structure is called tt^* geometry. The purpose of this paper is to define it and discuss it first in an abstract setting and then in the case of singularities.

tt^* geometry was established more than 10 years ago in the work of Cecotti and Vafa [CV1][CV2][CV3]. They considered moduli spaces of $N = 2$ supersymmetric field theories. A distinguished class of these field theories, the Landau-Ginzburg models, is closely related to singularities. Especially, the unfoldings of quasihomogeneous singularities were studied by Cecotti and Vafa. Their work deserves much more attention from the singularity community.

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tt^* geometry in the semisimple case turned up already in the work on holonomic quantum fields of Jimbo, Miwa, Mori, Sato ([JM] and references there, [CV2]).

Completely independently, a slightly weaker version of tt^* geometry was studied by Simpson [Si1][Si2][Si3] with the notion of harmonic bundles. But his techniques and results seem to be further away from the singularity case than the physicists' work. In his work the base spaces M are compact manifolds.

Sabbah [Sab4] greatly generalized the concepts of Simpson and proved with them a special case of a conjecture of Kashiwara [Ka].

Mochizuki [Mo1][Mo2] built on Sabbah's work and extended it. It seems [Mo2, Remark 1.7] that his results imply the general case of Kashiwara's conjecture.

The idea to generalize variations of Hodge structures in terms of meromorphic connections is also present in the work of Barannikov [Ba].

One way to describe tt^* geometry is in terms of a holomorphic vector bundle $H \to \mathbb{C} \times M$ with a flat connection ∇ on $H|_{\mathbb{C}^* \times M}$ with a pole of Poincaré rank 1 along $\{0\} \times M$, with a flat real structure and a flat pairing with certain properties [He2]. The case $M = \{pt\}$ is explained in section 2.

In the singularity case this is realized by a Fourier-Laplace transformation of the Gauss-Manin system of the unfolding parametrized by M, that means, essentially, by oscillating integrals [DS1][He2].

The same structure was used 20 years ago by K. Saito [SaK] and M. Saito [SaM] to establish on M Frobenius manifold structures. Now tt^* geometry enriches this with a real structure and a hermitian metric. I hope that the interplay of these structures will have many applications, for example on the moduli of singularities, on K. Saito's period maps, on two conjectures about the distribution of the spectral numbers ([He1, ch. 14] and [CV3, ch. 4.3]), on the relation to quantum cohomology and mirror symmetry.

Note added in proof: Conjecture 4.3 has now been proved by C. Sabbah in [Sab5, theorem 4.9]. The proof of most of theorem 3.8 has now been written up in [HS, chapters 6 and 9].

§2. Definitions

Definition 2.1. (a) A (TERP)-structure (Twistor Extension Real Pairing) of weight $w \in \mathbb{Z}$ is a tuple $(H \to \mathbb{C}, \nabla, H'_{\mathbb{R}}, P)$ with

$H \to \mathbb{C}$ a hol. vector bundle;

∇ a flat connection on $H|_{\mathbb{C}^*}$ with a pole of order ≤ 2 at 0;

$H'_{\mathbb{R}} \to \mathbb{C}^*$ a ∇-flat subbundle of $H|_{\mathbb{C}^*}$ of real vector spaces with
$H_z = (H'_{\mathbb{R}})_z \oplus i(H'_{\mathbb{R}})_z$ for $z \in \mathbb{C}^*$;
P a $\mathbb{C}$-bilinear $(-1)^w$-symmetric nondegenerate ∇-flat pairing

$$P : H_z \times H_{-z} \to \mathbb{C} \quad \text{for } z \in \mathbb{C}^*$$

such that

$$P : (H_{\mathbb{R}})_z \times (H_{\mathbb{R}})_{-z} \to i^w\mathbb{R}$$

and such that

$$P : \mathcal{O}(H)_0 \times \mathcal{O}(H)_0 \to z^w\mathcal{O}_{\mathbb{C},0}$$

is nondegenerate.

This generalizes part of a Hodge structure in the following sense. Define $H' := H|_{\mathbb{C}^*}$ and $H^\infty := \{$ global flat manyvalued sections in $H'\}$. It comes equipped with a real subspace $H^\infty_{\mathbb{R}} \subset H^\infty$, a monodromy operator $M_{mon} : H^\infty_{\mathbb{R}} \to H^\infty_{\mathbb{R}}$, and a pairing S (from P, see [He2, 7.2] for the definition). If the pole at 0 is logarithmic (i.e. a pole of order 1), then the pole corresponds to a decreasing M_{mon}-invariant filtration $F^\bullet$ on H^∞ (which encodes the growth at 0 of sections in H). In this sense the pole of order ≤ 2 at 0 generalizes the notion of a (Hodge) filtration $F^\bullet$ on H^∞.

In order to generalize the notion of the filtration $\overline{F^{w-\bullet}}$ in the case of a Hodge structure of weight w, one has to do the following. Define

$$\gamma : \mathbb{P}^1 \to \mathbb{P}^1, \quad z \to \frac{1}{\overline{z}}$$

and define a $\mathbb{C}$-antilinear map

$$\begin{aligned} \tau \quad : \quad & H_z \to H_{\gamma(z)} \qquad \text{for } z \in \mathbb{C}^* \\ & a \mapsto \nabla\text{-flat shift to } H_{\gamma(z)} \text{ of } \overline{z^{-w}a} \end{aligned}$$

(one takes the ∇-flat shift from z to $\gamma(z)$ along the path within $\mathbb{R}_{>0} \cdot z$). Then $\tau^2 = \mathrm{id}$. Glue $H \to \mathbb{C}$ and $\overline{\gamma^* H} \to \mathbb{P}^1 - \{0\}$ with τ to a bundle $\hat{H} \to \mathbb{P}^1$. It is a holomorphic bundle with a pole of order ≤ 2 at ∞. The pole at ∞ generalizes $\overline{F^{w-\bullet}}$.

The condition that the filtrations $F^\bullet$ and $\overline{F^{w-\bullet}}$ are opposite is generalized as follows.

Definition 2.1. (b) A $(TERP(w))$-structure $(H, \nabla, H'_{\mathbb{R}}, P)$ is a (tr.TERP)-structure if $\hat{H} \to \mathbb{P}^1$ is a trivial bundle.

This generalizes the notion of a Hodge structure.

In the case of a (tr.TERP)-structure the fiber H_0 and the space $\Gamma(\mathbb{P}^1, \mathcal{O}(\hat{H}))$ are canonically isomorphic. By construction, the map τ acts on $\Gamma(\mathbb{P}^1, \mathcal{O}(\hat{H}))$ and induces a $\mathbb{C}$-antilinear involution

$$\kappa : H_0 \to H_0.$$

The pairing P gives rise to a $\mathbb{C}$-bilinear symmetric nondegenerate pairing g on H_0,

$$\begin{aligned} g \quad : \quad & H_0 \times H_0 \to \mathbb{C} \\ & (a,b) \mapsto z^{-w} P(\widetilde{a}, \widetilde{b}) \quad \mod z\mathcal{O}_{\mathbb{C},0} \end{aligned}$$

where $\widetilde{a}, \widetilde{b} \in \mathcal{O}(H)_0$ with $\widetilde{a}(0) = a, \widetilde{b}(0) = b$. Then define a hermitian pairing $h := g(.,\kappa.)$ on H_0.

Definition 2.1. (c) A (tr.TERP)-structure is a (pos.def.tr.TERP)-structure if h is positive definite.

This generalizes the notion of a polarized Hodge structure (PHS).

Lemma 2.2. *Let $(H, \nabla, H'_{\mathbb{R}}, P)$ be a (tr.TERP)-structure.*
Then there exist endomorphisms $\mathcal{U} : H_0 \to H_0$ and $\mathcal{Q} : H_0 \to H_0$ such that

$$\nabla_{z\partial_z} = \frac{1}{z}\mathcal{U} - \mathcal{Q} + \frac{w}{2}\,\mathrm{id} - z\kappa\mathcal{U}\kappa$$

on $\Gamma(\mathbb{P}^1, \mathcal{O}(H)) \cong H_0$.

In the case of a (pos.def.tr.TERP)-structure, $\mathcal{Q}$ is a hermitian endomorphism with real eigenvalues symmetric around 0. In the case of a PHS it corresponds to $\bigoplus_p (p - \frac{w}{2})\,\mathrm{id}\,|_{H^{p,w-p}}$. The physicists called $\mathcal{Q}$ *a new supersymmetric index* [CFIV].

One can also define the notion of a variation of (TERP)-structures [He2], in terms of a vector bundle $H \to \mathbb{C} \times M$ with a flat connection ∇ on $H|_{\mathbb{C}^* \times M}$ with a pole of Poincaré rank 1 along $\{0\} \times M$ (generalizing Griffiths transversality), a flat real subbundle $H'_{\mathbb{R}}$ and a flat pairing P.

If one then has at generic parameters (tr.TERP)-structures then h and $\mathcal{Q}$ vary real analytically in a most interesting way.

§3. Generalization of mixed Hodge structures

PMHS's correspond to nilpotent orbits of Hodge structures (theorem 3.2). Conjecture 3.7 below will generalize this correspondence to (TERP)-structures. First we review some facts on PMHS's [Sch][CKS].

Fix a reference PHS $(H^\infty, H^\infty_\mathbb{R}, S, F^\bullet)$ of weight w with Hodge filtration $F^\bullet$ and polarizing form S. The projective manifold

$$\check{D} := \{\text{filtrations } F^\bullet \subset H \mid \dim F^p = \dim F_0^p,\ S(F^p, F^{w+1-p}) = 0\}$$

contains as an open submanifold the classifying space for PHS's

$$D := \{F^\bullet \in \check{D} \mid F^\bullet \text{ is part of a PHS}\}.$$

Fix a nilpotent endomorphism $N : H^\infty_\mathbb{R} \to H^\infty_\mathbb{R}$ with $S(Na, b) + S(a, Nb) = 0$. It gives rise to a unique increasing filtration $W_\bullet$ on $H^\infty_\mathbb{R}$ with $N(W_l) \subset W_{l-2}$ and $N^l : \mathrm{Gr}^W_{w+l} \to \mathrm{Gr}^W_{w-l}$ an isomorphism [Sch, Lemma 6.4].

Definition 3.1. (a) The tuple $(H^\infty, H^\infty_\mathbb{R}, S, N, F^\bullet)$ with $F^\bullet \in \check{D}$ is a *PMHS* of weight w if $F^\bullet Gr^W_k$ is a Hodge structure of weight k, if $N(F^p) \subset F^{p-1}$, and if the induced Hodge structure on the primitive subspace

$$P_{w+l} := \ker(N^{l+1} : Gr^W_{w+l} \to \mathrm{Gr}^W_{w-l-2})$$

is polarized by $S_l := S(., N^l.)$.

(b) The pair $(F^\bullet, N)$ with $F^\bullet \in \check{D}$ *gives rise to a nilpotent orbit* (of Hodge structures) if $N(F^p) \subset F^{p-1}$ and

$$e^{i\xi N} F^\bullet \in D \text{ for } \xi \in \mathbb{C} \text{ with } \mathrm{Re}\xi \gg 0.$$

Theorem 3.2. [Sch][CKS] *The pair $(F^\bullet, N)$ is part of a PMHS $\iff$ it gives rise to a nilpotent orbit.*

The inclusion $\Leftarrow$ is a main consequence [Sch, theorem 6.16] of Schmid's Sl_2-orbit theorem; the inclusion $\Rightarrow$ is proved in [CKS, corollary 3.13]. The theorem gives a very nice geometric characterization of PMHS's.

The definitions 3.3 and 3.6 and conjecture 3.7 will generalize definition 3.1 and theorem 3.2 to (TERP)-structures.

For $x \in \mathbb{R}_{>0}$ define $\pi_x : \mathbb{C} \to \mathbb{C},\ z \mapsto \frac{1}{x} z$.

Definition 3.3. A (TERP)-structure $(H, \nabla, H'_\mathbb{R}, P)$ *gives rise to a nilpotent orbit* if $\pi_x^*(H, \nabla, H'_\mathbb{R}, P)$ is a (pos.def.tr.TERP)-structure for $x \gg 0$. (Here $x \sim e^{\mathrm{Re}\xi}$ in definition 3.1 (b).)

The generalization of PMHS's is much more involved and requires a description of the Stokes structure of the order 2 pole at 0 of (H, ∇).

Consider a (TERP)-structure $(H, \nabla, H'_\mathbb{R}, P)$ of rank n with pole part $\mathcal{U} := [z\nabla_{z\partial_z}] : H_0 \to H_0$ with set of eigenvalues $\{u_1, ..., u_k\}$. In the

following we will always make the assumption, called
(No ramification): *The pair* $(\mathcal{O}(H)_0, \nabla)$ *is formally isomorphic to a sum*

$$\bigoplus_{i=1}^{k} e^{-u_i/z} \otimes \mathcal{R}_i,$$

where $\mathcal{R}_i$ *is a free* $\mathcal{O}_{\mathbb{C},0}$*-module with flat connection with regular singularity at 0.*

Define

$$\begin{aligned}
H'^* &:= \text{ dual bundle to } H' = H|_{\mathbb{C}^*}, \\
D^+ &:= \{z \in \mathbb{C}^* \mid -\frac{\pi}{2} < \arg z < \frac{\pi}{2} + \varepsilon \text{ (mod } 2\pi)\}, \quad (\varepsilon > 0 \text{ small}), \\
D^- &:= \{z \in \mathbb{C}^* \mid \frac{\pi}{2} < \arg z < \frac{3\pi}{2} + \varepsilon \text{ (mod } 2\pi)\}, \\
\mathcal{A}_{\pm}^{\leq 0} &:= \{f \in \mathcal{O}(D^{\pm}) \mid f \text{ has an asymptotic expansion of the type} \\
&\qquad \sum_{\mathrm{Re}(\alpha) \geq \alpha_0} \sum_{p} a_{\alpha,p} z^{\alpha} (\log z)^p\}.
\end{aligned}$$

(See [Mal2, IV.3, page 61] for the definition of the sheaf $\mathcal{A}^{\leq 0}$.) Then the Stokes structure which distinguishes $(\mathcal{O}(H)_0, \nabla)$ in its formal equivalence class can be described by the following splittings (Birkhoff, Hukuhara, Turrittin, Jurkat, Sibuya, Deligne, Malgrange, ... [Mal1][Mal2, IV]):

$$\Gamma^{flat}(D^{\pm}, H'^*) = \bigoplus_{i=1}^{k} \Gamma_i^{\pm}$$

where

$$\Gamma_i^{\pm} := \{\gamma \in \Gamma^{flat}(D^{\pm}, H'^*) \mid \forall\ \omega \in \mathcal{O}(H)_0 \quad \langle \omega, \gamma \rangle \in e^{-u_i/z} \cdot \mathcal{A}_{\pm}^{\leq 0}\}.$$

The (TERP)-structure is said to have *compatible* real structure and Stokes structure if

$$\Gamma_i^{\pm} = \mathbb{C} \cdot (\Gamma_i^{\pm} \cap \Gamma^{flat}(D^{\pm}, H'^*_{\mathbb{R}})).$$

Remark 3.4. In the singularity case (section 4), the oscillating integrals, more precisely, the Fourier-Laplace transform of the Gauss-Manin system of a function, gives rise to such a pair $(\mathcal{O}(H)_0, \nabla)$ (theorem 4.1). The u_i are the critical values of the function. $\Gamma_i^{\pm}$ is the space of complex linear combinations of the Lefschetz thimbles starting at critical points with the (common) critical value u_i. The functions $\langle \omega, \gamma \rangle$ are oscillating integrals. Here the compatibility of real structure and Stokes structure is trivial, because the Lefschetz thimbles are real. The Lefschetz thimbles provide even a $\mathbb{Z}$-lattice which is compatible with the Stokes structure.

Lemma 3.5. *(a) If real structure and Stokes structure are compatible, then $\mathcal{R}_i$ is a (TERP)-structure with regular singularity at 0.*

(b) [He2, 7.3] *It induces in a canonical way a tuple $(H^\infty_{(i)}, H^\infty_{\mathbb{R}(i)}, S_{(i)}, M_{s(i)}, N_{(i)}, F^\bullet_{(i)})$ with $M_{s(i)}$ and $N_{(i)}$ commuting semisimple and nilpotent endomorphisms of $H^\infty_{\mathbb{R}(i)}$ and $F^\bullet_{(i)}$ a decreasing filtration on $H^\infty_{(i)}$.*

Definition 3.6. A (TERP(w))-structure with (No ramification) is a *(mixed.TERP)-structure* if real structure and Stokes structure are compatible and if the regular singular pieces $\mathcal{R}_i$ induce PMHS's $(H^\infty_{(i)}, H^\infty_{\mathbb{R}(i)}, S_{(i)}, M_{s(i)}, N_{(i)}, F^\bullet_{(i)})$ of weight w.

Conjecture 3.7. *A (TERP(w))-structure with (No ramification) is a (mixed.TERP)-structure $\iff$ it induces a nilpotent orbit.*

Theorem 3.8. *(a) $\Rightarrow$ is true.*
Then for $x \to \infty$ the eigenvalues of $\mathcal{Q}$ in $\pi_x^*(H, \nabla, H'_{\mathbb{R}}, P)$ tend to $\bigcup_i \mathrm{Exponents}(\mathcal{R}_i) - \frac{w}{2}$ (definition of $\mathrm{Exponents}(\mathcal{R}_i)$ in [He2, 7.3]*).*

(b) $\Leftarrow$ is true if the (TERP)-structure has a regular singularity (i.e. if its pole part $\mathcal{U}$ is nilpotent).

(c) $\Leftarrow$ is true if $\operatorname{rk} H = 2$.

Part (a) in the case $\mathcal{U}$ semisimple is due to Dubrovin [Du, proposition 2.2], part (a) in the case $\mathcal{U}$ nilpotent is proved in [He2, 7.6]. The whole proof will appear elsewhere.

Some remarks concerning the proof:

(a) Case $\mathcal{U}$ nilpotent: [He2, theorem 7.20], using [CKS, corollary 3.13] and additional estimations.
Case $\mathcal{U}$ semisimple: [Du, proposition 2.2], the case of trivial Stokes structure ($\Gamma_i^+ = \Gamma_i^-$) is simple, the general case is rewritten as a Riemann boundary value problem and is solved with a singular integral equation.
General case: combination of both cases [HS, chapter 9]

(b) [HS, chapter 6], the proof uses [Mo2, theorem 12.1].

(c) The case $\operatorname{rk} H = 2$ and $\mathcal{U}$ semisimple is considered implicitly in [IN] and is reduced there to the radial sinh-Gordon equation

$$(\partial_x^2 + \frac{1}{x}\partial_x)\ \alpha(x) = \sinh\ \alpha(x).$$

Nilpotent orbits correspond to real solutions without singularities for $x \to \infty$. These are analyzed in [MTW] and [IN].

Let $(H, \nabla, H'_{\mathbb{R}}, P)$ be a (TERP)-structure. It is also interesting to look at $\pi_x^*(H, \nabla, H'_{\mathbb{R}}, P)$ for $x \to 0$. Theorem 3.9 below is the analogue for this limit to theorem 3.8 (a)+(b) in the case $\mathcal{U}$ nilpotent. Sabbah [Sab2] defined a tuple $(H^\infty, H^\infty_{\mathbb{R}}, S, M_s, N, F^\bullet_{Sabbah})$ by looking at the behaviour at $z = \infty$ of sections in $\Gamma(\mathbb{C}, \mathcal{O}(H))$ with moderate growth at $z = \infty$.

Theorem 3.9. *This tuple is a PMHS if and only if* $\pi_x^*(H, \nabla, H'_{\mathbb{R}}, P)$ *is a (pos.def.tr.TERP)-structure for* $x > 0$ *close to 0.*

In that case, the eigenvalues of $\mathcal{Q}$ *tend for* $x \to 0$ *to* Exponents(*this PMHS*) $- \frac{w}{2}$.

This result and its proof are close to theorem 3.8 (a)+(b).

The case of (mixed.TERP(w))-structures with $\mathcal{U}$ semisimple is especially nice. Such (mixed.TERP(w))-structures are uniquely characterized by $w \in \mathbb{Z}$, the eigenvalues $u_1, ..., u_n$ of $\mathcal{U}$, and a Stokes matrix $S \in M(n \times n, \mathbb{R})$ with $S_{ij} = 0$ if $i > j$ and $S_{ii} = 1$. Any such data give rise to a (mixed.TERP(w))-structure.

Conjecture 3.10. *If* $S + S^{tr}$ *is positive definite then this is a (pos.def.tr.TERP)-structure for any* $u_1, ..., u_n$.

This conjecture is true in rank 2 because of [MTW][IN]. In the case of the Stokes matrices of the ADE singularities it would follow from conjecture 4.3.

If $S + S^{tr}$ is not positive definite then it depends on the values $u_1, ..., u_n$ whether the (mixed.TERP)-structure is a (pos.def.tr.TERP)-structure. Dubrovin's result [Du, proposition 2.2] says that it is a (pos.def.tr.TERP)-structure if $|u_i - u_j|$ is sufficiently big for all $i \neq j$.

§4. The case of singularities

We consider simultaneously the following two cases.

Case I: $f : (\mathbb{C}^{n+1} \to (\mathbb{C}, 0)$ a holomorphic function germ with an isolated singularity at 0 and Milnor number μ.

Case II: $f : Y \to \mathbb{C}$ a regular function on an affine manifold Y ($\dim Y = n + 1$), such that f has isolated singularities and is M-tame (definition in [NS]); then

$$\mu = \sum_{x \in \ Crit(f)} \mu(f, x).$$

In both cases a semiuniversal unfolding F exists (cf. [DS1] for the meaning of this in case II),

$$\begin{aligned} F \ &: \ B \times M \to \mathbb{C}, \\ F_t \ &: \ B \times \{t\} \to \mathbb{C}, \ t \in M, \end{aligned}$$

with $F_0 = f$, with $M \subset \mathbb{C}^\mu$ a neighborhood of 0, and with (case I) B a small ball in $\mathbb{C}^{n+1}$, respectively (case II) $B = Y \cap ($ large ball in $\mathbb{C}^N)$, where $Y \subset \mathbb{C}^N$ is a closed embedding.

Theorem 4.1. *In both cases one obtains on M a variation of (mixed.TERP)-structures of rank μ from the Fourier-Laplace transform of the Gauss-Manin system of F.*

Some remarks concerning the proof: The study of the Gauss-Manin system [SaK][SaM] and its Fourier-Laplace transform in terms of oscillating integrals [Ph1][Ph2] is classical in case I; part of it is reviewed in [He2, 8.1]. A very careful more algebraic treatment of the Fourier-Laplace transform and of the (TERP)-structures for both cases is given in [DS1]. In [DS2, ch. 6 + Appendix] this is connected with the Lefschetz thimbles and oscillating integrals.

For any fixed parameter t one obtains a (TERP)-structure $(\mathcal{O}(H_t)_0, \nabla)$. The dual bundle $H_t'^*$ is a bundle of linear combinations of Lefschetz thimbles. Evaluating holomorphic sections of H_t on flat sections of $H_t'^*$ gives oscillating integrals.

The compatibility of real structure and Stokes structure is trivial, because Lefschetz thimbles are real. That the local singularities come equipped with PMHS's via the regular singular pieces $\mathcal{R}_i$ is essentially due to Varchenko [Va] and Steenbrink [St][SchSt]. The polarizing form of the PMHS is discussed in [Loe, Cor. 3] and [He1, 10.5+10.6][He2, 7.2+8.1].

The nilpotent orbits of these (mixed.TERP)-structures (theorem 3.8 (a)) have a nice geometric meaning: for $x \in \mathbb{R}_{>0}$

$$\pi_x^*((TERP)(F_t) \cong (TERP)(x \cdot F_t).$$

The real 1-parameter unfoldings $\{x \cdot F_t \mid x \in \mathbb{R}_{>0}\}$ correspond to the orbits in M of $E + \overline{E}$ ($\sim x\partial_x$), where E is the Euler field on M. The flow of $E + \overline{E}$ on M corresponds to the renormalization group flow of the physicists [CV1][CV3].

Above, M is a (small or large) ball in $\mathbb{C}^\mu$. But in [He2, remark 8.5] it is shown that one can extend M to a manifold which is complete with respect to the flow of E and that the variation of (mixed.TERP)-structures extends to this manifold. A part of the following corollary was still a conjecture (1.4 and 8.3) in [He2].

Corollary 4.2. *(of theorem 3.8 (a) and theorem 4.1)*
Going sufficiently far along $E+\overline{E}$ in (this extension of) M, the (TERP)-structures are (pos.def.tr.TERP)-structures.

The following conjecture seems to be a theorem for the physicists in the case of functions which correspond to Landau-Ginzburg models.

Conjecture 4.3. *In case II, the (TERP)-structure of f is a (pos.def.tr.TERP)-structure.*

If the conjecture is true it would give a very distinguished class of (pos.def.tr.TERP)-structures. It would be comparable to the fact that the cohomology of compact Kähler manifolds carries Hodge structures. (One could speculate that the tameness of f at infinity replaces the compactness of Kähler manifolds.) It would give together with conjecture 3.7 and theorem 3.9 a good explanation for all the PMHS's associated to hypersurface singularities.

Consider for f as in case II the family of functions $x \cdot f : Y \to \mathbb{C}$, $x \in \mathbb{R}_{>0}$. Conjecture 4.3 is true for $x \gg 0$ because the (TERP)-structure of f is a (mixed.TERP)-structure and because of theorem 3.8 (a). The other way round, the conjectures 4.3 and 3.7 together would give a new proof that this is a (mixed.TERP)-structure.

Conjecture 4.3 is true for $x > 0$ close to 0 because of theorem 3.9 and because Sabbah's tuple $(H^\infty, H^\infty_{\mathbb{R}}, S, N, F^\bullet_{Sabbah})$ for the (TERP)-structure of f is a PMHS. In [Sab1] Sabbah proved that it is a MHS (see also [Sab2] for a more explicit statement). Recently (not yet available in march 2004) he proved that it is a PMHS.

For example, if $Y = \mathbb{C}^{n+1}$ and f is quasihomogeneous, then the deformations of weight < 1 are all M-tame functions and are parametrized by a space $\mathbb{C}^m$ (for some $m \leq \mu$). With conjecture 4.3 one would obtain a variation of (pos.def.tr.TERP)-structures on $\mathbb{C}^m$. This might be useful for Torelli problems or Schottky problems.

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Thom polynomials

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Abstract.

By (generalized) Thom polynomials we mean universal cohomology characteristic classes that express Poincaré duals to the singularity loci appearing in various context: singularities of maps, hypersurface singularities, complete intersection singularities, Lagrange and Legendre singularities, multisingularities, etc. In these notes we give a short review of the whole theory with a special account of discoveries of last years. We discuss existence of Thom polynomials, methods of their computations, relation between Thom polynomials for different classifications. Some of the theorems announced here are new and their proofs are not published yet. Some of known results acquire a new interpretation.

§1. Introduction

Theorems of the global singularity theory relate global topological invariants of manifolds, bundles, etc. to the geometry of singularities of various differential geometry structures. The classical example is the Poncaré theorem that relates the Euler characteristic of a manifold to the singular points of a generic vector field on it. Many classical relations in algebraic geometry like Rieman-Hurwitz or Plücker formulas for algebraic curves can also be considered as theorems of global singularity theory.

As a separate theory, the global singularity theory appeared in the 60s after R. Thom's observation that the cohomology classes Poincaré dual to the cycles of singularities of smooth maps can be expressed as

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universal polynomials (called later Thom polynomials) in the Stiefel-Whitney classes of manifolds [39]. Though Thom used topological arguments, the computation of particular Thom polynomials have been accomplished in 60–70s using the algebraic geometry methods of blowups, residue intersections, etc. (see references in the review [4]).

The first step towards the general study of multisingularities has been done by S. Kleiman [24, 25]. His theory of multiple points has been constructed entirely in the framework of the intersection theory. For technical reasons, Kleiman's formulas can be applied if the map admits only singularities of corank 1 (or if the singularities of corank greater than 1 can be ignored by dimensional reasons) but even in this case Kleiman's theory found many interesting applications [6, 7].

Topological methods in the study of global properties of singularities have been developing independently by two groups. In Moscow, Vassiliev [41] inspired by the ideas of Arnold [2] has created the theory of characteristic classes for real Lagrange and Legendre singularities. He introduced the universal complex of singularity classes which allows one to chose those singularity classes in the real problems for which Poincaré dual cohomology class is well defined. The Vassiliev universal complex has been generalized by M. Kazarian [17, 18, 19, 20] to the characteristic spectral sequence which contains all cohomological information about adjacencies of singularities.

About the same time in Budapest A. Szücs developed his theory of cobordisms of maps with prescribed collections of allowed singularities [35, 36, 37]. He constructed classifying spaces for this type of cobordisms by gluing the classifying spaces of the symmetry groups of various singularities. Following Szücs, R. Rimányi [31, 32, 33] showed that the collection of allowed singularities can be extended at least to the set of all stable map germs. He noticed also that the gluing construction provides an effective method for computing Thom polynomials which does not require the detailed geometric study of the singularities. He demonstrated the efficiency of his restriction method by computing Thom polynomials for essentially all classified singularities.

The two approaches have been developing quite independently until the Oberwolfach Conference 2000 in Singularity theory where A. Szücs introduced the author to his theory. Since that time the two approaches combined providing a very strong counterpart to the methods of intersection theory. It is clear now that the global theory of multisingularities is related to the cobordism theory in the same way as the global theory of monosingularities is related to the theory of characteristic classes of vector bundles. The universal formulas for the classes of multisingularities have been obtained in [22]. These formulas allowed the author to solve in a unified form many enumerative problems that have not been solved by the methods of algebraic geometry.

The discoveries of the last years changed the face of the global singularity theory dramatically. Although the construction of the theory is not completed yet, its general pattern seems to be more or less clear. In these notes I present my own view of the modern state of the theory. Some of the theorems announced here are new and their proofs are not published yet. Some of known results acquire a new interpretation. I have tried to present main formulas in the form ready to be applied to specific enumerative geometric problems. They may provide rich experimental material for further research, even without rigorous justification.

The paper is organized as follows. In Section 2 we discuss the existence theorems for Thom polynomials in different context: singularities of maps (Sect. 2.1 and 2.2), Lie group action (Sect. 2.3), stable $\mathcal{K}$-singularities (Sect. 2.4), isolated hypersurface singularities (Sect. 2.5), and multisingularities (Sect. 2.6 and 2.7). In Sect. 2.8 we discuss some aspects of global singularity theory which are common for all these classifications.

Section 3 is devoted to the detailed study of the structure of Thom polynomials. We introduce the notion of a localized Thom polynomial which is the usual Thom polynomial written in a special additive basis well adjusted to the classification of singularities by corank. It allows one to single out the terms in the Thom polynomial for which closed formulae could be given. In Sect. 3.1 and 3.2 we present formulae for localized terms of Thom polynomials related to singularities of maps. In Sect 3.3 and 3.4 we extend this computation to the case of Lagrange, Legendre, and isolated hypersurface singularities.

Some terms of the Thom polynomials can be computed using the method of resolution of singularities discussed in Section 3. The simplest way to compute remaining terms is to apply the restriction method suggested by Rimányi. It is discussed in Section 4. The method itself is described in Sect. 4.1. In Sect. 4.2 we explain some details of the extension of this method to the study of multisingularities. The results of computations are presented in Sect. 4.3. More complete tables of computed Thom polynomials are available in [23].

§2. Existence of Thom polynomials

2.1. Characteristic classes of singularities

In the most general form the problems of the global singularity theory are often formulated in the following way. Suppose we are given a parameter (or moduli) space M whose points parameterize some geometric objects: varieties, maps, fields, configurations, etc. Generic parameter values correspond to non-degenerate objects. These values form an open subspace $M_0 \subset M$. The complement $M \setminus M_0$ consists of degenerate objects. It is stratified according to the possible degeneracy types. Local singularity theory studies local behavior of these degenerations, normal forms, adjacencies, etc. Denote by $M(\alpha) \subset M$ the closure of the locus of points with prescribed degeneracy type α. Then the problem is to find the cohomology class

$$[M(\alpha)] \in H^*(M)$$

Poincaré dual to the cycle $M(\alpha)$. For example, if $M(\alpha)$ consists of finite number of points, then the problem is just to find the number of these points.

The general answer to this problem suggested by singularity theory is as follows. To each classification problem $\mathcal{S}$ one associates an appropriate 'classifying space' $B\mathcal{S}$. The classifying space is equipped with the natural stratification. The strata of this stratification are labelled by various singularity classes of the classification.

Consider the underlying manifold M of a particular geometric problem. Assume that the degeneracies associated with the points of M are classified with respect to the given classification $\mathcal{S}$. Then one constructs the 'classifying map'

$$\varkappa : M \to B\mathcal{S}$$

such that the partition on M is induced from the partition on $B\mathcal{S}$ by the map $\varkappa$.

The cohomology ring $\mathcal{C}(\mathcal{S}) = H^*(B\mathcal{S})$ of the classifying space is considered as the ring of 'universal characteristic classes' associated with the classification $\mathcal{S}$. The characteristic homomorphism

$$\varkappa^* : \mathcal{C}(\mathcal{S}) \to H^*(M).$$

is a topological invariant of M. It usually can be computed independently of the study of the singularities of the stratification on M.

Then the general principle says:

each singularity type α determines a universal characteristic class $\mathrm{Tp}_\alpha \in \mathcal{C}(\mathcal{S})$ so that the class $[M(\alpha)]$ is given by this class evaluated at the given parameter space M:

$$[M(\alpha)] = \varkappa^*(\mathrm{Tp}_\alpha).$$

The class Tp_α expressed in terms of the multiplicative generators of the ring $\mathcal{C}(\mathcal{S})$ is referred to as the (generalized) *Thom polynomial* of the singularity α. It can be defined simply as the cohomology class $\mathrm{Tp}_\alpha \in H^*(B\mathcal{S}) = \mathcal{C}(\mathcal{S})$ Poincaré dual to the closure of the stratum of the singularity α on the classifying space.

Thus the solution of the initial problem consists of the following steps.

Step 1. Identify the singularity theory problem $\mathcal{S}$ that reflects the classification of points on M.

Step 2. Determine the ring of universal characteristic classes $\mathcal{C}(\mathcal{S})$ corresponding to this classification problem.

Step 3. Find the Thom polynomial $\mathrm{Tp}_\alpha \in \mathcal{C}(\mathcal{S})$ for a particular singularity type α.

Step 4. Compute the characteristic homomorphism $\varkappa^* : \mathcal{C}(\mathcal{S}) \to H^*(M)$ and the required cohomology class $[M(\alpha)] = \varkappa^*(\mathrm{Tp}_\alpha) \in H^*(M)$.

Every step in this program is usually non-trivial and can be done independently. The aim of these notes is to show how this program can be accomplished in various particular geometric problems of counting singularities and multisingularities of maps, complete intersection singularities, Lagrange and Legendre singularities, critical points of functions, etc.

It is known that the same types of local singularities can appear in a stable way in quite different situations. For example, the famous 'swallowtail' singularity A_k could appear in the context of critical point function singularities, complete intersection singularities, caustic, wave front singularities, and many others. Therefore, the choice of a classification problem is not well formalized and can vary according to the preferences of the author. The variety of known classifications in singularity theory is enormous. Some of them appearing in complex problems, in a sense, the basic ones, are listed in Table 1. One can notice that even for well-studied classifications the final answer for the topology of the classifying space is not evident at all. The detailed explanation of the entries of this table is discussed in the main body of the paper.

2.1. Example. Consider a nonsingular projective subvariety $V \subset \mathbb{C}P^d$, $\dim V = r$. We study the tangency singularities of V with respect to various s-dimensional projective subspaces. All these subspaces form the Grassmann manifold $N = G_{s+1,d+1}$. Denote by $M \subset V \times G_{s+1,d+1}$ the incidence subvariety formed by the pairs (x, λ) such that $x \in \lambda$ and let

$$f : M \to N, \quad (x, \lambda) \mapsto \lambda$$

be the natural projection to the second factor. If V satisfies certain genericity condition, then the map f possesses only standard singularities studied in singularity theory. Thus the singularity theory in

Table 1. **Characteristic classes in singularity theory**

Classification $\mathcal{S}$	Classifying space $B\mathcal{S}$	Characteristic classes, $H^*(B\mathcal{S})$
$(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$	$BU(m) \times BU(n)$	$\mathbb{Z}[c_1, ..., c_m, c_1', ..., c_n']$
Orbits of G-action on an affine space V	BG	Characteristic classes of G-bundles
Stable $\mathcal{K}_\ell$-classification of map germs, $\ell = n - m$	BU	$\mathbb{Z}[c_1, c_2, \dots]$
Classification of critical points; IHS	Stable Lagrange (Legendre) Grassmannian $\Lambda \quad (\Lambda^{\mathrm{leg}})$	Lagrange (Legendre) characteristic classes
Multisingularities	Classifying space of complex cobordisms $\Omega^{2m} MU(m+\ell)$, $m \to \infty$	Landweber-Novikov operations $U^{2\ell}(\cdot) \to H^*(\cdot)$
Any classification	'Generalized Pontryagin-Thom-Szücs construction'	Splitting $H^*(B\mathcal{S}) = \bigoplus_\alpha H^*(BG_\alpha)$

question is the classification of map germs $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$, where $m = \dim M = r + s(d-s)$ and $n = \dim N = (s+1)(n-d)$.

By a *singularity class* we mean any non-singular semialgebraic (not necessary closed) subvariety in the k-jet space of map germs $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ which is invariant with respect to the coordinate change of the source and the target manifolds, where k is some large integer. The singularity class may consist of a unique orbit of the group of left-right equivalence, or it may contain a continuous family of non-equivalent orbits.

The ring of characteristic classes for this classification is the ring of polynomials in two groups of variables $c_1, \dots, c_m$ and $c'_1, \dots, c'_n$. The characteristic classes corresponding to these variables are the corresponding Chern classes of the manifolds M and N,

$$\varkappa^* c_i = c_i(M), \quad \varkappa^* c'_j = f^* c_j(N).$$

R. Thom formulated the following general statement.

2.2. Theorem ([39, 13]). *The cohomology class on $H^*(M)$ dual to the closure of a particular singularity α is given by a universal polynomial Tp_α in the classes $c_1(M), \dots, c_m(M), f^*c_1(N), \dots, f^*c_n(N)$.*

The proof and the computation of particular Thom polynomials is discussed below. The theorem above holds also in the real case with $\mathbb{Z}_2$-cohomology and the Stiefel-Whitney classes instead of Chern classes. It holds also in the algebraic situation for an arbitrary algebraically closed ground field and Chow groups instead of cohomology. It seems that the algebraic geometry proof that does not relay on topological arguments has never been published. Therefore, we present it here.

2.2. Proof of the existence theorem for Thom polynomials

The standard topological argument used in the proof of Theorem 2.2 is as follows. On the first step we notice that the class $[M(\alpha)]$ is a pull-back of the class dual to the corresponding singularity locus in the jet bundle space $J^k(M,N)$ under the jet extension map $j^k f : M \to J^k(M,N)$. The natural projection $J^k(M,N) \to M \times N$ has contractible fibers

$$V = J^k_0(\mathbb{C}^m, \mathbb{C}^n)$$

and the map $j^k f$ lifts the graph map $\Gamma_f = \mathrm{id} \times f : M \to M \times N$. Therefore, it suffices to prove that the Poincaré dual of the locus $J^k(M,N)(\alpha)$ in the cohomology ring $H^*(J^k(M,N)) = H^*(M \times N)$ is given by a universal polynomial in the Chern classes $p_1^* c_i(M)$ and $p_2^* c_j(N)$, where p_i, $i = 1, 2$, is the projection of $M \times N$ to the corresponding factor. Remark that as a result we have obtained a reformulation of the existence theorem for Thom polynomials that does not involve the original map f at all.

On the second step we notice that $J^k(M,N)$ forms a fiber bundle space over $M \times N$ whose structure group G of k-jets of left-right changes is homotopy equivalent to the group of linear changes,

$$G = J^k_0 \mathrm{Diff}(\mathbb{C}^m) \times J^k_0 \mathrm{Diff}(\mathbb{C}^n) \simeq GL(m, \mathbb{C}) \times GL(n, \mathbb{C}) \simeq U(m) \times U(n).$$

Therefore, this bundle is a pull-back of the corresponding classifying bundle BV over the classifying space BG of G-bundles:

$$\begin{tikzcd}
& J^k(M,N) \arrow[r, "\varkappa"] \arrow[d, "V"'] & BV \arrow[d, "V"'] \\
M \arrow[ur, "j^k f"] \arrow[r, "\Gamma_f"] & M \times N \arrow[r] & BG
\end{tikzcd}$$

$$\begin{tikzcd}
H^*(M) & H^*(J^k(M,N)) \arrow[l, "j^k f^*"'] \arrow[d, equal] & H^*(BV) \arrow[l, "\varkappa^*"'] \arrow[d, equal] \\
& H^*(M \times N) & H^*(BG)
\end{tikzcd}$$

$$\begin{tikzcd}
{[M(\alpha)]} & {[J^k(M,N)(\alpha)]} \arrow[l, mapsto, "j^k f^*"'] & \mathrm{Tp}_\alpha \arrow[l, mapsto, "\varkappa^*"']
\end{tikzcd}$$

Respectively, the cohomology class $[J^k(M,N)(\alpha)]$ under consideration is the pull-back of the corresponding class in $H^*(BV)$. Thus, Tp_α is a universal characteristic class

$$\mathrm{Tp}_\alpha \in H^*(U(m) \times U(n)) = \mathbb{Z}[c_1, \ldots, c_m, c'_1, \ldots, c'_n]$$

and Theorem follows.

Let us show how the topological argument above could be adjusted to the algebraic situation. We need to indicate explicitly an algebraic model for the classifying space BG and an algebraic replacement for the classifying map $\varkappa$.

Fix some K large enough and set $BV_K = G'_{m,K} \times G''_{K-n,K}$, where

—$G'_{m,K}$ is the variety of all k-jets of germs at the origin of non-singular m-dimensional submanifolds in $\mathbb{C}^K$;

—$G''_{n,k}$ is the variety of k-jets of germs at the origin of non-singular foliations in $\mathbb{C}^K$ with n-codimensional fibers.

It is easy to see that BV_K is a non-singular quasiprojective variety. There is a natural projection $BV_K \to G_{m,K} \times G_{K-n,K}$ sending an m-submanifold to its tangent space at the origin and a foliation to the tangent space of the fiber at the origin. This projection is a fibration whose fibers are affine spaces. Therefore,

$$H^*(BV_K) \simeq H^*(G_{m,K} \times G_{K-n,K}),$$

and this stabilizes to the polynomial ring $\mathbb{Z}[c_1, \dots, c_m, c'_1, \dots, c'_n]$ with the growth of K. Similar statement clearly holds for Chow groups as well.

To each point of BV_K one associates the singularity of the natural projection from the m-dimensional submanifold to the n-dimensional parameter space of the fibers of the foliation. More precisely, only k-jet of this singularity is well defined. Thus, for each singularity type α one associates the corresponding locus $BV_K(\alpha)$ in BV_K.

Let us *define* the Thom polynomial Tp_α as the class of the closure of $BV_K(\alpha)$ in the cohomology (or Chow) group of BV_K. By definition, the element Tp_α is independent of K if K is large enough and is expressed as certain polynomial in the generators c_i, c'_j.

To complete the proof of the algebraic version of Theorem 2.2 we need, for any two given non-singular quasiprojective varieties M and N, to construct a map $\varkappa$ from $J^k(M,N)$ to BV_K that classifies singularities. In the algebraic context such a map can not be constructed in general, but it can be constructed after a suitable modification of the source $J^k(M,N)$.

Denote by $\bar{J}^k_K(M,N)$ the variety whose points are parameterized by the tuples (x,y,j,p) where $x \in M$ and $y \in N$ are some points, j is the k-jet of a map germ $(M,x) \to (\mathbb{C}^K, 0)$, and p is the k-jet of a map germ $(\mathbb{C}^K, 0) \to (N,y)$. Since the choices for j and p form an affine space we get that the cohomology (or Chow) groups of $\bar{J}^k_K(M,N)$ are isomorphic to those of $M \times N$. Passing to the composition $p \circ j : (M,x) \to (N,y)$ determines a natural morphism $\bar{J}^k_K(M,N) \to J^k(M,N)$.

Now, denote by $J^k_K(M,N) \subset \bar{J}^k_K(M,N)$ the open subvariety formed by the tuples (x,y,j,p) such that *j is injective and p is surjective.* Remark that the complement $\bar{J}^k_K(M,N) \backslash J^k_K(M,N)$ has codimension growing to infinity together with K. Therefore, passing to $J^k_K(M,N)$ does affect the cohomology (or Chow groups) in any specified in advance finite range of dimensions, if K is large enough. Over $J^k_K(M,N)$ we have the evident classifying map $\varkappa : J^k_K(M,N) \to BV_K$. This map associates with the injection i and surjection p the (k-jet of the) submanifold $j(M)$ and the foliation formed by the fibers of p, respectively. Thus we get the diagram of mappings

$$M \times N \longleftarrow J^k(M,N) \longleftarrow \bar{J}^k_K(M,N) \longleftarrow J^k_K(M,N) \xrightarrow{\varkappa} BV_K$$

and the induced diagram of homomorphisms in cohomology. Since the first three arrows induce an isomorphism, the universality of the Thom polynomial Tp_α follows from functorial properties of the pull-back homomorphism.

2.3. Remark (cf. [40]). The concept of the classifying space is commonly used in topology. For example, any complex vector bundle $E \to M$ can be induced from the classifying one over BU by some continuous map $\varkappa : M \to BU$. Therefore, from the topological point of view the Chern classes are just pull-backs of certain properly chosen generators in the cohomology ring of the classifying space. In algebraic setting this definition can not be used directly and the Chern classes are usually introduced by means of a direct geometric construction, see e.g. [10]. Let us show that the idea of the classifying map can be applied to the algebraic situation as well.

Let $\bar{M}_K$ be the total space of the bundle $\mathrm{Hom}(E, \mathbb{C}^K)$ and $M_K \subset \bar{M}_K$ be the open submanifold formed by those maps $f_x : E_x \to \mathbb{C}^K$, $x \in M$, which are injective. Let $\pi : M_K \to M$ be the natural projection. The maps f_x provide an embedding of the pull-back bundle $E_K = \pi^* E$ over M_K to the trivial bundle $\mathbb{C}^K$. Thus the fibers of E_K can be treated as n-dimensional subspaces in $\mathbb{C}^K$, where $n = \mathrm{rk}\, E$. This provides the classifying map $\varkappa : M_K \to G_{n,K}$. As above, we get the diagram of maps

$$M \longleftarrow \bar{M}_K \longleftarrow M_K \xrightarrow{\varkappa} G_{n,K}$$

The first two arrows induce isomorphisms in cohomology for K large enough. Therefore, $\varkappa$ can be used to define Chern classes via the characteristic homomorphism

$$\varkappa^* : H^d(G_{n,K}) \to H^d(M_K) \simeq H^d(M), \quad K \gg d.$$

2.3. Classifying space for Lie group action

Many classification problems in singularity theory can be formulated as the classification of orbits of a smooth action of a given Lie group G on a given manifold V. For example, in the case of the classification of map germs $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ up to the right-left equivalence one has

$$V = J_0^k(\mathbb{C}^m, \mathbb{C}^n)$$

is the space of k-jets of map germs at the origin and

$$G = J_0^k \mathrm{Diff}(\mathbb{C}^m) \times J_0^k \mathrm{Diff}(\mathbb{C}^m)$$

is the group of k-jets of changes of coordinates in the source and target manifolds, respectively, where k is a fixed sufficiently large integer.

In what follows we always assume that *the manifold V is topologically trivial*, that is, as a manifold it is isomorphic to an affine space of appropriate dimension. The theory of linear representations provides a lot of examples of such actions. Remark that in the case of left-right equivalence of map germs the action is not linear.

The construction for the classifying space BV associated with the classification of G-orbits on V and its cohomology group $H^*(BV)$ repeats the well-known Borel's construction of G-equivariant cohomology $H^*_G(V)$. Since V is a contractible topological space, we have

$$H^*(BV) = H^*_G(V) \simeq H^*_G(\mathrm{pt}) = H^*(BG).$$

In other words, the group of characteristic classes associated with the classification of G-orbits on V is actually the group of characteristic classes of G-bundles.

In more details, consider the classifying principle G-bundle $EG \to BG$. It means that EG is a topologically trivial space equipped with the free G-action, and BG is the orbit space of this action. It is well known that the cohomology ring of the classifying space $H^*(BG)$ serves as the ring of characteristic classes of G-bundles. Consider the diagonal G-action on the product space $V \times EG$.

2.4. Definition. *The classifying space of* (G, V)*-action* is defined as the orbit space

$$BV = V \times_G BE = (V \times BE)/G.$$

The fibers of the natural projection $\pi : BV \to BG$ are isomorphic to V. Therefore, BV can be interpreted as the total space of the bundle over BG with the fiber V and the structure group G associated with the principle G-bundle $EG \to BG$. Since π has contractible fibers, it induces the mentioned above isomorphism $H^*(BV) \simeq H^*(BG)$.

The spaces EG, BG, BV are usually infinite-dimensional. In practice, it is more convenient to replace them by smooth finite-dimensional approximations. Namely, consider the sequence of smooth principle G-bundles $EG_K \to BG_K$ with $K \to \infty$, such that the manifold EG_K is K-connected. Then the manifolds $BV_K = V_K \times_G EG$ can be considered as finite-dimensional approximations for the classifying space BG and the isomorphisms

$$H^p(BV_K) \simeq H^p(BG_K) \simeq H^p(BV) \simeq H^p(BG)$$

hold in the stable range of dimensions (that is for any fixed p and large enough K).

The partition $V = \bigcup \alpha$ by the orbits determines the corresponding partition $BV = \bigcup B\alpha$ of the classifying space. If $\alpha \subset V$ is an orbit, we

set
$$B\alpha = \alpha \times_G EG \subset BV.$$

2.5. Definition. The *symmetry group* G_α of the orbit $\alpha \subset V$ is the stationary subgroup of any representative $x \in \alpha$ (this group is independent, up to an isomorphism, of the point $x \in \alpha$).

In singularity theory the orbit α is considered as a singularity class of the given classification, and the fixed representative $x \in \alpha$ is its 'normal form'.

2.6. Lemma. *The stratum $B\alpha \subset BV$ corresponding to the orbit α is homotopy equivalent to the classifying space BG_α of its symmetry group.*

Proof. By definition, we have
$$B\alpha = (\alpha \times EG)/G = (\{x\} \times EG)/G_\alpha.$$

The group G_α acts free on $(\{x\} \times EG) \simeq EG$. Since this space is topologically trivial, it can serve as the total space of the principle classifying G_α-bundle. Therefore, $(\{x\} \times EG)/G_\alpha \simeq BG_\alpha$. $\square$

The lemma can be reformulated by saying that *the classifying space BV is glued from the classifying spaces of symmetry groups of various orbits*,

$$BV = \bigcup_\alpha BG_\alpha. \tag{1}$$

2.7. Remark. In singularity theory one considers various *stable* classification problems which are not reduced to the study of a unique Lie group action. Among those are the stable classification of $\mathcal{K}$-singularities, the classification of critical points of functions with respect to the stable $\mathcal{R}$- or $\mathcal{K}$-equivalence, the classification of multisingularities etc. *With a proper definition of the symmetry group, the statement above about the gluing of the classifying space from the classifying spaces of symmetry groups remains true for all those classifications.* This assertion will be detailed in the subsequent sections.

2.4. Thom polynomials for stable classification of maps

Computations show that in many cases the Thom polynomial depends actually on certain combinations $c_i(f) = c_i(f^*TN - TM)$ of the classes $c_i(M)$, $f^*c_j(N)$, given by the formal expansion

$$1 + c_1(f) + c_2(f) + \dots = \frac{1 + f^*c_1(N) + f^*c_2(N) + \dots}{1 + c_1(M) + c_2(M) + \dots}.$$

This observation can be reformulated as follows. Fix some integer $\ell \in \mathbb{Z}$ and consider the so called *stable $\mathcal{K}$-classification* of map germs $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+\ell}, 0)$ where m may vary. By definition, the $\mathcal{K}$-singularity class of the map germ f is the isomorphism class of the zero level germ $f^{-1}(0)$ equipped with the local algebra of functions on it

$$Q_f = \mathcal{O}_{\mathbb{C}^m,0}/f^*\mathfrak{m}_{\mathbb{C}^{m+\ell},0}.$$

For $\ell \geq 0$ the $\mathcal{K}$-classification is actually the classification of finite-dimensional local algebras while for $\ell \leq 0$ it is the classification of $(-\ell)$-dimensional ICIS's (isolated complete intersection singularities).

There is the following simple interpretation of the classifying space for this classification. Choose some large integers $k \gg 0$ and $d \gg m \gg 0$. Consider the manifold $\mathcal{G}(m, d, k)$ of k-jets of germs of m-dimensional submanifolds in $(\mathbb{C}^d, 0)$. The points of this manifold are classified according to the $\mathcal{K}$-singularities of the projection to the fixed coordinate subspace $\mathbb{C}^{m+\ell} \subset \mathbb{C}^d$. We define the classifying space of stable $\mathcal{K}$-singularities as the limit space of $\mathcal{G}(m, d, k)$ with $m \to \infty$, $(d - m) \to \infty$, and $k \to \infty$.

The projection sending a germ of a submanifold to its tangent space at the origin has contractible fibers. Therefore, the space $\mathcal{G}(m, d, k)$ is homotopy equivalent to the usual Grassmannian $G_{m,d}$. Thus the ring of universal characteristic classes associated with the stable $\mathcal{K}$-classification is isomorphic (for each $\ell \in \mathbb{Z}$) to the cohomology ring of the stable Grassmannian that is to the ring of polynomials in the variables $c_1, c_2, \dots$. In other words, the existence theorem for Thom polynomials of $\mathcal{K}$-singularities can be formulated as follows.

2.8. Theorem ([8]). *The cohomology class dual to the cycle of a $\mathcal{K}$-singularity α of a generic holomorphic map f is given by a universal polynomial* Tp_α *in the classes $c_i(f)$.*

2.5. Thom polynomials for isolated hypersurface singularities

Two function germs $f_i : (\mathbb{C}^{m_i}, 0) \to (\mathbb{C}, 0)$, $i = 1, 2$, are called *stably $\mathcal{K}$-equivalent*, if after adding suitable nondegenerate quadratic forms in new variables and after the multiplication by non-vanishing functions they can be brought one to the other by a change of coordinates in the source space. By equivalence of hypersurfaces we mean $\mathcal{K}$-equivalence of functions providing the equations of these hypersurfaces. The classification of IHS's (isolated hypersurface singularities) is one of the most studied classification problem in singularity theory. Nevertheless the theory of characteristic classes associated with this classification has appeared only recently [21]. It turns out that the theory of characteristic classes related to the stable classification of IHS's is the theory of Legendre characteristic classes. For the first glance, the definition below looks unmotivated. The explanation will be given in the subsequent sections.

2.9. Definition. The ring $\mathcal{L}$ of universal *Legendre characteristic classes* is the quotient ring of polynomials in the variables u, a_1, a_2, $\dots$, $\deg u = 1$, $\deg a_i = i$, over the ideal of relations generated by the homogeneous components of the formal expansion

$$(1 + a_1 + a_2 + \dots)\left(1 - \frac{a_1}{1+u} + \frac{a_2}{(1+u)^2} - \dots\right) = 1. \tag{2}$$

If we set formally $1 + a_1 + a_2 + \dots = c(U)$ for a virtual bundle U of virtual rank 0 and $u = c_1(I)$ for a line bundle I, then (2) can be written as

$$c(U + U^* \otimes I) = 1. \tag{3}$$

The additive basis of $\mathcal{L}$ is formed by the monomials of the form $u^k a_1^{i_1} a_2^{i_2} \dots$ with $i_j \in \{0, 1\}$.

2.10. Theorem. *The ring of characteristic classes associated to the classification of IHS's is the ring of Legendre characteristic classes.*

This theorem means that whenever we have a manifold M whose points are classified according to various IHS's types, we have also a natural characteristic homomorphism $\varkappa^* : \mathcal{L} \to H^*(M)$ such that the cohomology class dual to the locus of a given singularity α is given by a universal Legendre characteristic class $\mathrm{Tp}_\alpha \in \mathcal{L}$ determined uniquely by α and evaluated on the homomorphism $\varkappa^*$.

2.11. Example. Consider the diagram

$$H \hookrightarrow W \xrightarrow{\pi} N \tag{4}$$

where the first arrow is a smooth embedding of a hypersurface and the second one is a smooth locally trivial bundle. Denote by $M \subset H$ the locus formed by the tangency points of H with the fibers of π. If certain genericity condition holds, then M is smooth of dimension $\dim M = \dim N - 1$. The points of M are classified according to the singularities of the hypersurfaces cut out by H on the fibers of π.

2.12. Example. The singularities considered in the previous example are determined completely by the composition $H \to N$. More general, consider an arbitrary smooth map $f : H \to N$ such that *the dimension of the cokernel of its derivative is not greater than* 1 at any point. The fibers of this map have dimension $-\ell = \dim H - \dim N$ and the embedded dimension of their singularities is $-\ell + 1$ i.e. these are isolated hypersurface singularities. The parameter space M in this situation is the locus $M \subset H$ of all singular points of the fibers of f i.e. it is the critical set of f.

2.13. Example. In the situation of the previous example, with any point $x \in M$ one can associate the tangent hyperplane $f_* T_x H \subset T_{f(x)} N$. This gives an embedding $i : M \to PT^*N$. This embedding is *Legendrian*: the manifold $i(M)$ is tangent to the natural contact distribution on PT^*N. More general, consider arbitrary Legendrian submanifold $M \subset PT^*N$. A *Legendrian mapping* is the projection of a Legendrian submanifold $M \subset PT^*N$ to the base N of the projectivized cotangent bundle. Singularities of Legendrian mappings are classified according to the classes of stable $\mathcal{K}$-equivalence of functions [5]. Therefore, to each point of M there corresponds an equivalence class of IHS's.

The Legendre characteristic classes in all three examples above are cohomology classes in $H^*(M)$ *defined by* $u = c_1(I)$ *and* $a_i = c_i(f^*TN - TM - I)$*, where* $I \simeq \mathcal{O}_{PT^*N}(1)$ *is the conormal bundle of the contact structure on* PT^*M. In Example 2.11 the bundle I can also be defined as the restriction to M of the line bundle of the divisor $H \subset W$. Verification of identity (3) is a nice exercise in the theory of characteristic classes.

2.6. Characteristic classes of multisingularities

In this section, by a local singularity we mean a class of stable $\mathcal{K}$-singularity of map germs $(\mathbb{C}^*, 0) \to (\mathbb{C}^{*+\ell}, 0)$, where $\ell \in \mathbb{Z}$ is fixed. A *multisingularity* $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ is a finite collection of local singularities. In what follows we assume that the collection $\underline{\alpha}$ contains no classes of submersion (this condition implies an additional restriction for $\ell \leq 0$ only).

The classification of multisingularities can be considered as an independent problem of singularity theory with its own table of normal forms, adjacencies, bifurcation diagrams etc. This implies that the general approach discussed in Sect. 2.1 can be applied to the case of multisingularities as well. It turns out that the construction for the classifying space of multisingularities and its cohomology ring is related to the theory of cobordisms and cohomological operations [22]. The formulated below existence theorem of universal expressions for characteristic classes of multisingularities appears as a corollary of this construction.

To a given map $f : M \to N$ one associates a number of multisingularity loci. First, we define $M(\underline{\alpha}) \subset M^r$ as the closure of the locus r-tuples of pairwise different of points $(x_1, \dots, x_r)$, such that $f(x_1) = \cdots =$

$f(x_r)$ and such that f acquires the singularity α_i at x_i for $i = 1, \dots, r$. This definition is applicable only to the maps satisfying certain genericity condition. For the general case the definition should be corrected. If the genericity condition holds, then $M(\underline{\alpha})$ is a subvariety of expected dimension

$$\dim M(\underline{\alpha}) = \dim M - (r-1)\ell - \sum \operatorname{codim} \alpha_i,$$

where the codimension $\operatorname{codim} \alpha_i$ of the local singularity α_i is counted in the jet space of map germs $(\mathbb{C}^*, 0) \to (\mathbb{C}^{*+\ell}, 0)$.

Consider the natural projections $p : M^r \to M$ to the first factor and $q = f \circ p_M : M^r \to N$ to N, respectively. If the multisingularity type $\underline{\alpha}$ has no classes of submersion, then the restriction to $M(\underline{\alpha})$ of these projections is finite and we can consider the corresponding multisingularity loci on M and N, respectively. Denote by $\overline{m}_{\underline{\alpha}}$, $\overline{n}_{\underline{\alpha}}$ the cohomology classes dual to these loci considered as singular varieties *equipped with their reduced structures*,

$$\overline{m}_{\underline{\alpha}} = [pM(\underline{\alpha})] \in H^*(M), \qquad \overline{m}_{\underline{\alpha}} = [qM(\underline{\alpha})] \in H^*(N).$$

If the symbol $(\alpha_1, \dots, \alpha_r)$ of the multisingularity $\underline{\alpha}$ contains repeating entries, then the projections p and q of the locus $M(\underline{\alpha})$ to its images are not one-to-one, and it is natural to consider the classes $\overline{m}_{\underline{\alpha}}$, $\overline{n}_{\underline{\alpha}}$ with their natural multiplicity given as the degree of the corresponding projection. Thus, we set

$$m_{\underline{\alpha}} = |\operatorname{Aut}(\underline{\alpha})|\, \overline{m}_{\underline{\alpha}} = p_*[M(\underline{\alpha})], \quad n_{\underline{\alpha}} = |\operatorname{Aut}(\underline{\alpha}')|\, \overline{n}_{\underline{\alpha}} = q_*[M(\underline{\alpha})],$$

where $\underline{\alpha}' = (\alpha_2, \dots, \alpha_r)$ and $|\operatorname{Aut}(\underline{\alpha})|$ is the order of the permutation subgroup $\operatorname{Aut}(\underline{\alpha}) \subset S(r)$ whose elements preserve the collection $\underline{\alpha}$. These definitions imply the equalities

$$f_* m_{\underline{\alpha}} = n_{\underline{\alpha}}, \qquad f_* \overline{m}_{\underline{\alpha}} = k_1\, \overline{n}_{\underline{\alpha}},$$

where k_1 is the number of appearances of α_1 in the collection $\underline{\alpha}$.

Recall that $f_* : H^*(M) \to H^*(N)$ is the *push-forward*, or *Gysin homomorphism*. It is defined as the composition of the Poincaré duality in M, usual homomorphism f_* in homology, and Poincaré duality in N. In order this homomorphism to be defined, one needs to assume that the map f is *proper*. The homomorphism f_* shifts the (complex) grading of the even-dimensional cohomology by ℓ. It is not multiplicative. Instead the usual *projection formula* holds,

$$f_*(f^* a) \smile b = a \smile f_* b, \quad a \in H^*(N) \quad b \in H^*(M).$$

In other words, f_* is a homomorphism of $H^*(N)$-modules, where the action of $H^*(N)$ on $H^*(M)$ is defined via f^*. Because of that, we often drop the indication on f^* in the notation of classes on M and instead of $f^*a \smile b$ we write often $a \smile b$ or just $a\,b$.

To any monomial $c^I(f) = c_1^{i_1}(f)c_2^{i_2}(f)\dots$ in the relative Chern classes $c_i(f) = c_i(f^*TN - TM)$ we associate the push-forward *Landweber-Novikov class*

$$s_I(f) = f_*(c^I(f)) \in H^*(N).$$

Landweber-Novikov classes are well known in cobordism theory. In the original definition they take values in complex cobordisms, here we use their images in cohomology only. In the case $\ell \leq 0$ the class $s_{i_1,i_2,\dots}(f)$ vanishes for $\sum k\,i_k < -\ell$ by dimensional reason. Besides, if $\sum k\,i_k = -\ell$, then $s_{i_1,i_2,\dots}(f) \in H^0(N)$ is equal to the corresponding characteristic number of a generic fiber of f. Except these evident relations Landweber-Novikov classes are multiplicatively independent for different monomials c^I. It means that for any polynomial in the Landweber-Novikov classes there is a sample map for which this polynomial gives a non-trivial class.

Now we are ready to formulate the principal theorem in the theory of characteristic classes of multisingularities. Shortly, this theorem says that the Thom polynomial for a multisingularity is a polynomial in Landweber-Novikov classes. In this theorem, $\ell \in \mathbb{Z}$ is a fixed integer. If $\ell \leq 0$, we assume that the collection of local singularities α_i forming the given multisingularity type $\underline{\alpha}$ contains no classes of submersion.

2.14. Theorem ([22])**.** 1. *For every collection $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ of local singularities the cohomology class of the corresponding multisingularity $n_{\underline{\alpha}}$ in the target* (*respectively, the class $m_{\underline{\alpha}}$ in the source*) *manifold is given by a universal polynomial with rational coefficients in the Landweber-Novikov classes $s_I(f)$ of the map* (*respectively, in the relative Chern classes $c_i(f)$ and the pull-backs $f^*s_I(f)$ of the Landweber-Novikov classes*)*.*

2. *The multisingularity polynomial has in fact the following specific form. To every multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ there corresponds a universal polynomial $R_{\underline{\alpha}}$* (*called* residue polynomial) *in the Chern classes $c_1, c_2, \dots$ such that the multisingularity classes $m_{\underline{\alpha}}$, $n_{\underline{\alpha}}$ are determined by the residue polynomials $R_{\underline{\alpha}_J} = R_{\alpha_{j_1},\dots,\alpha_{j_k}}$ of various subcollections $\underline{\alpha}_J = (\alpha_{j_1}, \dots, \alpha_{j_k}) \subset (\alpha_1, \dots, \alpha_r)$ forming the given multisingularity $\underline{\alpha}$*

by the following explicit formulas

$$m_{\underline{\alpha}} = \sum_{J_1\sqcup\dots\sqcup J_k=\{1,\dots,r\}} R_{\underline{\alpha}_{J_1}}\, f^*f_*R_{\underline{\alpha}_{J_2}} \dots f^*f_*R_{\underline{\alpha}_{J_k}}, \tag{5}$$

$$n_{\underline{\alpha}} = f_*m_{\underline{\alpha}} = \sum_{J_1\sqcup\dots\sqcup J_k=\{1,\dots,r\}} f_*R_{\underline{\alpha}_{J_1}} \dots f_*R_{\underline{\alpha}_{J_k}}, \tag{6}$$

where the polynomials $R_{\underline{\alpha}_J}$ are evaluated on the relative Chern classes $c_i = c_i(f) = c_i(f^*TN - TM)$. The sum is taken over all possible partitions of the set $\{1,\dots,r\}$ into a disjoint union of non-empty non-ordered subsets $\{1,\dots,r\} = J_1 \sqcup \dots \sqcup J_k$, $k \geq 1$. The subset containing the element $1 \in \{1,\dots,r\}$ is denoted by J_1.

Moreover, the residue polynomial $R_{\underline{\alpha}}$ is independent of the order of local singularities α_i forming the collection $\underline{\alpha} = (\alpha_1,\dots,\alpha_r)$.

For example, if the collection $\underline{\alpha} = \{\alpha\}$ contains only one element ($r = 1$), then $R_\alpha = m_\alpha$ is the corresponding Thom polynomial of the local singularity α.

Combining the terms on the right hand side expressions we arrive at the following recursive relations equivalent to (5–6).

$$m_{\underline{\alpha}} = R_{\underline{\alpha}} + \sum_{1\in J\subsetneq\{1,\dots,r\}} R_{\underline{\alpha}_J}\, f^*n_{\underline{\alpha}_{\overline{J}}}, \tag{7}$$

$$n_{\underline{\alpha}} = f_*m_{\underline{\alpha}} = f_*R_{\underline{\alpha}} + \sum_{1\in J\subsetneq\{1,\dots,r\}} f_*R_{\underline{\alpha}_J}\, n_{\underline{\alpha}_{\overline{J}}}, \tag{8}$$

where the sum is taken over all proper subsets $J \subsetneq \{1,\dots,r\}$ containing the element 1, and $\overline{J} = \{1,\dots,r\} \setminus J$.

The combinatorial expression (6) for the multisingularity classes $n_{\underline{\alpha}}$ can be rewritten in the following way by means of generation functions. Assume that we study multisingularities formed by the local singularities (perhaps, with repetitions) from a finite list $\alpha_1,\dots,\alpha_r$ of pairwise different ones. Then the following formal identity holds

$$1+\sum_{k_1,\dots,k_r} n_{\alpha_1^{k_1}\dots\alpha_r^{k_r}} \frac{t_1^{k_1}}{k_1!}\cdots\frac{t_r^{k_r}}{k_r!} = \exp\left(\sum_{k_1,\dots,k_r} f_*(R_{\alpha_1^{k_1}\dots\alpha_r^{k_r}})\frac{t_1^{k_1}}{k_1!}\cdots\frac{t_r^{k_r}}{k_r!}\right). \tag{9}$$

(The author is grateful to S. Lando for this remark.)

2.15. Example. Assume that the multisingularity $\underline{\alpha} = (\alpha,\dots,\alpha) = (\alpha^r)$ contains r copies of the same singularity α. Then the class $\overline{m}_{\alpha^r}$

of *multiple singularity* α can be determined by the following recursive formula

$$(10) \qquad \overline{m}_{\alpha^r} = q_r + \sum_{k=1}^{r-1} q_{r-k}\overline{n}_{\alpha^k}, \quad f_*\overline{m}_{\alpha^r} = r\,\overline{n}_{\alpha^r},$$

where $q_k = \frac{1}{(k-1)!}R_{\alpha^k}$ are certain polynomials in relative Chern classes of the map. Over $\mathbb{Q}$ this relation follows from (7). Conjecturally the polynomials q_k have always integer coefficients and (10) holds in the integer cohomology group.

At present, the first statement of Theorem is proved under certain restrictions by topological argument from cobordism theory, see [22] and the next section. It is a challenge to find an intersection theory proof of this statement and especially of its various conjectural generalizations formulated in [22].

Let us show that the second statement is a consequence of the first one. For simplicity we shell prove the relation (9) equivalent to (6). The relation (5) is proved by similar argument. Consider the generating series

$$\mathcal{N}(f) = 1 + \sum_{k_1,\dots,k_r} n_{\alpha_1^{k_1}\dots\alpha_r^{k_r}} \frac{t_1^{k_1}}{k_1!}\cdots\frac{t_r^{k_r}}{k_r!} = 1 + \sum_{k_1,\dots,k_r} \overline{n}_{\alpha_1^{k_1}\dots\alpha_r^{k_r}} t_1^{k_1}\dots t_r^{k_r}.$$

By the first statement of Theorem, each coefficient in this series is a polynomial in Landweber-Novikov classes. So the coefficients of the logarithm $\log(\mathcal{N}(f))$ are. We need to show that the coefficients of $\log(\mathcal{N}(f))$ depend *linearly* in the Landweber-Novikov classes.

The generating series $\mathcal{N}(f)$ satisfies the following remarkable property. Assume that M has two connected components, $M = M_1 \sqcup M_2$. Then denoting by f_i the restriction of f to M_i we have

$$\mathcal{N}(f) = \mathcal{N}(f_1)\mathcal{N}(f_2).$$

Indeed, every multisingularity locus of the map f consists of many components numbered by possible distributions of local singularities between M_1 and M_2. These components correspond one-to-one to the summands in the right hand side of the equality provided by the multiplication rule for generating functions.

The equality is applied as follows. Starting from the given map $f : M \to N$ we construct a series of new maps $f^{(d)} : M^{(d)} \to N$, $d = 1, 2, \dots$, in the following way. The source manifold $M^{(d)}$ of $f^{(d)}$ is the disjoint union of d copies of M, and the restriction of $f^{(d)}$ to each component of $M^{(d)}$ is defined to be a small perturbation of the original map f. The Landweber-Novikov classes of $f^{(d)}$ are given by $s_I(f^{(d)}) = d\, s_I(f)$. On the other hand, from the equality above we have

$$\mathcal{N}(f^{(d)}) = (\mathcal{N}(f))^d, \qquad \log(\mathcal{N}(f^{(d)})) = d\,\log(\mathcal{N}(f)).$$

It follows that $\log(\mathcal{N}(f))$ has no terms of order greater then 1 in Landweber-Novikov classes since such terms would contribute to the terms in $\log(\mathcal{N}(f^{(d)}))$ of order greater then 1 in d. Equality (9) is proved.

In the applications related to the enumeration of isolated multisingularities of hypersurfaces one needs a Legendrian version of Theorem 2.14. It is formulated in a similar way. The only difference is that the residue classes $R_{\underline{\alpha}}$ of the Legendre (or isolated hypersurface) multisingularity is an element of the ring $\mathcal{L}$ of universal Legendre characteristic classes (and thus it determines a cohomology class on the source manifold of a Legendre map).

2.16. Theorem ([22]). *To every collection $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ of stable isolated hypersurface singularity classes there corresponds a universal Legendre characteristic class $R_{\underline{\alpha}} \in \mathcal{L}$ such that for any generic proper holomorphic Legendre map $f : M \to PT^*N \to N$ the corresponding multisingularity classes $m_{\underline{\alpha}}$, $n_{\underline{\alpha}}$ are given by the formulas* (5–8).

2.7. Multisingularities and cobordisms

In this section we show that the classical Thom's construction for the classifying space of cobordisms is the best fit for the study of multisingularities. Strangely this fact have not been noticed for such a long time!

Consider a differentiable map $f : M \to N$ of *real* compact manifolds. This map can be treated as a representative of the cobordism class $[f] \in O^\ell(N)$, $\ell = \dim N - \dim M$. Therefore, it corresponds to a classifying map from N to the classifying space of cobordisms. Recall the Thom's construction for this map. First, represent f as a composition of an embedding and a projection

$$M \hookrightarrow \mathbb{R}^K \times N \to N.$$

The normal bundle of the embedding is classified by certain map $M \to BO(K+\ell)$. This map extends to a map $U \to EO(K+\ell)$ of the total spaces of the corresponding rank $K+\ell$ bundles, where we identified the total space of the normal bundle of M with a tubular neighborhood U of the submanifold $M \subset \mathbb{R}^K \times N$. Finally, denoting by $MO(K+\ell)$ the Thom space of the bundle $EO(K+\ell)$ we extend the constructed map to the whole $\mathbb{R}^K \times N$ by sending the complement of U to the marked point in $MO(K+\ell)$. Thus constructed map $h : \mathbb{R}^K \times N \to MO(K+\ell)$ can be treated as a family of maps

$$h_y : S^k \to MO(K+\ell)$$

parameterized by points $y \in N$ or as a map

$$\varkappa : N \to \mathbf{N}_\ell$$

to the corresponding iterated loop space $\mathbf{N}_\ell = \Omega^K MO(K+\ell)$. The limit $\lim_K \Omega^K MO(K+\ell)$ is called the *classifying space of ℓ-dimensional cobordisms.*

The cobordism class on N is determined uniquely by the homotopy class of $\varkappa$. For example, the source manifold M of the map can be recovered as the inverse image of the 'zero section' $BO(K+\ell) \subset MO(K+\ell)$ under the associated map $\mathbb{R}^K \times N \to MO(K+\ell)$. In other words, for any $y \in N$ the preimages $f^{-1}(y)$ are in one-to-one correspondence with the intersection points of $h_y(S^K)$ with the zero section. To any such intersection point $h_y(x)$ we associate its $\mathcal{K}$-singularity type, that is the singularity type of the 'projection of S^K to the fiber of $EO(K+\ell)$ along the zero section'. The following statement is almost evident.

2.17. Lemma. *The stable $\mathcal{K}$-singularity type of the intersection point $h_y(x)$ coincides with the $\mathcal{K}$-singularity type of the map f at the corresponding point $x \in M$.*

Remark that the usage of stable $\mathcal{K}$-classification is essential here: the manifolds participating in the two map germs $(M,x) \to (N,y)$ and $(S^K,x) \to (\mathbb{R}^{K+\ell},0)$ of the lemma have different dimensions (but equal relative dimension ℓ). Besides, the second map is not well defined and only its $\mathcal{K}$-singularity type can be determined.

The elements of the classifying space $\mathbf{N}_\ell$ are continuous maps $g : S^K \to MO(K+\ell)$. Without loss of generality we can replace the infinite dimensional space $BO(K+\ell)$ by its smooth finite dimensional approximation $G_{K+\ell,K_1}$, $K_1 \gg K+\ell$. Moreover, we may assume that the maps g forming the classifying space are differentiable in a neighborhood of the zero section. Thus, the classifying space $\mathbf{N}_\ell$ is classified by the multisingularity types of the intersection of $g(S^K)$ with the zero section. Therefore, the lemma can be reformulated by saying that the *classifying map $\varkappa : N \to \mathbf{N}_\ell$ preserves the partitions by the multisingularity types.* As a result, we arrive at the following conclusion:

2.18. Corollary. *The classifying space $\mathbf{N}_\ell = \Omega^K MO(K+\ell)$ of cobordisms serves also as the classifying space of multisingularities.*

The proof of the first assertion of Theorem 2.14 uses the complex version $\mathbf{N}_\ell^{\mathbb{C}}$ of the classifying space of multisingularities. Ignoring some technical difficulties we claim that the homotopy type of this space is given by

$$\mathbf{N}_\ell^{\mathbb{C}} = \Omega^{2K} MU(K+\ell), \quad K \gg 0,$$

which is the classifying space of complex cobordisms. The cohomology ring of this space can be computed explicitly, at least in the case of rational coefficients. It is a polynomial ring whose generators correspond to Landweber-Novikov classes. This leads to the formulation of Theorem 2.14, see [22].

2.8. Generalized Pontryagin-Thom-Szücs construction

The topological type of the classifying space $B\mathcal{S}$ of singularities depends heavily on the particular classification $\mathcal{S}$. However all these spaces, in particular those considered in the previous sections, have many common features. The most important one is the following splitting that we call the *generalized Pontryagin-Thom-Szücs construction*:

$$B\mathcal{S} = \bigcup_\alpha B\alpha, \quad B\alpha \sim BG_\alpha, \tag{11}$$

where G_α is the symmetry group of the corresponding singularity α.

The notion of a 'singularity theory classification' can be axiomatized as follows. By a *classification* $\mathcal{S}$ we mean a (finite or infinite) list of symbols α called 'singularity classes'. Every singularity class is assigned a number $\operatorname{codim}\alpha$ (its codimension) and, in addition, the following data:

- a *bifurcation diagram* of this singularity, that is the germ of a $(\operatorname{codim}\alpha)$-dimensional manifold T_α equipped with the partition into the strata labelled by the singularity classes of smaller codimensions;
- a *symmetry group* G_α acting on T_α preserving the partition into the strata.

These data must satisfy some natural compatibility conditions for adjacent singularities. For example, a normal slice to any stratum in T_α (together with the induced partition) must be diffeomorphic to the bifurcation diagram of the corresponding singularity. We do not formulate the compatibility conditions explicitly; they are always automatically satisfied for all 'natural' classifications.

The homotopy type of the classifying space $B\mathcal{S}$ is determined uniquely by the classification.

Indeed, the condition $B\alpha \sim BG_\alpha$ determines the topology of the strata; and the geometry of bifurcation diagrams determines the way how these strata are glued for adjacent singularities.

The classifying property of $B\mathcal{S}$ is formulated as follows. Assume that we are given a parameter space M whose points are classified according to the given classification $\mathcal{S}$. Consider some stratum $M(\alpha) \subset M$. The structure group of the normal bundle to this stratum is reduced to G_α. Therefore, it is classified by some map $M(\alpha) \to BG_a$. These maps glue together to provide a map

$$\varkappa : M \to B\mathcal{S}.$$

in other words, the *partition on M is induced from the classifying space $B\mathcal{S}$ by certain classifying map $\varkappa$.*

The detailed realization of the general picture formulated above meets evident technical difficulties. These difficulties have been overcame by A. Szücs in his theory of τ-maps developed in a series of papers [35, 36, 37]. By a τ-map we mean a differentiable map that admits only (multi)singularities from a given list τ of allowed ones. In one of the most general form this theory is described in the joint paper with R. Rimanyi [33]. In this theory the classifying space τY is constructed by gluing the classifying spaces of symmetry groups of multisingularities from τ. With small changes the same construction can be applied to any 'abstract' classification, not necessary related to singularities of maps. It is assumed in the paper [33] that the classification of singularities is discrete that is a neighborhood of any point intersects only finitely many orbits. This technical restriction is not essential. It can be dropped using the notion of a cellular classification introduced in [41].

On the other hand, in the previous sections we have used the alternative *a priori* constructions for the classifying spaces of particular classifications.

Both approaches to the construction of the classifying space are equivalent.

It a consequence of a 'general nonsense': the uniqueness of the classifying space is guarantied by its universality that can be verified under either approach.

Example. The classifying space τY for τ-maps can be obtained from the classifying space $\mathbf{N}_\ell = \Omega^K MO(K+\ell)$, $K \gg 0$, of cobordisms by selecting the strata in $\mathbf{N}_\ell$ corresponding to the allowed singularities. The classifying property is almost evident: consider a τ-map $M \to N$ and repeat the Thom's construction for this map without regarding its singularities; then Lemma 2.17 assures that the resulting classifying map $\varkappa : N \to \mathbf{N}_\ell$ automatically takes values in the union τY of required strata. This interpretation of τY allows one to avoid technical difficulties arising in the gluing construction. Another advantage of the *a priori* construction is that the space obtained in this way is smooth (although non-compact). It is quite hard to achieve this by the gluing construction. Besides, we obtain a clear answer to the question what is the limit of the spaces τY when τ contains *all* multisingularities.

Thus instead of 'gluing' $B\mathcal{S}$ from BG_α's we prefer to speak about 'cutting' $B\mathcal{S}$ into BG_α's. Moreover, the validity of the splitting (11) can be used to give the correct definition for the symmetry group. Here are a few examples.

- If $\mathcal{S}$ is the classification of orbits of some Lie group G action then G_α is the stabilizer of (any point of) the orbit.
- In the case of stable $\mathcal{K}_\ell$-classification the symmetry group is the (maximal compact subgroup of) the symmetry group of the k-jet of any stable representative $f_0 : (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+\ell}, 0)$ *with the smallest possible* m (equal to the codimension of the singularity).
- The symmetry group of a stable class of critical points of functions is the symmetry group of any representative $f_0 : (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ *with the smallest possible* m (equal to the corank of the singularity).
- For the classification of multisingularities the symmetry group of the multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ is the semidirect product of the symmetry groups G_{α_i} of local $\mathcal{K}$-singularities α_i and the subgroup in $S(r)$ of automorphisms of the multi-index $\underline{\alpha}$.

The cohomological information on the topology the splitting (11) is formulated in terms of the characteristic spectral sequence. Assume for simplicity that there are finitely many singularity classes (the general case is considered in [18, 19]). Consider the open increasing filtration on $B\mathcal{S}$ whose pth term F_p is formed by the singularity strata of codimension at most p,

$$F_p = \bigcup_{\operatorname{codim}\alpha \le p} B\alpha \subset B\mathcal{S}.$$

The spectral sequence $E_r^{p,q}$ associated with this filtration converges to the cohomology of the classifying space $H^*(B\mathcal{S})$. It is called the *characteristic spectral sequence*. The complement $F_p \setminus F_{p-1}$ is a smooth (in general, non-closed) submanifold in $B\mathcal{S}$ of codimension p formed by p-codimensional singularities. It follows that the initial term $E_1^{p,*}$ of this sequence is the cohomology group of the Thom space of the normal bundle to this manifold. Using the Thom isomorphism we get the following

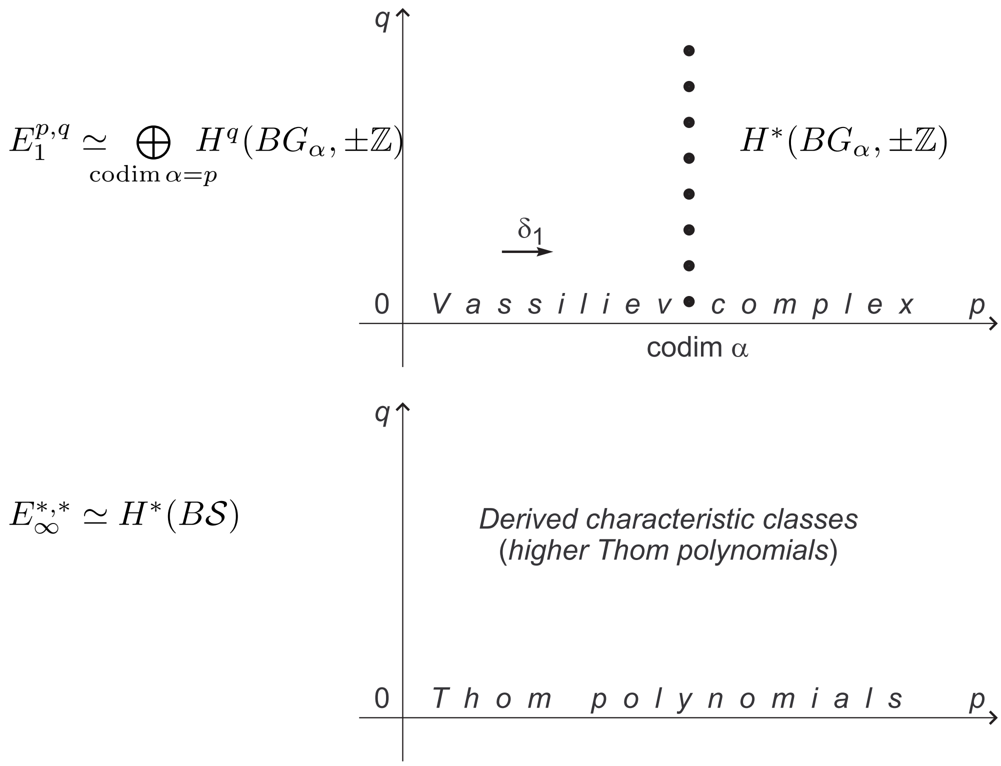

Fig. 1. **Characteristic spectral sequence**

description of the initial term:

$$E_1^{p,q} \simeq \bigoplus_{\operatorname{codim}\alpha=p} H^q(BG_\alpha, \pm\mathbb{Z}).$$

Here $\pm\mathbb{Z}$ is the coefficient system on $B\alpha = BG_\alpha$ that is locally isomorphic to $\mathbb{Z}$ and that is determined by the action of the group G_α on the orientation of the bifurcation diagram of the singularity α (for the details, see [18, 19]).

The *Vassiliev complex* is the row $(E_1^{*,0}, \delta_1)$ of the initial term. It allows one to select linear combinations of the strata for which the dual cohomology class is correctly defined.

The limit term $E_\infty^{*,*}$ defines a natural filtration on $H^*(B\mathcal{S})$. The pth term of this filtration is generated by the characteristic classes that can be represented by cycles supported on the union of strata of codimension greater than or equal to p.

The term $E_\infty^{p,0}$ corresponds to the fundamental cycles of strata of codimension p, that is, to Thom polynomials. The terms $E_\infty^{p,q}$ with $q > 0$ are *higher Thom polynomials*, or *derived characteristic classes* of singularities. They have the following meaning.

Let M be the parameter space of some geometric problem, and α be a singularity class of codimension p. The normal bundle to $M(\alpha)$ has the structure group G_α. Therefore, every characteristic class χ of the group G_α defines a cohomology class $\chi(\alpha) \in H^q(M(\alpha))$ on the locus $M(\alpha)$. Assume that $M(\alpha)$ is closed. Then the push-out class $i_*(\chi(\alpha)) \in H^{p+q}(M)$ is well defined, where i is the embedding. It corresponds to the term $E_1^{p,q}$ of the spectral sequence. The derived characteristic class is the result of an attempt to extend the definition of the class $i_*(\chi(\alpha))$ to the general situation. It is not always possible (only if all differentials vanish on this element of $E_1^{p,q}$). Even if this is possible, this extension is not unique (it is defined only modula higher terms of the filtration supported on the strata of codimension $> p$). Examples of derived characteristic classes are given in the subsequent sections.

The characteristic spectral sequence has especially simple description in the complex problems, where all topology is often concentrated in even dimensions and the sequence degenerates at the initial term by dimensional reason. In this case it implies the following splitting of the cohomology group of the classifying space,

$$H^n(B\mathcal{S}) \simeq \bigoplus_\alpha H^{n-\operatorname{codim}\alpha}(BG_\alpha). \tag{12}$$

This splitting implies an interesting relation between the Poincaré series of the cohomology groups $H^*(B\mathcal{S})$ and $H^*(BG_\alpha)$. The author used this relation many times to check various conjectures about classifications, the structure of symmetry groups and the topology of the classifying spaces.

§3. Localized Thom polynomials

A localized Thom polynomial is the Thom polynomial written in a special additive basis well adjusted to the classification of singularities by corank. It allows one to single out the terms in the Thom polynomial for which closed formulae could be given. It provides also the correspondence between the Thom polynomials for singularities with the same name appearing in different classifications. Finally, the concept of localized Thom polynomials is crucial in the application of the restriction method to the computation of the residue classes of complete intersection multisingularities.

3.1. Porteous-Thom classes and their derived classes

One of the historically first examples of the computed Thom polynomials are those for the so called Porteous-Thom singularities. Let M

be the source manifold of a generic holomorphic map

$$f : M \to N.$$

Denote by $\Sigma^r = \Sigma^r(f) \subset M$ the locus of points where the derivative of the map f has at least r-dimensional kernel, $r \geq \max(0, \ell)$, where $\ell = \dim N - \dim M$. More generally, let E, F be two complex vector bundles over the same base M and $\varphi : E \to F$ be a generic morphism. Then one can define the locus Σ^r for his morphism by similar conditions. In the case when M is the source of a holomorphic map f we can set $E = TM$, $F = f^*TN$, and $\varphi = f_* : TM \to f^*TN$ is the derivative map. If the genericity condition holds, then Σ^r is a subvariety of (complex) codimension

$$\operatorname{codim} \Sigma^r = r\,(r + \ell), \qquad \ell = \operatorname{rk} F - \operatorname{rk} E.$$

By Theorem 2.8, the dual of the locus Σ^r is expressed as a universal polynomial in the classes $c_i = c_i(f) = c_i(F - E)$.

3.1. Theorem ([27]). *The cohomology class Poincaré dual to the locus Σ^r is given by the following determinant*

$$[\Sigma^r] = \det \|c_{r+\ell-i+j}\|_{i,j=1,\dots,r}, \qquad c_i = c_i(f) = c_i(F - E).$$

Assume for a moment that the locus Σ^{r+1} is empty. Then (provided the genericity condition holds) the locus Σ^r is smooth. Denote by $p_r : \Sigma^r \to M$ the embedding. The Gysin homomorphism $p_{r*} : H^*(\Sigma^r) \to H^*(M)$ allows us to push-forward to M cohomology classes defined on Σ^r. Over Σ^r one has the natural kernel bundle K and cokernel bundle Q of ranks r and $r+\ell$, respectively. These bundles form the exact sequence (defined on Σ^r only)

$$0 \to K \to E \xrightarrow{\varphi} F \to Q \to 0.$$

Let $R(v, u)$ be arbitrary polynomial in formal variables $v_1, \dots, v_r$, $u_1, \dots, u_{r+\ell}$. Set $v_i = c_i(K)$, $u_j = c_j(Q)$.

3.2. Proposition. *If Σ^{r+1} is empty, then the push-forward class $p_{r*}R(c(K), c(Q))$ can be expressed as a universal polynomial* (*determined by ℓ, r, and R*) *in the relative Chern classes $c_i = c_i(f) = c_i(F - E)$.*

The polynomial representing the class $p_{r*}R(c(K), c(Q))$ is called the *derived Thom polynomial* of the singularity Σ^r. Since it is a polynomial in the classes $c_i(F - E)$, it can be considered for any map *not necessary satisfying the condition* $\Sigma^{r+1} = \varnothing$. The derived Thom polynomial is not defined uniquely but only up to a class that can be represented by a cycle supported on Σ^{r+1}. This ambiguity can be fixed, for example, by the following geometric construction. Consider the Grassmann bundle $G_r(E)$ formed by all r-dimensional subspaces λ in the fibers of the bundle $E \to M$.

3.3. Definition. The *standard resolution* $\widetilde{\Sigma}^r$ *of the singularities* of the locus Σ^r is the submanifold in the space of the Grassmann bundle $G_r(E)$ formed by all pairs of the form (x, λ), $x \in M$, $\lambda \subset E_x$, such that $\lambda \subset \ker f$.

If the genericity condition for the morphism $\varphi : E \to F$ holds, then $\widetilde{\Sigma}^r$ is smooth. Denote by $p_r : \widetilde{\Sigma}^r \to M$ the natural projection.

Denote by K the restriction to $\widetilde{\Sigma}^r$ of the tautological rank r bundle over $G_r(E)$. Denote also by Q the (virtual) bundle $Q = F - E + K$. In the case when $\Sigma^{r+1} = \varnothing$ the map p carries $\widetilde{\Sigma}^r$ isomorphically to Σ^r and the bundles K, Q over $\widetilde{\Sigma}^r$ correspond to similar bundles over Σ^r under this isomorphism. This justifies our notation.

Denoting $v_i = c_i(K)$, $u_j = c_j(Q)$ we see that $R(v, u)$ can be considered as a cohomology class on $\widetilde{\Sigma}^r$. This extends the definition of the class $p_{r*}R(v, u) \in H^*(M)$ to the case of arbitrary morphism $E \to F$ not necessary satisfying the condition $\Sigma^{r+1} = \varnothing$.

The explicit form of the class $p_{r*}R(v, u)$ can be obtained as follows. From the definition of Q, we have $c(Q) = c(F-E)\, c(K)$, or

$$u_k = \sum_{i+j=k} c_i\, v_j, \qquad c_i = c_i(F-E). \tag{13}$$

In view of the projection formula it remains to compute the push-forward class $p_{r*}R(v, u)$ in the case when R depends on the variables v_i only. According to the splitting principle we set formally

$$c(K) = 1 + v_1 + \cdots + v_r = \prod_{i=1}^{r} (1 - t_i) \tag{14}$$

and express the polynomial R in terms of $t_1, \ldots, t_r$ using these relations.

3.4. Theorem. *The homomorphism* $p_{r*} : H^*(\widetilde{\Sigma}^r) \to M$ *is given on the monomials in* t_i *by the formula*

$$p_{r*} t_1^{s_1} \ldots t_r^{s_r} = \det \| c_{r+\ell-i+s_i+j}(F - E) \|_{i,j=1,\ldots,r}. \tag{15}$$

Some versions of this formula can be found in [15, 12] The relation of this theorem should be understood formally since the classes t_i are not defined on $\widetilde{\Sigma}^r$. The determinantal expression on the right hand side is known as the *Schur polynomial*.

3.2. Localized Thom polynomials

Relations (13–15) have the following formal treatment. Consider the polynomial rings of universal characteristic classes

$$H^*(BU) = \mathbb{Z}[c_1, c_2, \dots],$$
$$H^*(BU(r) \times BU(r+\ell)) = \mathbb{Z}[v_1, \dots, v_r, u_1, \dots, u_{r+\ell}].$$

We consider the grading on these rings by setting $\deg c_i = \deg v_i = \deg u_i = i$ so that for a homogeneous cohomology class a one has $a \in H^{2\deg a}(\cdot)$. These rings are related by the natural multiplicative homomorphism

$$p_r^* : H^*(BU) \longrightarrow H^*(BU(r) \times BU(r+\ell))$$

given on the generators c_i by the formal expansion

$$p_r^* : 1 + c_1 + c_2 + \dots \longmapsto \frac{1 + u_1 + \dots + u_{r+\ell}}{1 + v_1 + \dots + v_\ell}.$$

Besides, we consider the homomorphism of $H^*(BU)$-modules

$$p_{r*} : H^*(BU(r) \times BU(r+\ell)) \longrightarrow H^*(BU)$$

given by the explicit formulae (13–15). (The action of $H^*(BU)$ on $H^*(BU(r) \times BU(r+\ell))$ is determined via p_r^*.)

3.5. Theorem. *The homomorphisms* p_{r*}, $r \geq \max(0, -\ell)$, *provide a natural splitting*

$$H^*(BU) \simeq \bigoplus_r H^*(BU(r) \times BU(r+\ell)). \tag{16}$$

In other words, every polynomial P in variables c_i has a unique presentation in the form

$$P = \sum_r p_{r*} R^{(r)}, \tag{17}$$

where $R^{(r)}$ is a polynomial of degree $\deg R^{(r)} = \deg P - r\,(r+\ell)$ *in variables $v_1, \dots, v_r, u_1, \dots, u_{r+\ell}$.*

The right hand side of (17) is called the *localized form* of the polynomial P. Its terms $p_{r*}(R^{(r)})$ are determined uniquely by P and by the number ℓ. These terms have the following meaning. Consider a morphism of vector bundles $\varphi : E \to F$ over some base M with $\operatorname{rk} F - \operatorname{rk} E = \ell$. Then the term $p_{r*} R^{(r)} \in H^*(BU) = \mathbb{Z}[c_1, c_2, \dots]$ evaluated on the relative Chern classes $c_i = c_i(F - E)$ can be represented by a cycle supported on the locus Σ^r of the morphism φ. For example, if $P = \mathrm{Tp}_\alpha$ is the Thom polynomial of some singularity α having the kernel rank r, then the polynomials $R^{(i)} = R_\alpha^{(i)}$ vanish for $i < r$.

The localized form clarifies also the structure of the residue polynomials of multisingularities. The following assertion is verified on hundreds of computed examples. However, we still have no formal proof of this fact.

3.6. Conjecture. *For any multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ the residue cohomology class $R_{\underline{\alpha}} \in H^*(M)$ of a map $f : M \to N$ can be represented by a cycle supported on the intersection $\bigcap_{i=1}^r M(\alpha_i) \subset M$. In particular, the number k_0 of the first non-zero term in the localized residue polynomial $R_{\underline{\alpha}} = \sum_{k \geq k_0} p_{k*} R_{\underline{\alpha}}^{(k)}$ is equal to the biggest kernel rank of the singularities α_i.*

The splitting of the Theorem is a particular case of the splitting (12). Consider linear maps forming the space $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^{m+\ell})$ with ℓ fixed and $m = 1, 2, \dots$. The classification of such maps can be considered as an independent classification problem. The singularity classes for this classification are the classes Σ^r of maps of kernel rank r. By the *symmetry group* G_{Σ^r} of the class Σ^r we mean the stationary subgroup of any representative $x \in \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^{m+\ell})$ *with the smallest possible* m (that is, with $m = r$). It is clear that such a representative is exactly the zero map $x = 0 \in \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r+\ell})$. The stationary group for this element contains all linear transformations of the source and the target space. Therefore,

$$G_{\Sigma^r} \sim U(r) \times U(r + \ell),$$

and the splitting (16) follows from (12). □

The construction for the homomorphisms p_{r*} has some variations. For example, we could consider the resolution of the locus Σ^r using the Grassmann bundle $G_{r+\ell}(F)$ or even combine the two methods. This would lead to another choice for the homomorphism $p_{r*} : H^*(BU(r) \times BU(r+\ell)) \to H^*(BU)$ such that the difference of the two choices is supported on Σ^{r+1}. More formally, consider the decreasing filtration on $H^*(BU)$ whose rth term is formed by polynomials in Chern classes c_i that can be represented by cycles supported on Σ^r. Then the right hand side of (16) represents the adjoint graded space of this filtration and the equality (17) provides a particular splitting of this filtration. It follows that the first nonzero term in the localized form of a polynomial P has more invariant meaning with respect to the other terms.

3.7. Example. The term $p_{0*} R_{\underline{\alpha}}^{(0)}$ can be non-trivial only in the case when every local singularity α_i of the multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$

is the class of *immersion* (that is, the class A_0 of non-singular map germs). In the latter case one has

$$R^{(0)}_{A_0^r} = (r-1)!\,(-1)^{r-1}u_\ell^{r-1}, \qquad p_{0*}R^{(0)}_{A_0^r} = (r-1)!\,(-1)^{r-1}c_\ell^{r-1}.$$

This equality together with (10) is equivalent to the known *Herbert-Ronga formula for the classes of multiple points of immersions*,

$$\overline{m}_r = f^*\overline{n}_{r-1} - c_\ell \overline{m}_{r-1},$$

where we denote by $\overline{m}_r = \frac{1}{(r-1)!} m_{A_0^r}$ and $\overline{n}_r = \frac{1}{r!} n_{A_0^r}$ the classes of the corresponding reduced cycles of multiple points.

3.8. Example. There has been a number of papers studying Thom polynomials for *corank* 1 *maps* that is for maps such that the kernel rank of the derivative does not exceed 1 at any point, see [27, 24, 25, 6, 30, 26, 1]. The results of these papers can be interpreted as the study of the term $p_{1*}R^{(1)}$ of these Thom polynomials. Indeed, if the map has no points with singularities of corank greater, then 1 then $\Sigma^r = \varnothing$ for $r \geq 2$ and all terms of the localized Thom polynomial except the first one vanish for such a map.

Our findings on the Thom polynomials for such maps can be summarized as follows (all necessary ingredients for obtaining formulas of Theorems 3.9 and 3.12 below are contained implicitly in [24, 25, 6]). Let $\ell \geq 0$. Denote by A_k the $\mathcal{K}$-singularity class of maps with local algebra isomorphic to $\mathbb{C}[x]/x^{k+1}$ i.e. the singularity class of the Thom-Boardman type $\Sigma^{1,\dots,1}$ (k units). The polynomial $R^{(1)}_{A_k}$ of the first term in the localized Thom polynomial $\mathrm{Tp}_{A_k} = \sum p_{r*}R^{(r)}_{A_k}$ depends on the variables $v_1, u_1, \dots, u_{\ell+1}$. Set $t = -v_1$.

3.9. Theorem. *The term $R^{(1)}_{A_k}$ of the localized Thom polynomial of the singularity A_k is given by*

$$R^{(1)}_{A_k} = \sigma_2\sigma_3\dots\sigma_k,$$

where

$$\sigma_p = u_{\ell+1} + p\,t\sum_{i=0}^{\ell} p^i t^i u_{\ell-i} = c_{\ell+1} + (p-1)t\sum_{i=0}^{\ell} p^i t^i c_{\ell-i}, \qquad t = -v_1.$$

For applications of this theorem remark that the homomorphism p_{1*} has especially simple form:

$$p_{1*}t^s = c_{s+\ell+1}.$$

3.10. Corollary. *If $f : M \to N$ is a generic corank 1 map, then the cohomology class dual to the singularity locus A_k can be obtained as follows. One should expand all brackets in the product*

$$\prod_{p=2}^{k} \left(c_{\ell+1} + (p-1)t \sum_{i=0}^{\ell} p^i t^i c_{\ell-i} \right)$$

and formally replace any occurrence of t^s with $s \geq 0$ by $c_{\ell+1+s} = c_{\ell+1+s}(f^*TN - TM)$.

In the case $\ell = 0$ this assertion is proved in [26].

The residue polynomials of multisingularities of corank one maps can also be written in a closed form. To describe these polynomials we introduce the following notation. Consider the ring homomorphism

$$\rho : \mathbb{Z}[c_1, c_2, \dots] \to \mathbb{Z}[t, c_1, \dots, c_{\ell+1}]$$

given on the generators by $\rho(c_i) = c_i$ for $i \leq \ell + 1$ and $\rho(c_{\ell+1+j}) = c_{\ell+1} t^j$. The following lemma can be formally derived from the formula of Theorem 3.4. Let P be arbitrary polynomial in the variables c_k. Consider its localized form $P = p_{*0} R^{(0)} + p_{*1} R^{(1)} + \dots$.

3.11. Lemma. *The homomorphism ρ vanishes on the terms $p_{s*} R^{(s)}$ for $s \geq 2$. Moreover, the terms $p_{*0} R^{(0)} + p_{*1} R^{(1)}$ are completely determined by the image $\rho(P)$ of this homomorphism.*

In what follows we set

$$Q_{p_1,\dots,p_r} = \rho(R_{A_{p_1-1},\dots,A_{k_r-1}}).$$

Due to Lemma 3.11, the polynomial $Q_{p_1,\dots,p_r}$ describes the initial terms of the residue polynomial for the multisingularity $(A_{p_1-1}, \dots, A_{k_r-1})$. With this notation Theorem 3.9 asserts that for the case of a monosingularity ($r = 1$) this polynomial is given by

$$Q_p = \rho(R_{A_p-1}) = \sigma_1 \sigma_2 \dots \sigma_{p-1}. \tag{18}$$

3.12. Theorem. *The terms $p_{0*} R^{(0)}_{\underline{\alpha}} + p_{1*} R^{(1)}_{\underline{\alpha}}$ of the localized residue polynomial of a given multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ can be nontrivial only if every singularity α_i has the form A_k for some $k \geq 0$. For the multisingularity $(A_{p_1-1}, \dots, A_{p_r-1})$ the corresponding localized terms are given by the following formula*

(19)

$$Q_{p_1,\dots,p_r} = \frac{1}{t^{r-1}} \sum_{\{1,\dots,r\} = J_1 \sqcup \dots \sqcup J_k} (-1)^{r-k} (k-1)! \, \sigma_0^{k-1} \, Q_{|\underline{p}_{J_1}|} \dots Q_{|\underline{p}_{J_1}|}.$$

In this relation, $|\underline{p}_J|$ denotes $\sum_{i\in J} p_i$. One can verify that the sum on the right hand side is divisible by t^{r-1} that is $Q_{p_1,\dots,p_r}$ is indeed a polynomial in variables $t, c_1, \dots, c_{\ell+1}$.

3.13. Corollary. *For a generic corank one map, the residue class of the multisingularity $(A_{p_1-1}, \dots, A_{p_r-1})$ can be obtained from the polynomial $Q_{p_1,\dots,p_r}$ in the following way. One should expand all brackets, replace any occurrence of $c_{\ell+1}t^k$ by $c_{\ell+1+k}$, and finally replace c_i by the relative Chern class $c_i(f) = c_i(f^*TN - TM)$.*

3.14. Example. For $\ell \le 0$ the first term of the localized Thom polynomial (with $r = 1-\ell$) has also a special meaning. If $f : (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+\ell}, 0)$ is a map germ of kernel rank $1-\ell$ (i.e. of cokernel rank 1), then the fiber $f^{-1}(0)$ is the germ of ICIS of embedded dimension $1-\ell$. It means that this fiber is actually the germ of an IHS (isolated hypersurface singularity). The IHS's admit a stabilization allowing to compare hypersurfaces of different dimensions, see Sect. 2.5. The theory of characteristic classes associated with the stable classification of IHS's is the theory of Legendre characteristic classes. Thus the term $R_\alpha^{1-\ell}$ of the localized Thom polynomial for a given cokernel rank 1 ICIS α is determined by the Thom polynomial for the corresponding IHS. The same is applied to the term $R_{\underline{\alpha}}^{1-\ell}$ of the residue polynomial for arbitrary multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$. Recall that the ring $\mathcal{L}$ of universal Legendre characteristic classes is generated by the classes u, a_i which are subject to relations (2).

3.15. Theorem. *The polynomial $R_{\underline{\alpha}}^{(1-\ell)}(v_1, \dots, v_{1-\ell}, u_1)$ of the first localized term in the residue polynomial of the complete intersection multisingularity $\underline{\alpha}$ is nontrivial only if all singularities α_i forming the multisingularity $\underline{\alpha}$ are hypersurface singularities. If this is true, then this polynomial can be obtained from the Legendre residue polynomial of the hypersurface multisingularity $\underline{\alpha}$ by the change of variables determined by $u = u_1$ and*

$$1 + a_1 + a_2 + \dots = \frac{(1+u_1)^{1-\ell} - (1+u_1)^{-\ell} v_1 + \dots \pm v_{1-\ell}}{1 + v_1 + \dots + v_{1-\ell}}.$$

Remark that the stabilization of IHS's does not extend to the complete intersection singularities of cokernel rank greater than 1: for different $\ell \le 0$ the classifications of $(-\ell)$-dimensional ICIS's are quite different.

3.3. Symmetric and Lagrange degeneracy loci

In this section we summarize some results on symmetric and Lagrange degeneracy loci. These results can be considered as symmetric analogues of the results on Porteous-Thom classes. Most of the relations of this section are known, see [14, 16, 10, 11, 12, 29]. Nevertheless, our presentation of these relations is quite different.

Consider the following problem. Assume we are given a complex vector bundle $V \to M$ over some smooth base and a generic self-conjugate morphism $\varphi : V \to V^*$. We may consider φ as a family of quadratic forms on the fibers of V or as a section of the bundle $\operatorname{Sym}^2 V^*$. The problem is to determine the cohomology classes dual to the locus $\Omega^r \subset M$ formed by the points at which φ has at least r-dimensional kernel.

The most efficient solution to this problem uses the language of symplectic geometry. Recall that the *symplectic structure* on a vector space E of even dimension $2n$ is a non-degenerate skew-symmetric bilinear form. The standard example is the space of the form $E = V \oplus V^*$ where the value of the symplectic form on the vectors $\xi \oplus \eta$, $\xi' \oplus \eta'$ is given by $\langle \xi', \eta \rangle - \langle \xi, \eta' \rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing between vectors and covectors.

A subspace $L \subset E$ of the middle dimension n is called *Lagrangian* if it is isotrope i.e. if $L^\perp = L$ where the orthogonal complement is considered with respect to the symplectic structure. All Lagrange subspaces of the fixed symplectic space E form the *Lagrange Grassmannian* Λ_n. Remark that a linear map $V \to V^*$ is self-adjoint iff its graph is Lagrangian. Thus the Lagrange Grassmannian can be considered as the natural compactification of the space of quadratic forms. In particular, $\dim \Lambda_n = n\,(n{+}1)/2$.

Now, consider more general problem formulated as follows. Consider a vector bundle E of even rank $2n$ over some smooth base M. Assume that the fibers of E are equipped with a symplectic structure smoothly depending on the point of the base. Let V, W be two Lagrange subbundles of E i.e. subbundles whose fibers are Lagrangian. We look for the cohomology class dual to the locus $\Omega^r \subset M$ formed by the points $x \in M$ at which the fibers V_x, W_x have at least r-dimensional intersection. Following [41] we call

$$[\Omega^r] \subset H^*(M)$$

Arnold-Fuks classes.

The problem on a self-adjoint map $\varphi : V \to V^*$ is a particular case of this one: for the symplectic bundle E one should take $E = V \oplus V^*$ and for Lagrange subbundles one should take the bundle $V \oplus \{0\}$ and the graph of the morphism φ, respectively.

3.16. Definition. The ring $\mathcal{L}^{\mathrm{Lag}}$ of *universal Lagrange characteristic classes* is the quotient ring of polynomials in variables $a_1, a_2, \ldots,$

$\deg a_i = i$, modulo the ideal generated by the relations:

$$a_k^2 - 2a_{k+1}a_{k-1} + 2a_{k+2}a_{k-2} - \dots \pm 2a_{2k} = 0. \tag{20}$$

The ring $\mathcal{L}^{\mathrm{Lag}}$ is the cohomology ring of the stable Lagrange Grassmannian $\Lambda = \lim_n \Lambda_n$. The generators a_i correspond (up to a sign) to the Chern classes of the tautological rank n bundle U over Λ_n, namely, we set $a_i = c_i(U^*) = (-1)^i c_i(U)$. The relations can be written also in the form

$$(1 + a_1 + a_2 + \dots)(1 - a_1 + a_2 - \dots) = 1 \quad \text{or} \quad c(U + U^*) = 1.$$

In this form they immediately follow from the natural isomorphism $E/U \simeq U^*$ provided by the symplectic structure. The relations allow one to expand all powers of the variables a_i. The additive basis of $\mathcal{L}^{\mathrm{Lag}}$ is formed by the monomials of the form $a_1^{i_1} a_2^{i_2} \dots$ with $i_j \in \{0, 1\}$.

The Lagrange analogue of Schur polynomials are the so called *Schur Q-polynomials*. These are certain elements $Q_{\lambda_1,\dots,\lambda_r} \in \mathcal{L}^{\mathrm{Lag}}$ defined for any sequence $\lambda = (\lambda_1, \dots, \lambda_r)$ of positive integers by the following conditions.

- if $r = 1$, we set $Q_k = a_k$;
- if $r = 2$, we set

$$Q_{k,l} = a_k^2 - 2a_{k+1}a_{l-1} + 2a_{k+2}a_{l-2} - 2a_{k+3}a_{l-3} + \dots;$$

- for any even $r \geq 4$ we set

$$Q_{\lambda_1,\dots,\lambda_r} = \mathrm{Pf}\, |Q_{\lambda_i,\lambda_j}|_{1 \leq i,j \leq r};$$

- for any odd $r \geq 3$ we set

$$Q_{\lambda_1,\dots,\lambda_r} = \sum_{k=1}^{r} (-1)^{k-1} a_{\lambda_k}\, Q_{\lambda_1,\dots,\lambda_{k-1},\lambda_{k+1},\dots,\lambda_r}$$

Here Pf is the Pfaffian. Recall that the Pfaffian of a skew-symmetric matrix $\omega = \|\omega_{i,j}\|$ of even order $2n$ is given, by definition, by the equality

$$\mathrm{Pf}\, \|\omega_{i,j}\| = \sum \pm \omega_{i_1,i_2} \cdots \omega_{i_{2n-1},i_{2n}},$$

where the sum runs over all $(2n-1)!!$ ways to represent $\{1, 2, \dots, 2n\}$ as a union of n pairs $\{i_1, i_2\} \cup \dots \cup \{i_{2n-1}, i_{2n}\}$ and $\pm$ is the sign of the permutation $(1, 2, \dots, 2n) \mapsto (i_1, i_2, \dots, i_{2n})$.

The equality $Q_{k,l} = -Q_{l,k}$ for $k + l$ odd follows directly from the definition, and for $k + l$ even it follows from the identity (20). Moreover, the polynomial $Q_{\lambda_1,\dots,\lambda_r}$ depends skew-symmetrically on the indices λ_i. This follows from the fact that the Pfaffian is skew-symmetric with respect to simultaneous permutations of rows and columns of the matrix. In particular, $Q_{\lambda_1,\dots,\lambda_r}=0$ if one has $\lambda_i=\lambda_j$ for some $i\neq j$.

3.17. Remark. The distinction between the cases of even and odd r is apparent. In fact, there is the following explicit formula due to V. Kryukov:

$$Q_{\lambda_1,\dots,\lambda_r}=\sum_{i_1,\dots,i_r} w_{i_1,\dots,i_r}\, a_{\lambda_1+i_1}\dots a_{\lambda_r+i_r}, \tag{21}$$

where the coefficients $w_{i_1,\dots,i_r}$ do not depend on $\lambda_1,\dots,\lambda_r$ and are given by the formal expansion

(22)
$$\begin{aligned}\sum w_{i_1,\dots,i_r}\,\tau_1^{i_1}\dots\tau_r^{i_r} &= \prod_{1\le i<j\le r}\frac{\tau_j-\tau_i}{\tau_j+\tau_i}=\prod_{1\le i<j\le r}\frac{1-\tau_i\tau_j^{-1}}{1+\tau_i\tau_j^{-1}}\\ &=\prod_{1\le i<j\le r}(1-2\,\tau_i\tau_j^{-1}+2\,\tau_i^2\tau_j^{-2}-2\,\tau_i^3\tau_j^{-3}+\dots).\end{aligned}$$

Equation (22) is considered in the ring of infinite series in generators $\tau_1/\tau_2,\dots,\tau_{r-1}/\tau_r$, or, which is equivalent, in the completion of the ring of Loran polynomials with respect to an auxiliary grading such that the degree of the monomial $\tau_1^{i_1}\dots\tau_r^{i_r}$ is equal to $\sum_{k=1}^r k\,i_k$.

We are able now to formulate principle results on Lagrange and symmetric degeneracy problems. Let M be a manifold. Consider a symplectic vector bundle E and two its Lagrange subbundles V,W as at the beginning of this section.

3.18. Definition. *Lagrange characteristic classes of the triple* (E,V,W) are the Chern classes $a_i=c_i(E-V-W)=c_i(V^*-W)=c_i(W^*-V)$.

The identity (20) follows immediately from the equalities $E/V\simeq V^*$, $E/W\simeq W^*$ provided by the non-degeneracy of the symplectic structure.

For a self-adjoint map $V\to V^*$ one has $V\simeq W$ so that the Lagrange characteristic classes in this case are $a_i=c_i(V^*-V)$.

3.19. Theorem. *The Arnold-Fuks class* $[\Omega^r]\in H^*(M)$ *of the triple* (E,V,W) *is a universal Lagrange characteristic class given by an appropriate Schur* Q*-polynomial:*

$$[\Omega^r]=Q_{r,r-1,\dots,1}.$$

The *derived Arnold-Fuks classes* can also be defined. The singularity locus of the variety Ω^r coincides with Ω^{r+1}. Consider the *standard resolution* $\widetilde{\Omega}^r$ of Ω^r defined as the subvariety of $G_r(E)$ formed by all pairs (x, K_x), where $x \in M$ and K_x is an r-dimensional subspace of the intersection $V_x \cap W_x \subset E_x$. Denote by K the restriction to $\widetilde{\Omega}^r$ of the tautological rank r bundle over $G_r(E)$ and by $p_r : \widetilde{\Omega}^r \to M$ the natural projection.

Let R be an arbitrary polynomial in variables $v_1, \ldots, v_r$. Denote by $R(c(K)) \in H^*(\widetilde{\Omega})$ the cohomology class obtained by setting $v_i = c_i(K)$.

3.20. Theorem. *The push-forward class $p_{r*}R(c(K))$ is expressed as a universal Lagrange characteristic class uniquely determined by r, R and evaluated for the given triple (E, V, W).*

More explicitly, set formally $c(K) = \prod_{i=1}^r (1 - t_i)$, substitute the corresponding symmetric functions in $-t_i$ to R and expand all brackets. Then the homomorphism p_{r_*} is given on the resulting monomials in the variables t_i by the following explicit formula

$$p_{r*}t_1^{s_1} \ldots t_r^{s_r} = Q_{r+s_1, r-1+s_2, \ldots, 1+s_r}.$$

3.21. Theorem. *The collection of homomorphisms p_{r*} provides the universal splitting*

$$\mathcal{L}^{\mathrm{Lag}} = \bigoplus_r H^*(BU(r)).$$

In other words, any universal Lagrange characteristic class $P \in \mathcal{L}^{\mathrm{Lag}}$ can be presented uniquely in the form

$$P = \sum_r p_{r*}R^{(r)},$$

where $R^{(r)}$ is a polynomial of degree $\deg R^{(r)} = \deg P - r\,(r+1)/2$ *in the variables $v_1, \ldots, v_r$.*

The splitting of the Theorem is a particular case of the splitting (12). Namely, consider the classification of quadratic forms *in arbitrary number of variables*, where the forms $Q(x)$ and $Q'(x, y) = Q(x) + y^2$ are considered as stably equivalent, where $x \in \mathbb{C}^n$, $y \in \mathbb{C}$. The classifying space for this classification is the stable Lagrange Grassmannian $\Lambda = \lim_n \Lambda_n$. The singularity classes for this classification are the classes Ω^r of forms with kernel rank r. By the *symmetry group* G_{Ω^r} of the class Ω^r we mean the stationary subgroup of any quadratic form $x \in \Omega^r$ depending on *the smallest possible n number of variables.* It is clear that this representative is exactly the zero form and $n = r$. The stationary group for this element contains all linear transformations of the space $\mathbb{C}^r$. Therefore,

$$G_{\Omega^r} \sim U(r)$$

and the splitting of Theorem follows from (12). □

3.4. Twisted Lagrange degeneracy loci and localized Legendre characteristic classes

In applications, instead of Lagrange and symmetric degeneracy loci, one meets more often their twisted analogues that are called Legendre degeneracy loci.

Let $V \to M$ be a complex vector bundle. Consider a family φ of quadratic forms on the fibers of V that take values not in numbers but in the fibers of a supplementary line bundle $I \to M$. One can treat φ as a self-adjoint morphism $V \to V^* \otimes I$ or as a section of the bundle $\operatorname{Sym}^2 V^* \otimes I$.

Similarly, one can consider a vector bundle $E \to M$ equipped with the symplectic form on its fibers that takes values in the fibers of a line bundle I. Lagrange subbundles of this twisted symplectic bundle are defined similarly to the non-twisted case. If V, W are two Lagrange subbundles in E, then one defines in a similar way the degeneracy loci Ω^r and the corresponding Arnold-Fuks class $[\Omega^r] \in H^*(M)$.

Moreover, similarly to the non-twisted case one can consider the resolution subvariety $\widetilde{\Omega}^r \subset G_r(E)$, the tautological rank r bundle K over $\widetilde{\Omega}^r$ and the derived Arnold-Fuks class $p_{r*}R \in H^*(M)$ where R is an arbitrary polynomial in $v_i = c_i(K)$.

The twisted analogue of the Lagrange characteristic classes are Legendre ones. Recall (see Sect 2.5) that the ring $\mathcal{L}$ of Legendre characteristic classes is generated by the generators $u, a_1, a_2, \ldots$ that are subject to relations (2).

Lagrange characteristic classes can be obtained from Legendre ones by setting $u = 0$. Conversely, one can show that over $\mathbb{Q}$ as well as over any ring containing $1/2$ there is an isomorphism

$$\mathcal{L}[\tfrac{1}{2}] \simeq \mathbb{Z}[u] \otimes \mathcal{L}^{\mathrm{Lag}}[\tfrac{1}{2}]. \tag{23}$$

On should remark that this splitting does not hold over integers. This remark is especially important in the real problems where the Chern classes are replaced by the Stiefel-Whitney classes, all coefficients are reduced modulo 2 and the division by 2 is forbidden.

3.22. Definition. Legendre characteristic classes associated with the twisted symplectic bundle E and its Lagrange subbundles V, W are the Chern classes $u = c_1(I)$ and

$$a_i = c_i(E - V - W) = c_i(V^*\otimes I - W) = c_i(W^*\otimes I - V).$$

Respectively, the Legendre characteristic classes associated with a twisted self-adjoint morphism $V \to V^*\otimes I$ are $u = c_1(I)$ and $a_i = c_i(V^*\otimes I - V)$.

The identity (3) for these classes follows immediately from the isomorphisms $E/V \simeq V^*\otimes I$, $E/W \simeq W^*\otimes I$ implied by the non-degeneracy of the symplectic structure.

The twisted version of theorems of the previous section holds also true. One should replace only Lagrange characteristic classes by their Legendre analogues.

3.23. Theorem. *Every twisted Arnold-Fuks class as well as any its derived class is expressed as a universal Legendre characteristic class. Moreover, the collection of homomorphisms p_{r*} provides the universal splitting*

$$\mathcal{L} \simeq \bigoplus_r H^*(BU(r) \times BU(1)).$$

In other words, any Legendre characteristic class P has a unique representation in the following localized form

$$P = \sum_r p_{r*} R^{(r)},$$

where $R^{(r)}$ are polynomials in the classes $v_i = c_i(K)$, $i = 1, \ldots, r$, and $u = c_1(I)$ of degree $\deg R^{(r)} = \deg P - r(r+1)/2$.

The importance of the splitting of this theorem is in the fact that for any Legendre or twisted symmetric degeneracy problem the term $p_{r*}R^{(r)}$ is represented by a cycle supported on the corresponding locus Ω^r.

The explicit formulae for the homomorphisms p_{r*} follow from the isomorphism (23). Namely, on can use the following trick borrowed from [14]. First consider the case when $I = J^{\otimes 2}$, where J is another line bundle with $c_1(J) = c_1(I)/2 = u/2$. In this case the twisted symplectic structure on E induces the non-twisted symplectic structure on $E' = E \otimes J^*$. The subbundles V, W of E induce the Lagrange subbundles $V' = V\otimes J^*$ and $W' = W\otimes J^*$ of E'. Moreover, the degeneracy locus Ω^r and its resolution $\widetilde{\Omega}^r$ for the triple (E, V, W) coincide with those for the triple (E', V', W').

Therefore we can apply the formulas of the previous section to find the direct images of the characteristic classes of the tautological bundle

K'. Since the Chern classes $c_i(K) = c_i(K' \otimes J)$ are expressed as polynomials in the classes $c_i(K')$ and $u/2 = c_1(J)$, this allows us to compute the direct image of any polynomial in the classes $c_i(K)$. The formulas obtained in this way can be applied to arbitrary line bundle I since they are universal. Remark that the intermediate steps in the derivation of these formulas use the division by 2 but the final expression for the derived classes has only integer coefficients since the group of Legendre characteristic classes is torsion free.

There are Legendre analogues of Theorems 3.6, 3.9, 3.12. In particular, there is an explicit formula for the initial terms of the localized form of a residue polynomial for a Lagrange multisingularity. These terms can be obtained from the trivial observation that the case $\ell = 0$ satisfies both $\ell \leq 0$ and $\ell \geq 0$. Namely, the terms $p_{0*}R^{(0)}_{\underline{\alpha}} + p_{1*}R^{(1)}_{\underline{\alpha}}$ for a Lagrange multisingularity $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$ are non-trivial only if $\alpha_i = A_{p_i}$ for $i = 1, \dots, r$ and some $p_i \geq 1$. By Theorem 3.15, these terms are determined by the corresponding formulas for the 0-dimensional complete intersection multisingularities $(A_{p_1}, \dots, A_{p_r})$ that is by the formulas of Theorem 3.12 with $\ell = 0$.

§4. Computation of Thom polynomials

4.1. Restriction method

The Porteous-Thom singularities are determined by the 1-jet of the map germ. The Thom polynomial for these singularities have been computed in [27] by resolving these singularities and applying the known formulas for the Gysin homomorphism, see Sect. 3.1. The resolution method can be applied also for certain singularity classes determined by higher order jets, see eg. [28, 34, 21]. Nevertheless, for more complicated singularities finding an appropriate resolution is not easy and the method meets serious technical difficulties.

Quite recently R. Rimányi [31] suggested a much more simple indirect method that uses the following idea. Since the existence of the Thom polynomial is established, it remains to find the coefficients of this polynomial. For that it is sufficient to consider a number of examples for which both the Chern classes of the map and the classes dual to the singularity loci are known. Every such example provides linear relations on the coefficients of the Thom polynomial. With an appropriate choice of the examples these relations could determine the polynomial completely. Rimányi has shown that this method can be efficiently applied to compute Thom polynomials for essentially all classified classes of complex singularities. In the real problems the method can also be applied though with less efficiency [9].

For the collection of the test maps one can use the following ones. Let $f_0 : \mathbb{C}^m \to \mathbb{C}^n$ be the 'normal form' of a certain singularity class α. The symmetry group G_α of this singularity acts on the source and the target spaces $\mathbb{C}^m$, $\mathbb{C}^n$, respectively. Denote by BG_α the classifying space of the group G_α (or some of its smooth finite-dimensional approximations). Denote by M and N the total spaces of the vector bundles $E \to BG_\alpha$ and $F \to BG_\alpha$ with the fibers $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, corresponding to these actions. Finally, let $f : M \to N$ be the fibred map that coincides with f_0 on each fiber.

The relative Chern classes $c(f) = c(TN)/c(TM) = c(F)/c(E)$ of the test map can usually be easily computed using the splitting principle. Some information is known also on the classes dual to the singularity loci. For example, the locus of the singularity α is the zero section of the bundle $E \to BG_\alpha$ and its dual is the Euler characteristic class $e(E) = c_m(E)$ of the bundle. Besides, for any class β which is not adjacent to α the corresponding singularity locus is empty and so the dual cohomology class is equal to zero.

Relations arising from these test examples is usually sufficient to compute the Thom polynomials of all necessary singularities. To see when this method can lead to the desired answer let us turn back to the splitting (1). Assume that we know the complete classification of singularities up to a given codimension p. Assume that the classification problem under consideration is a complex one and that there are only finitely many singularity classes of codimension below p. In this case the splitting (12) holds and every test example allows us to compute the corresponding summand of the Thom polynomial provided by this splitting. This argument explains the applicability of the method in complex problems.

As an example let us show the computation of the Thom polynomial for the 'pleat' singularity $\Sigma^{1,1}$ of a map between two manifolds of equal dimension. Let $\xi \to B$ be a line bundle over some smooth base B, say, the tautological line bundle over $B = \mathbb{C}P^n$ for some $n \geq 2$. Set $t = c_1(\xi) \in H^2(B)$. Consider the following quasihomogeneous normal forms of the simplest singularities Σ^1 and $\Sigma^{1,1}$:

$$x \mapsto x^2, \qquad (x, y) \mapsto (x^3 + xy, y).$$

These formulas can be interpreted as the fibred maps $\xi \to \xi^{\otimes 2}$ and $\xi \oplus \xi^{\otimes 2} \to \xi^{\otimes 3} \oplus \xi^{\otimes 2}$, respectively, of the total spaces of the corresponding vector bundles. The source and the target manifolds are homotopy equivalent to B so we can identify $H^*(M) \simeq H^*(N) \simeq H^*(B)$. With these identifications, the relative Chern classes of the test maps are

$$\frac{1+2t}{1+t} = 1 + t - t^2 + \dots, \qquad \frac{(1+3t)(1+2t)}{(1+t)(1+2t)} = 1 + 2t - 2t^2 + \dots,$$

respectively. Besides, the Euler class of the bundle $\xi \oplus \xi^{\otimes 2}$ is equal to $2t^2$. It follows that the unknown coefficients a, b of the desired Thom polynomial $a\,c_1^2 + b\,c_2$ satisfy the relations

$$0 = a\,t^2 + b\,(-t^2), \qquad 2\,t^2 = a\,(2t)^2 + b\,(-2t^2).$$

These equations lead to the unique solution $a = b = 1$ which means that the Tom polynomial of the singularity $\Sigma^{1,1}$ is equal to $c_1^2 + c_2$.

4.2. Computation of the residue polynomials for multisingularities

The restriction method considered above can be applied to the study of the characteristic classes of multisingularities. To apply the formulas of the section 2.6 we need every test map to be *proper* (this restriction is not needed for the study of monosingularities). The maps of the previous section satisfy this condition if *the relative dimension* $\ell = \dim N - \dim M$ *is non-negative.* Thus the direct application of the restriction method provides the computation of the residue polynomials of multisingularities with $\ell \geq 0$.

A new feature in this computation is that of the Gysin homomorphism $f_* : H^*(M) \to H^*(N)$ for the test maps. Assume that M and N are the total spaces of the vector bundles E and F, respectively, over some smooth base B. Let $f : M \to N$ be a fibred map whose restriction to each fiber coincides with the standard proper quasihomogeneous map $f_0 : \mathbb{C}^m \to \mathbb{C}^n$ in some coordinates. Because of the isomorphism $H^*(M) \simeq H^*(N) \simeq H^*(B)$ we can consider the homomorphism f_* as acting in the cohomology group $H^*(B)$. Due to the projection formula this homomorphism acts as the multiplication by the class $f_*(1) \in H^{2\ell}(B)$. This class can by found using the following lemma.

4.1. Lemma. *The class $f_*(1)$ satisfies the relation*

$$f_*(1)\,c_m(E) = c_n(F).$$

In applications, the cohomology ring $H^*(B)$ has no zero divisors, therefore, the relation of the lemma determines the class $f_*(1)$ and the homomorphism f_* uniquely.

Proof. The top Chern classes $c_m(E)$ and $c_n(F)$ are the cohomology classes Poincaré dual to the zero sections of the bundles E and F, respectively. In other words, $c_m(E) = i_*(1)$ and $c_n(F) = j_*(1)$, where $i : B \to M$ and $j : B \to N$ are the zero section embeddings. Since $j = f \circ i$, the relation of the lemma follows from the identity

$$f_*\, i_* = j_*.$$

Let now $\ell < 0$. Let the holomorphic map $f : M \to N$ be one of the test maps of the previous section used for the computation of the Thom polynomials. Then all level sets of f have positive dimensions. Therefore, this map cannot be proper, the homomorphism f_* is not defined, and the formulas of the section 2.6 cannot be applied directly. To extend the restriction method to this case we use the following observation. Set $\ell = \dim N - \dim M$. Fix some integer $s \geq \max(0, -\ell)$. Let $\Sigma^s = \Sigma^s(f) \subset M$ be the locus of the corresponding Porteous-Thom singularity. In what follows we make the following weaker assumption on the map f:

4.2. Assumption. The restriction of the map f to the locus Σ^s is proper.

If this assumption holds, then the homomorphism f_* is well defined on those classes that can be represented by cycles supported on Σ^s. More precisely, let $p_s : \tilde{\Sigma}^s \to M$ be the natural resolution of Σ^s from Sect. 3.1. Then the map $f \circ p_s : \widetilde{\Sigma}^s \to N$ is proper and the homomorphism $(f \circ p_s)_* : H^*(\widetilde{\Sigma}^s) \to H^*(N)$ is well defined. In other words, *the homomorphism f_* is well defined on the image of the homomorphism $p_{s*} : H^*(\widetilde{\Sigma}^s) \to H^*(M)$.*

In particular, consider some element $P \in H^*(M)$ that is represented as a polynomial in the relative Chern classes $c_i = c_i(f)$. Assume that the localized form of Sect. 3.2 for this polynomial $P = \sum_k p_{k*} R^{(k)}$ has nontrivial terms with $k \geq s$ only. Then the homomorphism f_* is well defined on such a class. Namely, we set

$$f_* P = \sum_k (f p_k)_* R^{(k)}.$$

This assumption suggests the following conjectural sharpening of Conjecture 3.6.

4.3. Conjecture. *Let $f : M \to N$ be a holomorphic map, not necessary proper, that satisfies Assumption* 4.2. *Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$ be a multisingularity type such that each individual singularity α_i of this collection has corank greater than or equal to s. Then every residue polynomial in the right hand side expressions of* (5–6) *is supported on*

Σ^s, *the homomorphism* f_* *is well defined on these polynomials and relations* (5–6) *hold.*

Now we observe that for $\ell < 0$ the test maps considered in the previous section satisfy Assumption 4.2 with $s = -\ell + 1$. This allows us to apply the restriction method to finding the residue polynomial of multisingularities of maps with $\ell < 0$.

4.3. Tables of computed polynomials

In this section we present some of the results on the computation of Thom polynomials for local singularities and residue polynomials for multisingularities. More complete tables occupy many pages. They are available on [23]. The polynomials are represented both in the localized form and in terms of the Chern classes. In the localized form, we skip the expressions for the terms given by the explicit formulas of Theorems 3.9, 3.12, and 3.15. The tables below allow the reader to estimate to what extend these formulas determine the residue polynomials.

The standard notation for the singularity classes is taken mostly from [3, 4]. All singularities in the studied range of dimensions are simple (have no modula in the normal form). Remark that the stable classification of $\mathcal{K}$-singularities is independent for different values ℓ of the relative dimension of the map. Therefore, one should not confuse with similar notation of different singularity classes appearing for different ℓ. Similarly, the homomorphism p_{r*} of Sect. 3.2 is different for different ℓ. By the codimension of a multisingularity we mean its complex codimension in the source manifold, that is, the degree of the corresponding residue polynomial.

Table 2 represents the residue polynomials of Legendre multisingularities (or hypersurface singularities) up to codimension 4. Up to codimension 6 the formulas can be found in [23].

Table 2: **Residue polynomials for Legendre multisingularities**

$$
\begin{aligned}
R_{A_1} &= p_{0*}(1) = 1\\
R_{A_2} &= p_{1*}(1) = a_1\\
R_{A_1^2} &= -p_{0*}(u) - 3\,p_{1*}(1) = -u - 3\,a_1\\
R_{A_3} &= p_{1*}(u - 3\,v_1) = u\,a_1 + 3\,a_2\\
R_{A_1A_2} &= -6\,p_{1*}(u - 2\,v_1) = -6\,(u\,a_1 + 2\,a_2)\\
R_{A_1^3} &= 2\,p_{0*}(u^2) + 2\,p_{1*}R^{(1)}_{A_1^3} = 2\,(u^2 + 19\,a_1u + 30\,a_2)\\
R_{A_4} &= p_{1*}R^{(1)}_{A_4} + 3\,p_{2*}(1) = a_1u^2 + 4\,a_2u + 3\,a_1a_2 + 6\,a_3\\
R_{D_4} &= p_{2*}(1) = -u\,a_2 + a_1a_2 - 2\,a_3
\end{aligned}
$$

$$
\begin{aligned}
R_{A_2^2} &= p_{1*}R^{(1)}_{A_2^2} - 21\,p_{2*}(1) = -3\,(3\,a_1u^2 + 8\,a_2u + 7\,a_1a_2 + 6\,a_3)\\
R_{A_1A_3} &= p_{1*}R^{(1)}_{A_1A_3} - 24\,p_{2*}(1) = -4\,(2\,a_1u^2 + 5\,a_2u + 6\,a_1a_2 + 3\,a_3)\\
R_{A_1^2A_2} &= p_{1*}R^{(1)}_{A_1^2A_2} + 144\,p_{2*}(1) = 24\,(3\,a_1u^2 + 7\,a_2u + 6\,a_1a_2 + 3\,a_3)\\
R_{A_1^4} &= -6\,p_{0*}(u^3) + p_{1*}R^{(1)}_{A_1^4} - 1026\,p_{2*}(1)\\
&= -6\,(u^3 + 111\,a_1u^2 + 239\,a_2u + 171\,a_1a_2 + 78\,a_3)\\
R_{A_5} &= p_{1*}R^{(1)}_{A_5} + p_{2*}(16\,u - 27\,v_1)\\
&= a_1u^3 - 4\,a_2u^2 + 16\,a_1a_2u - 12\,a_3u + 27\,a_1a_3 + 6\,a_4\\
R_{D_5} &= 2\,p_{2*}(2\,u - 3\,v_1) = -2\,(2\,a_2u^2 - 2\,a_1a_2u + 7\,a_3u - 3\,a_1a_3 + 6\,a_4)\\
R_{A_2A_3} &= p_{1*}R^{(1)}_{A_2A_3} - 6\,p_{2*}(28\,u - 39\,v_1)\\
&= -6\,(2\,a_1u^3 - 10\,a_2u^2 + 28\,a_1a_2u - 39\,a_3u + 39\,a_1a_3 - 18\,a_4)\\
R_{A_1A_4} &= p_{1*}R^{(1)}_{A_1A_4} - 70\,p_{2*}(2\,u - 3\,v_1)\\
&= -10\,(a_1u^3 - 4\,a_2u^2 + 14\,a_1a_2u - 16\,a_3u + 21\,a_1a_3 - 6\,a_4)\\
R_{A_1D_4} &= -4\,p_{2*}(5\,u - 6\,v_1) = 4\,(5\,a_2u^2 - 5\,a_1a_2u + 16\,a_3u - 6\,a_1a_3 + 12\,a_4)\\
R_{A_1A_2^2} &= p_{1*}R^{(1)}_{A_1A_2^2} + 18\,p_{2*}(74\,u - 95\,v_1)\\
&= 18\,(7\,a_1u^3 - 20\,a_2u^2 + 74\,a_1a_2u - 96\,a_3u + 95\,a_1a_3 - 50\,a_4)\\
R_{A_1^2A_3} &= p_{1*}R^{(1)}_{A_1^2A_3} + 18\,p_{2*}(76\,u - 99\,v_1)\\
&= 2\,(56\,a_1u^3 - 220\,a_2u^2 + 684\,a_1a_2u - 951\,a_3u + 891\,a_1a_3 - 522\,a_4)\\
R_{A_1^3A_2} &= p_{1*}R^{(1)}_{A_1^3A_2} - 2400\,p_{2*}(5\,u - 6\,v_1)\\
&= -48\,(28\,a_1u^3 - 55\,a_2u^2 + 250\,a_1a_2u - 318\,a_3u + 300\,a_1a_3 - 180\,a_4)\\
R_{A_1^5} &= 24\,p_{0*}(u^4) + p_{1*}R^{(1)}_{A_1^5} + 72\,p_{2*}(1621\,u - 1830\,v_1)\\
&= 24\,(u^4 + 671\,a_1u^3 - 701\,a_2u^2 + 4863\,a_1a_2u\\
&\qquad - 5844\,a_3u + 5490\,a_1a_3 - 3420\,a_4)
\end{aligned}
$$

The classification of $\mathcal{K}$-singularities of maps of relative dimension $\ell = -1$, that is, of one-dimensional complete intersection singularities starts with the classification of plane curve singularities. Up to codimension 5, the two classifications coincide and every residue polynomial of the complete intersection multisingularity is determined due to Theorem 3.15 by the residue polynomial of the corresponding hypersurface multisingularity. In codimension 6, there appears the simplest space curve singularity, S_5. The residue polynomials of the codimension 5 multisingularities are presented in Table 3. Up to codimension 8, these polynomials are available in [23].

Table 3: **Residue polynomials for multisingularities with $\ell = -1$**

$$
\begin{aligned}
R_{A_5} &= p_{2*}R^{(2)}_{A_5} + 6\,p_{3*}(1)\\
&= 2\,(15\,c_1^6 + 5\,c_2c_1^4 + 25\,c_3c_1^3 - 26\,c_2^2c_1^2 - c_4c_1^2 - 15\,c_2c_3c_1\\
&\qquad - 6\,c_5c_1 + 3\,c_2^3 + 3\,c_3^2 - 3\,c_2c_4)\\
R_{D_5} &= p_{2*}R^{(2)}_{D_5} - 4\,p_{3*}(1)\\
&= 2\,(2\,c_1^6 - c_2\,c_1^4 - 9\,c_3c_1^3 + 6\,c_2^2c_1^2 - 3\,c_4c_1^2 + 6\,c_2c_3c_1\\
&\qquad + c_5c_1 - 2\,c_2^3 - 2\,c_3^2 + 2\,c_2c_4)\\
R_{S_5} &= p_{3*}(1) = c_2^3 - 2\,c_1c_3c_2 - c_4c_2 + c_3^2 + c_1^2c_4\\
R_{A_2A_3} &= p_{2*}R^{(2)}_{A_2A_3} - 12\,p_{3*}(1)\\
&= -6\,(50\,c_1^6 - 13\,c_2c_1^4 + 23\,c_3c_1^3 - 34\,c_2^2c_1^2 - 7\,c_4c_1^2\\
&\qquad - 14\,c_2c_3c_1 - 7\,c_5c_1 + 2\,c_2^3 + 2\,c_3^2 - 2\,c_2c_4)\\
R_{A_1A_4} &= p_{2*}R^{(2)}_{A_1A_4} - 20\,p_{3*}(1)\\
&= -20\,(13\,c_1^6 - c_2c_1^4 + 11\,c_3c_1^3 - 14\,c_2^2c_1^2 - 3\,c_4c_1^2 - 5\,c_2c_3c_1\\
&\qquad - 2\,c_5c_1 + c_2^3 + c_3^2 - c_2c_4)\\
R_{A_1D_4} &= p_{2*}R^{(2)}_{A_1D_4} + 8\,p_{3*}(1)\\
&= -4\,(5\,c_1^6 - 4\,c_2c_1^4 - 21\,c_3c_1^3 + 15\,c_2^2c_1^2 + 6\,c_2c_3c_1\\
&\qquad + c_5c_1 - 2\,c_2^3 - 2\,c_3^2 + 2\,c_2c_4)\\
R_{A_1A_2^2} &= p_{2*}R^{(2)}_{A_1A_2^2} + 108\,p_{3*}(1)\\
&= 36\,(71\,c_1^6 - 35\,c_2c_1^4 + 33\,c_3c_1^3 - 42\,c_2^2c_1^2 - 9\,c_4c_1^2\\
&\qquad - 15\,c_2c_3c_1 - 6\,c_5c_1 + 3\,c_2^3 + 3\,c_3^2 - 3\,c_2c_4)\\
R_{A_1^2A_3} &= p_{2*}R^{(2)}_{A_1^2A_3} + 72\,p_{3*}(1)\\
&= 2\,(1260\,c_1^6 - 545\,c_2c_1^4 + 425\,c_3c_1^3 - 692\,c_2^2c_1^2 - 173\,c_4c_1^2\\
&\qquad - 214\,c_2c_3c_1 - 97\,c_5c_1 + 36\,c_2^3 + 36\,c_3^2 - 36\,c_2c_4)\\
R_{A_1^3A_2} &= p_{2*}R^{(2)}_{A_1^3A_2} - 864\,p_{3*}(1)\\
&= -48\,(501\,c_1^6 - 332\,c_2c_1^4 + 215\,c_3c_1^3 - 241\,c_2^2c_1^2 - 56\,c_4c_1^2\\
&\qquad - 78\,c_2c_3c_1 - 27\,c_5c_1 + 18\,c_2^3 + 18\,c_3^2 - 18\,c_2c_4)\\
R_{A_1^5} &= p_{2*}R^{(2)}_{A_1^5} + 9408\,p_{3*}(1)\\
&= 24\,(10368\,c_1^6 - 8561\,c_2c_1^4 + 5045\,c_3c_1^3 - 4285\,c_2^2c_1^2 - 1125\,c_4c_1^2\\
&\qquad - 1456\,c_2c_3c_1 - 379\,c_5c_1 + 391\,c_2^3 + 393\,c_3^2 - 390\,c_2c_4 - c_6)
\end{aligned}
$$

For the case of multisingularities of maps of equally dimensional manifolds ($\ell = 0$) and maps of relative dimension $\ell = 1$, the residue polynomials are given in Tables 4 and 5, respectively. These polynomials are computed up to codimension 8, see [23]. In the tables, we present these polynomials up to codimension 5 and 6, respectively. By A_k, $I_{a,b}$, and J_6 we denote singularity classes with local algebras isomorphic to $\mathbb{C}[x]/x^{k+1}$, $\mathbb{C}[x,y]/(x\,y,\ x^a{+}y^b)$, and $\mathbb{C}[x,y]/(x^2,\ x\,y,\ y^2)$, respectively.

Table 4: **Residue polynomials for multisingularities with $\boldsymbol{\ell = 0}$**

$$
\begin{aligned}
R_{A_1} &= p_{1*}(1) = c_1\\
R_{A_2} &= p_{1*}(u_1 - 2\,v_1) = c_1^2 + c_2\\
R_{A_1^2} &= -2\,p_{1*}(2\,u_1 - 3\,v_1) = -2\,(2\,c_1^2 + c_2)\\
R_{A_3} &= p_{1*}((u_1 - 3\,v_1)\,(u_1 - 2\,v_1)) = c_1^3 + 3\,c_2c_1 + 2\,c_3\\
R_{A_1A_2} &= -6\,p_{1*}((u_1 - 2\,v_1)^2) = -6\,(c_1^3 + 2\,c_2c_1 + c_3)\\
R_{A_1^3} &= 8\,p_{1*}(5\,u_1^2 - 17\,v_1u_1 + 15\,v_1^2) = 8\,(5\,c_1^3 + 7\,c_2c_1 + 3\,c_3)\\
R_{A_4} &= p_{1*}R^{(1)}_{A_4} + 2\,p_{2*}(1) = c_1^4 + 6\,c_2c_1^2 + 9\,c_3c_1 + 2\,c_2^2 + 6\,c_4\\
R_{I_{2,2}} &= p_{2*}(1) = c_2^2 - c_1c_3\\
R_{A_2^2} &= p_{1*}R^{(1)}_{A_2^2} - 12\,p_{2*}(1) = -3\,(3\,c_1^4 + 12\,c_2c_1^2 + 13\,c_3c_1 + 4\,c_2^2 + 8\,c_4)\\
R_{A_1A_3} &= p_{1*}R^{(1)}_{A_1A_3} - 8\,p_{2*}(1) = -4\,(2\,c_1^4 + 9\,c_2c_1^2 + 11\,c_3c_1 + 2\,c_2^2 + 6\,c_4)\\
R_{A_1^2A_2} &= p_{1*}R^{(1)}_{A_1^2A_2} + 48\,p_{2*}(1) = 24\,(3\,c_1^4 + 10\,c_2c_1^2 + 10\,c_3c_1 + 2\,c_2^2 + 5\,c_4)\\
R_{A_1^4} &= p_{1*}R^{(1)}_{A_1^4} - 288\,p_{2*}(1) = -48\,(14\,c_1^4 + 37\,c_2c_1^2 + 33\,c_3c_1 + 6\,c_2^2 + 15\,c_4)\\
R_{A_5} &= p_{1*}R^{(1)}_{A_5} + 2\,p_{2*}(5\,u_1 - 11\,v_1)\\
&= c_1^5 + 10\,c_2c_1^3 + 25\,c_3c_1^2 + 10\,c_2^2c_1 + 38\,c_4c_1 + 12\,c_2c_3 + 24\,c_5\\
R_{I_{2,3}} &= 2\,p_{2*}(u_1 - 2\,v_1) = -2\,(c_3c_1^2 - c_2^2c_1 + c_4c_1 - c_2c_3)\\
R_{A_2A_3} &= p_{1*}R^{(1)}_{A_2A_3} - 72\,p_{2*}(u_1 - 2\,v_1)\\
&= -12\,(c_1^5 + 7\,c_2c_1^3 + 13\,c_3c_1^2 + 6\,c_2^2c_1 + 17\,c_4c_1 + 6\,c_2c_3 + 10\,c_5)\\
R_{A_1A_4} &= p_{1*}R^{(1)}_{A_1A_4} - 60\,p_{2*}(u_1 - 2\,v_1)\\
&= -10\,(c_1^5 + 8\,c_2c_1^3 + 17\,c_3c_1^2 + 6\,c_2^2c_1 + 22\,c_4c_1 + 6\,c_2c_3 + 12\,c_5)\\
R_{A_1I_{2,2}} &= -2\,p_{2*}(5\,u_1 - 8\,v_1) = 2\,(5\,c_3c_1^2 - 5\,c_2^2c_1 + 3\,c_4c_1 - 3\,c_2c_3)\\
R_{A_1A_2^2} &= p_{1*}R^{(1)}_{A_1A_2^2} + 72\,p_{2*}(7\,u_1 - 13\,v_1)\\
&= 18\,(7\,c_1^5 + 40\,c_2c_1^3 + 65\,c_3c_1^2 + 28\,c_2^2c_1 + 76\,c_4c_1 + 24\,c_2c_3 + 40\,c_5)
\end{aligned}
$$

$$
\begin{aligned}
R_{A_1^2A_3} &= p_{1*}R^{(1)}_{A_1^2A_3} + 48\,p_{2*}(9\,u_1 - 17\,v_1)\\
&= 8\,(14\,c_1^5 + 88\,c_2c_1^3 + 157\,c_3c_1^2 + 54\,c_2^2c_1 + 179\,c_4c_1 + 48\,c_2c_3 + 90\,c_5)\\
R_{A_1^3A_2} &= p_{1*}R^{(1)}_{A_1^3A_2} - 192\,p_{2*}(19\,u_1 - 34\,v_1)\\
&= -48\,(28\,c_1^5 + 139\,c_2c_1^3 + 211\,c_3c_1^2 + 76\,c_2^2c_1\\
&\qquad + 221\,c_4c_1 + 60\,c_2c_3 + 105\,c_5)\\
R_{A_1^5} &= p_{1*}R^{(1)}_{A_1^5} + 768\,p_{2*}(41\,u_1 - 71\,v_1)\\
&= 384\,(42\,c_1^5 + 176\,c_2c_1^3 + 244\,c_3c_1^2 + 82\,c_2^2c_1\\
&\qquad + 236\,c_4c_1 + 60\,c_2c_3 + 105\,c_5)
\end{aligned}
$$

Table 5: **Residue polynomials for multisingularities with $\boldsymbol{\ell = 1}$**

$$
\begin{aligned}
R_{A_0} &= p_{0*}(1) = 1\\
R_{A_0^2} &= -p_{0*}(u_1) = -c_1\\
R_{A_1} &= p_{1*}(1) = c_2\\
R_{A_0^3} &= 2\,p_{0*}(u_1^2) + 2\,p_{1*}(1) = 2\,(c_1^2 + c_2)\\
R_{A_0A_1} &= 2\,p_{1*}(-u_1 + 2\,v_1) = -2\,(c_1c_2 + c_3)\\
R_{A_0^4} &= -6\,p_{0*}(u_1^3) - 6\,p_{1*}(3\,u_1 - 5\,v_1) = -6\,(c_1^3 + 3\,c_1c_2 + 2\,c_3)\\
R_{A_2} &= p_{1*}R^{(1)}_{A_2} = c_2^2 + c_1c_3 + 2\,c_4\\
R_{A_0^2A_1} &= p_{1*}R^{(1)}_{A_0^2A_1} = 2\,(3\,c_1^2c_2 + 2\,c_2^2 + 7\,c_1c_3 + 6\,c_4)\\
R_{A_0^5} &= 24\,p_{0*}(u_1^4) + p_{1*}R^{(1)}_{A_0^5} = 24\,(c_1^4 + 6\,c_1^2c_2 + 2\,c_2^2 + 9\,c_1c_3 + 6\,c_4)\\
R_{A_1^2} &= 2\,p_{1*}(-u_1 + 3\,v_1)\,(2\,u_2 - 3\,u_1v_1 + 6\,v_1^2)\\
&= -2\,(2\,c_1c_2^2 + c_1^2c_3 + 4\,c_2c_3 + 5\,c_1c_4 + 6\,c_5)\\
R_{A_0A_2} &= p_{1*}R^{(1)}_{A_0A_2} = -3\,(c_1c_2^2 + c_1^2c_3 + 2\,c_2c_3 + 4\,c_1c_4 + 4\,c_5)\\
R_{A_0^3A_1} &= p_{1*}R^{(1)}_{A_0^3A_1} = -24\,(c_1^3c_2 + 2\,c_1c_2^2 + 4\,c_1^2c_3 + 3\,c_2c_3 + 8\,c_1c_4 + 6\,c_5)\\
R_{A_0^6} &= -120\,p_{0*}(u_1^5) + p_{1*}R^{(1)}_{A_0^6}\\
&= -120\,(c_1^5 + 10\,c_1^3c_2 + 10\,c_1c_2^2 + 25\,c_1^2c_3 + 12\,c_2c_3 + 38\,c_1c_4 + 24\,c_5)\\
R_{A_3} &= p_{1*}R^{(1)}_{A_3} + p_{2*}(1)\\
&= c_2^3 + 3\,c_1c_2c_3 + c_3^2 + 2\,c_1^2c_4 + 7\,c_2c_4 + 10\,c_1c_5 + 12\,c_6\\
R_{J_6} &= p_{2*}(1) = c_3^2 - c_2c_4
\end{aligned}
$$

$$\begin{aligned}
R_{A_0A_1^2} &= p_{1*}R^{(1)}_{A_0A_1^2} + 24\,p_{2*}(1)\\
&= 8\,(2\,c_1^2c_2^2 + c_2^3 + c_1^3c_3 + 9\,c_1c_2c_3 + 3\,c_3^2 + 8\,c_1^2c_4\\
&\qquad + 9\,c_2c_4 + 21\,c_1c_5 + 18\,c_6)\\
R_{A_0^2A_2} &= p_{1*}R^{(1)}_{A_0^2A_2} + 18\,p_{2*}(1)\\
&= 6\,(2\,c_1^2c_2^2 + c_2^3 + 2\,c_1^3c_3 + 10\,c_1c_2c_3 + 3\,c_3^2 + 13\,c_1^2c_4\\
&\qquad + 11\,c_2c_4 + 30\,c_1c_5 + 24\,c_6)\\
R_{A_0^4A_1} &= p_{1*}R^{(1)}_{A_0^4A_1} + 408\,p_{2*}(1)\\
&= 24\,(5\,c_1^4c_2 + 20\,c_1^2c_2^2 + 5\,c_2^3 + 30\,c_1^3c_3 + 67\,c_1c_2c_3\\
&\qquad + 17\,c_3^2 + 103\,c_1^2c_4 + 55\,c_2c_4 + 178\,c_1c_5 + 120\,c_6)\\
R_{A_0^7} &= 720\,p_{0*}(u_1^6) + p_{1*}R^{(1)}_{A_0^7} + 12240\,p_{2*}(1)\\
&= 720\,(c_1^6 + 15\,c_1^4c_2 + 30\,c_1^2c_2^2 + 5\,c_2^3 + 55\,c_1^3c_3 + 79\,c_1c_2c_3\\
&\qquad + 17\,c_3^2 + 141\,c_1^2c_4 + 55\,c_2c_4 + 202\,c_1c_5 + 120\,c_6)
\end{aligned}$$

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Quasi-convex decomposition in o-minimal structures. Application to the gradient conjecture

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Abstract.

We show that every subset of $\mathbb{R}^n$ definable in an o-minimal structure can be decomposed into a finite number of definable sets that are quasi-convex i.e. have comparable, up to a constant, the intrinsic distance and the distance induced from the embedding. We apply this result to study the limits of secants of the trajectories of gradient vector field ∇f of a C^1 definable function f defined in an open subset of $\mathbb{R}^n$. We show that if the o-minimal structure is polynomially bounded then the limit of such secants exists, that is an analog of the gradient conjecture of R. Thom holds. Moreover we prove that for $n = 2$ the result is true in any o-minimal structure.

§ 0. Introduction

Let f be a real analytic function on an open set $U \subset \mathbb{R}^n$ and let ∇f be its gradient in the Euclidean metric. Let $x(t)$ be a trajectory of ∇f. Then, after Łojasiewicz [16], if $x(t)$ has a limit point $x_0 \in U$, then the length of $x(t)$ is finite and $x(t) \to x_0$ as $t \to \infty$. Moreover, then the trajectory cannot spiral, that is the limit of secants

$$\lim_{t\to\infty} \frac{x(t) - x_0}{|x(t) - x_0|}$$

exists. The last result, known as the gradient conjecture of R. Thom, has been proven recently in [14]. The main purpose of this paper is to

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study this conjecture in the o-minimal set-up, that is for f that is C^1 and definable in an o-minimal structure.

Recall that the o-minimal structures are natural generalizations of the semi-algebraic or the subanalytic geometry satisfying important finiteness properties. The reader that is not familiar with this notion may refer to various introductory references as for instance [4], [5]. An o-minimal structure is called polynomially bounded if for each continuous function $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, $\varphi^{-1}(0) = 0$, definable in the structure, there is a constant $N > 0$ such that $|\varphi(r)| \geq |r|^N$. In o-minimal polynomially bounded structures the classical Łojasiewicz Inequalities (with exponents) hold. On the other hand the o-minimal structures that are not polynomially bounded contain the exponential function, cf. [17], and hence many flat functions. In what follows we suppose that we have fixed an o-minimal structure and the functions we consider are definable in this structure.

The trajectories of the gradient vector field of definable functions have been studied in [9], where an analog of Łojasiewicz's result of finiteness of length was proven. Thus we may place ourselves in the following set-up. We suppose that $f : U \to \mathbb{R}$ is a C^1 definable function defined in an open bounded definable $U \subset \mathbb{R}^n$. We consider a trajectory $x(s)$ of ∇f parameterized by the arc-length s. Since its length is finite the trajectory $x(s)$ has a unique limit point x_0, that is $x(s) \to x_0$ as $s \to s_0$, and either $x_0 \in U$, and then $\nabla f(x_0) = 0$, or $x_0 \in \overline{U} \setminus U$. In both cases we shall study the limits of secants

$$\lim_{s \to s_0} \frac{x(s) - x_0}{|x(s) - x_0|}.$$

Even in the subanalytic case this set-up is more general than the classical analytic one but of course the main difficulty to extend the gradient conjecture to this case is the presence of flat functions. In this paper, we were able to extend most of the properties of the trajectories of the gradient established in [14], but we came short of proving the conjecture in general. Our main results are the following

(1) *The length of the trajectory has the same asymptotic as the distance to the limit point*

$$\frac{|x(s) - x_0|}{|s - s_0|} \to 1 \text{ as } s \to s_0.$$

(2) *The gradient conjecture holds for $n = 2$. More precisely, in this case the trajectory is definable in an o-minimal structure, maybe bigger than the one that contains f.*
(3) *The gradient conjecture holds for polynomially bounded o-minimal structures.*

Moreover, similarly to [14], we were able to "capture" the trajectories arriving to a fixed limit point $x_0 \in \overline{U}$ into a finite number of sets. First of all only finitely many limiting values of f, $\lim_{s \to s_0} f(x(s))$, are allowed along the trajectories of ∇f that tend to x_0, see remark 6.2. Furthermore, if we suppose $|\nabla f| \geq 1$, that we can do by section 3, then we can describe the asymptotic behavior of f at the limit point more precisely. There exists a finite number of definable functions $\{\varphi(r)\}$ of one real variable r, where r stands for radius $r = |x - x_0|$, such that on each trajectory that tends to x_0, $f(x(s)) \sim \varphi(|x(s) - x_0|)$ for exactly one such function φ. We shall call these functions the characteristic functions associated to f at x_0. A more precise result on the asymptotic behavior of f along the trajectory is given in section 7.

Some parts of our argument are similar to that of [14]. Let us stress here the main differences. The characteristic exponents of [14] characterizing the possible asymptotic behavior of f along trajectories are replaced by characteristic functions. In order to show their existence we cannot use the argument of finitude of exponents as in [14] since it does not make sense in general. Instead we use a geometric argument on the structure of definable sets. Namely we show that each definable set can be decomposed into a finite union of quasi-convex cells, as explained in section 1 below. In a polynomially bounded case if $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is definable continuous then φ/r is locally integrable. This is not the case in general. We have to carefully distinguish those φ for which φ/r is integrable, we call them small, and the other ones, that we call unit-like. Many our arguments, in particular the proof of conjecture for $n = 2$, relies on the properties of small functions. We stopped short of carrying out the proof of the gradient conjecture for o-minimal structure because of the existence of small functions with unit-like square root.

The paper is organized as follows. In section 1 we show that each definable set can be decomposed in a finite union of quasi-convex cells, that is such cells in $\mathbb{R}^n$ for which the induced Euclidean distance is comparable, up to a constant, with the intrinsic one (i.e. along the cell). This part is quite technical. The reader interested mainly in the properties of the trajectories of gradient can go directly to section 2. In this section we study the germs of continuous definable functions $\varphi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and the question of integrability of φ/r. In section 3 we show analogs of Łojasiewicz and Bochnak-Łojasiewicz Inequalities for gradient in the o-minimal set-up. The characteristic functions are introduced in section 4. We show that there are finitely many such functions and that each trajectory $x(s)$ of ∇f with the origin as the limit point has to finally end-up in one of finitely many sets $U_\varphi = \{x | c\varphi(x) < |f(x)| < C\varphi(x)\}$, $C, c > 0$, φ being a characteristic function. This holds under the assumption $|\nabla f| \geq 1$, that can be always achieved by replacing f by $\Psi \circ f$ without affecting the trajectories of the gradient. Subsequently the function $F(x) = \frac{f(x)}{\varphi(r)}$ is used as a control function in the sense of Thom. The estimates along trajectories are carried out in sections 5 and 7. It is convenient, for each characteristic function, to make another change of target coordinate , that is to replace f by a function of the from $\Phi \circ f$, so that the corresponding characteristic function φ becomes equivalent to the distance to the origin r. This simplifies many formulae. After such a change we show not only that $F = \frac{f}{\varphi}$ is bounded from zero and infinity on the trajectory but also that it approaches a fixed value and only finitely many such values are allowed. These values, called asymptotic critical values, are studied in section 6. As application we show in section 8 the o-minimal gradient conjecture for $n = 2$ and in section 9 for the polynomially bounded structures.

The result of the first section has been obtained independently by W. Pawłucki [21]. During the redaction of this paper we also learned that some other of the results proven is this paper were obtained during a workshop at the Fields Institute (Toronto) by M. Aschenbrenner, S. Kuhlmann, C. Miller, D. Novikov, P.Speisseger, and S. Starchenko. In particular, we were informed that the gradient conjecture holds in the polynomially bounded o-minimal structures. The case $n = 2$ was stated as an open problem at this meeting.

Notation and convention.

We often write r instead of $|x|$ which is the Euclidean norm of x. We use the standard notation $\varphi = o(\psi)$ or $\varphi = O(\psi)$ to compare the asymptotic behavior of φ and ψ, usually when we approach the origin.

Sometimes we write $\varphi \ll \psi$ instead of $\varphi = o(\psi)$. We write $\varphi \sim \psi$ if $\varphi = O(\psi)$ and $\psi = O(\varphi)$, and $\varphi \simeq \psi$ if $\frac{\varphi}{\psi}$ tends to 1.

Erratum to [14]**:**

The formula on the line -6 on page 783 should be replaced by:

$$\frac{dF}{d\tilde{s}} = \frac{|\nabla' f|}{r^{l-1}} + \frac{|\partial_r f|}{r^{l-1}} \frac{|\partial_r f|}{|\nabla' f|} O(r^{2\omega}) = O(r^{\eta}) + \frac{|\partial_r f|}{|\nabla' f|} O(r^{2\omega}).$$

§1. L-regular cells

Consider $\mathbb{R}^n$ equipped with the canonical scalar product. We say that $A \subset \mathbb{R}^n$ verifies *the Whitney property with constant* $M > 0$, if any two points $x, y \in A$ can be joined in A by a piecewise smooth arc of length $\leq M|x - y|$. Following M. Gromov [6] one could also say that A is *quasi-convex*, or more precisely that A is M*-quasi-convex*. Any bounded semianalytic set can be covered by a finite number of quasi-convex (and semianalytic) sets as proven by the second named author [19] using the regular projections of T. Mostowski [18]. The construction proposed in [19] (extended in [20] to subanalytic sets) does not allow to estimate the constant M. Next the first named author [10] proved, by a different argument, that any bounded subanalytic subset can be decomposed (more precisely stratify) into a finite union of M-quasi-convex (and subanalytic) sets, with the constant M depending only on n - the dimension of the ambient space. This result was improved in [12], where it is shown that for any $M > 1$ such a finite decomposition into M-quasi-convex sets exists.

The construction from [10] can be adapted for o-minimal structures and actually can be done with parameters (which we need in the sequel). We shall explain it in this section.

We define, by induction on n, a class of subsets of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ let us write $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We say that $A \subset \mathbb{R}^n$ is a *standard L-regular cell in* $\mathbb{R}^n$ *with constant* C, if $A = \{0\}$ for $n = 0$, and for $n > 0$ the set A is of one of the following forms:

(graph) $\quad A = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\, x_n = h(x'),\, x' \in A'\}$

(we write often h instead of A), or
(*band*)

$$A = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ f(x') < x_n < g(x'),\ x' \in A'\} = (f, g)$$

where A' is a standard L-regular cell in $\mathbb{R}^{n-1}$ with constant C, $f, g, h : A' \to \mathbb{R}$ are C^1 functions such that $f(x') < g(x')$ for $x' \in A'$. Moreover we require that

$$\|df(x')\| \le C,\ \|dg(x')\| \le C,\ \|dh(x')\| \le C \tag{1.1}$$

for all $x' \in A'$. We call A' *the base* of the cell A, and in the case of band the graphs of f and g *the horizontal part of the boundary* of A.

By induction, we obtain that A is a C^1 submanifold of $\mathbb{R}^n$ (not closed in general). So it make sense to define df, dg, dh and also their norms (with respect to the norm induced on tangent space to A' at x'). If in the above we drop the condition (1.1), but we still assume that the functions f, g, h are C^1 we say that the set A is a *standard C^1 cell* in $\mathbb{R}^n$. If the functions f, g, h are only continuous we shall say that A is a *standard cell* in $\mathbb{R}^n$.

Finally we say that $B \subset \mathbb{R}^n$ is an *L-regular cell in $\mathbb{R}^n$ with constant C,* if there exists an orthogonal change of variables $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(B)$ is a standard L-regular cell (with constant C) in $\mathbb{R}^n$. By convention the empty set is an L-regular set (with any constant), also it will be convenient not to distinguish between function and its graph.

It is easily seen by induction that

Lemma 1.1. *Any L-regular cell in $\mathbb{R}^n$ with constant C is M-quasi-convex, where $M = (C+1)^{n-1}$. Moreover $\overline{A}$ is also M-quasi-convex.*

As a piece of terminology we recall that by a *decomposition* we always understand a disjoint union. We say that a decomposition $\mathbb{R}^N = \bigcup_{i \in I} B^i$ is *compatible* with a collection $A^k \subset \mathbb{R}^N$, $k \in K$, if $B^i \cap A^k = \emptyset$ or $B^i \subset A^k$ for any $i \in I$, $k \in K$. We also say that a decomposition $\mathbb{R}^N = \bigcup_{i \in I} B^i$ is a *stratification* if each B^i is a C^1 submanifold and $\dim(\overline{B^i} \setminus B^i) < \dim B^i$, and moreover that this decomposition is compatible with the collection $\overline{B^i}, i \in I$.

Notation. For $B \subset \mathbb{R}^n \times \mathbb{R}^p$ and $t \in \mathbb{R}^p$ we write $B_t = \{x \in \mathbb{R}^n : (x, t) \in B\}$.

Now we state the main result on a decomposition of a definable set into a finite number of quasi-convex sets.

Theorem 1.2. *There exists $M = M(n) > 0$ such that any set $A \subset \mathbb{R}^n \times \mathbb{R}^p$ definable in an o-minimal structure can be decomposed into a finite (and disjoint) union $A = \bigcup_{i \in I} B^i$, such that for each $t \in \mathbb{R}^p$, every set B^i_t has the Whitney property with constant M (i.e. is M-quasi-convex). So, in particular, $A_t = \bigcup_{i \in I} B^i_t$ for each $t \in \mathbb{R}^p$.*

Corollary 1.3. *Let $A \subset \mathbb{R}^n \times \mathbb{R}^p$ be a family of definable sets such that each A_t, $t \in T \subset \mathbb{R}^p$, is connected. Then there is a constant $C > 0$ such that for every $t \in T$ and $x, x' \in A_t$ there is a definable continuous curve ξ joining x and x' in A_t such that*

$$\operatorname{length}(\xi) \leq C \operatorname{diam}(A_t),$$

where $\operatorname{diam}(A_t)$ *stands for the diameter of* A_t.

What we actually prove below is more precise than theorem 1.2, namely we have:

Proposition 1.4. *Let $A^k \subset \mathbb{R}^n \times \mathbb{R}^p$, $k \in K$, be a finite collection of definable sets in an o-minimal structure. Then there exists finitely many disjoint definable sets $B^i \subset \mathbb{R}^n \times \mathbb{R}^p$, $i \in I$, and linear orthogonal mappings $\varphi^i : \mathbb{R}^n \to \mathbb{R}^n$, $i \in I$, such that:*

a) *for every $t \in \mathbb{R}^p$, each $\varphi^i(B^i_t)$ is a standard L-regular cell in $\mathbb{R}^n$ with constant C. The constant $C = C_n$ depends only on n.*

b) *For every $t \in \mathbb{R}^p$, the family $B^i_t \subset \mathbb{R}^n$, $i \in I$, is a stratification of $\mathbb{R}^n$.*

c) *For any $k \in K$ there exists $I_k \subset I$ such that $A^k_t = \bigcup_{i \in I_k} B^i_t$, for every $t \in \mathbb{R}^p$.*

Remark 1.5. Clearly, for a fixed $t \in \mathbb{R}^p$ some of B^i_t may be empty.

Proposition 1.4 will be proved at the end of this section. Before we give some preliminaries on the distances between linear subspaces and we recall some basic facts on cell decompositions in o-minimal structures. We establish also the definability of tangent mapping (with parameters).

1.1. Distances between linear subspaces

In this subsection by a line or a hypersurface in $\mathbb{R}^n$ we mean a linear subspace of dimension 1 and $n-1$ respectively. First we recall the definition of the angle (or the distance) between linear subspaces. If

P, S are (vector) lines we denote by $\delta(P,S)$ the sine of the angle between P and S, in other words $\delta(P,S) = \sqrt{1-\langle p,s\rangle^2}$, where $|p| = |s| = 1$, $p \in P$ and $s \in S$.

Let X be a linear subspace in $\mathbb{R}^n$, let P be a line in $\mathbb{R}^n$. We define the angle between P and X as

$$\delta(P,X) = \inf\{\delta(P,S); S \textit{ is a line in } X\}$$

Finally, if Y is a linear subspace in $\mathbb{R}^n$ we put

$$\delta(Y,X) = \sup\{\delta(P,X); P \textit{ is a line in } Y\}$$

Let us denote by $\mathbb{G}_{d,n}$ the grassmanian of all d-dimensional linear subspaces of $\mathbb{R}^n$ equipped with the natural structure of real algebraic variety. Then, it is easily seen by the Tarski-Seidenberg theorem that:

Lemma 1.6. *The function $\mathbb{G}_{d,n} \times \mathbb{G}_{e,n} \ni (Y,X) \longmapsto \delta(Y,X) \in \mathbb{R}$ is continuous and semialgebraic. Moreover, if $d = e$, then δ is a distance on $\mathbb{G}_{d,n}$, compatible with the standard topology on $\mathbb{G}_{d,n}$.*

Remark 1.7. Let X be a linear subspace and P a line in $\mathbb{R}^n$. Denote by $P^{\perp}$ the orthogonal complement of P and by π the orthogonal projection on $P^{\perp}$. Let $c > 0$. Assume that $\delta(P,X) > c$, then X is the graph of a linear mapping

$$\xi : P^{\perp} \cap \pi(X) \to P$$

satisfying $\|\xi\| \leq C < +\infty$, where $C = \frac{\sqrt{1-c^2}}{c}$.

Given a finite system $X_1, ...X_r$ of hyperplanes of $\mathbb{R}^n$. Then we may find, in a uniform way, a line P transverse ot each X_i. More precisely we have the following fact of metric-combinatorial nature that will be crucial in the proof of proposition 1.4.

Lemma 1.8. *For any two positive integers r, n there exist constants $\tau = \tau(r,n) > 0$ and $c = c(r,n) > 0$ such that for given $X_1, ...X_r$ hyperplanes in $\mathbb{R}^n$, there exists a line P such that, if $Y_1, ..., Y_r$ are hyperplanes verifying $\delta(Y_i, X_i) < \tau$, $i = 1, ...r$, then*

$$\delta(P, Y_i) > c \quad for\ each\ i = 1, ..., r. \tag{1.2}$$

Proof. We fix n, and consider the metric d on the sphere S^{n-1} induced by δ i.e. $d(p,q) = \delta(\mathbb{R}p, \mathbb{R}q)$ for $p, q \in S^{n-1}$. Let us denote

$X_i^\tau = \{p \in S^{n-1} : \, dist(p, X_i \cap S^{n-1}) < \tau\}$, where as usually $dist(p, Z) = \inf\{d(p,q) : \, q \in Z\}$. Note that $\delta(Y_i, X_i) < \tau$ means that $Y_i \cap S^{n-1} \subset X_i^\tau$ and $\delta(P, Y_i) > c$ means that $B(p,c) \cap Y_i \cap S^{n-1} = \emptyset$, where $p \in P \cap S^{n-1}$.

Claim 1.9. *For any $r \in \mathbb{N}$, there exists $\tau_r > 0$ and $c_r > 0$ such that the complement of $\bigcup_{i=1}^r X_i^\tau$ in S^{n-1} contains a ball of radius c_r (in the metric d).*

The claim implies lemma 1.8. Indeed, the line passing by the center of the ball has the property desired in (1.2). We show the claim by induction on r. The case $r = 1$ is obvious. Let us denote τ_r and c_r corresponding constants in the claim for r hyperplanes. Let $B(p, c_r)$ be a ball in S^{n-1} which is disjoint with each $X_i^{\tau_r}$, $i = 1, ...r$. Put $\tau_{r+1} = c_{r+1} = \min\{\tau_r, c_r\}/3$, then the set $B(p, c_r) \setminus X_{r+1}^{\tau_{r+1}}$ contains a ball of radius c_{r+1}. Q.E.D.

1.2. Cell decompositions in families

Recall that a finite decomposition $\mathbb{R}^N = \bigcup_{i \in I} B^i$ is called *a cell decomposition (resp. a C^1 cell decomposition)* if each B^i is a standard (resp. a C^1 standard) cell in $\mathbb{R}^N$, and the collection $\pi(B^i)$, $i \in I$, is a cell decomposition of $\mathbb{R}^{N-1}$, where $\pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$ is the projection parallel to the x_N-axis. We say that a decomposition is definable if all its members are definable (in some fixed o-minimal structure). We have the following fundamental result in the theory of o-minimal structures due to Steinhorn, Pillay and Knight [22],[8] (see also [4], [3]):

Theorem 1.10 (Cell decomposition)**.** *For any finite collection A^k, $k \in K$, of definable sets in $\mathbb{R}^N$ there exists a definable C^1 cell decomposition $\mathbb{R}^N = \bigcup_{i \in I} B^i$ compatible with the collection A^k, $k \in K$.*

Remark 1.11. The basic result (proved in [22],[8]) is the existence of a cell decomposition (without any smoothness assumption). The existence of C^1 decomposition is due to van den Dries and is valid in the C^k class for any finite k (cf. [4]). Moreover this decomposition can be refined to a stratification (loc.cit.).

What we need in the sequel is a decomposition with parameters (we rather say in a family). We say that that a definable set $A \subset \mathbb{R}^n \times \mathbb{R}^p$ is *a definable family of standard cells in* $\mathbb{R}^n$ if: each A_t is either empty or is a standard cell in $\mathbb{R}^n$ and the type of A_t does not depend on $t \in \mathbb{R}^p$.

(We say that two cells $A_1, A_2 \subset \mathbb{R}^n$ *are of the same type* if they are both graphs or both bands over their bases which are of the same type.)

Clearly if $A \subset \mathbb{R}^{n+p}$ is a standard cell in $\mathbb{R}^{n+p} = \mathbb{R}^p \times \mathbb{R}^n$ then A is a definable family of standard cells in $\mathbb{R}^n$ (cf. eg. [4] chap 3). Hence all claims of existence of decomposition into a definable family of standard cells (or C^1 cells) in $\mathbb{R}^n$ follows from theorem 1.10.

We shall often use the following construction of cell decomposition in an o-minimal structure.

The CD (cell decomposition) construction:

Let $A^k \subset \mathbb{R}^{n+1} \times \mathbb{R}^p$, $k \in K$, be a finite collection of disjoint definable sets. Denote by $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the projection which forgets the coordinate x_{n+1}. Suppose that for each $t \in \mathbb{R}^p$ every A^k_t (if nonempty) is a C^1 submanifold of $\mathbb{R}^{n+1}$ of dimension d and moreover that π restricted to A^k_t is an immersion. Each $\pi^{-1}(x) \cap A^k_t$ is discrete, so it must be finite, by o-minimality. Now, for every $r \in \mathbb{N}$ and $k \in K$ the set

$$\Sigma^k_r = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^p;\ \pi^{-1}(x) \cap A^k_t \text{ has } r \text{ elements}\}$$

is definable, moreover $\Sigma^k_r = \emptyset$ for r larger than some r_0. This is due to the fundamental property of o-minimal structures; if the fibers of a definable mapping have only isolated points, then there exists a uniform bound for the number of points in each fiber (cf. eg. [4],[3]).

Let $B^l \subset \mathbb{R}^n \times \mathbb{R}^p$, $l \in L$, be a finite collection of definable families of standard C^1 cells in $\mathbb{R}^n$ compatible with the family Σ^k_r; $r \le r_0$, $k \in K$, and such that, for each $t \in \mathbb{R}^p$, the collection B^l_t; $l \in L$, is a cell decomposition of $\mathbb{R}^n$. Fix $l \in L$ such that B^l_t is non-empty and hence is a C^1 submanifold of $\mathbb{R}^n$. We claim that all connected components of

$$\pi^{-1}(B^l_t) \cap A^k_t,\ k \in K$$

are the graphs of C^1 functions $f^j_t : B^l_t \to \mathbb{R}$, $1 \le j \le r$. Indeed, $\Gamma = \pi^{-1}(B^l_t) \cap A^k_t$ is a C^1 submanifold of $\mathbb{R}^{n+1}$ and the projection $\pi|_\Gamma : \Gamma \to B^l_t$ is a local diffeomorphism. Since $B^l_t \subset \Sigma^k_r$ for some $r \in \mathbb{N}$, the number of points in the fiber is constant, and it follows that $\pi|_\Gamma$ is a finite (r-sheeted) covering. Moreover, it is a diffeomorphism on each connected component of Γ, because B^l_t is simply connected (in fact homeomorphic to a ball). So the family

$$\pi^{-1}(B^l_t) = \bigcup_{1 \le j \le r} f^j_t \cup \bigcup_{0 \le j \le r} (f^j_t, f^{j+1}_t),$$

form a (standard) C^1 cell decomposition of $\pi^{-1}(B^l_t)$. We shall call this collection *subordinate to the collection* A^k, $k \in K$. (Recall that $(f^j_t, f^{j+1}_t) = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R};\ f^j_t(x) < x_{n+1} < f^{j+1}(x),\ x \in B^l_t\}$.) The functions are ordered in the way that $f^j_t < f^{j+1}_t$ and $f^0_t \equiv -\infty$, $f^{r+1}_t \equiv +\infty$. Moreover each function (or rather its graph) f^j_t, $1 \le j \le r$, is contained in some A^k_t, where k may depend on t. Subdividing, if necessary, the set B^l, we may assume that $f^j_t \subset A^k_t$, where $k = k(j)$ does not depend on $t \in \mathbb{R}^p$. Of course for some t the set A^k_t may be empty and then by convention we set $f^j_t = \emptyset$, $1 \le j \le r$.

Remark 1.12. Note that by construction the horizontal parts of boundaries of cells are also cells.

1.3. Controlling tangents

First let us observe that each C^1 cell has a definable tubular neighborhood. More precisely

Lemma 1.13 (Definable tubular neighborhoods)**.** *Let $A \subset \mathbb{R}^n \times \mathbb{R}^p$ be a definable family of standard C^1 cells of dimension d. Then there is a definable family of submersions*

$$\rho_t : \Omega_t \to A_t,\ t \in \mathbb{R}^p$$

such that $\Omega_t \subset \mathbb{R}^n$ is an open neighborhood of A_t and each ρ_t is the identity on A_t.

Proof. We sketch the construction only in the case without parameters. The reader may easily check that it works also with parameters. Let $A \subset \mathbb{R}^n$ be a standard C^1 cell of dimension $d < n$. We proceed by induction on n. Let $\rho' : \Omega' \to A'$ be a definable tubular neighborhood (in $\mathbb{R}^{n-1}$) of the base A' of cell A. In the case A is a band

$$A = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ f(x') < x_n < g(x'),\ x' \in A'\}$$

we put $\rho(x', x_n) = (\rho'(x'), x_n)$ for $x' \in \Omega', x_n \in \mathbb{R}$, and $\Omega = \rho^{-1}(A)$.

In the case of graph

$$A = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ x_n = h(x'),\ x' \in A'\}$$

we set $\rho(x', x_n) = (\rho'(x'), h(\rho'(x)))$ for $x' \in \Omega', x_n \in \mathbb{R}$, and $\Omega = \rho^{-1}(A) = \Omega' \times \mathbb{R}$. Q.E.D.

Lemma 1.14 (Definability of the tangent map)**.** *Let $A \subset \mathbb{R}^n \times \mathbb{R}^p$ be a definable family of standard C^1 cells of dimension d. Then the mapping*

$$\sigma : A \ni (x,t) \mapsto T_x A_t \in \mathbb{G}_{d,n}$$

is definable, where $T_x A_t$ stands for the tangent space to A_t at x.

Proof. We proceed by induction on n. We may suppose that $n > 0$ and $d < n$. We construct (by induction) a definable family of mappings

$$\varphi_t : \Omega_t \to \mathbb{R}^{n-d},\ t \in \mathbb{R}^p$$

such that Ω_t is an open neighborhood of A_t, $\varphi_t^{-1}(0) = A_t$ and φ_t is submersive on Ω_t.

The case of graph; each non empty A_t is the graph of a C^1 function $h_t : A'_t \to \mathbb{R}$, where $A' = \bigcup A'_t \subset \mathbb{R}^{n-1} \times \mathbb{R}^p$ is a definable family of C^1 cells (of dimension d) in $\mathbb{R}^{n-1}$. By lemma 1.13 each h_t can be extended to C^1 function in an open neighborhood Ω'_t of A'_t, moreover this can be done in a definable family. By induction we have family $\varphi'_t : \Omega'_t \to \mathbb{R}^{n-d-1}$, $t \in \mathbb{R}^p$, corresponding to A' . Clearly we may suppose that φ'_t and h_t are defined on the same Ω'_t. We put

$$\varphi_t(x', x_n) = (\varphi_t(x'), x_n - h_t(x'))$$

for $(x', x_n) \in \Omega'_t \times \mathbb{R} = \Omega_t$.

The case of band is similar and is left to the reader.

The derivative of φ_t i.e. the mapping

$$\varphi^{(1)} : (x,t) \mapsto d\varphi_t(x) \in L^*(\mathbb{R}^n, \mathbb{R}^{n-d})$$

is definable (cf. eg. [4] Chap 7.). (Here by $L^*(\mathbb{R}^n, \mathbb{R}^{n-d})$ we mean the space of linear epimorphisms from $\mathbb{R}^n$ to $\mathbb{R}^{n-d}$.) The mapping $L^*(\mathbb{R}^n, \mathbb{R}^{n-d}) \ni \phi \mapsto \ker \phi \in \mathbb{G}_{d,n}$ is semialgebraic, hence definable in any o-minimal structure. So our σ is definable as a composition of definable maps. Q.E.D.

Our next goal is to control the variation of tangent spaces to cells. Recall that we have the metric δ on the grasmannian $\mathbb{G}_{d,n}$. Let $\varepsilon > 0$, we say that Γ, a d-dimensional C^1 submanifold of $\mathbb{R}^n$, is *ε-flat* if for any $x, y \in \Gamma$ we have

$$\delta(T_x, T_y) \le \varepsilon.$$

For each $\varepsilon > 0$ we fix a finite covering $\mathbb{G}_{d,n} = \bigcup \Theta^{\varepsilon}_{\nu}$, where each $\Theta^{\varepsilon}_{\nu}$ is an open ball of diameter (with respect to δ) less then ε. Let A_t be a definable C^1 submanifold in $\mathbb{R}^n$, of dimension d and let $\sigma : A_t \to \mathbb{G}_{d,n}$ denote the tangent mapping. Then each nonempty $\sigma^{-1}(\Theta^{\varepsilon}_{\nu})$ is ε-flat. Moreover, by lemma 1.14, it is a definable set. Indeed each $\Theta^{\varepsilon}_{\nu}$ is semialgebraic (cf. lemma 1.6) and the inverse image of a definable set, by a definable map, is definable. Having this observation it is now routine to prove the following:

Proposition 1.15. *Given $\varepsilon > 0$, and let $A^k \subset \mathbb{R}^n \times \mathbb{R}^p$, $k \in K$, be a finite collection of definable sets. Then there exists finitely many disjoint definable sets $B^i \subset \mathbb{R}^n \times \mathbb{R}^p$, $i \in I$, such that:*

a) *for each $i \in I$, (B^i_t) is a definable family of ε-flat standard C^1 cells of dimension d. More precisely; for every $i \in I$ there exists ν_i such that*

$$T_x B^i_t \in \Theta^{\varepsilon}_{\nu_i},\ (x,t) \in B^i;$$

b) *For every $t \in \mathbb{R}^p$ the collection $B^i_t \subset \mathbb{R}^n$, $i \in I'$, is a stratification of $\mathbb{R}^n$;*

c) *For any $k \in K$ there exists $I_k \subset I$ such that $A^k_t = \bigcup_{i \in I_k} B^i_t$, for every $t \in \mathbb{R}^p$.*

1.4. Proof of Proposition 1.4

We proceed by induction on n. The case $n = 0$ is trivial. Suppose that Proposition 1.4 holds for $n-1$. We argue now by induction on $d = \max\{\dim A^k_t\}$. For the sake of clarity we prove only the decomposition part, i.e. statements a) and c). The refinement to a stratification is routine (cf. [4]). At first we deal with the non-open cells, that is $d < n$, then we decompose the open ones.

Case of non-open cells.

Fix an $\varepsilon < 1/2$ and assume that we are given $A^k \subset \mathbb{R}^n \times \mathbb{R}^p$, $k \in K$, a finite collection of definable sets. Let B^i be one of the sets given by proposition 1.15, let $d < n$ be the dimension of nonempty B^i_t. We shall prove that:

Lemma 1.16. *There exists a finite collection of definable families of cells $D^l \subset \mathbb{R}^{n-1} \times \mathbb{R}^p$, $l \in \Lambda$, and linear orthogonal mappings $\varphi^l : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $l \in \Lambda$, such that:*

(1) *for every* $l \in \lambda$ *each (nonempty)* $\varphi^\lambda(D_t^l)$ *is a standard L-regular cell in* $\mathbb{R}^{n-1}$*;*

(2) *For every* $i \in I$ *there is* $\Lambda_i \subset \Lambda$ *such that* $B_t^i = \bigcup_{\lambda \in \Lambda_i} \pi^{-1}(D_t^\lambda)$, *for every* $t \in \mathbb{R}^p$.

(Here $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ *denotes the projection on the first* $n-1$ *variables.)*

Proof. Since all tangent spaces to B_t^i are in a ball of diameter less than $\varepsilon < 1/2$ there exists a line P and $c = c(\varepsilon) > 0$ such that

$$\delta(P, T_{(x,t)}B_t^i) > c, \ (x,t) \in B^i.$$

Let $\varphi^i : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal mapping which sends P to x_n-axis and $P^\perp$ to $\mathbb{R}^{n-1}$ (the first $n-1$ coordinates). According to remark 1.7, the set $\varphi^i(B^i)$ is locally the graph of a C^1 function defined on a submanifold in $\mathbb{R}^{n-1}$. Moreover, there exists $C < \infty$ (depending only on c) such that the norm of the differential of this function is bounded by C. Now it is enough to apply the induction hypothesis and the CD construction to obtain lemma 1.16. Q.E.D.

Case of open cells.

The main difficulty is to decompose an open cell into finitely many L-regular cells. This will be done in two steps: in the first one, using proposition 1.15, we construct a decomposition into C^1 cells such that the boundary of each open cell is contained in a union of at most $2n$ ε-flat submanifolds of dimension $n-1$. Then, in the second step, we apply to each such cell lemma 1.8. If $\varepsilon \leq \tau(2n, n)$ then there exists a line P that makes angle with any tangent space to the boundary of the cell larger than some $c > 0$. After changing the coordinates in the way that P becomes the x_n-axis we apply the CD construction. This will give us C^1 cells with the horizontal parts of the boundary that are graphs of C^1 functions with differential of norm smaller than $C < \infty$ (cf. remark 1.7). Now by induction we may subdivide (in $\mathbb{R}^{n-1}$) the base of each above cell into L-regular cells. Hence the proof will be achieved. Now we explain the details.

Step 1. Let us fix $\varepsilon = \tau(2n, n)$ of lemma 1.8. Let $A^k \subset \mathbb{R}^n \times \mathbb{R}^p$, $k \in K$, be a finite collection of definable sets. By proposition 1.15, theorem 1.10, and the CD construction there exists a finite collection of disjoint definable families $B^l \subset \mathbb{R}^n \times \mathbb{R}^p$, $l \in \Lambda$, with properties we explain below.

Fix $l \in \Lambda$ such that each (nonempty) cell $B = B^l_t$ is open (we skip l, t for a moment to simplify the notation), that is of the form:

(1.3)
$$B = B_n = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ f_n(x') < x_n < g_n(x'),\ x' \in B_{n-1}\}$$

and by induction:

$$(1.4) \qquad B_i = \{(x'_i, x_i) \in \mathbb{R}^{i-1} \times \mathbb{R};\ f_i(x'_i) < x_i < g_i(x'_i),\ x'_i \in B_{i-1}\},$$

$i = 1, \dots, n-1$. We may assume that each B_i, $i = 1, \dots, n-1$, is open in $\mathbb{R}^i$ and every f_i, g_i, $i = 1, \dots, n-1$, is a C^1 function such that its graph (in $\mathbb{R}^i$) is ε-flat. More precisely; independently of $t \in \mathbb{R}^p$, there exist $\tilde{\Theta}^f_i$, $\tilde{\Theta}^g_i$ two open balls, of diameter ε, in the grasmannian $\mathbb{G}_{i-1,i}$, such that the tangent spaces to the graph of f_i (resp. g_i) belong to $\tilde{\Theta}^f_i$ (resp. $\tilde{\Theta}^g_i$). Note that if $\tilde{X}, \tilde{Y} \in \mathbb{G}_{i-1,i}$ and $X = \tilde{X} \times \mathbb{R}^{n-i}$, $Y = \tilde{Y} \times \mathbb{R}^{n-i}$, then

$$(1.5) \qquad \delta(X, Y) = \delta(\tilde{X}, \tilde{Y}),$$

since $\mathbb{R}^i \times 0$ and $0 \times \mathbb{R}^{n-i}$ are orthogonal. This implies that there exist Θ^f_i, Θ^g_i two open balls of diameter ε, in the grasmannian $\mathbb{G}_{n-1,n}$ such that independently of $t \in \mathbb{R}^p$ we have

(1.6)
$$\{X = \tilde{X} \times \mathbb{R}^{n-i};\ \tilde{X} \in \tilde{\Theta}^f_i\} \subset \Theta^f_i,\ \{X = \tilde{X} \times \mathbb{R}^{n-i};\ \tilde{X} \in \tilde{\Theta}^g_i\} \subset \Theta^g_i.$$

Denote by ∂B the boundary of B. Then clearly, by (1.3) and (1.4),

$$(1.7) \qquad \partial B \subset \bigcup_{i=1}^{n} f_i \times \mathbb{R}^{n-i} \cup \bigcup_{i=1}^{n} g_i \times \mathbb{R}^{n-i}$$

Hence the tangent spaces to ∂B belong to the union of balls Θ^f_i, $i = 1, \dots, n$, and Θ^g_i, $i = 1, \dots, n$. Indeed we can take the decomposition (1.7) of the boundary of B.

So we have proved the following:

Lemma 1.17. *For every $l \in \Lambda$ such that B^l_t is open there exist $2n$ balls of diameter ε in the grassmanian $\mathbb{G}_{n-1,n}$ such that for each $t \in \mathbb{R}^p$ any tangent space to the boundary of B^l_t belongs to one of these balls.*

Step 2. Recall $\varepsilon \le \tau(2n, n)$ of lemma 1.8 and we work with a fixed definable family B^l such that for each $t \in \mathbb{R}^p$ the set B^l_t is open (possibly empty) in $\mathbb{R}^n$, and B^l_t satisfies lemma 1.17. By lemma 1.8 there exist a line P and $c > 0$ such that if $Y \in \mathbb{G}_{n-1,n}$ is a tangent space to the boundary of B^l_y, then $\delta(P, Y) > c$. After a linear orthogonal change of variables in $\mathbb{R}^n$ we may assume that P is the x_n-axis. Applying the CD construction to ∂B^l_t, decomposed as in (1.7), we obtain finitely many disjoint definable families $D^s \subset \mathbb{R}^{n-1} \times \mathbb{R}^p$, $s \in S$, and such that B^l_t, $t \in \mathbb{R}^p$, is a union of the sets of the form

$$(f, g) = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ f(x') < x_n < g(x'),\ x' \in D^s_t\}$$

and

$$h = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R};\ x_n = h(x'),\ x' \in D^s_t\},$$

with C^1 functions $f, g, h : D^s_t \to \mathbb{R}$. By remark 1.7 the norm of differential of each f, g, h is bounded by a constant C which depends only on n. On the other hand we may assume by induction, that after an orthogonal change of coordinates in $\mathbb{R}^{n-1}$ (independent of $t \in \mathbb{R}^p$), each D^s_t is an L-regular cell in $\mathbb{R}^{n-1}$, with constant C. So h and (f, g) are standard L-regular cells in $\mathbb{R}^n$, with constant C.

This ends the proof of proposition 1.4.

§2. Definable Functions of One Variable

First we shall recall some elementary properties of germs at 0 of definable functions. We denote $\mathbb{R}_{\geq 0} = \{r \in \mathbb{R}; r \geq 0\}$ and the variable in $\mathbb{R}_{\geq 0}$ will be usually denoted by r.

Lemma 2.1. *Let $\varphi(r)$ and $\psi(r)$ be two continuous definable functions $(\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$, not identically equal to 0. Suppose $\psi \geq \varphi$. Fix $c > 1$. Then for r sufficiently small*

$$\psi'(r) \geq \varphi'(r) \tag{2.1}$$

$$c\frac{\varphi'(r)}{\varphi(r)} \geq \frac{\psi'(r)}{\psi(r)}. \tag{2.2}$$

Proof. Since $\psi - \varphi \geq 0$ and $(\psi - \varphi)(0) = 0$, $\psi - \varphi$ is increasing for small r and the first inequality follows. Similarly, $\rho(r) = \frac{(\varphi(r))^c}{\psi(r)}$ is non-negative and $\rho(r) \to 0$ as $r \to 0$. Hence ρ has to be increasing and

$$0 \leq \rho' = \frac{c\varphi^{c-1}\varphi'\psi - \varphi^c\psi'}{\psi^2} = \frac{\varphi^c}{\psi}\Big(\frac{c\varphi'}{\varphi} - \frac{\psi'}{\psi}\Big),$$

as claimed. Q.E.D.

Remark 2.2. If, moreover, $\varphi(r)/\psi(r) \to 1$ as $r \to 0$, then $\varphi(r)/\psi(r)$ is decreasing and

$$c\frac{\varphi'}{\varphi} \geq \frac{\psi'}{\psi} \geq \frac{\varphi'}{\varphi}.$$

Definition 1. Let $\varphi(r)$ be the germ at 0 of a continuous definable function $(\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}, 0)$. We shall say that φ is *small* if there is a continuous definable function $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$, such that

$$\frac{|\varphi|}{r} \leq \psi'. \tag{2.3}$$

In particular, if φ is small then $\frac{\varphi}{r}$ is integrable and $\varphi(r) \to 0$ as $r \to 0$.

We shall say that φ is *unitlike* if there is a continuous function $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$, C^1 for $r > 0$, such that

$$\varphi = \frac{r\psi'}{\psi}. \tag{2.4}$$

Clearly in a polynomially bounded o-minimal structure all continuous definable $\varphi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}, 0)$ are small. This is not the case for the other o-minimal structures, see example 1 below.

Lemma 2.3. *Let* $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ *be continuous definable. Then* $\frac{r\psi'}{\psi}$ *is bigger than any small function.*

Let $\varphi_1(r)$ *and* $\varphi_2(r)$ *be two continuous definable functions* $(\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}, 0)$, *not identically equal to* 0. *Suppose* $\varphi_2(r) \geq r$. *Then the function*

$$\frac{\varphi_1'\varphi_2}{\varphi_1\varphi_2'} \tag{2.5}$$

is bigger than any small function.

Proof. Let $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ be definable. Then $(\log \psi)' = \frac{\psi'}{\psi}$ is not integrable and hence $\frac{r\psi'}{\psi}$ is bigger than any small function. By lemma 2.1, $\frac{\varphi_2'}{\varphi_2} \leq 2\frac{1}{r}$, and hence $\frac{\varphi_1'\varphi_2}{\varphi_1\varphi_2'}$ is bigger than any small function. Q.E.D.

We have a more precise result that, however, we do not use in this paper.

Proposition 2.4. *Each continuous definable $\varphi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ is either small or unitlike. Moreover, φ is small iff $\frac{\varphi}{r}$ is integrable and then there is $\psi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ such that*

$$\frac{\varphi}{r} = \psi'. \tag{2.6}$$

The functions ψ of (2.6) and (2.4) belong to the Pfaffian closure of the o-minimal structure containing φ.

Proof. Let $\varphi : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$. Fix $a > 0$ small and consider

$$f(r) = \int_a^r \frac{\varphi(t)}{t}\, dt. \tag{2.7}$$

By [23], f is definable in the Pfaffian closure of the o-minimal structure containing φ. If $f(r)$ is bounded then φ is small and we may take in (2.3), $\psi(r) = f(r) - f(0)$.

Suppose $f(r)$ is not bounded that is $f(r) \to -\infty$ as $r \to 0$. Then the structure is not polynomially bounded and hence contains the exponential and the logarithmic functions, see [17]. Then we may take in (2.4), $\psi = e^f$. Q.E.D.

Example 1. Let $\alpha(r) = (-\ln r)^{-1}$ for $r > 0$ and $\alpha(0) = 0$. Then, $\alpha(r)$ satisfies

$$r\alpha'(r) = \alpha^2(r). \tag{2.8}$$

In particular, α^2 is small and $\alpha = \frac{r\alpha'}{\alpha}$ is unitlike.

§3. Łojasiewicz Inequalities in o-minimal Structures

We recall the main result of [9].

Theorem 3.1. *Let $f : U \to \mathbb{R}$ be a differentiable definable function defined in an open bounded $U \subset \mathbb{R}^n$. Then there exist $c > 0, \rho > 0$, and a continuous definable change of target coordinate $\Psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that*

$$|\nabla(\Psi \circ f)(x)| \geq c, \tag{3.1}$$

for $x \in U$ and $f(x) \in (-\rho, \rho)$.

Let us recall briefly after [9] the construction of Ψ. We suppose for simplicity that $f \geq 0$. Choose a definable curve $\gamma(t) : (\mathbb{R}_{\geq 0}, 0) \to \bar{U}$, such that $\gamma(t) \in U$ for $t > 0$, $f(\gamma(t)) = t$, and that

$$|\nabla f(\gamma(t))| \leq 2 \inf\{|\nabla f(x)|; f(x) = t\}, \tag{3.2}$$

in $\bar{U}$. Such a curve exists by the o-minimal version of the curve selection lemma and the fact that the right hand side of (3.2) is a definable function strictly bigger than 0 for $t > 0$ and sufficiently small, see [9]. Change the parameter by $\gamma(s) = \gamma(s(t))$ so that $|\frac{d\gamma}{ds}(0)| = 1$ and $\gamma(s)$ is definable of class C^1 (for instance we may use the distance to $\gamma(0)$ as the parameter). Then we define Ψ as the inverse function of $s \to f(\gamma(s))$ that is

$$\Psi(f(\gamma(s))) = s.$$

Hence for arbitrary $x \in U$, $t = f(x)$ close to 0, and $s = s(t)$,

$$|\nabla(\Psi \circ f)(x)| \geq \tfrac{1}{2}|\nabla(\Psi \circ f)(\gamma(t))| \geq 1/4\langle\nabla(\Psi \circ f)(\gamma(s)), \gamma'(s)\rangle = 1/4, \tag{3.3}$$

as required.

Corollary 3.2. *([9], Theorem 2) Let $f : U \to \mathbb{R}$ be a C^1-definable function defined in an open bounded $U \subset \mathbb{R}^n$. Then there exists a constant A such that all the trajectories of ∇f have length bounded by A. In particular, each trajectory $x(t)$ has a unique limit point $x_0 \in \overline{U}$, that is there is $t_0 \in \mathbb{R} \cup \{\infty\}$ such that*

$$\lim_{t \to t_0} x(t) = x_0$$

and $\nabla f(x_0) = 0$ if $x_0 \in U$.

We have as well an o-minimal version of Bochnak-Łojasiewicz Inequality [2].

Proposition 3.3. *Let $f : U \to \mathbb{R}$ be a differentiable definable function defined in an open $U \subset \mathbb{R}^n$. Suppose $0 \in \bar{U}$ and $f(x) \to 0$ as $x \to 0$. Then there exists a continuous definable change of target coordinate $\Phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and constants $c_\Phi > 0, \rho > 0$, such that*

$$|x||\nabla(\Phi \circ f)| \geq c_\Phi |\Phi \circ f|, \tag{3.4}$$

for $x \in U$, close to the origin, and $f(x) \in (-\rho, \rho)$.

Proof. Again we suppose $f \geq 0$ leaving the general case to the reader. Define $\varphi_0(r) = \sup_{|x|=r} f(x)$. Let Φ be the inverse function of φ_0. Then

$$(\Phi \circ f)(x) \leq r. \tag{3.5}$$

Let γ be a definable curve going to the origin and parameterized by r. Then $|\gamma'(r)| \to 1$ as $r \to 0$. By the choice of parameterization and (3.5), $(\Phi \circ f)(\gamma(r)) \leq r$. Denote $\psi(r) = (\Phi \circ f)(\gamma(r))$. By Lemma 2.1, for any $c > 1$,

$$c\frac{\psi'(r)}{\psi(r)} \geq \frac{1}{r}. \tag{3.6}$$

On the other hand

$$\psi'(r) = \langle \nabla(\Phi \circ f), \gamma'(r)\rangle \leq 2|\nabla(\Phi \circ f)|. \tag{3.7}$$

The proposition follows from (3.6) and (3.7) by the curve selection lemma. Q.E.D.

The actual constants in both (3.1) and (3.4) can be made arbitrarily small. For instance for (3.4) it suffices to replace $\Phi \circ f$ by its power $(\Phi \circ f)^\alpha$.

Remark 3.4. Unlike in the analytic case, in general, it is not possible to find a definable change of target coordinate which gives both Łojasiewicz type inequalities. We may take as example $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ given in polar coordinates by

$$f(r, \theta) = \alpha(r)\sin\theta,$$

where α is the function of example 1. Indeed, suppose $\Phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is the change of target coordinate such that $\Phi \circ f$ satisfies both inequalities. In what follows we suppose Φ increasing and restrict ourselves to the set $(\Phi \circ f) \geq 0$. Then

$$\begin{aligned} |\nabla(\Phi \circ f)| &= (\Phi' \circ f)|(\alpha'(r)\sin\theta, r^{-1}\alpha(r)\cos\theta)| \\ &= (\Phi' \circ f)r^{-1}|(\alpha^2(r)\sin\theta, \alpha(r)\cos\theta)|. \end{aligned}$$

For $\sin\theta = 1, \cos\theta = 0$, the Bochnak-Łojasiewicz Inequality gives

$$r|\nabla(\Phi \circ f)| = (\Phi' \circ f)\alpha^2 = r(\Phi \circ \alpha)'(r) \geq \tilde{c}(\Phi \circ \alpha)(r)$$

that is

$$\frac{(\Phi \circ \alpha)'}{\Phi \circ \alpha} \geq \frac{\tilde{c}}{r}.$$

and by integration (or Lemma 2.1)

$$r^c \geq (\Phi \circ \alpha)(r), \tag{3.8}$$

for $c < \tilde{c}$. Since $\tilde{c} > 0$ may choose $c > 0$ as well.

On the other hand, consider the set $\{\theta < r\}$. Then $\sin\theta \to 0$ and $\cos\theta \to 1$ as $r \to 0$. By Łojasiewicz Inequality (3.1)

$$r^{-1}\alpha(r)\Phi'(\alpha(r)\sin\theta) \geq c. \tag{3.9}$$

Define $\gamma(r,\theta)$ by $\alpha(r)\sin\theta = \alpha(\gamma(r,\theta))$. Then (3.9) is equivalent to

$$\alpha(r)(\Phi \circ \alpha)'(\gamma(r,\theta)) \geq c_1 r\alpha'(\gamma(r,\theta)),$$

that gives by (3.8)

$$c\alpha(r)(\gamma(r,\theta))^{c-1} \geq c_1 r\alpha'(\gamma(r,\theta)).$$

Equivalently

$$c\alpha(r)/r \geq c_1\alpha'(\gamma(r,\theta))/(\gamma(r,\theta))^{c-1}$$

that is impossible since the right hand side $\alpha'(\gamma(r,\theta))/(\gamma(r,\theta))^{c-1} = \alpha^2(\gamma(r,\theta))/(\gamma(r,\theta))^c$ tends to ∞ as $\theta \to 0$ and r is fixed and the left hand side does not depend on θ.

Remark 3.5. Suppose that there is a positive exponent a and a constant $c > 0$ such that $r^a \geq |f(x)| \geq cr$. Then f itself, without any change of target coordinate , satisfies both inequalities. Indeed, by construction, it suffices to check these inequalities on a definable curve and in this case they are obvious.

§4. Characteristic Functions

In this section we suppose that $f : U \to \mathbb{R}$ is a differentiable definable function defined in an open $U \subset \mathbb{R}^n$, $0 \in \bar{U}$. We shall assume f bounded. The gradient ∇f of f splits into its radial component $\frac{\partial f}{\partial r}\frac{\partial}{\partial r}$ and the spherical one $\nabla' f = \nabla f - \frac{\partial f}{\partial r}\frac{\partial}{\partial r}$. Fix $\varepsilon > 0$ and consider

$$V^\varepsilon = \{0 \leq |x| \leq r_0; f(x) \neq 0, |f(x)| \geq \varepsilon r|\nabla' f(x)|\},$$

where $r_0 > 0$ is small. By the local conical structure of definable sets we may suppose that, for $r_0 > 0$ sufficiently small, $V^\varepsilon \ni x \to |x| \in (0, r_0]$ is a topologically trivial fibration. In particular for $0 < r \le r_0$, the inclusion $S(r) \cap V^\varepsilon \subset V^\varepsilon$, where $S(r) = \{x; |x| = r\}$, is a homotopy equivalence. Let V be a connected component of V^ε. Denote

$$\varphi(r) = \varphi_V(r) = \inf\{|f(x)|; x \in V \cap S(r)\}. \tag{4.1}$$

Proposition 4.1. *There exists $C > 0$ such that*

$$\varphi(|x|) \le |f(x)| \le C\varphi(|x|), \qquad \text{for } x \in V. \tag{4.2}$$

In particular $\varphi(r) > 0$ for $r > 0$.

Proof. By corollary 1.3 there exists a constant $M > 0$ such that for every $x, x' \in V$, that satisfy $|x| = |x'| = r$, there is a continuous definable curve $\xi(t)$ joining x and x' in $V \cap S(r)$ and of length $\le Mr$. Then, by the definition of V^ε,

$$|\frac{d}{dt} f(\xi(t))| = |\langle \nabla' f, \xi'(t)\rangle| \le |\nabla' f||\xi'(t)| \le \varepsilon^{-1}\frac{|f|}{r}|\xi'(t)|.$$

Hence

$$|\frac{d}{dt} \ln |f(\xi(t))|| \le \frac{\varepsilon^{-1}}{r}|\xi'(t)|.$$

Finally, by integration of both sides along curve $\xi(t)$, $|\ln|f(x)| - \ln|f(x')||$ $\le M' = M\varepsilon^{-1}$, which gives

$$|\frac{f(x)}{f(x')}| \le e^{M'}.$$

The proposition follows by the curve selection lemma. Q.E.D.

We shall call the (finite) set of functions φ_V defined by (4.1), where V goes over the connected components of V^ε, *the characteristic functions of f*. They depend on the choice of ε though it may be shown that the number of connected components of V^ε at the origin stabilizes as $\varepsilon \to 0$. Each of these connected components give rise to a family of characteristic functions $\varphi_{\varepsilon,V}$. It can be shown that they can be compared as follows: if $\varepsilon' < \varepsilon$ then there exists $C = C(\varepsilon', \varepsilon)$ such that $\varphi_{\varepsilon',V} \le \varphi_{\varepsilon,V} \le C(\varepsilon', \varepsilon)\varphi_{\varepsilon',V}$. In what follows shall consider ε fixed and small and we will be interested mostly in those connected components V of V^ε such that $\varphi_V(r) \to 0$ as $r \to 0$. Let V be such a component and let $\gamma(t), t \geq 0$ be a definable curve such that $\gamma(t) \to 0$ as $t \to 0$ and $\gamma(t) \in V$ for $t \neq 0$. In order to simplify the notation we reparametrize γ by the distance to the origin, that is to say $|\gamma(t(r))| = r$. Write in spherical coordinates $\gamma(r) = r\theta(r)$, $|\theta(r)| \equiv 1$. Then $r|\theta'(r)| \to 0$ as $r \to 0$. Moreover, $r|\theta'(r)|$ is small in sense of definition 1. Denote $\psi(r) = |f(\gamma(r))|$. Then

$$\frac{df(\gamma(r))}{dr} = \frac{|f|}{r}\frac{r\psi'}{\psi} \geq \varepsilon|\nabla' f|\frac{r\psi'}{\psi} \gg r|\theta'(r)||\nabla' f|,$$

since, by lemma 2.3, $\frac{r\psi'}{\psi}$ is much bigger than $r|\theta'(r)|$. In particular,

$$\frac{df(\gamma(r))}{dr} = \partial_r f + \langle \nabla' f, r\theta'(r)\rangle \simeq \partial_r f. \tag{4.3}$$

We shall consider as well

$$W^\varepsilon = \{x; f(x) \neq 0, |\partial_r f| \geq \varepsilon|\nabla' f|\}.$$

Unlike V^ε, the sets W^ε do not change if we replace f by $\Psi \circ f$, for any definable change of target coordinate Ψ at $0 \in \mathbb{R}$.

Proposition 4.2.

$$W^\varepsilon \cap \{x;\ |f(x)| \geq |x|\} \subset V^{\varepsilon'} \cap \{x;\ |f(x)| \geq |x|\} \quad \textit{if} \quad \varepsilon' < \varepsilon. \tag{4.4}$$

Let W be a connected component of $W^\varepsilon \cap \{x, |f(x)| \geq |x|\}$ and define $\varphi(r) = \inf\{|f(x)|; x \in W \cap S(r)\}$. There exists $C > 0$ such that

$$\varphi(|x|) \leq |f(x)| \leq C\varphi(|x|), \qquad \textit{for } x \in W. \tag{4.5}$$

Proof. It suffices to check (4.4) on definable curves. Fix a definable curve $\gamma(r)$ in $W^\varepsilon \cap \{x||f(x) \geq |x|\}$ parameterized by the distance to the origin. Denote $\psi(r) = |f(\gamma(r))|$. Suppose first $\psi(r) \to 0$ as $r \to 0$. Then $\psi \geq r$ and hence by lemma 2.1, $\psi \geq cr\psi'$, where we may take $1 > c > \frac{\varepsilon'}{\varepsilon}$. Then, by (4.3),

$$\psi \geq cr\psi' \geq \varepsilon' r|\nabla' f|,$$

as claimed. The proof for the curves on which $|f(\gamma(r))| \to c_0 > 0$ is similar since (4.3) holds for the curves in W^ε.

The last claim of the proposition follows from (4.4) and proposition 4.1. Q.E.D.

§5. Estimates on a trajectory. I

Let $f : U \to \mathbb{R}$ be a C^1 definable function defined in an open and bounded $U \subset \mathbb{R}^n$ and let $x(t)$ be a trajectory of ∇f with limit point $x_0 \in \overline{U}$, cf. corollary 3.2. We shall suppose, for simplicity of notation, that $x_0 = 0$ and usually we parameterize $x(t)$ by its arc-length s, starting from point $p_0 = x(0)$. Then

$$\dot{x} = \frac{dx}{ds} = \frac{\nabla f}{|\nabla f|}.$$

By corollary 3.2 the length of $x(s)$ is finite. Denote it by s_0. Then

$$x(s) \to 0 \quad \text{as } s \to s_0.$$

Our purpose is to study the geometric behavior of $x(s)$ as it approaches its limit point. We shall also assume that

$$f(x(s)) \to 0 \quad \text{as } s \to s_0.$$

Note that it means in particular, as being increasing, that f has negative along the trajectory.

By theorem 3.1 we may assume that $|\nabla f| \geq 1$ that we shall do. Then

$$|f(x(s))| \geq \text{length}\{x(s'); s \leq s' < s_0\} \geq |x(s)|. \tag{5.1}$$

Fix a definable $\varphi(r) : (\mathbb{R}_{\geq}, 0) \to (\mathbb{R}_{\geq}, 0)$ and consider $F = \frac{f}{\varphi(r)}$. Then

$$\begin{aligned} \tfrac{dF(x(s))}{ds} &= \langle \frac{\nabla f}{|\nabla f|}, \frac{\nabla' f}{\varphi} + \big(\frac{\partial_r f}{\varphi} - \frac{\varphi' f}{\varphi^2}\big)\partial_r \rangle \\ &= \frac{1}{|\nabla f|\varphi}\Big(|\nabla' f|^2 + |\partial_r f|^2 - \partial_r f \frac{\varphi' f}{\varphi}\Big) \\ &= \frac{1}{|\nabla f|\varphi}\Big(|\nabla' f|^2 + |\partial_r f|^2 \big(1 - \frac{\varphi' f}{\varphi \partial_r f}\big)\Big) \end{aligned} \tag{5.2}$$

Lemma 5.1. *Let $\varphi(r) \geq r$ and let $F = \frac{f}{\varphi(r)}$. Suppose $\varepsilon < 1$. Then in the complement of $V^\varepsilon = \{x; |f| \geq \varepsilon r |\nabla' f|\}$*

$$\frac{dF(x(s))}{ds} \geq \tfrac{1}{2}\frac{|\nabla f|}{\varphi}. \tag{5.3}$$

Proof. By (5.2), it is sufficient to show that

$$|\nabla f|^2 \geq 2f\partial_r f \frac{\varphi'}{\varphi}$$

on the complement of V^ε. Since $\varphi \geq r$ we have by lemma 2.1

$$\varepsilon \frac{\varphi'}{\varphi} \leq \frac{1}{r}. \tag{5.4}$$

Consequently, since we are away of V^ε,

$$|\nabla f|^2 \geq 2|\nabla' f||\partial_r f| \geq 2\varepsilon^{-1}\frac{|f|}{r}|\partial_r f| \geq 2f\partial_r f\frac{\varphi'}{\varphi}$$

as required. Q.E.D.

Corollary 5.2. *The trajectory $x(s)$ passes through V^ε in any neighborhood of the origin.*

Proof. Let $q > 0$ and consider $\varphi(r) = r^{1+q}$. Then $r\frac{\varphi'}{\varphi} = 1+q$ (5.4) is satisfied for $\varepsilon < (1+q)^{-1}$. Consequently the statement of lemma 5.1 holds for $F = \frac{f}{\varphi}$. Suppose, contrary to our claim, that $x(s)$ stays away of V^ε. Then, by lemma 5.1, $F = \frac{f}{r^{1+q}}$ is increasing on the trajectory. Hence it is bounded (recall $f(x(s))$ is negative). That is there exists a constant $C > 0$ such that

$$|f(x(s))| \leq C|x(s)|^{1+q},$$

which contradicts (5.1). Q.E.D.

Fix $\varepsilon < 1$. By Proposition 4.1 there is a finite family of functions of one variable $\{\varphi(r)\}$ such that

$$V^\varepsilon = \bigcup V^\varepsilon_\varphi, \tag{5.5}$$

so that $V^\varepsilon_\varphi \subset U_\varphi = \{x | c\varphi < |f| < C\varphi\}$. We regroup together the φ's with the same asymptotic behavior at 0, that is in the same equivalence classe of relation $\varphi_1(r) \sim \varphi_2(r)$. Thus we may actually assume that the U_φ's are mutually disjoint and so is the union in (5.5).

Fix one of such φ satisfying $\varphi(r) \geq r$ and consider $F = \frac{f}{\varphi}$. Recall that F is negative on the trajectory. Define

$$\partial^- U_\varphi = \{x; F(x) = -C\}, \qquad \partial^+ U_\varphi = \{x; F(x) = -c\}.$$

Then, by lemma 5.1, $F(x(s))$ is strictly increasing on $\partial^- U_\varphi \cup \partial^+ U_\varphi$. That is to say, the trajectory may enter U_φ only through $\partial^- U_\varphi$ and leave it only through $\partial^+ U_\varphi$. If the latter happens then the trajectory leaves U_φ definitely and never enters it again. Hence, by corollary 5.2,

Corollary 5.3. *The trajectory $x(s)$ has to end up in one of $U_\varphi = \{x | c\varphi < |f| < C\varphi\}$.*

Note that $\varphi(r) \geq r$ by (5.1). We shall fix such φ. Now we have the following strenghtened versions of lemma 5.1 and corollary 5.2.

Lemma 5.4. *Let $F = \frac{f}{\varphi}$. Then for any $\varepsilon > 0$ there is $c' > 0$ such that in the complement of W^ε in U_φ*

$$\frac{dF(x(s))}{ds} \geq c' \frac{|\nabla f|}{\varphi} \geq \frac{c'}{\varphi}. \tag{5.6}$$

Proof. Fix $\varepsilon > 0$. By (5.2), it is sufficient to show that there is $c > 1$ such that

$$|\nabla f|^2 \geq c f \partial_r f \frac{\varphi'}{\varphi} \tag{5.7}$$

on $U_\varphi \setminus W^\varepsilon$. This we show on an arbitrary definable curve $\gamma(r)$ in $U_\varphi \setminus W^\varepsilon$. Again we denote $\psi(r) = |f(\gamma(r))|$ and write in the spherical coordinates $\gamma(r) = r\theta(r)$. Then,

$$\psi'(r) = |\partial_r f + \langle \nabla' f, r\theta'(r) \rangle| \leq |\partial_r f| + |\nabla' f||r\theta'(r)|, \tag{5.8}$$

where $r\theta'(r) \to 0$.

Suppose first that $|\partial_r f| \gg |\nabla' f||r\theta'(r)|$ as $r \to 0$. Then, since we are away of W^ε,

$$|\nabla f|^2 \geq (1 + \varepsilon^{-2})|\partial_r f|^2 \geq (1 + \tilde{\varepsilon}^{-2})|\partial_r f|\psi', \tag{5.9}$$

for any $\tilde{\varepsilon} > \varepsilon$. An even stronger bound holds if $|\partial_r f| \gg |\nabla' f||r\theta'(r)|$ fails. Indeed, then $|\partial_r f| \ll |\nabla' f|$ and $|\nabla f|^2 \simeq |\nabla' f|^2 \gg |\partial_r f|\psi'$.

On the other hand, by lemma 2.1, for any $c > 1$ and on U_φ

$$c\psi' \geq \psi \frac{\varphi'}{\varphi}.$$

This and (5.9) show (5.7). The proof is complete. Q.E.D.

Corollary 5.5. *The trajectory $x(s)$ passes through W^ε in any neighborhood of the origin.*

Proof. Suppose, contrary to our claim, that $x(s)$ stays away of W^ε. Let $\varphi(r)$ be such that $x(s)$ stays in U_φ for s close to s_0. Then, there exists a constant $\tilde{c} > 0$, such that

$$\frac{dF(x(s))}{ds} \geq c' \frac{|\nabla f|}{\varphi} \geq \tilde{c} \frac{df}{ds} |f|^{-1} = \tilde{c} \frac{d(-\ln|f|)}{ds}.$$

But this is impossible since F is bounded and $-\ln|f|$ is not on $x(s)$. Q.E.D.

Let W be the union of those connected components of W^ε such that the trajectory $x(s)$ passes through them in any neighborhood of the origin. W is non-empty by corollary 5.5. Denote

$$\varphi(r) = \inf_{W \cap S(r)} |f(x)|.$$

Let Φ be the inverse function of φ and consider

$$\tilde{f}(x) = -\Phi(-f(x)). \tag{5.10}$$

Then, by the definition of φ,

$$|\tilde{f}(x)| \geq |x|, \qquad \text{for } x \in W. \tag{5.11}$$

Proposition 5.6. *There is a $C > 0$ such that on the trajectory $x(s)$ and for $x(s)$ sufficiently close to the origin*

$$-Cr \leq \tilde{f}(x(s)) \leq -r. \tag{5.12}$$

Proof. By definition of φ, $F = \frac{f}{\varphi} \leq -1$ on W and by lemma 5.4, $F(x(s))$ is strictly increasing in the complement of W. If $F(x(s)) > -1$ for one s then it rests bigger than -1 which contradicts the fact that the trajectory crosses W in any neighborhood of the origin. Thus, on the trajectory,

$$|f(x(s))| \geq \varphi(|x(s)|).$$

This implies $r \leq |\tilde{f}(x(s))|$.

By proposition 4.2 applied to W and $\tilde{f}$, $|\tilde{f}| \leq Cr$ on W. Now the second inequality of (5.12) follows from (5.11) and the fact that $F(x(s)) = \frac{f}{\varphi}$ is increasing in the complement of W. Q.E.D.

§6. Asymptotic critical values

Let F be a C^1 definable function F defined on an open definable set U such that $0 \in \overline{U}$. We say that $a \in \mathbb{R}$ is *an asymptotic critical value of F at the origin* if there exists a sequence $x \to 0$, $x \in U$, such that

(a) $|x||\nabla F(x)| \to 0$,
(b) $F(x) \to a$.

Proposition 6.1. *(see also* [1]*)*
The set of asymptotic critical values is finite.

Proof. Let $X = \{(x,t); F(x) - t = 0\}$ be the graph of F. Consider X and $T = \{0\} \times \mathbb{R}$ as a pair of strata in $\mathbb{R}^n \times \mathbb{R}$. Then the (w)-condition of Kuo-Verdier at $(0,a) \in T$ reads

$$1 = |\partial/\partial t(F(x) - t)| \le C|x||\partial/\partial x(F(x) - t)| = Cr|\nabla F|.$$

In particular, $a \in \mathbb{R}$ is an asymptotic critical value if and only if the condition (w) fails at $(0,a)$. The set of such a's is finite by the genericity of (w) condition, see [15] or [1]. Q.E.D.

Remark 6.2. Suppose $x(s)$ is a trajectory of ∇f and let $a = \lim_{s \to s_0} f(x(s))$. Then a is an asymptotic critical value of f. Indeed, suppose contrary to our claim that $r|\nabla f(x)| \ge c > 0$ for $f(x)$ close to a and we may assume $a = 0$. By corollary 5.5, $x(s)$ passes through W^ε in any neighborhood of the origin. Let $\gamma(r)$ be a definable curve in W^ε such that $f(\gamma(r)) \to 0$ as $r \to 0$. Denote, as before, $\psi(r) = f(\gamma(r))$. Then, by (4.3) and since we are in W^ε,

$$r|\psi'(r)| \simeq r|\partial_r f| \ge \varepsilon' r|\nabla f| \ge \varepsilon' c > 0$$

that is impossible since the left-hand side is small.

In particular, only finitely many values of f are allowed as limits along the trajectories of the gradient.

One may ask whether we have an analogue of Łojasiewicz Inequality (3.1) for asymptotic critical values. More precisely, whether for an asymptotic critical value a there exists a continuous definable change of target coordinate $\Psi : (\mathbb{R}, a) \to (\mathbb{R}, 0)$ such that

$$r|\nabla(\Psi \circ F)| \ge c > 0, \tag{6.1}$$

at least if $F(x)$ is close to a. This is not the case in general, but it holds if we approach the singularity "sufficiently slowly".

Proposition 6.3. *Let F be as above and let $a \in \mathbb{R}$. Let $\eta(r)$ be small in sense of definition 1. Then there exists a continuous definable change of parameter $\Psi : (\mathbb{R}, a) \to (\mathbb{R}, 0)$ and a constant $c_a > 0$ such that* (6.1) *holds on $\{x \in U; |\partial_r F| \le \eta(r)|\nabla' F|, |F(x) - a| \le c_a\}$.*

Proof. The proof follows the main ideas of the proof of Łojasiewicz Inequality (3.1). We may assume that a is an asymptotic critical value of F. Choose first $c_a > 0$ so that there is no other asymptotic critical value in $\{t \in \mathbb{R}; |t-a| \le c_a\}$. For simplicity of notation we suppose also $a = 0$, $c_a = c_0$. We may also suppose $F \ge 0$, otherwise we replace F by F^2.

Denote $U_0 = \{x \in U; |\partial_r F| \le \eta(r)|\nabla' F|, |F(x) - a| \le c_a\}$. Choose a definable curve $\gamma(t) \not\equiv 0$ such that $F(\gamma(t)) = t$, and

$$r|\nabla F(\gamma(t))| \le 2 \min\{r|\nabla F(x)|; F(x) = t\},$$

in U_0. Such a curve exists by the o-minimal version of curve selection lemma.

Let $x_0 = \lim_{t\to 0} \gamma(t)$. Suppose first that $x_0 \ne 0$. By [9] there exists Ψ such that $\nabla(\Psi \circ F) \ge 1$. Therefore, by the choice of γ,

$$r|\nabla(\Psi \circ F)(x)| \ge \tfrac{1}{2} r|\nabla(\Psi \circ F)(\gamma(F(x))| \ge c|x_0| > 0. \tag{6.2}$$

So suppose $x_0 = 0$. In this case we may use r as the parameter on γ, $\gamma(r) = \gamma(t(r))$, and write as before $\gamma(r) = r\theta(r)$ in spherical coordinates. Define $\psi(r) = F(\gamma(r))$. Then

$$|\psi'(r)| = |\partial_r F + \langle \nabla' F, r\theta' \rangle| \le \tilde{\eta}(r)|\nabla' F|, \tag{6.3}$$

where $\tilde{\eta} = \eta + r|\theta'|$ is small. In particular there exists a germ of continuous definable function $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that

$$\tilde{\eta}(r) \le r|h'(r)|. \tag{6.4}$$

Then $\Psi := h \circ \psi^{-1}$ satisfies the statement. Indeed, by (6.3) and (6.4),

$$r|\nabla(\Psi \circ F)(\gamma(r)| = r\frac{h'(r)}{\psi'(r)}|\nabla F(\gamma(r))| \ge 1.$$

Hence for x close to the origin, $t = f(x)$,

$$r|\nabla(\Psi \circ F)(x)| \geq \tfrac{1}{2} r |\nabla(\Psi \circ F)(\gamma(t)| \geq \tfrac{1}{2},$$

as required. Q.E.D.

Consider F of the form $F = \frac{f}{\varphi(r)}$. Then

$$\partial_r F = \frac{\partial_r f}{\varphi(r)}\Big(1 - \frac{f\varphi'}{\varphi \partial_r f}\Big), \quad \nabla' F = \frac{\nabla' f}{\varphi(r)}. \tag{6.5}$$

Proposition 6.4. *Suppose that there is an exponent $N > 0$ such that $r^N \leq \varphi(r) \leq r^{1/N}$. Then $a \neq 0$ is an asymptotic critical value of $F = \frac{f}{\varphi(r)}$ if and only if there exists a sequence $x \to 0$, $x \neq 0$, such that*

(a') $\frac{|\nabla' f(x)|}{|\partial_r f(x)|} \to 0$,
(b') $F(x) \to a$

Proof. The proof is similar to that of Proposition 5.3 of [14] and is left to the reader. Q.E.D.

§7. Estimates on a trajectory. II

Let $x(s)$ be a trajectory and let W the union of connected components of W^ε (for any fixed $\varepsilon > 0$) such that $x(s)$ passes through them in any neighborhood of the origin. Restricting ourselves to a smaller neighborhood of the origin, if necessary, we may suppose that the trajectory stays away of $W^\varepsilon \setminus W$. Recall after proposition 5.6 that we may assume that

$$x(s) \in U_C = \{x; -C\varphi(r) \leq \tilde{f}(x) \leq -c\varphi(r)\}, \tag{7.1}$$

$0 < c < C < \infty$ and $\varphi(r) \sim r$, and $\tilde{f}(x)$ is given by (5.10). (Actually by propostion 5.6 we may assume $c = 1$ and $\varphi(r) = r$ but we do not need it.) In particular $\tilde{f}$ on U_C satisfies both Łojasiewicz and Bochnak-Łojasiewicz Inequalities, see remark 3.5. In order to simplify the notation we shall write f for $\tilde{f}$. Define

$$F(x) = \frac{f(x)}{\varphi(r)}, \tag{7.2}$$

Then

$$\frac{dF(x(s))}{ds} = \frac{1}{\varphi(r)|\nabla f|}\Big(|\nabla' f|^2 + |\partial_r f|^2 (1 - \frac{f\varphi'}{\varphi(r)\partial_r f})\Big)$$

By lemma 5.4

$$\frac{dF(x(s))}{ds} \geq c' \frac{|\nabla f|}{\varphi(r)} \geq c'' \frac{1}{r} \quad \text{on } U_C \setminus W. \tag{7.3}$$

Lemma 7.1. *There exists a continuous definable function* $\tilde\omega$*,* $\tilde\omega(r) \to 0$ *as* $r \to 0$*, such that*

$$|1 - \frac{f\varphi'}{\varphi\partial_r f}| \leq \tfrac{1}{2}\tilde\omega(r) \quad \textit{on } W. \tag{7.4}$$

Moreover, $\tilde\omega$ *may be chosen small in sense of definition 1.*

Proof. Let $\gamma(r)$ be a definable curve such that $|(1-\frac{f\varphi'}{\varphi\partial_r f})(\gamma(r)| \geq \frac{1}{2}\sup_{W\cap S(r)} |(1 - \frac{f\varphi'}{\varphi\partial_r f})|$. Denote $\psi(r) = f(\gamma(r))$. Then, by (4.3)

$$\psi'(r) = \partial_r f + < \nabla' f, r\theta'(r) >, \tag{7.5}$$

and $r|\theta'(r)|$ is small. Consequently, since recall $|\nabla' f| \leq \varepsilon^{-1}|\partial_r f|$ on W,

$$(1 - \frac{f\varphi'}{\varphi\partial_r f}) = \frac{\varphi\partial_r f - \varphi' f}{\varphi\partial_r f} = \frac{\varphi\psi' - \varphi'\psi}{\varphi\psi'} + \tau(r), \tag{7.6}$$

and

$$|\tau(r)| \leq 2\frac{|\nabla' f|}{|\partial_r f|} r|\theta'| \leq 2\varepsilon^{-1} r|\theta'| \tag{7.7}$$

is small. Note that $\psi'(r) \sim 1$. Hence

$$\frac{\varphi\psi' - \varphi'\psi}{\varphi\psi'} = (\frac{\psi}{\varphi})' \frac{\varphi}{\psi'} \tag{7.8}$$

is small. This ends the proof of lemma. Q.E.D.

We list below some other properties of f on W which follows from (4.3). By (4.3), we get $\partial_r f \simeq \varphi' \sim 1$ on any definable curve in W. Thus, by the curve selection lemma, for any constant $c_1 < 1$, and some positive constants C', c'

$$-C' \leq -c_1^{-1}\varphi' \leq \partial_r f \leq -c_1\varphi' \leq -c' < 0 \quad \text{on W.} \tag{7.9}$$

In particular, $\partial_r f$ is negative on W.

We shall show in the proposition below that $F(x(s))$ has a limit as $s \to 0$. For this we use an auxiliary function $g = F - \alpha(r)$ where $\alpha : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ satisfies $\tilde{\omega} \leq {C'}^{-1}\varphi\alpha'$. Such an α exists since $\tilde{\omega}$ is small.

Proposition 7.2. *Let $\alpha : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ be a continuous definable function and suppose $\tilde{\omega} \leq {C'}^{-1}\varphi\alpha'$. Then the function $g(x) = F(x) - \alpha(r)$ is strictly increasing on the trajectory $x(s)$. In particular $F(x(s))$ has a nonzero limit*

$$F(x(s)) \to a_0 < 0, \qquad \text{as } s \to s_0. \tag{7.10}$$

Furthermore, a_0 has to be an asymptotic critical value of F at the origin.

Proof. First we show that $g(x(s))$ is increasing for $x(s) \in U_C \setminus W$. Recall that on $U_C \setminus W$, $|\partial_r f| < \varepsilon |\nabla' f|$ and (7.3) holds. On the other hand

$$|\frac{d\alpha}{ds}| = |\alpha'(r)\frac{\partial_r f}{|\nabla f|}| \leq \varepsilon|\alpha'(r)| \ll r^{-1}. \tag{7.11}$$

Consequently, in this case,

$$\frac{dg}{ds}(x(s)) \geq c' r^{-1}.$$

This shows that g is increasing on $U_C \setminus W$ as claimed.

In general we have

$$\frac{dg}{ds}(x(s)) = \frac{1}{\varphi|\nabla f|}\Big(|\nabla' f|^2 + |\partial_r f|^2\big(1 - \frac{f\varphi'}{\varphi\partial_r f}\big)\Big) - \alpha'(r)\frac{\partial_r f}{|\nabla f|}. \tag{7.12}$$

Now we consider $x(s) \in W$. By (7.9) and by the choice of α

$$\alpha'(r)\frac{|\partial_r f|}{|\nabla f|} \geq C'\varphi^{-1}\tilde{\omega}\frac{|\partial_r f|}{|\nabla f|} \geq \frac{1}{\varphi|\nabla f|}\Big(|\partial_r f|^2\big(1 - \frac{f\varphi'}{\varphi\partial_r f}\big)\Big),$$

and hence the right-hand side expression in (7.12) is positive (recall $\partial_r f$ is negative on W).

Thus, since $g(x(s))$ is increasing, negative, and bounded from zero on U_C, it has a limit $a_0 < 0$. We shall show that a_0 is an asymptotic critical value of F.

Suppose, by contradiction, that $F(x(s)) \to a$ and a is not an asymptotic critical value of F at the origin. Then, by Proposition 6.4, there is $\tilde{c} > 0$ such that

$$|\nabla' f(x(s))| \geq \tilde{c}|\partial_r f(x(s))|,$$

for s close to s_0. Hence on W

$$\frac{dF}{ds} = \frac{|\nabla' f|^2}{\varphi|\nabla f|} + \frac{|\partial_r f|^2}{\varphi|\nabla f|}(1 - \frac{f\varphi'}{\varphi\partial_r f}) \geq c'\frac{1}{r}. \tag{7.13}$$

A similar bound holds on $U_C \setminus W$ by (7.3).

But (7.13) is not possible since $|\frac{dr}{ds}| \leq 1$. Indeed, (7.13) implies $\frac{dF}{dr} \geq c\frac{1}{r}$ with the right-hand side not integrable which contradicts the fact that F is bounded on the trajectory. This ends the proof. Q.E.D.

Corollary 7.3. *Let $\sigma(s)$ denote the length of the trajectory between $x(s)$ and the origin. Then*

$$\frac{\sigma(s)}{|x(s)|} \to 1 \text{ as } s \to s_0.$$

Proof. The proof follows from Proposition 7.2 and is similar to the one of Corollary 6.5 of [14]. Q.E.D.

§8. Gradient Conjecture on the Plane

In this section we show the following finiteness result.

Theorem 8.1. *Let $f : U \to \mathbb{R}$ be a differentiable definable function, where $U \subset \mathbb{R}^2$ is open definable and $0 \in \bar{U}$. Let $x(t)$ be a trajectory of ∇f such that $x(t) \to 0, f(x(t)) \to 0$ as $t \to 0^-$. Given a definable curve $\Gamma \subset U$. Then, there is $\varepsilon > 0$ such that the set $\{x(t); -\varepsilon < t < 0\}$ either lies entirely in Γ or does not intersect Γ at all.*

Proof. By a standard argument, see the proof of Proposition 2.1 of [14], it suffices to show that the trajectory cannot spiral, that is the statement of theorem holds for at least one curve Y, $0 \in \overline{Y}$. Indeed, consider an arbitrary definable curve $\Gamma \subset U$ parameterized in polar coordinates (r, θ) by $\gamma(r) = r\theta(r)$. Write

$$f(r, \theta) = f(r\cos\theta, r\sin\theta).$$

Denote $\partial_\theta f = \partial f/\partial\theta$. Then $|\partial_\theta f| = r|\nabla' f|$ and $\partial_\theta f$ is positive if and only if $\nabla' f$ is directed anti-clockwise. If Γ is not a trajectory itself, that is if ∇f is not tangent to Γ, then, near the origin, the trajectories of ∇f cross Γ only in one direction. Fix a point $x_0 = \gamma(r) = r\theta(r)$ and the orthonormal basis of $\mathbb{R}^2$ with the first vector being $\frac{x_o}{\|x_o\|} = \theta(r)$. Comparing in this basis the tangent vector $(1, r\theta'(r)$ to the curve Γ and the gradient $(\partial_r f, r^{-1}\partial_\theta f)$ of f we see that the trajectories cross Γ anti-clockwise if and only if

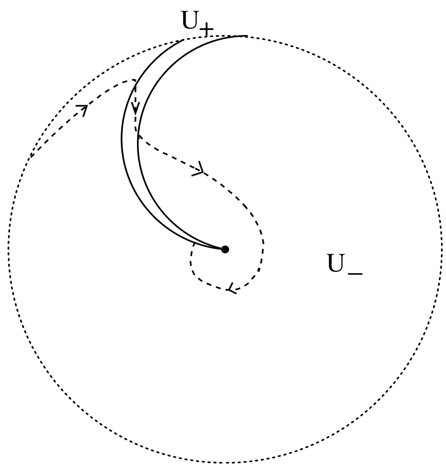
$$\partial_\theta f > r^2\theta'(r)\partial_r f(r). \tag{8.1}$$

Thus if the trajectory does not spiral and is not contained in Γ then, in a small neighborhood of the origin, it may cross Γ only once. In particular if U does not contain a punctured disc of the form $\{0 < r < r_0\}$ then any trajectory going to the origin cannot spiral otherwise it would hit the boundary of U. Thus we may suppose that U contains a punctured disc centered at the origin.

Divide U into two pieces

$$U_+ = \{\partial_\theta f \geq 0\}, \quad U_- = \{\partial_\theta f \leq 0\}.$$

Both of them are non-empty as germs at the origin since $f(r,\theta)$ is periodic for r fixed. On U_- the trajectory moves clockwise and on U_+ anti-clockwise. It is clear that the trajectory cannot spiral if each $U_\pm$ contains a non-empty sector of the form $\{\theta_1 < \theta < \theta_2\}$. This is the case for f analytic, see [14]. But for f definable in an o-minimal structure or even for f subanalytic it may happen that one of $U_\pm$ does not contain a sector, see the picture below.



(One may construct such example easily by choosing two definable curves $r = \gamma_1(\theta)$, $r = \gamma_2(\theta)$ and definable $f(r, \theta)$, periodic in θ, and such that $\partial_\theta f(r, \theta) \geq 0$ exactly on $\gamma_1(\theta) \leq r \leq \gamma_2(\theta)$.)

In what follows we shall assume that U_- contains a non-empty sector but U_+ not necessarily. If we show that U_+ contains a definable curve which $x(t)$ crosses anti-clockwise then we are done.

Lemma 8.2. *Let $\Gamma \in U_+$ be a germ at the origin of a definable curve parameterized by $\gamma(r)$. If*

$$r \to \lambda_\gamma(r) = \frac{|\nabla' f(\gamma(r))|}{|\partial_r f(\gamma(r))|}$$

is not small then the trajectories of ∇f cross Γ anti-clockwise.

Proof. Let $\gamma(r) = r\theta(r)$. It suffices to show (8.1). By lemma 2.3 $\lambda_\gamma(r) \gg r\theta'(r)$ and hence we have

$$\partial_\theta f = \lambda_\gamma(r) r |\partial_r f| \gg r^2 \theta'(r) \partial_r f,$$

as required. Q.E.D.

Thus in what follows it suffices to suppose that

$$\lambda(r) = \sup_{x \in S(r) \cap U_+} \frac{|\nabla' f|}{|\partial_r f|} \tag{8.2}$$

is small. Then, in particular, $\lambda(r) \to 0$ as $r \to 0$. Thus $U_+ \subset W^\varepsilon$ for any $\varepsilon > 0$.

Suppose, contrary to our claim, that there exists a trajectory $x(t)$ of ∇f which spirals. By the previous sections we may suppose that

$$F(x(t)) = \frac{f(x(t))}{|x(t)|}$$

goes to -1 as $t \to 0$. The trajectory $x(t)$, since it spirals, has to cross infinitely many times any component of W^ε. Thus on W^ε, and hence on $U_+ \subset W^\varepsilon$, $f \simeq r$ and, by (7.9), $\partial_r f \simeq -1$.

Denote

$$\psi(r) = \min_{x \in S(r)} f(x) = \min_{x \in S(r) \cap U_+} f(x) \tag{8.3}$$

$$\varphi(r) = \max_{x \in S(r)} f(x) = \max_{x \in S(r) \cap U_+} f(x). \tag{8.4}$$

Lemma 8.3. *Under the above assumptions* $\frac{\varphi(r)-\psi(r)}{r}$ *is small.*

Proof. $\partial_r f \simeq -1$ on U_+. Hence $|\partial_\theta f| = r|\nabla' f| \simeq r\lambda(r)$. By integration in U_+,

$$|f(r,\theta_1) - f(r,\theta_2)| = |\int_{\theta_1}^{\theta_2} \partial_\theta f \, d\theta| \leq C(\theta_2 - \theta_1) r\lambda(r).$$

Therefore

$$\frac{\varphi(r) - \psi(r)}{r} \leq \tilde{C}\lambda(r),$$

and the right-hand side is small by assumption. Q.E.D.

Lemma 8.4. *Suppose* $\frac{\varphi(r)-\psi(r)}{r}$ *small and assume that* U_- *contains a non-empty sector* $\{\theta_1 \leq \theta \leq \theta_2\}$*. Then the set*

$$\{x \in U_-; |\partial_\theta f| \leq \frac{3(\varphi(r) - \psi(r))}{\theta_2 - \theta_1}\}$$

contains a non-empty sector.

Proof. Otherwise

$$|f(r,\theta_1) - f(r,\theta_2)| = -\int_{\theta_1}^{\theta_2} \partial_\theta f \, d\theta \geq 2(\varphi(r) - \psi(r))$$

that contradicts the definition of φ and ψ. Q.E.D.

Let U_0 be a sector satisfying the statement of lemma 8.4. On this sector $|\nabla' f|$ is bounded by $\frac{3(\varphi(r)-\psi(r))}{r(\theta_2-\theta_1)}$ that is small and hence $|\nabla' f| \to 0$ as $r \to 0$. Therefore, by remark 3.5, $|\nabla f| \simeq |\partial_r f| \sim 1$. This means that U_0 is contained in W^ε for any $\varepsilon > 0$. Consider the part of the trajectory that is in U_0. Since the trajectory spirals we may find such a part in any neighborhood of the origin. Since $U_0 \subset W^\varepsilon$, $\partial_r f < 0$ and r is strictly decreasing on the trajectory. Parameterizing the trajectory by r

$$\frac{d\theta}{dr} = \frac{|\nabla' f|}{r|\partial_r f|} \simeq \frac{|\partial_\theta f|}{r^2} \leq C\frac{1}{r}\frac{\varphi(r) - \psi(r)}{r}$$

and the right-hand side is integrable by lemma 8.3. This means that the trajectory cannot cross U_0 if it remains in a small neighborhood of the origin $\{0 < r < r_0\}$. Indeed, by integrability, on the part of the trajectory that is in $U_0 \cap \{0 < r < r_0\}$ the difference of the maximum and the minimum of θ goes to 0 as $r_0 \to 0$.

This ends the proof. Q.E.D.

Corollary 8.5. *If f is defined in an o-minimal structure $\tilde{\mathbb{R}}$ then the trajectory $x(t)$ is definable in the pfaffian closure of $\tilde{\mathbb{R}}$.*

Proof. This follows directly from [23]. Indeed, it suffices to show that the image L of $x(t)$ is a Rolle leaf. For this we fix U a definable "horn" neighborhood of that contains $L \setminus 0$, that is divided by $L \setminus 0$ into two connected components, and f is C^1 on U. The existence of such U follows from theorem 8.1. Clearly, $L \setminus 0$ is a Rolle leaf of

$$\omega = \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial x} dy$$

by the Rolle-Khovanskii Lemma [7]. Q.E.D.

§9. Gradient Conjecture for Polynomially Bounded o-minimal Structures

In this section we place ourselves in the situation described in section 7. We may suppose that $\varphi \equiv r$ which we shall do just for simplicity of notation. We shall also make the following additional assumption:

Assumption. There exists a continuous definable function ω, small in sense of definition 1, such that

$$|1 - \frac{f}{r\partial_r f}| \le \tfrac{1}{2}\omega^2(r) \quad \text{on } W. \tag{9.1}$$

By lemma 7.1 such ω exists for any polynomially bounded o-minimal structure. Indeed, in this case it suffices to take $\omega = \sqrt{\tilde{\omega}}$. On the other hand example 1 shows that, in an o-minimal structure which is not polynomially bounded, $\tilde{\omega}$ small does not imply necessarily that $\sqrt{\tilde{\omega}}$ is small.

Theorem 9.1. *Let $x(s)$ be a trajectory of $\frac{\nabla f}{|\nabla f|}$, $x(s) \to 0$ as $s \to s_0$. Denote by $\tilde{x}(s)$ the projection of $x(s)$ onto the unit sphere, $\tilde{x}(s) = \frac{x(s)}{|x(s)|}$. Then $\tilde{x}(s)$ is of finite length.*

Proof. Let $F = \frac{f}{r}$ be given by section 7. Then we may suppose that the trajectory is contained in $U_C = \{x; -C \le F(x) \le -1\}$ and, by proposition 7.2, that (7.10) holds, and that $\lim_{s \to s_0} F(x(s)) \to a_0 \le -1$.

We use the arc-length parameterization $\tilde{s}$ of $\tilde{x}(s)$ given by

$$\frac{ds}{d\tilde{s}} = \frac{r|\nabla f|}{|\nabla' f|}, \tag{9.2}$$

Reparametrize $x(s)$ using $\tilde{s}$ as parameter. Then

$$\frac{dF}{d\tilde{s}} = \frac{1}{|\nabla' f|}\Big(|\nabla' f|^2 + |\partial_r f|^2 \big(1 - \frac{f}{r\partial_r f}\big)\Big) = r|\nabla' F| + r\partial_r F \frac{\partial_r f}{|\nabla' f|}, \tag{9.3}$$

where $\nabla' F = \frac{\nabla' f}{r}$.

Lemma 9.2. *There exists a continuous definable change of parameter* $\Psi : (\mathbb{R}, a_0) \to (\mathbb{R}, 0)$ *and a constant* $c' > 0$ *such that*

$$\frac{d\Psi(F(x(s)) - a_0)}{d\tilde{s}} \geq c' \tag{9.4}$$

holds on $\{x \in U_C;\, \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$.

Proof. On the set $\{x \in U_C; \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$, by formulae (6.5),

$$|\partial_r F| = |\frac{\partial_r f}{r}\big(1 - \frac{f}{r\partial_r f}\big)| \leq \frac{\omega^2 |\partial_r f|}{r} \leq \omega |\nabla' F|.$$

By assumption ω is small and we may use Proposition 6.3. Thus there is Ψ such that (6.1) holds. We shall show that Ψ satisfies the statement of lemma.

First we suppose that we are also in W that is in the set $\{x \in W; \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$. Then, by (9.1)

$$\frac{|\partial_r f|^2}{|\nabla' f|}\big(1 - \frac{f}{r\partial_r f}\big) \leq \tfrac{1}{2}\omega^2 \frac{|\partial_r f|^2}{|\nabla' f|} \leq \tfrac{1}{2}|\nabla' f|.$$

Consequently

$$\frac{dF}{d\tilde{s}} \geq |\nabla' f| + \frac{|\partial_r f|^2}{|\nabla' f|}\big(1 - \frac{f}{r\partial_r f}\big) \geq \tfrac{1}{2} r|\nabla' F|,$$

and the lemma follows from (6.1).

A similar argument works on $U_C \setminus W$ since, by (7.3),

$$\frac{dF}{d\tilde{s}} \geq c' r|\nabla' F| = c'|\nabla' f| \geq const > 0.$$

This ends the proof. Q.E.D.

Given $\alpha : (\mathbb{R}_{\geq 0}, 0) \to (\mathbb{R}_{\geq 0}, 0)$ such that

$$\omega \leq \tilde{c}^{-1} r\alpha', \tag{9.5}$$

where $\tilde{c}$ will be specified later. Define $\tilde{\alpha} = \Psi \circ \alpha$. We consider $g = \Psi(F - a_0) - \tilde{\alpha}(r)$ as a control function. Then

$$\frac{dg}{d\tilde{s}}(x(s)) = \Psi' |\nabla' f| + \frac{\partial_r f}{|\nabla' f|} \Psi' \Big(\partial_r f \big(1 - \frac{f}{r \partial_r f}\big) - r\tilde{\alpha}'(r) \Big). \tag{9.6}$$

We may also suppose that $\Psi' \geq 1$.

Lemma 9.3. *There is a constant $c' > 0$ such that*

$$\frac{dg}{d\tilde{s}} \geq c' \tag{9.7}$$

holds on $\{x \in U_C ; \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$.

Proof. On W, $\partial_r f$ is negative and hence

$$-\frac{d\tilde{\alpha}}{d\tilde{s}} = -r\tilde{\alpha}'(r) \frac{\partial_r f}{|\nabla' f|} \geq 0. \tag{9.8}$$

Thus the statement follows from lemma 9.2.

On $U_C \setminus W$

$$\Big|\frac{d\tilde{\alpha}}{d\tilde{s}}\Big| = \Big|r\tilde{\alpha}'(r) \frac{\partial_r f}{|\nabla' f|}\Big| \leq \varepsilon |r\tilde{\alpha}'(r)| = o(1)$$

and the lemma follows again from lemma 9.2. Q.E.D.

It remains to show that (9.7) holds on $U_C \setminus \{x \in U_C ; \omega \leq \frac{|\nabla' f|}{|\partial_r f|}\}$ that is contained in W. We denote it by $W(\omega)$ that is $W(\omega) = \{x \in W ; \omega > \frac{|\nabla' f|}{|\partial_r f|}\}$. Firstly we note that on $W(\omega)$

$$-\frac{d\tilde{\alpha}}{d\tilde{s}} = r\tilde{\alpha}'(r) \frac{|\partial_r f|}{|\nabla' f|} | \geq \Psi' \frac{r\alpha'(r)}{\omega} \geq \tilde{c}\Psi' \geq \tilde{c}. \tag{9.9}$$

On the other hand

$$r\alpha' \geq \tilde{c}\omega \gg \tfrac{1}{2}\omega^2 \partial_r f \big(1 - \frac{f}{r\partial_r f}\big) \tag{9.10}$$

This shows that $-\frac{d\tilde{\alpha}}{d\tilde{s}} = -\frac{\partial_r f}{|\nabla' f|} \Psi' r\tilde{\alpha}'(r)$ dominates in the second term of the right hand side of (9.6). Since the first part cannot be negative we get, by (9.9), (9.7) as required.

This ends the proof of the theorem. Q.E.D.

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Homotopy groups of complements to ample divisors

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Abstract.

Homotopy groups of the complements to divisors with ample components on non-singular projective varieties are considered as the modules over the fundamental group. We prove a vanishing theorem and consider the calculation of supports of these modules by relating them to the cohomology of local systems. We review previous work on the local study of isolated non-normal crossings. As an application, we obtain information about the support loci of homotopy groups of arrangements of hyperplanes.

§1. Introduction

An interesting problem in the study of the topology of algebraic varieties is to understand the fundamental group of the complement to a divisor on a non-singular algebraic variety in terms of the geometry of the divisor. Works of Abhyankar ([1]) and Nori ([31]) show that, if C is an irreducible curve on a non-singular algebraic surface X, then for some effective constant $F(C)$ depending on the local type of singularities of C, the inequality $C^2 > F(C)$ implies that the kernel of the map $\pi_1(X - C) \to \pi_1(X)$ belongs to the center of $\pi_1(X - C)$. For example, if X is simply connected, then $\pi_1(X - C)$ is abelian. Historically, such results were originated in the so-called Zariski problem and we refer to [16] for a survey. The case of non-abelian fundamental groups of complements, notably when $X = \mathbf{P}^2$, is also very interesting. The geometric information, such as the dimensions of the linear systems defined by singularities of the curve, becomes essential in descriptions of fundamental groups and their invariants (cf. [37], [21] [24]). Recently, analogous questions about fundamental groups of the complements in the case when X is symplectic began to attract attention as well (cf. [4]).

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In the present work, we shall show that, in appropriate settings, the relationship between the topology of the complement and the geometry of the divisor can be extended to some higher homotopy groups. Some work in this direction already was done. In [22], we show that if V is a hypersurface in $\mathbf{C}^{n+1}$ with isolated singularities whose compactification in $\mathbf{P}^{n+1}$ is transversal to the hyperplane at infinity, then the first homotopy groups of the complement are the following:

$$(1) \qquad \pi_1(\mathbf{C}^{n+1} - V) = \mathbf{Z}, \quad \pi_i(\mathbf{C}^{n+1} - V) \text{ for } 2 \leq i \leq n-1$$

Moreover, the next homotopy group $\pi_n(\mathbf{C}^{n+1} - V)$ depends on the local type of the singularities and also on the geometry of a collection of singularities as a finite subset in $\mathbf{C}^{n+1}$. It can also be described via a generalization of the van Kampen procedure in terms of pencils of hyperplane sections (cf. [22], [8]). Recently, homotopy groups of arrangements were considered in [13] and [32].

Below, we shall extend these results in two directions. On the one hand, we shall consider complements on arbitrary algebraic varieties rather than just in projective space. The latter case, however, appears to be the most important one due to a variety of interfaces with other areas – e.g. the study of arrangements of hyperplanes. On the other hand, we do not assume here that V has isolated singularities, but rather that the divisor D has normal crossings except for finitely many points. The effect of this is that the fundamental group, which plays the key role in the description of higher homotopy, may be abelian rather than cyclic, as is the case in (1), and the theory which we obtain is *abelian* rather than *cyclic*.

In the next section, we prove the triviality of the action of the fundamental group on higher homotopy groups in certain situations (cf. Theorem 2.1). This implies that all information about homotopy groups in these cases is homological (π_1 in these situations is automatically abelian). In some instances, as result of homological calculations, one obtains a *vanishing* of homotopy groups in certain range. In particular if D is a divisor in $\mathbf{P}^{n+1}$ having only isolated non normal crossings and the number of components greater than $n+1$ then $\pi_i(\mathbf{P}^{n+1} - D) = 0$ in the range $2 \leq i \leq n-1$. The results of section 2 also isolate first non-trivial homotopy group in the sense that it is a non-trivial π_1-module.

In Section 3 we define our main invariant of the homotopy group, i.e. a sub-variety of the spectrum of the group ring of the fundamental group which is the support of the first non trivial homotopy group considered as the module over π_1. We call these sub-varieties *characteristic* and show that they are related to the jumping loci for the cohomology of local systems. The latter have a very restricted structure (cf. [2]),– i.e.

they are unions of translated by points of finite order subgroups which also suggest the numerical data that describe these varieties completely.

The methods of obtaining numerical data specifying the characteristic varieties from the geometry of the divisor are discussed in the Section 6. This is done by using the Hodge theory of abelian covers, which is studied in section 5, and relies on our local study of the isolated non normal crossings in [26] and [14]. The results of these papers are discussed in the Section 4 where, among other things, we compare cyclic theory of isolated singularities with abelian theory of isolated-non normal crossings having more than one components. In Section 7, we review cases to which the results of Section 6 can be applied. In particular, the Kummer configuration yields an arrangement of planes in $\mathbf{P}^3$ with non-trivial π_2 of the complement which we calculate. In the final section, we show the relationship between the invariants of the homotopy groups and the motivic zeta function of Denef-Loeser.

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§2. Action of the fundamental group on homotopy groups.

In this section we discuss homotopy groups, in a certain range of dimensions, for a class of quasi-projective varieties. This is done in two steps. Firstly, we show that these varieties support a trivial action of π_1 (in particular are *nilpotent* in certain range). Secondly, we use homological calculations to determine these homotopy groups and to describe cases when homotopy groups vanish.

Recall that homotopy groups $\pi_n(X, x)$ of a topological space X are $\pi_1(X, x)$-modules, with the action given by the "change of the base point" (cf. [34]). In the case when $\pi_i(X) = 0$ for $1 < i < n$, this action on $\pi_n(X)$, which is isomorphic to $H_n(\tilde{X})$ where $\tilde{X}$ is the universal cover of X, coincides with the action of the fundamental group on the homology of the universal cover via deck transformations. A topological space is called k-simple if the action of the fundamental group on $\pi_i(X)$ is trivial for $i \leq k$ (cf. [34]).

Examples of k-simple spaces appearing naturally in algebraic geometry are the following. Locally, they come up when one looks at the complement to a union of germs of divisors in $\mathbf{C}^{n+1}$ forming an isolated non-normal crossing. This situation was studied in [26]. More generally (cf. [14]), instead of divisors in $\mathbf{C}^{n+1}$, one can look at the union of germs of divisors in a germ of a complex space Y having a link which is $(\dim Y - 2)$-connected. Examples of such local singularities are provided by the cones over a normal crossings divisor in $\mathbf{P}^n_{\mathbf{C}}$, in particular by cones over generic arrangements of hyperplanes. These k-simple spaces ($k = \dim Y - 2$) are, of course, Stein spaces and their theory will be reviewed in Section 4. Global k-simple examples are given by the following quasi-projective varieties:

Theorem 2.1. *Let X be a simply connected projective manifold and $D = \bigcup D_i$ be a divisor with normal crossings such that its all components D_i are smooth and ample. Then $\pi_1(X - D)$ is abelian and its action on $\pi_i(X - D)$ is trivial for $2 \leq i \leq \dim X - 1$.*

The proof is similar to the one presented in the local case in [26]. It uses the reduction to the case of normal crossings divisors using Lefschetz hyperplane section theorem and then surjectivity of $\pi_i(D_i - \bigcup_{j \neq i} D_j) \to \pi_i(X - D)$ which follows from ampleness of the components D_i.

This theorem reduces the calculation of the homotopy group to the calculation of *homology* of the complements. The latter can be done using the exact sequence:

$$H_2(X) \to H_2(X, X - D) \to H_1(X - D) \to H_1(X) \to H_1(X, X - D) \tag{2}$$

and the isomorphisms

$$H_j(X, X - D) = H^{2n+2-j}(D)$$

We obtain hence:

Corollary 2.2. *Let $H = \mathbf{Z}^N$ be a free abelian group generated by components of the divisor D. Let*

$$h : H_2(X, \mathbf{Z}) \to H$$

given by $a \to \sum (a, D_i) D_i$ where $a \in H_2(X), D_i \in H^2(X)$ and (a, D) is the Kronecker pairing. Then $\pi_1(X - D) = \operatorname{Coker} h$. For example, if $X = \mathbf{P}^{n+1}$ and one of the components D_i $(i = 1, .., r+1)$ is a hyperplane, then $\pi_1(X - D) = \mathbf{Z}^r$. Let X be a hypersurface in $\mathbf{P}^{n+1}$ and D be a union of $r + 1$-hyperplanes. Then $H_1(X - D) = \mathbf{Z}^r$.

The following result can be used for the calculation of the homology of some branched covers of X:

Corollary 2.3. *Let $D_i \in |\mathcal{L}_i^{m_i}|(i = 1, ..., r)$ such that D_i are divisors on X having isolated non-normal crossings and D_i is the zero set of $f_i \in H^0(X, \mathcal{O}(D_i))$. Let $s_i \in H^0(X, \mathcal{L}_i) \neq 0$. Then $\bar{U}_{m_1,..,m_r}$ given in the total space of $\oplus \mathcal{L}_i$ by $s_i^{m_i} = f_i$ is the cover corresponding the surjection: $\phi : H_1(X - D) \to G = \oplus_{i=1} \mathbf{Z}/m_i\mathbf{Z}$ The projection $\bar{U}_{m_1,..,m_r} \to X$ induces the isomorphism: $H_i(\bar{U}_{m_1,...,m_r}) \to H_i(X)$ for $i \leq n-1$.*

The next theorem is an abelian version of the result in [22] and identifies “the first non-trivial homotopy group” in the sense of [22].

Theorem 2.4. (a) *Let $X = \mathbf{P}^{n+1}$ and D be an arrangement of $r+1$ hypersurfaces as in Corollary 2.2* (*i.e., such that one of the hypersurfaces has degree 1*) *and having finitely many non-normal crossings. Then $\pi_i(\mathbf{P}^{n+1} - D) = 0$ for $2 \leq i \leq n-1$. If all intersections are the normal crossings, then the $\pi_n(\mathbf{P}^{n+1} - D) = 0$.*
(b) *Let V be a complete intersection in $\mathbf{P}^N$ and* $\dim V = n + 1$. *Let D be the arrangement of $r + 1$ hyperplane sections of V having isolated non-normal crossings. Then $\pi_1(V - D) = \mathbf{Z}^r$ and $\pi_i(V - D) = 0$ for $2 \leq i \leq n-1$.*

Proof. Consider first (a). The claimed vanishing is a consequence of the Lefschetz hyperplane section theorem (cf. [18]) and the second part of (a). The first part follows by induction, with the inductive step being the vanishing of $\pi_n(\mathbf{P}^{n+1} - D)$ where D is an arrangement of hypersurfaces with normal crossings. Taking into account the triviality of the action of $\pi_1(\mathbf{P}^{n+1} - D)$ on π_n, the claim is a consequence of the exact sequence (cf. [6]):

$$H_{n+1}(\mathbf{P}^{n+1} - D) \to H_{n+1}(\mathbf{Z}^r) \to \pi_n(\mathbf{P}^{n+1} - D)_{\mathbf{Z}^r} \to \tag{3}$$

$$\to H_n(\mathbf{P}^{n+1} - D) \to H_n(\mathbf{Z}^r) \to 0$$

and the calculation of the homology of $\mathbf{P}^{n+1} - D$. The latter can be done using Mayer Vietoris spectral sequence (cf. [26]). The proof of (b) is similar.

§3. Characteristic varieties of homotopy groups

In this section, we study the support of the first homotopy group of quasi-projective varieties from Section 2 on which the action of π_1 fails to be trivial. This support is a subvariety of $\operatorname{Spec} \mathbf{C}[\pi_1]$, which we call *the characteristic variety.* We show that in the range $2 \leq i \leq k-1$, in which the action of π_1 on π_i is trivial, the homology H_i of the local systems, corresponding to the points of the algebraic group $\operatorname{Spec} \mathbf{C}[\pi_1]$ different from the identity, is trivial. Moreover, the first homotopy group outside this range, i.e. π_k, determines the homology H_k of the local systems. Vice versa, the (co)homology of local systems determines the support of $\pi_k \otimes \mathbf{C}$ as π_1-module. This yields, in the algebro-geometric context, a "linear" structure of the characteristic varieties.

Theorem 3.1. *Let X be a topological space such that its fundamental group $\pi_1(X) = A$ is abelian. Assume that for an ideal $\wp$ in $\mathbf{C}[A]$ the localization of the homotopy groups is trivial for $2 \le i < k$: $\pi_i(X)_\wp = 0$. Then $H_i(\tilde{X})_\wp = 0$ for $1 \le i < k$ and $H_k(\tilde{X})_\wp = \pi_k(X)_\wp$*

Sketch of the proof. The universal cover $\tilde{X}$ of X is a simply connected space on which A acts freely. For such a space, the group A acts on $H_j(\tilde{X}, \mathbf{C})$ for any j and on the homotopy groups $\pi_j(\tilde{X}, \tilde{x}_0) = \pi_j(X, x_0)$ $(j \ge 2)$ so that the Hurewicz map: $\pi_j(\tilde{X}) \to H_j(\tilde{X})$ is $\pi_1(X)$-equivariant (cf. [34] Ch.7, Cor. 3.7).

Let us consider a simply connected CW-complex Y on which an abelian group A acts freely. The group A then acts on the homotopy groups via composition of the map $\pi_n(Y, x) \to \pi_n(Y, a(x))$ and the identification $\pi_n(Y, a(x))$ and $\pi_n(Y, x)$, which is independent of the choice of a path connecting x and $a(x)$ due to $\pi_1(Y) = 0$. The claim is that, if $\pi_i(Y)_\wp = 0$ for $1 < i \le n-1$, then $\pi_n(Y)_\wp = H_n(Y)_\wp$. The theorem above will follow for $Y = \tilde{X}$ and $G = \pi_1(X)$.

The claim can be obtained by induction over n as follows. Consider the fibration of path space $Maps(I, Y) \to Y \times Y$. This fibration is equivariant (where the action on $Y \times Y$ is diagonal). Space $Maps(I, Y)$ is homotopy equivalent to Y. We have the spectral sequence:

$$E^2_{p,q} : H_p(Y \times Y, H_q(\Omega Y)) \to H_{p+q}(Y)$$

This spectral sequence is equivariant. The action on the homology of fiber is given by $av = p_* a_*(v)$ where $a_* : H_i(\Omega_x Y) \to H_i(\Omega_{gx} Y)$ and p_* is the natural identification of the homology of different fibers in a fibration with a simply-connected base. Localizing at $\wp$, due to inductive assumption on Y, we obtain that the terms with $0 < p \le n-1$ and $0 < q \le n-2$ are zeros. In localized spectral sequence we can identify the map $H_n(Y)_\wp \to E^{n,0}_\infty = \operatorname{Ker} d^n_{n,0} : H_n(Y \times Y)_\wp \to H_{n-1}(\Omega Y)_\wp$ with the map $i_\Delta : H_n(Y)_\wp \to H_n(Y \times Y)_\wp$ corresponding to the diagonal embedding. Moreover, $d^{n,0}_n$ is surjective (since $H_{n-1}(Y)_\wp = 0$). Hence we have an exact sequence:

$$0 \to \operatorname{Im}(i_\Delta)_\wp \to H_n(Y \times Y)_\wp \to H_{n-1}(\Omega Y)_\wp \to 0$$

and since cokernel of $H_n(Y)_\wp \to H_n(Y \times Y)_\wp = H_n(Y)_\wp \oplus H_n(Y)_\wp$ is isomorphic to $H_n(Y)_\wp$, due assumed vanishing, we obtain that $H_n(Y)_\wp = H_{n-1}(\Omega Y)_\wp = \pi_n(Y)_\wp$.

We shall apply this theorem to $(n-1)$-simple spaces. For such a space the support of $\pi_i(X) \otimes_{\mathbf{Z}} \mathbf{C}$ as a $\mathbf{C}[\pi_1(X)]$ module belongs for $2 \le i \le n-1$ to the maximal ideal of the identity of the group $\operatorname{Spec} \mathbf{C}[\pi_1(X)] = \operatorname{Char}[\pi_1(X)]$. This maximal ideal is just the augmentation ideal of the group ring. Hence the localization at a prime ideal not belonging to the maximal ideal of the identity satisfies (after tensoring with $\mathbf{C}$) the assumption of Theorem 3.1. This allows, for $(n-1)$-simple spaces, to express the homology of the local systems in terms of the homotopy groups $\pi_n(X)$:

Theorem 3.2. *Let $\rho \in \operatorname{Char} \pi_1(X)$ be a character of the fundamental group different from the identity and let $\mathbf{C}_\rho$ be $\mathbf{C}$ considered as $\mathbf{C}[\pi_1(X)]$ module via the character ρ. Then*

$$H_i(X, \rho) = 0 \ (i \le n-1) \quad H_n(X, \rho) = \pi_n(X) \otimes_{\mathbf{C}[\pi_1(X)]} \mathbf{C}_\rho$$

Proof. The proof is similar to the one in the case when X is a complement to a plane curve (cf. [24]) and the local case (cf. [26]). Consider the spectral sequence (cf. [7], ch.XVI, th.8.4):

$$H_p(\pi_1(X), H_q(\tilde{X})_\rho) \Rightarrow H_{p+q}(X, \rho)$$

where $H_*(\tilde{X})_\rho$ is the homology of the complex $C(\tilde{X}) \otimes_{\mathbf{Z}} \mathbf{C}$ with the action of $\pi_1(X)$ given by $g(e \otimes \alpha) = g \cdot e \otimes \rho(g^{-1})\alpha$. We can localize this spectral sequence at the maximal ideal $\wp_\rho$ of $\operatorname{Spec} \mathbf{C}[\pi_1(X)]$ corresponding to the character ρ. The resulting spectral sequence has $E_2^{i,j} = 0$ for $1 \le j \le n-1$. The exact sequence of low degree terms yields: $H_n(X, \rho) = H_n(\tilde{X}) \otimes_{\mathbf{C}[\pi_1(X)]} \mathbf{C}_\rho$ which together with Theorem 3.1 proves the claim.

Now we are ready to define the main invariant.

Definition 3.3. *The k-th characteristic variety $V_k(\pi_n(X))$ of the homotopy group $\pi_n(X)$ is the zero set of the k-th Fitting ideal of $\pi_n(X)$, i.e. the zero set of minors of order $(n-k+1) \times (n-k+1)$ of Φ in a presentation*

$$\Phi : \mathbf{C}[\pi_1(X)]^m \to \mathbf{C}[\pi_1(X)]^n \to \pi_n(X) \to 0$$

of $\pi_1(X)$ module $\pi_n(X)$ via generators and relations. Alternatively (cf. Theorem 3.2) outside of $\rho = 1$, $V_k(\pi_n(X))$ is the set of characters $\rho \in \operatorname{Char}[\pi_1(X)]$ such that $\dim H_n(X, \rho) \ge k$.

Theorem 3.2 combined with the results of [2] yields the following strong structure property (for possibly non-essential characters):

Theorem 3.4. *The characteristic variety $V_k(\pi_n(X-D))$ is a union of translated subgroups S_j of the group* $\mathrm{Char}\,\pi_1(X-D)$ *by unitary characters ρ_j:*

$$V_k(\pi_n(X-D)) = \bigcup \rho_j S_j$$

This is an immediate consequence of the interpretation 3.2 and the following theorem applied to a resolution $\hat{X}$ of non-normal crossings of D:

Theorem 3.5 (Arapura [2])**.** *Let $\hat{X}$ be a projective manifold such that $H^1(\hat{X}, \mathbf{C}) = 0$. Let $\hat{D}$ be a divisor with normal crossings. Then there exists a finite number of unitary characters $\rho_j \in \mathrm{Char}\,\pi_1(\hat{X} - \hat{D})$ and holomorphic maps $f_j : \hat{X} - \hat{D} \to T_j$ into complex tori T_j such that the set $\Sigma^k(\hat{X} - \hat{D}) = \{\rho \in \mathrm{Char}\,\pi_1(\hat{X} - \hat{D}) \mid \dim H^k(\hat{X} - \hat{D}, \rho) \geq 1\}$ coincides with $\bigcup \rho_j f_i^* H^1(T_j, \mathbf{C}^*)$. In particular, Σ^k is a union of translated by unitary characters subgroups of* $\mathrm{Char}\,\pi_1(X-D)$.

The components of Σ^1 can *all* be obtained using the maps $X - D$ onto the curves with negative Euler characteristics (cf. [2]). In the case $k > 1$, maps onto quasi-projective algebraic varieties with abelian fundamental group and vanishing π_i for $2 \leq i \leq k-1$ allow one to construct components of $V(\pi_k)$ (cf. Example 7.4 below).

§4. Review of local theory of isolated non-normal crossings

Local theory of isolated singularities of holomorphic functions provides a beautiful interplay between algebraic geometry and topology and in particular the topology of (high dimensional) links (cf. [30]). The main structure is the Milnor fibration $\partial B_\epsilon - V_f^0 \cap \partial B_\epsilon \to S^1$, where V_f^0 is the zero set of a holomorphic function $f(x_1, .., x_{n+1})$ and B_ϵ is a ball of a small radius ϵ about $\mathcal{O}$ (the fibration exist even in the non-isolated case). If the singularity of f at $\mathcal{O}$ is isolated, then the fiber M_f of this fibration (the Milnor fiber) is homotopy equivalent to a wedge of spheres: $S^n \vee ... \vee S^n$. Going around the circle, which is the base of Milnor's fibration, yields the monodromy: $H^n(M_f) \to H^n(M_f)$. It has as its eigenvalues only the roots of unity $\exp(2\pi i\kappa)$ $(\kappa \in \mathbf{Q})$. Moreover, there are several ways to pick a particular value of the logarithm κ of an eigenvalue of the monodromy so that the corresponding rational number will have some geometric significance. One of the ways to do this depends on the existence of a Mixed Hodge structure (cf. [35]) on $H^n(M_f)$. The value of the logarithm is selected so that its integer part is determined by the degree of the component of $Gr_*^F H^n(M_f)$ (graded space associated with the Hodge filtration) on which particular eigenvalue of the semi-simple part of the monodromy appears.

Some of the data above can be obtained by considering the infinite cyclic cover of $\partial B_\epsilon - V_f^0 \cap \partial B_\epsilon$ instead of Milnor fibration. Such a cover is well-defined since $H_1(\partial B_\epsilon - V_f^0 \cap \partial B_\epsilon, \mathbf{Z}) = \mathbf{Z}$ for $n > 1$. For example, the universal cyclic cover is diffeomorphic to the product $M_f \times \mathbf{R}$. The monodromy can be identified with the deck transformation of the infinite cover.

With such reformulation, the Milnor theory can be extended to the case of germs of isolated non-normal crossings in $\mathbf{C}^{n+1}$ (cf. [26]), i.e. germs of functions $f_1 \cdot ... \cdot f_r$ such that the intersection points of divisors $f_1 = 0, ..., f_r = 0$ are normal crossings except for the origin $\mathcal{O}$ (more general case of germs of complex spaces with isolated singularities considered in (cf. [14]). The results, using infinite covers as a substitute for the Milnor fiber, are parallel to the above mentioned results in the isolated singularities case. Notice, however, that though the theory of Milnor fibers is applicable to germs of INNC, much less detailed information can be obtained since these singularities are not isolated for $n > 1$. For example, the Milnor fiber is not even simply-connected (cf., below however, where quite a bit of information about the Milnor fiber can be obtained as a consequence of the present approach).

Let D be a germ of INNC which belongs to a ball B_ϵ about $\mathcal{O}$ and which has r irreducible components. We have the isomorphism $H_1(\partial B_\epsilon - D, \mathbf{Z}) = \mathbf{Z}^r$ and hence the universal abelian cover of $\partial B_\epsilon - D$ has $\mathbf{Z}^r$ as the covering group. The replacement of the Milnor fiber in this abelian situation is the universal abelian cover $\partial \widetilde{B_\epsilon - D}$. Notice that a locally trivial fibration of $\partial B_\epsilon - D$ over a torus does not exist in general since typically $\partial \widetilde{B_\epsilon - D}$ has the homotopy type of an infinite complex. We have the following (cf. [26]):

Theorem 4.1. *For $n > 1$, the fundamental group $\pi_1(\partial B_\epsilon - D)$ is free abelian. The universal (abelian) cover $\partial \widetilde{B_\epsilon - D}$ is $(n-1)$-connected. In particular, $H_n(\partial \widetilde{B_\epsilon - D}, \mathbf{Z})$ is isomorphic to the homotopy group $\pi_n(\partial B_\epsilon - D)$. The latter isomorphism is the isomorphism of $\mathbf{Z}[\pi_1(\partial B_\epsilon - D)]$-modules where the module structure on the homology is given by the action of $\pi_1(\partial B_\epsilon - D)$ on the universal cover via deck transformations and the action on the homotopy is given by the Whitehead product (cf. [34]).*

Notice that the case when D is a divisor with normal crossings is "a non-singular" case since the universal cover is contractible. The simplest example of INNC is given in $\mathbf{C}^{n+1}$ by the equation $l_1 \cdot ... \cdot l_r = 0$, where

l_i are *generic* linear forms (i.e. a cone over a generic arrangement of hyperplanes in $\mathbf{P}^n$). Since the complement to a generic arrangement of r hyperplanes in $\mathbf{P}^n$ has a homotopy type of n-skeleton of the product of $r-1$-copies of the circle S^1 (in minimal cell decomposition in which one has $\binom{r-1}{i}$ cells of dimension i) one can calculate the module structure on the π_n of such skeleton. Its universal cover is obtained by removing the $\mathbf{Z}^{r-1}$ orbits of all open faces of a dimension greater than n in the unit cube in $\mathbf{R}^{r-1}$. Hence $\pi_n(\partial B_\epsilon - D) = H_n(\widetilde{\partial B_\epsilon - D}, \mathbf{Z})$ ($\widetilde{\partial B_\epsilon - D}$ is the universal cover). The chain complex of the universal cover of $(S^1)^{r-1}$ can be identified with the Koszul complex of the group ring of $\mathbf{Z}^{r-1} = \mathbf{Z}^r/(1,...,1)$ (so that the generators of $\mathbf{Z}^r$ correspond to the standard generators of $H_1(\partial B_\epsilon - D)$). The system of parameters of this Koszul complex is $(t_1-1,..,t_r-1)$. Hence $H_n(\widetilde{\partial B_\epsilon - D}, \mathbf{Z}) = \operatorname{Ker}\Lambda^n R \to \Lambda^{n-1}R$ where $R = \mathbf{Z}[t_1,..,t_r]/(t_1\cdot ...\cdot t_r - 1)$. As a result, one has the following presentation:

$$\begin{aligned}(4)\quad &\Lambda^{n+1}(\mathbf{Z}[t_1,t_1^{-1},...,t_r,t_r^{-1}]/(t_1...,t_r-1)^r) \to \\ &\Lambda^n(\mathbf{Z}[t_1,t_1^{-1},...,t_r,t_r^{-1}]/(t_1...,t_r-1)^r) \to \pi_n(\mathbf{C}^{n+1} - \bigcup D_i) \to 0\end{aligned}$$

In particular, the support of the π_n is the subgroup $t_1\cdot ...\cdot t_r = 1$.

We summarize the similarities between the case of hypersurfaces with isolated singularities and INNC in the table 1 in the next page (with 4.1 justifying the first three rows):

In the case of isolated singularities one has the isomorphism: $\pi_n(\partial B_\epsilon - D) = H_n(\widetilde{\partial B_\epsilon - D})$ as $\mathbf{Z}[t,t^{-1}]$-modules, where the module structure on the right is given by the monodromy action. In particular, it is a torsion module and its support is a subset of $\operatorname{Char}\mathbf{Z} = \mathbf{C}^*$ consisting of the eigenvalues of the monodromy of Milnor fibration. Monodromy theorem ([30]) is equivalent to the assertion that eigenvalues are the torsion points of $\mathbf{C}^*$. A generalization of this is the following:

Conjecture 4.2. *The support of $\pi_n(\partial B_\epsilon - D)$ is a union of translated subgroups of* $\operatorname{Char}\pi_1(\partial B_\epsilon - D)$ *by points of finite order.*

4.2 is a local analog of the result 3.4 in the quasi-projective case. Now let us describe a partial result in the direction of 4.2 describing some components of a characteristic variety which satisfy 4.2, and which also will explain last two rows in the above table.

As already was mentioned, the cohomology group of the Milnor fiber $H^n(M_f, \mathbf{C})$ of an isolated singularity support a Mixed Hodge structure (cf. [35]). The monodromy splits into the product of the semi-simple and

Isolated singularities	**INNC**
Milnor fiber	Infinite abelian cover
Homology of Milnor fiber	π_n
Monodromy	π_1 – module structure on π_n
Eigenvalues of monodromy	Characteristic varieties of π_{n}
Monodromy theorem	Translated subgroup Property
Multiplier Ideals	Multivariable Ideals of Quasiadjucntion
Spectrum	Polytopes of quasiadjunction

Table 1

the unipotent part. The semi-simple part leaves the Hodge filtration invariant. The latter allows one to split the eigenvalues into groups corresponding to the components of $Gr^F H^n(M_f, \mathbf{C})$, depending on the graded piece on which the eigenvalue appears. As a consequence, one can assign a rational number to each eigenvalue, i.e., its logarithm so that its integer part is determined by the group to which the eigenvalue belongs (we refer to [35] for the exact description). In other words, we obtain a lift of the support of the homotopy group of the Milnor fiber into the universal cover of the subgroup of unitary characters of $\mathbf{Z}$ (the eigenvalues of the monodromy having a finite order are unitary).

In the abelian (local) case, we have the following. Let us consider the universal cover of the subgroup $\mathrm{Char}^u(\pi_1(\partial B_\epsilon - D))$ of unitary characters. It is isomorphic to $\mathbf{R}^r$ and one can take the unit cube as the fundamental domain of the covering group (i.e. $\mathbf{Z}^r$). We assign an element in the fundamental domain to a unitary character χ having finite order using the following interpretation of the unitary characters from $V_k(\pi_n(\partial B_\epsilon - D))$ (cf. [26] Prop. 4.5).

Proposition 4.3. *Let $G = \oplus_{i=1}^{i=r}\mathbf{Z}/m_i\mathbf{Z}$ be a finite quotient of $\pi_1(\partial B_\epsilon - D)$ and let $\chi \in \mathrm{Char}(\pi_1(\partial B_\epsilon - D))$ which is the image of a character of G. Then the link $X_{m_1,..,m_r}$ of the isolated complete intersection singularity:*

$$z_1^{m_1} = f_1(x_1, ..., x_{n+1}), ..., z_r^{m_r} = f_r(x_1, ..., x_{n+1}) \tag{5}$$

is a $n-1$-connected $2n+1$-manifold, which is a cover of ∂B_ϵ branched over INNC D. The condition: $\chi \in V_k(\pi_n(\partial B_\epsilon - D))$ and χ is essential (cf. 5.3) is equivalent to

$$k = \dim\{v \in H_n(X_{m_1,...,m_r}) \mid gv = \chi(g)v \forall g \in G\}$$

Note that the covering map $X_{m_1,..,m_r} \to \partial B_\epsilon$ is just a projection $(z_1, ..., z_r, x_1, .., x_{n+1}) \to (x_1, ..., x_{n+1})$. Next, we shall use the Mixed Hodge structure on the cohomology of the link (5) (cf. [36]). The Hodge filtration

$$F^0H^n(X_{m_1,...m_r}) \supset ... \supset F^nH^n(X_{m_1,...m_r}) \supset 0$$

is preserved by the group G. The logarithms of characters which appear on the subspace $F^nH^n(X_{m_1,...m_r})$ (i.e., the vectors $\log\chi = (\xi_1, .., \xi_r)$ with $0 \le \xi_i < 1, \forall i$ such that $\exp(2\pi i\xi_1), ..., \exp(2\pi i\xi_r)$ is a character χ of $H_1(\partial B_\epsilon - D)$ in coordinates given by the generators $H_1(\partial B_\epsilon - D)$) form a polytope in the sense of the following

Definition 4.4. *A polytope in the unit cube $\mathcal{U} = \{\mathbf{x} = (x_1, ..., x_n) \mid 0 \le x_i \le 1 \forall i\}$ is a subset of $\mathcal{U}$ formed by the solutions of a system of inequalities $\mathbf{a_k} \cdot \mathbf{x} \le c_k$ for some constants c_k (resp. vectors $\mathbf{a_k}$) such that $\mathbf{a_k} = (a_k^1, ..., a_k^i, ..., a_k^n), 0 \le a_k^i \in \mathbf{Q}$ and $0 \le c_k \in \mathbf{Q}, \forall i, k$. A face of a polytope $\mathcal{P}$ is a subset of its boundary $\partial\mathcal{P}$ which has the form $\partial\mathcal{P} \cap H$ for a hyperplane H different from one of $2n$ hyperplanes $x_i = 0, 1$.*

We have the following:

Theorem 4.5. *To each germ of INNC D and $l, 0 \le l \le n$ corresponds a collection $\mathbf{P}_l$ of polytopes $\mathcal{P}_{k,l} \in \mathbf{P}_l$ such that a vector $\log\chi \in \mathbf{Q}^r$ in unit cube belongs to one of the polytopes $\mathcal{P}_{k,l}$ if and only if $\dim\{v \in F^l/F^{l+1}H^n \mid gv = \chi(g)v\} = k$. In particular, $V_k(\pi_n(\partial B_\epsilon - D)) = \bigcup_k \exp\mathcal{P}_{k,l}$ where $\mathbf{R}^r \to \mathrm{Char}^u H_1(\partial B_\epsilon - D)$ is the exponential map.*

In the cyclic case, each of $\mathcal{P}_{k,l}$ is a rational number ξ such that $\exp(2\pi i\xi)$ is an eigenvalue of the monodromy having a multiplicity k which appears on $F^l/F^{l+1}H^n(M_f)$, i.e. is an element of the spectrum having a multiplicity k (in the case $l = n$ one obtains the constant of quasi-adjunction from [20]).

Remark 4.6. *In the case $n = 1$, i.e., the case of reducible plane curves, we have the polytopes of quasi-adjunction studied in* [25]. *In particular, these polytopes are related to the multi-variable log-canonical thresholds and multiplier ideals* (*cf. remark 2.6 and section 4.2 respect. in* [25]). *Similar relations exist in the case of INNC discussed here. In particular, to each face $\mathcal{F}$ of a polytope of quasiadjunction for INNC corresponds the ideal of quasiadjunction $\mathcal{A}_{\mathcal{F}}$ in the local ring of the singular point of INNC used below* (*cf.* (6.3)).

In the case of isolated singularities, there are very explicit and beautiful calculations of the eigenvalues of the monodromy and spectrum of singularities. We would like to pose the following problem:

Problem 4.7. *Calculate the characteristic varieties of INNC with $\mathbf{C}^*$-actions and in the case when D_i are generic for their Newton polytopes. What are the polytopes described in Theorem 4.5?*

This should be a generalization of the case, discussed above, of the cone over a generic arrangement and the example in [26] of the cone over a divisor with normal crossings in $\mathbf{P}^n$.

§5. Homology of abelian covers

In this section, we return to the global case of divisors with ample components having only isolated non-normal crossings.

5.1. Topology of unbranched covers

The characteristic varieties $\mathrm{Char}_i(\pi_n(X - D))$ contain information about both branched and unbranched abelian covers.

Lemma 5.1. *Let G be a finite abelian quotient of $\pi_1(X-D)$ and let U_G be corresponding unbranched covers of $X-D$. Let $\chi \in \mathrm{Char}(\pi_1(X - D))$ be a pull back of a character of G* (*we shall considered it as a character of the latter*). *Let $H_n(U_G)_\chi = \{v \in H_n(U_G) \mid g \cdot v = \chi(g)v(g \in G)\}$ Then $H_n(X - D, \mathcal{L}_\chi) = H_n(U_G)_\chi$. In particular, $\chi \in \mathrm{Char}\, G \subset \mathrm{Char}_i(\pi_n)$ if and only if $H_n(U_G)_\chi \geq i$.*

A proof can be obtained, for example, from the exact sequence of low degree non-vanishing terms in the spectral sequence of the action of the group $K = \mathrm{Ker}\,\pi_1(X-D) \to G$ on the universal cover $\widetilde{X-D}$ (for which we have $\widetilde{X-D}/K = U_G$):

$$H_p(K, H_q(\widetilde{X-D})) \Rightarrow H_{p+q}(U_G)$$

(cf. [7]). This is a spectral sequence of $\mathbf{C}[\pi_1(X-D)]$-modules where the $\mathbf{C}[\pi_1(X-D)]$-module structure on $\mathbf{C}[G]$ module $H_{p+q}(U_G)$ comes via surjection: $\mathbf{C}[\pi_1(X-D)] \to \mathbf{C}[G]$. The localization of this spectral sequence at a point χ of $\mathrm{Char}\,G \subset \mathrm{Char}\,\pi_1(X-D)$ yields the claim using 3.2, since the localization of $H_n(U_G)$ at χ has the same χ-eigenspace as $H_n(U_G)$.

Now, let us consider the effect of adding (ample) components to D.

Lemma 5.2. *Let D' an ample divisor such that $D \cup D'$ is a divisor with isolated non-normal crossings. Then the homomorphism of $\pi_1(X-D)$ modules: $\pi_i(X-D\cup D') \to \pi_i(X-D)$ is surjective for $1 \le i \le \dim X - 1$. In particular, if one considers $\mathrm{Spec}\,\mathbf{C}[\pi_1(X-D)]$ as a subset in $\mathrm{Spec}\,\mathbf{C}[\pi_1(X-D\cup D')]$, then the intersection of $V_k(\pi_n(X-D\cup D'))$ with $\mathrm{Spec}\,\mathbf{C}[\pi_1(X-D)]$ contains $V_k(\pi_n(X-D))$.*

Sketch of the proof. Let $T(D')$ be a small neighborhood of D' in X. Then by the Lefschetz theorem, $\pi_i(T(D') - D' \cap D)$ surjects onto $\pi_i(X-D)$. On the other hand, this map can be factored through $\pi_i(X-D\cup D')$ which yields the claim.

Lemma 5.2 suggests the following definition:

Definition 5.3. *The components of $V_k(X-\bar{D})$ where $\bar{D}$ is a union of a proper collection of D_i's forming D and which are considered as subsets in $\mathrm{Spec}\,\mathbf{C}[\pi_1(X-D)]$ called the non-essential components of $V_k(X-D)$. The remaining components are called essential.*

A character χ is called essential if $\chi(\gamma) \ne 1$ for each element $\gamma \in \pi_1(X-D)$ which is a boundary of a small 2-disk transversal to one of irreducible components of D.

We shall see in the next section that only essential characters contribute to the homology of branched covers.

5.2. Hodge theory of branched covers.

The relationship between the homology of branched and unbranched covers is more subtle in the present case than in the case of plane curves considered in [24] and the local case of Section 4. One of the reasons is that there is no prefer non-singular model for the abelian global case. Only the birational type of branched cover is an invariant of $X-D$, and hence the Betti numbers of branched covers depend upon compactification of the unbranched cover. However, the Hodge numbers $h^{i,0}$ are birational invariants (in the case $\dim X = 2$, they determine the relevant part of homology of branched cover completely due to the relation $b_1 = 2h^{1,0}$) and one can expect a relation between the Hodge numbers $h^{i,0}$ and the homology of unbranched covers.

Recall that the cohomology of unitary local systems supports a mixed Hodge structure (cf. [38]). We shall denote

$$h^{p,q,k}(\mathcal{L}) = \dim Gr_F^p Gr_{\bar{F}}^q Gr_{p+q}^W (H^k(\mathcal{L}))$$

the dimension of the corresponding Hodge space. In the case of a rank one local system having a finite order, one has the following counterpart of 5.1:

Theorem 5.4. *Let, as in 5.1, $\chi \in \mathrm{Char}(\pi_1(X - D))$ be a character of a finite quotient G of $\pi_1(X - D)$. Let $\bar{U}_G$ be a G-equivariant non-singular compactification of U_G and let $H^{p,q}(\bar{U}_G)_\chi$ be the χ-eigenspace of G acting on $H^{p,q}(\bar{U}_G)$. Then*

$$h^{n,0,n}(\mathcal{L}_\chi) = h^{n,0}(\bar{U}_G)_\chi$$

Sketch of the Proof. The functoriality of the Hodge structure on cohomology of local systems yields that the isomorphism in 3.2 is compatible with the Hodge structure: $h^{n,0,n}(X - D, \mathcal{L}_\chi) = h^{n,0,n}(U_G)_\chi$ where in the RHS are the Hodge numbers of the Deligne's MHS on the cohomology of non-singular quasi-projective manifold (cf. [11]). Let $E = \bar{U}_G - U_G$, which we assume is a divisor with normal crossings. In the exact sequence of MHS: $H^n(\bar{U}_G, U_G) \to H^n(\bar{U}_G) \to H^n(U_G)$, which splits into corresponding sequences of χ-eigenspaces, the image of right homomorphism is $W_n H^n(U_G)$ (cf. [11] 3.2.17). This result is a consequence of the identity: $\mathrm{Ker}\, H^n(\bar{U}_G) \to H^n(U_G) \cap H^{n,0} = 0$ To see the latter, notice that using the duality $H^{n+2}(E) \times H^n(\bar{U}_G, U_G) \to \mathbf{C}(-n-1)$ ($\mathbf{C}(-k)$ is the Hodge-Tate) we obtain $h^{n,0,0}(\bar{U}_G, U_G) = h^{n+1,1,n+2}(E)$. On the other hand, for each smooth component E_i of E one has $h^{i,j,n+2} \neq 0$ only when $0 \le i, j \le n$ and the Mayer Vietoris sequence of MHS yields the same conclusion for E. Hence $H^n(\bar{U}_G) \to H^n(U_G)$ is injective on $H^{n,0}$ and the result follows.

§6. Conjecture and results on the structure of characteristic varieties.

Now we return to the situation discussed in Section 2 and consider the complements to divisors D with isolated non-normal crossings on projective manifolds X ($\dim X = n+1$). Our goal is to calculate the components of $V_i(\pi_n(X-D))$. The procedure described below is a generalization of the one outlined in [24].

Let us assume that $H_1(X-D) = \mathbf{Z}^r$ (i.e., to avoid mainly notational complications, assume that $H_1(X-X, \mathbf{Z})$ is torsion free) and consider the covering corresponding to the homomorphism $H_1(X-D) \to G = \oplus_{i=1}^{i=r} \mathbf{Z}_{m_i}$. Let k_i be the order in G of the element of $H^{2n}(D)$ corresponding to D_i so that we have the surjective map $H^{2n}(D) \to \oplus_{i=1,..N} \mathbf{Z}_{k_i}$ and also the surjection $G' \to G$ where $G' = \oplus \mathbf{Z}_{k_i}$. Let $K = \ker G' \to G$. We have the following diagram (the left column is the part of the sequence (2)):

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ H_1(X-D, \mathbf{Z}) & \to & G \\ \uparrow & & \uparrow \\ H^{2n}(D, \mathbf{Z}) & \to & G' \\ \uparrow & & \uparrow \\ \operatorname{Im} H_2(X, \mathbf{Z}) & \to & K \\ \uparrow & & \uparrow \\ 0 & & 0 \end{array}$$

Dualizing, we obtain:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ H^1(X-D, \mathbf{R}) & \to & \operatorname{Char} H_1(X-D) & \leftarrow & \operatorname{Char} G \\ \downarrow & & \downarrow & & \downarrow \\ H_{2n}(D, \mathbf{R}) & \to & \operatorname{Char} H^{2n}(D) & \leftarrow & \operatorname{Char} G' \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Hom}(\operatorname{Im} H_2(X), \mathbf{R}) & \to & \operatorname{Char} \operatorname{Im} H_2(X) & \leftarrow & \operatorname{Char} K \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array} \tag{6}$$

with the maps from the left to the middle column on (6) induced by the universal covering map $\mathbf{R} \to S^1$. The left column itself is the part of the cohomology sequence of the pair $(X, X-D)$.

Consider the preimage of $\operatorname{Char} G' \subset \operatorname{Char} H^{2n}(D, \mathbf{Z})$ under the map $H^{2n}(D, \mathbf{Q}) \to H^{2n}(D, S^1)$ and select the fundamental domain for the action of the kernel of the latter map i.e. the action of $H^{2n}(D, \mathbf{Z})$ on $H^{2n}(D, \mathbf{Q})$. We shall assume that this domain is the unit cube $\mathcal{U}$: $\{(j_1, ..., j_N) \mid 0 \le j_i < 1\}$ in $\mathbf{Q}^N$ ($N = rkH^{2n}(D, \mathbf{Q})$) with coordinates corresponding to the components of D. Selection of the fundamental domain allows to attach to each $\chi \in \operatorname{Char} G'$ unique element in $\mathcal{U}$. The preimage of $\mathrm{Char}\, G$ is a subgroup of $H_{2n}(D, \mathbf{R})$. The image of this subgroup in $\mathrm{Char}\,\mathrm{Im}\, H_2(X, \mathbf{Z})$ is trivial and hence belongs to

$$\mathrm{Hom}(\mathrm{Im}\, H_2(X, \mathbf{Z}), \mathbf{Z}) \subset H^2(X, \mathbf{Z}).$$

In particular, any character χ of G determines the element $\mathcal{L}_\chi$ of $\mathrm{Pic}(X)$. These bundles satisfy: $\mathcal{L}_\chi \otimes \mathcal{L}_{\chi^{-1}} = \otimes \mathcal{O}(D_s)$ where D_s is the collection of irreducible components of D such that $\chi(\gamma_{D_s}) \neq 1$ (γ_{D_s} is the image in $H_1(X - D)$ of the generator of the summand of $H^{2n}(D)$ corresponding to D_s). One can show that if $\tilde{X}_G \to X$ is a branched cover then the divisorial components of $f_*(\mathcal{O}_{\tilde{X}_G})$ are $\oplus_\chi L^*_\chi$. We have: $f_*(\Omega^{n+1}_{\tilde{X}_G}) = \oplus_\chi \Omega^{n+1}_X \otimes L_{\chi^{-1}}$. Also, for a given $\mathcal{D} \in \mathrm{Pic}(X)$ the collection of lifts of characters $\chi \in \mathrm{Char}\, H^{2n}(D)$ to $H_{2n}(D, \mathbf{R})$ such that $\mathcal{L}_{\chi^{-1}} = \mathcal{D}$ form an affine subspace $L_{\mathcal{D}}$ of $H_{2n}(D, \mathbf{R})$.

Example 6.1. *Let $X = \mathbf{P}^{n+1}$ and let D be an arrangement of $r+1$ hyperplanes $H_i, i = 1, ..., r$ (i.e., $H_1(\mathbf{P}^{n+1} - D, \mathbf{Z}) = \mathbf{Z}^r$). The characters of $H^{2n}(D, \mathbf{Z})$ which factor through $(\mathbf{Z}/n\mathbf{Z})^r$ correspond to the collections $x_i \in \mathbf{Z}, i = 1, , , , r+1, 0 \leq x_i < n$ such that $\sum x_i \equiv 0 \mod n$. Let us consider a covering X_G with the Galois group $G = (\mathbf{Z}/n\mathbf{Z})^r$ corresponding to the homomorphism $H_1(\mathbf{P}^{n+1} - D, \mathbf{Z}) \to \mathbf{Z}/n\mathbf{Z}^{r+1}/(\mathbf{Z}/n\mathbf{Z})$ (quotient by the diagonally embedded cyclic subgroup K of $G' = (\mathbf{Z}/n\mathbf{Z})^{r+1}$) sending the boundary of a small disk transversal to H_i to a generator of the i-th summand. We have $f_*(\mathcal{O}_{X_G}) = \oplus \mathcal{L}_\chi$ with $\mathcal{L}_\chi = \mathcal{O}((-\sum \frac{x_i}{n})H)$. Moreover, $\mathcal{L}_{\chi^{-1}} = \mathcal{O}(\sum(1 - \frac{x_i}{n}))H$. Taking ramification into account, the assignment the characters of $H_1(X - D, \mathbf{Z})$ to elements of $H^{2n}(D)$ can be done so that to $(x_1, ..., x_{r+1})$ corresponds the character $\exp(\sum(1 - \frac{x_i+1}{n}))$ and so that: $f_*(\Omega^{n+1}) = \oplus_{x_1,..,x_{r+1}} \Omega^{n+1}_{\mathbf{P}^{n+1}} \otimes \mathcal{O}((1 - \frac{x_1+1}{n})H)$ for $(x_1, ..., x_{r+1}), 0 \leq x_i < n$ selected so that $\sum \frac{x_i+1}{n} \in \mathbf{Z}$. We have $\mathrm{Pic}(X) = \mathbf{Z}$ and the preimage of $\mathcal{O}(l) \in \mathrm{Pic}(X)$ is the hyperplane in $H^{2n}(X, \mathbf{R})$ corresponding to the latter lift. It is given by $\frac{x_1+1}{n} + ... + \frac{x_r+1}{n} = l$ where x_i's are the coordinates corresponding to the basis of $H^{2n}(X, \mathbf{Z})$ given by the cycles dual to D_i's*

Now, with each $S \in \mathcal{S}$, we associate a polytope in the unit cube in $\mathbf{R}^r$ as follows. For any $S \in \mathcal{S}$, one has the map $H_1(B_\epsilon - D) \to H_1(X - D)$ and hence the map $\mathrm{Char}\, H_1(X - D) \to \mathrm{Char}\, H_1(B_\epsilon - D)$. The latter lifts to the map of universal covers: $\mathbf{R}^r \to \mathbf{R}^s$ where s is the number of components of D containing S. This can be described in coordinates as follows. A vector $\Xi : (\kappa_1, .., \kappa_r) \in \mathbf{Q}^r (0 \leq \kappa_i < 1)$ for any collection $(j_1, ..., j_s)$ determines the vector: $\Xi^{j_1, .., j_s} = (\{\sum a_{i,j_1} \kappa_i\}, ..., \{\sum a_{i,j_1} \kappa_i\}) \in \mathbf{Q}^r$ ($\{\}$ is the fractional part of a rational number). For each $S \in \mathcal{S}$, we consider subsets $\mathcal{P}^{gl}_S$ consisting of vectors $\Xi = (\kappa_1, .., \kappa_r)$ such that $\Xi^{j_1,..,j_s} \in \mathcal{P}_S$ where $D_{j_1}, ., D_{j_s}$ are the components of D passing through $S \in \mathcal{S}$ and $\mathcal{P}_S \in \mathbf{Q}^s$ is a face of a polytope of quasi-adjunction of INNC formed by $D_{j_1}, .., D_{j_s}$.

Definition 6.2. *Let $\mathcal{S} \subset X$ be the collection of non-normal crossings of the divisor D. Global polytope of quasi-adjunction corresponding to $\mathcal{S}$ is $\bigcap_{S \in \mathcal{S}} \mathcal{P}_S^{gl}$. A global face of quasi-adjunction is a face of a global polytope of quasiadjunction. A divisor $\mathcal{D} = \sum \alpha_i D_i \in \operatorname{Pic} X, \alpha_i \in \mathbf{Q}$ is called contributing if the corresponding subset $L_{\mathcal{D}}$ of the elements of the universal cover (cf. definition before Example 6.1) contains a global face of quasi-adjunction $\mathcal{F}$ and $H^1(\mathcal{A}_{\mathcal{F}} \otimes \Omega_X^{n+1} \otimes \mathcal{D}) \neq 0$. Here $\mathcal{A}_{\mathcal{F}}$ is the ideal of quasiadjunction corresponding to the face $\mathcal{F}$ (cf. Remark 4.6). A global face of quasi-adjunction $\mathcal{F}$ is contributing if there is a contributing divisor such that the corresponding subspace $L_{\mathcal{D}}$ contains $\mathcal{F}$.*

Conjecture 6.3. *Zariski closure of* $\exp(\mathcal{F}) \subset \operatorname{Char} H_1(U, \mathbf{Z})$ *is a component of characteristic variety V_k where $k = \dim H^1(\mathcal{A}_{\mathcal{F}} \otimes \Omega_X^{n+1} \otimes \mathcal{D})$ if $\mathcal{F}$ is a contributing face of a polytope of quasiadjunction.*

I don't know if such components are *all* essential components of the characteristic variety.

The supporting evidence is the following. This conjecture is shown in [24] in the case of curves and in the case $X = \mathbf{P}^{n+1}$ and D is a hypersurface with isolated singularities in [23]. Both of these results can be generalized as follows (the proof, based on the methods used in these two papers will appear elsewhere).

Theorem 6.4. *Let $D \subset \mathbf{P}^{n+1}$ be a union of hypersurfaces D_0, D_1,..., D_r of degrees $1, d_1, ..., d_r$ respectively, which is a divisor with isolated non-normal crossings. Let $\mathcal{F}$ be a face of global polytope of quasi-adjunction, i.e. a face of an intersection of polytopes of quasi-adjunction corresponding to a collection $\mathcal{S}$ of non-normal crossings of D. Let $d_1x_1 + ... + d_rx_r = l$ be a hyperplane containing the face of quasiadjunction $\mathcal{F}$. If $H^1(\mathcal{A}_{\mathcal{F}} \otimes \mathcal{O}(l-3)) = k$, then the Zariski closure of* $\exp(\mathcal{F}) \subset \operatorname{Char} H_1(\mathbf{P}^{n+1} - D)$ *is a component of $V_k(\pi_n(X - D))$.*

A consequence of the conjecture is the corollary.

Conjecture 6.5. *Let $\chi \in \operatorname{Char} \pi_1(X - D)$ be a character of a finite quotient of G of the fundamental group. Let, as in Lemma 5.4, $\bar{U}_G$ be a G-equivariant compactification of the unbranched cover of $X - D$ with the Galois group G and let $h_\chi^{n,0}(\bar{U}_G)$ be the χ-eigenspace of the G acting on $H^{n,0}(U_G)$. Then $h_\chi^{n,0}(\bar{U}_G) = 0$ unless the lift of χ belongs to a contributing global face of quasi-adjunction $\mathcal{F}$ and in which case one has:*

$$h_\chi^{n,0}(\bar{U}_G) = \dim H^1(\mathcal{A}_{\mathcal{F}} \otimes \Omega_X^n \otimes \mathcal{O}(D))$$

where $D \in \mathrm{Pic}(X)$ is the divisor corresponding to the lift of character χ.

In the case when there are bundles $\mathcal{L}_i$ such that $\mathcal{L}_i^{n_i} = \mathcal{O}(D_i)$ and the cover corresponds to the group $\mathbf{Z}_{n_1} \times ... \times \mathbf{Z}_{n_r}$, using the arguments similar to those used in [39], one obtains (in agreement with the conjecture) the following: the eigenspace of the action of G corresponding to the dimension of the eigenspace corresponding to $(e^{2\pi i \frac{p_1}{n_1}}, ..., e^{2\pi i \frac{p_r}{n_r}})$ is $\dim H^1(\mathcal{A}_{\mathcal{F}} \otimes \Omega_X^{n+1} \otimes \mathcal{L}_1^{p_1} \otimes ... \otimes \mathcal{L}_r^{p_r})$. In the case when $r = 1$ the condition that $(\frac{p_1}{n_1}, ..., \frac{p_r}{n_r})$ belongs to the face of quasi-adjunction becomes the condition that $\frac{p}{n}$ is an element of the spectrum of one of the singularities of the divisor D and one obtains the result from [39].

§7. Examples

7.1. Local examples

Example 7.1. *Germs of curves.*

In the case of curves, the support of $H_1(\widetilde{\partial B_\epsilon - D}, \mathbf{C})$ is the zero set of the Alexander polynomial. There are extensive calculations of this invariant using knot-theoretical methods (cf. [15]). Hodge decomposition is considered in [25]. For example, for the singularity $x^r - y^r = 0$, the characteristic variety is $t_1 \cdots t_r = 1$ (cf. the calculation for the cone over the generic arrangement in Section 4). The faces of the polytopes of quasi-adjunction are the hyperplanes $x_1 + ... + x_r = l$, $(l = 1, .., r-2)$.

Example 7.2. *Cones.*

A generalization of the example of arrangements of hyperplanes considered in Section 4 is given by a union of non-singular hypersurfaces in $\mathbf{P}^n$ which form a divisor with normal crossings (cf. [26]). If the degrees of hypersurfaces are $d_1, .., d_r$ respectively then $V_1 = \mathrm{Supp}(\pi_n(\mathbf{C}^{n+1} - D) \otimes \mathbf{C})$ is given by $t_1^{d_1} \cdot ... \cdot t_r^{d_r} - 1 = 0$.

7.2. Global examples

Example 7.3. *Plane curves*

We refer to [24] for examples of characteristic varieties for pencils quadrics (Ceva arrangement of four lines) and pencils of cubics (arrangement of nine lines dual to inflection points of a non-singular cubic and the arrangement of 12 lines containing its inflection points). Papers [27], [28] and [10] describe a combinatorial method to detect components of characteristic variety and in [9] a generalization to arrangements of rational curves is considered. Papers [3] and [5] contain applications of characteristic varieties to geometric problems.

Example 7.4. *Arrangement in* $\mathbf{P}^3$ *with isolated non-normal crossings for which* π_2 *of the complement which support has non-trivial essential components.*

Consider the arrangement $D_{8,4}$ of hyperplanes in $\mathbf{P}^3$ which is an (8,4) configuration (cf. [17]). It includes a plane containing 4 generic points $Q_1, ..., Q_4$, six generic planes $H_{i,j}$ each passing through the line Q_iQ_j and also the plane containing the four coplanar (by Desargue theorem) points $H_{i,j} \cap H_{j,k} \cap H_{i,k}$. Recall (cf. [17]) that this configuration contains eight planes and eight points such that every plane contains four points and every point belongs to exactly four planes. Denoting eight points by $1, 2, 3, 4, 1', 2', 3', 4'$ and eight planes by $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{1}', \mathbf{2}', \mathbf{3}', \mathbf{4}'$ the incidence relation is given by the diagram:

$$\begin{array}{cc} 1 & 1' \\ 2 & 2' \\ 3 & 3' \\ 4 & 4' \end{array} \qquad \begin{array}{cc} \mathbf{1} & \mathbf{1}' \\ \mathbf{2} & \mathbf{2}' \\ \mathbf{3} & \mathbf{3}' \\ \mathbf{4} & \mathbf{4}' \end{array}$$

where the plane in position (i, j) contains all points in row i and column j except for the point in position (i, j).

This arrangement of eight hyperplanes has only isolated non-normal crossings. From 2.2, we infer that $H_1(\mathbf{P}^3 - D_{8,4}) = \mathbf{Z}^7$. Moreover we have the rational map:

$$\Pi : \quad \mathbf{P}^3 \to \mathbf{P}(H^0(\mathbf{P}^3, \mathcal{I}(2)))^* = \mathbf{P}^2$$

where $\mathcal{I}$ is the ideal sheaf of the collection of eight points in $\mathbf{P}^3$ forming this configuration. The indeterminacy points are the eight points of configuration. In order to calculate the Π-image of the hyperplanes of the arrangement, notice that the points in the target of the map correspond to the pencils of quadrics in the web, the image of a point is the pencil of quadric in the web containing this point and the lines correspond to quadrics in $H^0(\mathbf{P}^3, \mathcal{I}(2))$ i.e. are the collections of pencils containing a quadric. In particular, the image of a point P in a hyperplane $H \in D_{8,4}$ is a pencil of quadrics from $H^0(\mathbf{P}^3, \mathcal{I}(2))$ containing P. This pencil contains the quadric among the four quadrics containing P, mentioned earlier. Hence the image of P belongs to the union L of four lines in $\mathbf{P}(H^0(\mathbf{P}^3, \mathcal{I}(2)))^*$ corresponding to above four quadrics. Therefore we have a regular map: $\Pi : \mathbf{P}^3 - D_{8,4} \to \mathbf{P}^2 - L$.

Let us calculate the cohomology of local systems $\Pi^*(\mathcal{L})$, where $\mathcal{L}$ is a local system on $\mathbf{P}^2 - L$. We have the Leray spectral sequence: $H^p(\mathbf{P}^2 - L, R\Pi^q_*(\Pi^*\mathcal{L})) \Rightarrow H^{p+q}(\mathbf{P}^3 - D_{8,4}, \Pi^*\mathcal{L})$. Using $R\Pi^q_*(\mathcal{L}) = \mathcal{L} \otimes R^q_*(\mathbf{C})$ and looking at the critical set of Π, one checks that $H^0(\mathbf{P}^2 -$

$L, \mathcal{L} \otimes R\Pi^1_*(\mathbf{C})) = H^1(\mathbf{P}^2 - L, \mathcal{L} \otimes R\Pi^1_*(\mathbf{C})) = 0$ for a Zariski dense set of local systems. Hence this spectral sequence degenerates for those local systems $\mathcal{L}$ on $\mathbf{P}^2 - L$. This yields that $H^2(\mathbf{P}^3 - D_{8,4}, \Pi^*(\mathcal{L})) = H^2(\mathbf{P}^2 - L, \mathbf{C}) = \mathbf{C}^3$ for a Zariski dense set of local systems $\mathcal{L}$ on $\mathbf{P}^2 - L$.

The above calculation shows that, since

$$\pi_2(\mathbf{P}^2 - L) \otimes \mathbf{C} = \mathbf{C}[t_1, t_{,2}, t_3, t_4]/(1 - t_1 t_2 t_3 t_4)$$

the support of the homotopy group: $\pi_2(\mathbf{P}^3 - D_{8,4}) \otimes \mathbf{C}$ has a 3-dimensional component. Projections from each of eight vertices of this configuration yield linear maps of $\mathbf{P}^3 - D_{8,4}$ onto the complement in $\mathbf{P}^2$ to four lines in a general position and hence a 3-dimensional component. We obtain hence in $\operatorname{Spec} \mathbf{C}[H_1(\mathbf{P}^3 - D_{8,3})]$ nine 3-dimensional components of the support of $\pi_2(\mathbf{P}^3 - D_{8,3}) \otimes \mathbf{C}$.

The component corresponding to the web of quadrics can be detected using Theorem 6.4. Indeed local face of quasiadjucntion has the form $x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1$ where x_{i_j} correspond to the planes containing one of the above 8 points. The intersection of these hyperplanes in $\mathbf{R}^8$ has dimension 3 and belongs to the hyperplane $x_1 + ... + x_8 = 2$ since adding relations $x_{i_1} + x_{i_2} + x_{i_3} + x_{i_4} = 1$ for all 8 points yield the relation $4x_1 + ... + 4x_8 = 8$ since each point belongs to 4 hyperplanes. This intersection will be contributing iff $\dim H^1(\mathcal{J}(8-4-2) \neq 0$ where $\mathcal{O}_{\mathbf{P}^3}/J$ has support at the above 8 points and and the stalk of J at each of those is the maximal ideal of $\mathcal{O}_{\mathbf{P}^3}$. The sheaf $\mathcal{J}$ has Koszul resolution:

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-6) \to \mathcal{O}_{\mathbf{P}^3}(-4)^3 \to \mathcal{O}_{\mathbf{P}^3}(-2)^3 \to \mathcal{J} \to 0$$

which yields $\dim H^1(\mathbf{P}^3, \mathcal{J}(2)) = 1$ i.e. the above face of quasiadjunction is contributing.

§8. Betti and Hodge realizations of multi-variable motivic zeta function

The purpose of this section is to relate the motivic zeta function of Denef and Loeser in the case of local INNC to the invariants considered in Section 4.

Recall that, to a smooth variety X over $\mathbf{C}$ and r holomorphic functions $f_i : X \to \mathbf{C}$, one associates a multi-variable motivic zeta-function $Z_{f_1,...,f_r}(T_1, .., T_r)$ which is a formal series in $\mathcal{M}_{X_0 \times \mathbf{G}_m^r}[[T_1, ..., T_r]]$. Here, as in [12], $X_0 = \bigcap_i f_i^{-1}(0)$, $\mathbf{G}_m$ is the multiplicative group of the field $\mathbf{C}$ and for a variety S the ring $\mathcal{M}_S$ is obtained from the Grothendieck group $K_0(Var_S)$ of varieties over S by inverting the class $\mathbf{L}$ of $\mathbf{A}^1_k \times S \in K_0(Var_S)$. More precisely, denote $\mathcal{L}(X)$ (resp. $\mathcal{L}_n(X)$) the arc space of

X (resp. arc space mod n) whose points are the maps Spec $\mathbf{C}[[t]] \to X$ (resp. Spec $\mathbf{C}[[t]]/(t^{n+1}) \to X$). Let

$$\mathcal{X}_{n_1,...,n_r} = \{\phi \in \mathcal{L}_n(X), n = \sum n_j \mid \mathrm{ord}_t \phi^*(f_j) = n_j \quad j = 1, ..., r\}$$

and $ac(f) = (ac(f_1), ..., ac(f_r)) : \mathcal{X}_{n_1,..,m_r} \to \mathbf{G}_m^r$ assigns to an arc in $\mathcal{X}_{n_1,..,n_r}$ the vector which j-th component is the coefficient of t^{n_j} in $\phi^*(f_j)$. Together with $\pi_0 : \mathcal{X}_{n_1,..,n_r} \to X$ which assigns to an arc the image in X of its closed point Spec $\mathbf{C} \to$ Spec $\mathbf{C}[[t]]$, this makes $\mathcal{X}_{n_1,..,n_r}$ into $\mathbf{G}_m^r \times X_0$-manifold. Then

(7)

$$Z_{f_1,...,f_r}(T_1, .., T_r) = \sum_{n_1,..,n_r, n_i \in \mathbf{N}} [\mathcal{X}_{n_1,...,n_r}/X_0 \times \mathbf{G}_m^r] \mathbf{L}^{(d \sum n_i)} T_1^{n_1} \cdots T_r^{n_r}$$

One has the canonical maps (resp. Betti and Hodge realizations): $e_{top} : K_0(Var_{\mathbf{C}}) \to \mathbf{Z}$ and $e_h : K_0(Var_{\mathbf{C}}) \to \mathbf{Z}[u, v]$ induced by the maps assigning to a variety V its topological euler characteristic and the E-function $\sum_i (-1)^i \dim Gr_F^p Gr_{p+q}^W H^i(V) u^p v^q$ (both F and W filtration are coming from Deligne's Mixed Hodge structure on V). We also will use the equivariant refinement of e_{top} and e_h defined for $V \in Var_{\mathbf{C}}$ supporting an action of a finite group G via biholomorphic transformations. For $\chi \in \mathrm{Char}\, G$, those refinements pick the corresponding eigenspaces:

(8)

$$e_{top,\chi}(V) = \sum (-1)^i \dim H^i(V)_\chi \quad \text{and} \quad e_{h,\chi}^m(V) = \sum_i (-1)^i Gr_F^m H^i(V)_\chi$$

The function (7) can be expressed in terms of a resolution of singularities of $f_1, , , , f_r$ as follows (cf. [12]). Let $Y \to X$ be a resolution of singularities of D, i.e. the union of the exceptional set $\bigcup_{i \in J} E_i$ and the proper preimage of an INNC D is a normal crossings divisor. For $I \subset J$, let $E_I^\circ = \cap_{i \in I} E_i - \cup_{j \in J - I} E_j$, $a_{i,k}$ (resp. c_k) is the order along the exceptional component of the pull-back on Y of function f_i (resp. the order of the pull back of the differential $dx_1 \wedge ... \wedge dx_{n+1}$). Let U_i be the complement to the zero section of the normal bundle to E_i in Y, and U_I is the fiber product of $U_i|_{E_I^\circ}$ over E_I. Then:

(9)

$$Z_{f_1,..f_r}(T_1, .., T_r) = \sum_{I \subset J} [U_J/\mathbf{G}_m^r \times X_0] \Pi_{i \in I} \frac{\mathbf{L}^{-c_i - 1} T_1^{a_{i,1}} \cdots T_r^{a_{i,r}}}{1 - \mathbf{L}^{-c_i - 1} T_1^{a_{i,1}} \cdots T_r^{a_{i,r}}}$$

We have the following:

Theorem 8.1. *Betti realization of $Z_{f_1,...,f_r}(T_1, ..., T_r)$ determines the essential components of the characteristic variety V_1. More precisely,* *for an essential χ:*

$$V_1 = \{\chi \mid e_{top,\chi} \lim_{T_i \to \infty} Z_{f_1,\ldots,f_r}(T_1, \ldots, T_r) \neq 0\} \tag{10}$$

Proof. One can deduce this from C. Sabbah's results in [33] similarly to [19] since, due to the vanishing theorem 4.1, the multi-variable zeta function studied in [33] determines the support of the $\pi_1(B_\epsilon - D)$ module $\pi_n(B_\epsilon - D)$.

In the cyclic case, the Hodge realization of the motivic zeta function is equivalent to the spectrum (cf. [12]). At least in the case of curves, one has the Hodge version in the abelian case as well (as was suggested in [29]):

Theorem 8.2. *For $n = 1$, the Hodge realization of* (7) *determines the polytopes of quasiadjunction.*

Proof. Let $X_{m_1,\ldots,m_r}$ be the link of an abelian cover $\mathcal{V}_{m_1.,,.,m_r}$ given by the equations (5) with $n = 1$. A resolution of this complete intersection singularity in the category of spaces with quotient singularities (in the case of surfaces with ADE singularities) can be obtained as the normalization $\widetilde{V_{m_1,..,m_r}}$ of $\mathcal{V}_{m_1,..,m+r} \times_{B_\epsilon} Y_D$, where $Y_D \to B_\epsilon$ is an embedded resolution of the singularities of D. The exceptional locus $\tilde{E}$ of the resolution of (5) supports the action of the group $G = \mathbf{Z}_{m_1} \times \ldots \times \mathbf{Z}_{m_r}$. We have the following sequence of MHS (cf. [36]): $0 \to H^1_E(\widetilde{V_{m_1,..,m_r}}) \to H^1(E) \to H^1(X_{m_1,\ldots,m_r}) \to 0$, which in the case $n = 1$ yields the equivariant isomorphism $H^1(X_{m_1,\ldots,m_r}) = H^1(\tilde{E})$ of MHSs. Since the MHS on $H^2(\tilde{E})$ is pure, we have:

$$\dim F^1 H^1(X_{m_1,..,m_r})_\chi = \dim Gr^1_F H^1(L)_\chi = e^1_{h,\chi}(\tilde{E}) \tag{11}$$

The latter is determined by the Hodge realization of (9), since the pullback of $[U_i]$ via the map $\mathcal{M}_{\mathbf{G}^r_m \times X_0} \to \mathcal{M}_{\mathbf{G}^r_m \times X_0}$ corresponding to the map $\mathbf{G}^r_m \times X_0 \to \mathbf{G}^r_m \times X_0$ given by $z_i = u_i^{m_i}$ is equivalent to the unbranched cover of $\partial B_\epsilon - D$, which is preimage of $\mathbf{G}^r_m \subset \mathbf{C}^r$ for the projection of (5) onto the space of z-coordinates. In particular, it determines the class of the exceptional set $\tilde{E}_i$ in $\mathcal{M}_\mathbf{C}$. It follows from (11) that $\dim F^1 H^1(X_{m_1,..,m_r})_\chi \geq 1$ iff $e^1_{h,\chi}(\tilde{E}) \geq 1$. QED.

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Massey products of complex hypersurface complements

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Abstract.

It was shown by Kohno that all higher Massey products in the rational cohomology of a complex hypersurface complement vanish. We show that in general there exist non-vanishing triple Massey products in the cohomology with finite field coefficients.

§1. Introduction

The study of the topology of hypersurface complements is a classical subject in algebraic geometry. Most of what is known about these spaces is related to invariants of their rational homotopy type. In this paper, we attempt to show that their $\mathbb{F}_p$-homotopy type captures in general more information than the $\mathbb{Q}$-homotopy type, where $\mathbb{F}_p$ is the prime field of p elements.

Let X be the complement to a hypersurface S in $\mathbb{C}P^d$. Then we have the following results due to Kohno [12, 13]: Massey products in $H^*(X,\mathbb{Q})$ of length ≥ 3 vanish. Moreover, the Malcev Lie algebra of $\pi_1(X)$ and the completed holonomy Lie algebra of $H^{\leq 2}(X,\mathbb{Q})$ are isomorphic. Thus, the $\mathbb{Q}$-completion of $\pi_1(X)$ is completely determined by the $\mathbb{Q}$-cohomology algebra of X. In the case when S is a hyperplane arrangement X is $\mathbb{Q}$-formal by Morgan [17], that is the entire $\mathbb{Q}$-homotopy type of X is determined by the algebra $H^*(X,\mathbb{Q})$. In this context, it seems natural to pose the following questions: Are there non-vanishing Massey products in $H^*(X,\mathbb{F}_p)$ for all primes p? Is X a $\mathbb{F}_p$-formal space, particularly when X is a hyperplane arrangement complement?

Massey products are known to be obstructions to formality, see [5, 7]. So, if the answer to the first question was yes, then the space X would

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not be $\mathbb{F}_p$-formal. For compact Kähler manifolds the above questions were answered by Ekedhal in [6] by constructing such manifolds M with non-vanishing triple products in $H^*(M, \mathbb{F}_p)$. Thus, a compact Kähler manifold although is $\mathbb{Q}$-formal by [5], in general it may not be $\mathbb{F}_p$-formal. The case of non-compact complex algebraic varieties is already different over $\mathbb{Q}$ from the compact case. As pointed out by Morgan in [17] such varieties may not be $\mathbb{Q}$-formal.

The main result of this paper settles in affirmative the existence of non-vanishing Massey products in the $\mathbb{F}_p$-cohomology of a hyperplane arrangement complement for all odd primes p, thus showing that arrangement complements are not $\mathbb{F}_p$-formal in general.

Theorem 1.1. *For every odd prime p, the complement X to the complex reflection arrangement $\mathcal{A}$ in $\mathbb{C}^3$ associated with the unitary reflection group $G(p,1,3)$ has, modulo indeterminacy, non-vanishing Massey products in $H^2(X, \mathbb{F}_p)$.*

The cohomology operations that came to be known as Massey products were introduced by W. S. Massey in [14]. Since then, they became important tools in algebraic topology, being especially used as means of distinguishing spaces with the same cohomology but different homotopy type. In general they are rather complicated objects, since they are in fact sets of cohomology classes. But, in certain cases, they turn out to be cosets as shown by May in [16], the simplest instance being that of the Massey products of three cohomology classes. In this paper we will only consider triple Massey products of cohomology classes of degree 1 in the cohomology algebra in degrees at most 2. In fact all the computations will take place in the group cohomology of the fundamental group $\pi_1(X)$ of our hypersurface complement. By the Lefschetz-Zariski classical theorem a generic 2-dimensional section of X captures all that topological information.

The hypersurfaces S that we will be our main focus are the hyperplane arrangements. Firstly because their complements are $\mathbb{Q}$-formal as discussed above. Secondly because the integral cohomology of their complements is known to be torsion-free, see [19]. In general, for a hypersurface S consisting of non-linear irreducible components, $H^*(X, \mathbb{Z})$ will have torsion, and thus, at least conceptually, the chances of getting non-vanishing Massey products in $H^*(X, \mathbb{F}_p)$ are already greater. However, it is possible for a non-linear hypersurface to have torsion-free $H^*(X, \mathbb{Z})$ as long as sufficiently many components of it are hyperplanes. We will briefly consider an example of such a non-linear hypersurface that nevertheless has triple non-vanishing Massey products.

The arrangement $\mathcal{A}$ of Theorem 1.1 is the full monomial arrangement $\mathcal{A}(r,1,3)$ in $\mathbb{C}^3$, with $r = p \geq 3$. These arrangements are members of a series of complex reflection arrangements, $\mathcal{A}(r,1,d)$, associated to the full monomial reflection group $G(r,1,d)$, see [1, 4, 19]. For $r \geq 1$ and $d \geq 2$ let $\mathcal{A}(r,1,d)$ be defined by the polynomial

$$Q = z_1 \cdots z_d \cdot \prod_{1 \leq i < j \leq d} (z_i^r - z_j^r).$$

Note that the arrangements $\mathcal{A}(1,1,d)$ and $\mathcal{A}(2,1,d)$ are the Coxeter arrangements of type A and respectively B.

For $\mathcal{A}$ a complex hyperplane arrangement in $\mathbb{C}^d$, with complement X and group $G = \pi_1(X)$, it is known that the rings $H^{*\leq 2}(X,\mathbb{K})$ and $H^{*\leq 2}(G,\mathbb{K})$ are isomorphic for $\mathbb{K}$ a field or $\mathbb{Z}$, see for example [15]. Moreover, the complement to $\mathcal{A}(r,1,d)$ is a $K(\pi,1)$ with π the pure braid group $P(r,1,d)$ associated to $G(r,1,d)$, see Orlik and Solomon [18]. Taking advantage of this, we use the cochains of G rather than those of X to compute Massey products. We will use a presentation of $P(r,1,d)$ obtained by Cohen [2].

A key rôle in the computations is played by the so-called resonance varieties of the arrangement, see [8, 15]. The resonance variety $\mathcal{R}(\mathcal{A},\mathbb{K})$ over a field $\mathbb{K}$ of an arrangement $\mathcal{A}$ is the subvariety of $H^1(X,\mathbb{K})$ encoding the vanishing cup products:

$$\mathcal{R}(\mathcal{A},\mathbb{K}) = \left\{\lambda \in H^1(X,\mathbb{K}) \mid \exists\, \mu \notin \mathbb{K}\lambda \text{ such that } \lambda \cup \mu = 0\right\}.$$

The knowledge of the classes in $H^1(X,\mathbb{K})$ that cup zero is especially needed for calculating a triple Massey product $\langle \alpha, \beta, \gamma \rangle$ as that is well-defined only when $\alpha \cup \beta = \beta \cup \gamma = 0$. In [8], Falk gives a combinatorial recipe to detect posible essential components of $\mathcal{R}(\mathcal{A},\mathbb{K})$. For $\mathcal{A} = \mathcal{A}(r,1,3)$, the classes used to define the non-vanishing Massey products belong to such components arising when $\mathbb{K} = \mathbb{F}_p$, for the special primes p dividing r. It can be shown that $\mathcal{A}(r,1,3)$ presents non-vanishing $\mathbb{F}_p$-Massey products for all primes p and all multiples r of p (multiples of 4 if $p = 2$). Here only the case $r = p$ is treated.

The paper is organized as follows. In Section 2 we define the triple Massey products of a 2-complex associated to a finitely presented group, and explain how they can be computed from the presentation. In Section 3 we introduce the monomial arrangements and give presentations by generators and relators of the fundamental groups of their complements. In Section 4 we exhibit non-vanishing triple Massey products in the $\mathbb{F}_p$-cohomology of the complements to 3-dimensional monomial arrangements, for p an odd prime. We also present a non-linear arrangement of curves in $\mathbb{C}P^2$ whose complement has non-vanishing triple Massey products over $\mathbb{F}_2$. In the last section we pose some further questions that we intend to explore elsewhere.

§2. Massey products of CW-complexes

The results on Massey products that we need may be found in the works of Porter [20], Turaev [21], and Fenn and Sjerve [9, 10]. In these papers the Massey products of 1-cohomology classes are computed in terms of the so-called Magnus coefficients, via the free calculus of Fox. Unless otherwise specified, all the homology and cohomology groups will have coefficients in $\mathbb{F}_p$, the integers modulo a prime p.

Definition 2.1. Let X be a space of the homotopy type of a CW-complex. If α, β, γ in $H^1(X)$ are such that $\alpha \cup \beta = \beta \cup \gamma = 0$ then the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined as follows: Choose representative 1-cocycles α', β', γ' and cochains x, y in $C^1(X)$ such that $dx = \alpha' \cup \beta'$ and $dy = \beta' \cup \gamma'$. Then $z = \alpha' \cup y + x \cup \gamma'$ is a 2-cocycle. The cohomology classes $z \in H^2(X)$ constructed in this way are only determined up to $\alpha \cup H^1(X) + H^1(X) \cup \gamma$, and they form a set denoted by $\langle \alpha, \beta, \gamma \rangle$.

As pointed out by May [16], the indeterminacy is a vector space, and so $\langle \alpha, \beta, \gamma \rangle$ can be thought of as a coset modulo $\alpha \cup H^1(X) + H^1(X) \cup \gamma$. The triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is said to be vanishing if this coset is trivial.

In this paper X will always be a $K(G, 1)$ for G a finitely presented group. We will identify from now on the cohomology of X with that of G.

Let $G = \langle x_1, \dots, x_n \mid R_1, \dots, R_m \rangle$ be a presentation for $G = \pi_1(X)$. Assume that R_l is a commutator and that the presentation is minimal. By Hopf's formula the homology classes of the relators R_l form a basis in $H_2(G, \mathbb{Z}) = \mathbb{Z}^m$. Morover, the generators x_i determine a basis of $H_1(G, \mathbb{Z}) = \mathbb{Z}^n$. Let e_i be the dual basis in $H^1(G, \mathbb{Z}) = \mathbb{Z}^n$.

Let F be the free group on $x_1, \dots, x_n$. If w is a word in F then its Fox derivative $\partial_j(w)$ is computed by the following rules: $\partial_j(1) = 0$, $\partial_j(x_i) = \delta_{i,j}$, and $\partial_j(uv) = \partial_j(u)\epsilon(v) + u\partial_j(v)$, where $\epsilon : \mathbb{Z}F \to \mathbb{Z}$ is the augmentation of the group ring $\mathbb{Z}F$.

Let $I = (i_1, \dots, i_q)$ be a multi-index with i_j taking values in $1, \dots, n$. The Magnus I-coefficient of a word w is defined by $\epsilon_I^{(0)}(w) = \epsilon\partial_I(w)$, where $\partial_I(w) = \partial_{i_1} \dots \partial_{i_q}(w)$. The $\mathbb{F}_p$-valued Magnus coefficients $\epsilon_I^{(p)}(w)$ of w are defined simply by taking integers modulo the prime p. Most of the time we will drop the reference to it and simply write $\epsilon_I(w)$ for

$\epsilon_I^{(p)}(w)$. We will usually refer to $\epsilon_{i,j}(w)$ and $\epsilon_{i,j,k}(w)$ as a double, and respectively triple Magnus coefficient.

The following result [9, 10, 20, 21] will be used to compute triple Massey products. Let α, β, γ be cohomology classes in $H^1(G)$ such that $\alpha \cup \beta = \beta \cup \gamma = 0$. Denote by $(\cdot, \cdot)$ the Kronecker pairing between cohomology and homology. Then we have:

Theorem 2.2. *The Massey product $\langle \alpha, \beta, \gamma \rangle$ of $\alpha = \sum \alpha_i e_i$, $\beta = \sum \beta_j e_j$, $\gamma = \sum \gamma_k e_k$ contains ξ, where:*

$$(\xi, R_l) = \sum_{1 \leq i,j,k \leq n} \alpha_i \beta_j \gamma_k \cdot \epsilon_{i,j,k}(R_l).$$

From now on by $\langle \alpha, \beta, \gamma \rangle$ we will understand the coset of the class ξ modulo the indeterminacy $\alpha \cup H^1(G) + H^1(G) \cup \gamma$. The Massey product $\langle \alpha, \beta, \gamma \rangle = \xi$ is functorial with respect to maps of spaces, as shown by Fenn and Sjerve in [9, 10].

The following formulae can readily deduced from the definitions and they will be used to compute the Magnus coefficients of a commutator word.

$$\epsilon_{k,l}([u,v]) = \epsilon_k(u)\epsilon_l(v) - \epsilon_k(v)\epsilon_l(u), \text{ where } [u,v] = uvu^{-1}v^{-1}. \tag{2.1}$$

$$\begin{aligned}\epsilon_{k,l,m}([u,v]) = {} & \epsilon_k(u)\epsilon_{l,m}(v) - \epsilon_m(u)\epsilon_{k,l}(v) + \epsilon_{k,l}(u)\epsilon_m(v) - \epsilon_k(v)\epsilon_{l,m}(u) + \\ & (\epsilon_k(v)\epsilon_l(u) - \epsilon_k(u)\epsilon_l(v)) \cdot (\epsilon_m(u) + \epsilon_m(v)).\end{aligned} \tag{2.2}$$

We will also need formulae for products of conjugated generators:

$$\epsilon_k(x_{i_1}^{w_1} \cdots x_{i_j}^{w_j}) = \sum_{a=1}^{j} \epsilon_k(x_{i_a}) = \sum_{a=1}^{j} \delta_{k,i_a}. \tag{2.3}$$

$$\epsilon_{k,l}(x_{i_1}^{w_1} \cdots x_{i_j}^{w_j}) = \sum_{a=1}^{j} (\epsilon_k(w_a)\delta_{l,i_a} - \epsilon_l(w_a)\delta_{k,i_a}) + \sum_{1 \leq a < b \leq j} \delta_{k,i_a}\delta_{l,i_b}, \tag{2.4}$$

where $x^a = axa^{-1}$ and $\delta_{i,j}$ is Kronecker's delta.

§3. Monomial arrangements and their groups

We introduce in this section our main examples of hypersurfaces whose complements have non-vanishing Massey products in the $\mathbb{F}_p$-cohomology. They are the complex reflection arrangements $\mathcal{A}(r,1,d)$ associated with the monomial reflection group $G(r,1,d)$. Their complements are $K(\pi,1)$ spaces and for all practical purposes we will identify their cohomology with that of their fundamental groups. We will describe here the group presentations that will be used in the Massey products computation.

3.1. Arrangements groups

We start with a brief overview of the fundamental group of hyperplane complements. Let $\mathcal{A}$ be a hyperplane arrangement in the affine space $\mathbb{C}^d$ and X its complement. Let us recall now the most salient features of the fundamental group $G = \pi_1(X)$ as a finitely presentable group. For all the details see [19]. First G is generated by the meridians γ_H around each hyperplane $H \in \mathcal{A}$. Each codim 2 intersection $H_{i_1} \cap \cdots \cap H_{i_n}$ of hyperplanes in $\mathcal{A}$ determines $n-1$ relations: $g_1 g_2 \cdots g_n = g_2 \cdots g_n \cdot g_1 = \cdots = g_n \cdot g_1 \cdots g_{n-1}$, where g_j is some conjugate of the generator $x_j = g_{H_j}$. We denote by $[g_1, \ldots, g_n]$ the family of commutator relators $[g_1 \ldots g_i, g_{i+1} \ldots g_n]$, with $1 \leq i < n$.

Thus we are lead to compute the Magnus coefficients of relators in families of the form: $\left[x_{i_1}^{w_1}, \ldots, x_{i_n}^{w_n}\right]$. Note that the indices i_j are all distinct. Denote by $R_{I,w}^j$ the commutator $\left[x_{i_1}^{w_1} \ldots x_{i_j}^{w_j}, x_{i_{j+1}}^{w_{j+1}} \ldots, x_{i_n}^{w_n}\right]$.

The Magnus coefficients of order 2 of $R_{I,w}^j$ are given by:

$$\epsilon_{k,l}\left(R_{I,w}^j\right) = \sum_{a=1}^{j} \sum_{b=j+1}^{n} (\delta_{k,i_a}\delta_{l,i_b} - \delta_{k,i_a}\delta_{k,i_b}). \tag{3.1}$$

It is easily seen that:

$$\epsilon_{k,l}\left(R_{I,w}^j\right) = \begin{cases} 1 & \text{if } k = i_a \text{ and } l = i_b, \\ & \text{for some } 1 \leq a \leq j \text{ and } j+1 \leq b \leq n \\ -1 & \text{if } k = i_b \text{ and } l = i_a, \\ & \text{for some } 1 \leq a \leq j \text{ and } j+1 \leq b \leq n \\ 0 & \text{otherwise.} \end{cases}$$

The Magnus coefficients of order 3 of $R_{I,w}^j$ are given by:

(3.2)

$$\epsilon_{k,l,m}\left(R^j_{I,w}\right) = \begin{cases} \epsilon_m(w_b) & \text{if } k = i_a, \\ & l = i_b,\ m \notin I \\ -\epsilon_m(w_a) & \text{if } k = i_b, \\ & l = i_a,\ m \notin I \\ \epsilon_l(w_b) + \epsilon_k(w_{a'}) - \epsilon_l(w_a) + \delta_{a\le a'} & \text{if } k = i_a, \\ & l = i_{a'},\ m = i_b \\ \epsilon_l(w_a) + \epsilon_k(w_{b'}) - \epsilon_l(w_b) + \delta_{b\le b'} & \text{if } k = i_b, \\ & l = i_{b'},\ m = i_a \\ \epsilon_k(w_b) - \epsilon_m(w_b) - 1 & \text{if } k = i_a, \\ & l = i_b,\ m = i_{a'} \\ \epsilon_l(w_{b'}) - \epsilon_l(w_a) - \epsilon_m(w_b) + \delta_{b\le b'} - 1 & \text{if } k = i_a, \\ & l = i_b,\ m = i_{b'} \\ \epsilon_k(w_a) - \epsilon_m(w_a) + 1 & \text{if } k = i_b, \\ & l = i_a,\ m = i_{b'} \\ \epsilon_l(w_{a'}) - \epsilon_l(w_b) - \epsilon_m(w_a) + \delta_{a\le a'} + 1 & \text{if } k = i_b, \\ & l = i_a,\ m = i_{a'} \\ 0 & \text{otherwise.} \end{cases}$$

where always $1 \le a, a' \le j$ and $j+1 \le b, b' \le n$.

3.2. Monomial arrangements

We introduce now our main class of examples. For $r \ge 1$ and $d \ge 2$ let $\mathcal{A}(r,1,d)$ be the arrangement defined by:

$$Q = z_1 \cdots z_d \cdot \prod_{1\le i<j\le d} (z_i^r - z_j^r).$$

The complement of $\mathcal{A}(r,1,d)$ is a $K(\pi,1)$ with π the pure braid group $P(r,1,d)$ associated to the full monomial complex reflection group $G(r,1,d)$, see [1, 18]. The group $P(r,1,d)$ admits an iterated semidirect product structure: $P(r,1,d) = F_{n_l} \rtimes \cdots \rtimes F_{n_1}$, where $n_i = (i-1)r+1$ for $1 \le i \le r$, as shown in [1].

A presentation for $P(r,1,d)$ was obtained by Cohen in [2]. Following that paper, let us first describe the codim 2 intersections among the hyperplanes of $\mathcal{A}(r,1,d)$:

$$H_i \cap H_{i,j}^{(1)} \cap \cdots \cap H_{i,j}^{(r-1)} \cap H_j \cap H_{i,j}^{(r)} \tag{3.3}$$

$$H_k \cap H_{i,j}^{(q)}, \text{if } k \neq i \text{ or } k > j \tag{3.4}$$

$$H_{i,j}^{(q)} \cap H_{k,l}^{(s)}, \text{if } i,j,k,l \text{ distinct}, \tag{3.5}$$

$$H_{i,j}^{(q)} \cap H_{j,k}^{(s)} \cap H_{i,k}^{(t)}, \text{ if } t = q + s \pmod{r}, \tag{3.6}$$

where $H_i = \{z_i = 0\}, 1 \leq i \leq 3$, and $H_{i,j}^{(q)} = \{z_i = \zeta^q z_j\}$, where $\zeta = \exp(2\pi i/r)$, $1 \leq i < j \leq 3$ and $1 \leq q \leq r$.

We focus now on the case $d = 3$. In [2] a presentation of $P(r, 1, 3)$ is given having $3r + 3$ generators, say $x_1, \ldots, x_{3r+3}$, and $2r^2 + 6r + 3$ relators. We group these relators in nine families corresponding to the types of the codimension 2 intersections.

(3.7)
$$A = \left[x_{3r+1}, x_1, \ldots, x_{r-1}, x_{3r+2}, x_r\right],$$
(3.8)
$$B = \left[x_{3r+1}, x_{2r+1}, \ldots, x_{3r-1}, x_{3r+3}, x_{3r}\right],$$
(3.9)
$$C = \left[x_{3r+2}, x_{r+1}^{x_r x_{3r+1} x_1 x_2 \cdots x_{r-1} x_{3r+1}^{-1}}, \ldots, x_{2r-1}^{x_r x_{3r+1} x_1 x_{3r+1}^{-1}}, x_{3r+3}, x_{2r}\right],$$
(3.10)
$$D_{1,s} = \left[x_{3r+1}, x_{r+i}\right], 1 \leq s \leq r,$$
(3.11)
$$D_{2,s} = \left[x_{3r+3}, x_i\right], 1 \leq s \leq r,$$
(3.12)
$$D_{3,s} = \left[x_{3r+2}^{x_i x_{i+1} \cdots x_{r-1}}, x_{2r+i}^{x_{2r}}\right], 1 \leq s \leq r,$$
(3.13)
$$T_s = \left[x_s, x_{2r+s}, x_{2r}\right], 1 \leq s \leq r,$$
(3.14)
$$U_{t,s} = \left[x_s, x_{2r-t}, x_{2r+s-t}^{x_{2r-t+1} \cdots x_{2r-1}}\right], 1 \leq t < s \leq r,$$
(3.15)
$$V_{s,t} = \left[x_s^{x_{3r+1}}, x_{2r-t}, x_{3r+s-t}^{x_{2r-t+1} \cdots x_{2r-1}}\right], 1 \leq s \leq t < r.$$

Now, recall that the notation $R = [x_{i_1}^{w_1}, \ldots, x_{i_n}^{w_n}]$ stands for the following set of commutators: $\{R^j = [x_{i_1}^{w_1} \ldots x_{i_j}^{w_j}, x_{i_{j+1}}^{w_{j+1}} \ldots, x_{i_n}^{w_n}] \mid 1 \leq j <$

$n\}$. Thus, in agreement with the notations of (3.7)-(3.15), the relators in $P(r,1,3)$ will be denoted by: A^j, B^j, C^j, where $j = 1, \dots, r+1$, and $D_{1,s}, D_{2,s}, D_{3,s}$, where $j = 1$ and is omitted, and finally $T_s^j, U_{t,s}^j, V_{s,t}^j$, where $j = 1, 2$.

§4. Non-vanishing triple Massey products

In this section we will present non-vanishing triple Massey products in the $\mathbb{F}_p$ cohomology of certain hypersurface complements. All such products will be of the form $\langle \alpha, \alpha, \beta \rangle$ with α and β linearly independent. The main example will consist of the monomial arrangements introduced in the previous section. We will also give an example of a non-linear arrangement of curves with the desired non-vanishing property.

4.1. Resonance varieties

We first determine the vanishing cup products in $H^2(X, \mathbb{F}_p)$, for X the complement of a monomial arrangement $\mathcal{A}$, using an invariant of a cohomology ring introduced by Falk in [8]. The resonance variety $\mathcal{R}(\mathcal{A}, \mathbb{F}_p)$ of an arrangement $\mathcal{A}$ is the subvariety of $H^1(X, \mathbb{F}_p)$ defined by:

$$\mathcal{R}(\mathcal{A}, \mathbb{F}_p) = \left\{ \lambda \in H^1(X, \mathbb{K}) \mid \exists\, \mu \notin \mathbb{K}\lambda \text{ such that } \lambda \cup \mu = 0 \right\} .$$

In [8] it is shown how one can construct components of $\mathcal{R}(\mathcal{A}, \mathbb{F}_p)$ from the so-called neighborly partitions of the arrangement $\mathcal{A}$. The neighborly partitions of the monomial arrangements $\mathcal{A} = \mathcal{A}(r,1,3)$ have been determined in [4]. The most interesting for us is the partition $\Pi = (H_3, H_{12}^{(i)} \mid H_2, H_{13}^{(j)} \mid H_1, H_{23}^{(k)})$ giving rise to a component C_Π of $\mathcal{R}(\mathcal{A}, \mathbb{F}_p)$ having the following equations:

$$\begin{aligned}
&\lambda_1 + \cdots + \lambda_r = \lambda_{r+1} + \cdots + \lambda_{2r-1} = \lambda_{2r+1} + \cdots + \lambda_{3r-1} = 0\\
&\lambda_i + \lambda_{2r} + \lambda_{2r+i} = 0, \quad 1 \le i \le r,\\
&\lambda_i + \lambda_{2r-j} + \lambda_{2r+i-j} = 0, \quad 1 \le j < i \le r,\\
&\lambda_i + \lambda_{2r-j} + \lambda_{3r+i-j} = 0, \quad 1 \le i \le j < r,\\
&\lambda_{3r+1} = \lambda_{3r+2} = \lambda_{3r+3} = 0
\end{aligned} \tag{4.1}$$

It is easily seen that $\dim C_\Pi = 3$ if p divides r (or 4 divides r, if $p = 2$), and $\dim C_\Pi = 2$, otherwise.

4.2. Massey products of monomial arrangements

We prove here the main result, showing that, in general, Massey products in the positive characteristic cohomology of a hypersurface complement may not vanish modulo indeterminacy, although over the rationals they always do so.

Theorem 4.1. *For every odd prime p the complement X of the arrangement $\mathcal{A}(p,1,3)$ in $\mathbb{C}^3$ of degree $3p+3$ has non-vanishing triple Massey products in $H^2(X,\mathbb{F}_p)$.*

Proof. We will show that a certain triple product $\langle\alpha,\alpha,\beta\rangle$ does not vanish modulo its indeterminacy. The cohomology classes α and β are given in coordinates by

$$a:\ \alpha_i = 1, \alpha_{r+i} = -1, \alpha_{2r+i} = \alpha_{3r+1} = \alpha_{3r+2} = \alpha_{3r+3} = 0,$$

and respectively by

$$\beta:\ \beta_i = 0, \beta_{r+i} = 1, \beta_{2r+i} = -1, \beta_{3r+1} = \beta_{3r+2} = \beta_{3r+3} = 0,$$

where $1 \leq i \leq r$. Clearly the points α and β satisfy the equations (4.1), so they belong to C_Π, and moreover $\alpha \cup \beta = 0$. Using (3.2) we can express $\langle\alpha,\alpha,\beta\rangle$ in the basis of $H^2(X,\mathbb{F}_p)$ given by the duals of the relators (3.7)-(3.15), abusing the notation for the sake of simplicity.

(4.2)
$$\langle\alpha,\alpha,\beta\rangle = \sum_{j=1}^{p}(j-1)C^j + (p-1)C^{p+1} - \sum_{s=1}^{p} T_s^2 + \sum_{1\leq t<s\leq p} t\,U_{t,s}^1 + \sum_{1\leq t<s\leq p} U_{t,s}^2 + \sum_{1\leq s\leq t<p} t\,V_{s,t}^1 + \sum_{1\leq s\leq t<p} V_{s,t}^2.$$

Next, using (3.1), we obtain the indeterminacy $\alpha\cup H^1(X)+H^1(X)\cup\beta$. If $a=\sum a_i e_i$ and $b=\sum b_i e_i$ are arbitrary classes in $H^1(X)$ then we find the following expression for $\alpha \cup a + b \cup \beta$:

$$
\begin{aligned}
&\left(\sum_{s=1}^{p} a_s + a_{3p+1} + a_{3p+2}\right)\left(\sum_{j=1}^{p}(j-1)A^j + (p-1)A^{p+1}\right)+ \\
&\left(\sum_{s=1}^{p} b_{2p+s} + b_{3p+1} + b_{3p+3}\right)\left(\sum_{j=1}^{p}(j-1)B^j + (p-1)B^{p+1}\right)- \\
&\left(\sum_{s=1}^{p}(a_{p+s} + b_{p+s}) + a_{3p+2} + a_{3p+3} + b_{3p+2} + b_{3p+3}\right) \\
&\left(\sum_{j=1}^{p}(j-1)C^j + (p-1)C^{p+1}\right) + (a_{3p+1} + b_{3p+1})\sum_{s=1}^{p} D_{1,s}- \\
&a_{3p+3}\sum_{s=1}^{p} D_{2,s} - b_{3p+2}\sum_{s=1}^{p} D_{3,s} + \sum_{s=1}^{p}(a_s + a_{2p+s} + a_{2p})\left(T_s^1 + T_s^2\right) + \\
&\sum_{s=1}^{p}(b_s + b_{2p+s} + b_{2p})\,T_s^2 + \sum_{1\le t<s\le p}(a_s + a_{2p-t} + a_{2p+s-t})\,U_{t,s}^1- \\
&\sum_{1\le t<s\le p}(b_s + b_{2p-t} + b_{2p+s-t})\,U_{t,s}^2 + \sum_{1\le s\le t<p}(a_s + a_{2p-t} + a_{3p+s-t})\,V_{s,t}^1- \\
&\sum_{1\le s\le t<p}(b_s + b_{2p-t} + b_{3p+s-t})\,V_{s,t}^2.
\end{aligned}
\tag{4.3}
$$

We want to show that the triple Massey product $\langle\alpha, \alpha, \beta\rangle$ does not vanish modulo indeterminacy. Suppose that it does vanish, and so there exist a and b in $H^1(X)$ such that $\langle\alpha, \alpha, \beta\rangle$ is of the form $\alpha \cup a + b \cup \beta$. This leads to the following set of equations over $\mathbb{F}_p$:

$$\sum_{s=1}^{p} a_s + a_{3p+1} + a_{3p+2} = \sum_{s=1}^{p} b_{2p+s} + b_{3p+1} + b_{3p+3} = 0, \tag{4.4}$$

$$\sum_{s=1}^{p}(a_{p+s} + b_{p+s}) + a_{3p+2} + a_{3p+3} + b_{3p+2} + b_{3p+3} = -1, \tag{4.5}$$

$$a_{3p+1} + b_{3p+1} = a_{3p+3} = b_{3p+2} = 0, \tag{4.6}$$

$$a_s + a_{2p+s} + a_{2p} = 0,\ b_s + b_{2p+s} + b_{2p} = -1, \tag{4.7}$$

$$a_s + a_{2p-t} + a_{2p+s-t} = t,\ b_s + b_{2p-t} + b_{2p+s-t} = -1, \tag{4.8}$$

$$a_s + a_{2p-t} + a_{3p+s-t} = t,\ b_s + b_{2p-t} + b_{3p+s-t} = -1, \tag{4.9}$$

where the ranges of the indices in (4.7), (4.8), and (4.9) are those in (4.3).

Now from (4.7), (4.8), and (4.9) we can readily see that we must have:

$$\sum_{s=1}^{p} a_s = \sum_{s=1}^{p} a_{p+s} = \sum_{s=1}^{p} a_{2p+s} = 0 \text{ and } \sum_{s=1}^{p} b_s = \sum_{s=1}^{p} b_{p+s} = \sum_{s=1}^{p} b_{2p+s} = 0.$$

From these equations combined with (4.4), (4.5) and (4.6) we obtain:

$$a_{3p+1}+a_{3p+2} = b_{3p+1}+b_{3p+3} = a_{3p+1}+b_{3p+1} = 0, \quad a_{3p+2}+b_{3p+3} = -1.$$

But this system of equations clearly has no solution.

Q.E.D.

Remark 4.2. We will show elsewhere that in fact any triple Massey product in $H^2(P, \mathbb{F}_p)$ of the form $\langle \alpha, \alpha, \beta \rangle$ with α and β (not proportional) in $C_\Pi \subset \mathcal{R}(P, \mathbb{F}_p) \subset H^1(P, \mathbb{F}_p)$ does not vanish modulo the indeterminacy $\alpha \cup H^1(P, \mathbb{F}_p) + H^1(P, \mathbb{F}_p) \cup \beta$, if $p \mid r$ (or $4 \mid r$, if $p = 2$), where $P = P(r, 1, 3)$. Thus it will follow that for every prime p and multiple $N \geq 3$ of p (of 4 if $p = 2$) there exists a line arrangement $\mathcal{A}$ in $\mathbb{C}^2$ of degree $3N + 3$ whose complement X has non-vanishing triple Massey products in $H^2(X, \mathbb{F}_p)$.

4.3. Curves with non-linear components

Let $\mathcal{C} = Q_2 \cup T_1 \cup T_2 \cup T_3$ be the curve in $\mathbb{C}P^2$ of degree 5, consisting of a smooth irreducible curve Q_2 of degree 2 and three lines T_1, T_2, T_3 tangent to Q_2. As explained by Kaneko, Tokunaga and Yoshida in [11], this curve is related with the discriminant of a certain crystallographic group, thus is of the same nature as the above reflection arrangements. In [11] a presentation for the fundamental group of the complement to $\mathcal{C}$ in $\mathbb{C}P^2$ is determined:

$$\pi_1(\mathbb{C}P^2 \setminus \mathcal{C}) = \langle x_1, x_2, x_3 \mid [x_3 x_i x_3, x_i], i = 1, 2, [x_3 x_1 x_3^{-1}, x_2] \rangle.$$

An easy computation with double Magnus coefficients shows that all $\mathbb{F}_2$ cup products $e_i \cup e_j$ vanish except for $e_1 \cup e_2$. Moreover, by computing triple Magnus coefficients we can see that the Massey products $\langle \alpha, \alpha, \beta \rangle$ over $\mathbb{F}_2$ do not vanish, if $\alpha \notin \mathbb{F}_2 \cdot (e_1 + e_2 + e_3)$.

Remark 4.3. It is possible to generalize this example to a curve $\mathcal{C} = Q_d \cup T_1 \cup \cdots \cup T_n$ of degree $d + n$, where Q_d is a smooth irreducible curve of degree $d \geq 2$ and $T_1 \ldots T_n$ are $n \geq d+1$ tangent lines to Q. Then the complement X of $\mathcal{C}$ will have non-vanishing triple Massey products of the form $\langle \alpha, \alpha, \beta \rangle$ in $H^2(X, \mathbb{F}_p)$, for every prime p dividing d.

§5. Further questions

We end the paper by raising a few questions:

(1) Note that the above arrangements exhibiting non-vanishing Massey products do not admit linear equations over the reals! Is it true that real complexified arrangements never have non-vanishing Massey products? Computational evidence suggest that in this case Massey products in $H^2(X, \mathbb{F}_p)$ indeed all vanish.
(2) All non-orientable matroids realizable over some $\mathbb{Q}(\alpha)$ lead to complex arrangements with non-vanishing Massey products?
(3) Is there an analogue of Kohno's result over $\mathbb{F}_p$? Is it true that non-vanishing of higher Massey products over $\mathbb{F}_p$ implies that the $\mathbb{F}_p$-completion of $\pi_1(X)$ is not isomporhic to the completed holonomy algebra of $H^{\leq 2}(X, \mathbb{F}_p)$?
(4) Do non-linear curves (with enough cohomology) always present non-vanishing Massey products?
(5) Are there any good criteria for $\mathbb{F}_p$-formality of X? In this context, what is the rôle played by the lower p-central series of $\pi_1(X)$?

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On degree of mobility for complete metrics

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Abstract.

The degree of mobility of a Riemannian metric g is the dimension of the space of Riemannian metrics sharing the same geodesics with g. We prove that the degree of mobility of an irreducible Riemannian metric on a closed manifold is at most two, unless the sectional curvature is positive constant.

§1. Introduction

1.1. Main question

Let (M^n, g) be a Riemannian manifold.

Definition 1. *A* **BM-structure** *on* (M^n, g) *is a smooth self-adjoint* $(1,1)$*-tensor* L *such that, for every point* $x \in M^n$*, for every vectors* $u, v, w \in T_xM^n$*, the following equation holds:*

$$g((\nabla_u L)v, w) = \frac{1}{2} g(v,u) \cdot d\mathrm{trace}_L(w) + \frac{1}{2} g(w,u) \cdot d\mathrm{trace}_L(v),$$

where trace_L *is the trace of* L*.*

Definition 2. *Let* g*,* $\bar g$ *be Riemannian metrics on* M^n*. They are* **projectively equivalent***, if they have the same (unparameterized) geodesics.*

The relation between BM-structures and projectively equivalent metrics is given by

Theorem 1 ([9])**.** *Let* g *be a Riemannian metric. Suppose* L *is a self-adjoint positive-definite* $(1,1)$*-tensor. Consider the metric* $\bar g$ *defined by*

$$\bar g(\xi, \eta) = \frac{1}{\det(L)} g(L^{-1}\xi, \eta) \quad (1)$$

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(for every tangent vectors ξ and η with the common foot point.)

Then, the metrics g and $\bar{g}$ are projectively equivalent, if and only if L is a BM-structure on (M^n, g).

The set of all BM-structures on (M^n, g) will be denoted by $\mathcal{B}(M^n, g)$. It is a linear vector space. The dimension of $\mathcal{B}(M^n, g)$ is called the **degree of mobility** of the metric. It is at least one, since $\mathcal{B}(M^n, g)$ contains the identity tensor $\mathrm{Id} \stackrel{\mathrm{def}}{=} \mathrm{diag}(1, 1, 1, ..., 1)$.

Main question: *How big can be the dimension of the space $\mathcal{B}(M^n, g)$?*

In other words, how big is the space of the metrics projectively equivalent to the given one?

1.2. History of the question

The history of the theory of projectively equivalent metrics goes back to works of Beltrami [2], Dini [16], Levi-Civita [29] and Weyl [61, 62]. The question how big is the space of the metrics projectively equivalent to the given one was considered by Lie [31] and Fubini [18, 19].

It is known that, locally, the degree of mobility of a metric is less than $\frac{(n+1)(n+2)}{2} + 1$, and is equal to $\frac{(n+1)(n+2)}{2}$ for spaces of constant curvature only, see [65, 53, 28]. The most power tools in the local study of the degree of mobility are the theory of concircular vector fields developed in Yano [65], and the theory of $V(K)$ spaces developed in Solodovnikov [54, 55, 56, 57]. Combining these two theories, Shandra [52] obtained that, locally, if the dimension n of the manifold is greater than two, the degree of mobility of a metric of nonconstant curvature can take the values

$$\frac{m(m+1)}{2} + l$$

only, where $1 \le m \le n$ and $1 \le l \le \left[\frac{n+1-m}{3}\right]$. For every such "admissible" value $D_{mobility}$ there exists a metric on the disk such that the degree of mobility is precisely $D_{mobility}$. For dimension two, it follows from [28, 33] that the degree of mobility can take the values $1, 2, 3, 4, 6$.

A more detailed historical overview of the local side of the question can be found in the surveys [1, 50].

The goal of this paper is to study the degree of mobility globally, i.e. when the manifold is closed or complete. Most results on the degree of mobility of closed manifolds require additional geometric assumptions written as a tensor equation. A typical result is that, under certain tensor assumptions, the degree of mobility is precisely 1, see, for example, [14, 63, 64, 20, 51].

1.3. Main Result

Theorem 2. *Let (M^n, g) be a connected complete Riemannian manifold of dimension greater than one. Suppose* $\dim(\mathcal{B}(M^n, g)) \geq 3$.

Then, if a complete Riemannian metric $\bar{g}$ is projectively equivalent to g, then g has positive constant sectional curvature, or $\bar{g}$ is affine equivalent to g.

Recall that two metrics are said to be **affine equivalent**, if their Levi-Civita's connections coincide.

All assumption in the theorem are important: we can construct counterexamples, if one of the assumptions is omitted.

It is easy to understand whether a complete metric admits affine equivalent one which is not proportional to it. In this case, the holonomy group of the manifold must be reducible [30, 25], which implies that the universal cover of the manifold with the lifted metric is the Riemannian product of two Riemannian manifolds. Thus, a direct consequence of Theorem 2 is the following

Corollary 1. *Let (M^n, g) be a closed connected Riemannian manifold with irreducible holonomy group. Suppose* $\dim(\mathcal{B}(M^n, g)) \geq 3$. *Then, g has constant positive sectional curvature.*

In dimension 2, in view of Theorem 3, Corollary 1 follows from results of [26, 27, 24].

It is worse to mention that the converse of Corollary 1 is not always true. Of cause, the space $\mathcal{B}$ is huge for the round sphere. But for certain quotients of the round sphere, the space $\mathcal{B}$ can have dimension one. This phenomena appears already in dimension 3, see [42].

In the present paper we will prove Theorem 2 assuming that the dimension n of the manifold is greater than 2. If $n = 2$, in view of Theorem 3, under the assumption that the manifold is closed, Theorem 2 follows from [27, 24]. Without this assumption, Theorem 2 (for dimension 2) is nontrivial. It is announced in [44, 45]. Its proof uses methods from the global theory of Liouville metrics developed in [7, 8, 22], and can be found in [47, 48].

Our prove of Theorem 2 (for dimension ≥ 3) uses the following methods:

- The classical one is the local theory of projectively equivalent metrics. It is due to Beltrami [2], Dini [16], Levi-Civita [29], Fubini [18], Eisenhart [17], Cartan [13] , Weyl [61, 62] and Solodovnikov [54]. We will formulate a part of their results in Theorems 4, 5, 6, 7, 8.

- The newer one were introduced in [32, 58, 59, 36, 35]: the main observation is that, for a given Riemannian metric g, the existence of a projectively equivalent metric allows one to construct commuting integrals for the geodesic flow of g, see Theorem 3 in Section 2.1. This technique has been used quite successfully in finding topological obstruction that prevent a closed manifold from possessing (nontrivial) BM-structure, see [34, 39, 40, 37, 42, 43, 46, 49], and for the study of the degree of mobility for the metric of ellipsoid, see [41].
- And the general idea came from the singularity theory. The role of singularities play the points where the eigenvalues of the BM-structure bifurcate. In Section 3.1, we describe behavior of the metric near the simplest singular points. In Sections 3.2 and 4, we will show that the simplest singular points always exist. In Section 3.3, we will explain how the structure near singular points can be extended to the whole manifold.

1.4. Acknowledgements

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§2. New and classical instruments of the proof

2.1. Integrals for geodesic flows of metrics admitting BM-structure.

The relation between BM-structures and integrable geodesic flows is observed on the level of geodesic equivalence in [32] and is as follows:

Let L be a self-adjoint $(1,1)$-tensor on (M^n, g). Consider the family S_t, $t \in \mathbb{R}$, of $(1,1)$-tensors

$$S_t \stackrel{\text{def}}{=} \det(L - t\ \mathrm{Id})\,(L - t\ \mathrm{Id})^{-1}. \tag{2}$$

Remark 1. *Although $(L - t\ \mathrm{Id})^{-1}$ is not defined for t lying in the spectrum of L, the tensor S_t is well-defined for every t. Moreover, S_t is a polynomial in t of degree $n-1$ with coefficients being (1,1)-tensors.*

We will identify the tangent and cotangent bundles of M^n by g. This identification allows us to transfer the natural Poisson structure from T^*M^n to TM^n.

Theorem 3 ([58, 32, 59]). *If L is a BM-structure, then, for every $t_1, t_2 \in \mathbb{R}$, the functions*

$$I_{t_i} : TM^n \to \mathbb{R}, \quad I_{t_i}(v) \stackrel{\text{def}}{=} g(S_{t_i}(v), v) \tag{3}$$

are commuting integrals for the geodesic flow of g.

Remark 2. *Integrable systems of slightly less general type were recently studied in* [3, 4, 5, 21, 15]*).*

Since L is self-adjoint, its eigenvalues are real. At every point $x \in M^n$, let us denote by $\lambda_1(x) \leq ... \leq \lambda_n(x)$ the eigenvalues of L at the point.

Corollary 2 ([43, 59, 38]). *Let (M^n, g) be a connected Riemannian manifold such that every two points can be connected by a geodesic. Suppose L is a BM-structure on (M^n, g). Then, for every $i \in \{1, ..., n-1\}$, for every $x, y \in M^n$, the following statements hold:*

(1) $\lambda_i(x) \leq \lambda_{i+1}(y)$.
(2) *If $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.*

At every point $x \in M^n$, denote by $N_L(x)$ the number of different eigenvalues of the BM-structure L at x.

Definition 3. *A point $x \in M^n$ will be called typical with respect to the BM-structure L, if*

$$N_L(x) = \max_{y \in M^n} N_L(y).$$

Corollary 3 ([36]). *Let L be a BM-structure on a connected Riemannian manifold (M^n, g). Then, almost every point of M is typical with respect to L.*

2.2. Results of Beltrami, Levi-Civita and Solodovnikov

Theorem 4. *Let Riemannian metrics g and $\bar{g}$ on M^n be projectively equivalent. If g has constant sectional curvature, then $\bar{g}$ has constant sectional curvature as well.*

For dimension two, Theorem 4 was proven by Beltrami [2]. For dimension greater than two, a proof can be found in Eisenhart [17].

Corollary 4. *Let projectively equivalent metrics g and $\bar{g}$ on M^n (of dimension $n > 1$) be complete. If g has constant negative sectional curvature, $\bar{g}$ is proportional to g. If g is flat, $\bar{g}$ is affine equivalent to g.*

This statements is a folklore, in the sense that we did not find a classical reference for it, although certain authors use it as a known fact. We will be grateful if anybody gives us this reference.

Let us explain a proof of Corollary 4 by using newer methods. If both metrics are flat, Corollary 4 is equivalent to the statement that every diffeomorphism of $\mathbb{R}^n$ that takes straight lines to straight lines is a composition of linear transformation and translation. Its proof can be found in almost every advanced textbook on linear algebra and analytic geometry.

If g has constant negative sectional curvature, it is sufficient to prove Corollary 4 in dimension two only, since in every two-dimensional direction there exists a totally geodesic complete submanifold. If $\bar g$ is flat, the statement is trivial, since in the Euclidean space the parallel postulate of Euclid holds, and in the hyperbolic space not. If both metrics have constant negative curvature, Corollary 4 was proven in [10], see his lemma on page 59. The geometric idea behind the proof of Bonahon is the nontrivial observation from metric geometry (see [11, 12, 23] for the proof of this observation) that, for hyperbolic 2-spaces, the only isometry that preserves the boundary at infinity is the identity. Since the boundary at infinity can be defined by using unparameterized geodesics, Corollary 4 becomes to be trivial.

In view of Theorem 1, the next theorem is equivalent to the classical Levi-Civita's Theorem from [29].

Theorem 5 (Levi-Civita's Theorem)**.** *The following statements hold:*

(1) *Let L be a BM-structure on (M^n, g). Let $x \in M^n$ be typical. Then, there exists a coordinate system $\bar x = (\bar x_1, ..., \bar x_m)$ (in a neighborhood $U(x)$ containing x), where $\bar x_i = (x_i^1, ..., x_i^{k_i})$, $(1 \le i \le m)$, such that L is diagonal*

$$\operatorname{diag}(\underbrace{\phi_1, ..., \phi_1}_{k_1}, ..., \underbrace{\phi_m, ..., \phi_m}_{k_m}), \tag{4}$$

and the quadratic form of the metric g have the following form:

$$\begin{aligned} g(\dot{\bar x}, \dot{\bar x}) \quad &= \quad P_1(\bar x) A_1(\bar x_1, \dot{\bar x}_1) + \ P_2(\bar x) A_2(\bar x_2, \dot{\bar x}_2) + \cdots + \\ &+ \quad P_m(\bar x) A_m(\bar x_m, \dot{\bar x}_m), \end{aligned} \tag{5}$$

where $A_i(\bar x_i, \dot{\bar x}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar x}_i$ with coefficients depending on $\bar x_i$,

$$P_i \stackrel{\text{def}}{=} (\phi_i - \phi_1) \cdots (\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i) \cdots (\phi_m - \phi_i),$$

and $0 < \phi_1 < \phi_2 < ... < \phi_m$ *are smooth functions such that*

$$\phi_i = \begin{cases} \phi_i(\bar{x}_i), & \text{if} \quad k_i = 1 \\ \text{constant}, & \text{otherwise.} \end{cases}$$

(2) *Let g be a Riemannian metric and L be a (1,1)-tensor. If in a neighborhood $U \subset M^n$ there exist coordinates $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ such that g and L are given by formulae (4, 5), then the restriction of L to U is a BM-structure for the restriction of g to U.*

Corollary 5 ([9],[39])**.** *The Nijenhuis torsion of a BM-structure vanishes.*

Remark 3. *In Levi-Civita's coordinates from Theorem 5, the metric $\bar{g}$ given by (1) has the form*

$$\begin{aligned} \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) &= \rho_1 P_1(\bar{x}) A_1(\bar{x}_1, \dot{\bar{x}}_1) + \rho_2 P_2(\bar{x}) A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &+ \rho_m P_m(\bar{x}) A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned}$$

where

$$\rho_i = \frac{1}{\phi_1^{k_1} ... \phi_m^{k_m}} \frac{1}{\phi_i}$$

The metrics g and $\bar{g}$ are affine equivalent if and only if all functions ϕ_i are constant.

Let p be a typical point with respect to the BM-structure L. Fix $i \in 1, ..., n$ and a small neighborhood U of p. At every point of U, consider the eigenspace V_i with the eigenvalue ϕ_i. If the neighborhood is small enough, it contains only typical points and V_i is a distribution. Denote by $M_i(p)$ the integral manifold containing p.

Levi-Civita's Theorem says that the eigenvalues ϕ_j, $j \neq i$, are constant on $M_i(p)$, and that the restriction of g to $M_i(p)$ is proportional to the restriction of g to $M_i(q)$, if it is possible to connect q and p by a line orthogonal to M_i. We will need the second observation later and formulate it as

Corollary 6. *Let L be a BM-structure for connected (M^n, g). Suppose the curve $\gamma : [0, 1] \to M^n$ contains only typical points and is orthogonal to $M_i(p)$ at every point $p \in Image(\gamma)$. Let the multiplicity of the eigenvalue ϕ_i at every point of the curve be greater than one. Then, the restriction of the metric to $M_i(\gamma(0))$ is proportional to the restriction of the metric to $M_i(\gamma(1))$. (i.e. there exists a diffeomorphism of a small neighborhood $U_i(\gamma(0)) \subset M_i(\gamma(0))$ to a small neighborhood*

$U_i(\gamma(1)) \subset M_i(\gamma(1))$ taking the restriction of the metric g to $M_i(\gamma(0))$ to a metrics proportional to the restriction of the metric g to $M_i(\gamma(1))$).

Definition 4. *Let (M^n, g) be a Riemannian manifold. We say that the metric g has a* **warped decomposition** *near $x \in M^n$, if a neighborhood U^n of x can be split in the direct product of disks $D^{k_0} \times ... \times D^{k_m}$, $k_0 + ... + k_m = n$, such that the metric g has the form*

$$g_0 + \sigma_1 g_1 + \sigma_2 g_2 + ... + \sigma_m g_m, \tag{6}$$

where the ith *metric g_i is a Riemannian metric on the corresponding disk D^{k_i}, and functions σ_i are functions on the disk D^{k_0}. The metric*

$$g_0 + \sigma_1 dy_1^2 + \sigma_2 dy_2^2 + ... + \sigma_m dy_m^2 \tag{7}$$

on $D^{k_0} \times \mathbb{R}^m$ is called the **adjusted metric***.*

We will always assume that k_0 is at least 1.

Comparing formulae (5,6), we see that if L has at least one simple eigenvalue at a typical point, Levi-Civita's Theorem gives us a warped decomposition near every typical point of M^n: the metric g_0 collects all $P_i A_i$ from (5) such that ϕ_i has multiplicity one, the metrics g_1,...,g_m coincide with A_j for multiple ϕ_j, and $\sigma_j = P_j$.

Definition 5 ([54, 55])**.** *Let K be a constant. A metric g is called a* **$V(K)$-metric** *near $x \in M^n$ ($n \geq 3$), if there exist coordinates in a neighborhood of x such that g has the Levi-Civita form (5) such that the adjusted metric has constant sectional curvature K.*

The definition above is independent of the choice of the presentation of g in Levi-Civita's form:

Theorem 6 ([54, 55])**.** *Suppose g is a $V(K)$-metric near $x \in M^n$. Assume $n \geq 3$. The following statements hold:*

(1) *If there exists another presentation of g (near x) in the form (5), then the sectional curvature of the adjusted metric constructed for this other decomposition is constant and is equal to K.*

(2) *Consider the metric (5). For every $i = 1, ..., m$, denote*

$$\frac{g(grad(P_i), grad(P_i))}{4\,P_i} + KP_i \tag{8}$$

by K_i. Then, the metric (6) has constant sectional curvature if and only if for every $i \in 0, ..., m$ such that $k_i > 1$ the metric A_i has constant sectional curvature K_i. More precisely, if the *metric (6) is a $V(K)$-metric, if $k_1 > 1$ and if the metric A_1 has constant curvature K_1, then the metric $g_0 + P_1 A_1$ has constant curvature K.*

(3) *For a fixed presentation of g in the Levi-Civita form (5), for every i such that $k_i > 1$, K_i is a constant.*

Since the papers [54, 55] are not widely accessible, we will comment the proof of this theorem. The first statement of Theorem 6 is proven in §3 of [54]. In the form sufficient for our paper, it appeared already in [60]; although it is hidden there.

The second statement is in §8 of [54]. One can understand the second statement with the help of projective Weyl tensor from [62]. We will give the definition in Section 3.3, see formula (16) there. It is known [62], that (in dimension ≥ 3) the projective Weyl tensor vanishes if and only if the metric has constant sectional curvature. Now, it is possible to show by direct calculations that the projective Weyl tensor vanishes for a metric of form (5), if and only if the sectional curvatures of all A_i are equal to the corresponding K_i. In Section 3.3, we will do these calculations for one component of the projective Weyl tensor; the calculations for the other components are similar.

The third statement can be found in §8 of [54]. Its proof is similar to the standard proof of the fact that (for dimensions ≥ 3) if a metric has constant sectional curvature at every point, then the constant does not depend on the point.

The relation between $V(K)$-metrics and BM-structures is given by

Theorem 7 ([54, 56, 57]). *Let (D^n, g) be a disc of dimension $n \geq 3$ with two BM-structures L_1 and L_2 such that every point of the disc is typical with respect to both structures and the BM-structures* Id, L_1, L_2 *are linearly independent. Then, g is a $V(K)$-metric near every point.*

Its corollary is

Theorem 8 (Fubini's Theorem). *Let (D^n, g) be a disc of dimension $n \geq 3$ with two BM-structures L_1 and L_2 such that $N_{L_1} = N_{L_2} = n$ at every point. If the BM-structures* Id, L_1, L_2 *are linearly independent, then g has constant sectional curvature.*

For dimension 2, Fubini's Theorem is wrong. First counterexamples can be found in [28]. We will give new counterexamples in [47]. Fubini's Theorem was proven by Fubini [18] for dimension 3, and was announced there and in [19] for arbitrary dimension ≥ 3. One can check that Fubini's proof for dimension 3 can be applied to every dimension ≥ 3 without essential changes.

In Section 3.4, we will explain how Solodovnikov's Theorem 7 follows from Fubini's Theorem 8.

§3. Singularity theory for BM-structures

We will need the following technical lemma. For every fixed $v = (\xi_1, \xi_2, ..., \xi_n) \in T_xM^n$, the function (3) is a polynomial in t. Consider the roots of this polynomial. From the proof of Lemma 1, it will be clear that they are real. We denote them by

$$t_1(x,v) \le t_2(x,v) \le ... \le t_{n-1}(x,v).$$

Lemma 1. *Suppose λ is an eigenvalue of L of multiplicity k at $x \in M^n$. Then, for every $v \in T_xM^n$, λ is a root of $I_t(v)$ of multiplicity at least $k-1$.*

Proof: By definition, the tensor L is self-adjoint with respect to g. Then, for every $x \in M^n$, there exist "diagonal" coordinates in T_xM^n where the metric g is given by the diagonal matrix $\mathrm{diag}(1,1,...,1)$ and the tensor L is given by the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. Then, the tensor (2) reads:

$$\begin{aligned} S_t &= \det(L - t\mathrm{Id})(L - t\mathrm{Id})^{(-1)} \\ &= \mathrm{diag}(\Pi_1(t), \Pi_2(t), ..., \Pi_n(t)), \end{aligned}$$

where the polynomials $\Pi_i(t)$ are given by the formula

$$\Pi_i(t) \stackrel{\mathrm{def}}{=} (\lambda_1 - t)(\lambda_2 - t)...(\lambda_{i-1} - t)(\lambda_{i+1} - t)...(\lambda_{n-1} - t)(\lambda_n - t).$$

Hence, for every $v = (\xi_1, ..., \xi_n) \in T_xM^n$, the polynomial $I_t(x,v)$ is given by

$$I_t = \xi_1^2\Pi_1(t) + \xi_2^2\Pi_2(t) + ... + \xi_n^2\Pi_n(t). \tag{9}$$

We see that, if λ is an eigenvalue of multiplicity k, every Π_i contains the factor $(\lambda - t)^{k-1}$. Lemma is proven.

3.1. Behavior of BM-structure near simplest non-typical points.

Within this section we assume that L is a BM-structure on a connected (M^n, g). As in Section 2.1, we denote by $\lambda_1(x) \le ... \le \lambda_n(x)$ the eigenvalues of L, and by $N_L(x)$ the number of different eigenvalues of L at $x \in M^n$.

Theorem 9. *Suppose the eigenvalue λ_1 is not constant, the eigenvalue λ_2 is constant and $N_L = 2$ in a typical point. Let p be a non-typical point. Then, the following statements hold:*

(1) *The spheres of small radius with center in p are orthogonal to the eigenvector of L corresponding to λ_1, and tangent to the eigenspace of L corresponding to λ_2. In particular, the points where $\lambda_1 = \lambda_2$ are isolated.*

(2) *The restriction of the metric to the spheres has constant sectional curvature.*

Proof: Since λ_1 is not constant, it is a simple eigenvalue in every typical point. Since $N_L = 2$, the roots $\lambda_2, \lambda_3, ..., \lambda_n$ coincide at every point and are constant. We denote this constant by λ. By Lemma 1, at every point $(x, \xi) \in T_xM^n$, the number λ is a root of multiplicity at least $n-2$ of the polynomial $I_t(x, \xi)$. Then,

$$I'_t(x,\xi) := \frac{I_t(x,\xi)}{(\lambda - t)^{n-2}}$$

is a linear function in t and, for every fixed t, is an integral of the geodesic flow of g. Denote by $\tilde{I} : TM \to \mathbb{R}$ the function

$$\tilde{I}(x,\xi) := I'_\lambda(x,\xi) := (I'_t(x,\xi))_{|t=\lambda}\,.$$

Since λ is a constant, the function $\tilde{I}$ is an integral of the geodesic flow of g. At every tangent space T_xM^n, consider the coordinates such that the metric is given by $\mathrm{diag}(1, ..., 1)$ and L is given by $\mathrm{diag}(\lambda_1, \lambda, ..., \lambda)$. By direct calculations we see that the restriction of $\tilde{I}$ to T_xM^n is given by (we assume $\xi = (\xi_1, \xi_2, ..., \xi_n)$)

$$\tilde{I}_{|T_xM^n}(\xi) = (\lambda_1(x) - \lambda)(\xi_2^2 + ... + \xi_n^2).$$

Thus, for every geodesic γ passing through p, the value of $\tilde{I}(\gamma(\tau), \dot{\gamma}(\tau))$ is zero. Then, for every typical point of such geodesic, since $\lambda_1 < \lambda$, the components $\xi_2, ..., \xi_n$ of the velocity vector vanish. Then, the velocity vector is an eigenvector of L with the eigenvalue λ_1.

Then, the points where $\lambda_1 = \lambda$ are isolated: otherwise we can pick two such points p_1 and p_2 lying in a ball with radius less than the radius of injectivity. Then, for almost every point q of the ball, the geodesics connecting this point with the points p_1 and p_2 intersect transversally at q. Then, the point q is non-typical; otherwise the eigenspace of λ_1 contains the velocity vectors of geodesics and is no more one-dimensional. Finally, almost every point of the ball is not typical, which contradicts Corollary 3. Thus, the points where $\lambda_1 = \lambda$ are isolated.

It is known (Lemma of Gauß), that the geodesics passing through p intersect the spheres of small radius with center in p orthogonally. Since the velocity vectors of such geodesics are eigenvectors of L with eigenvalue λ_1, then the eigenvector with eigenvalue λ_1 is orthogonal to the spheres of small radius with center in p. Since L is self-adjoint, the spheres are tangent to the eigenspaces of λ. The first statement of Theorem 9 is proven.

The second statement of Theorem 9 is trivial, if $n = 2$. In order to prove the second statement for $n \geq 3$, we will use Corollary 6. The role of the curve γ from Corollary 6 plays the geodesic passing through p. We put $i = 2$. By the first statement of Theorem 9, $M_i(x)$ are spheres with center in p. Then, by Corollary 6, for every sufficiently small spheres S_{ϵ_1} and S_{ϵ_2} with center in p, the restriction of g to the first sphere is proportional to the restriction of g to the second sphere. Since for very small ϵ the metric in a ϵ-ball is very close to the Euclidean metric, the restriction of g to the ϵ-sphere is close to the round metric of the sphere. Thus, the restriction of g to every (sufficiently small) sphere has constant sectional curvature. Theorem 9 is proven.

Theorem 10. *Suppose $N_L = 3$ at a typical point and there exists a point where $N_L = 1$. Then, there exist points p_1, p_n such that $\lambda_1(p_1) < \lambda_2(p_1) = \lambda_n(p_1)$ and $\lambda_1(p_n) = \lambda_2(p_n) < \lambda_n(p_n)$.*

Proof: Suppose $\lambda_1(p_2) = \lambda_2(p_2) = ... = \lambda_n(p_2)$ and the number of different eigenvalues of L at a typical point equals three. Then, by Corollary 2, the eigenvalues $\lambda_2 = ... = \lambda_{n-1}$ are constant. We denote this constant by λ. Take a ball B of small radius with center in p_2. We will prove that this ball has a point p_1 such that $\lambda_1(p_1) < \lambda_2 = \lambda_n(p_1)$; the proof that there exists a point where $\lambda_1 = \lambda_2 < \lambda_n$ is similar. Take $p \in B$ such that $\lambda_1(p) < \lambda$ and $\lambda_1(p)$ is a regular value of the function λ_1. Denote by $\check{M}_1(p)$ the connected component of $\{q \in M^n : \ \lambda_1(q) = \lambda_1(p)\}$ containing the point p. Since $\lambda_1(p)$ is a regular value, $\check{M}_1(p)$ is a submanifold of codimension 1. Then, there exists a point $p_1 \in \check{M}_1(p)$ such that the distance from this point to p_2 is minimal over all points of $\check{M}_1(p)$.

Let us show that $\lambda_1(p_1) < \lambda = \lambda_n(p_1)$. The inequality $\lambda_1(p_1) < \lambda$ is fulfilled by definition, since $p_1 \in \check{M}_1(p)$. Let us prove that $\lambda_n(p_n) = \lambda$.

Consider the shortest geodesic γ connecting p_2 and p_1. We will assume $\gamma(0) = p_1$ and $\gamma(1) = p_2$. Consider the values of the roots $t_1 \leq ... \leq t_{n-1}$ of the polynomial I_t at points of the geodesic orbit $(\gamma, \dot{\gamma})$. Since I_t are integrals, the roots t_i are independent of the point of the orbit. Since the geodesic pass through the point where $\lambda_1 = ... = \lambda_n$, by Lemma 1, we have

$$t_1 = ... = t_{n-1} = \lambda. \tag{10}$$

Since the distance from p_1 to p_2 is minimal over all points of $\check{M}_1$, the velocity vector $\dot{\gamma}(0)$ is orthogonal to $\check{M}_1$. In view of Corollary 5, the sum of eigenspaces of L corresponding to λ and λ_n is tangent to $\check{M}_1$. Hence, the vector $\dot{\gamma}(0)$ is an eigenvector of L with eigenvalue λ_1.

At the tangent space $T_{p_1}M^n$, choose a coordinate system such that L is diagonal $\mathrm{diag}(\lambda_1, ..., \lambda_n)$ and g is Euclidean $\mathrm{diag}(1, ..., 1)$. In this coordinate system, $I_t(\xi)$ is given by (we assume $\xi = (\xi_1, ..., \xi_n)$)

$$(\lambda - t)^{n-3}\big((\lambda_n - t)(\lambda - t)\xi_1^2 + (\lambda_n - t)(\lambda_1 - t)(\xi_2^2 + ... + \xi_{n-1}^2) \\ +(\lambda_1 - t)(\lambda - t)\xi_n^2\big).$$

Since $\dot{\gamma}(0)$ is an eigenvector of L with eigenvalue λ_1, the last $n-1$ components of $\dot{\gamma}(0)$ vanish, so that $t_{n-1} = \lambda_n$. Comparing this with (10), we see that $\lambda_n(p_1) = \lambda$. Theorem 10 is proven.

3.2. Splitting Lemma

Definition 6. *A local-product structure on M^n is the triple (h, B_r, B_{n-r}), where h is a Riemannian metrics and B_r, B_{n-r} are transversal foliations of dimensions r and $n-r$, respectively (it is assumed that $1 \le r < n$), such that every point $p \in M^n$ has a neighborhood $U(p)$ with coordinates*

$$(\bar{x}, \bar{y}) = \big((x_1, x_2, ...x_r), (y_{r+1}, y_{r+2}, ..., y_n)\big)$$

such that the x-coordinates are constant on every leaf of the foliation $B_{n-r} \cap U(p)$, the y-coordinates are constant on every leaf of the foliation $B_r \cap U(p)$, and the metric h is block-diagonal such that the first $(r \times r)$ block depends on the x-coordinates and the last $((n-r) \times (n-r))$ block depends on the y-coordinates.

A model example of manifolds with local-product structure is the direct product of two Riemannian manifolds (M_1^r, g_1) and (M_2^{n-r}, g_2). In this case, the leaves of the foliation B_r are the products of M_1^r and the points of M_2^{n-r}, the leaves of the foliation B_{n-r} are the products of the points of M_1^r and M_2^{n-r}, and the metric h is the product metric $g_1 + g_2$.

Below we assume that

(a) L is a BM-structure for a connected (M^n, g).

(b) There exists r, $1 \le r < n$, such that $\lambda_r < \lambda_{r+1}$ at every point of M^n.

We will show that (under the assumptions (a,b)) we can naturally define a local-product structure (h, B_r, B_{n-r}) such that (the tangent spaces to) the leaves of B_r and B_{n-r} are invariant with respect to L, and such that the restrictions $L_{|B_r}$, $L_{|B_{n-r}}$ are BM-structures for the metrics $h_{|B_r}$, $h_{|B_{n-r}}$, respectively.

At every point $x \in M^n$, denote by V_x^r the subspaces of T_xM^n spanned by the eigenvectors of L corresponding to the eigenvalues $\lambda_1, ...,$ λ_r. Similarly, denote by V_x^{n-r} the subspaces of T_xM^n spanned by the eigenvectors of L corresponding to the eigenvalues $\lambda_{r+1}, ..., \lambda_n$. By assumption, for every i, j such that $i \le r < j$, we have $\lambda_i \ne \lambda_j$ so that V_x^r and V_x^{n-r} are two smooth distributions on M^n. By Corollary 5, the distributions are integrable so that they define two transversal foliations B_r and B_{n-r} of dimensions r and $n - r$, respectively.

By construction, the distributions V_r and V_{n-r} are invariant with respect to L. Let us denote by L_r, L_{n-r} the restrictions of L to V_r and V_{n-r}, respectively. We will denote by χ_r, χ_{n-r} the characteristic polynomials of L_r, L_{n-r}, respectively. Consider the (1,1)-tensor

$$C \stackrel{\text{def}}{=} ((-1)^r \chi_r(L) + \chi_{n-r}(L))$$

and the metric h given by the relation

$$h(u, v) \stackrel{\text{def}}{=} g(C^{-1}(u), v)$$

for every vectors u, v. (In the tensor notations, the metrics h and g are related by $g_{ij} = h_{i\alpha} C_j^\alpha$).

Lemma 2 (Splitting Lemma). *The following statements hold:*

(1) *The triple (h, B_r, B_{n-r}) is a local-product structure on M^n.*

(2) *For every leaf of B_r, the restriction of L to it is a BM-structure for the restriction of h to it. For every leaf of B_{n-r}, the restriction of L to it is a BM-structure for the restriction of h to it.*

Proof: First of all, h is a well-defined Riemannian metric. Indeed, take an arbitrary point $x \in M^n$. At the tangent space to this point, we can find a coordinate system such that the tensor L and the metric g are diagonal. In this coordinate system, the characteristic polynomials χ_r, χ_{n-r} are given by

$$\begin{array}{rcl} (-1)^r \chi_r & = & (t - \lambda_1)(t - \lambda_2)...(t - \lambda_r) \\ \chi_{n-r} & = & (\lambda_{r+1} - t)(\lambda_{r+2} - t)...(\lambda_n - t). \end{array} \tag{11}$$

Then, the (1,1)-tensor

$$C \;=\; ((-1)^r \chi_r(L) + \chi_{n-r}(L))$$

is given by the diagonal matrices

$$\text{(12)}\quad \operatorname{diag}\left(\prod_{j=r+1}^{n}(\lambda_j-\lambda_1),\dots,\prod_{j=r+1}^{n}(\lambda_j-\lambda_r),\right.$$
$$\left.\prod_{j=1}^{r}(\lambda_{r+1}-\lambda_j),\dots,\prod_{j=1}^{r}(\lambda_n-\lambda_j)\right),$$

We see that the tensor is diagonal and that all diagonal components are positive. Then, the tensor C^{-1} is well-defined and h is a Riemannian metric.

By construction, B_r and B_{n-r} are well-defined transversal foliations of supplementary dimensions. In order to prove Lemma 2, we need to verify that, locally, the triple (h, B_r, B_{n-r}) is as in Definition 6, that the restriction of L to a leaf is a BM-structure for the restriction of h to the leaf.

It is sufficient to verify these two statements at almost every point of M^n. More precisely, it is known that the triple (h, B_r, B_{n-r}) is a local-product structure if and only if the foliations B_r and B_{n-r} are orthogonal and totally geodesic. Clearly, if the foliations and the metric are globally given and smooth, if the foliations are orthogonal and totally-geodesic at almost every point, then they are orthogonal and totally-geodesic at every point.

Similarly, since the foliations and the metric are globally-given and smooth, if the restriction of L satisfies Definition 1 at almost every point, then it satisfies Definition 1 at every point.

Consider Levi-Civita's coordinates $\bar{x}_1, \dots, \bar{x}_m$ from Theorem 5. As in Levi-Civita's Theorem, we denote by $\phi_1 < \dots < \phi_m$ the different eigenvalues of L. In Levi-Civita's coordinates, the matrix of L is diagonal

$$\operatorname{diag}\left(\underbrace{\phi_1,\dots,\phi_1}_{k_1},\dots,\underbrace{\phi_m,\dots,\phi_m}_{k_m}\right)=\operatorname{diag}(\lambda_1,\dots,\lambda_n).$$

Consider s such that $\phi_s = \lambda_r$ (clearly, $k_1 + \dots + k_s = r$). Then, by constructions of the foliations B_r and B_{n-r}, the coordinates $\bar{x}_1, \dots, \bar{x}_s$ are constant on every leaf of the foliation B_{n-r}, the coordinates $\bar{x}_{s+1}, \dots, \bar{x}_m$ are constant on every leaf of the foliation B_r. The coordinates $\bar{x}_1, \dots, \bar{x}_s$ will play the role of x-coordinates from Definition 6, and the coordinates $\bar{x}_{s+1}, \dots, \bar{x}_m$ will play the role of y-coordinates from Definition 6.

Using (12), we see that, in Levi-Civita's coordinates, C is given by

$$\mathrm{diag}\left(\underbrace{\prod_{j=s+1}^{m}(\phi_j-\phi_1)^{k_j},\dots,\prod_{j=s+1}^{m}(\phi_j-\phi_1)^{k_j}}_{k_1},\dots,\underbrace{\prod_{j=s+1}^{m}(\phi_j-\phi_s)^{k_j},\dots,\prod_{j=s+1}^{m}(\phi_j-\phi_s)^{k_j}}_{k_s},\right.$$

$$\left.\underbrace{\prod_{j=1}^{s}(\phi_{s+1}-\phi_j)^{k_j},\dots,\prod_{j=1}^{s}(\phi_{s+1}-\phi_j)^{k_j}}_{k_{s+1}},\dots,\underbrace{\prod_{j=1}^{s}(\phi_m-\phi_j)^{k_j},\dots,\prod_{j=1}^{s}(\phi_m-\phi_j)^{k_j}}_{k_m}\right).$$

Thus, h is given by

$$h(\dot{\bar{x}},\dot{\bar{x}}) \quad = \quad \tilde{P}_1A_1(\bar{x}_1,\dot{\bar{x}}_1) \quad +\dots+ \quad \tilde{P}_sA_s(\bar{x}_s,\dot{\bar{x}}_s) \tag{13}$$

$$+ \quad \tilde{P}_{s+1}A_{s+1}(\bar{x}_{s+1},\dot{\bar{x}}_{s+1}) \quad +\dots+ \quad \tilde{P}_mA_m(\bar{x}_m,\dot{\bar{x}}_m), \tag{14}$$

where the functions $\tilde{P}_i$ are as follows: for $i\le r$, they are given by

$$\tilde{P}_i \quad \stackrel{\mathrm{def}}{=} \quad (\phi_i-\phi_1)\dots(\phi_i-\phi_{i-1})(\phi_{i+1}-\phi_i)\dots(\phi_s-\phi_i) \prod_{\substack{j=1\\ j\ne i}}^{s} |\phi_i-\phi_j|^{k_j-1}.$$

For $i>r$, the functions $\tilde{P}_i$ are given by

$$\tilde{P}_i \quad \stackrel{\mathrm{def}}{=} \quad (\phi_i-\phi_{s+1})\dots(\phi_i-\phi_{i-1})(\phi_{i+1}-\phi_i)\dots(\phi_m-\phi_i) \prod_{\substack{j=s+1\\ j\ne i}}^{m} |\phi_i-\phi_j|^{k_j-1}.$$

Clearly, $|\phi_i-\phi_j|^{k_j-1}$ can depend on the variables $\bar{x}_i$ only; moreover, if ϕ_i is multiple, $|\phi_i-\phi_j|^{k_j-1}$ is a constant. Then, the products

$$\prod_{\substack{j=1\\ j\ne i}}^{s} |\phi_i-\phi_j|^{k_j-1}$$

can be hidden in A_i, so that the the restriction of the metric to the leaves of B_r has the form from Levi-Civita's Theorem, and, therefore, the restriction of L is a BM-structure for it. We see that the leaves of B_r are orthogonal to leaves of B_{n-r}, and that the restriction of h to B_r (B_{n-r}, respectively) is precisely the first row of (13) (second row of (14), respectively) and depends on the coordinates $\bar{x}_1, ..., \bar{x}_s$ ($\bar{x}_{s+1}, ..., \bar{x}_m$, respectively) only. Lemma 2 is proven.

Let p be a typical point for g with respect to BM-structure L. Fix $i \in 1, ..., n$. At every point of M^n, consider the eigenspace V_i with the eigenvalue λ_i. V_i is a distribution near p. Denote by $M_i(p)$ its integral manifold containing p.

Remark 4. *The following statements hold:*

(1) *If $\lambda_i(p)$ is multiple, the restriction of g to $M_i(p)$ is proportional to the restriction of h to $M_i(p)$.*

(2) *The restriction of L to B_r does not depend on the coordinates $y_{r+1}, ..., y_n$ (which are coordinates $\bar{x}_{s+1}, ..., \bar{x}_m$ in the notations in proof of Lemma 2). The restriction of L to B_{n-r} does not depend on the coordinates $x_1, ..., x_r$ (which are coordinates $\bar{x}_1, ..., \bar{x}_s$ in the notations in proof of Lemma 2).*

Combining Lemma 2 with Theorem 9, we obtain

Corollary 7. *Let L be BM-structure on connected (M^n, g). Suppose there exist $i \in 1, ..., n$ and $p \in M^n$ such that:*

- *λ_i is multiple (with multiplicity $k \geq 2$) at a typical point.*
- *$\lambda_{i-1}(p) = \lambda_i(p) < \lambda_{i+k}(p)$,*
- *The eigenvalue λ_{i-1} is not constant.*

Then, for every typical point $q \in M^n$ which is sufficiently close to p, $M_i(q)$ is diffeomorphic to the sphere and the restriction of g to $M_i(q)$ has constant sectional curvature.

Indeed, take a small neighborhood of p and apply Splitting Lemma 2 two times: for $r = i + k - 1$ and for $r = i - 1$. We obtain a metric h such that locally, near p, the manifold with this metric is the Riemannian product of three discs with BM-structures, and BM-structure is the direct sum of these BM-structures. The second component of such decomposition satisfies the assumption of Theorem 9; applying Theorem 9 and Remark 4 we obtain what we need.

Arguing as above, combining Lemma 2 with Theorem 10, we obtain

Corollary 8. *Let L be a BM-structure for connected (M, g). Suppose the eigenvalue λ_i has multiplicity k at a typical point. Suppose there exists a point where the multiplicity of λ_i is greater than k. Then, there exists a point where the multiplicity of λ_i is precisely $k + 1$.*

Combining Lemma 2 with Corollary 2, we obtain

Corollary 9. *Let L be a BM-structure for connected (M^n, g). Suppose the eigenvalue λ_i has multiplicity k_i at a typical point and multiplicity $k_i + d$ at a point $p \in M^n$. Then, there exists a point $q \in M^n$ in a small neighborhood of p such that the eigenvalue λ_i has multiplicity $k_i + d$ in p, and such that*

$$N_L(q) = \max_{x \in M^n} (N_L(x)) - d.$$

We saw that under hypotheses of Theorems 9,10, the set of typical points is connected. As it was shown in [34], in dimension 2 the set of typical points is connected as well. Combining these observations with Lemma 2, we obtain

Corollary 10. *Let L be a BM-structure on connected (M^n, g). Then, the set of typical points of L is connected.*

3.3. If ϕ_i is not isolated and if $\dim(\mathcal{B}(M^n, g)) \geq 3$, then A_i has constant sectional curvature K_i.

In this section we assume that (M^n, g) is connected and complete and L is a BM-structure for M^n. As usual, we denote by $\lambda_1(x) \leq ... \leq \lambda_n(x)$ the eigenvalues of L at $x \in M^n$.

Definition 7. *An eigenvalue λ_i is called isolated, if $\lambda_i(p_1) = \lambda_j(p_1)$ implies $\lambda_i(p_2) = \lambda_j(p_2)$ for every point p_2.*

As in Section 3.2, at every point $p \in M^n$, we denote by V_i the eigenspace of L with the eigenvalue $\lambda_i(p)$. V_i is a distribution near every typical point; by Corollary 5, it is integrable. We denote by $M_i(p)$ the connected component containing p of the intersection of the integral manifold with a small neighborhood of p.

Theorem 11. *Suppose λ_i is a non-isolated eigenvalue. Then, for every typical point p, the restriction of g to $M_i(p)$ has constant sectional curvature.*

It could be easier to understand this Theorem using the language of Levi-Civita's Theorem 5: denote by $\phi_1 < \phi_2 < ... < \phi_m$ the different eigenvalues of L at a typical point. Theorem 11 says that, if ϕ_i is non-isolated, then A_i from Levi-Civita's Theorem has constant sectional curvature.

Proof of Theorem 11: If eigenvalue λ_i is simple at a typical point, M_i is one dimensional and the statement is trivial; below we assume that λ_i is multiple. Let $k_i > 1$ be the multiplicity of λ_i at a typical point.

Then, λ_i is constant. Take a typical point p. We assume that λ_i is not isolated; without loss of generality, we can suppose $\lambda_i(p_1) = \lambda_{i+k_i-1}(p_1)$ for some point p_1. By Corollary 8, without loss of generality, we can assume $\lambda_{i-1}(p_1) = \lambda_i(p_1) < \lambda_{i+k_i}(p_1)$. By Corollary 9, we can also assume that $N_L(p_1) = N_L(p) - 1$.

Consider a geodesic $\gamma : [0,1] \to M^n$ connecting p_1 and p, $\gamma(0) = p$ and $\gamma(1) = p_1$. Since it is sufficient to prove Theorem 11 at almost every typical point, arguing as in proof of Corollary 2 in [43], without loss of generality, we can assume that p_1 is the only non-typical point of the geodesic segment $\gamma(\tau),\ \tau \in [0,1]$.

Take a point $q := \gamma(1-\epsilon)$ of the segment, where $\epsilon > 0$ is small enough. By Corollary 7, the restriction of g to $M_i(q)$ has constant sectional curvature.

Let us prove that the geodesic segment $\gamma(\tau),\ \tau \in [0, 1-\epsilon]$ is orthogonal to $M_i(\gamma(\tau))$ at every point.

Indeed, consider the function

$$\tilde{I} : TM^n \to \mathbb{R};\ \tilde{I}(x,\xi) := \left(\frac{I_t(x,\xi)}{(\lambda_i - t)^{k_i-1}}\right)_{|t=\lambda_i}.$$

Since the multiplicity of λ_i at every point is at least k_i, the function $\left(\frac{I_t(x,\xi)}{(\lambda_i-t)^{k_i-1}}\right)$ is polynomial in t of degree $n-k_i$ and is an integral for every fixed t; since λ_i is a constant, the function $\tilde{I}$ is an integral.

At the tangent space to every point of geodesic γ consider the coordinates such that $L = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ and $g = \mathrm{diag}(1, ..., 1)$. In this coordinates, $I_t(\xi)$ is given by (9). Then, the integral $\tilde{I}(\xi)$ is the sum (we assume $\xi = (\xi_1, ..., \xi_n)$)

$$\left(\sum_{\alpha=i}^{i+k_i-1}\left(\xi_\alpha^2 \prod_{\substack{\beta=1\\ \beta\neq i,i+1,...,i+k_i-1}}^{n}(\lambda_\alpha-\lambda_i)\right)\right) \tag{15}$$

$$+\left(\sum_{\substack{\alpha=1\\ \alpha\neq i,i+1,...,i+k_i-1}}^{n}\left(\xi_\alpha^2 \prod_{\substack{\beta=1\\ \beta\neq i+1,...,i+k_i-1\\ \beta\neq\alpha}}^{n}(\lambda_\alpha-\lambda_i)\right)\right).$$

Since the geodesic passes through the point where $\lambda_{i-1} = \lambda_i = ... = \lambda_{i+k_i-1}$, all products in the formulae above contain the factor $\lambda_i - \lambda_i$, and, therefore, vanish, so that $\tilde{I}(\gamma(0), \dot{\gamma}(0)) = 0$. Since $\tilde{I}$ is an integral, $\tilde{I}(\gamma(\tau), \dot{\gamma}(\tau)) = 0$ for every τ. Let us show that it implies that the geodesic is orthogonal to M_i at every typical point, in particular, at points lying on the segment $\gamma(\tau),\ \tau \in [0,1]$.

Clearly, every term in the sum (16) contains the factor $\lambda_i - \lambda_i$, and, therefore, vanishes. Then, the integral $\tilde{I}$ is equal to (15).

At a typical point, we have

$$\lambda_1 \leq ... \leq \lambda_{i-1} < \lambda_i = ... = \lambda_{i+k_i-1} < \lambda_{k_i} \leq ... \leq \lambda_n.$$

Then, all products

$$\prod_{\substack{\beta=1 \\ \beta \neq i, i+1, ..., i+k_i-1}}^{n} (\lambda_\alpha - \lambda_i)$$

have the same sign and are nonzero. Then, all components ξ_α, $\alpha \in i, ..., k_i - 1$ vanish. Thus, γ is orthogonal to M_i at every typical point.

Finally, by Corollary 6, the restriction of g to $M_i(p)$ is proportional to the restriction of g to $M_i(q)$ and, hence, has constant sectional curvature. Theorem is proven.

Theorem 12. *Suppose* $\dim(\mathcal{B}(M^n, g)) \geq 3$. *Let* ϕ_i *be a non-isolated eigenvalue of L such that its multiplicity at a typical point is at least two. Then, the sectional curvature of* A_i *is equal to* K_i.

Recall that the definition of K_i is in the second statement of Theorem 6.

Proof of Theorem 12: Let us denote by $\bar{K}_i$ the sectional curvature of the metric A_i. By assumptions, it is constant in a neighborhood of every typical point. Since by Corollary 10, the set of typical points is connected, $\bar{K}_i$ is independent of a typical point. Similarly, since K_i is locally-constant by Theorem 6, K_i is independent of a typical point. Thus, it is sufficient to find a point where $\bar{K}_i = K_i$.

Without loss of generality, we can suppose that there exists $p_1 \in M^n$ such that $\lambda_r(p_1) = \lambda_{r+1}$.

By Corollary 9, without loss of generality we can assume that the multiplicity of λ_{r+1} is $k_i + 1$ in p_1, and that $N_L(p_1) = m - 1$. Take a typical point p in a small neighborhood of p_1.

Then, by Corollary 7, the submanifold $M_{r+1}(p)$ is homeomorphic to the sphere. Since it is compact, there exists a set of local coordinates charts on it such that there exist constants const and CONST such that, in every chart $(x_i^1, ..., x_i^{k_i})$, for every $\beta \in \{1, .., k_i\}$, the entry $(A_i)_{\beta\beta}$ lies between const and CONST, (i.e. CONST $\geq A_i(\frac{\partial}{\partial x_i^\beta}, \frac{\partial}{\partial x_i^\beta}) \geq$ const.)

By shifting these local coordinates along the vector fields $\frac{\partial}{\partial x_j^k}$, where $j \neq i$, for every typical point p' in a neighborhood of p_1, we obtain coordinate charts on $M_{r+1}(p')$ such that CONST $\geq (A_i)_{\beta\beta} \geq$ const.

Let us calculate the projective Weyl tensor W for g in these local coordinate charts. Recall that the projective Weyl tensor is given by the formula

$$W^i_{jkl} := R^i_{jkl} - \frac{1}{n-1}\left(\delta^i_l\, R_{jk} - \delta^i_k\, R_{jl}\right) \tag{16}$$

We will be interested in the components (actually, in one component) of W corresponding to the coordinates $\bar{x}_i$. In what follows we reserve the Greek letter α, β for the coordinates from $\bar{x}_i$, so that, for example, $g_{\alpha\beta}$ will mean the component of the metric staying on the intersection of column number $r + \beta$ and row number $r + \alpha$.

As we will see below, the formulae will include only the components of A_i. To simplify the notations, we will not write subindex i near A_i, so for example, $g_{\alpha\beta}$ is equal to $P_i\, A_{\alpha\beta}$.

Let calculate the component $W^\alpha_{\beta\beta\alpha}$. In order to do it by formula (16), it is necessary to calculate $R^\alpha_{\beta\beta\alpha}$ and $R_{\beta\beta}$. These was done in §8 of [54]. Rewriting the results of Solodovnikov in our notations, we obtain

$$R^\alpha_{\beta\beta\alpha} = \left(\bar{K}_i - (K_i - K\, P_i)\right) A_{\beta\beta},$$

$$R_{\beta\beta} = \left((k_i - 1)\, \bar{K}_i + K\,(n-1)\, P_i - (k_i - 1)\, K_i\right)\, A_{\beta\beta}.$$

Substituting these expressions in (16), we obtain

$$W^\alpha_{\beta\beta\alpha} = (\bar{K}_i - K_i)\frac{n - k_i}{n-1} A_{\beta\beta}.$$

We see that, if $\bar{K}_i \neq K_i$, the component $W^\alpha_{\beta\beta\alpha}$ is bounded from zero.

Now if we consider a sequence of typical points converging to p_1, the component $W^\alpha_{\beta\beta\alpha}$ converge to zero, since the length of $\frac{\partial}{\partial x_i^\beta}$ goes to zero. Finally, $\bar{K}_i = K_i$. Theorem is proven.

3.4. Geometric sense of the adjusted metric

Consider the metric (7) on the product

$$D^{k_0} \times ... \times D^{k_m}.$$

Take a point $P = (p_0, ..., p_m) \in D^{k_0} \times ... \times D^{k_m}$. At every disk D^{k_i}, $i = 1, ..., m$, consider a geodesic segment $\gamma_i \in D^{k_i}$ passing through p_i. Consider the product

$$M_A := D^{k_0} \times \gamma_1 \times \gamma_2 \times ... \times \gamma_m$$

as a submanifold of $D^{k_0} \times ... \times D^{k_m}$. As it easily follows from Definition 4,

- M_A is a totally geodesic submanifold.
- The restriction of the metric (7) to M_A is (isometric to) the adjusted metric.

Now let us explain how one can proof Theorem 7. Our proof is slightly different from the original proof of Solodovnikov [54] (which is correct and very good written).

If the dimension of M_A is one, Theorem 7 follows from Definition 4. Suppose the dimension of M_A is two. Consider two BM-structures L_1 and L_2 such that L_1, L_2 and Id are linearly independent, and such that the number of different eigenvalues of each BM-structure at each point is precisely two. Then, without loss of generality, locally there exists a coordinate system $(x_1, ..., x_n)$ such that

$$L = \mathrm{diag}(\lambda_1(x_1), \underbrace{\lambda_2, ..., \lambda_2}_{n-1}),$$

and g is given by the formula

$$(\lambda_1(x_1) - \lambda_2)(dx_1^2 + A_2), \tag{17}$$

where A_2 is a metric on the disk of dimension $(n-1)$ with coordinates $(x_2, ..., x_n)$, λ_2 is a constant, and λ_1 is a function of x_1. Consider the Ricci-tensor R^i_j of the metric (17). By direct calculation, it is possible to see that

- At every point, R^i_j has at most two different eigenvalues.
- If R^i_j has two eigenvalues, one eigenvalue has multiplicity 1. The corresponding eigenvector is $\frac{\partial}{\partial x_1}$.
- If R^i_j has precisely one eigenvalue in a neighborhood of a point, then the sectional curvature of the adjusted metric is constant near the point.

Combining these three observation, we see that the sectional curvature of the adjusted metric is constant, or L_1 and L_2 are diagonal in the same coordinate system. In the latter case, the formula (17) for the metric shows that L_1, L_2 and Id are linear dependent. Theorem 7 is proven under the assumption that M_A is two-dimensional.

Now let the dimension of M_A be greater than 2. Consider two BM-structures L_1 and L_2 such that L_1, L_2 and Id are linearly independent and have only typical points on (D^n, g). Without loss of generality, since Id is also a BM-structure, we can think that L_1 and L_2 are positive-definite. If $N_{L_1} = n$, Theorem 7 follows from Theorem 8. Suppose $N_{L_1} < n$. Then, as we already explained after Definition 4, Theorem 5 applied to the BM-structure L_1 gives us a warped decomposition $D^n = D^{k_0} \times ... \times D^{k_m}$. Consider the constructed above submanifold

$$M_A := D^{k_0} \times \gamma_1 \times \gamma_2 \times ... \times \gamma_m$$

for this warped decomposition.

By construction, every tangent space to M_A is invariant with respect to L_1. By the second part of Levi-Civita's Theorem 5, the restriction of L_1 to M_A is a BM-structure for the restriction $g_{|M_A}$ of g to M_A. The number of its different eigenvalues at P coincides with the number of different eigenvalues of L_1 and, therefore, equals the dimension of M_A.

Let us show that L_2 generates one more BM-structure on M_A. Since L_2 is positive-definite, by Theorem 1, it generates a metric g_2 projectively equivalent to g. Since M_A is totally geodesic, $g_{2|M_A}$ is geodesically equivalent to $g_{|M_A}$. Then, by Theorem 1, it generates one more BM-structure for $g_{|M_A}$. We denote this BM-structure by $\bar{L}_2$.

Thus, in view of Fubini's Theorem 8, our goal is to prove that, for a certain choice of geodesic segments $\gamma_1, ..., \gamma_m$, these two BM-structures (on M_A) and the trivial BM-structure Id are linearly independent.

By construction, the metric $g_{1|M_A}$ does not depend on the choice of geodesic segments γ_k: the results are isometric. Suppose the BM-structures $L_{1|M_A}$, $\bar{L}_2$ and Id are linearly dependent for every choice of the geodesic segments. Then, for every choice of the geodesic segments, $\bar{L}_2$ is a linear combination of $L_{1|M_A}$ and Id. Clearly, the coefficients of the linear combination do not depend on the choice of geodesic segments γ_k. (To see it, it is sufficient to consider the length of the integral curve of the eigenvector v_i corresponding to a nonconstant λ_i. The integral curve lies in M_A and its length does not depend on the choice of the geodesic segments γ_k.) Then, the eigenspaces of L are invariant with respect to the BM-structure L_2. Hence, the metrics g_1, g_2 have the form from Remark 3 in the same coordinate system. Then, L_2 is linear dependent of L_1 and Id. We obtained a contradiction. Thus, the adjusted metric has constant sectional curvature. Theorem 7 is proven.

§4. Proof of Theorem 2

Assume $\dim(\mathcal{B}(M^n,g)) \geq 3$, where (M^n,g) is a connected complete Riemannian metric of dimension $n \geq 3$. Suppose a complete Riemannian metric $\bar g$ is projectively equivalent to g. Denote by L the BM-structure from Theorem 1. By Theorem 7, for every typical point, the sectional curvature of the adjusted metric is constant.

Denote by m the number of different eigenvalues of L in a typical point. The number m does not depend on the typical point. If $m = n$, Theorem 2 follows from Fubini's Theorem 8 and Corollary 4.

Thus, we can assume $m < n$. Denote by m_0 the number of simple eigenvalues of L at a typical point. By Corollary 2, the number m_0 does not depend on the typical point. Then, by Levi-Civita's Theorem 5, the metric g has the following warped decomposition near every typical point p:

$$g = g_0 + \left|\prod_{i=1}^{m_0}(\phi_{m_0+1} - \phi_i)\right| g_{m_0+1} + \dots + \left|\prod_{i=1}^{m_0}(\phi_m - \phi_i)\right| g_m. \tag{18}$$

Here the coordinates are $(\bar y_0, ..., \bar y_m)$, where $\bar y_0 = (y_0^1, ..., y_0^{m_0})$ and for $i > 1$ $\bar y_i = (y_i^1, ..., y_i^{k_i})$. For $i > 0$, every metric g_{m_0+i} depends on the coordinates $\bar y_i$ only. Every function ϕ_i depends on y_0^i for $i \leq m_0$ and is constant for $i > m_0$.

Let us explain the relation between Theorem 5 and the formula above. The term g_0 collects all one-dimensional terms of (5). The coordinates $\bar y_0 = (y_0^1, ..., y_0^{m_0})$ collect all one-dimensional $\bar x_i$ from (5). For $i > m_0$, the coordinate $\bar y_i$ is one of the coordinates $\bar x_j$ with $k_j > 1$. Every metric g_{m_0+i} for $i > 1$ came from one of the multidimensional terms of (5) and is proportional to the corresponding A_j. The functions ϕ_i are eigenvalues of L; they must not be ordered anymore: the indexing can be different from (4). Note that, by Corollary 2, this re-indexing can be done simultaneously in all typical points.

Since the dimension of the space $\mathcal{B}(M^n,g)$ is greater than two, by Theorem 7, g is a $V(K)$ metric.

According to Definition 7, a multiple eigenvalue ϕ_i of L is **isolated**, if there exists no nonconstant eigenvalue ϕ_j such that $\phi_j(q) = \phi_i$ at some point $q \in M^n$. If every multiple eigenvalue of L is non-isolated, then applying Theorems 11,12,6 we obtain that g has constant sectional curvature.

Thus, we can assume that there exist isolated eigenvalues. Without loss of generality, we can assume that (at every typical point) the re-indexing is made in such a way that the first multiple eigenvalues

$\phi_{m_0+1}, ..., \phi_{m_1}$ are non-isolated and the last multiple eigenvalues ϕ_{m_1+1}, ..., ϕ_m are isolated. By assumption, $m_1 < m$.

We will prove that in this case all eigenvalues of L are constant. By Remark 3, it implies that the metrics g, $\bar{g}$ are affine equivalent.

Let us show that the sectional curvature of the adjusted metric g is nonpositive. We suppose that it is positive and will find a contradiction.

At every point q of M^n, denote by $V_0 \subset T_q M^n$ the direct product of the eigenspaces of L corresponding to the eigenvalues $\phi_1, ..., \phi_{m_1}$. Since the eigenvalues $\phi_{m_1+1}, ..., \phi_m$ are isolated by the assumptions, the dimension of V_0 is constant, and V_0 is a distribution. By Corollary 5, V_0 is integrable. Take a typical point $p \in M^n$ and denote by M_0 the integral manifold of the distribution containing this point. The restriction $g_{|M_0}$ of the metric g to M_0 is complete.

Consider the direct product $M_0 \times \mathbb{R}^{m-m_1}$ with the metric

$$g_{|M_0} + \left|\prod_{i=1}^{m_0}(\phi_{m_1+1} - \phi_i)\right| dt_{m_1+1}^2 + ... + \left|\prod_{i=1}^{m_0}(\phi_m - \phi_i)\right| dt_m^2, \tag{19}$$

where $(t_{m_1+1}, ..., t_m)$ are the standard coordinates on $\mathbb{R}^{m-m_1}$. Since the eigenvalues $\phi_{m_1+1}, ..., \phi_m$ are isolated, (19) is a well-defined Riemannian metric. Since $g_{|M_0}$ is complete, the metric (19) is complete. By definition, the metric is the adjusted metric for the warped decomposition (18). Hence, the sectional curvature of the adjusted metric is positive constant. Then, the product $M_0 \times \mathbb{R}^{m-m_1}$ must be compact, which contradicts the fact that $\mathbb{R}^{m-m_1}$ is not compact. Finally, the sectional curvature of the adjusted metric is not positive.

Now let us prove that all eigenvalues of L are constant. Without loss of generality, we can assume that the manifold is simply connected. We will construct a totally geodesic submanifold M_A, which is a global analog of the submanifold M_A from Section 3.4. At every point $x \in M^n$, consider $V_{m_1+1}, ..., V_m \subset T_x M^n$, where V_{m_1+i} is the eigenspace of the eigenvalue ϕ_{m_1+i}. Since the eigenvalues ϕ_{m_1+i} are isolated, $V_{m_1+1}, ..., V_m$ are distributions. By Corollary 5, they are integrable. Denote by $M_{m_1+1}, M_{m_1+2}, ..., M_m$ the corresponding integral submanifolds.

Since the manifold is simply connected, then, by [6], it is homeomorphic to the product $M_0 \times M_{m_1+1} \times M_{m_1+2} \times ... \times M_m$. Clearly, the metric g on

$$M^n \simeq M_0 \times M_{m_1+1} \times M_{m_1+2} \times ... \times M_m$$

has the form

$$g_{|M_0} + \left|\prod_{i=1}^{m_0}(\phi_{m_1+1} - \phi_i)\right| g_{m_1+1} + \ldots + \left|\prod_{i=1}^{m_0}(\phi_m - \phi_i)\right| g_m, \tag{20}$$

where every g_k is a metric on M_k. Take a point

$$P = (p_0, p_{m_1+1}, \ldots, p_m) \in M_0 \times M_{m_1+1} \times M_{m_1+2} \times \ldots \times M_m.$$

On every M_{m_1+k}, $k = 1, \ldots, m - m_1$, pick a geodesic γ_{m_1+k} (in the metric g_{m_1+k}) passing through p_k. Denote by M_A the product

$$M_0 \times \gamma_{m_1+1} \times \ldots \times \gamma_m.$$

M_A is an immersed totally geodesic manifold. More precisely, the natural immersion of $M_0 \times \mathbb{R}^{m-m_1}$ (endowed with the metric (19)) into M^n is isometric and totally geodesic. Locally, in a neighborhood of every point, M_A coincides with M_A from Section 3.4 constructed for the warped decomposition (20). The restriction of the metric g to M_A is isometric to the adjusted metric and, therefore, has nonpositive constant sectional curvature. Then, by Corollary 4, the restriction of $\bar{g}$ to M_A is affine equivalent to the restriction of g to M_A. Then, by Remark 3, all ϕ_i are constant. Then, the metric g is affine equivalent to the metric $\bar{g}$. Theorem 2 is proven.

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Valuations, and moduli of Goursat distributions

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Abstract.

Goursat distributions are subbundles in the tangent bundles to manifolds having the flag of consecutive Lie squares of ranks not depending on a point and growing always by 1. It is known that moduli of the local classification of these objects (distributions determine their flags, and vice versa) are not functional, only continuous numeric, and appear in codimensions two and higher; singularities of codimension one are all simple. In the present work we show that *most* of the codimension-two singularities of Goursat flags is not simple. As to the *precise* modalities of those singularities, we give them at paper's end in the conjectural mode.

§1. Introduction, main result, and infinitesimal symmetries

A distribution D of corank $r \geq 2$ on a smooth or analytic manifold M (a codimension–r subbundle of TM) is Goursat when its Lie square $[D,\, D]$ is a distribution of constant corank $r-1$, the Lie square of $[D,\, D]$ is of constant corank $r-2$, and so on until reaching the full tangent bundle TM. A Goursat flag of length r is any such D together with the nested sequence of its consecutive Lie squares through TM inclusively. (Without loss of generality, it could have been assumed that D is of rank 2. Locally it would lead to the same theory, for in any general Goursat distribution D there locally splits off an integrable subdistribution of rank $\operatorname{rk} D - 2$.)

Those distributions appear naturally, among others, as the outcome of series of so-called Cartan prolongations (see [1, 6] for details) of rank-2 distributions on different manifolds, starting from the full tangent

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bundle of a 2-dimensional surface. They generalize the well-known, also Cartan's, distributions on the jet spaces of functions $\mathbb{R} \longrightarrow \mathbb{R}$ (sometimes also called *contact systems*) in that they admit singularities, discovered in 1978 by Giaro-Kumpera-Ruiz – see their *exceptional model* (1) – and started to be systematically investigated in [5], while they retain the basic flag property of Cartan's distributions.

In fact, original Cartan structures are obtained in the prolongation procedures when at each prolongation step one avoids the vertical directions. And the vertical directions – that can be chosen in prolongations at will, either in row or intermittently with non-vertical ones – account for a rich pattern of singularities hidden in flags. Upon closer inspection there emerges, [4, 6], Jean-Montgomery-Zhitomirskii stratification of germs of flags into *geometric classes*, with strata encoded by words (of length equal to flag's length) over the alphabet $\{\mathrm{G}, \mathrm{S}, \mathrm{T}\}$: Generic, Singular, Tangent.

1.1. Geometric classes of Goursat flags.

The canonical geometric definition of them deals with a given Goursat flag

$$TM = D^0 \supset D^1 \supset D^2 \supset \cdots \supset D^{r-1} \supset D^r = D$$

($[D^j,\, D^j] = D^{j-1}$) around a fixed point $p \in M$ and firstly precises which members of it (excepting D^1 and D^2) are at p in singular positions. Namely, D^j is at p in singular position when it coincides at p with the *Cauchy characteristics* $L(D^{j-2})$ of D^{j-2}: $D^j(p) = L(D^{j-2})(p)$.

(In general, for any distribution D, $L(D)$ is the module, or sheaf of modules, of such vector fields v with values in D that preserve D, $[v,\, D] \subset D$. And one of first observations, see [6] for inst., is that for D – Goursat, $L(D)$ is a regular corank *two* subdistribution of D, $\mathrm{rk}\, L(D) = \mathrm{rk}\, D - 2$. Thus, needless to say, D^j and $L(D^{j-2})$ in the definition above have the same ranks.)

At this moment it is very useful to have under eyes the so-called *sandwich diagram* excerpted from [6]:

$$\begin{array}{cccccccccc}
D^1 \supset & D^2 & \supset & D^3 & \supset \cdots \supset & D^{r-1} & \supset & D^r & & \\
& \cup & & \cup & & \cup & & \cup & & \\
& L(D^1) & \supset & L(D^2) & \supset \cdots \supset & L(D^{r-2}) & \supset & L(D^{r-1}) & \supset L(D^r)\,. &
\end{array}$$

All direct inclusions in this diagram are of codimension one. The squares built by [drawn] inclusions can be perceived as certain 'sandwiches': for instance, in the utmost left sandwich $L(D^1)$ and D^3 are as if fillings, while D^2 and $L(D^2)$ constitute the covers (of different dimensions, one has to admit).

For instance, the [Lie] square of the [Lie] square, i. e., D^1, of the Goursat object D^3 described on $\mathbb{R}^5(x^1, \dots, x^5)$ by the Pfaffian equations

$$dx^2 - x^3 dx^1 = dx^3 - x^4 dx^1 = dx^1 - x^5 dx^4 = 0\,, \tag{1}$$

is given by $dx^2 - x^3 dx^1 = 0$, and its Cauchy characteristics are $\mathrm{span}(\partial_4, \partial_5)$. This 2–plane coincides with D^3 at 0, and – more generally – at all points of the hypersurface $\{x^5 = 0\}$. Therefore, this D^3 is in singular position by far not at isolated points, but **in codomension 1**.

In general D^3, $D^4, \dots, D^r$ can be in singular positions at a point (in different sandwiches on the diagram above) one independently of another and it gives rise to 2^{r-2} rough invariant classes of flag's germs (termed 'Kumpera–Ruiz classes' in [6]). Thus, at this moment, the local behaviour at p is encoded by a word of length r over $\{*, \mathrm{S}\}$ starting with two $*$, possibly having more $*$'s, and having S as the j-th letter ($3 \le j \le r$) precisely and only when D^j is at p in singular position.

Secondly, heading towards geometric classes and labels over $\{\mathrm{G}, \mathrm{S}, \mathrm{T}\}$, one plainly replaces all $*$ before the first S (if any) by letters G and then turns to strings of $*$ standing behind, or past, letters S (if there are such strings). Let us depart from a given letter S, being the j-th letter in the word *and* having a string of $*$'s past it. In the eventual label [of the geometric class of the flag's germ at p] this S is followed by T when $D^{j+1}(p)$ is tangent to the locus (always being a regular hypersurface in M, as in the example above) of the previous singularity 'D^j in singular position', while it is followed by G when $D^{j+1}(p)$ is not tangent to that locus (and, as a matter of course, not in the singular position at p – see the detailed discussion below).

At this point it is important to explain that this tangent position of D^{j+1} with respect to the hypersurface

$$H \;=\; \{\, q \in M :\; D^j(q) = L(D^{j-2})(q) \,\}, \tag{2}$$

when materializing at p, implies that $D^{j+1}(p)$ itself is not in the singular position. Or, in other words, that the presently being defined meaning of 'S**T**' does not *conflict* with the previously introduced meaning of 'S**S**'. Indeed, choosing any fixed local vector field V with values in $L(D^{j-1})$ and independent of $L(D^j)$, $L(D^{j-1}) = (V) \oplus L(D^j)$, the flow φ_V^t of V preserves D^{j-1}, hence also D^{j-2} and $L(D^{j-2})$: $\left(\varphi_V^t\right)_* L(D^{j-2}) = L(D^{j-2})$ for small $|t| > 0$. On the other hand, recalling, V takes values in D^j ($\supset L(D^{j-1})$) but not in $L(D^j)$. Consequently, $\left(\varphi_V^t\right)_* D^j \neq D^j$ for $|t| > 0$ small and, altogether, the flow of V does not preserve the defining equation (2) of H. In fact, upon analyzing more carefully the *speed* of $\left(\varphi_V^t\right)_* D^j$ deviating from D^j, V is not tangent to H at p. Hence, H being of codimension one, V is transverse, and all the more so is $L(D^{j-1})(p)$. A new singular position at p (i. e., the second S in row in a code) would then imply the transversality to the locus (H) of the previous singular position, while the [($j+1$)-st member'] position encoded by T is tangent to H.

In turn, T can follow ST, further selecting the germs with $D^{j+2}(p)$ tangent to the locus $\widetilde{H}$ (now being regular of codimension 2 in M) of the geometry ...ST. Such a new ST**T** does not conflict with a hypothetical ST**S** – because the latter would imply the relative (within H) transversality of $D^{j+2}(\cdot)$ to $\widetilde{H}$. When there are sufficiently many $*$'s after the reference letter S, another T can follow ...STT, again causing no conflict with a potential ...STT**S** in view of another tangency as opposed to relative transversality, and so on. Since the moment of interruption of such sequence of tangencies, one plainly replaces all remaining $*$'s by G's, till the next letter S.

All thus emerging labels are geometrically realizable; the only restriction in them, clear from the geometric behaviours being encoded, are the two necessary G's in the beginning and that T cannot go directly after G. Therefore, for length 2 there is but one class GG, for length 3 – only GGG and GGS (the latter having as a unique local model (1)), for length 4 – GGGG, GGSG, GGST, GGSS, GGGS. A straightforward recurrence yields that there exist u_{2r-3} (Fibonacci number) geometric classes of the germs of flags of length r. They are, obviously, pairwise disjoint and invariant under the action of local diffeomorphisms between manifolds. The class GG...G is the fattest single orbit,[1] as established in now classical papers [2, 13] (those contributions were widely popularized in the 1920s by Goursat in his book *Leçons sur le problème de Pfaff*). The well-known local representative is, for the length r, the *chained model* of that length – the germ at 0 of

$$dx^2 - x^3 dx^1 = dx^3 - x^4 dx^1 = \cdots = dx^{r+1} - x^{r+2} dx^1 = 0\,, \tag{3}$$

still actively in use in control theory and differential geometry.

Clearly, the geometric classes approximate from above orbits of the local classification of Goursat germs. As a matter of record, we just note that this approximation is 100% precise for lengths $\leq$ 6, but too rough from length 7 onwards ([7]). Before passing to finer issues, let us note one remarkable property of geometric classes' stratification.

Namely, the materialization of any given stratum is, if non-empty, an embedded submanifold of **codimension** equal to the **number** of letters S and T in its code. For inst., without these letters, there comes the unique open stratum of generic germs; all of them, recalling, equivalent to the relevant chained model (3).

Notwithstanding so regular a definition, there exist continuous numerical moduli of the local classification of Goursat distributions. In the first turn, in [12, 7], such moduli were found in codimension three. Later, in [11], real invariants were produced already in codimension two. While they are absent in codimension one, for it turns out, [8], that the codimension-one strata, i. e., all GG...GSG..G, are single orbits of the local classification. Moreover, because

[1] and the only one open, cf. a remark on codimensions below

these strata are adjacent only to the generic stratum, the codimension-one singularities of Goursat are all simple.

The present work extends substantially [11] and, at the same time, contributes to a vast project (first stated in 1999 by the authors of [6]) of finding all *simple* singularities of Goursat flags.

1.2. Codimension–two singularities of Goursat.

Codimension-two singularities are of two essentially different types.

• Either precisely two different members of a flag are simultaneously in singular positions at the same point; this is the concatenation SGG. . . GS of two singularities S separated by a number $j \geq 0$ of intermediate flag's members being in generic positions G.

•• Or else only one flag's member, say of corank k, is at a point in singular position, and its 'Lie square root' of corank $k+1$ is tangent at that point to the locus (a hypersurface in M) of the corank–k member being in singular position. Those latter singularities, as we know from sec. 1.1, are labelled ST. In virtue of [9], the ST singularities are simple, possibly modulo subvarieties of codimension three.

Passing to the class •, it is generally conjectured that the singularities SS (that is, with $j = 0$ letters G in between the S's) are all simple, too. However, the situation in geometric classes having two singularities S really separated by a number of G's, is different. The results presented in this paper, together with those of [11], justify that, excepting possibly the classes with the first S at the earliest admissible position No 3, for all *positive* numbers j of intermediate positions G, a modulus hides already in the flag' member three steps past the second singularity S. In fact, we are going to prove that

Theorem 1. *Fix any segment* [G. . . G] *of* $j \geq 1$ *letters* G*. Excepting (possibly) only the geometric classes starting with exactly two* G*'s before that segment, in each geometric class* GG. . . GS[G. . . G]SGGG. . . *with at least three* G*'s in the end, there sits at least one modulus of the local classification. Therefore, such classes' modalities are all not smaller than one.*

Therefore, in codimension two there are but *few* simple geometric classes: only 'ST', conjecturally 'SS', GG. . . GS[G. . . G]S... with at most two G's past the second S, and plausibly all GGS[G. . . G]S..., with, this time, any number of G's at the end.

This statement becomes more precise when one fixes flag's length r. Then the number of these simple and supposedly simple classes is $2(r-3)+(r-7)4+3+2+1 = 6r-28$, while the number of all codimension-two geometric classes of length r is $(r-3)+(r-3)+(r-4)+\cdots+2+1 = \frac{r^2-3r}{2}$. The remaining codimension-two classes, that assuredly are not simple by Theorem 1, constitute therefore the

$$1 - \frac{(6r-28)2}{r^2-3r} = \frac{r^2-15r+56}{r^2-3r}$$

fraction of all codimension-two classes in that length, an overwhelming majority.

Remark 1. The assertions of Thm. 1 can be equivalently stated in terms of the mentioned systematization of prolongations of Goursat germs from [6]. In the language of that reference work these reformulations go as follows.
For any $k \geq 4$, $j \geq 1$, and any germ, at a point p, of a Goursat flag

$$\cdots \supset D^{k+1+j} \supset D^{k+2+j} \supset D^{k+3+j} \supset D^{k+4+j} \tag{4}$$

sitting in the class $\mathrm{G}_{k-1}\mathrm{SG}_j\mathrm{SGGG}$, **firstly**, the prolongation pattern of the germ of D^{k+2+j} at p is 1: there is only one fixed point $L(D^{k+1+j})(p)/L(D^{k+2+j})(p)$ on the circle $S^1(D^{k+2+j})(p)$, and only two orbits in it: this fixed point and all the remaining of the circle; the prolonged germ D^{k+3+j} sits in that second orbit (this corresponds to all values b in (5) below being equivalent, hence equivalent to the value 0). And **secondly** – the main property being established in the present paper – the prolongation pattern of the germ at p of D^{k+3+j} is either 2c or 3, and, consequently, there appears a modulus in the local classification of the one-step prolongations of D^{k+3+j} (the value c cannot be moved by *flows* of symmetries of (5) understood not as a germ).

1.3. Basic preliminaries needed in the proof.

It follows from the works [5, 6] and previous contributions by the author that any Goursat germ D^{k+4+j} sitting in the geometric class $\mathrm{G}_{k-1}\mathrm{SG}_j\mathrm{SG}_3$ can be written down in certain *Kumpera-Ruiz* coordinates $x^1, x^2, \dots, x^{k+6+j}, \dots, x^n$ as the germ at $0 \in \mathbb{R}^n$ of a system of Pfaffian equations

$$\begin{aligned}
dx^2 - x^3 dx^1 &= 0\,,\\
dx^3 - x^4 dx^1 &= 0\,,\\
* \qquad &*\\
dx^k - x^{k+1} dx^1 &= 0\,,\\
dx^1 - x^{k+2} dx^{k+1} &= 0\,,\\
dx^{k+2} - \left(1 + x^{k+3}\right) dx^{k+1} &= 0\,,\\
dx^{k+3} - x^{k+4}\, dx^{k+1} &= 0\,,\\
* \qquad &*\\
dx^{k+1+j} - x^{k+2+j} dx^{k+1} &= 0\,,\\
dx^{k+1} - x^{k+3+j} dx^{k+2+j} &= 0\,,\\
dx^{k+3+j} - \left(1 + x^{k+4+j}\right) dx^{k+2+j} &= 0\,,\\
dx^{k+4+j} - \left(b + x^{k+5+j}\right) dx^{k+2+j} &= 0\,,\\
dx^{k+5+j} - \left(c + x^{k+6+j}\right) dx^{k+2+j} &= 0\,,
\end{aligned} \tag{5}$$

with *certain* real constants b and c. (In these constants resides much of the difficulty and geometric complication of D^{k+4+j}, and we will cope with

them rather heavily.) The directions of the remaining, invisible in (5) variables $x^{k+7+j}, \ldots, x^{n-1}$, x^n do span the Cauchy-characteristic subdistribution $L(D^{k+4+j})$ (see sec. 1.1 for the definition) by which one can always factor out.
In all the sequel we assume this done; in other words, in what follows, $n = k + 6 + j$. This factoring out simplification, that loses – this is critical – no local geometry of a Goursat flag, amounts to saying that, without loss of generality, we assume the smallest member D^{k+4+j} of a flag to be of **rank two**. The ambient dimension after factoring out $(k + 6 + j)$ exceeds then just by two the flag's length $(k + 4 + j)$.

We will strive to move the parameters b, c. That is, to conjugate germs (5) displaying different values of b and c. The work [7] shows how involved it is to do on the level of *finite* symmetries. Only after that contribution we realized that the level of *infinitesimal* symmetries, of a non-local object encompassing nearby germs, was more promising.
How to describe the vector fields infinitesimally preserving a given Goursat distribution like D^{k+4+j} above? And so, each rank two Goursat germ is equivalent to the result of a sequence of certain projective extensions (called Cartan prolongations, described in detail in [1, 6]) started from the differential system (a contact structure) $\omega^1 = dx^2 - x^3 dx^1 = 0$ living on $\mathbb{R}^3(x^1, x^2, x^3)$. And the infinitesimal symmetries of $\omega^1 = 0$ are generated by all C^∞ (or analytic, depending on the chosen category) functions $f(x^1, x^2, x^3)$ – a deep and basic thing observed long time ago by S. Lie. Those generating functions are nowadays called *contact hamiltonians*.

In view of the mentioned stepwise extensions yielding D^{k+4+j}, the i. s.'s of D^{k+4+j} turn out, fortunately if not unexpectedly, to be sequences of fairly simple parallel prolongations of the i. s.'s of the relevant sequence of Cartan prolongations of *that* Darboux structure. Consequently, they inherit the property of being locally 1–1 parametrized by C^∞ or C^ω functions in three variables.
However, the parametrization depends sensitively on the distribution of inversions of differentials in the pseudo-normal form for D (i. e., depends on the word over $\{*, S\}$ preliminarily encoding the germ' local geometry – see sec. 1.1 – or still else, depends on which members of the flag of D are in singular positions at the reference point). Therefore, one has to deal in general with a vast *binary tree* of different parametrizations. This is a disadvantage, yet for D^{k+4+j} in a concrete pseudo-normal form as above one can advance rather far.

These 'infinitesimal' tools are given in more detail in, for inst., [8] or [9]. Here we just recapitulate that, having a D of rank two, in a Kumpera-Ruiz pseudo-normal form (originating from [5]) in the ambient dimension $r + 2$, one denotes by $\mathcal{Y}_f$ its infinitesimal symmetry induced by a function $f(x^1, x^2, x^3)$

and deliberately puts in relief in $\mathcal{Y}_f$ the first three components,

$$\mathcal{Y}_f \;=\; A\partial_1 + B\partial_2 + C\partial_3 + \sum_{l=4}^{r+2} F^l \partial_l \tag{6}$$

– because, understandingly, the vector field $A\partial_1 + B\partial_2 + C\partial_3$ is an infinitesimal symmetry of $dx^2 - x^3 dx^1 = 0$. Hence the classical expressions of Lie: $A = -f_3\,,\quad B = f - x^3 f_3\,,\quad C = f_1 + x^3 f_2$.

Prior to write the infinitesimal symmetries of D^{k+4+j} in our situation, we need the following three vector fields

$$\begin{aligned} y &= \partial_1 + x^3\partial_2 + x^4\partial_3 + \cdots + x^{k+1}\partial_k\,,\\ Y &= x^{k+2} y + \partial_{k+1} + X^{k+3}\partial_{k+2} + x^{k+4}\partial_{k+3} + \cdots + x^{k+2+j}\partial_{k+1+j}\,,\\ \widehat{Y} &= x^{k+3+j} Y + \partial_{k+2+j} + X^{k+4+j}\partial_{k+3+j} + X^{k+5+j}\partial_{k+4+j} + X^{k+6+j}\partial_{k+5+j}\,. \end{aligned} \tag{7}$$

With these notations, the first group of components of $\mathcal{Y}_f$ contains, on top of functions A, B, C,

$$F^4 \;=\; yC - x^4 yA\,,\quad F^l \;=\; yF^{l-1} - x^l yA \qquad \text{for}\ \ 5 \le l \le k+1\,. \tag{8}$$

In the second group of components,

$$\begin{aligned} F^{k+2} &= x^{k+2}\big(yA - YF^{k+1}\big)\,, \qquad F^{k+3} \;=\; YF^{k+2} - X^{k+3}\,YF^{k+1}\,,\\ F^l &= YF^{l-1} - x^l\,YF^{k+1} \qquad \text{for}\ \ k+4 \le l \le k+2+j;\\ F^{k+3+j} &= x^{k+3+j}\big(YF^{k+1} - \widehat{Y}F^{k+2+j}\big)\,,\\ F^{k+4+j} &= \widehat{Y}F^{k+3+j} - X^{k+4+j}\,\widehat{Y}F^{k+2+j}\,,\\ F^{k+5+j} &= \widehat{Y}F^{k+4+j} - X^{k+5+j}\,\widehat{Y}F^{k+2+j}\,,\\ F^{k+6+j} &= \widehat{Y}F^{k+5+j} - X^{k+6+j}\,\widehat{Y}F^{k+2+j}\,. \end{aligned} \tag{9}$$

These formulas confirm and re-establish a basic property of Kumpera-Ruiz coordinates (for whatever local Goursat object of corank r) that, in such coordinates, for $4 \le \nu \le r{+}2$, the component F^ν depends only on x^1, $x^2, \dots, x^{\nu-1}$, x^ν. In our situation, they also help to quickly find the first $k+3$ components of $\mathcal{Y}_f$ **at zero**,

$$\mathcal{Y}_f\,|0 \;=\; -f_3\partial_1 + f\partial_2 + \sum_{j=3}^{k+1} f_{1_{j-2}}\partial_j - \big(2f_2 + (2k-1)f_{13}\big)\partial_{k+3} + \cdots \tag{10}$$

(the ∂_{k+3} component is also standard, cf., for inst., the formula (5) in [11]). Note the absence of the ∂_{k+2} component in (10), explained by the fact that the hypersurface $x^{k+2} = 0$ is invariant under all symmetries of D^{k+4+j}, let alone those embeddable in flows (for the same reason, cf. the proof of Prop. 1 below, each such i. s. $\mathcal{Y}_f\,|0$ has no ∂_{k+3+j} component as well). The next component ∂_{k+4} at 0 is computationally more delicate (albeit simple in the outcome).

1.4. Computation of $F^{k+4}\,|0$.

During all the computation we use the recursive formulas (9):

$$\begin{aligned} F^{k+4}\,|0 &= YF^{k+3} - x^{k+4}\,YF^{k+1}\,|0 = Y^2F^{k+2} - X^{k+3}Y^2F^{k+1}\,|0 \\ &= 2X^{k+3}Y\Big(yA - YF^{k+1}\Big) - X^{k+3}Y^2F^{k+1}\,|0 \\ &= 2X^{k+3}YyA - 3X^{k+3}Y^2F^{k+1}\,|0\,. \qquad (11) \end{aligned}$$

To proceed, an operational expression for F^{k+1} is needed,

Lemma 1. $F^{k+1} = y^{k-2}C - (k-2)x^{k+1}yA \quad \mathrm{mod}\,(x^k,\, x^{k-1}, \dots,\, x^4)\,.$

This lemma follows from the following formula than can easily be proved by induction on k, departing from F^4, already expressed in (8):

$$F^{k+1} = y^{k-2}C - \binom{k-2}{1}x^{k+1}yA - \binom{k-2}{2}x^ky^2A - \cdots - \binom{k-2}{k-2}x^4y^{k-2}A\,.$$

Differentiating the RHS in Lem. 1 twice with respect to Y,

$$Y^2F^{k+1} = Y^2y^{k-2}C - 2(k-2)YyA \quad \mathrm{mod}\,(x^{k+2},\, x^{k+1}, \dots,\, x^4)\,.$$

After evaluating the RHS here at 0 and substituting to (11),

$$F^{k+4}\,|0 \;=\; \big(2 + 6(k-2)\big)X^{k+3}YyA - 3X^{k+3}Y^2y^{k-2}C\,|0\,. \qquad (12)$$

The first summand on the RHS in (12) is being made transparent immediately,

$$YyA\,|0 \;=\; \begin{cases} -f_{33}\,|0 & \text{when } k=3\,, \\ 0 & \text{when } k \geq 4\,. \end{cases}$$

As for the second summand, a more careful approach is needed, because the derivative $y^{k-2}C$ consists, for k big, of a rich array of terms.[2] Trying to see through them, we note that, in C, there clearly is the term f_1, in yC there is f_{11} (cf. Ex. 1 below) and likewise there is $f_{1_{k-1}}$ in $y^{k-2}C$. In fact, the structure of terms building up $y^{k-2}C$, together with the particular form of the vector field Y, imply that the exemplified term $f_{1_{k-1}}$ is the *only* one that contributes under Y^2 to the value of $Y^2y^{k-2}C$ at 0,

$$f_{1_{k-1}} \xrightarrow{Y} x^{k+2}f_{1_k} + \cdots \xrightarrow{Y} X^{k+3}f_{1_k} + \cdots\,,$$

[2] A deeper analysis of $y^{k-2}C$ is given in sec. 6.1 of the preprint No 39, *Simple codimension–two singularities of Goursat flags, I*, available at http://www.mimuw.edu.pl/english/research/reports/imat

with the last $\cdots$ vanishing at 0. Thus $Y^2 y^{k-2} C\,|0 = X^{k+3} f_{1_k}\,|0$. Substituting all these data to (12),

$$F^{k+4}\,|0 \;=\; \begin{cases} -3\bigl(X^{k+3}\bigr)^2 f_{1_k} - 8X^{k+3} f_{33}\,|0 & \text{when } k=3\,, \\ -3\bigl(X^{k+3}\bigr)^2 f_{1_k}\,|0 & \text{when } k \geq 4\,. \end{cases} \tag{13}$$

This information will be instrumental later in the proof of Thm. 1.

§2. Annihilation of the constant b

For transparency reasons it is useful to work, instead of the object (5), with a 'universal' distribution, E, that displays *no* constants shifting the last two variables, $E =$

$$\begin{aligned}&\bigl(dx^2 - x^3dx^1,\ \ dx^3 - x^4dx^1,\ \dots,\ dx^k - x^{k+1}dx^1,\ \ dx^1 - x^{k+2}dx^{k+1},\\ &\quad dx^{k+2} - X^{k+3}dx^{k+1},\ dx^{k+3} - x^{k+4}dx^{k+1},\ \dots,\ dx^{k+1+j} - x^{k+2+j}dx^{k+1},\\ &\quad dx^{k+1} - x^{k+3+j}dx^{k+2+j},\ \ dx^{k+3+j} - X^{k+4+j}dx^{k+2+j},\\ &\qquad dx^{k+4+j} - x^{k+5+j}dx^{k+2+j},\ \ dx^{k+5+j} - x^{k+6+j}dx^{k+2+j}\bigr)\,,\end{aligned} \tag{14}$$

$X^{k+3} = 1 + x^{k+3}$, $X^{k+4+j} = 1 + x^{k+4+j}$. The reason is that the symmetries under consideration will keep all but the last two coordinates of $0 \in \mathbb{R}^{k+6+j}$, while the two last ones will be moved.

In the annihilation of b there will be used certain concrete contact hamiltonians. Namely, $f(x^1, x^2, x^3) = \bigl(x^1\bigr)^l x^2$ when $j = 2l$ is even, and $f = \bigl(x^1\bigr)^{k+l}$ when $j = 2l+1$ is odd. This dependence on the parity of a distance parameter j should not be surprising, comparing, for inst., with the arguments in the codimension-one situation (Sec. 4 in [8]). We are going to reduce the constant b to 0, changing also – this is inevitable, cf. [11] – the value of c, but preserving the normalizations already achieved in (5). The value of b will be moved to 0 gradually, using the indicated flow of symmetries of the *non-local* object (14).

An important auxiliary question is why this would not perturb the zero constants standing by $x^{k+5}, x^{k+6}, \dots, x^{k+2+j}$, as well as the constants one standing by x^{k+3} and x^{k+4+j}. It is so because

Proposition 1. *In either case of j even or odd, the corresponding infinitesimal symmetry $\mathcal{Y}_f$, with f proposed above, has at 0 the first $k+4+j$ components zero.*

Most of the present section is devoted to a proof of this statement. Concerning the first $k+4$ components of $\mathcal{Y}_f$, it is clear, in view of (10) and (13). Concerning $F^{k+3+j}\,|0$, it is also clear, for the $(k+1+j)$-th letter in the code is S and the hypersurface $x^{k+3+j} = 0$ is, naturally, invariant. Likewise, as regards $F^{k+4+j}\,|0$, this component corresponds to a place in the code going directly after a letter S and, as such, is a multiple of the additive constant one standing by the variable x^{k+4+j}. (In the absence of that constant, the germ would represent an 'ST' singularity that should be preserved by all symmetries of any its representative, and thus the $(k+4+j)$-th component would vanish.) It equals 1 times an integer combination of the basic partials $f_2\,|0$ and $f_{13}\,|0$. In fact, after a straightforward computation like in [8, 10],

$$F^{k+4+j}\,|0 \;=\; (2j+3)f_2 + \big(2+(k-1)(2j+3)\big)f_{13}\,|0\,. \tag{15}$$

Yet, trapped in between, there are $j-2$ remaining components and one should cope with them, too. In order to avoid many separate (and laborious) formulas, we propose a particular *valuation* $w(\cdot)$ assigning integer values, or *multiplicities*, to all Kumpera-Ruiz variables[3] and allowing to assign integer *abstract weights* to all terms in the polynomial expansion for $F^{\nu}(f)$, whichever is ν. In such a way, moreover, that each $F^{\nu}(f)$, a polynomial in x^3, $x^4, \dots,$ x^{k+2}, $X^{k+3}, \dots$ with coefficients – integer combinations of partials of f, gets its own abstract weight; cf. also sec. 2.3 in [10].

For it turns out that all terms of a given polynomial expansion have the *same* weights. Such is a surprising 'additional value' brought in by that valuation. There is, however, a price to it: the polynomials are to be understood particularly: variables shifted by a constant, like X^{k+3}, should be treated as indivisible entities; a valuation is assigned to a letter, irrespectively of its being small or capital.

In the proof of Prop. 1, upon getting the abstract weights of F^{k+5} through F^{k+2+j}, we will be in a position to rule out the presence of certain terms in their expansions, and that will do.

2.1. Definition of the valuation w.

An algebraic machinery underlying this definition concerns the auxiliary vector fields y, Y, $\widehat{Y}$ defined in (7) that are crucial in the formulas for the components of $\mathcal{Y}_f$. Not entering into details, roughly speaking we stipulate their being quasihomogeneous of order $-(2j+3)$, -2, and -1, respectively. We underline, however, that it is rather an algebraic, not analytic, quasihomogeneity; this is constantly being reminded of in the adjective 'abstract' (abstract weights). Here is that instrumental valuation.

$$w(x^1) = 2j+3\,, \quad w(x^2) = 2+(k-1)(2j+3)\,, \quad w(x^3) = 2+(k-2)(2j+3)\,,$$

$$w(x^4) = 2+(k-3)(2j+3)\,, \;\dots\dots,\; w(x^k) = 2+2j+3, \quad w(x^{k+1}) = 2\,,$$

$$w(x^{k+2}) = 2j+1,\; w\big(X^{k+3}\big) = 2j-1,\; w(x^{k+4}) = 2j-3, \dots,\; w(x^{k+1+j}) = 3,$$

$$w(x^{k+2+j}) = 1\,, \quad w(x^{k+3+j}) = 1\,, \quad w\big(X^{k+4+j}\big) = 0\,,$$

---

[3] not to be confused with nonholonomic orders of variables as functions! These arithmetic, not analytic, tools are in use since 1999.

$$w(x^{k+5+j}) = -1\,, \qquad w(x^{k+6+j}) = -2\,.$$

Having these multiplicities of letters, to each formal monomial $x^I X^J f_K$ (multiindices vary within their pertinent ranges; for J it is $\{k+3,\, k+4+j\}$, for K it is $\{1,\, 2,\, 3\}$, etc.) we attach its abstract weight

$$w\big(x^I X^J f_K\big) \overset{\text{def}}{=} \sum_{k\in K} w(x^k) - \sum_{j\in J} w(X^j) - \sum_{i\in I} w(x^i)\,. \tag{16}$$

Note that these abstract weights are attached irrespectively of the concrete nature of f which could even be, say, identically zero.

Example 1. $F^4 \;=\; yC - x^4 yA \;=\; y(f_1 + x^3 f_2) - x^4 y(-f_3) \;=$

$$f_{11} + 2x^3 f_{12} + 2x^4 f_{13} + \big(x^3\big)^2 f_{22} + 2x^3 x^4 f_{23} + \big(x^4\big)^2 f_{33}$$

has, as one instantly checks, all displayed monomial terms of abstract weight $2(2j+3)$.

Example 2. Let us analyze the terms appearing on the RHS of (13). The monomial $\big(X^{k+3}\big)^2 f_{1_k}$ has weight $kw(x^1) - 2w(X^{k+3}) = k(2j+3) - 2(2j-1)$. The auxiliary monomial appearing for $k=3$, $X^{k+3} f_{33}$, has weight $2(2+(3-2)(2j+3)) - (2j-1) = 3(2j+3) - 2(2j-1)$ coinciding, for this value of k, with the previous one.

These examples are instances of a more general, already invoiced, fact.

Proposition 2. *All monomials building up each fixed F^ν have one and the same abstract weight that becomes, by definition, the abstract weight $w(F^\nu)$ of that F^ν. In particular, $w(F^{k+4+\nu}) = w(F^{k+4}) + 2\nu = k(2j+3) - 2(2j-1) + 2\nu$ for $\nu = 1,\, 2, \ldots,\, j-2$.*
Moreover, $w(F^{k+3+j}) = w(x^2) - 1$, $w(F^{k+4+j}) = w(x^2)$, $w(F^{k+5+j}) = w(x^2) + 1$, $w(F^{k+6+j}) = w(x^2) + 2$.

One proves this proposition constantly using the recursive formulas (9) for the components of an i. s., plus the quasihomogeneity of the fields y, Y, $\widehat{Y}$. For the indicated particular $j-2$ components, in the recurrences there appears uniquely the field Y which is quasihomogeneous of order -2, whence that arithmetical progression with step 2 on the level of abstract weights. □

2.2. Proof of Proposition 1.

Assume first $j = 2l$ even. In view of Prop. 2, it is a matter of arithmetics that, for $\nu = 1,\, 2, \ldots,\, j-2$,

$$w\big(F^{k+4+\nu}\big) \;=\; w\big(f_{1_l 2}\big) - (l+1)(2j-1) - (2j-2-2\nu)\,.$$

Note that $2j-2-2\nu$ takes values in the set $\{2,\, 4, \ldots,\, 2j-4\}$. Therefore, no term $\big(X^{k+3}\big)^\mu f_{1_l 2}$, having its abstract weight equal to $w\big(f_{1_l 2}\big) - \mu(2j-1)$, may appear in the expansion of $F^{k+4+\nu}$.

Assume now $j = 2l + 1$ odd. It is also only simple arithmetics that, for $\nu = 1, 2, \dots, j-2$,

$$w\big(F^{k+4+\nu}\big) \;=\; w\big(f_{1_{k+l}}\big) - (l+2)(2j-1) - (2j-2-2\nu)\,.$$

And, this time, no term $\big(X^{k+3}\big)^{\mu} f_{1_{k+l}}$, having its abstract weight $w\big(f_{1_{k+l}}\big) - \mu(2j-1)$, appears in the expansion of $F^{k+4+\nu}$.

The proof of Prop. 1 is now complete. □

2.3. The reduction at work.

Recalling, we are going to analyze $F^{k+5+j}(f)\,|\,0$ in two different situations. Firstly for $j = 2l$, when $f = \big(x^1\big)^l x^2$, and later for $j = 2l+1$, when $f = \big(x^1\big)^{k+l}$. The value at 0 is important, but we will also need the values of this component of the infinitesimal symmetry at all points of the x^{k+5+j}–axis (as it was the case in codimension one in [8], and codimension two in ([9])).

Observation 1. *In either case of j even or odd, $F^{k+5+j}(f)\,|\,0$ is a positive integer $N(k,j)$.*

Idea of proof. In the even $j = 2l$ case, one directly indicates, in the formal polynomial $F^{k+5+j}(f)$, a term $(\cdot)\big(X^{k+3}\big)^{l+1}\big(X^{k+4+j}\big)^2 f_{1_l\,2}$, with $(\cdot)$ being a positive integer. Moreover, as one can also check, this is the only term in $F^{k+5+j}(f)$ using this concrete partial of f. And, clearly, only this partial does not vanish at 0, when evaluated on the proposed contact hamiltonian. Whence the statement with $N(k,j) = l!(\cdot)$.

In the odd $j = 2l+1$ case, in the polynomial in question there is a term $(\cdot)\big(X^{k+3}\big)^{l+2}\big(X^{k+4+j}\big)^2 f_{1_{k+l}}$ with, again, a positive integer $(\cdot)$, and this is the unique term displaying this partial of f – the only partial sensitive to (or: not-vanishing on) the proposed hamiltonian, implying the statement with, this time, $N(k,j) = (k+l)!(\cdot)$. □

Dealing with the infinitesimal symmetries of (14), we need to know the values of F^{k+5+j} not only at 0, but also at all points of the x^{k+5+j}-axis. Yet this causes no complications. It is a matter of course that this function is affine in x^{k+5+j}. In fact, after standard computations that we omit here (compare with (15)),

$$\begin{aligned} &F^{k+5+j}(0,0,\dots,0,x^{k+5+j}) \;= \\ &\qquad F^{k+5+j}(0) + \Big((3j+4)f_2 + \big(3+(k-1)(3j+4)\big)f_{13}\Big)(0)\cdot x^{k+5+j} \qquad (17)\\ &\qquad = N(k,j) \end{aligned}$$

by Obs. 1, for f in either case: the coefficient standing by x^{k+5+j} vanishes because so do $f_2\,|\,0$ and $f_{13}\,|\,0$ (the partial f_{13} vanishes identically; f_2 vanishes identically in the odd case, and equals $\big(x^1\big)^l$ in the even case).

Moreover, with our final objective being the next variable x^{k+6+j} (supposed to conceal a modulus) we need to know the next, and last, component F^{k+6+j}

at all its arguments of the form $(0, 0, \ldots, 0, x^{k+5+j}, x^{k+6+j})$, similarly to the situation $j = 1$ discussed in [11]. Much like in that reference, nearly repeating the arguments from there, one sees that the latter function does not depend on x^{k+6+j} and is just affine in x^{k+5+j},

$$F^{k+6+j}(0,\, 0,\, \ldots,\, 0,\, x^{k+5+j},\, x^{k+6+j}) \;=\; A(k,j) + B(k,j)x^{k+5+j}\,,$$

for certain real (even integer) constants A, B (in the present work, unlike in [11], we are not interested in their precise values). Let us recapitulate our informations. With the above chosen f, we have the first $k + 4 + j$ components of $\mathcal{Y}_f$ vanishing at 0, the next component effectively known on the x^{k+5+j}–axis, and the still next effectively known on the $(x^{k+5+j},\, x^{k+6+j})$-plane. Therefore, the integral curve $\gamma(\cdot)$ of $\mathcal{Y}_f$ passing at time $t = 0$ through $(0, 0, \ldots, 0, b, c) \in \mathbb{R}^{k+6+j}$ is easily tractable. Indeed, it reads

$$\gamma(t) \;=\; \Big(0,\ 0,\ \ldots,\ 0,\ \ b + N\,t,\ \ c + \big(A + bB\big)t + \frac{BN}{2}t^2\Big) \tag{18}$$

and is defined for all values of t. We are interested in the time $t = -\frac{b}{N}$ when the one before last coordinate of γ vanishes. This point on the curve equals, after a short computation,

$$\gamma\Big(-\frac{b}{N}\Big) \;=\; \Big(0,\, 0,\, 0, \ldots,\, 0,\, 0,\ c - \frac{2bA + b^2B}{2N}\Big)\,.$$

Retreating now from the universal object E back to the Goursat germs (5), we summarize the present section in

Corollary 1. *It is possible to annihilate the constant b in* (5) *at the expense of passing from the value c of the next (and, in the occurrence, last) constant to a new value $c - \frac{2bA+b^2B}{2N}$, with A, B, and $N > 0$ depending only on k and j.*

Attention. For the sake of simplicity, we will use the same letter c for the last constant in (5) also after the annihilation of b.

§3. Proof of Theorem 1

From now on we assume that $j \geq 2$, since the geometric classes with the segment 'SGS' ($j = 1$) have already been treated in the work [11]. We assume also, cf. the wording of Theorem, that the first letter S in the class' code has number $k \geq 4$; the second S has thus number $k + j + 1$. It is very important that the cases $k = 3$ are put aside.
The moduli emerging in the situations with $j = 1$ are understood to the end: it is known that they are of the type 3 of the systematization of [6], cf. Rem. 1. Recalling after that remark, in the classes with $j \geq 2$ we will only show that the modulus' type is either 2c or 3.

We will explore the possibility of changing only the last constant c in (5) when keeping previously secured simplifications *and* having, after Sec. 2, $b = 0$. Prior to that, however, we are to work more with the distribution E given by (14). Recalling, that object has no additive constant next to x^{k+6+j}, and the only variables in its description that are shifted by constants are X^{k+3} and X^{k+4+j}. This is advantageous; to the distribution that does display the constant c we will come back only in sec. 3.3.

Lemma 2. *Concerning the infinitesimal symmetries of E, for any contact hamiltonian f, $F^{k+6+j}(f)\,|0 \;=\; (\cdot)X^{k+3}\big(X^{k+4+j}\big)^2 f_{1_k}\,|0 \;=(\cdot) f_{1_k}\,|0$, where $(\cdot)$ is a certain integer.*

This lemma is critical for the paper and its proof will be long. Starting it now, one notes that only the terms of the form $\big(X^{k+3}\big)^{\mu}\big(X^{k+4+j}\big)^{\nu} f_K$, with certain $\mu,\,\nu \geq 0$, *can* contribute to the value of F^{k+6+j} at 0. It is also clear that $K \neq \{3\}$ (because $w(F^{k+6+2}) = w(x^2) + 2 > w(f_3)$), and that any possible $2 \in K$ is arithmetically equivalent to 1 and 3 taken together, as $w(x^2) = w(x^1) + w(x^3)$, and the same will hold for the other valuation, v, to be defined below in sec. 3.1, see (21)).
Therefore, without loss of generality, one can assume that either

- • K contains uniquely $d \geq 1$ integers 1, or else
- •• K contains one 3 and $d \geq 1$ integers 1, or else
- ••• K contains $d \geq 0$ integers 1 and $e \geq 2$ integers 3.

The alternative • is the easiest one and we can handle it right now. Suppose that such a term shows up in the polynomial expansion of F^{k+6+j} and look at the difference $w(f_{1_d}) - w(F^{k+6+j})$ which has to be non-negative, hence necessarily $d \geq k$, and which has to be a multiple of $w\big(X^{k+3}\big) = 2j-1$, that is to say,

$$(d-k+1)(2j+3) - 4 \;\equiv\; 0 \mod (2j-1)\,. \tag{19}$$

For $d_1 = k$ we get a first solution, yielding the associated-to-it value $\mu_1 = 1$. What are the subsequent solutions? For $d = k+1$ the LHS of (19) is congruent to $4 \cdot 1$, for $d = k+2$ — congruent to $4 \cdot 2$, and so on. Since 4 and $2j-1$ are coprime, not sooner than for $d_2 = k + 2j - 1$ we get the second solution, with the associated to it $\mu_2 = 2j+4$. Such value of μ is far too high, the highest theoretically conceivable power of X^{k+3} being $j+4$: it is only $\big(X^{k+3}\big)^1$ in F^{k+3}, it is at most $\big(X^{k+3}\big)^2$ in F^{k+4}, and so on, at most $\big(X^{k+3}\big)^{j+4}$ (or, only in terms contributing at 0, a lower power of X^{k+3}) in F^{k+6+j}. The subsequent solutions have still bigger μ's, and we thus end • with the unique possibility

$$d = k\,, \qquad \mu = 1\,, \qquad \nu = ?\,. \tag{20}$$

Before tackling (and, in the outcome, disposing of) the alternative ••• we need to introduce another valuation. Its necessity is felt already in (20): the valuation w is sensitive to X^{k+3}, but insensitive to X^{k+4+j}, attributing it the multiplicity zero. Because of that its weak point, at this moment we do not know the value(s) of ν in (20). Our next valuation v, not surprisingly, will be sensitive to X^{k+4+j} and insensitive to X^{k+3}.

3.1. Definition of the valuation v.

This new valuation is entirely abstract, or: algebraic; it was found much time after the valuation w. Together with w, it allows to catch the moduli of the geometric classes 'SG...GS' in a crossing fire.

$$v(x^1) = -2\,, \quad v(x^2) = -2k+1\,, \quad v(x^3) = -2k+3\,,$$

$$v(x^4) = -2k+5\,, \ \ldots\ldots, \ \ v(x^k) = -3, \quad v(x^{k+1}) = -1\,,$$

$$v(x^{k+2}) = -1,\ v\big(X^{k+3}\big) = 0,\ v(x^{k+4}) = 1,\ v(x^{k+5}) = 2\ \ \ldots\,,$$

$$v(x^{k+2+j}) = j-1\,, \quad v(x^{k+3+j}) = -j\,, \quad v\big(X^{k+4+j}\big) = -2j+1\,,$$

$$v(x^{k+5+j}) = -3j+2\,, \qquad v(x^{k+6+j}) = -4j+3\,.$$

With this valuation, the vector fields y, Y, $\widehat{Y}$ are quasihomogeneous of orders, respectively, 2, 1, and $-j+1$. Note also the fundamental property

$$v(x^2) \;=\; v(x^1) + v(x^3)\,, \tag{21}$$

as well as the fact that the analogue of Prop. 2 holds. That is, the v-abstract weights of the components of infinitesimal symmetries are well-defined. In particular, $v(F^{k+4+j}) = 2j-2k$, $v(F^{k+5+j}) = 3j-2k-1$,

$$v(F^{k+6+j}) \;=\; 4j-2k-2\,. \tag{22}$$

3.2. Dispensing with the alternatives ••• and •• – the end of proof of Lemma 2.

The alternative ••• becomes now tractable, modulo an elementary (if not completely trivial)

Observation 2. *Any term* $\big(X^{k+3}\big)^{\mu}\big(X^{k+4+j}\big)^{\nu} f_K$ *that contributes to the value* $F^{k+6+j}\,|\,0$ *has the exponent* $\nu < 3$.

Idea of proof. What is needed, is just to express the function F^{k+6+j} recursively back via F^{k+4+j} (the first component with the factor X^{k+4+j} present) and F^{k+2+j}, and then analyze carefully all the appearing terms, focusing on those contributing at 0.

Let us suppose now the presence in F^{k+6+j} of a term $\big(X^{k+3}\big)^{\mu}\big(X^{k+4+j}\big)^{\nu} f_{1_d\,3_e}$, with $d \geq 0$ and $e \geq 2$. This term has its v-abstract weight equal to (22),

$$-\,2d - 2ek + 3e - \nu(-2j+1) \;=\; 4j-2k-2\,,$$

or else

$$-2d - 2(e-1)k + 3e \;=\; (2-\nu)(2j-1)\,. \tag{23}$$

The RHS of this equation is, in view of Obs. 2, non-negative, whereas the LHS is negative, for $3e < 8(e-1) \le 2(e-1)k$. This contradiction shows that the alternative $\bullet\bullet\bullet$ is void.

Remark 2. The LHS of (23) is often negative also for $e=1$ (and not only for $e \ge 2$). It equals then $-2d+3$ and is negative for $d \ge 2$, while the RHS is always non-negative in virtue of Obs. 2.

In turn, passing to the alternative $\bullet\bullet$, suppose that such a term $\big(X^{k+3}\big)^{\mu}\big(X^{k+4+j}\big)^{\nu} f_{1_d\,3}$ is being present and look at the 'old' quantity $w(f_{1_d\,3}) - w(F^{k+6+j})$ which must be a non-negative multiple of $w\big(X^{k+3}\big) \;=\; 2j-1$. That is,

$$2 + (k+d-2)(2j+3) - 4 - (k-1)(2j+3) \;\equiv\; 0 \mod (2j-1)\,,$$

or else

$$(d-1)(2j+3) - 2 \;\equiv\; 0 \mod (2j-1)\,. \tag{24}$$

For $d=1$ the LHS of this congruence is -2. For $d=2$ it is congruent to $2\cdot 1$, for $d=3$ — congruent to $2\cdot 3$, for $d=4$ — to $2\cdot 5$, and so on. Since 2 and $2j-1$ are coprime, it is straightforward to see that the smallest natural integer solution of (24) is $d = j+1$; the LHS of (24) is then congruent to $2(2j-1)$, that is congruent to 0. In view of Rem. 2 above, this, and all the remaining (bigger) solutions of (24) lead to no contributing term in $F^{k+6+j}\,|\,0$. The alternative $\bullet\bullet$ is void.

Knowing that much, we are now in a position to find the unknown ν in (20). Indeed, the v-abstract weights say that, for a term $X^{k+3}\big(X^{k+4+j}\big)^{\nu} f_{1_k}$ to appear in the polynomial F^{k+6+j}, there must hold $4j-2k-2 \;=\; -2k - \nu(-2j+1)$, yielding $\nu = 2$. At long last, Lemma 2 is proved. $\square$

3.3. The end of proof of Theorem 1.

Now that the last component at 0 of the infinitesimal symmetries of the distribution E, (14), is known, we ask the same question for the specific D written down in (5) and, recalling, having the constant $b=0$. That is, differing from E only by the constant c standing in the last Pfaffian equation.
The situation is similar to those leading earlier to the formulas (15) and (17). To the value ascertained in Lem. 2 one should add a correction term involving only the partials $f_2\,|\,0$ and $f_{13}\,|\,0$. The computation of that correction is fully algorithmized and yields

$$F^{k+6+j}(f)\,|\,0 \;=\; (\cdot)f_{1_k} + c\Big((4j+5)f_2 + \big(4 + (k-1)(4j+5)\big)f_{13}\Big)\;|\,0\,. \tag{25}$$

Consider now any infinitesimal symmetry, of a *non*-local object given by the equations (5) now with $b=0$, such that its first $k+5+j$ components vanish at 0. That is, on the level of germs, stipulated is the preservation of all normalizations made until now save, hypothetically, the preservation of the value of c: the finite symmetries that emerge by integration preserve all but the last coordinate of the point $0 \in \mathbb{R}^{k+6+j}$. 'But the last' is in the hypothetical mode and we will instantly see that this possibility is void.

In fact, we consider the contact hamiltonians f such that all but the last components of $\mathcal{Y}_f\,|0$ vanish and look at the formulas (10) and (15). The RHS in the latter is zero and so is the coefficient standing next to ∂_{k+3} in the former. On top of that, the matrix of integer coefficients creating the mentioned expressions is invertible,

$$\begin{vmatrix} -2 & -(2k-1) \\ 2j+3 & 2+(k-1)(2j+3) \end{vmatrix} = 2j-1\,.$$

Hence $f_2\,|0 = f_{13}\,|0 = 0$. In view of (13), also $f_{1_k}\,|0 = 0$. By (25), therefore, $F^{k+6+j}(f)\,|0 = 0$. The preservation of all but the last coordinates of 0 implies the same for the last one – the preservation of the point 0 as such!

It is impossible to perturb the value of c in (5) by means of the *embeddable* symmetries of (5) understood as a finite object. The prolongation pattern at the last step No $k+4+j$, visualised in (4), is thus – out of the five alternatives of [6] – either 2c or 3, meaning the emergence of a modulus of the local classification. Theorem 1 is now proved. $\square$

§4. Conjectured precise modalities of the 'SG...GS' classes

In this section, in contradistinction to the preceding part, $j \geq 1$ (we merge now with the domain of [11]. That is, in the codes of codimension-two geometric classes of Goursat flags there stand $j \geq 1$ letters G in between the two letters S.) The classes with the two S just neighbouring are consequently not discussed in the present work.

We **conjecture** since long, and recently even more so in the light of partially related to this field works [14, 3, 15] of Ishikawa and Zhitomirskii, that [but see also Rem. 3 below]

⋆ the modality of all classes $\mathrm{GGSG}_j\mathrm{SG}_l$, $l \geq 1$, is zero — they are all simple;

⋆ the modality of classes $\mathrm{GGGSG}_j\mathrm{SG}_l$ is one from $l = 3$ onwards;

⋆ the modality of classes $\mathrm{GGGGSG}_j\mathrm{SG}_l$ is one for $l = 3, 4, 5, 6$, and is two from $l = 7$ onwards;

⋆ the modality of classes $\mathrm{GGGGGSG}_j\mathrm{SG}_l$ is one for $l = 3, 4, 5, 6$, is two for $l = 7, 8, 9, 10$, and is three from $l = 11$ onwards,

and so on.

Remark 3. Precisely speaking, not 100% of these statements is in the conjectural mode. Namely, these for $l = 3$, with at least three G's at the code' beginning, are proved: for $j = 1$ in [11] and for $j \geq 2$ in the present work.

The moduli exemplified in [11] are of type 3/[6], and the same we *suppose* for those of Thm. 1 related to $j \geq 2$. Yet, by the infinitesimal methods alone, it is not possible to distinguish between the module types 2c and 3 of [6] (cf. in this respect also Sec. 4 in [10]).

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Semidifférentiabilité et version lisse de la conjecture de fibration de Whitney

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Abstract.

For controlled stratified maps $f : \mathcal{X} \to \mathcal{X}'$ between two stratified spaces, we define what it means for f to be semi-differentiable, horizontally-C^1 and $\mathcal{F}$-semi-differentiable (where $\mathcal{F}$ is a foliation).

When $\mathcal{X}'$ is a smooth manifold, f is always semi-differentiable.

In general, semi-differentiability is equivalent to f being horizontally-C^1 with bounded differential.

Horizontally-C^1 regularity depends on the existence of (a)-regular horizontal stratified foliations of $\mathcal{X}$ and $\mathcal{X}'$, which gives a smooth version of the stratified fibration whose existence was conjectured by Whitney for analytic varieties in 1965, and implies a horizontally-C^1 version of Thom's first isotopy theorem.

We obtain finally the corresponding theorems for the finer property of $\mathcal{F}$-semi-differentiability.

§1. Introduction

Dans $[\mathrm{MT}]_{1,2}$, nous avons considéré le problème de l'extension continue contrôlée d'un champ de vecteurs, donné sur une (ou plusieurs) strate(s) d'une stratification régulière, à toutes les strates supérieures.

Le cas d'un relèvement (extension) contrôlé est classique, et il est bien connu [Ma], [GWPL] que les flots relevés sur une stratification au moins (b)-régulière, définissent à tout instant $t \in \mathbb{R}$ des homéomorphismes stratifiés qui sont lisses sur chaque strate, mais qui ne donnent pas en général une application C^1 (exemple de la famille des quatre droites:

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[Wh], [GWPL]). C'est la raison pour laquelle les principaux théorèmes de la théorie des stratifications régulières sont des résultats de nature topologique et non différentiable. Le *premier théorème d'isotopie de Thom*, donnant la stabilité topologique de la fibre d'une submersion stratifiée contrôlée $f : \mathcal{X} \to M$ à valeurs dans une variété, en est l'exemple le plus célèbre; il est obtenu par la technique "standard" de relèvement contrôlé de champs de vecteurs v_i qui donne des flots globaux ϕ_i qui ne sont pas toujours C^1.

Nous avons considéré dans $[\mathrm{MT}]_{1,2}$ des extensions de champs qui de plus sont *continues*; notre question devient alors: *"quel type de régularité (en plus de la continuité* [Ma]*) obtient-on pour les flots relevés ?"*.

Cet article est consacré à ce problème.

Quand on relève un champ ζ_X défini sur une strate X de dimension 1, les trajectoires du flot du champ relevé définissent un feuilletage "horizontal" de dimension 1, $\mathcal{F} = \{F_\beta\}$ de type $C^{0,\infty}$: i.e. des courbes lisses dont les espaces tangents, en coïncidant avec ceux de la distribution canonique $\mathcal{D}_X(y) = \{\mathcal{D}_{XY}(y)\}_{Y \geq X}$ $[\mathrm{MT}]_2$, tendent vers T_xX quand $y \to x$. En considérant alors un relèvement continu $\zeta = \{\zeta_Y\}_{Y \geq X}$ du champ ζ_X, comme nous le montrons au §4 (théorème 4 et corollaires), une régularité de type C^1-affaibli :

$$\text{“} \lim_{(y_n,v_n)\to(x,v)} \phi_{Y*y_n}(v_n) = \phi_{X*x}(v) \quad , \quad \forall\, Y > X \text{ ”}$$

reste valable au moins pour les suites de vecteurs $v_n \in \mathcal{D}_X(y_n)$.

Cette observation est la motivation cruciale de cet article où nous donnons des réponses aux questions suivantes :

– *Quand* $\dim X > 1$ *quel est l'analogue du feuilletage horizontal* $\mathcal{F} = \{F_\beta\}_\beta$ *?*

– *Peut-on obtenir que ses feuilles tendent vers T_xX de manière C^1 quand $y \to x$?*

– *Que peut-on dire de la différentiabilité du flot relevé (à l'instant t) $\phi_Y : Y \to Y$ ($Y > X$) le long de la "direction horizontale" ?*

Ce problème rappelle une conjecture de H. Whitney [Wh] pour des stratifications (b)-régulières d'une variété analytique. Whitney conjecture pour toute strate X et pour tout $x_0 \in X$ l'existence d'un feuilletage $\{F_\beta\}_\beta$ de même dimension que X vérifiant la condition $\lim_{z\to x} T_zF_\beta = T_xX$. Au §2.3, nous définissons cette propriété pour des stratifications réelles (définition 6), comme la (a)-*régularité (autour de x_0) du feuilletage horizontal* $\{F_\beta\}$.

Pour des stratifications réelles (b)-régulières, l'existence d'un tel "bon" feuilletage a été conjecturée par le deuxième auteur en 1993 ;

nous montrons au §5 qu'elle est une condition nécessaire et suffisante pour que les flots des champs relevés soient horizontalement-C^1, et plus généralement qu'elle est nécessaire pour que d'autres morphismes horizontalement-C^1 puissent exister. Nous réinterpretons alors la conjecture de Trotman comme la *version lisse de la conjecture de fibration de Whitney* ([Wh], §9, remarque 3). On peut voir $[\mathrm{MT}]_{1,3}$ pour plus de détails concernant les autres travaux sur les conjectures de Whitney et Trotman, et une indication de leurs multiples conséquences. Nous préparons actuellement (décembre 2006) avec A. du Plessis une étude approfondie de ces conjectures (lesquelles semblent bien être vraies).

Dans le §2, nous introduisons les notions de *semidifférentiabilité*, de régularité *horizontalement-C^1* et de $\mathcal{F}$-semidifférentiabilité pour un morphisme stratifié $f : \mathcal{X} \to \mathcal{X}'$; on précise les relations entre elles (théorème 1) et on montre que la réciproque d'un homéomorphisme horizontalement-C^1 est encore horizontalement-C^1 (théorème 2).

Une question importante reste ouverte :

– *Quand est-ce qu'un morphisme stratifié est horizontalement-C^1 ?*

Toute la théorie développée dans la suite est dédiée à ce problème. Dans le §3 nous analysons le cas particulier où le morphisme considéré est le flot stratifié $\phi = \{\phi_Y : Y \to Y\}_{Y \geq X}$ d'un champ de vecteurs $\zeta = \{\zeta_Y\}_{Y \geq X}$ obtenu par relèvement continu contrôlé d'un champ ζ_X défini sur une strate X.

Dans le §4.1 nous supposons l''involutivité d'une distribution canonique $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ relative à la strate X $[\mathrm{MT}]_{1,2}$ [1] et montrons que les flots (à l'instant t) $\phi = \{\phi_Y : Y \to Y\}_{Y \geq X}$ du champ ζ relevé continu contrôlé du champ ζ_X sont horizontalement-C^1 en tout point x_0 autour duquel $\mathcal{D}_X$ est involutive (théorèmes 3 et 4).

On remarque que si $\dim X \in \{1, \dim A - 1\}$, le flot ϕ est toujours horizontalement-C^1 en tout point de X (corollaires 4 et 5).

Dans le §4.2 on donne les résultats correspondants concernant la $\mathcal{F}$-semidifférentiabilité de $\phi = \{\phi_Y : Y \to Y\}_{Y \geq X}$ (théorèmes 5 et 6).

Dans le §4.3 nous donnons quelques caractérisations de l'involutivité de la distribution canonique $\mathcal{D}_X$ en termes du feuilletage $\mathcal{H}$ induit par la trivialisation topologique locale $H : \mathbb{R}^l \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0})$ d'une projection $\pi_X : T_X \to X$ (théorème 7).

Dans le §5 nous considérons le cas général où la distribution canonique n'est pas nécessairement intégrable. Dans le §5.1, après avoir énoncé

[1]D. Trotman et A. du Plessis ont vérifié en 1994 qu'en général une distribution canonique $\mathcal{D}_X$ n'est pas involutive. M. Field [Fi] avait remarqué cette difficulté à D. Trotman, sous une forme équivalente, dans une lettre de 1976.

la *version lisse de la conjecture de fibration de Whitney* (i.e. la conjecture du feuilletage (a)-régulier), nous remplaçons l'intégrabilité de $\mathcal{D}_X$ par l'hypothèse plus faible de (a)-régularité d'un feuilletage horizontal $\mathcal{H}$ (transverse aux fibres de la projection π_X) et qui équivaut à l'involutivité d'une nouvelle distribution canonique. On retrouve ainsi les mêmes théorèmes qu'au §4, et en particulier que les flots des champs relevés sont horizontalement-C^1 (théorème 8).

On donne ensuite des conditions suffisantes en termes de ces feuilletages pour qu'un morphisme stratifié arbitraire $f : \mathcal{X} \to \mathcal{X}'$ soit horizontalement-C^1 (théorème 9) et on conclut la section en énonçant une version horizontalement-C^1 du premier théorème d'isotopie de Thom (théorème 10). Dans le §5.2 on établit enfin les versions $\mathcal{F}$-semidifférentiable des théorèmes 8, 9 et 10 du §5.1 (théorèmes 11, 12, 13).

Les démonstrations des théorèmes 10 et 13 du §5 omises, seront publiées séparément.

§2. Régularités de morphismes stratifiés.

Dans ce paragraphe, nous introduisons et étudions différentes conditions de régularité pour un morphisme stratifié $f : \mathcal{X} \to \mathcal{X}'$.

On introduit d'abord la notion de *semidifférentiabilité*, qui dans le cas de deux strates $X < Y$ signifie que la différentielle $f_{X*} \cup f_{Y*} : TX \cup TY \to TX' \cup TY'$ d'un morphisme stratifié $f_X \cup f_Y : X \cup Y \to X' \cup Y'$ soit continue.

Le contrôle par rapport aux deux systèmes de données de contrôle (S.D.C.) de $\mathcal{X}$ et $\mathcal{X}'$ est supposé par définition[2], et implique qu'un morphisme stratifié $f : \mathcal{X} \to M$ à valeurs dans une variété lisse M est toujours semidifférentiable.

La semidifférentiabilité se préserve par composition, mais ne se préserve pas par application réciproque. Nous introduisons alors la notion de morphisme horizontalement-C^1 et démontrons que la semidifférentiabilité de f équivaut à ce que f soit horizontalement-C^1 et ait des dérivées bornées (théorème 1) et que la réciproque f^{-1} d'un homéomorphisme stratifié $f : A \to A'$ horizontalement-C^1 est horizontalement-C^1 (théorème 2).

On conclut la section en introduisant la notion plus fine de $\mathcal{F}$-semidifférentiabilité (le long un feuilletage $\mathcal{F}$) de f.

[2]On ne sait pas en général s'il est possible de construire deux S.D.C. par rapport auxquels $f : \mathcal{X} \to \mathcal{X}'$ soit contrôlée ; mais ceci est bien connu quand f est une submersion et $\mathcal{X}' = M$ est une variété [Ma].

2.1. Morphismes semidifférentiables.

Dans tout le paragraphe $\mathcal{X} = (A, \Sigma)$ et $\mathcal{X}' = (A', \Sigma')$ seront deux stratifications (a)-régulières de $A \subseteq \mathbb{R}^n$ et $A' \subseteq \mathbb{R}^m$ au sens de Whitney [Wh].

Une variété et/ou une application entre deux variétés sera dite *"lisse"* quand elle est de classe C^1.

Définition 1. *Un morphisme stratifié est une application continue $f : A \to A'$, stratifiée, i.e. qui envoie chaque strate X de A dans une unique strate X' de A' et telle que la restriction $f_X : X \to X'$ de f est lisse (non nécessairement une submersion).*

Grâce à la condition de frontière $X \subseteq \overline{Y}$ et la continuité de f, pour toutes strates $X < Y$ de A, $f(X) \subseteq f(\overline{Y}) \subseteq \overline{f(Y)}$. Donc les strates X' et Y' vérifient aussi $X' \subseteq \overline{Y'}$, i.e. $X' \leq Y'$.

Les stratifications $\mathcal{X}$ et $\mathcal{X}'$ de cette section seront munies de deux systèmes de données de contrôle $\mathcal{T} = \{(T_X, \pi_X, \rho_X)\}_{X \in \Sigma}$ et $\mathcal{T}' = \{(T_{X'}, \pi_{X'}, \rho_{X'})\}_{X' \in \Sigma'}$ [Ma] et tout morphisme stratifié f sera *contrôlé par rapport aux systèmes $\mathcal{T}$ et $\mathcal{T}'$*, i.e. :

$$\forall\, X < Y \;,\; \exists\, \epsilon > 0 \quad \text{tel que :}$$

$$\forall\, y \in T^{\epsilon}_{XY} = T_X(\epsilon) \cap Y \text{ on ait : } \begin{cases} \pi_{X'} f_Y(y) = f_X \pi_X(y) \\ \rho_{X'} f_Y(y)) = \rho_X(y). \end{cases}$$

Si la stratification est seulement (a)-régulière et le système $\mathcal{T}$ se réduit à la famille des projections, alors la condition de contrôle devient $\pi_{X'} f_Y(y) = f_X \pi_X(y)$ ([MT]$_2$, §3 remarque 1). Remarquons que $\forall\, X < Y$, la condition de frontière et la (a)-régularité de $\mathcal{X}$ et $\mathcal{X}'$ impliquent que $TX \subseteq \overline{TY}$ et $TX' \subseteq \overline{TY'}$ dans $\mathbb{R}^{2n}$ et $\mathbb{R}^{2m}$.

Définition 2. *Un morphisme stratifié $f : \mathcal{X} \to \mathcal{X}'$ sera dit semidifférentiable en $x \in X$ (resp. en $(x, v) \in TX$), si pour toute strate $Y > X$ (et donc $Y' \geq X'$) l'application différentielle $f_{X*} \cup f_{Y*} : TX \cup TY \to TX' \cup TY'$ est continue en tout point $(x, v) \in \{x\} \times T_x X \subseteq TX$ (resp. en (x, v)), i.e. la condition de limite est vérifiée :*

$$\forall\, \{(y_n, v_n)\}_n \subseteq TY\,,$$
$$\lim_{n\to\infty} (y_n, v_n) = (x, v) \in TX \;\;\Rightarrow\;\; \lim_{n\to\infty} f_{Y*\,y_n}(v_n) = f_{X*\,x}(v).$$

On dira que f *est semidifférentiable sur* X si f est semidifférentiable en tout $x \in X$ et que f *est semidifférentiable* (*sur* $\mathcal{X}$) si f est semidifférentiable sur toute strate $X \in \Sigma$.

Des exemples élémentaires montrent que la semidifférentiabilité en un point x (sur une strate X ou sur A tout entier) d'un morphisme stratifié $f : \mathcal{X} \to \mathcal{X}'$ est en général plus faible que la C^1-régularité de f en x (sur X ou sur A). Si la stratification $\mathcal{X}'$ se réduit à une variété lisse, la semidifférentiabilité découle de la condition de contrôle :

Proposition 1. *Toute application $f : \mathcal{X} \to M$ à valeurs dans une variété M et π-contrôlée par rapport à la famille des projections d'un S.D.C. est semidifférentiable.*

Preuve. Pour tout couple de strates adjacentes $X < Y$ de $\mathcal{X}$, soit $\{(y_n, v_n)\}_n$ une suite dans TY telle que $\lim_n(y_n, v_n) = (x, v) \in TX$.

La stratification but de $f : \mathcal{X} \to M$, se réduisant à une variété $\mathcal{X}' = (M, \{M\})$, la condition de π-contrôle se réduit à $f_Y = f_X \circ \pi_{XY}$. Donc $f_{Y*} = f_{X*} \circ \pi_{XY*}$ et la relation $\lim_n f_{Y*y_n}(v_n) = f_{X*x}(v)$ découle immédiatement de la propriété que

$$\lim_n f_{X*x_n}(\pi_{XY*y_n}(v_n)) = f_{X*x}(v) \qquad \text{où} \quad x_n = \pi_{XY}(y_n)$$

valable pour les applications lisses $f_X : X \to M$ et $\pi_{XY} : T_{XY} \to X$. Q.E.D.

Les stratifications (b) et (c)-régulières admettent toujours des S.D.C. dont les projections et les fonctions distance sont lisses [Ma], [Be]. Cependant quand une projection $\pi_X : T_X \to X$ n'est pas C^1, comme elle est π-contrôleé par définition de S.D.C., on a :

Remarque 1. *Toute projection $\pi_X : T_X \to X$ du S.D.C. est semidifférentiable sur X.*

Les remarques 2 et 3 qui suivent sont élémentaires :

Remarque 2. *Si $f : \mathcal{X} \to \mathcal{X}'$ est une application semidifférentiable il existe une application "différentielle" f_* de f, continue sur le "fibré tangent généralisé" $T\mathcal{X} = \cup_{X\in\Sigma} TX$ à la stratification $\mathcal{X}$:*

$$f_* = \cup_X f_{X*} \ : \ T\mathcal{X} = \cup_{X\in\Sigma} TX \ \longrightarrow \ T\mathcal{X}' = \cup_{X'\in\Sigma'} TX' \ .$$

Remarque 3. *Si $f : \mathcal{X} \to \mathcal{X}'$ et $g : \mathcal{X}' \to \mathcal{X}''$ sont semidifférentiables alors $g \circ f : \mathcal{X} \to \mathcal{X}''$ l'est aussi.*

La proposition 3 anticipe des observations sur la convergence de f_* le long des directions des fibres des projections du S.D.C. qu'on développera à la section 2.2.

Proposition 2. *Pour qu'un morphisme stratifié $f : \mathcal{X} \to \mathcal{X}'$ soit semidifférentiable en un point $x \in X$ il faut que pour toute strate $Y > X$ la restriction $f_{Y*y|\ker \pi_{XY*y}} : \ker \pi_{XY*y} \to \ker \pi_{X'Y'*y'}$ soit de norme*

$$g_{XY}(y) := ||f_{Y*y|\ker \pi_{XY*y}}|| \quad \textit{bornée autour de } x \textit{ dans } X \cup Y.$$

Preuve. S'il existe une suite $\{y_n\} \subseteq Y$ convergente vers $x \in X$, telle que la suite $||f_{Y*y_n|\ker \pi_{XY*y_n}}||$ soit divergente, il existe alors une suite de vecteurs $v_n \in \ker \pi_{XY*y_n} = T_{y_n}\pi^{-1}_{XY}(x_n) \subseteq T_{y_n}Y$ $(x_n = \pi_{XY}(y_n))$ telle que :

i) $||f_{Y*y_n|\ker \pi_{XY*y}}(v_n)|| = ||f_{Y*y_n|\ker \pi_{XY*y_n}}|| \cdot ||v_n||$;
ii) $||v_n|| = ||f_{Y*y_n|\ker \pi_{XY*y}}||^{-\frac{1}{2}}$.

Comme $\{||f_{Y*y_n|\ker \pi_{XY*y_n}}||\}_n$ est une suite divergente, de *ii)* on a $\lim_n v_n = 0 \in T_xX$ et $\lim_n(y_n, v_n) = (x, 0) \in TX$.

Or $\lim_n ||v_n|| = 0$, mais grâce à *i)* et *ii)* on trouve :

$$\begin{aligned} ||f_{Y*y_n|\ker \pi_{XY*y}}(v_n)|| &= ||f_{Y*y_n|\ker \pi_{XY*y_n}}|| \cdot ||v_n|| \\ &= ||f_{Y*y_n|\ker \pi_{XY*y}}||^{\frac{1}{2}} \to \infty \end{aligned}$$

ce qui implique que f n'est pas semidifférentiable en x. Q.E.D.

De la proposition 2, par des exemples élémentaires on a facilement :

Remarque 4. *L'application réciproque $f^{-1} : \mathcal{X}' \to \mathcal{X}$ d'un homéomorphisme semidifférentiable $f : \mathcal{X} \to \mathcal{X}'$ n'est pas en général semidifférentiable.*

2.2. Morphismes horizontalement-C^1.

La proposition 2 suggère de séparer l'analyse de la convergence le long de “la direction verticale” (celle du sous-fibré $\ker \pi_{X*}$) et celle le long d'une “direction horizontale”.

Pour une stratification (c)-régulière [Be], les auteurs du présent article ont montré le théorème suivant ($[\mathrm{MT}]_2$ théorème 4, §5) et donné la définition qui suit (voir aussi $[\mathrm{MT}]_1$, théorèmes 1 et 2) :

Théorème. *Soient $\mathcal{X}$ un espace stratifié (c)-régulier dans une variété C^1 M et $\mathcal{F} = \{(\pi_X, \rho_X) : T_X \to X \times [0, \infty[\,\}_{X \in \Sigma}$ un système de données de contrôle de $\mathcal{X}$.*

Pour toute strate X de $\mathcal{X}$, il existe une distribution stratifiée continue $\mathcal{D}_X : T^\epsilon_X \to \mathbb{G}^l_n$ où $l = \dim X$, $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ et $\forall Y \geq X$, $\mathcal{D}_{XY} = \mathcal{D}_{X|T_{XY}}$ avec $T_{XY} = T_X \cap Y$, telle que :

i) $\forall Y \geq X$ *la restriction* $\mathcal{D}_{XY}$ *est un sous-fibré de* $\ker \rho_{XY*}$*;*

ii) $\mathcal{D}_{XX}(x) = T_x X$, $\forall x \in X$*;*

iii) $T_y Y = \mathcal{D}_{XY}(y) \oplus \ker \pi_{XY*y}$ *est une somme directe* $\forall\, y \in T^{\epsilon}_{XY}$*;*

iv) la restriction $\pi_{XY*y|} : \mathcal{D}_{XY}(y) \to T_x X$ *où* $x = \pi_{XY}(y)$ *est un isomorphisme;*

v) pour tout champ de vecteurs C^1 ξ_X *sur* X*,la formule :*

$$\xi_Y(y) = \mathcal{D}_{XY}(y) \cap \pi^{-1}_{XY*y}(\xi_X(x)) \quad , \quad x = \pi_X(y)$$

définit un relèvement stratifié continu contrôlé $\xi = \{\xi_Y\}_{Y \geq X}$ *de* ξ_X *sur* $T^{\epsilon}_X = \cup_{Y \geq X} T^{\epsilon}_{XY}$.

Définition. *Toute distribution* $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ *vérifiant les conditions* $i), \ldots, v)$ *dans le théorème précédent est dite une distribution canonique (relative a la strate* X*).*

Définition 3. *Un morphisme stratifié* $f : \mathcal{X} \to \mathcal{X}'$ *sera dit horizontalement-*C^1 *en un point* x *d'une strate* $X \in \Sigma$ *s'il existe une distribution canonique locale* $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ *autour de* x *dans* A *(*[MT]$_{1,2}$, [Mu]*) telle que :*

$$\lim_{(y_n, v_n) \to (x, v)} f_{Y*y_n}(v_n) = f_{X*x}(v) \quad , \quad \forall\ (x, v) \in TX$$

soit vérifiée pour chaque suite $\{(y_n, v_n)\} \subseteq Y$ *convergente vers* (x, v) *avec* $v_n \in \mathcal{D}_{XY}(y_n)$*. On appellera horizontaux les vecteurs* $v_n \in \mathcal{D}_X(y_n)$ *et verticaux les* $v_n \in \ker \pi_{XY*y_n}$*.*

Un morphisme stratifié f est donc horizontalement-C^1 en $x \in X$ ssi $\forall\, Y > X$

$$f_{X*} \cup f_{Y*|\mathcal{D}_{XY}} : TX \cup \mathcal{D}_{XY} \to TX' \cup TY'$$

est continue en x.

On dira que f est *horizontalement-*C^1 *sur une strate* X (resp. *sur* (A, Σ)) si f est horizontalement-C^1 en tout point $x \in X$ (resp. $\forall x \in X$ et $\forall\, X \in \Sigma$).

Remarque 5. *Tout morphisme stratifié semidifférentiable* f *est horizontalement-*C^1*.*

Théorème 1. *Un morphisme stratifié contrôlé* $f : \mathcal{X} \to \mathcal{X}'$ *est semidifférentiable en un point* $x \in X$ *si et seulement s'il est horizontalement-*C^1 *en* x *et s'il existe un voisinage* U_x *de* x *dans* A *tel que pour toute strate* $Y > X$*, la fonction* $g_{XY}(y) = ||f_{Y*y|\ker \pi_{XY*y}}||$ *est bornée dans* $U_x \cap Y$*.*

Preuve. La remarque 5 et la proposition 2 montrent l'implication *"seulement si"*.

Pour toute strate $Y > X$ et pour toute suite $\{(y_n, v_n)\} \subseteq TY$ telle que $\lim_n (y_n, v_n) = (x, v) \in TX$, décomposons [MT]$_{1,2}$ tout vecteur $v_n \in T_{y_n} Y = \mathcal{D}_{XY}(y_n) \oplus \ker \pi_{XY*y_n}$ en une somme directe $v_n = v_n^h + v_n^v$ de ses composantes horizontale et verticale de sorte que:

$$\lim_{n\to\infty} v_n^h = v \in T_x X \quad \text{et} \quad \lim_{n\to\infty} v_n^v = 0 \in T_x X\,.$$

Comme f est horizontalement-C^1 en x, on a $\lim_n f_{Y*y_n}(v_n^h) = f_{X*_x}(v)$.

D'autre part, comme les différentielles le long des fibres des projections sont bornées autour de x, on peut écrire

$$0 \le ||f_{Y*y_n}(v_n^v)|| \le ||f_{Y*y_n | \ker \pi_{XY*y_n}}|| \cdot ||v_n^v|| \le M \cdot ||v_n^v||$$

et en déduire que $\lim_n f_{Y*y_n}(v_n^v) = 0$. En décomposant via f_* les images $f_{Y*y_n}(v_n)$ nous concluons que f est semidifférentiable en x :

$$\begin{aligned}\lim_{n\to\infty} f_{Y*y_n}(v_n) &= \lim_{n\to\infty} f_{Y*y_n}(v_n^h) + \lim_{n\to\infty} f_{Y*y_n}(v_n^v)\\ &= f_{X*_x}(v) + 0 = f_{X*_x}(v)\,.\end{aligned}$$

Q.E.D.

Remarque 6. *Une projection π_X d'un S.D.C., étant toujours semidifférentiable sur X, est également horizontalement-C^1 sur X.*

Quand $\mathcal{X}$ est (c)-régulière sur X au sens de K. Bekka [Be] on a aussi facilement :

Proposition 3. *Soit $\mathcal{X}$ une stratification (c)-régulière munie d'un S.D.C. $\mathcal{T} = \{(\pi_X, \rho_X) : T_X \to X \times [0, \infty[\,\}_{X\in\Sigma}$, tel que chaque voisinage tubulaire T_X est muni de la stratification induite de Σ et $[0,\infty[$ est stratifié par $\{0\}\cup]0,\infty[$. Alors toute fonction distance $\rho_X : T_X \to \{0\}\cup]0,\infty[$ est horizontalement-C^1 sur X.*

Preuve. Soit $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y\ge X}$ une distribution canonique obtenue à partir du S.D.C. $\mathcal{T}$ auquel ρ_X appartient.

Rappelons que $\forall Y > X$, $\mathcal{D}_{XY}$ est par définition un sous-fibré vectoriel de $\ker \rho_{XY*}$.

Etant donnée une suite $\{(y_n, v_n)\} \subseteq TY$ telle que $\lim_n (y_n, v_n) = (x, v) \in TX$ et dont les vecteurs v_n sont horizontaux $v_n \in \mathcal{D}_{XY}(y_n)$, à partir de l'inclusion $\mathcal{D}_{XY}(y_n) \subseteq \ker \rho_{XY*y_n}$ on trouve que $v_n \in \ker \rho_{XY*y_n}$, donc $\lim_n \rho_{XY*y_n}(v_n) = 0 = \rho_{XX*x}(v)$. Q.E.D.

Remarque 7. *Une fonction* $\rho_X : T_X \to \{0\}\cup]0,\infty[$ *est semidifférentiable sur* X *ssi* $\forall\, Y > X$ *la fonction* $g_{XY}(y) = ||\rho_{Y*y|\ker \pi_{XY*y}}||$ *est bornée.*

Définition 4. *Un homéomorphisme stratifié est un morphisme stratifié contrôlé* $f : \mathcal{X} = (A,\Sigma) \to \mathcal{X}\,' = (A,\Sigma)$ *tel que* $f : A \to A'$ *soit un homéomorphisme et* $\forall\, X \in \Sigma$ *la restriction* $f_X : X \to X'$ *soit un difféomorphisme.*

Dans ce cas f induit sur $\mathcal{X}\,'$ un S.D.C. $\mathcal{T}' = f_*(\mathcal{T})$ (§3, [MTP]) pour lequel $f^{-1} : \mathcal{X}\,' \to \mathcal{X}$ est un morphisme stratifié contrôlé.

Théorème 2. *Si* $f : \mathcal{X} \to \mathcal{X}\,'$ *est un homéomorphisme stratifié horizontalement-*C^1 *sur un voisinage* U *de* x *dans* X*, alors* $\forall\, z' \in f(U)$ *tel que* $g(z') = ||f_{*z'}^{-1}||$ *soit bornée dans un voisinage* $V_{z'}$ *de* z' *dans* A'*, l'homéomorphisme* f^{-1} *est horizontalement-*C^1 *en* z'*.*

La preuve du théorème 2 réside dans la proposition ci-dessous :

Proposition 4. *Si* $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ *est une distribution canonique définie dans un voisinage* W *de* x *dans* A *et* $f : \mathcal{X} \to \mathcal{X}\,'$ *est un homéomorphisme stratifié horizontalement-*C^1 *par rapport à* $\mathcal{D}_X$ *sur un voisinage* $U \subseteq W \cap X$ *de* x *dans* X*, alors la distribution* $\mathcal{D}_{X'} = \{\mathcal{D}_{X'Y'}\}_{Y' \geq X'}$ *définie dans le voisinage* $W' = f(W)$ *de* $x' = f(x) \in X'$ *dans* A'*, par* $\mathcal{D}_{X'Y'}(y') = f_{Y*y}(\mathcal{D}_{XY}(y))\ ,\forall\, y' = f(y),$ *est une distribution canonique locale dans le voisinage* W' *de* x'*.*

Preuve. Comme f est un homéomorphisme stratifié contrôlé, il est immédiat que :

i) $\forall Y' \geq X'$, $\mathcal{D}_{X'Y'}$ est un sous-fibré de $\ker \rho_{X'Y'*}$ de même dimension que TX;

ii) $\mathcal{D}_{X'X'}(z') = T_{z'}X'$ pour tout $z' \in U' \subseteq X'$;

iii) $T_{y'}Y' = \mathcal{D}_{X'Y'}(y) \oplus \ker \rho_{X'Y'*y'}$ est une somme directe $\forall y' \in Y'$ dans $V' = f(V)$;

iv) $\pi_{X'Y'*y'|} : \mathcal{D}_{X'Y'}(y') \to T_{z'}X'$ où $z' = \pi_{X'Y'}(y')$ est un isomorphisme;

v) pour tout champ de vecteurs $\xi_{X'}$ sur X', la formule

$$\xi_{Y'}(y') := \mathcal{D}_{X'Y'}(y') \cap \pi^{-1}_{X'Y'*y'}(\xi_{X'}(z'))\ , \quad \text{où } z = \pi_{X'}(y')\,,$$

définit un relèvement $\xi' = \{\xi_{Y'}\}_{Y' \geq X'}$ de $\xi_{X'}$ contrôlé par rapport à $\mathcal{T}'$.

Afin de montrer que $\mathcal{D}_{X'}$ est une distribution canonique autour de x', soit $\{y'_n\} \subseteq W' = f(W')$ telle que $\lim_n y'_n = z'$ et montrons que $\lim_n \mathcal{D}_{X'}(y'_n) = T_{z'}X'$.

L'application f étant bijective on peut écrire $y'_n = f(y_n)$ avec $\lim_n y_n = z$, $T_{z'}X' = f_{X*z}(T_zX)$, et $v' = f_{X*z}(v)$ avec $v \in T_zX$.

Comme $\mathcal{D}_X$ est une distribution canonique et $\lim_n y_n = z$, il existe une suite $v_n \in \mathcal{D}_{XY}(y_n)$ telle que $\lim_n v_n = v$ et d'autre part, f étant horizontalement-C^1 en z', on trouve $\lim_{y_n \to z'} f_{Y*y_n}(v_n) = f_{X*z'}(v) = v'$.

Alors la suite de vecteurs $v'_n = f_{Y*y_n}(v_n) \in f_{Y*y_n}(\mathcal{D}_{XY}(y_n)) = \mathcal{D}_{X'Y'}(y'_n)$ vérifie $\lim_n v'_n = v'$, donc $T_{z'}X' = \lim_n \mathcal{D}_{X'Y'}(y'_n)$. Q.E.D.

Preuve (du théorème 2). Soit $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ la distribution canonique locale par rapport à laquelle f est horizontalement-C^1 en x. Considérons autour de $x' = f(x)$ la distribution canonique locale $\mathcal{D}_{X'} = \{\mathcal{D}_{X'Y'}(y) = f_{Y*y}(\mathcal{D}_{XY}(y))\}_{Y' \geq X'}$ et montrons que f^{-1} est horizontalement-C^1 autour de x' par rapport à $\mathcal{D}_{X'}$.

Fixons $z' \in U'$, $v' \in T_{z'}X'$, une strate $Y' > X'$ et une suite $\{(y'_n, v'_n)\}$ dans Y' telle que $\lim_n (y'_n, v'_n) = (z', v')$ avec $v'_n \in \mathcal{D}_{X'Y'}(y'_n)$.

Comme f est un homéomorphisme stratifié, nous pouvons écrire :

$$z' = f(z), \qquad X' = f(X), \qquad Y' = f(Y)$$

$$z \in X, \qquad X, Y \in \Sigma, \qquad Y > X$$

$$v' = f_{X*z}(v), \quad y'_n = f(y_n), \quad v'_n = f_{Y*y_n}(v_n)$$

$$v \in T_zX, \qquad y_n \in Y, \qquad v_n \in \mathcal{D}_{XY}(y_n)$$

de sorte que la condition *"f^{-1} est horizontalement-C^1 en z'"* devient

$$\lim_{n\to\infty} f^{-1}_{Y'*y'_n}(v'_n) = f^{-1}_{X'*z'}(v') \quad \Leftrightarrow \quad \lim_{n\to\infty} v_n = v\,.$$

Comme $f^{-1}_{Y'*}$ est bornée dans $V_{z'}$, la suite $\{v_n\}$ est également bornée

$$||v_n|| \leq ||f^{-1}_{Y'*y'_n}(v'_n)|| \leq ||f^{-1}_{Y'*y'_n}|| \cdot ||v'_n|| \leq M \cdot (||v'|| + 1)\,.$$

Montrons alors que $\{v_n\}$ s'accumule sur un point unique $v = f^{-1}_{X'*z'}(v')$. En fait, $\forall u \in T_zX$ limite d'une sous-suite convergente $\{v_{n_h}\}_h$ de $\{v_n\}$, comme f est horizontalement-C^1 en z, on déduit de l'égalité $u = \lim_h v_{n_h}$ que :

$$f_{X*z}(u) = \lim_{h\to\infty} f_{Y*y_h}(v_{n_h}) = \lim_{h\to\infty} v'_{n_h} = v'$$

et donc que $u = f^{-1}_{X'*z'}(v') = v$.

On conclut alors que $\{v_n\}$ a un unique point d'accumulation v, donc $\lim_n v_n = v$ et f^{-1} est horizontalement-C^1 en $z' \in U' = f(U)$. Q.E.D.

2.3. Morphismes $\mathcal{F}$-semidifférentiables.

Nous introduisons maintenant pour des morphismes stratifiés une notion de régularité, associée à un feuilletage, plus fine que la semi-différentiabilité et que la régularité horizontalement-C^1, et qui généralise ces notions et la condition (a_f) de Thom.

Nous noterons $\mathcal{F} = \{F_\beta\}_\beta$ un feuilletage en sous-variétés F_β lisses de dimension h ($\leq \dim \mathcal{X}$) d'un sous-ensemble ouvert U du support A de $\mathcal{X} = (A, \Sigma)$ et on supposera U stratifié par la stratification naturelle $\Sigma_U = \{Y \cap U\}_{Y \in \Sigma}$ induite par Σ sur U. Enfin, $\forall\, y \in U$, F_y notera la feuille de $\mathcal{F}$ qui passe par y ; on a alors $\mathcal{F} = \{F_y\}_{y \in U}$.

Définition 5. *On dit que $\mathcal{F}$ est un feuilletage stratifié compatible avec Σ (ou avec Σ_U) si toute feuille F_y de $\mathcal{F}$ est contenue dans la strate Y de Σ qui contient y.*

Les feuilletages que nous considérons seront toujours compatibles avec la stratification Σ fixée au départ et on parlera donc simplement de *"feuilletage stratifié"*.

Un feuilletage stratifié $\mathcal{F}$ de U détermine alors une famille de feuilletages $\{\mathcal{F}_Y\}_{Y \in \Sigma}$ où $\forall\, Y \in \Sigma$, $\mathcal{F}_Y = \{F_y\}_{y \in U \cap Y}$ est l'ensemble des feuilles contenues dans la strate Y. $\mathcal{F}$ est dit *de classe* $C^{0,1}$ si pour toute Y le feuilletage $\mathcal{F}_Y$ est de classe $C^{0,1}$ (les variétés C^1 F_y, "varient de manière C^0"). On considérera toujours des feuilletages de classe $C^{0,1}$.

Soient maintenant x un point de U et $X \in \Sigma$ la strate contenant x.

Définition 6. *Le feuilletage stratifié $\mathcal{F}$ sera dit (a)-régulier en x si pour toute suite $\{y_n\} \subseteq U$ telle que $\lim_n y_n = x$ et telle que la limite $\lim_n T_{y_n} F_{y_n}$ existe on a :*

$$\lim_n T_{y_n} F_{y_n} = \tau \in \mathbb{G}_h^n \quad \Longrightarrow \quad \begin{cases} \tau \subset T_x X & si \quad h < \dim X \\ \tau = T_x X & si \quad h = \dim X \\ \tau \supset T_x X & si \quad h > \dim X. \end{cases}$$

Le feuilletage $\mathcal{F}$ sera dit (a)-régulier sur X s'il est (a)-régulier en tout point $x \in X$. Enfin $\mathcal{F}$ sera dit (a)-régulier quand il est (a)-régulier sur toute strate X.

Remarque 8. *Si $\dim \mathcal{F} = \dim X = h$, $\mathcal{F}$ est (a)-régulier en $x \in X$ (resp. sur X) si et seulement si (resp. $\forall x \in X$) $\lim_{y \to x} T_y F_y = T_x X$.*

Dans le cas d'un feuilletage induit par une submersion stratifiée, la notion de feuilletage (a)-régulier généralise la condition (a_f) de Thom :

Remarque 9. *Une submersion stratifiée surjective, $f : \mathcal{X} \to \mathcal{X}'$, de corang constant h, vérifie la condition (a_f) de Thom en un point x d'une strate X si et seulement si $\forall Y > X$ le feuilletage stratifié $\mathcal{F}(f) = \{f^{-1}(a')\}_{a' \in A'}$ est (a)-régulier en x.*

Preuve. Comme f est une submersion stratifiée de corang constant h, alors $\mathcal{F}(f) = \{f^{-1}(a')\}_{a' \in A'}$ est un feuilletage stratifié de variétés lisses de dimension h.

Donc la condition (a_f) de Thom en $x \in X$ est valable si et seulement si la limite $\lim_{y \to x} T_y f^{-1}(y') = T_x X$, i.e. si $\mathcal{F}(f)$ est (a)-régulier en x. Q.E.D.

Définition 7. *Le fibré tangent $T\mathcal{F}$ à un feuilletage stratifié $\mathcal{F}$ est le h-sous-fibré de TU des vecteurs tangents aux feuilles de $\mathcal{F}$, i.e. :*

$$T\mathcal{F} = \cup_{y \in U} \{y\} \times T_y F_y \ .$$

Soit $\mathcal{F}$ un feuilletage stratifié (a)-régulier d'un voisinage ouvert U d'un point x d'une strate X et soit $f : \mathcal{X} \to \mathcal{X}'$ un morphisme stratifié.

Définition 8. *On dira que f est $\mathcal{F}$-semidifférentiable en x si, pour chaque $(x, v) \in \{x\} \times T_x X \subseteq T(X \cap U)$ et $\forall \{(y_n, v_n)\}_n \subseteq T\mathcal{F}$ telle que* $\lim_n (y_n, v_n) = (x, v)$, *on a aussi* $\lim_n f_{Y_n * y_n}(v_n) = f_{X * x}(v)$ *où, $\forall n \in \mathbb{N}$, Y_n désigne la strate de Σ_U qui contient y_n. De façon évidente on définit la $\mathcal{F}$-semidifférentiabilité sur X (ou sur $X \cap U$) et sur $\mathcal{X}$.*

La $\mathcal{F}$-semidifférentiabilité généralise la semidifférentiabilité et la régularité horizontalement-C^1. Si on note $\forall Y \in \Sigma$, $\{Y\}$ le feuilletage trivial de la strate Y on a évidemment:

Remarque 10. *La stratification Σ est (a)-régulière (en $x \in U \cap X$) si et seulement si chaque feuilletage trivial $\{U \cap Y\}$ est (a)-régulier (en $x \in U \cap X$).*

Proposition 5. *Un morphisme stratifié $f : \mathcal{X} \to \mathcal{X}'$ est semidifférentiable (en $x \in X$) si et seulement si pour toute strate $Y > X$ il est $\{Y\}$-différentiable (en x).*

Preuve. Si $\mathcal{F} = \{Y\}$ alors $T\mathcal{F} = \cup_{y \in Y} \{y\} \times T_y Y = TY$. Q.E.D.

Soient U un voisinage ouvert de $x \in X$ dans A, $\mathcal{F} = \{F_y\}_{y \in U}$ un feuilletage (a)-régulier de dimension $\dim F_y = \dim X$ transverse aux fibres de la projection π_X du S.D.C. de $\mathcal{X}$ et $\mathcal{D}(\mathcal{F}) = \{\mathcal{D}_{XY}\}_{Y \geq X}$ la distribution d'espaces tangents aux feuilles de $\mathcal{F}$.

Proposition 6. *Un morphisme stratifié $f : \mathcal{X} \to \mathcal{X}'$ est horizontalement-C^1 en x par rapport à $\mathcal{D}(\mathcal{F}) = \{\mathcal{D}_{XY}\}_{Y \geq X}$, si et seulement s'il est $\mathcal{F}$-semidifférentiable en x.*

Preuve. Si $\dim \mathcal{F} = \dim X$ et si $\mathcal{F}$ est (a)-régulier alors la distribution $\mathcal{D}(\mathcal{F}) := \{\mathcal{D}_{XY}\}_{Y \geq X}$ définie $\forall Y > X$ par $\mathcal{D}_{XY}(y) := T_y F_y$ est une distribution canonique sur U et le sous-fibré $\cup_{y \in U}\{y\} \times \mathcal{D}_X(y)$ coïncide avec l'espace tangent $T\mathcal{F}$ au feuilletage $\mathcal{F}$. Q.E.D.

§3. Le cas du flot d'un champ relevé.

A partir de ce paragraphe, nous étudierons la régularité d'un flot continu contrôlé obtenu par relèvement aux strates supérieures d'un champ de vecteurs défini sur un squelette de la stratification A.

Soient $\mathcal{X}$ une stratification (c)-régulière d'un sous-ensemble $A \subseteq \mathbb{R}^n$, ζ_X un champ de vecteurs sur une strate X de $\mathcal{X}$ ayant un flot global Φ_X, $\zeta = \{\zeta_Y\}_{Y \geq X}$ le relèvement continu contrôlé de ζ_X [MT]$_{1,2}$ sur un voisinage stratifié T_X et $\Phi = \{\Phi_Y\}_{Y \geq X}$ son flot.

Il est bien connu [Ma] que la seule hypothèse de *contrôle* de $\zeta = \{\zeta_Y\}$ suffit pour que $\Phi : \mathbb{R} \times T_X \to T_X$ soit un prolongement continu de $\Phi_X : \mathbb{R} \times X \to X$.

Remarque 11. $\Phi : \mathbb{R} \times T_X \to T_X$ *est C^1 par rapport à $t \in \mathbb{R}$.*

Preuve. ζ étant continu sur X, $\frac{\partial}{\partial t}\Phi = \zeta \circ \Phi$ l'est aussi. Q.E.D.

Considérons le relèvement contrôlé ζ qui soit de plus *continu* [MT]$_2$. Quelle amélioration de la régularité de $\Phi_* = \cup_{Y \geq X} \Phi_{Y*}$ a-t-on par rapport aux variables autres que $t \in \mathbb{R}$?

Fixons une strate X de $\mathcal{X}$ et le voisinage tubulaire T_X stratifié par la stratification $\Sigma_{T_X} = \{T_{XY}\}_{Y \geq X}$ induite de Σ. Notons, pour simplifier les notations, $T_X \equiv A$, $\Sigma_{T_X} \equiv \Sigma$ et donc tout $T_{XY} \equiv Y$. Fixons $t \in \mathbb{R}$.

L'homéomorphisme stratifié: $\Phi_t = \cup_{Y \geq X} \Phi_{Y\,t} : T_X \to T_X$ a pour restriction à $Y \geq X$ une application lisse $\phi_Y = \Phi_{Yt} : Y \to Y$ ($Y \equiv T_{XY}$) qui *a priori*, se prolonge de manière seulement continue sur le *lieu singulier* $X \subseteq \overline{Y}$.

Soit $\{y_n\}_n \subseteq Y$ une suite telle que $\lim_n y_n = x \in X$ et $\lim_n T_{y_n} Y = \tau$. On va étudier la convergence de la suite des différentielles $\phi_{Y*y_n} : T_{y_n} Y \to T_{y'_n} Y$ où $y'_n = \phi_Y(y_n)$.

Soit $\pi_X : T_X \to X$ la projection du système de données de contrôle et soit $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ la *distribution canonique* relative à X [MT]$_{1,2}$.

Rappelons alors que le champ $\zeta = \{\zeta_Y\}_{Y \geq X}$ relevé contrôlé continu est défini sur toute strate $Y > X$ par la formule:

$$\zeta_Y(y) = \pi_{XY}{}_{*y}^{-1}(\zeta_X(x)) \cap \mathcal{D}_{XY}(y) \ , \quad \text{où} \quad x = \pi_{XY}(y).$$

Fixons $x_0 \in X$. La stratification $\mathcal{X}$ étant (c)-régulière, la submersion stratifiée $\pi_X : T_X \to X$ admet pour un voisinage U_{x_0} de x_0 dans X une trivialisation topologique locale, i.e. un homéomorphisme stratifié

$$H : U_{x_0} \times {\pi_X}^{-1}(x_0) \to {\pi_X}^{-1}(U_{x_0})$$

lisse sur les strates et qui est "l'identité" sur U_{x_0} $(H(p, x_0) = p \ , \ \forall\, (p, x_0) \in \mathbb{R}^l \times x_0 \equiv X$. Ainsi, ${\pi_X}^{-1}(U_{x_0})$ est structuré par un *feuilletage vertical* $\mathcal{V}$ dont les feuilles sont les fibres de la projection π_X et par un *feuilletage horizontal* $\mathcal{H} = \mathcal{H}_{x_0}$ supplémentaire à $\mathcal{V}$. Notons :

$$\mathcal{V} = \left\{N_y = {\pi_X}^{-1}(\pi_X(y))\right\}_{y \in {\pi_X}^{-1}(U_{x_0})},$$
$$\mathcal{H} = \left\{M_{y_0} = H(\mathbb{R}^l \times y_0)\right\}_{y_0 \in {\pi_X}^{-1}(x_0)} .$$

Remarque 12. *Le feuilletage $\mathcal{H}_{x_0}$ est compatible avec la stratification de T_X.*

Remarque 13. *L'application $\phi = \{\phi_Y\}$ est compatible avec le feuilletage vertical $\mathcal{V}$:*

$$\phi_Y(N_y) = N_{y'} \quad \forall\, y, y' \in Y \, avec \;\; y' = \phi_Y(y) \, .$$

Preuve. Grâce à la condition de π_X-contrôle, pour toute strate $Y > X$ et $\forall\, y \in Y$, on a $\pi_{XY*y}(\zeta_Y(y)) = \zeta_X(\pi_{XY}(y))$ ce qui équivaut à $\pi_{XY}(\phi_Y(y)) = \phi_X(\pi_{XY}(y))$. Alors ϕ_Y préserve les fibres de π_{XY} : $T_{XY} \to X$ (feuilles verticales dans Y) et on a : $\phi_Y(N_y) \subseteq N_{y'}$. Par symétrie, on a aussi $\phi_Y^{-1}(N_{y'}) \subseteq N_y$ et donc $\phi_Y(N_y) = N_{y'}$. Q.E.D.

L'étude de la convergence des applications $\phi_{Y*y} : T_yY \to T_{y'}Y$ peut alors être séparée selon les "composantes verticales"

$$\phi_{Y*y|T_yN_y} : T_yN_y \to T_{y'}N_{y'}$$

i.e.

$$\phi_{Y*y|\ker \pi_{XY*y}} : \ker \pi_{XY*y} \to \ker \pi_{XY*y'}$$

qui sont l'application différentielle de

$$\phi_{Y|\ {\pi_{XY}}^{-1}(x)} \ : {\pi_{XY}}^{-1}(x) \to {\pi_{XY}}^{-1}(x')$$

où $x = \pi_{XY}(y)$ et $x' = \pi_{XY}(y')$ et les "composantes horizontales":

$$\phi_{Y*y|\mathcal{D}_{XY}(y)} : \mathcal{D}_{XY}(y) \to T_{y'}Y\,.$$

Cependant ici deux précisions s'imposent :

i) On ne peut pas considérer la convergence des restrictions aux feuilles horizontales de $\mathcal{H}$, $\phi_{Y*y|T_yM_y} : T_yM_y \to T_{y'}M_{y'}$, car $\mathcal{H} = \{M_y\}_y$ n'étant pas nécessairement (a)-régulier sur U_{x_0}, la $\mathcal{H}$-semidifférentiabilité de $\phi = \{\phi_Y\}_Y$ pourrait ne pas avoir de sens.

ii) On ne peut pas écrire $\phi_{Y*y|\mathcal{D}_{XY}(y)} : \mathcal{D}_{XY}(y) \to \mathcal{D}_{XY}(y')$, car on n'a pas nécessairement $\phi_{Y*y}(\mathcal{D}_{XY}(y)) \subseteq \mathcal{D}_{XY}(y')$,

On verra au §4 que la propriété $\phi_{Y*y}(\mathcal{D}_{XY}(y)) \subseteq \mathcal{D}_{XY}(y')$ est équivalente à l'involutivité de la distribution canonique $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y\geq X}$ et au §5 que la (a)-régularité du feuilletage $\mathcal{H}$ est la condition nécessaire et suffisante pour que les flots des champs relevés soient horizontalement-C^1 et $\mathcal{H}$-semidifférentiables. De plus, sous l'hypothèse d'involutivité de $\mathcal{D}_X$ on trouvera $\mathcal{D}_{XY}(y) = T_yM_y$ ce qui *unifie* les deux choix ci-dessus et permet d'obtenir la bonne régularité des flots stratifiés.

Depuis la proposition 2 §2 on a :

Corollaire 1. *S'il existe une strate Y et une suite $\{y_n\}_n$ convergente en un point $x \in U_{x_0}$ telles que les restrictions verticales $\phi_{Y|\ {\pi_{XY}}^{-1}(x)} : {\pi_{XY}}^{-1}(x) \to {\pi_{XY}}^{-1}(x')$ aient des différentielles non-bornées, alors le flot $\phi_Y : Y \to Y$ relevé de $\phi_X : X \to X$ n'est pas semidifférentiable en x (mais il peut être horizontalement-C^1).*

Une situation récurrente est par exemple celle de l'*Escargot de Kuo* (ci-dessous) où le difféomorphisme $\phi_{Y|\ {\pi_{XY}}^{-1}(x)}$ transforme une fibre ${\pi_{XY}}^{-1}(x)$, à courbure inférieurement bornée en une fibre ${\pi_{XY}}^{-1}(x')$ à courbure arbitrairement grande. Ceci entraîne une divergence de la norme des différentielles pour $y \to x$ [Wi] : pour toute suite $\{y_n\}_n \subseteq {\pi_{XY}}^{-1}(x)$, la suite des normes $\{||\phi_{Y*y_n|N_{y_n}}||\}_n$ n'est pas bornée.

Example. *(Escargot de Kuo). Soit $A = X \cup Y \subseteq \mathbb{R}^3$ stratifié par $X =$ "l'axe des x" et une strate 2-dimensionelle Y contenant une surface Y' obtenue à partir d'une spirale avec un enroulement infini autour de l'axe des x d'équation en coordonnées polaires du type $\rho = h(\theta)$ (avec $\lim_{\theta\to\infty} h(\theta) = 0$) dans un plan $\{x = 1\}$ orthogonale à X en la réduisant le long de Ox jusqu'à la contracter en un point $x_0 = (0, 0, 0)$ (figure 1).*

En coordonnées cylindriques (x, ρ, θ), on peut représenter le morceau enroulé de Y par $Y' = \{(x, \rho, \theta) \in \mathbb{R}^3 \,\big|\, \rho = g(x) \cdot h(\theta),\ x \in [0, 1[\,,\ \theta \in [0, \infty[\,\}$. Selon les fonctions d'enroulement $h(\theta)$ et de contraction $g(x)$

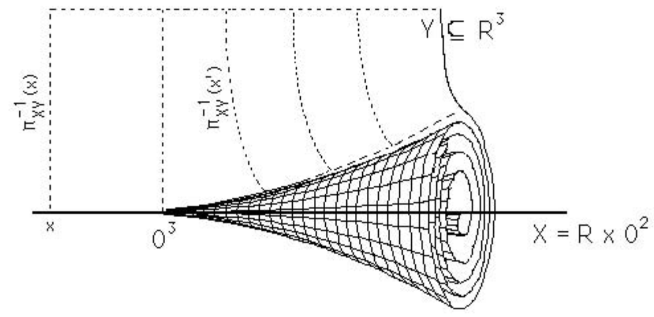

Fig. 1

on peut obtenir des escargots (c) ou (b)-réguliers (voir [Mu] pour plus de détails) mais jamais (w)-réguliers $[\mathrm{OT}]_{1,2}$.

La "géométrie de la strate" Y a donc des conséquences importantes sur la nature (des normes) d'une suite de restrictions $\{||\phi_{Y*y_n|\ker \pi_{XY*y_n}}||\}_n$. Précisons aussi que la différentielle d'un flot relevé ϕ_Y peut être non bornée, même quand Y n'a pas de pathologies de courbure (voir [Kuo] pour un exemple de calcul explicite).

Pour des stratifications Lipschitziennes, les relèvements des champs peuvent être obtenus avec des flots à dérivées bornées [Pa] et grâce aux propositions 4 et 6 :

Proposition 7. *Si $\mathcal{X}$ est une stratification Lipschitzienne admettant un S.D.C. alors $\phi = \{\phi_Y\}_Y$ est semidifférentiable en $x \in U_{x_0}$ si et seulement si $\phi = \{\phi_Y\}_Y$ est horizontalement-C^1 en $x \in U_{x_0}$.*

Dans les prochaines sections nous considérerons la convergence du flot $\phi = \{\phi_Y\}_{Y \geq X}$ relevé de ϕ_X le long des directions horizontales.

§4. Le cas où la distribution canonique est involutive.

Dès maintenant, toute extension contrôlée de champ de vecteurs sera considéré obtenue par la méthode du relèvement continu dans la distribution canonique $\mathcal{D}_X$ $[\mathrm{MT}]_{1,2,3}$, ceci aussi pour les champs v_i et leurs flots ϕ_i qui définissent la trivialisation topologique H.

L'involutivité d'une distribution canonique locale $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ est suffisante pour que le feuilletage horizontal

$$\mathcal{H} = \mathcal{H}_{x_0} = \{M_{y_0} = H(U_{x_0} \times y_0) \quad | \quad y_0 \in \pi_X^{-1}(x_0)\}$$

soit (a)-régulier autour de x_0 (proposition 8 et corollaire 2).

Dans ce cas nous démontrons alors que tout flot relevé $\Phi_t = \{\phi_Y : Y \to Y\}_{Y \geq X}$ est horizontalement-C^1 (théorèmes 3 et 4) sur U_{x_0} et même $\mathcal{H}$-semidifférentiable sur $\pi_X^{-1}(U_{x_0})$ (théorèmes 5 et 6).

Si $\dim X \in \{1, \dim A - 1\}$, l'involutivité de $\mathcal{D}_X$ est toujours vérifiée et donc les flots des champs relevés dans $\mathcal{D}_X$ sont horizontalement-C^1 sur U_{x_0} (corollaires 3 et 4) et $\mathcal{H}$-semidifférentiables sur $\pi_X^{-1}(U_{x_0})$ (corollaires 5 et 6).

4.1. $\phi = \{\phi_Y : Y \to Y\}$ est horizontalement-C^1 sur X.

Supposons qu'une distribution canonique $\mathcal{D}_X$ soit involutive dans un voisinage $\pi_X^{-1}(U_{x_0})$ dans A. A. du Plessis et D. Trotman ont construit en 1994 un exemple montrant que ce n'est pas vrai en général.

Notre problème étant local, nous ne considérerons que la stratification induite de T_X sur le voisinage $\pi_X^{-1}(U_{x_0})$ de x_0 dans A et identifierons alors U_{x_0} avec X et $\pi_{XY}^{-1}(U_{x_0}) = \pi_X^{-1}(U_{x_0}) \cap Y$ avec $Y > X$.

D'autre part $U_{x_0} \equiv X$ étant le domaine d'une carte on peut aussi supposer $U_{x_0} \equiv \mathbb{R}^l \times 0^{n-l}$ et $x_0 \equiv 0^n$.

Rappelons que pour une stratification qui vérifie une des conditions de régularité (c), (b), ou *Lipschitz* et dont les strates sont de classe C^k $(k \geq 2)$, l'homéomorphisme H de trivialisation topologique locale de la projection π_X est obtenu de la manière suivante.

Soient $E_1, \ldots, E_l$ les champs de vecteurs constants sur $\mathbb{R}^l \times 0 = X$ et $v_1, \ldots, v_l$ les champs relevés continus dans la distribution $\mathcal{D}_X$.

Les champs $v_i = v_i(y)$ étant contrôlés, leurs flots $\phi_1, \ldots, \phi_l$ existent $\forall\, t \in \mathbb{R}$, et l'application

$$H : \mathbb{R}^l \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0}) \,,$$

$$H((t_1, \ldots, t_l), y_0) = \phi_l(t_l, \ldots, \phi_1(t_1, y_0)..) = y$$

est un homéomorphisme stratifié lisse sur les strates.

$\mathcal{D}_X$ étant involutive par hypothèse, il existe un feuilletage horizontal $\mathcal{H}' = \{M'_y\}_y$ tel que les espaces tangents aux variétés intégrales maximales sont précisément les espaces tangents à la distribution :

$$T_y M'_y = \mathcal{D}_X(y) \quad , \quad \forall\, y \in \pi_X^{-1}(U_{x_0}) \equiv T_X \,.$$

Lemme 1. *Si la distribution canonique* $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ *est involutive,* $\forall\, i, j \leq l$*,* $[v_i, v_j] = 0$ *et en particulier les flots* ϕ_i *commutent entre eux. Réciproquement il est évident que* $[v_i, v_j] = 0 \;\; \forall i, j \leq l$ *entraîne l'involutivité de* $\mathcal{D}_X$*.*

Preuve. Fixons $Y > X$ et soit $y \in Y$.

Par définition les relevés $v_1(y), \ldots, v_l(y)$ sont tangents à $\mathcal{D}_{XY}(y)$, et comme $\mathcal{D}_X$ est involutive pour tout $i, j \leq l$ alors $[v_i, v_j](y)$ est tangent à $\mathcal{D}_{XY}(y)$. Donc $[v_i, v_j](y)$ coïncide avec sa composante horizontale.

Les champs $\{v_h\}_h$ étant des relèvements π_X-contrôlés, $\pi_{XY*}(v_h) = E_h \,\forall\, h \leq l$, d'où:

$$\pi_{XY*}[v_i, v_j](y) = [\pi_{XY*}(v_i), \pi_{XY*}(v_j)](y) = [E_i, E_j](y) = 0\,,$$

donc $[v_i, v_j](y) \in \ker \pi_{XY*y}$, i.e. sa composante horizontale est nulle.
Q.E.D.

Proposition 8. *Si $\mathcal{D}_X$ est involutive, alors les champs relevés v_i sont les images des champs canoniques par la trivialisation H :*

$$v_i(y) = H_{*(t_1,\ldots,t_l,y_0)}(E_i) \quad , \quad \forall\, i = 1, \ldots, l\,.$$

De plus le feuilletage intégral $\mathcal{H}'$ tangent à $\mathcal{D}_X$ coïncide avec le feuilletage horizontal $\mathcal{H} = \{M_{y_0} = H(\mathbb{R}^l \times y_0)\}_{y_0 \in \pi_{XY}{}^{-1}(x_0)}$ déterminé par H.

Preuve. Fixons une strate $Y > X$ et un point $y \in Y$ $(\equiv \pi_{XY}^{-1}(U_{x_0}))$.

Il suffira de montrer que les espaces tangents aux feuilles $T_y M_y$ et $T_y M'_y$ coïncident.

Pour tout $y = H(t_1, \ldots, t_l, y_0) \in Y$ la variété intégrale horizontale $M_y = M_{y_0}$ définie par H et passant par y passe aussi par y_0 et donc $T_y M_y$ est engendré par les vecteurs $\{H_{*(t_1,\ldots,t_l,y_0)}(E_i)\}_i$.

En notant $w_i(y) = H_{*(t_1,\ldots,t_l,y_0)}(E_i)$, $\forall\, i = 1, \ldots, l$ on a :

$$\begin{aligned} T_y M_y = T_y M_{y_0} &= T_y H(\mathbb{R}^l \times y_0) \\ &= H_{*(t_1,\ldots,t_l,y_0)}(\mathbb{R}^l \times x_0) = \big[w_1(y), \ldots, w_l(y)\big]. \end{aligned}$$

Par le lemme 1, les flots ϕ_i commutent, et on peut écrire (où $\widehat{\phi}$ signifie "omission de ϕ")

$$H(t_1, \ldots, t_l, y_0) = \phi_i(t_i, \phi_l(t_l, \ldots, \widehat{\phi_i(t_i}, \ldots, \phi_1(t_1, y_0) \ldots)\,.$$

Alors pour tout $i = 1, \ldots, l$ on a :

$$w_i(y) = \frac{\partial}{\partial \tau}\bigg|_{\tau = t_i} \phi_i(\tau, \phi_l(t_l, \ldots, \widehat{\phi_i(t_i}, \ldots, \phi_1(t_1, y_0) \ldots) =$$

$$v_i(\phi_i(t_i, \phi_l(t_l, \ldots, \widehat{\phi_i(t_i}, \ldots, \phi_1(t_1, y_0) \ldots) = v_i(H(t_1, \ldots, t_l, y_0)) = v_i(y)$$

d'où

$$T_y M_y = \left[\, w_1(y), \ldots, w_l(y)\,\right] = \left[\, v_1(y), \ldots, v_l(y)\,\right] = \mathcal{D}_{XY}(y) = T_y M'_y\,.$$

Q.E.D.

Corollaire 2. *Si $\mathcal{D}_X$ est involutive, $\mathcal{H}$ est (a)-régulier sur X.*

Preuve. De $\mathcal{H} = \mathcal{H}'$ on déduit :

$$\lim_{y\to x} T_y M'_y = \lim_{y\to x} T_y M_y = \lim_{y\to x} \mathcal{D}_{XY}(y) = T_x X\,.$$

Q.E.D.

Théorème 3. *Si la distribution canonique $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y\geq X}$ est involutive, alors $\forall\, Y > X$ le difféomorphisme ϕ_Y préserve les variétés intégrales du feuilletage horizontal $\mathcal{H}$, et $\forall\, y \in Y$, $y' = \phi_Y(y)$, l'isomorphisme $\phi_{Y*y} : T_y Y \to T_{y'} Y$ vérifie $\phi_{Y*y}(\mathcal{D}_{XY}(y) = \mathcal{D}_{XY}(y')$ (i.e. préserve la distribution $\mathcal{D}_{XY}$), et se décompose en une somme directe :*

$$\phi_{Y*y} = \phi^h_y \oplus \phi^v_y \ : \ \mathcal{D}_{XY}(y) \oplus \ker \pi_{XY\,*y} \xrightarrow{\ \phi^h_y \oplus \phi^v_y\ } \mathcal{D}_{XY}(y') \oplus \ker \pi_{XY\,*y'}\ .$$

La matrice $A(y)$ de l'isomorphisme $\phi^h_y : \mathcal{D}_{XY}(y) \to \mathcal{D}_{XY}(y')$ par rapport aux bases $\sigma = \{v_i(y)\,\}_{i=1}^l$ et $\sigma' = \{v_i(y')\,\}_{i=1}^l$ coïncide avec la matrice $A = A(x)$ de l'isomorphisme $\phi_{X*x} : T_x X \to T_{x'} X$ ($x = \pi_{XY}(y)$) par rapport à la base $\{E_i\}_{i=1}^l$.

Preuve. Soient $Y > X$ une strate, $y \in Y$ et ζ_Y le champ continu relevé de ζ_X sur Y. Comme ζ_Y est tangent à $\mathcal{D}_{XY}$, il est (proposition 1) tangent aux variétés du feuilletage $\mathcal{H}$ et chaque trajectoire de ζ_Y est contenue dans une unique variété intégrale maximale.

En particulier, $M_y = M_{y'}$, car $y' = \phi_Y(y) = (\Phi_Y)_t(y)$.

La même propriété étant vérifiée pour tout couple de points z, $z' = \phi_Y(z)$ correspondant par ϕ_Y, pour tout $z \in M_y$ on a $M_{z'} = M_z = M_y$.

Alors,

$$\phi_Y(M_y) = \cup_{z\in M_y} \left\{\, \phi_Y(z)\,\right\} \subseteq \cup_{z\in M_y} M_{z'} = \cup_{z\in M_y} M_y = M_y$$

et le difféomorphisme ϕ_Y préserve les variétés intégrales du feuilletage $\mathcal{H} = \mathcal{H}'$.

De même en raisonnant sur $\phi_Y^{-1} = (\Phi_Y)_{-t}$ on a $\phi_Y(M_y) = M_y$.

Comme

$$\begin{aligned}\phi_{Y*y}(\mathcal{D}_{XY}(y)) = \phi_{Y*y}(T_y M_y) &= T_{y'}\ \phi_Y(M_y)\\ &= T_{y'}M_y = T_{y'}M_{y'} = \mathcal{D}_{XY}(y')\end{aligned}$$

les sous-espaces supplémentaires $\mathcal{D}_{XY}(y')$ et $\ker \pi_{XY*y'}$ de T_yY se préservent par $\phi_{Y*y} : T_yY \to T_yY$, et on a la décomposition en somme directe :

$$\phi_{Y*y} : \mathcal{D}_{XY}(y) \oplus \ker \pi_{XY*y} \xrightarrow{\phi^h \oplus \phi^v} \mathcal{D}_{XY}(y') \oplus \ker \pi_{XY*y'} .$$

Soit $A(x) = A = (A^i_j)_{\substack{i\leq l\\ j\leq l}}$ (i = indice de colonne, j = indice de ligne), la matrice de l'isomorphisme $\Phi_{X*x} : T_xX \to T_{x'}X$ dans la base canonique $\{E_i\}_{i=1}^l$ de $\mathbb{R}^l \times 0$.

Pour démontrer la formule de transformation

$$(*)_Y : \qquad \phi_{Y*y}(v_i(y)) = \sum_{j=1}^{l} A^i_j v_j(y') \quad , \qquad \forall\, i = 1, \dots, l$$

comme les deux membres sont (maintenant!) des vecteurs de $\mathcal{D}_{XY}(y')$ et comme $\pi_{XY*y'} : \mathcal{D}_{XY}(y') \to \mathbb{R}^l \times 0 = T_{x'}X$ est un isomorphisme $[\mathrm{MT}]_{1,2}$, il suffit de vérifier que

$$\pi_{XY*y'}(\phi_{Y*y}(v_i(y))) \;=\; \pi_{XY*y}\Big(\sum_{j=1}^{l} A^i_j v_j(y')\Big).$$

En fait, on a

$$\pi_{XY*y'}\Big(\sum_{j=1}^{l} A^i_j v_j(y')\Big) = \sum_{j=1}^{l} A^i_j \pi_{XY*y'}(v_j(y')) = \sum_{j=1}^{l} A^i_j E_j\,,$$

et grâce à la condition de contrôle ce vecteur coïncide avec

$$\pi_{XY*y'}\big(\phi_{Y*y}(v_i(y)\big) = \phi_{X*x}\big(\pi_{XY*y}(v_i(y))\big) = \phi_{X*x}(E_i) = \sum_{j=1}^{l} A^i_j E_j.$$

Q.E.D.

Théorème 4. *Si la distribution canonique $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y\geq X}$ est involutive, alors le flot $\phi = \{\phi_Y : Y \to Y\}_{Y\geq X}$ (à l'instant $t \in \mathbb{R}$) est horizontalement-C^1 sur $X \equiv U_{x_0}$.*

Preuve. Soit $Y > X$ une strate et $\{(y_n, v_n)\}_n \subseteq TY$ une suite de vecteurs v_n sans composante verticale, i.e.

$$v_n^v = 0 \quad \text{et} \quad v_n = v_n^h \in \mathcal{D}_{XY}(y_n)\,.$$

Par continuité de l'application $\phi = \cup_{Y \geq X} \phi_Y : \cup_{Y \geq X} Y \to \cup_{Y \geq X} Y$ on a $\phi_Y(y_n) = \phi_X(x)$, et il suffit de montrer que :

$$\lim_{n \to \infty} \phi_{Y*y_n}(v_n) = \phi_{X*x}(v)\,.$$

Notons $A(y_n) = \left[\phi_{Y*y_n}\right]$ la matrice du théorème 3 et utilisons les notations analogues pour $A(x_n)$, $x_n = \pi_{XY}(y_n) \in X$. Notons aussi $A(y_n)^i$ et $A(x_n)^i$ leurs $i-$èmes colonnes.

Les relevés canoniques $v_1(y_n), \ldots, v_l(y_n)$ étant une base de $\mathcal{D}_{XY}(y_n)$ on peut écrire $v_n = \sum_{i=1}^l \lambda_i^n v_i(y_n)$ pour des $\lambda_i^n \in \mathbb{R}$ convenables.

Pour $v \in \mathbb{R}^l \times 0$, notons $v = \sum_{i=1}^l \lambda_i E_i$ avec $\lambda_1, \ldots, \lambda_l \in \mathbb{R}$. Les

champs de vecteurs v_i étant des relevés continus de champs canoniques E_i on a :

$$\lim_n v_i(y_n) = E_i \quad \text{et donc} \quad \lim_{n \to \infty} (\lambda_1^n, \ldots, \lambda_l^n) = (\lambda_1, \ldots, \lambda_l).$$

En notant $M \cdot W = \sum_{i=1}^k \mu_i w_i$ par le produit d'un k-uplet de scalaires $M = (\mu_1, \ldots, \mu_k)$ et un k-uplet de vecteurs $W = (w_1, \ldots, w_k)$ on peut écrire :

$$\phi_{Y*y_n}(v_n) = \sum_{i=1}^l \lambda_i^n \phi_{Y*y_n}(v_i(y_n)) = \sum_{i=1}^l \lambda_i^n \left[\phi_{Y*y_n}\right]^i \cdot (v_1(y_n), \ldots, v_l(y_n))$$

qui grâce au théorème 3 coïncide avec

$$\sum_{i=1}^l \lambda_i^n \left[\phi_{X*x_n}\right]^i \cdot (v_1(y_n), \ldots, v_l(y_n))$$

et donc on déduit que :

$$\lim_{n \to \infty} \phi_{Y*y_n}(v_n) = \lim_{n \to \infty} \sum_{i=1}^l \lambda_i^n \left[\phi_{X*x_n}\right]^i \cdot (v_1(y_n), \ldots, v_l(y_n)).$$

Alors on peut conclure que :

$$\begin{aligned}\lim_{n\to\infty} \phi_{Y*y_n}(v_n) &= \sum_{i=1}^{l} \lambda_i \big[\phi_{X*x}\big]^i \cdot (E_1,\ldots,E_l)\\ &= \sum_{i=1}^{l} \lambda_i \phi_{X*x}(E_i) = \phi_{X*x}(v),\end{aligned}$$

car $\lim_n \phi_{X*x_n} = \phi_{X*x}$, $\lim\limits_{n\to\infty}(\lambda_1^n,\ldots,\lambda_l^n) = (\lambda_1,\ldots,\lambda_l)$, et les champs relevés canoniques $v_1,\ldots,v_l$ de $E_1,\ldots,E_l$ vérifient

$$\lim_{n\to\infty}(v_1(y_n),\ldots,v_l(y_n)) = (E_1,\ldots,E_l)\,.$$

Q.E.D.

Corollaire 3. *Si* $\dim X = 1$, *toute distribution canonique locale ou globale* $\mathcal{D}_X$ *est involutive. Donc tout relèvement continu contrôlé* $\zeta = \{\zeta_Y\}_{Y\geq X}$ *d'un champ de vecteurs* ζ_X *défini sur une strate* X *de dimension* 1 *a un flot* $\phi=\{\phi_Y : Y\to Y\}_{Y\geq X}$ *horizontalement-*C^1 *sur* X.

Corollaire 4. *Si* $\dim X = \dim A - 1$, *la distribution canonique globale* $\mathcal{D}_X$ *est involutive. Donc tout relèvement continu contrôlé* $\xi = \{\xi_Y\}_{Y\geq X}$ *d'un champ de vecteurs* ξ_X *a un flot* $\phi = \{\phi_Y : Y\to Y\}_{Y\geq X}$ *horizontalement-*C^1 *sur* X.

Preuve. Si $\dim X = \dim A - 1$, alors, par (c)-régularité de la stratification, chaque système de donnée de contrôle $\mathcal{T} = \{(T_X,\pi_X,\rho_X)\}$ admet une seule distribution canonique $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y\geq X}$, qui s'obtient en prenant $\mathcal{D}_{XY}(y) = \ker \rho_{XY*y}$ pour toute $Y > X$.

Une telle distribution canonique globale $\mathcal{D}_X$ est nécessairement intégrable sur T_X tout entier, admettant comme variétés intégrales maximales les hypersurfaces de niveaux $\{\rho_{XY}^{-1}(\rho_{XY}(y)\}_y$ de $\rho_{XY} : T_{XY} \to [0,\infty[$. Le résultat découle alors du théorème 4. Q.E.D.

4.2. $\mathcal{H}$-semidifférentiabilité de $\phi = \{\phi_Y\}_{Y\geq X}$.

Dans cette section, nous améliorons les résultats du 4.1 en démontrant que le flot $\phi = \{\phi_Y : Y\to Y\}_{Y\geq X}$ d'un champ $\xi = \{\xi_Y\}_{Y\geq X}$ relevé continu contrôlé dans une distribution canonique $\mathcal{D}_X$ involutive est $\mathcal{F}$-semidifférentiable par rapport au feuilletage horizontal $\mathcal{H} = \{M_y = H(y_0\times\mathbb{R}^l)\}_{y\in\pi_X^{-1}(x_0)}$ de $\pi_X^{-1}(U_{x_0})$, i.e. il est $\mathcal{H}$-semidifférentiable.

Ceci signifie qu'étant donnée une strate $Z > X$ et une suite $\{(z_n,v_n)\}_n \subseteq TZ$ convergeant en un point $(y,v)\in TY$, où Y est une strate telle que

$Z > Y > X$ on a $\lim_n \phi_{Z*z_n}(v_n) = \phi_{Y*y}(v)$, au moins pour les suites v_n "horizontaux par rapport à X" ($X-$*horizontaux*), i.e. : $v_n \in \mathcal{D}_{XZ}(z_n)$.

Avec les mêmes hypothèses et notations qu'aux théorèmes 3 et 4 :

Théorème 5. *Pour toutes $Z > Y > X$ et $\forall\, z \in Z \equiv T_{XZ} \cap T_{YZ}$ en notant $z' = \phi_Z(z)$, $y = \pi_{YZ}(z)$ et $y' = \pi_Y(z') = \phi_Y(y')$, la matrice $A(z)$ de l'isomorphisme restriction $\phi_{Z*z|} : \mathcal{D}_{XZ}(z) \to \mathcal{D}_{XZ}(z')$ par rapport aux bases $\sigma_z = \{v_i(z)\}_i$ et $\sigma'_{z'} = \{v_i(z')\}_i$ coïncide avec la matrice $A = A(y)$ de l'isomorphisme restriction $\phi_{Y*y|} : \mathcal{D}_{XY}(y) \to \mathcal{D}_{XY}(y')$ par rapport aux bases canoniques $\sigma_y = \{v_i(y)\}_i$ et $\sigma'_{y'} = \{v_i(y')\}_i$.*

Preuve. Grâce au théorème 4, $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ étant involutive, on a bien deux isomorphismes de restriction

$$\phi_{Z*z|} : \mathcal{D}_{XZ}(z) \to \mathcal{D}_{XZ}(z') \quad \text{et} \quad \phi_{Y*y|} : \mathcal{D}_{XY}(y) \to \mathcal{D}_{XY}(y').$$

En notant $A = \left(A^i_j\right)_{\substack{i \leq l \\ j \leq l}}$, la thèse s'obtient en démontrant la formule:

$$(*)_Z \quad : \quad \phi_{Z*z}(v_i(z)) = \sum_{j=1}^{l} A^i_j v_j(z') \quad , \qquad i = 1, \ldots, l\,.$$

La distribution canonique $\mathcal{D}_X$ est (par construction) *multicompatible* avec une distribution canonique $\mathcal{D}_Y = \{\mathcal{D}_{YZ}\}_{Z \geq Y}$ ([MT]$_2$, Proposition dans §5) , c.à.d. $\mathcal{D}_{XZ}(z') \subseteq \mathcal{D}_{YZ}(z')$.

L'application $\pi_{YZ*z'} : \mathcal{D}_{YZ}(z') \to T_{y'}Y$ étant un isomorphisme, $\mathcal{D}_{XZ}(z')$ se projette isomorphiquement sur un sous-espace vectoriel de $T_{y'}Y$. Ce sous-espace est, grâce aux conditions de contrôle, précisément $\mathcal{D}_{XY}(y')$, et donc on a l'isomorphisme de restriction

$$\pi_{YZ*z'|} \; : \; \mathcal{D}_{XZ}(z') \; \to \; \mathcal{D}_{XY}(y')\,.$$

et la preuve de $(*)_Z$ découlera l'égalité des deux images via $\pi_{YZ*z'}$:

$$\pi_{YZ*z'}(\phi_{Z*z'}(v_i(z'))) = \pi_{YZ*z'}\Big(\sum_{j=1}^{l} A^i_j v_j(z')\Big).$$

En fait, comme ϕ_Z est π_Y-contrôlé et $v_i(z)$ est le π_{YZ}-relevé contrôlé de $v_i(y)$ on a :

$$\pi_{YZ*z'}(\phi_{Z*z}(v_i(z))) = \phi_{Y*y}(\pi_{YZ*z}(v_i(z))) = \phi_{Y*y}(v_i(y))$$

et de façon analogue on trouve :

$$\pi_{YZ*z'}\big(\sum_{j=1}^{l} A^i_j v_j(z')\big)= \sum_{j=1}^{l} A^i_j \pi_{YZ*z'}\big(v_j(z')\big)= \sum_{j=1}^{l} A^i_j v_j(y')\ .$$

On conclut alors car l'involutivité de $\mathcal{D}_X$ (théorème 1) entraîne

$$(*)_Y \quad : \quad \phi_{Y*y}(v_i(y)) = \sum_{j=1}^{l} A^i_j v_j(y') \quad , \qquad i = 1, \ldots, l\ .$$

Q.E.D.

Proposition 9. *Si $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y\geq X}$ est involutive, le feuilletage $\mathcal{H} = \{M_y = H(y_0 \times \mathbb{R}^l)\}$ est (a)-régulier sur $\pi_X^{-1}(U_{x_0})$.*

Preuve. Soit $Y > X$ une strate et fixons une strate $Z > Y$.

Pour tout $z \in Z \equiv T_{YZ}$, par l'involutivité de $\mathcal{D}_X$ et la proposition 1 on a : $T_z\mathcal{H} = T_z M_z = \mathcal{D}_{XZ}(z)$.

La distribution canonique étant continue sur $Y \equiv T_{XY}$, $\forall\, y' \in T_{XY}$ on a

$$\lim_{z\to y'} T_z\mathcal{H} = \lim_{z\to y'} \mathcal{D}_{XZ}(z) = \mathcal{D}_{XY}(y')$$

et comme $\mathcal{D}_{XY}(y') \subseteq T_{y'}Y$, on déuit la ($a$)-régularité de $\mathcal{H}$ sur Y.

Q.E.D.

Théorème 6. *Si $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y\geq X}$ est involutive et si $\mathcal{H} = \mathcal{H}_{x_0} = \{M_y = H(y_0 \times \mathbb{R}^l)\}_y$ désigne le feuilletage défini sur le voisinage $\pi_X^{-1}(U_{x_0})$ par la trivialisation topologique de la projection $\pi_X : T_X \to X$, alors le flot relevé $\phi = \{\phi_Y : Y \to Y\}$ est $\mathcal{H}$-semidifférentiable sur $\pi_X^{-1}(U_{x_0})$.*

Preuve. Pour que la notion de $\mathcal{H}$-semidifférentiabilité ait du sens il faut que le feuilletage $\mathcal{H}$ soit (a)-régulier et ceci est assuré par la proposition 9 ci-dessus.

En considérant alors une suite $\{(z_n, v_n)\} \in T\mathcal{H}$ $(y, v) \in \{y\}\times T_y\mathcal{H} \subseteq T_yY$, la preuve est analogue à celle du théorème 4. En fait, comme $v_n \in T_{z_n}\mathcal{H} = T_{z_n}M_{z_n} = \mathcal{D}_{XZ}(z_n)$ et ce dernier est engendré par les relevés canoniques $v_1(z_n), \ldots, v_l(z_n)$, nous pouvons écrire

$$v_n = \sum_{i=1}^{l} \lambda^n_i v_i(z_n) \quad \text{pour des } \lambda^n_1, \ldots, \lambda^n_l \in \mathbb{R} \text{ convenables}, \quad \forall n \in \mathbb{N}.$$

De même, comme $v \in T_y\mathcal{H} = T_yM_y = \mathcal{D}_{XY}(y)$ et que ce dernier est engendré par les vecteurs canoniques relevés $v_1(y), \dots, v_l(y)$ on peut également écrire :

$$v = \sum_{i=1}^{l} \lambda_i v_i(y) \quad \text{pour des } \lambda_1, \dots, \lambda_l \in \mathbb{R} \text{ convenables.}$$

Les relèvements $v_i(z)$ étant continus sur Y de $\lim\limits_{z_n \to y} v_i(z_n) = v_i(z)$ on a :

$$(1) : \qquad \lim_{n\to\infty} (\lambda_1^n, \dots, \lambda_l^n) = (\lambda_1, \dots, \lambda_l).$$

En notant alors $\forall\, n \in \mathbb{N}$, $y_n = \pi_{YZ}(z_n)$, et en utilisant des notations analogues à celles du théorème 2 pour les matrices (et leurs i-colonnes)

$$A(z_n) = \left[\phi_{Z*z_n|}\right] \quad , \quad A(y_n) = \left[\phi_{Y*y_n|}\right] \quad , \quad A(y) = \left[\phi_{Y*y|}\right]$$

$$A(z_n)^i = \left[\phi_{Z*z_n|}\right]^i \quad , \quad A(y_n)^i = \left[\phi_{Y*y_n|}\right]^i \quad , \quad A(y)^i = \left[\phi_{Y*y|}\right]^i$$

on peut écrire :

$$\phi_{Z*z_n}(v_n) = \sum_{i=1}^{l} \lambda_i^n \phi_{Z*z_n}(v_i(z_n)) = \sum_{i=1}^{l} \lambda_i^n \left[\phi_{Z*z_n|}\right]^i \cdot (v_1(z_n), \dots, v_l(z_n)) \, .$$

Par le théorème 3, les deux matrices suivantes coïncident :

$$[\phi_{Z*z_n|}] = A(z_n) = A(y_n) = [\phi_{Y*y_n|}]$$

et on a :

$$\phi_{Z*z_n}(v_n) = \sum_{i=1}^{l} \lambda_i^n \left[\phi_{Y*y_n|}\right]^i \cdot (v_1(z_n), \dots, v_l(z_n))$$

et

$$\lim_{n\to\infty} \phi_{Z*z_n}(v_n) = \lim_{n\to\infty} \sum_{i=1}^{l} \lambda_i^n \left[\phi_{Y*y_n|}\right]^i \cdot (v_1(z_n), \dots, v_l(z_n)) \, .$$

Finalement, par la continuité des relevés $v_i(z_n)$ qui convergent vers les champs $v_i(y)$ quand $z_n \to y$, la relation de limite (1) et la relation $\lim_{y_n \to y}\left[\phi_{Y*y_n|}\right] = \left[\phi_{Y*y|}\right]$ (car $\phi_Y : Y \to Y$ est lisse), on conclut que :

$$\begin{aligned} \lim_{n\to\infty} \phi_{Z*z_n}(v_n) &= \sum_{i=1}^{l} \lambda_i \left[\phi_{Y*y}\right]^i \cdot (v_1(y), \dots, v_l(y)) \\ &= \sum_{i=1}^{l} \lambda_i \phi_{Y*y} v_i(y) = \phi_{Y*y}(v) \, . \end{aligned}$$

Q.E.D.

Comme pour le cas horizontalement-C^1, en améliorant les corollaires 3 et 4 on trouve:

Corollaire 5. *Si* $\dim X = 1$*, toute distribution canonique locale ou globale* $\mathcal{D}_X$ *est involutive. En particulier tout flot* $\phi = \{\phi_Y : Y \to Y\}_{Y \geq X}$ *obtenu à partir d'un relèvement continu contrôlé d'un champ de vecteurs* ζ_X *défini sur une strate* X *de dimension* 1 *est* $\mathcal{H}$*-semidifférentiable sur* X*.*

Corollaire 6. *Si* $\dim X = \dim A - 1$*, la distribution canonique globale* $\mathcal{D}_X$ *est involutive. Donc tout relèvement continu contrôlé* $\xi = \{\xi_Y\}_{Y \geq X}$ *d'un champ de vecteurs* ξ_X *a un flot* $\phi = \{\phi_Y : Y \to Y\}_{Y \geq X}$ $\mathcal{H}$*-semidifférentiable.*

Preuve. Si $\dim X = \dim A - 1$ la distribution canonique $\mathcal{D}_X$ est unique (comme on l'a vu au corollaire 3), globalement définie sur T_X et intégrable, car $\forall Z > X$ elle coïncide avec $\mathcal{D}_{XZ}(y) = \ker \rho_{XZ*y}$. De plus, le feuilletage $\mathcal{H}$ a pour feuilles les hypersurfaces de niveaux $\{\rho_{XZ}^{-1}(\rho_{XZ}(y))\}_y$ de la fonction distance $\rho_{XY} : T_{XZ} \to [0, \infty[$.

Comme $\dim X = \dim A - 1$ et $Z > X$, on a $\dim Z = \dim A$, et donc il n'existe aucune strate Y vérifiant $Z > Y > X$. Le flot $\phi = \{\phi_Y : Y \to Y\}$ est alors $\mathcal{H}$-semidifférentiable si et seulement s'il est horizontalement-C^1. Q.E.D.

4.3. Quelques caractérisations de l'involutivité de $\mathcal{D}_X$.

Le domaine de la trivialisation H de la projection $\pi_X : T_X \to X$ est le produit $\mathbb{R}^l \times \pi_X^{-1}(x_0)$ dont la projection (*verticale*) sur $\mathbb{R}^l$ corresponde via H à π_X. De même, l'homéomorphisme H induit une *projection horizontale* π' correspondant à la projection pr_2 sur $\pi_X^{-1}(x_0)$:

$$\begin{array}{ccc} \mathbb{R}^l \times \pi_X^{-1}(x_0) & \xrightarrow{H} & \pi_X^{-1}(U_{x_0}) \\ {\scriptstyle pr_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ \pi_X^{-1}(x_0) & \xrightarrow{H|_{\pi_X^{-1}(x_0)}=id} & \pi_X^{-1}(x_0). \end{array}$$

Les deux feuilletages (horizontal et vertical) triviaux transverses de $\mathbb{R}^l \times \pi_X^{-1}(x_0)$ induisent via H deux feuilletages transverses dans $\pi_X^{-1}(U_{x_0})$: le feuilletage horizontal que nous avons noté $\mathcal{H}$ et le feuilletage vertical, ayant pour variétés intégrales les fibres de la projection π_{XY} (H étant contrôlée), que nous avons précédemment noté $\mathcal{V}$ au §3.

Sur toute strate $Y > X$, les traces de $\mathcal{H}$ et $\mathcal{V}$ induisent deux feuilletages transverses, $\mathcal{H}_Y$ et $\mathcal{V}_Y$, dont les feuilles $(M_y, {\pi_{XY}}^{-1}(x))$ se coupent en un unique point $\{y_x\} = M_y \cap {\pi_{XY}}^{-1}(x)$. Ce point y_x a pour variété intégrale verticale la fibre de π_{XY} passant par y_x, et pour variété intégrale horizontale la fibre M_y de π'.

Lemme 2. *Pour tout $Y > X$, et $y \in \pi_{XY}(U_{x_0})$, la projection horizontale π' envoie la variété intégrale horizontale M_y sur le point où elle rencontre la fibre ${\pi_{XY}}^{-1}(x_0)$:*

$$\pi' : \pi_X^{-1}(U_{x_0}) \to \pi_X^{-1}(x_0)\,,$$
$$\pi'(M_y) = M_y \cap {\pi_{XY}}^{-1}(x_0) = y_0 \quad \textit{où} \quad y = H(t_1, \ldots, t_l, y_0).$$

Donc, les feuilles de $\mathcal{H}$ coïncident avec les fibres de la projection horizontale π'.

Définition 9. *Un champ de vecteurs ζ sur Y sera dit contrôlé par rapport à la projection π' si pour tout $y, y' \in Y$ dans la même fibre de π', on a : $\pi'_{*y}(\zeta(y)) = \pi'_{*y'}(\zeta(y'))$.*

Ceci est valable si et seulement si il existe un champ de vecteurs η tangent à $\pi_X^{-1}(x_0)$ tel que $\pi'_{*y}(\zeta(y)) = \eta(\pi'(y))\,,\, \forall\, y \in \pi_X^{-1}(U_{x_0})$.

Un tel champ sera alors noté par $\eta(y_0) = \pi'_{*y}(\zeta(y))$, y étant un point arbitraire de la fibre $\pi'^{-1}(y_0)$.

Théorème 7. *Les conditions suivantes sont équivalentes :*

1) $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ est une distribution involutive dans $\pi_X^{-1}(U_{x_0})$.

2) Pour tout relèvement continu contrôlé $\zeta = \{\zeta_Y\}_{Y \geq X}$ d'un champ de vecteurs ζ_X sur X, et $\forall\, Y > X$,

$$\phi_{Y*y}(\mathcal{D}_{XY}(y)) = \mathcal{D}_{XY}(y') \quad , \quad \forall\, y, y' = \phi_Y(y) \in Y$$

et en particulier on a la décomposition en somme directe $\phi_{Y*y} = \phi^h \oplus \phi^v$.

3) Pour tout champ ζ_X sur X, le relèvement continu contrôlé $\zeta = \{\zeta_Y\}_{Y \geq X}$ du champ ζ_X dans $\mathcal{D}_X$ est contrôlé par rapport à la projection horizontale π' et on a $\pi'_*(\zeta_Y) = 0$.

4) $[v_i, v_j] = 0$ pour tout $i, j = 1, \ldots, l$.

Preuve. $(1 \Leftrightarrow 2)$. Voir les preuves des théorèmes 3 et 4 au §4.1.

$(1 = 2 \Rightarrow 3)$. Soit $Y > X$.

Par hypothèse d'involutivité, $\forall\ y = H(t_1, \ldots, t_l, y_0) \in Y$ on a $\mathcal{D}_{XY}(y) = T_y M_{y_0}$, par définition du relèvement canonique $\zeta_Y(y) \in \mathcal{D}_{XY}(y)$, et alors $\zeta_Y(y) \in T_y M_{y_0} = \ker \pi'_{*y}$. Donc $\pi'_{*y}(\zeta_Y(y)) = 0$.

$(3 \Rightarrow 4)$. Par hypothèse, chaque relèvement v_i (du champ canonique E_i), est π'-contrôlé : $\pi'_{*y}(v_i(y)) = \pi'_{*y_0}(v_i(y_0)) = 0$.

Donc $\pi'_{*y}([v_i, v_j]) = [\pi'_{*y}(v_i), \pi'_{*y}(v_j)] = 0$, i.e. les crochets de Lie $[v_i, v_j](y) \in \ker \pi'_{*y}$ sont tangents au feuilletage horizontal $\ker \pi'_{*y}$.

D'autre part, par la condition de π_X-contrôle on a :

$$\pi_{XY*y}([v_i, v_j]) = [\pi_{XY*y}(v_i), \pi_{XY*y}(v_j)] = [E_i, E_j] = 0, \forall\, i, j = 1, \ldots, l$$

et donc $[v_i, v_j](y) \in \ker \pi_{XY*y}$ est un champ vertical.

Par transversalité et complémentarité de dimension on conclut alors:

$$[v_i, v_j](y) \in \ker \pi'_{*y} \cap \ker \pi_{XY*y} = \{0\}$$

$(4 \Leftrightarrow 1)$. C'est le lemme 1 au §4.1. Q.E.D.

§5. Cas général : $\mathcal{D}_X$ non nécessairement involutive.

Notons $\pi_X^{-1}(U_{x_0}) = W$ par simplicité.

Dans ce paragraphe, on montre que si la distribution canonique $\mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y \geq X}$ n'est pas intégrable, la (a)-régularité du feuilletage horizontal $\mathcal{H} = \{M_y\}_{y \in W}$ (condition nécessaire mais pas suffisante pour l'involutivité de $\mathcal{D}_X$) peut alors remplacer l'hypothèse d'involutivité de $\mathcal{D}_X$ en vue d'obtenir des flots relevés et de façon plus générale des morphismes stratifiés horizontalement-C^1 et $\mathcal{H}$-semidifférentiables.

La distribution canonique $\mathcal{D}_X$ introduite dans $[\mathrm{MT}]_{1,2}$ est “canonique” dans le sens où elle vérifie des propriétés importantes obtenues dans $[\mathrm{MT}]_{1,2}$, mais elle n'est pas univoquement déterminée et dépend du S.D.C. considéré et d'une partition de l'unité.

La distribution $\mathcal{D}'_X = \mathcal{D}(\mathcal{H})$ tangent au feuilletage $\mathcal{H} = \{M_y\}_{y \in W}$, i.e. :

$$\mathcal{D}'_X(y) = T_y \mathcal{H}_y = T_y M_y \quad , \quad \forall\, y \in W = \pi_X^{-1}(U_{x_0})$$

ne coïncide pas en général avec $\mathcal{D}_X$ (sauf quand $\mathcal{D}_X$ est integrable) et vérifie toutes ces conditions de $[\mathrm{MT}]_{1,2}$, sauf éventuellement la condition de continuité. D'autre part, la (a)-régularité du feuilletage local $\mathcal{H} = \{M_y\}_{y \in W}$ équivaut à la continuité de $\mathcal{D}'_X$ sur W et donc, si $\mathcal{H}$ est (a)-régulier, $\mathcal{D}'_X$ peut être réinterprétée comme une distribution canonique locale définie dans le voisinage W de x_0 dans $\mathcal{X}$.

La différence par rapport aux résultats du §4, où $\mathcal{D}_X$ est supposée involutive, est que maintenant nous trouvons des flots de champs relevés qui sont horizontalement-C^1, non plus par rapport à $\mathcal{D}_X$, mais par rapport à $\mathcal{D}'_X$. Cela signifie en particulier que nous devons remplacer les champs $\zeta = \{\zeta_Y\}_{Y \geq X}$ et les flots $\phi = \{\phi_Y\}_{Y \geq X}$ précédents (relevés dans $\mathcal{D}_X$) respectivement par les champs $\xi = \{\xi_Y\}_{Y \geq X}$ et les flots $\psi = \{\psi_Y\}_{Y \geq X}$ correspondants relevés sur le feuilletage $\mathcal{H}$ (i.e. sur $\mathcal{D}'_X$). Grâce à la (a)-régularité de $\mathcal{H}$, le relevé ξ sera (de même que ζ) un relèvement continu et contrôlé de ζ_X (mais, ne disposant pas de l'involutivité de $\mathcal{D}_X$[3], ξ ne coïncidera pas avec ζ, ni ψ avec ϕ).

5.1. Régularité horizontalement-C^1.

Avant d'énoncer les théorèmes de régularité des morphismes stratifiés soumis à l'existence d'un feuilletage (a)-régulier, précisons que A. du Plessis et D. Trotman ont vérifié en 1994 que, même dans le cas d'une stratification (b)-régulière, une distribution canonique n'est pas nécessairement involutive.

Conjecture (D. Trotman, 1993). *Toute stratification (b)-régulière admet localement une distribution canonique involutive (et donc un feuilletage horizontal (a)-régulier).*

Une telle propriété pourrait aussi avoir lieu pour des stratifications (c)-régulières.

Théorème 8. *Les conditions suivantes sont équivalentes :*

1) Le relèvement contrôlé $\xi = \{\xi_Y : Y \to Y\}_{Y \geq X}$ tangent à $\mathcal{H} = \{M_y\}_{y \in W}$ de tout champ de vecteurs ξ_X sur X est continu sur U_{x_0} et a un flot $\psi = \{\psi^t_Y\}_{Y \geq X}$, horizontalement-$C^1$ sur U_{x_0}.

2) Les relèvements contrôlés w_i tangents à $\mathcal{H} = \{M_y\}_{y \in W}$ des champs canoniques E_i sont continus sur U_{x_0} pour tout $i = 1, \ldots, l$, et ont des flots $\psi_i = \{\psi^t_{iY} : Y \to Y\}_{Y \geq X}$ horizontalement-C^1 sur U_{x_0}.

3) L'homéomorphisme de trivialisation topologique de la projection $\pi_X : T_X \to X$,

$$H : \mathbb{R}^l \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0})$$

est horizontalement-C^1 sur $\mathbb{R}^l \times \{x_0\}$.

4) $\lim_{(t_1,\ldots,t_l,y_0) \to x} H_{*(t_1,\ldots,t_l,y_0)}(E_i) = E_i \,, \forall\, x \in X \equiv U_{x_0} \equiv \mathbb{R}^l \,, \forall\, i = 1, \ldots, l$ *;*

5) Le feuilletage horizontal $\mathcal{H} = \{M_y\}_{y \in W}$ induit par H est (a)-régulier sur U_{x_0} (i.e. il vérifie la conjecture de Trotman sur U_{x_0}) .

[3] Ce qui assurerait que $\mathcal{D}_X = \mathcal{D}'_X$ comme on l'a vu au §4.

Le lemme suivant est nécessaire :

Lemme 3. *Chaque champ $w_i(y) = H_{*(t_1,\ldots,t_l,y_0)}(E_i)$ est l'unique relèvement controlé du champ canonique E_i tangent aux feuilles du feuilletage horizontal $\mathcal{H}$.*

Preuve. Les champs $v_j(y)$ étant les relèvements contrôlés dans $\mathcal{D}_{XY}(y)$ leurs flots $\phi_j^\tau = \{\phi_{jY}^\tau : Y \to Y\}_{Y\geq X}$ sont aussi contrôlés et vérifient $\forall\, j \leq l$:

$$\pi_{XY*y}\,\phi_{jY*y}^\tau = \phi_{jX*x}^\tau\,\pi_{XY*y} \quad \text{où} \quad x = \pi_{XY}(y)$$

et $\phi_{jX*x}^\tau = id_{T_xX} = id_{\mathbb{R}^l}$ car chaque ϕ_{jX} est le flot "identique" du champ E_j sur X.

La trivialisation H définie par compositions des ϕ_j est alors contrôlée et on a

$$\begin{aligned}\pi_{XY*y}\big(w_i(y)\big) &= \pi_{XY*y}H_{*(t_1,\ldots,t_l,y_0)}(E_i)\\ &= H_{*x}\pi_{XY*y}(E_i) = id_{\mathbb{R}^l}(E_i) = E_i\,.\end{aligned}$$

Chaque $w_i(y)$ est donc π_X-contrôlé et de manière similaire il est ρ_X-contrôlé.

Pour tout $y = H(t_1,\ldots,t_l,y_0)$, en considérant le difféomorphisme restriction de H, $H_{|\mathbb{R}^l\times y_0} : \mathbb{R}^l\times\{y_0\} \xrightarrow{\simeq} H(\mathbb{R}^l\times y_0)$, à la variété intégrale $M_y = M_{y_0} = H(\mathbb{R}^l \times y_0)$ on a :

$$w_i(y) = H_{*(t_1,\ldots,t_l,y_0)}(E_i) = \big[H_{|\mathbb{R}^l\times y_0}\big]_{*(t_1,\ldots,t_l,y_0)}(E_i)$$

et donc chaque $w_i(y)$ est tangent à la feuille horizontale M_y de $\mathcal{H}$.

En conclusion, pour tout $Y > X$, si $y \in Y$ on a

$$w_i(y) \in \pi_{XY}{}^{-1}_{*y}(E_i) \cap T_y\,H(\mathbb{R}^l \times y_0)$$

où par transversalité et complémentarité de dimension dans T_yY, l'intersection $\pi_{XY}{}^{-1}_{*yy}(E_i) \cap T_y\,H(\mathbb{R}^l \times y_0)$ se réduit alors à un unique vecteur.

Q.E.D.

Preuve (du théorème 8). $(3 \Leftrightarrow 4)$. Considérons le domaine de H muni de la stratification produit de $\mathbb{R}^l$ et de $\pi_X^{-1}(x_0)$:

$$\mathbb{R}^l \times \pi_X^{-1}(x_0) = \mathbb{R}^l \times \{x_0\} \,\cup\, \big[\cup_{Y>X}\mathbb{R}^l \times \pi_{XY}^{-1}(x_0)\big]$$

et considérons sur $\mathbb{R}^l \times \pi_X^{-1}(x_0)$ le S.D.C. induit par l'homéomorphisme stratifié H.

Si $A = \mathbb{R}^l \times \{x_0\}$, alors toute strate $B > A$ de $\mathbb{R}^l \times \pi_X^{-1}(x_0)$ est du type $B = \mathbb{R}^l \times \pi_{XY}^{-1}(x_0)$ avec $Y > X$, et on a une distribution canonique évidente $\mathcal{D}_A = \{\mathcal{D}_{AB}\}_{B \geq A}$:

$$\mathcal{D}_{AB}(t_1, \ldots, t_l, y_0) = \mathbb{R}^l \times \{y_0\} \quad , \quad \forall\, (t_1, \ldots, t_l, y_0) \in B \;,\; \forall B > A$$

sur $\mathbb{R}^l \times \pi_X^{-1}(x_0)$ relative à la strate A.

Alors $\mathcal{D}_{AB}(t_1, \ldots, t_l, y_0) = [E_1, \ldots, E_l]$ et tout vecteur horizontal tangent à B est une combinaison linéaires de $E_1, \ldots, E_l$ et donc H est horizontalement-C^1 par rapport à $\mathcal{D}_A$ si et seulement si :

$$(*) \;:\; \lim_{(t_1,\ldots,t_l,y_0)\to x} H_{B*(t_1,\ldots,t_l,y_0)}(E_i) = H_{A*x}(E_i)\,, \quad \forall x \in A,\; \forall\, i \leq l\,.$$

Par l'identification $U_{x_0} \equiv \mathbb{R}^l \times 0^{n-l}$, la restriction H_A de H à A coïncide avec l'identité de la strate $A = \mathbb{R}^l \times \{x_0\}$. Donc $H_{A*x}(E_i) = E_i$ et on conclut grâce à (*).

($4 \Leftrightarrow 5$). Grâce au lemme 3, les champs de vecteurs images

$$H_{B*(t_1,\ldots,t_l,y_0)}(E_i) \quad \text{où} \quad y = H(t_1, \ldots, t_l, y_0)\;,\; \forall\, i = 1, \ldots, l$$

coïncident avec les champs $w_i(y)$ relevés contrôlés sur le feuilletage $\mathcal{H} = \{M_y\}_{y \in W}$.

Or, les feuilles $M_y = H(\mathbb{R}^l \times y_0)$ ont pour espaces tangents

$$\begin{aligned} T_y\mathcal{H}_y = T_yM_y &= H_{*(t_1,\ldots,t_l,y_0)}(\mathbb{R}^l \times 0) \\ &= \left[\left\{H_{*(t_1,\ldots,t_l,y_0)}(E_i)\right\}_{i \leq l}\right] = [\{w_i(y)\}_{i \leq l}\,] \end{aligned}$$

et donc *4)* est valable si et seulement si $\forall\, x \in A$ et $\forall\, i \leq l$:

$$\begin{aligned} \lim_{(t_1,\ldots,t_l,y_0)\to x} H_{*(t_1,\ldots,t_l,y_0)}(E_i) = E_i \quad &\Leftrightarrow \quad \lim_{y\to x} w_i(y) = E_i \\ &\Leftrightarrow \quad \lim_{y\to x} T_y\mathcal{H}_y = \mathbb{R}^l \times 0\,, \end{aligned}$$

i.e. si et seulement si $\mathcal{H}$ est (a)-régulier sur $X \equiv U_{x_0} \equiv \mathbb{R}^l \times 0$.

($4 = 5 \Rightarrow 1$). Comme $\mathcal{H}$ est (a)-régulier sur U_{x_0}, $\mathcal{D}'_X$ est alors continue sur $X \equiv U_{x_0}$ et une distribution canonique relative à la strate X.

En admettant comme feuilletage $\mathcal{H}$, $\mathcal{D}'_X$ est intégrable et donc par le lemme 3 les relevés des champs canoniques E_i dans $\mathcal{D}'_X$ sont les champs w_i continus sur X.

De même, pour tout champ de vecteurs ξ_X sur X, son relevé $\xi = \{\xi_Y\}_{Y \geq X}$ tangent à $\mathcal{H}$, coïncide avec le relevé canonique de ξ_X dans $\mathcal{D}'_X$ qui est continu sur X.

On conclut alors, grâce au théorème 4 du §4.1, que son flot (à tout instant $t \in \mathbb{R}$) $\psi = \{\psi_Y^t : Y \to Y\}_{Y \geq X}$ est horizontalement-C^1 sur X.

($1 \Rightarrow 2$). C'est évident en considérant $w_i(y)$ comme relèvement de $\xi_X = E_i$, pour tout $i = 1, \ldots, l$.

($2 \Rightarrow 4$). Les champs de vecteurs $w_1, \ldots, w_l$ coïncident (lemme 3) avec les relèvements contrôlés tangents à $\mathcal{H}$ et

$$T_y\mathcal{H}_y = T_yM_y = [w_1(y), \ldots, w_l(y)]\,.$$

Par hypothèse, $w_1, \ldots, w_l$ étant continus sur X on a $\lim\limits_{y \to x} w_i(y) = E_i$.

Donc : $\quad \lim\limits_{y \to x} T_y\mathcal{H}_y = \lim\limits_{y \to x}[w_1(y), \ldots, w_l(y)] = [E_1, \ldots, E_l] = T_xX$.

Q.E.D.

La remarque ci-dessous est élémentaire.

Remarque 14. *Le feuilletage $\mathcal{H}$ est (a)-régulier sur $X \equiv U_{x_0}$ si et seulement si le morphisme stratifié de projection horizontale π'*

$$\pi' : \pi_X^{-1}(U_{x_0}) \longrightarrow \pi_X^{-1}(x_0) \quad , \quad \pi'(M_{y_0}) = y_0$$

vérifie la condition (a_f) de Thom sur $X \equiv U_{x_0}$.

Nous introduisons maintenant la notion d'application π'-contrôlée résumant les propriétés essentielles qui nous ont permis de démontrer les théorèmes de régularité horizontalement-C^1 et $\mathcal{H}$-semidifférentiabilité des flots des champs relevés du §4.1 et §4.2. Ceci permettra d'obtenir des théorèmes analogues pour des morphismes stratifiés plus généraux.

Le feuilletage horizontal $\mathcal{H} = \mathcal{H}_{x_0}$ n'est pas intrinsèque car il dépend de $x_0 \in X$ "centre" de la trivialisation H et de l'ordre de composition des flots $\phi_1, \ldots, \phi_l$ qui définissent H (mais si $\mathcal{D}_X$ est involutive, $\mathcal{H}$ devient intrinsèque !). Par conséquent, la projection horizontale $\pi' : W = \pi_X^{-1}(U_{x_0}) \to \pi_X^{-1}(x_0)$ n'est pas intrinsèque non plus.

Définition 10. *Soient $f = \{f_Y\}_Y : \mathcal{X} \to \mathcal{X}'$ un morphisme stratifié, $X \in \Sigma$, $x_0 \in X$, $X' \in \Sigma'$ la strate telle que $f(X) \subseteq X'$, $x_0' = f_X(x_0)$, $W = \pi_X^{-1}(U_{x_0})$ et $W' = \pi_{X'}^{-1}(U'_{x_0'})$.*

On dira que f est π'-contrôlé (par rapport aux feuilletages horizontaux $\mathcal{H} = \{M_y\}_{y \in W}$ de A et $\mathcal{H}' = \{M'_y\}_{y' \in W'}$ de A') si pour toute feuille $M_y \in \mathcal{H}$, on a $f_Y(M_y) \subseteq M_{y'}$ où $y' = f_Y(y)$ (i.e. si f envoie chaque feuille horizontale de $\mathcal{H}$ dans une feuille de $\mathcal{H}'$).

D'après le lemme 2 du §4.3, $M_y = {\pi'}^{-1}_{XY}(\pi'_{XY}(y))$ et donc f est π'-contrôlé si et seulement s'il vérifie la *condition de contrôle horizontale*

$$f_Y\big({\pi'}^{-1}_{XY}\big(\pi'_{XY}(y)\big)\big) \subseteq {\pi'}^{-1}_{X'Y'}\big(\pi'_{X'Y'}\big(f_Y(y)\big)\big) \quad , \quad \forall\, y \in Y \ \text{ et } \ \forall\, Y \geq X\,.$$

Le théorème 9 étend à des morphismes stratifiés plus généraux les résultats du §4.1.

Théorème 9. *Soit $f = \{f_Y\}_{Y\in\Sigma} : \mathcal{X} \to \mathcal{X}'$ un morphisme stratifié contrôlé entre deux espaces stratifiés (c)-réguliers $\mathcal{X}$ et $\mathcal{X}'$.*

Soient $\mathcal{H} = \{M_y\}_{y\in W}$ et $\mathcal{H}' = \{M_{y'}\}_{y'\in W'}$ deux feuilletages stratifiés respectivement du voisinage $W = \pi_X^{-1}(U_{x_0})$ de $x_0 \in X$ dans A et du voisinage $W' = \pi_{X'}^{-1}(U'_{x'_0})$ de $x'_0 = f(x_0) \in X'$ dans A'.

Si $\mathcal{H}$ et $\mathcal{H}'$ sont (a)-réguliers sur U_{x_0} et $U'_{x'_0}$ et si f est π'-contrôlé par rapport à $\mathcal{H}$ et $\mathcal{H}'$, alors f est horizontalement-C^1 sur U_{x_0}.

Preuve. Supposons $A \subseteq \mathbb{R}^n$, $A' \subseteq \mathbb{R}^m$ et considérons les distributions tangentes aux feuilletages $\mathcal{H} = \{M_y\}_{y\in W}$ et $\mathcal{H}' = \{M_{y'}\}_{y'\in W'}$, définies localement sur W et W' par :

$$\mathcal{D}_X = \mathcal{D}(\mathcal{H}) \quad , \quad \mathcal{D}_X = \{\mathcal{D}_{XY}\}_{Y\geq X} \quad , \quad \mathcal{D}_{XY}(y) = T_y M_y$$

$$\mathcal{D}_{X'} = \mathcal{D}(\mathcal{H}') \quad , \quad \mathcal{D}_{X'} = \{\mathcal{D}_{X'Y'}\}_{Y'\geq X'} \quad , \quad \mathcal{D}_{X'Y'}(y') = T_{y'} M_{y'}\,.$$

Comme $\mathcal{H}$ et $\mathcal{H}'$ sont deux feuilletages stratifiés de dimensions $\dim \mathcal{H} = \dim X$ et $\dim \mathcal{H}' = \dim X'$, les feuilles $M_y \in \mathcal{H}$ et $M_{y'} \in \mathcal{H}'$ sont transverses aux projections π_X et $\pi_{X'}$, contenues dans les fibres des fonctions distances ρ_X et $\rho_{X'}$, et $\mathcal{H}$ et $\mathcal{H}'$ sont (a)-réguliers sur U_{x_0} et sur $U'_{x'_0}$, alors $\mathcal{D}_X$ et $\mathcal{D}_{X'}$ définissent deux distributions canoniques locales relatives respectivement aux strates U_{x_0} et $U'_{x'_0}$ de W et W'.

Fixons une strate $Y \geq X$ de Σ. Soit $Y' \geq X'$ la strate de Σ' contenant $f_Y(Y)$, et pour tout $y \in Y$ notons $y' = f(y) \in Y'$.

Comme $f = \cup_{Z\in\Sigma} f_Z$ est π'-contrôlée par rapport aux feuilletages $\mathcal{H}$ et $\mathcal{H}'$ nous avons $f_Y(M_y) \subseteq M_{y'}$ et donc pour tout $y \in Y$ l'application $f_{Y*y} : T_y Y \to T_{y'} Y'$ vérifie

$$f_{Y*y}(\mathcal{D}_{XY}(y)) = f_{Y*y}(T_y M_y) \subseteq T_{y'} M_{y'} = \mathcal{D}_{X'Y'}(y')\,.$$

Or, f est π-contrôlée d'où $f_Y(\ker \pi_{XY\,*y}) \subseteq \ker \pi_{X'Y'\,*y'}$ et f_{Y*y} se décompose en somme directe

$$f_{Y*y} = f_y^h \oplus f_y^v \ : \ \mathcal{D}_{XY}(y) \oplus \ker \pi_{XY\,*y} \ \longrightarrow \ \mathcal{D}_{X'Y'}(y') \oplus \ker \pi_{X'Y'\,*y'}\,.$$

Soit $p \in U_{x_0}$ et $p' = f(p)$.

Afin de montrer que f est horizontalement-C^1 en p, U_{x_0} et $U'_{x'_0}$ étant des domaines des systèmes de coordonnées locales de X et X', on prend $U_{x_0} \equiv \mathbb{R}^l \times 0^{n-l}$ et $U'_{x'_0} \equiv \mathbb{R}^{l'} \times 0^{m-l'}$.

Notons alors $\sigma = (E_1, \ldots, E_l)$ le champ de repères constants coordonnées de U_{x_0} et $\sigma_y = (v_1(y), \ldots, v_l(y))$ les champs de repères relevés continus contrôlés dans $\mathcal{D}_X$.

De même considérons les champs de repères $\sigma' = (E'_1, \ldots, E'_{l'})$ et $\sigma_{y'} = (v'_1(y'), \ldots, v'_{l'}(y'))$ et pour tout $y \in W$ notons $x = \pi_{XY}(y)$.

Comme f est π-contrôlée, on a l'égalité :

$$\pi_{X'Y'*y'} f_{Y*y} = f_{X*x} \pi_{XY*y} \quad , \quad \forall\, y \in Y \ \text{ et } \forall\, Y > X$$

grâce à laquelle il est facile de vérifier que la matrice $A(y)$ qui représente $f_y^h : \mathcal{D}_{XY}(y) \to \mathcal{D}_{X'Y'}(y')$ par rapport aux bases $\sigma_y = \{v_i(y)\}_{i=1}^{l}$ et $\sigma_{y'} = \{v'_j(y')\}_{j=1}^{l'}$ coïncide avec la matrice $A = A(x) = (A^i_j)_{i,j}$ qui représente l'application linéaire $f_{X*x} : T_x X \to T_{x'} X'$ par rapport aux bases canoniques $\sigma = (E_1, \ldots, E_l)$ et $\sigma' = (E'_1, \ldots, E'_{l'})$.

Cela s'obtient, comme pour le théorème 3 dans le §4.1, en observant que $\forall\, i = 1, \ldots, l$ les deux vecteurs $f_{Y*y}(v_i(y))$ et $\sum_{j=0}^{l'} A^i_j v'_j(y')$ appartiennent à $\mathcal{D}_{X'Y'}(y')$, qu'ils ont la même image via la restriction de la projection $\pi_{X'Y'*y'|} : \mathcal{D}_{X'Y'}(y') \to T_{x'} X'$ et que cette dernière est un isomorphisme car $\mathcal{D}_{X'}$ est une distribution canonique.

La preuve suit alors d'une répétition formelle de celle du théorème 4 du §4.1 grâce à la continuité en $p' = f(p)$ des relèvements canoniques $(v'_1, \ldots, v'_{l'})$ dans $\mathcal{D}_{X'}$. Q.E.D.

On a vu au §2 que si les projections d'un S.D.C. d'une stratification (A, Σ) sont des applications C^1, alors toute application contrôlée $f : (A, \Sigma) \to M$ à valeurs dans une variété M est semidifférentiable. Si de plus l'application est une submersion propre, le 1^{er} *Théorème d'isotopie de Thom* dit alors que l'application f est une fibration topologiquement localement triviale (voir par exemple [Ma]).

Notons $\forall\, \epsilon > 0$, $U^{\epsilon}_{x_0} = \{x \in U_{x_0} \mid d(x, X - U_{x_0}) > \epsilon\}$.

Si δ note le diamètre de U_{x_0}, alors pour ϵ suffisamment petit et $< 1/2\delta$, $U^{\epsilon}_{x_0}$ est un ouvert vérifiant $\overline{U^{\epsilon}_{x_0}} \subseteq U_{x_0}$, qui est encore un voisinage de x_0 dans la strate $X \in \Sigma$ et un domaine d'un système de coordonnées locales autour de x_0.

Les théorèmes 8 et 9 permettent de montrer que toute stratification (c)-régulière vérifiant autour d'un point x_0 la conjecture du feuilletage

(a)-régulier admet un isomorphisme de trivialisation topologique localement horizontalement-C^1 (pour la preuve ici omise voir [Mu]).

Théorème 10. (1^{er} théorème d'Isotopie horizontalement-C^1).
Soit $\mathcal{X} = (A, \Sigma)$ une stratification (c)-régulière, $X \in \Sigma$ et $x_0 \in X$ un point admettant un feuilletage $\mathcal{H}$=$\{M_y\}_{y\in W}$, (a)-régulier sur U_{x_0} du voisinage W=$\pi_X^{-1}(U_{x_0})$ de x_0 dans A.

Soit $f : (A, \Sigma) \to M$ une submersion stratifiée propre à valeurs dans une variété lisse. Pour tout $m_0 \in M$, pour tout domaine d'un système de coordonnées locales U_{m_0} de m_0 dans M et pour tout $U^\epsilon_{x_0} \subseteq U_{x_0}$, il existe alors un homéomorphisme stratifié

$$H \quad : \quad U_{m_0} \times f^{-1}(m_0) \longrightarrow f^{-1}(U_{m_0})$$

horizontalement-C^1 sur $U_{m_0} \times \left[f^{-1}(m_0) \cap U^\epsilon_{x_0}\right]$, et l'homéomorphisme stratifié réciproque

$$G = H^{-1} \quad : \quad f^{-1}(U_{m_0}) \longrightarrow U_{m_0} \times f^{-1}(m_0)$$

est horizontalement-C^1 sur $f^{-1}(U_{m_0}) \cap U^\epsilon_{x_0}$.

5.2. $\mathcal{H}$-semidifférentiabilité.

Dans les théorèmes 8, 9 et 10 de la section précédente, nous avons vu que l'existence d'un feuilletage local $\mathcal{H} = \{M_z\}_{z\in W}$ de $W = \pi_X^{-1}(U_{x_0})$, (a)-régulier sur U_{x_0} implique la régularité horizontalement-C^1 pour les flots des champs relevés et pour d'autres morphismes stratifiés plus généraux.

Dans cette section, nous précisons que si la (a)-régularité de $\mathcal{H}$ est valable sur le voisinage W, les théorèmes analogues à la section §5.1 deviennent valables en déduisant de plus la propriété de $\mathcal{H}$-semidifférentiabilité.

Rappelons que la $\mathcal{H}$-semidifférentiabilité donne, en plus de la régularité horizontalement-C^1, un contrôle des limites des restrictions $\lim_{z\to y'} f_{Z*z|T_zM_z} = f_{Y*y'|T_{y'}M_{y'}}$ quand z tend vers un point y' appartenant à une strate Y supérieure à X (voir le §2.3 pour la définition et le §4.2 pour quelques théorèmes). Nous avons alors :

Théorème 11. *Les conditions suivantes sont équivalentes :*

1) Le relèvement contrôlé $\xi = \{\xi_Y\}_{Y\geq X}$ tangent à $\mathcal{H} = \{M_z\}_{z\in W}$ de tout champ de vecteurs ξ_X est continu sur W et a un flot $\psi = \{\psi^t_Y\}_{Y\geq X}$ qui est $\mathcal{H}$-semidifférentiable.

2) Pour tout $i = 1, \ldots, l$, les relèvements contrôlés w_i des champs E_i tangents à $\mathcal{H} = \{M_z\}_{z\in W}$ sont continus sur W et ont des flots $\psi_i = \{\psi^t_{iY} : Y \to Y\}_{Y\geq X}$ qui sont $\mathcal{H}$-semidifférentiables.

3) L'homéomorphisme stratifié de trivialisation de π_X autour de x_0,

$$H : \mathbb{R}^l \times \pi_X^{-1}(x_0) \to \pi_X^{-1}(U_{x_0}) \quad \textit{est} \quad \mathcal{F}\textit{-semidifférentiable}$$

par rapport au feuilletage trivial $\mathcal{F} = \{\mathbb{R}^l \times y_0\}_{y_0 \in \pi_X^{-1}(x_0)}$ de $\mathbb{R}^l \times \pi_X^{-1}(x_0)$.

4) $\lim_{(t_1,\ldots,t_l,z_0)\to y'} H_{*(t_1,\ldots,t_l,z_0)}(E_i) = w_i(y')\,, \quad \forall\, i\,, \forall\, y' \in Y, \forall\, Y \geq X.$

5) Le feuilletage horizontal $\mathcal{H} = \{M_z\}_{z\in W}$ induit par H est (a)-régulier sur W.

Preuve. La démonstration s'obtient de manière complètement analogue à celle du théorème 8 au §5.1 en utilisant que la (a)-régularité du feuilletage $\mathcal{H}$ sur W équivaut à ce que les champs $w_1(z), \ldots, w_l(z)$ tangents à $\mathcal{H}$ soient continus sur les strates de W. Nous soulignons que la conclusion de la preuve de $(4 = 5 \Rightarrow 1)$ s'obtient en rappelant le théorème 6 du §4. 2 au lieu du théorème 4 du §4.1. Q.E.D.

Pour un morphisme stratifié plus général on obtient :

Théorème 12. *Soit $f = \{f_Y\}_{Y\in\Sigma} : \mathcal{X} \to \mathcal{X}\,'$ un morphisme stratifié contrôlé entre deux espaces stratifiés (c)-réguliers $\mathcal{X}$ et $\mathcal{X}\,'$.*

Soient $\mathcal{H} = \{M_y\}_{y\in W}$ et $\mathcal{H}\,' = \{M_{y'}\}_{y'\in W'}$ deux feuilletages stratifiés respectivement du voisinage $W = \pi_X^{-1}(U_{x_0})$ de $x_0 \in X$ dans A et du voisinage $W' = \pi_{X'}^{-1}(U'_{x'_0})$ de $x'_0 = f(x_0) \in X'$ dans A'.

Si $\mathcal{H}$ et $\mathcal{H}\,'$ sont (a)-réguliers et si f est π'-contrôlé par rapport à $\mathcal{H}$ et $\mathcal{H}\,'$, alors f est $\mathcal{H}$-semidifférentiable.

Preuve. On remarque (par rapport à la preuve du théorème 9, §5.1) que, les feuilletages $\mathcal{H}$ et $\mathcal{H}\,'$ étant maintenant (a)-réguliers sur W et W', les distributions canoniques $\mathcal{D}_X$ et $\mathcal{D}_{X'}$ induites sont continues sur les strates de W et W'.

Si on fixe des strates $Z > Y \geq X$, comme au théorème 9 du §5.1 on trouve que pour tout $z \in Z$ il existe une restriction de la différentielle $f_{Z*z|\mathcal{D}_{XZ}(z)} : \mathcal{D}_{XZ}(z) \to \mathcal{D}_{X'Z'}(z')$ avec laquelle, en utilisant la condition de contrôle $\pi_{Y'Z'*z'} f_{Z*z} = f_{Y*y} \pi_{YZ*z}$, $\forall\ z \in Z$ et $\forall Z > Y$ ainsi que le fait que la projection $\pi_{Y'Z'} : T_{YZ} \to Y'$ induise un isomorphisme de restriction $\pi_{Y'Z'*z'|} : \mathcal{D}_{X'Z'}(z') \to \mathcal{D}_{X'Y'}(y')$, on peut conclure de la même manière que dans le théorème 9 du §5.1 et le théorème 6 du §4.2. Q.E.D.

De même que pour les théorèmes 9 et 10, le théorème 12 permet de d'obtenir le suivant:

Théorème 13. (1^{er} théorème d'Isotopie $\mathcal{F}$-semidifférentiable [Mu]).

Soient $\mathcal{X}$ un espace stratifié (c)-régulier, X une strate de Σ et $x_0 \in X$ tels qu'il existe un feuilletage (a)-régulier $\mathcal{H} = \{M_y\}_{y \in W}$, du voisinage $W = \pi_X^{-1}(U_{x_0})$ de x_0 dans A.

Soit $f : (A, \Sigma) \to M$ une submersion stratifiée propre à valeurs dans une variété lisse.

Pour tout point m_0 dans M et pour tout domaine U_{m_0} d'un système de coordonnées locales de m_0 dans M et pour tout $U_{x_0}^{\epsilon} \subseteq U_{x_0}$, il existe un homéomorphisme stratifié

$$H : U_{m_0} \times f^{-1}(m_0) \to f^{-1}(U_{m_0}) \quad \textit{qui est } \mathcal{F}\textit{-semidifférentiable}$$

par rapport à $\mathcal{F} = U_{m_0} \times \mathcal{H}_{|f^{-1}(m_0) \cap \pi_X^{-1}(U_{x_0}^{\epsilon})}$ et dont l'homéomorphisme réciproque

$$G = H^{-1} : \; f^{-1}(U_{m_0}) \to U_{m_0} \times f^{-1}(m_0)$$

est $\mathcal{H}_{|f^{-1}(U_{m_0}) \cap \pi_X^{-1}(U_{x_0}^{\epsilon})}$-semidifférentiable.

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Submanifolds with a non-degenerate parallel normal vector field in euclidean spaces

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Abstract.

We consider the class of submanifolds M in an euclidean space $\mathbb{R}^n$ which admit a non-degenerate parallel normal vector field ν. The image of the associated Gauss map $G_\nu : M \to S^{n-1}$ defines an immersed hyperspherical submanifold M^ν which has the following property: if M has a contact of Boardman type $\Sigma^{i_1,\dots,i_k}$ with a hyperplane, then M^ν has the same contact type with the translated hyperplane. In particular, for a space curve α in $\mathbb{R}^3$, the spherical curve α^ν has the same flattenings and we deduce an extension of the Four Vertex Theorem. For an immersed surface M in $\mathbb{R}^4$, it admits a local non-degenerate parallel normal vector field if and only if it is totally semi-umbilic and has non zero gaussian curvature K. Moreover, G_ν preserves the inflections and the asymptotic lines between M and M^ν. As a consequence, we deduce an extension for this class of surfaces of the classical Loewner and Carathéodory conjectures for umbilic points of analytic immersed surfaces in $\mathbb{R}^3$.

§1. Introduction

Let $\xi : \mathbb{R}^n \to S^n \hookrightarrow \mathbb{R}^{n+1}$ denote the inverse of the stereographic projection and let M be an m-dimensional submanifold of $\mathbb{R}^n$. It has been shown by Sedykh [16] and Romero-Fuster [12, 13] that the contacts of M with the hyperspheres of $\mathbb{R}^n$ are the same as the contacts of its hyperspherical image $\xi(M)$ with the hyperplanes of $\mathbb{R}^{n+1}$. Since many of the differential-geometric aspects of a submanifold M of an euclidean space $\mathbb{R}^n$ can be translated in terms of contacts between hyperspheres or hyperplanes, they use this fact in order to obtain interesting relations between some special points of the submanifold M and its image $\xi(M)$.

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For instance, suppose that $\alpha : I \to \mathbb{R}^2$ defines a regular smooth curve in the plane. Then ξ transforms the vertices (points where $\kappa' = 0$) of α into the flattenings (points with $\tau = 0$) of the spherical space curve $\xi \circ \alpha$ in $\mathbb{R}^3$. This is due to the fact that a vertex corresponds to a point of contact of type $\Sigma^{1,1,1}$ (using Thom-Boardman notation) between the curve and a circle in the plane, while a flattening of a space curve is a point of contact of the same type $\Sigma^{1,1,1}$ with the osculating plane. One consequence is that it is possible to translate the classical Four Vertex Theorem in terms of flattenings and spherical space curves: every closed regular and simple space curve contained in the sphere S^2 has at least four flattenings. In fact, this observation led to the conjecture by Scherk in 1936 that every convex closed and simple space curve with non vanishing curvature has at least four flattenings, which was proved by Sedykh [15].

In the case of a smooth immersed surface M in $\mathbb{R}^3$ we have a similar history with respect to umbilics and the Carathéodory conjecture. The classical Carathéodory conjecture states that every smooth convex embedding of a 2-sphere in $\mathbb{R}^3$ must have at least two umbilics, i.e., points where the two principal curvatures coincide. This conjecture has a stronger local version, known as the Loewner conjecture, which states that the index of the principal foliation at any isolated umbilic of an immersed smooth surface in $\mathbb{R}^3$ is always ≤ 1. Since the sum of the indices of the umbilics of a compact immersed surface is equal to its Euler-Poincaré characteristic (according to the Poincaré-Hopf formula) it follows that the Loewner conjecture implies the Carathéodory conjecture, not only for a convex embedding of a 2-sphere, but for any immersion (not necessarily convex). The Loewner conjecture is known to be true in the analytic case (although there is a big controversy about the correct proof, see for instance [17, 7]).

Now, it is also possible to characterize an umbilic of M as a point which presents a contact of type $\Sigma^{2,2}$ between M and a sphere of $\mathbb{R}^3$. It follows that the map ξ will give a point of $\Sigma^{2,2}$ contact between the hyperspherical surface $\xi(M)$ and some hyperplane of $\mathbb{R}^4$. But this type of contact corresponds to an inflection of the surface in the sense of Little [8] (that is, a point where the two fundamental forms are collinear). In particular, we have that any analytic immersed surface M in $S^3 \subset \mathbb{R}^4$, homeomorphic to S^2, has at least two inflections. Moreover, it was also observed by Little that ξ also takes the principal foliation of M into the asymptotic foliation of $\xi(M)$ and it is also possible to translate Loewner conjecture: for any analytic immersed surface M in $S^3 \subset \mathbb{R}^4$, the index of the asymptotic foliation at any isolated inflection is always ≤ 1.

Now, it is natural to ask whether these results can be extended to general analytic surfaces immersed in $\mathbb{R}^4$. Since the asymptotic foliation is only defined in the convex part of the surface, it is obvious that we have to restrict ourselves to locally convex surfaces (that is, at any point there is a hyperplane which locally supports the surface). A proof of the Carathéodory conjecture for generic locally convex surfaces in $\mathbb{R}^4$ can be found in [4] (in fact, for a generic locally convex surface M, the index of an isolated inflection is always $\pm 1/2$ and hence, it must have at least $2\chi(M)$ inflections). In [6], they give a proof of the Loewner conjecture for a locally convex surface in $\mathbb{R}^4$ which satisfies some non-degeneracy condition with respect to the Newton polyhedra. Some results about the index of an isolated inflection of an immersed surface in $\mathbb{R}^4$ can be also found in [3].

In this paper, we consider the class of smooth submanifolds M immersed in $\mathbb{R}^n$ which admit a non-degenerate parallel normal vector field ν. This class appears in the literature in the context of differential geometry of submanifolds (see for instance [10]). We show that the image of the associated Gauss map $G_\nu : M \to S^{n-1}$ defines an immersed hyperspherical submanifold M^ν which has the following property: if M has a contact of Boardman type $\Sigma^{i_1,\dots,i_k}$ with a hyperplane, then M^ν has the same contact type with the translated hyperplane. Thus, for instance, in the case of a space curve α in $\mathbb{R}^3$, the spherical curve α^ν has the same flattenings and we deduce an extension of the Four Vertex Theorem.

For an immersed surface M in $\mathbb{R}^4$, it admits a non-degenerate parallel normal vector field if and only if it is totally semi-umbilic and has non zero gaussian curvature K. The semi-umbilic condition means that at any point of the surface, there is a non-zero normal vector ν such that the ν-principal curvatures are equal and it has been studied recently by Romero-Fuster and Sánchez-Bringas (see [14]). The totally semi-umbilic surfaces in $\mathbb{R}^4$ with $K \neq 0$ are an intermediate class between the class of hyperspherical surfaces and the class of locally convex surfaces. Moreover, we show that the Gauss map G_ν preserves the inflections and the asymptotic lines between M and the hyperspherical image M^ν. As a consequence, we obtain that Loewner and Carathéodory conjectures are also true for analytic totally semi-umbilic surfaces with $K \neq 0$.

§2. Contact with hyperspheres and hyperplanes

In this section, we recall basic definitions and properties of contact between submanifolds of an ambient manifold and the relationship with

$\mathcal{K}$-equivalence of map germs due to Montaldi [9]. We begin with the notion of contact.

Definition 2.1. Let M, N, M', N' be smooth submanifolds of $\mathbb{R}^n$ and let $x_0 \in M \cap N$ and $x'_0 \in M' \cap N'$. We say that *the contact of M and N at x_0 is of the same type as the contact of M' and N' at x'_0* if there are open neighbourhoods U of x_0 and U' of x'_0 in $\mathbb{R}^n$ and a diffeomorphism $\phi : U \to U'$ such that $\phi(U \cap M) = U' \cap M'$ and $\phi(U \cap N) = U' \cap N'$.

Now, we recall the concept of $\mathcal{K}$-equivalence between two smooth map germs.

Definition 2.2. Consider two smooth map germs $f : (M, x_0) \to (N, y_0)$ and $g : (M', x'_0) \to (N', y'_0)$ between smooth manifolds. We say that f, g are *$\mathcal{K}$-equivalent* if there exists a diffeomorphism

$$H : (M \times N, (x_0, y_0)) \to (M' \times N', (x'_0, y'_0))$$

such that:

(1) $H(x, y) = (h(x), \theta(x, y))$ for some map germs $h : (M, x_0) \to (M', x'_0)$ and $\theta : (M \times N, (x_0, y_0)) \to (N', y'_0)$.
(2) $\theta(x, y_0) = y'_0$ for any x in a neighbourhood of x_0 in M.
(3) $H(x, f(x)) = (h(x), g(h(x)))$ for any x in a neighbourhood of x_0 in M.

In order to see the relationship between contact and $\mathcal{K}$-equivalence we need to introduce some notations. Let M, N, M', N' be smooth submanifolds of $\mathbb{R}^n$ and let $x_0 \in M \cap N$ and $x'_0 \in M' \cap N'$. We assume that M, M' are locally given by the image of an embedding. That is, there are open neighbourhoods W_1 of x_0 in $\mathbb{R}^n$ and W_2 of x'_0 in $\mathbb{R}^n$, open subsets $U_1, U_2 \subset \mathbb{R}^m$ and smooth embeddings $f_1 : U_1 \to \mathbb{R}^n$ and $f_2 : U_2 \to \mathbb{R}^n$ such that $f_1(U_1) = M \cap W_1$ and $f_2(U_2) = M' \cap W_2$ (here m is the dimension of M and M'). We also denote $f_1(u_0) = x_0$ and $f_2(u'_0) = x'_0$.

For N, N' we assume that they are given locally in implicit forms. That is, there are smooth maps $g_1 : W_1 \to \mathbb{R}^p$ and $g_2 : W_2 \to \mathbb{R}^p$ such that $W_1 \cap N = g_1^{-1}(v_0)$, with v_0 a regular value of g_1 and $W_2 \cap N' = g_2^{-1}(v'_0)$, with v'_0 a regular value of g_2 (now, p is the codimension of N and N').

Theorem 2.3. [9] *With the above notation, it follows that the contact of M and N at x_0 is of the same type as the contact of M' and N' at x'_0 if and only if the map germs $g_1 \circ f_1 : (\mathbb{R}^m, u_0) \to (\mathbb{R}^p, v_0)$ and $g_2 \circ f_2 : (\mathbb{R}^m, u'_0) \to (\mathbb{R}^p, v'_0)$ are $\mathcal{K}$-equivalent.*

Definition 2.4. The map $g_1 \circ f_1 : U \to \mathbb{R}^p$ is called the *contact map* of M, N. It follows that its $\mathcal{K}$-singularity type at each point u_0 determines the contact of M, N at $x_0 = f_1(u_0)$ and does not depend on the choice of maps f_1, g_1.

In the case that we have a submanifold in Euclidean space $M \subset \mathbb{R}^n$, the most interesting contacts are those of M with hyperplanes and hyperspheres of $\mathbb{R}^n$, since they determine some of the geometrical invariants of M.

Assume that the embedding $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ locally parametrizes the submanifold $M \subset \mathbb{R}^n$ in a neighbourhood of $x_0 = f(u_0)$. For any $v \in S^{n-1}$, we consider the *height function* $h_v : U \to \mathbb{R}$, given by $h_v(u) = \langle f(u), v\rangle$. Then, h_v is the contact map of M and the hyperplane $\pi(x_0, v)$ of $\mathbb{R}^n$ through x_0 perpendicular to v. It is obvious that $u_0 \in U$ is a singular point of h_v if and only if v belongs to the normal subspace of M at x_0, that is, $\pi(x_0, v)$ is tangent to M at x_0.

Analogously, given $p \in \mathbb{R}^n$, we can also consider the *distance squared function* $d_p : U \to \mathbb{R}$, given by $d_p(u) = \|f(u) - p\|^2$. Now, d_p is the contact map between M and the hypersphere $S(p, R)$ of $\mathbb{R}^n$ centered at p with radius $R = d_p(u_0)$. Again, $u_0 \in U$ is a singular point of d_p if and only if p is in the (affine) normal subspace of M at x_0, that is, $S(p, R)$ is tangent to M at x_0.

Now, we recall the Thom-Boardman symbols $\Sigma^{i_1,\dots,i_k}$, which are a generalization of the rank of a map taking into account higher order derivatives and provide a useful invariant for $\mathcal{K}$-equivalence.

Let us denote by $\mathcal{E}_{m,x_0}$ the local ring of smooth function germs from $(\mathbb{R}^m, x_0)$ to $\mathbb{R}$. Given a $p \times q$ matrix U with entries in $\mathcal{E}_{m,x_0}$, we denote by $I_t(U)$ the ideal in $\mathcal{E}_{m,x_0}$ generated by the t-minors of U (by convention, $I_t(U) = \{0\}$ if $t > \min(p, q)$). In particular, if $f : (\mathbb{R}^m, x_0) \to (\mathbb{R}^p, y_0)$ is a smooth map germ, $I_t(Df)$ is the ideal generated by the t-minors of the jacobian matrix $Df = (\partial f_i/\partial x_j)$.

Definition 2.5. Let $f : (\mathbb{R}^m, x_0) \to \mathbb{R}^p$ be a smooth map germ and let $\mathbf{i} = (i_1, \dots, i_k)$ be a k-tuple of non-negative integer numbers. We define the *iterated jacobian extension* of f by induction on k. If $k = 1$, then $J_{i_1}(f) = I_{n-i_1+1}(Df)$. For $k > 1$, suppose that $J_{i_1,\dots,i_{k-1}}(f) = \langle g_1, \dots, g_r\rangle$, then

$$J_{i_1,\dots,i_k}(f) = J_{i_1,\dots,i_{k-1}}(f) + I_{n-i_k+1}(D(f, g)),$$

where $(f, g) = (f_1, \dots, f_p, g_1, \dots, g_r)$.

We say that f has *Boardman type* (or *Boardman symbol*) $\Sigma^{\mathbf{i}}$ if f has rank $n-i_1$ at x_0 and for $k>1$, $(f_1,\dots,f_p,g_1,\dots,g_r)$ has rank $n-i_k$ at x_0, being $g_1,\dots,g_r$ generators of the ideal $J_{i_1,\dots,i_{k-1}}(f)$.

Example 2.6. Given a smooth function $f: U\subset\mathbb{R}^m\to\mathbb{R}$, it has Boardman type $\Sigma^{m,\dots,m}$ at x_0 (with m repeated k times) if and only if all the partial derivatives of f at x_0 are zero up to order k.

We include now a result that will be used in next section.

Lemma 2.7. *Let $f,g:(\mathbb{R}^m,x_0)\to\mathbb{R}$ be two smooth function germs such that $J_m(f)=J_m(g)$. Then f,g have the same Boardman symbol $\Sigma^{i_1,\dots,i_k}$, for any $k\geq 1$.*

Proof. The ideals $J_m(f)=I_1(Df)$ and $J_m(g)=I_1(Dg)$ in $\mathcal{E}_{m,x_0}$ are generated by the partial derivatives $\partial f/\partial x_i$ and $\partial g/\partial x_i$ respectively. The assumption $J_m(f)=J_m(g)$ means that

$$\frac{\partial f}{\partial x_i}=\sum_{j=1}^m a_{ij}\frac{\partial g}{\partial x_j},$$

for some $a_{ij}\in\mathcal{E}_{m,x_0}$, with $\det(a_{ij})\neq 0$. We have that f,g have the same rank at x_0 and hence, the first Boardman number i_1 is the same for f,g.

If f,g are regular, then $i_1=m-1$, $J_{m-1}(f)=I_2(Df)=\{0\}$ and $J_{m-1}(g)=I_2(Dg)=\{0\}$. In particular, the Boardman symbol is $i_2=\dots=i_k=m-1$ for both f,g.

Assume now that f,g are singular and $i_1=m$. We will show by induction on k that f,g have the same Boardman symbol and the same iterated jacobian ideals. Assume that the Boardman numbers $i_1,\dots,i_{k-1}$ are equal for f,g and $J_{i_1,\dots,i_{k-1}}(f)=J_{i_1,\dots,i_{k-1}}(g)=\langle h_1,\dots,h_r\rangle$. The Boardman number i_k for f,g is determined in each case by the rank at x_0 of $(f,h_1,\dots,h_r)$ and $(g,h_1,\dots,h_r)$ respectively. We consider the matrices

$$A=\begin{pmatrix}\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_m}\\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_m}\\ \cdots & \cdots & \cdots\\ \frac{\partial h_r}{\partial x_1} & \cdots & \frac{\partial h_r}{\partial x_m}\end{pmatrix},\quad B=\begin{pmatrix}\frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_m}\\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_m}\\ \cdots & \cdots & \cdots\\ \frac{\partial h_r}{\partial x_1} & \cdots & \frac{\partial h_r}{\partial x_m}\end{pmatrix}.$$

Since $\partial f/\partial x_i$ and $\partial g/\partial x_i$ are 0 at x_0, it follows that A,B have the same rank at x_0 and the Boardman number i_k is the same for f,g.

On the other hand, by definition, $J_{i_1,\dots,i_k}(f)=J_{i_1,\dots,i_{k-1}}(f)+I_t(A)$ and $J_{i_1,\dots,i_k}(g)=J_{i_1,\dots,i_{k-1}}(g)+I_t(B)$, where $t=n-i_k+1$. Let M be a t-minor of A. If M does not contain the first row of A, then M is a t-minor of B. Otherwise, if M contains the first row of A, then $M \in J_m(f) = J_m(g) \subset J_{i_1,\dots,i_k}(g)$. This shows that $J_{i_1,\dots,i_k}(f) \subset J_{i_1,\dots,i_k}(g)$ and the opposite inclusion follows by symmetry. Q.E.D.

In general, it is not true that if $f, g : (\mathbb{R}^m, x_0) \to \mathbb{R}$ are two smooth function germs such that $J_m(f) = J_m(g)$, then they are $\mathcal{K}$-equivalent. For instance, consider $f, g : (\mathbb{R}^2, 0) \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ and $g(x, y) = x^2 - y^2$. In this case, $J_2(f) = J_2(g) = \langle x, y\rangle$, they have Boardman symbol $\Sigma^{2,0}$, but they are not $\mathcal{K}$-equivalent.

One of the well known properties of the Boardman symbol is that it is $\mathcal{K}$-invariant. Hence, it can be associated with each contact class between submanifolds.

Lemma 2.8. *Let $f, g : (\mathbb{R}^m, x_0) \to \mathbb{R}^p$ be two smooth map germs which are $\mathcal{K}$-equivalent. Then f, g have the same Boardman symbol $\Sigma^{i_1,\dots,i_k}$, for any $k \geq 1$.*

(See for instance [5] for a proof.)

Definition 2.9. Given submanifolds $M, N \subset \mathbb{R}^n$ and $x_0 \in M \cap N$, we say that they have *contact type* $\Sigma^{\mathbf{i}}$ if its contact map germ has Boardman type $\Sigma^{\mathbf{i}}$. The above lemma ensures that this definition does not depend on the choice of contact map germs.

Example 2.10. Here we present some known basic examples how the contacts of a submanifold in an euclidean space with hyperplanes or hyperspheres can be useful to characterize several special points in the differential geometry of curves and surfaces.

(1) Let $\alpha : I \to \mathbb{R}^2$ be a regular plane curve. Then α has a $\Sigma^{1,1}$ contact with a line $\pi(x_0, v)$ at $x_0 = \alpha(t_0)$ if and only if this point is an inflection (that is, $\kappa(t_0) = 0$) and v is the normal vector at such point.

(2) Analogously, α has a $\Sigma^{1,1,1}$ contact with a circle $S(p, R)$ at x_0 if and only if this point is a non-flat vertex (that is, $\kappa'(t_0) = 0$ and $\kappa(t_0) \neq 0$), and p and R are the centre and the radius of curvature of α at x_0 respectively. The case of a flat vertex (that is, $\kappa'(t_0) = \kappa(t_0) = 0$) corresponds to a $\Sigma^{1,1,1}$ contact with the tangent line.

(3) In the case of a regular space curve $\alpha : I \to \mathbb{R}^3$ with non vanishing curvature, α has a $\Sigma^{1,1,1}$ contact with a plane $\pi(x_0, v)$ at $x_0 = \alpha(t_0)$ if and only if this point is a flattening (that is, $\tau(t_0) = 0$) and v is the binormal vector at such point.

(4) Let M be a regular surface in $\mathbb{R}^3$. Then, M has a $\Sigma^{2,2}$ contact with a sphere $S(p, R)$ at x_0 if and only if this point is an non-flat

umbilic (that is, a point where the two principal curvatures are equal and distinct from zero) and p and R are the centre and the radius of principal curvature of M at x_0 respectively. The case of a flat umbilic (that is, when both principal curvatures are zero) corresponds to a $\Sigma^{2,2}$ contact with the tangent plane.

(5) Finally, we consider a regular surface M in $\mathbb{R}^4$. Then, M has a $\Sigma^{2,2}$ contact with a hyperplane $\pi(x_0, v)$ at x_0 if and only if this point is an inflection in the sense of Little [8] (that is, a point where the two second fundamental forms are linearly dependent) and v is the corresponding binormal vector.

We finish this section by showing that the contacts with hyperspheres and hyperplanes are related through the stereographic projection. Let $\xi : \mathbb{R}^n \to S^n \hookrightarrow \mathbb{R}^{n+1}$ denote the inverse of the stereographic projection, which is given by

$$\xi(x) = \frac{(2x, \|x\|^2 - 1)}{(\|x\|^2 + 1)}.$$

Since this map is conformal, it follows that it transforms any hypersphere $S(p, R)$ or hyperplane $\pi(x_0, v)$ of $\mathbb{R}^n$ into a $(n-1)$-sphere contained in S^n. We denote by $\pi_S(p, R)$ (respectively $\pi_S(x_0, v)$) the only hyperplane of $\mathbb{R}^{n+1}$ which has the property $\xi^{-1}(\pi_S(p, R)) = S(p, R)$ (respectively $\xi^{-1}(\pi_S(x_0, v)) = \pi(x_0, v)$).

It follows from the works by Romero Fuster [12, 13] and Sedykh [16] that the contact of a submanifold $M \subset \mathbb{R}^n$ and hyperplane $\pi(x_0, v)$ or hypersphere $S(p, R)$at $x_0 \in M$ is of the same type as the contact of $\xi(M)$ and $\pi_S(x_0, v)$ or $\pi_S(p, R)$ at $\xi(x_0)$ respectively. In fact, they show more, namely, that the family of distance squared functions of M in $\mathbb{R}^n$ is $\mathcal{K}$-equivalent to the family of height functions of $\xi(M)$ in $\mathbb{R}^{n+1}$.

Example 2.11. By looking at the examples of 2.10, we get some immediate consequences of this fact. For instance, if $\alpha : I \to \mathbb{R}^2$ is a regular plane curve, then $t_0 \in I$ is a vertex of α if and only if t_0 is a flattening of $\xi \circ \alpha : I \to S^2 \subset \mathbb{R}^3$.

In the case of a regular surface $M \subset \mathbb{R}^3$, it follows that $x_0 \in M$ is an umbilic if and only if $\xi(x_0)$ is an inflection of $\xi(M) \subset S^3 \subset \mathbb{R}^4$.

In general, which we can conclude is that in $\mathbb{R}^{n+1}$, the class of hyperspherical submanifolds (that is, submanifolds contained in some hypersphere of $\mathbb{R}^{n+1}$), presents the same contacts with hyperplanes of $\mathbb{R}^{n+1}$ as the submanifolds of $\mathbb{R}^n$ with respect to hyperspheres or hyperplanes.

Question 2.12. *Determine the submanifolds of* $\mathbb{R}^{n+1}$ *which have the same contacts with hyperplanes as the hyperspherical submanifolds.*

In the next section, we give a partial answer to this question, by considering submanifolds which admit a non-degenerate parallel normal vector field.

§3. Submanifolds with a non-degenerate parallel normal vector field

Let M be a smooth immersed m-dimensional submanifold in $\mathbb{R}^n$. We consider in M the riemannian metric induced by the euclidean metric of $\mathbb{R}^n$. Given a point $p \in M$, we have a decomposition $\mathbb{R}^n = T_pM \oplus T_pM^\perp$ and the corresponding orthogonal projections $\top : \mathbb{R}^n \to T_pM$ and $\perp : \mathbb{R}^n \to T_pM^\perp$. For vector fields X, Y tangent along M in a neighbourhood of p, we have

$$\nabla'_{X_p} Y = \top(\nabla'_{X_p} Y) + \perp(\nabla'_{X_p} Y),$$

where ∇' is the covariant derivative in $\mathbb{R}^n$. It follows that $\top(\nabla'_{X_p} Y) = \nabla_{X_p} Y$, where ∇ is the covariant derivative in M induced by the metric, while $\perp(\nabla'_{X_p} Y) = s(X_p, Y_p)$ is symmetric in X_p and Y_p (and independent of the extension Y of Y_p). This gives us the Gauss formula,

$$\nabla'_{X_p} Y = \nabla_{X_p} Y + s(X_p, Y_p).$$

Analogously, if ν is a normal vector field along M in a neighbourhood of p, we have a similar decomposition

$$\nabla'_{X_p} \nu = \top(\nabla'_{X_p} \nu) + \perp(\nabla'_{X_p} \nu).$$

The tangential component satisfies

$$\langle \top(\nabla'_{X_p} \nu), Y_p \rangle = \langle \nabla'_{X_p} \nu, Y_p \rangle = -\langle \nu_p, s(X_p, Y_p) \rangle,$$

and consequently, $\top(\nabla'_{X_p} \nu)$ depends only on X_p and ν_p. Now, for each normal vector $\nu_p \in T_pM^\perp$, we can define the self-adjoint linear map $A_{\nu_p} : T_pM \to T_pM$ by

$$A_{\nu_p}(X_p) = -\top(\nabla'_{X_p} \nu),$$

where ν is any normal vector field extending ν_p. We also define the second fundamental form II_{ν_p} as

$$\mathrm{II}_{\nu_p}(X_p, Y_p) = \langle A_{\nu_p}(X_p), Y_p \rangle = \langle s(X_p, Y_p), \nu_p \rangle, \quad X_p, Y_p \in T_pM.$$

For the normal component $\perp(\nabla'_{X_p} \nu)$, we will denote it by $D_{X_p} \nu$ so that it defines a connection on the normal bundle of M in $\mathbb{R}^n$ called the normal connection. With this notation, the decomposition of $\nabla'_{X_p}\nu$ can be written as

$$\nabla'_{X_p}\nu = -A_{\nu_p}(X_p) + D_{X_p}\nu,$$

which is called Weingarten equation.

Definition 3.1. We say that a normal vector field ν is *parallel* if $D_{X_p}\nu = 0$ for any $X_p \in T_pM$ and for any $p \in M$.

Definition 3.2. We say that a normal vector $\nu_p \in T_pM^\perp$ is *non-degenerate* if the self-adjoint linear map A_{ν_p} (or equivalently the second fundamental form II_{ν_p}) is non-degenerate. We say that a normal vector field ν is *non-degenerate* if it is non-degenerate at any point.

It follows from the definition that a parallel normal vector field ν has always constant length. Since D defines a connection on the normal bundle, we have

$$X_p\langle \nu, \nu\rangle = 2\langle \nu, D_{X_p}\nu\rangle = 0,$$

for any $X_p \in T_pM$. Hence $\langle \nu, \nu\rangle$ is a constant function.

If ν is a normal vector field on M with constant length, we can assume without loss of generality that it is unitary. Then, by translating the normal vector ν_p to the origin of $\mathbb{R}^n$ we have the Gauss map $G_\nu : M \to S^{n-1}$. That is, let $(x_1, \dots, x_n)$ be the standard coordinates of $\mathbb{R}^n$, so that $\nu = \sum_{i=1}^n \nu_i \frac{\partial}{\partial x_i}$ for some smooth functions ν_i on M. Then G_ν is equal to the map $(\nu_1, \dots, \nu_n)$. We characterize the parallel and non-degenerate conditions in terms of the differential or tangent map $G_{\nu *} : T_pM \to T_{\nu_p}S^{n-1}$.

Lemma 3.3. *Let ν be a unit normal vector field on M. Then,*

(1) *ν is parallel if and only if $G_{\nu *}(T_pM) \subset T_pM$, for any $p \in M$.*
(2) *ν is parallel and non-degenerate if and only if $G_{\nu *}(T_pM) = T_pM$, for any $p \in M$.*

Proof. For any $p \in M$ and for any $X_p \in T_pM$, it is obvious that

$$G_{\nu *}(X_p) = \nabla'_{X_p}\nu = -A_{\nu_p}(X_p) + D_{X_p}\nu.$$

Then, (1) and (2) follow directly from the definitions. Q.E.D.

In fact, in the case of a hypersurface, the constant length condition is also sufficient for a normal vector field to be parallel. Since the normal bundle is 1-dimensional in this case, it is obvious that $\langle \nu, D_{X_p}\nu\rangle = 0$ implies $D_{X_p}\nu = 0$, for any $X_p \in T_pM$. Thus, any hypersurface has always a local parallel normal vector field. Moreover, if it is orientable, then there is a global parallel normal vector field. Finally, it is also non-degenerate if and only if the gaussian curvature K is not zero.

In the case of a Frenet curve in $\mathbb{R}^n$, there always exists a parallel normal vector field. Let $\alpha : I \to \mathbb{R}^n$ be a smooth curve such that $\alpha'(t), \dots, \alpha^{(n-1)}(t)$ are linearly independent at any $t \in I$. We assume that α is parametrized by arc length and we denote by $\mathbf{e}_1, \dots, \mathbf{e}_n$ the Frenet frame and $\kappa_1, \dots, \kappa_{n-1}$ the curvatures. Let $\nu = \sum_{i=2}^n a_i \mathbf{e}_i$ be a normal vector field. Then, Frenet equations give

$$\begin{aligned}
\nu' =& \sum_{i=2}^{n-1} \big(a_i'\mathbf{e}_i + a_i(-\kappa_{i-1}\mathbf{e}_{i-1} + \kappa_i\mathbf{e}_{i+1})\big) + a_n'\mathbf{e}_n + a_n(-\kappa_{n-1}\mathbf{e}_{n-2}) \\
=& - a_2\kappa_1\mathbf{e}_1 + (a_2' - a_3\kappa_2)\mathbf{e}_2 + \sum_{i=3}^{n-1}(a_i' + a_{i-1}\kappa_{i-1} - a_{i+1}\kappa_i)\mathbf{e}_i \\
& + (a_n' + a_{n-1}\kappa_{n-1})\mathbf{e}_n.
\end{aligned}$$

Thus, ν is parallel if and only if $a_2, \dots, a_n$ are a solution of the following system of ordinary differential equations:

$$\begin{array}{lcl}
a_2' - a_3\kappa_2 & = & 0, \\
a_3' + a_2\kappa_2 - a_4\kappa_3 & = & 0, \\
\dots & & \\
a_{n-1}' + a_{n-2}\kappa_{n-2} - a_n\kappa_{n-1} & = & 0, \\
a_n' + a_{n-1}\kappa_{n-1} & = & 0.
\end{array} \tag{1}$$

Finally, note that if ν is parallel, then $\nu' = -a_2\kappa_1\mathbf{e}_1$. Hence, it is non-degenerate if and only if both a_2 and κ_1 are not zero (note that $\kappa_1 > 0$ if $n \geq 3$).

In general, if M is an immersed submanifold of dimension m in $\mathbb{R}^n$, with $1 < m < n$, a local parallel normal vector field does not always exist (see Section 5). However, it is obvious that if M is contained in a hyperplane $\pi(x_0, v)$ of $\mathbb{R}^n$, the constant normal vector field v is parallel. Analogously, if M is contained in a hypersphere $S(p, R)$ of $\mathbb{R}^n$, then the outward unit normal vector field of the hypersphere restricted to M is parallel and non-degenerate (in this case, G_ν is the inclusion map).

Corollary 3.4. *Let ν be a non-degenerate parallel unit normal vector field on M. Then the Gauss map $G_\nu : M \to S^{n-1}$ is an immersion whose image, $M^\nu = G_\nu(M)$, satisfies that $T_pM = T_{\nu_p}M^\nu$, for any $p \in M$.*

The condition $T_pM = T_{\nu_p}M^\nu$ can be seen as some kind of "parallelism" between M and the hyperspherical submanifold M^ν. Next proposition shows that this hyperspherical submanifold M^ν has the same contact type $\Sigma^{\mathbf{i}}$ with hyperplanes as the original submanifold M, thus giving a partial answer to Question 2.12.

Proposition 3.5. *Let ν be a non-degenerate parallel unit normal vector field on M. If M has contact type $\Sigma^{\mathbf{i}}$ with a hyperplane $\pi(p,v)$ at $p \in M$, then M^ν has the same contact type $\Sigma^{\mathbf{i}}$ with the translated hyperplane $\pi(\nu_p, v)$ at $\nu_p \in M^\nu$.*

Proof. Assume that M is locally parametrized in a neighbourhood of p by the immersion $g : U \subset \mathbb{R}^m \to \mathbb{R}^n$, with $g(u_0) = p$. Then, the contact between M and $\pi(p,v)$ is determined by the $\mathcal{K}$-class at u_0 of the height function $h_v : U \to \mathbb{R}$ given by $h_v(u) = \langle g(u), v\rangle$. Analogously, to study the contact between M^ν and $\pi(\nu_p, v)$ we consider $h_v^\nu : U \to \mathbb{R}$ given by $h_v^\nu(u) = \langle G_\nu(g(u)), v\rangle$.

Since ν is parallel, this means that

$$\frac{\partial G_\nu \circ g}{\partial u_i} = \nabla'_{\frac{\partial}{\partial u_i}} \nu \circ g = \sum_{j=1}^m a_{ij} \frac{\partial g}{\partial u_j},$$

for some smooth functions a_{ij}. Moreover, the fact that it is non-degenerate implies that $\det(a_{ij}) \neq 0$.

Hence, we also have that

$$\frac{h_v^\nu}{\partial u_i} = \sum_{j=1}^m a_{ij} \frac{h_v}{\partial u_j},$$

and the result is a consequence of Lemma 2.7, since this condition is equivalent to $J_m(h_v) = J_m(h_v^\nu)$ when considered as function germs from $(\mathbb{R}^m, u_0)$ to $\mathbb{R}$. Q.E.D.

In general, it is not true that M^ν has the same contact with hyperplanes as the original sumbanifold M. For instance, if $M \subset \mathbb{R}^3$ is a surface with gaussian curvature $K < 0$, then the corresponding height function is $\mathcal{K}$-equivalent to $x^2 - y^2$. However, the image of the Gauss map M^ν is an open subset of the sphere S^2 and the contact with the tangent plane is given by the height function $x^2 + y^2$.

§4. Curves in $\mathbb{R}^3$

Let $\alpha : I \to \mathbb{R}^3$ be a regular space curve with non-vanishing curvature, so that it has a well defined Frenet frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We also denote by κ, τ the curvature and the torsion of α respectively. Assume that α is parametrized by arc length. A normal unit vector field is given by $\nu = \cos\theta \mathbf{e}_2 + \sin\theta \mathbf{e}_3$.

Proposition 4.1. *A normal unit vector field* $\nu = \cos\theta \mathbf{e}_2 + \sin\theta \mathbf{e}_3$ *is parallel if and only if* $\theta' = -\tau$. *Moreover, it is also non-degenerate if and only if* $\cos\theta \neq 0$.

Proof. In this case, the system of differential equations (1) is

$$-\theta' \sin\theta - \tau \sin\theta = 0,$$
$$\theta' \cos\theta + \tau \cos\theta = 0,$$

which reduces to $\theta' = -\tau$. For the second part, just note that if ν is parallel, then $\nu' = -\kappa \cos\theta \mathbf{e}_1$, with $\kappa > 0$. Q.E.D.

As a consequence, we have that a parallel normal unit vector field always exists for a space curve and it is unique up to rotation in the normal plane. Moreover, since we can take the initial condition $\cos\theta_0 \neq 0$, we can choose the parallel vector to be non-degenerate in a neighbourhood of each point of the curve.

Assume that ν is parallel and non-degenerate. Then the "parallel" curve $\alpha^\nu : I \to S^2$ is nothing but the spherical indicatrix of ν. According to Proposition 3.5, α^ν has the same contact type $\Sigma^{\mathbf{i}}$ with planes than the original curve α. In particular, α has a $\Sigma^{1,1,1}$ contact with its osculating plane at a point if and only if α^ν has the same contact type $\Sigma^{1,1,1}$ with the translated plane at the corresponding point. Hence, $t \in I$ is a flattening (i.e., $\tau(t) = 0$) of α if and only if it is a flattening of α^ν.

The classical four vertex theorem for plane curves states that any regular closed and simple plane curve has at least four vertices. By taking stereographic projection this is equivalent to say that any regular closed and simple space curve contained in the sphere S^2 has at least four flattenings. This has been generalized in different ways by several authors (see [1, 2, 11, 15]) for convex space curves, although they do not use the same definition of convexity.

As a corollary of our computations we obtain one more different extension of the Four Vertex Theorem.

Corollary 4.2. *Let* $\alpha : I \to \mathbb{R}^3$ *be a regular, simple and closed space curve with non-vanishing curvature. Assume that the parallel curve* α^ν *is also regular, simple and closed. Then* α *has at least four flattenings.*

The regularity condition on α^ν is just the non-degeneracy of ν. Although we can choose ν so that it is non-degenerate in a neighbouhood of each point, it is not true that there always exists a parallel normal vector field which is globally non-degenerate. On the other hand, the condition that α^ν is closed is equivalent to the vanishing of the total torsion of α, that is, $\int_I \tau = 0$, which implies the existence of at least two flattenings.

§5. Totally semi-umbilic surfaces in $\mathbb{R}^4$

Let M be a smooth surface immersed in $\mathbb{R}^4$. Given a normal vector $\nu \in T_pM^\perp$, we define the ν-principal directions and the ν-principal curvatures to be the unit eigenvectors and corresponding eigenvalues for the self-adjoint linear map $A_\nu : T_pM \to T_pM$.

Definition 5.1. A point p of a smooth immersed surface M in $\mathbb{R}^4$ is said to be *semi-umbilic* if there is a non-zero normal vector $\nu \in T_pM^\perp$ such that the ν-principal curvatures are equal. We say that p is *umbilic* if the ν-principal curvatures are equal for any normal vector $\nu \in T_pM^\perp$. We say that M is *totally semi-umbilic* (respectively *totally umbilic*) if all its points are semi-umbilic (respectively umbilic).

Assume that M is locally parameterized as the image of a smooth immersion $\mathbf{x} : U \to \mathbb{R}^4$, where $U \subset \mathbb{R}^2$ is an open set. We denote by u, v the coordinates in $\mathbb{R}^2$ and by $\mathbf{x}_u, \mathbf{x}_v$. the partial derivatives of $\mathbf{x}$ with respect to these coordinates. Then, the first fundamental form is given in local coordinates by

$$\mathrm{I} = Edu^2 + 2Fdudv + Gdv^2,$$

where

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle.$$

Moreover, for any normal vector $\nu \in T_pM^\perp$, the second fundamental form can be expressed as

$$\mathrm{II}_\nu = a_\nu du^2 + 2b_\nu dudv + c_\nu dv^2,$$

with coefficients

$$a_\nu = \langle \mathbf{x}_{uu}, \nu \rangle, \quad b_\nu = \langle \mathbf{x}_{uv}, \nu \rangle, \quad c_\nu = \langle \mathbf{x}_{vv}, \nu \rangle.$$

Then, it follows that the ν-principal directions can be computed as the null directions of the quadratic form:

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ a_\nu & b_\nu & c_\nu \\ E & F & G \end{vmatrix}.$$

Example 5.2. Every hyperspherical surface M immersed in $\mathbb{R}^4$ is totally semi-umbilic. In fact, if M is contained in a hypersphere of $\mathbb{R}^4$ with center $p \in \mathbb{R}^4$ and radius $R > 0$ a simple computation shows that the principal curvatures with respect to some unit normal vector to the hypersphere are either both equal to $1/R$ or both equal to $-1/R$ (depending on the chosen normal vector).

Analogously, if M is contained in some hyperplane of $\mathbb{R}^4$, the principal curvatures with respect to any normal vector to the hyperplane are both equal to zero and hence, M is totally semi-umbilic.

Finally, note that there are semi-umbilic surfaces which are not contained in a hypersphere nor a hyperplane. For instance, consider two plane regular curves $\alpha : I \to \mathbb{R}^2$ and $\beta : J \to \mathbb{R}^2$. Then $\mathbf{x} = \alpha \times \beta : I \times J \to \mathbb{R}^4$ parameterizes a semi-umbilic surface. For simplicity, we assume that both α and β are parameterized by arc-length. Since $\mathbf{x}_u = (\alpha', 0)$ and $\mathbf{x}_v = (0, \beta')$, this implies that $E = G = 1$ and $F = 0$. Let us denote by $n_\alpha, n_\beta, \kappa_\alpha, \kappa_\beta$ the normal vectors and the curvatures of α, β respectively. If $\kappa_\alpha^2 + \kappa_\beta^2 > 0$, we consider $\nu = (\kappa_\beta n_\alpha, \kappa_\alpha n_\beta)$ so that both ν-principal curvatures are equal to $\kappa_\alpha \kappa_\beta$. Otherwise, if $\kappa_\alpha = \kappa_\beta = 0$, we consider $\nu = (n_\alpha, n_\beta)$ and the corresponding principal curvatures are both equal to zero.

We recall now the concept of curvature ellipse of an immersed surface M in $\mathbb{R}^4$. Given a point $p \in M$, we consider the unit circle in T_pM parameterized by the angle $\theta \in [0, 2\pi]$. Let γ_θ be the curve obtained by intersecting M with the hyperplane at p given by the direct sum of the normal plane $T_pM^\perp$ and the straight line in the tangent direction represented by θ. Such curve is called the *normal section* of M in the direction θ. The curvature vector $\eta(\theta)$ of γ_θ in p lies in $T_pM^\perp$. Varying θ from 0 to 2π, this vector describes an ellipse in $T_pM^\perp$, called the *curvature ellipse* of M at p. A tangent direction represented by the angle θ is called an *asymptotic direction* (or *conjugate direction* in the terminology of Little [8]) if $\eta(\theta)$ and $\frac{d\eta}{d\theta}(\theta)$ are collinear.

It is possible to characterize the asymptotic directions as those directions θ such that the line in $T_pM^\perp$ joining the origin p with $\eta(\theta)$ is tangent to the curvature ellipse at such point. Thus, if the curvature ellipse is not a radial segment, we can have three cases (Figure 1):

(1) The origin p lies outside the curvature ellipse. There are exactly two asymptotic directions and the point p is called *hyperbolic*.
(2) The origin p lies on the curvature ellipse. There is only one asymptotic direction and the point p is called *parabolic*.
(3) The origin p lies inside the curvature ellipse. There are no asymptotic directions and the point p is called *elliptic*.

Finally, in the case that the curvature ellipse degenerates to a radial segment, it follows that all the directions are asymptotic and the point

p is called an *inflection*. An inflection is said to be of real type when p belongs to the curvature ellipse and of imaginary type when it does not.

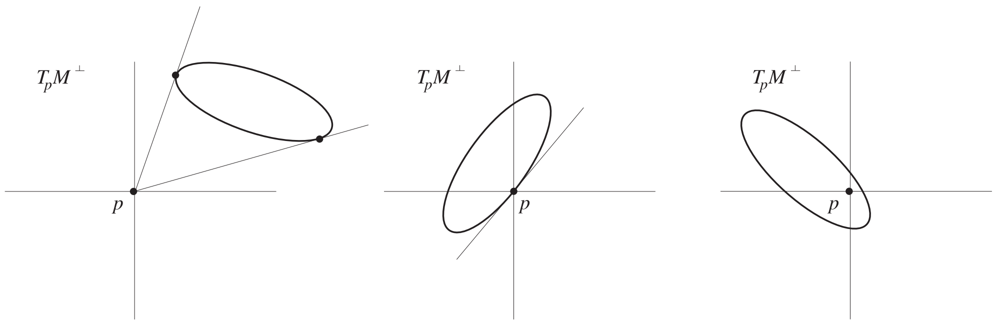

Fig. 1

In local coordinates, the asymptotic directions are computed by means of the quadratic equation

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ a_{\nu_1} & b_{\nu_1} & c_{\nu_1} \\ a_{\nu_2} & b_{\nu_2} & c_{\nu_2} \end{vmatrix} = 0,$$

being ν_1, ν_2 some orthonormal frame of the normal plane $T_pM^\perp$. Moreover, the inflections correspond to the singular points of the above differential equation, that is, the points where the matrix

$$\begin{pmatrix} a_{\nu_1} & b_{\nu_1} & c_{\nu_1} \\ a_{\nu_2} & b_{\nu_2} & c_{\nu_2} \end{pmatrix}$$

has rank ≤ 1.

The following theorem [14] gives characterizations of the semi-umbilic points of an immersed surface in $\mathbb{R}^4$ in terms of the curvature ellipse and asymptotic lines.

Theorem 5.3. *Let M be an immersed surface in $\mathbb{R}^4$ and let $p \in M$. The following are equivalent conditions:*

(1) *p is semi-umbilic.*
(2) *The curvature ellipse at p degenerates to a segment.*
(3) *There are two orthogonal asymptotic directions at p.*

Using this theorem, we give a characterization of a totally semi-umbilic surface in terms of the existence of a local parallel normal vector field.

Theorem 5.4. *Let M be an immersed surface in $\mathbb{R}^4$. Then M is totally semi-umbilic if and only if there is a parallel normal vector field defined in a neighbourhood of each point of M.*

Proof. Let $p \in M$ and assume that M is locally parameterized in a neighbourhood of p as the image of a smooth isothermal immersion $\mathbf{x} : U \to \mathbb{R}^4$, where $U \subset \mathbb{R}^2$ is an open set. This means that $E = G$ and $F = 0$. Now we take ν_1, ν_2 an orthonormal frame of the normal plane at each point, so that $\mathbf{x}_u, \mathbf{x}_v, \nu_1, \nu_2$ give an orthogonal frame of $\mathbb{R}^4$.

In order to make computations, we need to take the following coefficients:

$$\begin{aligned}
\nu_{1,u} &= \lambda_{11}\mathbf{x}_u + \lambda_{12}\mathbf{x}_v + \lambda_{13}\nu_2,\\
\nu_{1,v} &= \lambda_{21}\mathbf{x}_u + \lambda_{22}\mathbf{x}_v + \lambda_{23}\nu_2,\\
\nu_{2,u} &= \mu_{11}\mathbf{x}_u + \mu_{12}\mathbf{x}_v + \mu_{13}\nu_1,\\
\nu_{2,v} &= \mu_{21}\mathbf{x}_u + \mu_{22}\mathbf{x}_v + \mu_{23}\nu_1.
\end{aligned}$$

Since $\langle \nu_1, \nu_2 \rangle = 0$, it follows easily that $\mu_{13} = -\lambda_{13}$ and $\mu_{23} = -\lambda_{23}$.

Let now $\nu = A\nu_1 + B\nu_2$ be a normal vector field. We have that if ν is parallel then $\langle \nu, \nu \rangle =$ constant. Hence, we can assume, without loss of generality, that $A^2 + B^2 = 1$. With this assumption, it follows that ν is parallel if and only if

$$\det(\nu_u, \nu, \mathbf{x}_u, \mathbf{x}_v) = \det(\nu_v, \nu, \mathbf{x}_u, \mathbf{x}_v) = 0.$$

By direct computation we get that

$$\begin{aligned}
\det(\nu_u, \nu, \mathbf{x}_u, \mathbf{x}_v) &= (A_uB - B_uA - \lambda_{13}) \det(\nu_1, \nu_2, \mathbf{x}_u, \mathbf{x}_v),\\
\det(\nu_v, \nu, \mathbf{x}_u, \mathbf{x}_v) &= (A_vB - B_vA - \lambda_{23}) \det(\nu_1, \nu_2, \mathbf{x}_u, \mathbf{x}_v),
\end{aligned}$$

so that ν is parallel if and only if

$$\begin{aligned}
\lambda_{13} &= A_uB - B_uA,\\
\lambda_{23} &= A_vB - B_vA.
\end{aligned}$$

Finally, it is not difficult to see that this system of PDE's has a solution A, B with $A^2 + B^2 = 1$ if and only if $\lambda_{13,v} = \lambda_{23,u}$. In the second part of the proof, we see that such condition is equivalent to the fact that the asymptotic directions are orthogonal, and hence that M is totally semi-umbilic, by the above theorem.

In fact, since $\lambda_{13} = \langle \nu_{1,u}, \nu_2 \rangle$ and $\lambda_{23} = \langle \nu_{1,v}, \nu_2 \rangle$, we get

$$\begin{aligned}
\lambda_{13,v} - \lambda_{23,u} =& \langle \nu_{1,uv}, \nu_2 \rangle + \langle \nu_{1,u}, \nu_{2,v} \rangle - \langle \nu_{1,uv}, \nu_2 \rangle - \langle \nu_{1,v}, \nu_{2,u} \rangle\\
=& \langle \nu_{1,u}, \nu_{2,v} \rangle - \langle \nu_{1,v}, \nu_{2,u} \rangle\\
=& E(\lambda_{11}\mu_{21} + \lambda_{12}\mu_{22} - \lambda_{21}\mu_{11} - \lambda_{22}\mu_{12}).
\end{aligned}$$

In order to simplify the notation we change the notation for the coefficients of the second fundamental forms of ν_1, ν_2 in the following way:

$$\begin{aligned} a &= a_{\nu_1}, \quad b = b_{\nu_1}, \quad c = c_{\nu_1}, \\ e &= a_{\nu_2}, \quad f = b_{\nu_2}, \quad g = c_{\nu_2}. \end{aligned}$$

Then, since $\langle \mathbf{x}_u, \nu_i \rangle = \langle \mathbf{x}_v, \nu_i \rangle = 0$, we deduce that

$$\begin{aligned} a &= \langle \mathbf{x}_{uu}, \nu_1 \rangle = -\langle \mathbf{x}_u, \nu_{1,u} \rangle = -E\lambda_{11}, \\ e &= \langle \mathbf{x}_{uu}, \nu_2 \rangle = -\langle \mathbf{x}_u, \nu_{2,u} \rangle = -E\mu_{11}, \\ b &= \langle \mathbf{x}_{uv}, \nu_1 \rangle = -\langle \mathbf{x}_u, \nu_{1,v} \rangle = -E\lambda_{21} = -\langle \mathbf{x}_v, \nu_{1,u} \rangle = -E\lambda_{12}, \\ f &= \langle \mathbf{x}_{uv}, \nu_2 \rangle = -\langle \mathbf{x}_u, \nu_{2,v} \rangle = -E\mu_{21} = -\langle \mathbf{x}_v, \nu_{2,u} \rangle = -E\mu_{12}, \\ c &= \langle \mathbf{x}_{vv}, \nu_1 \rangle = -\langle \mathbf{x}_v, \nu_{1,v} \rangle = -E\lambda_{22}, \\ g &= \langle \mathbf{x}_{vv}, \nu_1 \rangle = -\langle \mathbf{x}_v, \nu_{1,v} \rangle = -E\mu_{22}. \end{aligned}$$

Using this, we conclude that $\lambda_{13,v} = \lambda_{23,u}$ if and only if

$$(a-c)f = (e-g)b.$$

On the other hand, if we look at the differential equation of the asymptotic lines

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ a & b & c \\ e & f & g \end{vmatrix} = \begin{vmatrix} b & c \\ f & g \end{vmatrix} dv^2 + \begin{vmatrix} a & c \\ e & g \end{vmatrix} dudv + \begin{vmatrix} a & b \\ e & f \end{vmatrix} du^2 = 0,$$

we see that there are two orthogonal asymptotic lines at each point if and only if

$$\begin{vmatrix} b & c \\ f & g \end{vmatrix} = -\begin{vmatrix} a & b \\ e & f \end{vmatrix},$$

which is in fact equivalent to the above condition $(a-c)f = (e-g)b$. Q.E.D.

Now we see that if we also impose the condition that the gaussian curvature K of M is not zero, then we can choose the local parallel vector field to be non-degenerate.

Theorem 5.5. *Let M be a totally semi-umbilic surface immersed in $\mathbb{R}^4$ with $K \neq 0$. Then, there is a non-degenerate parallel normal vector field defined in a neighbourhood of each point of M.*

Proof. Assume that M is locally parametrized in a neighbourhood of p as the image of a smooth isothermal immersion $\mathbf{x} : U \to \mathbb{R}^4$, where $U \subset \mathbb{R}^2$ is some open set. By Theorem 5.4, there is a parallel unit normal vector field ν and we take another unit normal vector field ξ so that ν, ξ is an orthonormal frame of the normal plane at each point and $\mathbf{x}_u, \mathbf{x}_v, \nu, \xi$ is an orthogonal frame of $\mathbb{R}^4$.

Since $\langle \nu, \xi \rangle = 0$ it follows that $\langle \xi_u, \nu \rangle = -\langle \xi, \nu_u \rangle = 0$ and $\langle \xi_v, \nu \rangle = -\langle \xi, \nu_v \rangle = 0$. This shows that ξ is also parallel. Therefore, we can write

$$\begin{aligned} \nu_u &= \lambda_{11}\mathbf{x}_u + \lambda_{12}\mathbf{x}_v, \\ \nu_v &= \lambda_{21}\mathbf{x}_u + \lambda_{22}\mathbf{x}_v, \\ \xi_u &= \mu_{11}\mathbf{x}_u + \mu_{12}\mathbf{x}_v, \\ \xi_v &= \mu_{21}\mathbf{x}_u + \mu_{22}\mathbf{x}_v, \end{aligned}$$

for some coefficients λ_{ij} and μ_{ij}. In the proof of Theorem 5.4 we showed that these coefficients are given by

$$\begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = -\frac{1}{E}\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix} = -\frac{1}{E}\begin{pmatrix} e & f \\ f & g \end{pmatrix},$$

where a, b, c and e, f, g are the coefficients of the second fundamental forms of the normal vectors ν and ξ respectively.

On the other hand, we can use the Gauss equation (see [8]) which gives the gaussian curvature K in terms of these coefficients:

$$K = \frac{1}{E^2}(ac - b^2 + eg - f^2).$$

In particular, if $K \neq 0$, it follows that either $ac - b^2 \neq 0$ or $eg - f^2 \neq 0$ and hence, either ν is an immersion or ξ is an immersion. Q.E.D.

Theorem 5.6. *Let M be a totally semi-umbilic surface immersed in $\mathbb{R}^4$ with $K \neq 0$. Assume that $G_\nu : W \to S^3$ is the Gauss map of a non-degenerate parallel unit normal vector field ν defined in a neighbourhood W of $p \in M$. Then G_ν preserves the inflections and the asymptotic directions between W and the image W^ν.*

Proof. We use the same notation as in the proof of the above theorem. Since ν is parallel, it follows that for any $q \in W$, the normal plane to W^ν at ν_q coincides with the normal plane to W at q. Thus, we can also take ν, ξ as an orthonormal frame of the normal plane of W^ν.

Remember that, according to the proof of the above theorem, we have

$$\begin{aligned}
\nu_u &= -\frac{1}{E}(a\mathbf{x}_u + b\mathbf{x}_v),\\
\nu_v &= -\frac{1}{E}(b\mathbf{x}_u + c\mathbf{x}_v),\\
\xi_u &= -\frac{1}{E}(e\mathbf{x}_u + f\mathbf{x}_v),\\
\xi_v &= -\frac{1}{E}(f\mathbf{x}_u + g\mathbf{x}_v),
\end{aligned}$$

where $\mathbf{x}_u, \mathbf{x}_v, \nu, \xi$ is the orthogonal frame adapted to the original surface M. Thus, it follows that the coefficients of the second fundamental form of W^ν with respect to ν are

$$\begin{aligned}
\langle \nu_{uu}, \nu\rangle &= -\langle \nu_u, \nu_u\rangle = -\frac{1}{E^2}\langle a\mathbf{x}_u + b\mathbf{x}_v, a\mathbf{x}_u + b\mathbf{x}_v\rangle = -\frac{1}{E}(a^2+b^2),\\
\langle \nu_{uv}, \nu\rangle &= -\langle \nu_u, \nu_v\rangle = -\frac{1}{E^2}\langle a\mathbf{x}_u + b\mathbf{x}_v, b\mathbf{x}_u + c\mathbf{x}_v\rangle = -\frac{1}{E}(ab+bc),\\
\langle \nu_{vv}, \nu\rangle &= -\langle \nu_v, \nu_v\rangle = -\frac{1}{E^2}\langle b\mathbf{x}_u + c\mathbf{x}_v, b\mathbf{x}_u + c\mathbf{x}_v\rangle = -\frac{1}{E}(b^2+c^2).
\end{aligned}$$

We compute now the coefficients of the second fundamental form with respect to ξ:

$$\begin{aligned}
\langle \nu_{uu}, \xi\rangle &= -\langle \nu_u, \xi_u\rangle = -\frac{1}{E^2}\langle a\mathbf{x}_u + b\mathbf{x}_v, e\mathbf{x}_u + f\mathbf{x}_v\rangle = -\frac{1}{E}(ae+bf),\\
\langle \nu_{uv}, \xi\rangle &= -\langle \nu_u, \xi_v\rangle = -\frac{1}{E^2}\langle a\mathbf{x}_u + b\mathbf{x}_v, f\mathbf{x}_u + g\mathbf{x}_v\rangle = -\frac{1}{E}(af+bg),\\
\langle \nu_{vv}, \xi\rangle &= -\langle \nu_v, \xi_v\rangle = -\frac{1}{E^2}\langle b\mathbf{x}_u + c\mathbf{x}_v, f\mathbf{x}_u + g\mathbf{x}_v\rangle = -\frac{1}{E}(bf+cg).
\end{aligned}$$

Now, it is easy to write down the differential equation for the asymptotic directions of W^ν,

$$\frac{1}{E^2}\begin{vmatrix} dv^2 & -dudv & du^2\\ a^2+b^2 & ab+bc & b^2+c^2\\ ae+bf & af+bg & bf+cg \end{vmatrix} = 0.$$

In the proof of Theorem 5.4, we showed that if M is semi-umbilic then $(a-c)f = (e-g)b$. By using this condition, it follows that

$$\frac{1}{E^2}\begin{vmatrix} dv^2 & -dudv & du^2\\ a^2+b^2 & ab+bc & b^2+c^2\\ ae+bf & af+bg & bf+cg \end{vmatrix} = \frac{ac-b^2}{E^2}\begin{vmatrix} dv^2 & -dudv & du^2\\ a & b & c\\ e & f & g \end{vmatrix}.$$

Finally, note that the fact that ν is non-degenerate implies that $ac-b^2 \neq 0$, which completes the proof. Q.E.D.

Finally, we consider the composition of the Gauss map of the parallel unit normal vector field ν with the stereographic projection from S^3 into $\mathbb{R}^3$. It follows that such a map transforms the asymptotic lines of M into the principal lines of its image in $\mathbb{R}^3$ and the inflections into the umbilics. Hence, we get as a direct consequence the following extension of the Loewner and Carathéodory conjectures for totally semi-umbilic analytic surfaces immersed in $\mathbb{R}^4$ with $K \neq 0$.

Corollary 5.7. *Let M be a totally semi-umbilic analytic surface immersed in $\mathbb{R}^4$ with $K \neq 0$. Then the index of the asymptotic foliation at an isolated inflection of M is always ≤ 1.*

Corollary 5.8. *Let M be a totally semi-umbilic analytic surface immersed in $\mathbb{R}^4$ with $K \neq 0$ and assume that M is homeomorphic to S^2. Then M has at least two inflections.*

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Weighted homogeneous polynomials and blow-analytic equivalence

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Abstract.

Based on the T. Fukui invariant and the recent motivic invariants proposed by S. Koike and A. Parusiński we give a simple classification of two variable quasihomogeneous polynomials by the blow-analytic equivalence.

§1. INTRODUCTION

Unlike the topological triviality of real algebraic germs, the C^1-equisingularity admits continuous moduli. For instance, the Whitney family $W_t(x, y) = xy(x - y)(x - ty)$, $t > 1$, has an infinite number of different C^1-types. Nevertheless, as was noticed by Tzee-Char Kuo, this family is blow-analytically trivial, that is, after composing with the blowing-up $\beta\colon M^2 \to \mathbf{R}^2$, $W_t \circ \beta$ becomes analytically trivial. T.-C. Kuo proposed new notions of blow-analytic equisingularity and the blow-analytic function (see [6, 3] for survey). Let $f\colon U \to \mathbf{R}$, U open in $\mathbf{R}^n$, be a continuous function. We say that f is blow-analytic, if there exists a sequence of blowing-up β such that the composition $f \circ \beta$ is analytic (for instance $f(x, y) = \frac{x^2y}{x^2+y^2}$ is blow-analytic but not C^1). A local homeomorphism $h\colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ is called blow-analytic if so are all coordinate functions of h and h^{-1}. Two function germs f_1, $f_2\colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ are blow-analytically equivalent if there is a blow-analytic homeomorphism h such that $f_1 = f_2 \circ h$.

Observation. Let f, $g\colon (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be weighted homogeneous polynomials with isolated singularities. It is known, for $n = 2$, 3, that if $(\mathbf{C}^n, f^{-1}(0))$ and $(\mathbf{C}^n, g^{-1}(0))$ are homeomorphic as germs at $0 \in \mathbf{C}^n$, then, their systems of weights coincide.

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We will consider real singularities. We can easily see that the notion of topological equivalence is too weak to consider the same problem for real analytic singularities. For example, consider $f(x,y) = x^3 + xy^6$ and $g(x,y) = x^3 + y^8$, they are topologically equivalent by Kuiper-Kuo Theorem (see [7, 8]). However, f and g have different weights. We replace the topological equivalence by the blow-analytic equivalence, and we will consider the following problem suggested by T. Fukui.

Problem 1 (T. Fukui, [2], Conjecture 9.2)**.** *Let $f, g\colon (\mathbf{R}^n,0) \to (\mathbf{R},0)$ be weighted homogeneous polynomials with isolated singularities. Suppose that f and g are blow-analytically equivalent. Then, do their systems of weights coincide?*

The purpose of this paper is to establish this conjecture for two variables. Namely, we will prove the following:

Theorem 1. *Let $f_i\colon (\mathbf{R}^2,0) \to (\mathbf{R},0)$ $(i = 1,2)$ be non-degenerate quasihomogeneous polynomials of type $(1; r_{i1}, r_{i2})$ such that $0 < r_{i2} \leq r_{i1}$. If f_1 and f_2 are blow-analytically equivalent, then either both f_1 and f_2 are nonsingular, or both are analytically equivalent to xy, or $(r_{11}, r_{12}) = (r_{21}, r_{22})$.*

We call a polynomial f quasihomogeneous of type $(d; w_1, \ldots, w_n) \in \mathbf{Q}^{n+1}$ if $i_1 w_1 + \cdots + i_n w_n = d$ for any monomial $\alpha x_1^{i_1} \ldots x_n^{i_n}$ of f. We say that a polynomial $f(x)$ is non-degenerate if $\{\frac{\partial f}{\partial x_1}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0\} \subset \{0\}$ as germs at the origin of $\mathbf{R}^n$.

We will next recall some important results on blow-analytic equivalence.

Theorem 2 (T. Fukui - L. Paunescu [4])**.** *Given a system of weights $w = (w_1, \ldots, w_n)$, let $f_t\colon (\mathbf{R}^n,0) \to (\mathbf{R},0)$ be an analytic function for $t \in I = [0,1]$. Suppose that for each $t \in I$, the weighted initial form of f_t with respect to w is the same weighted degree and has an isolated singularity at $0 \in \mathbf{R}^n$. Then $\{f_t\}_{t\in I}$ is blow-analytically trivial over I.*

T. Fukui ([2]) gave some invariants for blow-analytic equivalence. One of them is defined as follows:

For an analytic function $f\colon (\mathbf{R}^n,0) \to (\mathbf{R},0)$, set

$$A(f) = \{O(f \circ \lambda) \mid \lambda\colon (\mathbf{R},0) \to (\mathbf{R}^n,0)\ C^w arc\}.$$

Then we have

Theorem 3 (Fukui's invariant). *Suppose that analytic functions* $f, g\colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *are blow-analytically equivalent, then* $A(f) = A(g)$.

Recently in [5], S. Koike and A. Parusiński have defined motivic zeta functions (inspired by the work of Denef and Loser [1]) which are invariant for blow-analytic equivalence. We will briefly recall their definition of the zeta functions.

Denote by $\mathcal{L}$ the space of analytic arcs at the origin $0 \in \mathbf{R}^n$:

$$\mathcal{L} = \{\gamma\colon (\mathbf{R}, 0) \to (\mathbf{R}^n, 0) \mid \gamma \text{ is analytic }\}$$

and by $\mathcal{L}_k$ the space of truncated arcs:

$$\mathcal{L}_k = \{\gamma \in \mathcal{L} \mid \gamma(t) = v_1 t + \cdots + v_k t^k,\ v_i \in \mathbf{R}^n\}.$$

Given an analytic function $f\colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$. For $k \geq 1$ we denote

$$A_k(f) = \{\gamma \in \mathcal{L}_k \mid f \circ \gamma(t) = ct^k + \cdots,\ c \neq 0\}.$$

We define the zeta function of f by

$$Z_f(T) = \sum_{k \geq 1} (-1)^{-kn} \chi^c(A_k(f)) T^k$$

where χ^c denotes the Euler characteristic with compact support.

Then we have

Theorem 4 (S. Koike - A. Parusiński [5]). *Suppose that analytic functions* $f, g\colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *are blow-analytically equivalent, then* $Z_f = Z_g$.

Before starting the proof of Theorem 1, we will make one more remark, as follows.

Remark 5. *Let* $f\colon (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *be a non-degenerate quasihomogeneous polynomial of type* $(d;\, w_1, \ldots, w_n)$. *Taking a new representative of the blow-analytic class of* f *if necessary we can suppose that, for each* $\alpha \in \mathbf{N}^n$ *such that* $\langle \alpha\, ,\, w\rangle = \alpha_1 w_1 + \cdots + \alpha_n w_n = d$, *the coefficient term* $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ *is not zero in* $f(x)$.

Our remark is a simple consequence of Theorem 2 (we omit the details).

§2. PROOF OF THEOREM 1

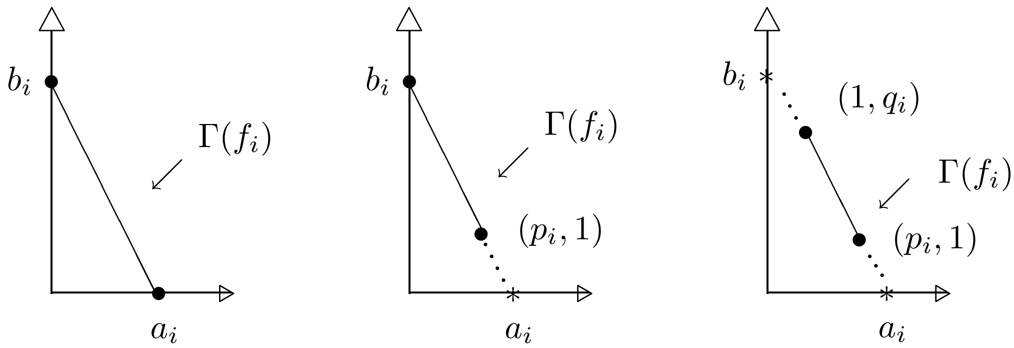
Let $f_i\colon (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ $(i = 1, 2)$ be non-degenerate quasihomogeneous polynomials of type $(1; r_{i1}, r_{i2})$. Setting

$$a_i = \frac{1}{r_{i1}} \text{ and } b_i = \frac{1}{r_{i2}} \quad \text{for } i = 1, 2.$$

Modulo a permutation coordinate of $\mathbf{R}^2$, we may assume that $a_i \le b_i$. Moreover, if $a_i < 2$, then f_i is analytically equivalent to $g(x, y) = x$ or xy by the Implicit Function Theorem. But $0 \in \mathbf{R}^2$ is a regular point of x and the polynomial xy is a weighted homogeneous of type $(1; \frac{1}{2}, \frac{1}{2})$. Given this, we can assume that

$$2 \le a_i \le b_i \quad \text{for } i = 1, 2. \tag{2.1}$$

Since f_i are non-degenerate quasihomogeneous polynomials, we have the following cases for Newton boundary $\Gamma(f_i)$ as in the following figure:



a_i, $b_i \in \mathbf{N}$ $\qquad$ $a_i \in \mathbf{N}$ or $b_i \in \mathbf{N}$ $\qquad$ $a_i \notin \mathbf{N}$, $b_i \notin \mathbf{N}$

These figures suggest that the proof of Theorem 1 should be divided into several steps, according to the possible cases for a_i and b_i:

Case 1. In this case, we suppose a_i, $b_i \in \mathbf{N}$ (i.e., f_i nearly convenient). Here $\mathbf{N}$ denotes the set of positive integers and let for any $a \in \mathbf{N}$, $\mathbf{N}_{\ge a} = \{k \in \mathbf{N} \mid k \ge a\}$. We first remark that the Fukui invariant of f_i can be computed easily as follows:

Assertion 6.

$$A(f_i) = \begin{cases} a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\} & \textit{if } f_i^{-1}(0) = \{0\}, \\ a_i\mathbf{N} \cup b_i\mathbf{N} \cup \mathbf{N}_{\ge [a_i, b_i]} \cup \{\infty\} & \textit{otherwise.} \end{cases} \tag{2.2}$$

Where $[a_i, b_i] = LCM(a_i, b_i)$.

Proof. Let $\lambda\colon (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be an analytic arc. Then $\lambda(t) = (X(t), Y(t))$ can be expressed in the following way:

$$X(t) = \alpha_u t^u + \alpha_{u+1} t^{u+1} + \cdots , \quad Y(t) = c_v t^v + c_{v+1} t^{v+1} + \cdots ,$$

where $\alpha_u, c_v \neq 0$ and $u, v \geq 1$. By the above Remark 5, we may assume that there exist the terms X^{a_i} and Y^{b_i} with non-zero coefficients in $f_i(X, Y)$.

We will first consider the case whereby $f_i^{-1}(0) = \{0\}$. If $u\,a_i \neq v\,b_i$, we have

$$f_i(X(t), Y(t)) = d_i\, t^{\min\{u\,a_i\,,\,v\,b_i\}} + \cdots , \; d_i \neq 0$$

then $O(f_i \circ \lambda) = \min\{u\,a_i\,,\, v\,b_i\} \in a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\}$. Thus it remains for us to consider the case $u\,a_i = v\,b_i$. In this case, we have

$$f_i(X(t), Y(t)) = f_i(\alpha_u\,,\, c_v)\, t^{u\,a_i} + \cdots ,$$

since $f_i(\alpha_u\,,\, c_v) \neq 0$. Therefore $A(f_i) \subseteq a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\}$. Any integer $s \in a_i\mathbf{N} \cup b_i\mathbf{N}$, for instance $s = k\,a_i$, is attained by the arc $\gamma(t) = (t^k, 0)$. Hence we have

$$A(f_i) = a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\}.$$

We will next consider the case whereby $f_i^{-1}(0) \neq \{0\}$. Similarly we have

$$a_i\mathbf{N} \cup b_i\mathbf{N} \cup \{\infty\} \subseteq A(f_i) \subseteq a_i\mathbf{N} \cup b_i\mathbf{N} \cup \mathbf{N}_{\geq [a_i, b_i]} \cup \{\infty\}.$$

Obviously we only have to prove that $\mathbf{N}_{\geq [a_i, b_i]} \subseteq A(f_i)$. Suppose that $k \in \mathbf{N}_{\geq [a_i, b_i]}$. Then there exists an arc γ through $0 \in \mathbf{R}^2$ such that $O(f \circ \gamma) = k$. Setting $[a_i, b_i] = n_i\,a_i = m_i\,b_i$, since f_i is non-degenerate and $f_i^{-1}(0) \neq \{0\}$, there exists a $(\alpha, c) \in f_i^{-1}(0)$ such that $(\frac{\partial f_i}{\partial X}(\alpha, c), \frac{\partial f_i}{\partial Y}(\alpha, c)) \neq (0, 0)$, we may assume that $\frac{\partial f_i}{\partial X}(\alpha, c) \neq 0$. Then it is easy to see that for any positive integers $[a_i, b_i] + s \in A(f)$, $s \in \mathbf{N}$, is attained by an arc $\gamma(t) = (\alpha t^{n_i} + t^{s+n_i}, c t^{m_i})$.

Evidently, this completes the proof of the Assertion. Q.E.D.

From Theorem 3, $A(f_1) = A(f_2)$. Thus, by the above Assertion, we have the following result:

$$\begin{aligned} &a_1 = a_2 \text{ same multiplicity for } f_i, \\ &b_1 = b_2 \text{ if } b_1 \notin a_1\mathbf{N} \text{ or } b_2 \notin a_2\mathbf{N}, \\ &b_1 = b_2 \text{ if } f_i^{-1}(0) \neq \{0\}. \end{aligned}$$

Manifestly, the Fukui invariant determines the weights except in the following case:

$$b_1 = k_1 a,\ b_2 = k_2 a \text{ and } f_i^{-1}(0) = \{0\},$$

where $a = a_1 = a_2$ is the smallest number in $A(f_i)$, and there remains to prove $k_1 = k_2$. In fact, assume that $k_1 \neq k_2$, for example $k_2 > k_1$. We will show that this gives rise to a contraduction by comparing the coefficients of the zeta functions. If $k_2 > k_1$ then we may write

$$\begin{aligned} A_{b_1}(f_2) &= \{\gamma(t) = (c_{k_1} t^{k_1} + \cdots + c_{b_1} t^{b_1}\ ,\ d_1 t^1 + \cdots + d_{b_1} t^{b_1}) \mid c_{k_1} \neq 0\} \\ &\simeq \mathbf{R}^* \times \mathbf{R}^{b_1 - k_1} \times \mathbf{R}^{b_1}. \end{aligned}$$

That is

$$\chi^c(A_{b_1}(f_2)) = (-2)\chi^c(\mathbf{R}^{b_1-k_1+b_1}) = (-2)(-1)^{2b_1-k_1}. \tag{2.3}$$

Also, since $f_1^{-1}(0) = \{0\}$, we obtain

$$\begin{aligned} A_{b_1}(f_1) &= \{\gamma = (u_{k_1} t^{k_1} + \cdots + u_{b_1} t^{b_1}, v_1 t^1 + \cdots + v_{b_1} t^{b_1}) \mid (u_{k_1}, v_1) \neq 0\} \\ &\simeq (\mathbf{R}^2 - \{0\}) \times \mathbf{R}^{b-k_1} \times \mathbf{R}^{b_1-1} \end{aligned}$$

which means

$$\chi^c(A_{b_1}(f_1)) = \chi^c(\mathbf{R}^2 - \{0\})\, \chi^c(\mathbf{R}^{2b_1-k_1-1}).$$

Since $\chi^c(\mathbf{R}^2 - \{0\}) = 0$ we get by (2.3) that $\chi^c(A_{b_1}(f_1)) \neq \chi^c(A_{b_1}(f_2))$. Therefore $Z_{f_1} \neq Z_{f_2}$, which contradicts Theorem 4. This ends the proof of Theorem 1 in the first case.

Case 2. In this case, we suppose $a_i \notin \mathbf{N}$, $b_i \in \mathbf{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exists the term $x^{p_i} y$ for some integers $p_i \geq 1$ with non-zero coefficients in $f_i(x, y)$. By Theorem 2 and (2.1), it is easy to see that for any integers $s \geq 1$, $f_i(x, y) + x^{p_i+s}$ is blow-analytically equivalent to $f_i(x, y)$. Then the Fukui invariant of f_i is determined by

$$A(f_i) = \{p_i + 1,\ p_i + 2,\ p_i + 3, \cdots\} \cup \{\infty\}. \tag{2.4}$$

Moreover $A(f_1) = A(f_2)$, and it follows that $p_1 = p_2$. Consequently it is sufficient to prove that $b_1 = b_2$. Indeed, suppose that $b_1 < b_2$. Then, we let

$$p = p_1 = p_2, \quad \Re_n = \{(r, s) \in \mathbf{N}^2 \mid rp + s = n\}$$

and

$$C^n_{r,s} = \{\gamma(t) = (u_r t^r + \cdots + u_n t^n\,,\, v_s t^s + \cdots + v_n t^n) \mid u_r, v_s \neq 0\}$$
$$\simeq (\mathbf{R}^*)^2 \times \mathbf{R}^{2n-r-s}.$$

Let us first compute $\chi^c(A_{b_1}(f_i))$. It is easy to see that for any positive integers $n < b_i$, we have that $A_n(f_i) = \bigcup_{(r,s)\in\Re_n} C^n_{r,s}$ (Remark that the union is disjoint). Thus, by the additivity of χ^c, we have

$$\chi^c(A_{b_1}(f_2)) = \sum_{(r,s)\in\Re_{b_1}} (-2)^2(-1)^{2b_1-r-s}. \tag{2.5}$$

Similarly if $b_1 - 1 \notin p\,\mathbf{N}$, we obtain

$$\chi^c(A_{b_1}(f_1)) = (-2)(-1)^{2b_1-d} + \sum_{(r,s)\in\Re_{b_1}} (-2)^2(-1)^{2b_1-r-s} \tag{2.6}$$

where d is the smallest number in $\{1,\dots,b_1\}$ such that $d\,p+1 > b_1$. It follows from (2.5) and (2.6) that $\chi^c(A_{b_1}(f_2)) \neq \chi^c(A_{b_1}(f_1))$. But this implies a contradiction, by comparing the coefficients of the zeta functions. Hence we have $b_1 - 1 \in p\,\mathbf{N}$. Now assume $b_1 = k\,p+1$. Then by elementary computation, we have

$$A_{b_1}(f_1) = C_{f_1} \cup (\cup_{(r,s)\in\Re_{b_1}\setminus\{(k,1)\}} C^{b_1}_{r,s}),$$

where

$$C_{f_1} = \{\gamma(t) = (u_k t^k + \cdots + u_{b_1} t^{b_1}\,,\, v_1 t^1 + \cdots + v_{b_1} t^{b_1}) \mid f_1(u_k, v_1) \neq 0\}$$
$$\simeq \{f_1 \neq 0\} \times \mathbf{R}^{2b_1-k-1},$$

Also, by the additivity of the Euler characteristic with compact support, we obtain

$$\chi^c(A_{b_1}(f_1)) = \chi^c(\{f_1 \neq 0\})(-1)^{2b_1-k-1} + \sum_{(r,s)\in\Re_{b_1}\setminus\{(k,1)\}} 4(-1)^{2b_1-r-s}.$$

Together with (2.5), it follows that

$$\chi^c(\{f_1 = 0\}) = -3. \tag{2.7}$$

We will next compute the $\chi^c(A_{b_1+1}(f_i))$, $(i = 1,2)$. Setting $m = kp+2 = b_1+1$. Then, by the above, $m - 1 \notin p\,\mathbf{N}$ and $m \leq b_2$, we can easily see the following

(2.8)
$$\chi^c(A_m(f_2)) = \begin{cases} \sum_{(r,s)\in\Re_m} 4(-1)^{2m-r-s} & \text{if } m < b_2, \\ -2(-1)^{2m-k-1} + \sum_{(r,s)\in\Re_m} 4(-1)^{2m-r-s} & \text{if } m = b_2 \end{cases}$$

Now we compute $\chi^c(A_m(f_1))$. Let $\lambda(t) = (X(t), Y(t))$ be an analytic arc defined by

$$X(t) = u_k t^k + \cdots + u_m t^m,$$
$$Y(t) = v_1 t + \cdots + v_m t^m.$$

We can write

$$f_1(X(t), Y(t)) = f_1(u_k, v_1)t^{m-1} + \langle\, \nabla f_1(u_k, v_1)\,;\, (u_{k+1}, v_2)\,\rangle\; t^m + \cdots ,$$

where

$$\langle\, \nabla f_1(u_k, v_1)\,;\, (u_{k+1}, v_2)\,\rangle = \frac{\partial f_1}{\partial x}(u_k, v_1)\, u_{k+1} + \frac{\partial f_1}{\partial y}(u_k, v_1)\, v_2.$$

Moreover, if $f_1(u_k, v_1) = 0$ and $\langle\, \nabla f_1(u_k, v_1)\,;\, (u_{k+1}, v_2)\,\rangle \neq 0,$ then we have $O(f_1 \circ \lambda) = m$. Let us put

$$\begin{aligned}
B_1 &= \{(u, v, w, z) \in (f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 \mid \langle\, \nabla f_1(u, v)\,;\, (w, z)\,\rangle \neq 0\,\},\\
B_2 &= \{(u, v, w, z) \in (f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 \mid \langle\, \nabla f_1(u, v)\,;\, (w, z)\,\rangle = 0\,\},\\
C_{\nabla f_1} &= \{(u_k t^k + \cdots + u_m t^m, v_1 t^1 + \cdots + u_m t^m) |\, (u_k, u_{k+1}, v_1, v_2) \in B_1\}\\
&\simeq B_1 \times \mathbf{R}^{2m-k-3},
\end{aligned}$$

Then, by the above, the $A_m(f_1)$ given by

$$A_m(f_1) = C_{\nabla f_1} \cup (\cup_{(r,s)\in \Re_m} C^m_{r,s}).$$

Thus the Euler characteristic with support compact of $A_{b_m}(f_1)$ equals

$$\chi^c(A_m(f_1)) = \chi^c(B_1)(-1)^{2m-k-3} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}. \tag{2.9}$$

By identification of the m-coefficients of both zeta functions of f_i for $i = 1, 2$, it follows from (2.8) and (2.9) that $\chi^c(B_1) = 0$ or -2. On the other hand, $(f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 = B_1 \cup B_2$. Therefore

$$\chi^c(f_1^{-1}(0) - \{0\}) = \chi^c(B_1) + \chi^c(B_2),$$

but $B_2 \simeq (f_1^{-1}(0) - \{0\}) \times \mathbf{R}$. This is clear because f_1 is non-degenerate, then we have

$$\chi^c(f_1^{-1}(0) - \{0\}) = \chi^c(f_1^{-1}(0) - \{0\})(-1) + \chi^c(B_1).$$

Since $\chi^c(B_1) = 0$ or -2, this yields

$$\chi^c(f_1^{-1}(0)) = 1 \text{ or } 0,$$

which contradicts (2.7). This ends the proof of Theorem 1 in the second case.

Remark 7. *If we drop the assumption that b_2 is an integer, then the above proof still holds.*

Case 3. In this case, we suppose $a_i \in \mathbf{N}$, $b_i \notin \mathbf{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exists the term xy^{q_i} for some integers $q_i \geq 1$ with non-zero coefficients in $f_i(x, y)$. For any real α we denote by $e(\alpha)$ the minimum positive integer n such that $n \geq \alpha$. By an argument similar to that of Assertion 6 and (2.4), we can compute the Fukui invariant of f_i as follows:

$$A(f_i) = a_i\mathbf{N} \cup \{e(b_i),\, e(b_i)+1, \cdots\} \cup \{\infty\}.$$

By Theorem 3, $A(f_1) = A(f_2)$. Then we have the following result:

$$a_1 = a_2 \text{ and } e(b_1) = e(b_2) \tag{2.10}$$

Suppose now $b_1 \neq b_2$. Then $q_1 \neq q_2$, but $\mid b_1 - b_2 \mid \geq \mid q_1 - q_2 \mid \geq 1$. It follows that $e(b_1) \neq e(b_2)$, which contradicts (2.10). This complete the proof of Theorem 1 in the third case.

Case 4. In this case, we suppose a_i, $b_i \notin \mathbf{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exist the terms $x^{p_i}y$ and xy^{q_i} for some integers $p_i \geq 1$ and $q_i \geq 1$ with non-zero coefficients in $f_i(x, y)$. Thus, the Fukui invariant of f_i can be written as

$$A(f_i) = \{p_i + 1,\, p_i + 2,\, p_i + 3, \cdots\} \cup \{\infty\},$$

which implies $p_1 = p_2$. Thus we only have to prove that $b_1 = b_2$. Indeed, let us assume that $b_1 < b_2$. Then we have $q_1 < q_2$ which implies $b_1 < e(b_1) < b_2$. Let us put

$$p = p_1 = p_2,\ m = e(b_1) \text{ and } \Re_m = \{(r, s) \in (\mathbf{N} - \{0\})^2 \mid rp + s = m\}.$$

We first observe that $m - 1 \notin p\mathbf{N}$. Otherwise, if $m - 1 = r\,p$, then we have:

$$b_1 < q_1 + r < r\,p + 1 < r\,a_1. \tag{2.11}$$

This is a consequence of $b_1 < m = r\,p + 1$ and also $(1\,,\,q_1)$ and $(p\,,\,1)$ are vertices of $\Gamma(f_1)$. But $m = \min\{n \in \mathbf{N} \mid n > b_1\}$, which contradicts (2.11). Hence we have $m - 1 \notin p\mathbf{N}$. Using this observation and by elementary computation we obtain the following result:

$$\begin{aligned} \chi^c(A_m(f_2)) &= \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}, \\ \chi^c(A_m(f_1)) &= (-2)^2(-1)^{m+q_1-1} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}. \end{aligned} \tag{2.12}$$

This means that $Z_{f_1} \neq Z_{f_2}$, which contradicts Theorem 4. This complete the proof of Theorem 1 in the fourth case.

In order to finish the proof of Theorem 1, it suffices to show the following lemmas.

Lemma 8. $a_1 \in \mathbf{N}$ *if and only if* $a_2 \in \mathbf{N}$.

Proof. Suppose that this is not the case. Namely, $a_1 \in \mathbf{N}$ and $a_2 \notin \mathbf{N}$. Since f_2 is non-degenerate, then there exists the term $x^{p_2}y$ for some integers $p_2 \geq 1$ with non-zero coefficients in $f_2(x,y)$. Again using the same argument in (2.4) one gets

$$A(f_2) = \{p_2+1,\, p_2+2,\, p_2+3, \cdots, \infty\},$$

Since $A(f_1) = A(f_2)$, then we have $a_1 = b_1 = p_2+1$, set $m = p_2+1$. We shall compute the $\chi^c(A_m(f_i))$ for $i = 1,2$, that is

$$\begin{aligned} A_m(f_2) &= \{\gamma(t) = (u_1t + \cdots + u_mt^m \,,\, v_1t + \cdots + v_mt^m) \mid u_1, v_1 \neq 0\} \\ &\simeq (\mathbf{R}^*)^2 \times \mathbf{R}^{2m-2}, \end{aligned}$$

so

$$\begin{aligned} A_m(f_1) &= \{\gamma(t) = (u_1t + \cdots + u_mt^m \,,\, v_1t + \cdots + v_mt^m) \mid f_1(u_1,v_1) \neq 0\} \\ &\simeq \{f_1 \neq 0\} \times \mathbf{R}^{2m-2}, \end{aligned}$$

and hence to

$$\chi^c(A_m(f_i)) = \begin{cases} (-2)^2(-1)^{2m-2} & \text{if } i = 2, \\ \chi^c(\{f_1 \neq 0\})(-1)^{2m-2} & \text{if } i = 1. \end{cases} \tag{2.13}$$

Since $\chi^c(A_m(f_1)) = \chi^c(A_m(f_2))$, then we have

$$\chi^c(\{f_1 = 0\}) = -3. \tag{2.14}$$

Using the same argument as Case 2, the $(m+1)$-coefficients of Z_{f_i} for $i = 1,2$ can be computed as follows:

$$\chi^c(A_{m+1}(f_1)) = \chi^c(B_1) \quad \text{and} \quad \chi^c(A_{m+1}(f_2)) = \begin{cases} -4 & \text{if } m \neq b_2, \\ -6 & \text{if } m = b_2. \end{cases}$$

We recall that:

$$\begin{aligned} B_1 &= \{(u,v,w,z) \in (f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 \mid \langle \nabla f_1(u,v)\,;\,(w,z)\rangle \neq 0\}, \\ B_2 &= \{(u,v,w,z) \in (f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 \mid \langle \nabla f_1(u,v)\,;\,(w,z)\rangle = 0\}. \end{aligned}$$

Finally, by comparing the $(m+1)$-coefficients of both zeta functions Z_{f_i}, it is evident that $\chi^c(B_1) = -4$ or -6, but $(f_1^{-1}(0) - \{0\}) \times \mathbf{R}^2 = B_1 \cup B_2$. It follows from the additivity of the Euler characteristic that $\chi^c(f_1^{-1}(0) - \{0\}) = \chi^c(B_1) + \chi^c(B_2)$. On the other hand, by $B_2 \simeq (f_1^{-1}(0) - \{0\}) \times \mathbf{R}$ (because f_1 is non-degenerate), then we have

$$\chi^c(f_1^{-1}(0)) = -1 \text{ or } -2,$$

which contradicts (2.14). This proves the lemma. Q.E.D.

Lemma 9. $b_1 \in \mathbf{N}$ *if and only if* $b_2 \in \mathbf{N}$.

Proof. Suppose now that $b_1 \in \mathbf{N}$ and $b_2 \notin \mathbf{N}$. Since f_2 is non-degenerate, then there exists the term xy^{q_2} for some integers $q_2 \geq 1$ with non-zero coefficients in $f_2(x, y)$.

We first consider $a_i \in \mathbf{N}$ for $i = 1, 2$. Then, by the same reason as above, we can compute the Fukui invariant of f_i as follows:

$$A(f_1) = a_1\mathbf{N} \cup b_1\mathbf{N} \cup \mathbf{N}_{\geq [a_1, b_1]} \cup \{\infty\},$$
$$A(f_2) = a_2\mathbf{N} \cup \mathbf{N}_{\geq e(b_2)} \cup \{\infty\}.$$

Since $A(f_1) = A(f_2)$, then we have the following result:

$$(2.15) \qquad a_1 = a_2, \;\; b_1 = k\,a_1, \;\; \text{and} \quad e(b_2) = b_1 \text{ or } b_1 + 1.$$

Since $b_1 = k\,a_1$, we may assume by Remark 5 that there exists the term $xy^{k(a_1-1)}$ with non-zero coefficients in $f_1(x, y)$. But $|b_2 - b_1| \geq |q_2 - k(a_1 - 1)| \geq 1$, which implies $b_2 \geq b_1 + 1$ or $b_1 \geq b_2 + 1$. It follows that $e(b_2) > b_1 + 1$ or $e(b_2) < b_1$, which contradicts (2.15), and ends the first part of the lemma.

Now we consider the case where $a_i \notin \mathbf{N}$ for $i = 1, 2$. Since f_i is non-degenerate, then there exists the term $x^{p_i}y$ for some integers $p_i \geq 1$ with non-zero coefficients in $f_i(x, y)$. It is easy to see that

$$A(f_i) = \{p_i + 1, \, p_i + 2, \, p_i + 3, \cdots\} \cup \{\infty\}.$$

Moreover $A(f_1) = A(f_2)$, and we get $p_1 = p_2$. Set

$$p = p_1 = p_2, \;\; m = e(b_2) \quad \text{and} \quad \Re_m = \{(r, s) \in \mathbf{N}^2 \mid rp + s = m\}.$$

As stated in Remark 7, we can exclude the case where $b_1 < b_2$ (because this is proved in exactly the same way as Case 2). Thus it remains to consider the case $b_2 < b_1$.

We next compute the m-coefficients of both zeta functions Z_{f_i} for $i = 1, 2$. For this, we can assert that $m - 1 \notin p\mathbf{N}$. Indeed, suppose that

$m-1 = \alpha\, p$ for some positive integer α. Since $b_2 < m = \alpha\, p + 1$ which implies $b_2 < q_2 + \alpha < \alpha p + 1$. This is clear because $(1\,,\, q_2) \in \Gamma(f_2)$. But $m = e(b_2)$ is equal to the smallest integer greater than b_2, which is a contradiction. Therefore we obtain that $m - 1 \notin p\,\mathbf{N}$, and so on by elementary computation, we have the following result:

$$\chi^c(A_m(f_2)) = (-2)^2(-1)^{m+q_2-1} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s}. \tag{2.16}$$

And

$$\chi^c(A_m(f_1)) = \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s} \qquad \text{if } m < b_1,$$

$$\chi^c(A_m(f_1)) = (-2)(-1)^{m+q_2} + \sum_{(r,s)\in\Re_m} (-2)^2(-1)^{2m-r-s} \qquad \text{if } m = b_1.$$

Now it suffices to note by the above equalities that $Z_{f_1} \neq Z_{f_2}$, which contradicts Theorem 4. This completes the proof. Q.E.D.

Theorem 1 is therefore proved.

Example 10. *Let k be an arbitrary integer greater than or equal to 4. We consider quasihomogeneous polynomial functions f_k, $g_k\colon (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ defined by*

$$f_k(x, y) = x^5 + x\,y^{2\,k}, \qquad g_k(x, y) = x^5 - y^{2\,k+2}.$$

Note that the weights of f_k and g_k are $(\frac{1}{5}\,,\, \frac{2}{5\,k})$ and $(\frac{1}{5}\,,\, \frac{1}{2\,k+2})$ respectively. Since f_k and g_k have different weights for $k > 4$, they are not blow-analytically equivalent by Theorem 1. However, f_k and g_k are topologically equivalent. In fact, the above $f_k(x, y) = x^5 + x\,y^{2\,k} \in J^{2\,k+1}_{\mathbf{R}}(2, 1)$ is C^0-sufficient by the Kuiper-Kuo Theorem (see [7, 8]). Therefore, f_k is topologically equivalent to $f_k - y^{2\,k+2}$. On the other hand, g_k and $g_k + x\,y^{2\,k}$ are blow-analytically equivalent by Theorem 2. Besides $f_k - y^{2\,k+2} = g_k + x\,y^{2\,k}$, hence the conclusion holds. Consequently, $f_k \in J^{2\,k+1}_{\mathbf{R}}(2, 1)$ is not blow-analytically sufficient for $k > 4$.

In the case $k = 4$, the weights of f_4 and g_4 are equal to $(\frac{1}{5}\,,\, \frac{1}{10})$. Furthermore, f_4 is blow-analytically equivalent to g_4. Indeed, consider the family $H_t\colon (\mathbf{R}^2, 0) \to (\mathbf{R}, 0)$ $(t \in [0, 1])$ defined by $H_t(x, y) = (1-t) f_4(x, y) + t\, g_4(x, y)$. It is easy to see that for each $t \in [0, 1]$, H_t has an isolated singularity at $0 \in \mathbf{R}^2$. Therefore, it follows from Theorem 2 that $\{H_t\}_{0\le t\le 1}$ is blow-analytically trivial over $[0, 1]$. In particular, $H_0 = f_4$ is blow-analytically equivalent to $H_1 = g_4$.

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Characteristic classes of singular varieties

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Abstract. This is a short and concise survey on recent results on the Milnor classes of global complete intersections. By definition the Milnor class of X equals the difference between the Chern-Schwartz-MacPherson and the Fulton-Johnson classes of X and we describe the results that express it in terms of the local and global invariants of the singular locus of X. In this survey we underline the characteristic cycle approach and its realtion to the vanishing Euler characteristic, as for instance to the Euler characteristic of the Milnor fibre in the hypersurface case.

We present some recent developments in the theory of characteristic classes of singular algebraic and analytic varieties. We would like, in particular, underline the characteristic cycle approach and the geometric insight given by this construction. For different approaches the reader may consult the excellent surveys [7] and [45].

Several different characteristic classes can be defined for a singular variety X: the Chern-Schwartz-MacPherson class $c_*(X)$, the Chern-Mather class $c_M(X)$, the Fulton class $c^F(X)$ and the Fulton-Johnson class $c^{FJ}(X)$. For nonsingular X they are all equal to the Poincaré dual of the Chern class $c(TX)$ of the tangent bundle. We present in this survey some results that answer the following question: how does the difference $c^F(X)-c_*(X)$ (or $c^{FJ}(X)-c_*(X)$) depend on the singularities of X?

The characteristic classes of singular varieties may be defined in different set-ups. For complex algebraic varieties they take values in the

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Chow group $A_*(\cdot)$. (They can be even defined for algebraic varieties over an arbitrary algebraically closed field of characteristic zero, cf. [21, 22].) For complex analytic spaces, assumed compact or such that they can be compactified, the characteristic classes take values in the Borel-Moore homology $H_*^{BM}(\cdot;\mathbb{Z})$. For complex algebraic varieties both approaches are linked by the cycle map $cl : A_*(\cdot) \to H_*^{BM}(\cdot;\mathbb{Z})$. For simplicity of exposition by variety we mean either a complex analytic space or an algebraic variety, and then by homology we mean $H_*^{BM}(\cdot;\mathbb{Z})$ or $A_*(\cdot)$ respectively.

We shall also include a brief review of Stiefel-Whitney classes that can be defined for real algebraic varieties or, in general, for Euler mod 2 triangulated spaces and that take values in $H_*^{BM}(\cdot;\mathbb{Z}_2)$.

If X is singular then the tangent bundle to X is not well-defined and therefore one cannot consider simply the characteristic classes of this bundle. Suppose X is a subvariety of a non-singular variety M. We recall briefely the definitions of the Fulton and Fulton-Johnson classes, see [16] for details. The idea is to find an object which plays the role of the normal bundle to X in M. Let $C_X M$ be the *normal cone* to X in M and let $\mathcal{J}$ be the ideal sheaf of X in $C_X M$. Denote by $\mathcal{N}_X M = \mathcal{J}/\mathcal{J}^2$ the *conormal sheaf* of X in M. Let p be the blow-up of X in M. The exceptional divisor of p can be identified with the projectivization of $C_X M$. The *Segre class* of X in M is given by:

$$s(X,M) = s(C_X M) = p_*\big(\sum c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C_X M]\big),$$

where $\mathcal{O}(1)$ denotes the canonical line bundle on $\mathbb{P}C_X M$.

The *Fulton class* of X is defined by

$$c^F(X) = c(TM|_X) \cap s(X,M).$$

The *Fulton-Johnson class* of X equals

$$c^{FJ}(X) = c(TM|_X) \cap s(\mathcal{N}_X M).$$

Both the Fulton and the Fulton-Johson classes are independent of the embedding of X into non-singular variety, cf. [16], Example 4.2.6. If X is regularly embedded in M (i. e. $\mathcal{J}$ is locally generated by a regular sequence) then the both classes coincide and

$$c^{FJ}(X) = c^F(X) = c(\tau_X) \cap [X],$$

where $\tau_X = TM|_X - N_X M$ is the *virtual tangent bundle to* X. In this case $N_X M = dual(\mathcal{J})$ is a vector bundle and is canonically isomorphic to $C_X M$, cf. [16], Appendix B7.

Another possibility is to recover the tangent bundle to singular X by the means of the Nash-blowing-up, see [26], [16], Example 4.2.9. Let $\nu : \tilde{X} \to X$ be the Nash blowing-up of X and let $\tilde{T}$ denote the vector bundle on $\tilde{X}$ that extends $\nu^* TX$. Then the *Chern-Mather class* $c_M(X)$ of X equals by definition

$$c_M(X) := \nu_*(c(\tilde{T}) \cap [\tilde{X}]).$$

In section 1 below we recall the definition of the Chern-Schwartz-MacPherson class $c_*(X)$ of X. Thus we have at least four different notions of characteristic classes that coincide for X nonsingular.

Example 0.1. (Hypersurface with an isolated singularity).
Let $L \to M$ a line bundle, M nonsingular, and let X be the zero scheme of a holomorphic section f of L. Suppose that, moreover, X has an isolated singularity $SingX = \{p_0\}$. Then

$$\begin{aligned} &c^{FJ}(X) = c^F(X) = c(TM - L) \cap [X] \\ &c_*(X) = c(TM - L) \cap [X] + (-1)^n \mu_n [p_0] \\ &c_M(X) = c(TM - L) \cap [X] + (-1)^n (\mu_n + \mu_{n-1})[p_0]. \end{aligned}$$

where $TM - L$ is the virtual tangent bundle of X, μ_n is the Milnor number of f at p_0 and μ_{n-1} is the Milnor number of the generic hyperplane section of f at p_0.

We shall study the general hypersurface case in section 2 below.

§1. Chern-Schwartz-MacPherson classes and characteristic cycles

We recall some of the basic results on Chern-Schwartz-MacPherson classes and characteristic cycles. For the details the reader is refered to [7, 17, 21, 26, 32, 36].

1.1. Constructible functions

For a variety X we denote by $F(X)$ the group of integer-valued *constructible functions on* X i.e. finite sums

$$\alpha = \sum_i n_i \mathbb{1}_{V_i}$$

where V_i are subvarieties of X. There are many interesting operations on constructible functions: sum, product, pull-back, push-forward, specialization, duality, and Euler integral inherited from sheaf theory by taking the index of a constructible complex of sheaves. Recall that for a constructible complex of sheaves $\mathcal{F}_\bullet$ on X its index is the stalkwise Euler characteristic $p \to \chi(\mathcal{F}_\bullet)(p) = \sum(-1)^i \dim H^i(\mathcal{F}_\bullet)_p$. It is a constructible function. Note that this definition is purely local so the global properties of $\mathcal{F}_\bullet$ are lost. The operations on constructible functions can be defined independently by means of Euler integral, see [42], [33], [20].

If X is compact then the *Euler integral* of α is defined as the weighted Euler characteristic: $\int \alpha\, d\chi := \sum_i n_i \chi(V_i)$. For a proper map $f : X \to Y$ the *proper push-forward* $f_* : F(X) \to F(Y)$ is given by

$$(f_*\alpha)(y) := \int_{f^{-1}(y)} \alpha\, d\chi.$$

Let $f : X \to S$ be a morphism to a curve and let s_0 be a nonsingular point of S. Denote $X_0 = f^{-1}(s_0)$. The *s*pecialization homomorphism $\mathrm{sp} : F(X) \to F(X_0)$, or nearby Euler characteristic, is given by the Euler integral on the Milnor fibre of f. That is, at $p \in X_0$ and for α as above

$$\mathrm{sp}\,(\alpha)(p) = \int_{F_p} \alpha\, d\chi = \sum_i n_i \chi(F_p \cap V_i), \tag{1}$$

where F_p is the Milnor fibre of f at p. That is, $F_p = f^{-1}(s) \cap B(p, \varepsilon)$, where, in local systems of coordinates, $B(p, \varepsilon)$ denotes the ball centered at p of radius ε and s is chosen so that $0 < |s - s_0| \ll \varepsilon \ll 1$.

1.2. Chern-Schwartz-MacPherson classes

The *Chern-Schwartz-MacPherson class* (the CSM class for short) c_* is the unique transformation from constructible functions $F(\cdot)$ to homology $H_*(\cdot)$ and satisfying:

(1) $f_* c_*(\alpha) = c_* f_*(\alpha)$ for a proper morphism $f : X \to Y$.
(2) $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$,
(3) $c_*(\mathbb{1}_X) = c(TX) \cap [X]$ for X nonsingular.

Its existence was conjectured by Deligne and Grothendieck and proven by MacPherson in [26]. They are, by the Alexander duality isomorphism, equal to the characteristic classes introduced by M.-H. Schwartz, cf. [38, 9]

By a theorem of Verdier [41] the CSM class commutes with specialization: $c_* \circ \mathrm{sp} = \mathrm{Sp} \circ c_*$, where $\mathrm{Sp} : H_*(X) \to H_*(X_0)$ is the specialization on homology, see [22] for the Chow group counterpart.

1.3. Characteristic cycles

Let M be a nonsingular variety of dimension n and let T^*M denote the cotangent bundle of M. We consider $\mathcal{L}(M)$: the free abelian group generated by the set of conical Lagrangian subvarieties of T^*M. Thus each element of $\mathcal{L}(M)$ is an integral combination of irreducible Lagrangian subvarieties that can be described as follows. Let V be a closed subvariety of M and let $Reg(V) = V \setminus Sing(V)$ denote the set of regular points of V. The *conormal space* to V in M

$$T^*_V M := Closure\left\{(x,\xi) \in T^*M \mid x \in Reg(V),\ \xi|_{T_x Reg(V)} \equiv 0\right\},$$

is a conical Lagrangian subvariety and each irreducible conical Lagrangian subvariety of T^*M is the conormal space of an irreducible subvariety of M. For a subvariety $X \subset M$ let $\mathcal{L}(X)$ denote the subgroup of $\mathcal{L}(M)$ given by the conical Lagrangian subvarieties of T^*M over X. We call an element of $\mathcal{L}(X)$ a *conical Lagrangian cycle over* X.

To a constructible function $\alpha \in F(X)$ we associate its *characteristic cycle* $\mathrm{Ch}(\alpha) \in \mathcal{L}(X)$ so that we get a group isomorphism $\mathrm{Ch} : F(X) \to \mathcal{L}(X)$. For instance, for a subvariety V, $\mathrm{Ch}(1\!\!1_V)$ can be defined by means of the characteristic cycle of a sheaf, cf. for instance [11], by

$$\mathrm{Ch}(1\!\!1_V) = \mathrm{Ch}(i_*\mathbb{C}_V),$$

where $i : V \hookrightarrow M$ is the inclusion. Then

$$T^*_V M = (-1)^{\dim V}\,\mathrm{Ch}(Eu_V),$$

where Eu_V denotes MacPherson's Euler obstruction [26]. (In literature there are two sign conventions in the definition of Ch that differ by $(-1)^{\dim M}$. We follow that of [21])

Let $f : (M,p) \to (\mathbb{C},0)$ be the germ of a holomorphic function and let $\alpha = \sum_i n_i 1\!\!1_{V_i}$ be a constructible function on M. Let $\mathrm{sp}\,\alpha(p)$ be the specialization of α to the zero fibre of f as defined in (1). The difference $\mathrm{sp}\,\alpha(p) - \alpha(p)$ can be interpreted as the vanishing Euler characteristic. Suppose that the graph $Gr(df)$ of df, considered as a section of T^*M, intersects $\mathrm{Ch}(\alpha)$ only at $(p, df(p))$. Then by the index formula for the sheaf vanishing cycles due to Lê, Dubson, and Sabbah, cf. [13] and (4.5) and (4.6) of [32], the local intersection number of the cycles $\mathrm{Ch}(\alpha)$ and $Gr(df)$ equals

$$(2) \qquad (\mathrm{Ch}(\alpha).Gr(df))_{(p,df(p))} = -(\mathrm{sp}\,\alpha(p) - \alpha(p)).$$

Thus one may interpret $\mathrm{Ch}(\alpha)$ as the set of such covectors $(p,\xi) \in T^*M$ that the Euler integral of the fibers of functions $f : (M,p) \to$

$(C,0)$ with $df(p)=\xi$ changes at p. To be more precise, fix a Whitney stratification $\{S_j\}$ of M, such that each V_i is the union of strata. Then, by Thom-Mather theory, there is no change of topology of fibers of $f|_{V_i}$ if $(p,\xi)\notin \bigcup T^*_{S_j}M$. In particular, $\mathrm{Ch}(\alpha)=\sum_i n_i T^*_{S_i}M$ with integer coefficients n_i. In general these coefficients may be zero or negative. By (2) they are determined by the vanishing Euler characteristic of such f that $Gr(df)$ intersects $T^*_{S_i}M$ at a generic point.

Example 1.1. Let $p\in X\subset M$. The coefficient of T^*_pM in $\mathrm{Ch}(\mathbb{1}_X)$ equals

$$1-\chi(lk_{\mathbb{C}}(X,p))$$

where $lk_{\mathbb{C}}(X,p)$ is the complex link of X at p (in local coordinates the intersection of X with generic hyperplane near p).

There are operations of proper push-forward and specialization on conical Lagrangian cycles defined geometrically. Ch is a natural transformation in the sense that it commutes with these operations and the corresponding operations on constructible functions, cf. [17], [21], [32],

(1) $f_*\,\mathrm{Ch}(\alpha)=\mathrm{Ch}\,f_*(\alpha)$ for proper morphisms $f:X\to Y$
(2) $\mathrm{Ch}(\alpha+\beta)=\mathrm{Ch}(\alpha)+\mathrm{Ch}(\beta)$
(3) $\mathrm{Ch}(\mathrm{sp}(\alpha))=\mathrm{Sp}(\mathrm{Ch}(\alpha))$.

By a formula of Sabbah [32], (1.2.1), for $\alpha\in F(X)$

$$c_*(\alpha):=(-1)^{n-1}c\,(TM|_X)\cap\pi_*\left(c(\mathcal{O}(1))^{-1}\cap[\mathbb{P}\,\mathrm{Ch}\,\alpha]\right), \tag{3}$$

where $\mathcal{O}(1)$ is the canonical line bundle on $\mathbb{P}T^*M$ and $\pi:\mathbb{P}T^*M|X\to X$ denotes the projection. Using Sabbah's own words "cela montre que la théorie des classes de Chern de [26] se ramène à une théorie de Chow sur T^*M, qui ne fait intervenir que des classes fondamentales".

The Chern-Mather class of V, see [26], equals

$$c_M(V)=c_*(Eu_V)=(-1)^{n-1-\dim V}c\,(TM|_V)\cap\pi_*\left(c(\mathcal{O}(-1))\cap[\mathbb{P}T^*_VM]\right). \tag{4}$$

Remark 1.2. The CSM class and the Euler obstruction are closely related to the geometry of polar varieties, see [24], and also [7] and the references therein.

§2. Characteristic cycles and Stiefel-Whitney classes

Characteristic cycles can be also defined in real analytic and algebraic geometry for semi-algebraic and subanalytic sets cf. [19], [20], [14], or even for sets defined in any o-minimal structure [34], see also an explicit construction in [36]. More precisely, given an oriented real analytic manifold M, we have a group isomorphism

$$\mathrm{Ch} : F(M) \to \mathcal{L}(M)$$

between the group of subanalytically constructible functions $F(M)$ on M and the group of subanalytic conical Lagrangian cycles $\mathcal{L}(M)$ in T^*M. (Here by conical we mean $\mathbb{R}_{>0}$-homogeneous.) The most important difference from the complex case is that the subanalytic Lagrangian conical cycles in T^*M are not necessarily combination of conormal spaces but usually more complicated subanalytic cycles of T^*M. Moreover usually the conormal space $T^*_V M$ is not a cycle. These differences are caused by the fact that for a subanalytic continuous function $f : (V,0) \to (\mathbb{R}, 0)$, $V \subset M$ subanalytic closed, or even for f and V real analytic, the vanishing Euler characteristic from the right (i.e., defined by the positive Milnor fiber) may not be equal to that from the left ((i.e., defined by the negative Milnor fiber). Note that in the real set-up there are more possible conventions on the sign, for instance the characteristic cycle constructed by Fu [14] corresponds to that of Kashiwara-Schapira [20] after the application of the antipodal map (multiplication by -1 in the fibers of T^*M).

Example 2.1. Let $V \subset M$ be subanalytic closed and let $\{S_i\}$ be a subanalytic Whitney stratification of V. Define

$$\Lambda^o := \bigsqcup \Lambda^o_{S_i}, \qquad \Lambda^o_{S_i} = T^*_{S_i} M \setminus \bigcup_{j\neq i} T^*_{S_j} M.$$

Decompose Λ^o into the connected components $\Lambda^o := \bigsqcup \Lambda^o_j$. Then

$$\mathrm{Ch}(\mathbb{1}_V) = \sum n_j \Lambda^o_j,$$

for some integers n_i that can be described topologically by the vanishing Euler characteristic, see the index formula below.

The analogue of the index formula (2), [20] Thm. 9.5.6, see also [19] and [37], has even more flavour of the Morse Theory. It says that for $V \subset M$ subanalytic closed, and a real analytic $f : (M,p) \to (\mathbb{R}, 0)$ such that $Gr(df)$ intersects $\mathrm{Ch}(\mathbb{1}_V)$ only at $(p, df(p))$

$$\begin{aligned}(Gr(df).\,\mathrm{Ch}(\mathbb{1}_V))_{(p,df(p))} &= \chi(B \cap \{x \in V, f(x) \le +\delta\}) \\ &\quad -\chi(B \cap \{x \in V, f(x) \le -\delta\}),\end{aligned}$$

where B denotes the ball of radius ε centered at p, $0 < \delta \ll \varepsilon \ll 1$. Given a conical Lagrangian cycle $\Lambda = \mathrm{Ch}(\alpha) \in \mathcal{L}(M)$. In order to recover the value $\alpha(p)$ at $p \in M$ it suffices to intersect Λ with $Gr(df)$, where $f : (M,p) \to (\mathbb{R},0)$ is a Morse function of index 0 (for instance $f(x) = x_1^2 + \cdots + x_n^2$ in local coordinates). Then

$$\alpha(p) = (Gr(df).\Lambda)_{(p,df(p))}$$

Remark 2.2. The operation inverse to Ch is related to MacPherson's Euler obstruction as follows. Let M be a complex manifold and V a complex analytic subvariety of M. Consider V as a subanalytic subset of M and M itself as an oriented real analytic manifold. Then T_V^*M is a real Lagrangian cycle. Let $p \in V$ and $f : (M,p) \to (\mathbb{R},0)$ be a real Morse function of index 0. Then $(Gr(df).T_V^*M))_{(p,df(p))} = (-1)^{\dim_{\mathbb{C}} V} Eu_V(p)$. This formula for the Euler obstruction is essentially the definition of MacPherson, where the intersection $Gr(df)$ is replaced by the intersection with the section given by the radial vector field.

2.1. Stiefel-Whitney classes

In 1935 Stiefel defined a characteristic class $w_i(X) \in H_i(X;\mathbb{Z}_2)$ for any smooth compact manifold. He conjectured that $w_i(X)$ is represented by the sum of all the i-simplices of the first barycentric subdivision of a triangulation of X. Stiefel's Conjecture was proved by Whitney in 1939. In 1969 Sullivan observed that Stiefel's definition can be applied to real analytic spaces since they are (mod 2) Euler spaces, that is to say, the link of each point has even Euler characteristic. Then, for a triangulated Euler space, the sum of all the i-simplices of the first barycentric subdivision is a $\mathbb{Z}_2$-cycle.

It was noticed in [15] that the Stiefel-Whitney classes of subanalytic sets can be defined via the characteristic cycles. We give below just a short account, for details the reader is refered to [15].

Remark 2.3. ([33], [20]) Verdier Duality on sheaves induces a duality on constructible functions. This duality can be written as

$$D\alpha(p) = \alpha(p) - \int_{S_p^\varepsilon} \alpha\, d\chi,$$

where S_p^ε is a small sphere centered at p. The corresponding duality on the conical Lagrangian cycles is given by the antipodal map that is by the multiplication by (-1) in the fibres of T^*M. Note that in the complex case the duality on constructible function and the one on conical Lagrangian cycles are the identity maps.

Let M be an oriented real analytic manifold. A subanalytically constructible function $\alpha \in F(M)$ is called *(mod 2) Euler* if it is self dual modulo 2 (equivalently its Euler integral along any small sphere is even). For such a function the projectivization of its characteristic cycle

$$\mathbb{P}Ch(\alpha) \subset \mathbb{P}T^*M$$

is a (mod 2)-cycle.

For a (mod 2) Euler constructible function $\alpha \in F(M)$ one may define its ith Stiefel-Whitney class by a formula corresponding to (3)

$$w_i(\alpha) = \pi_*(\gamma_M^{n-i-1} \cap [\mathbb{P}Ch(\alpha)])$$

where $\pi : \mathbb{P}T^*M \to M$ is the projection and

$$\gamma_M^k = \sum_j \pi^*(w^i(TM)) \cap \zeta_M^{k-j},$$

where $\zeta_M \in H^1(\mathbb{P}T^*M; \mathbb{Z}_2)$ is the first Stiefel-Whitney class of the tautological line bundle on $\mathbb{P}T^*M$.

Defined this way, Stiefel-Whitney homological classes satisfy the axioms analogous to the Deligne-Grothendieck axioms for the CSM-classes and the Verdier specialization property.

§3. Hypersurface case

Let M be a nonsingular compact complex analytic variety of pure dimension n and let L be a holomorphic line bundle on M. Take $f \in H^0(X, L)$ a holomorphic section of L such that the variety X of zeros of f is a reduced hypersurface in M.

Consider the constructible function $\chi : X \to \mathbb{Z}$ defined for $x \in X$ by $\chi(x) := \chi(F_x)$, where F_x denotes the Milnor fibre at x and $\chi(F_x)$ its Euler characteristic. Also, define $\mu := (-1)^{n-1}(\chi - \mathbb{1}_X)$, that is the signed vanishing Euler characteristic.

In this section, following [1], [2], [29], [3], we give the common descriptions of the CSM and Fulton classes of X as well as the results that present the contribution of the singularities of X to the difference of these two classes. Similarly to [29] our approach is based on the computation of characteristic cycle of X. For an account on different possible approaches see [4].

3.1. Local description of characteristic cycle

The characteristic cycle of X was calculated in [5] and [23] in terms of the blow-up of the Jacobian ideal of a local equation of X in M. More precisely, let $X \subset U \subset \mathbb{C}^n$ be the zero set of a holomorphic function $f : U \to \mathbb{C}$ and denote by $\pi : \mathrm{Bl}_{\mathcal{J}_f} U \to U$ the blowing-up of the Jacobian ideal of $\mathcal{J}_f = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$.

Let $\mathcal{X} = \pi^{-1}(X)$ denote the total transform of X by π and we denote the irreducible components of $\mathcal{X}$ by D_i and by C_i their projections onto U. Denote by $\mathcal{I}_{C_i}$ the ideal defining C_i and the multiplicities of $\mathcal{I}_{C_i}$, f, and $\mathcal{J}_f$ along D_i by n_i, m_i and p_i respectively. Note that D_i is contained in the exceptional divisor of π if and only if $p_i = 0$. It is known by the transversality of polar varieties that $m_i = n_i + p_i$, see [40], [5], and [31].

By [5], [23], we have the following explicit formulas

$$\mathrm{Ch}(\mathbb{1}_X) = (-1)^{n-1} \sum_i n_i T^*_{C_i} U \,;$$
$$\mathrm{Ch}(\chi) = \mathrm{Ch}(\mathrm{R}\,\Psi_f \mathbb{C}_U) = (-1)^{n-1} \sum_i m_i T^*_{C_i} U \,;$$
$$\mathrm{Ch}(\mu) = (-1)^{n-1} \mathrm{Ch}\big(\mathrm{R}\,\Phi_f \mathbb{C}_U\big) = \sum_i p_i T^*_{C_i} U \,.$$

(Here $\mathrm{R}\,\Psi_f$ and $\mathrm{R}\,\Phi_f$ denote the complexes of nearby and vanishing cycles respectively.)

$\mathrm{Bl}_{\mathcal{J}_f} U$ can be interpreted geometrically by means of the *relative conormal space* $T^*_f \subset T^*U$

$$T^*_f := \text{ Closure } \{(x, \eta) \subset T^*U; df(x) \neq 0, \exists_\lambda \text{ such that } \eta = \lambda df(x)\}.$$

Let $\tilde{f} : T^*_f \to \mathbb{C}$ denote the composition of the projection $T^*_f \to U$ and f. Then $\tilde{f}^{-1}(c)$, for a regular value c, equals the conormal space to $f^{-1}(c)$. Thus by Lagrangian specialization, cf. [25], [18], $\tilde{f}^{-1}(0)$ is a conical Lagrangian subvariety of T^*U. It is equal to $\mathrm{Ch}(\chi)$ since Ch commutes with specialization. Moreover the total transform $\mathcal{X}$ of X by π, is the set of limits of the direction of the gradient $[\frac{\partial f}{\partial z_1}(x) : \ldots : \frac{\partial f}{\partial z_n}(x)]$ and hence equals, at least as a set, the projectivization of $\tilde{f}^{-1}(0)$. Thus $\tilde{f}^{-1}(0)$ is the union of conormals $T^U_{C_i}$ and each $D_i = \mathbb{P}T^*_{C_i} U$. In particular, we may rewrite the above formulas as follows

$$\begin{aligned} [\mathbb{P}\,\mathrm{Ch}(\mathbb{1}_X)] &= (-1)^{n-1} \left([\mathcal{X}] - [\mathcal{Y}]\right) \,; \\ [\mathbb{P}\,\mathrm{Ch}(\chi)] &= (-1)^{n-1} [\mathcal{X}] \,; \\ [\mathbb{P}\,\mathrm{Ch}(\mu)] &= [\mathcal{Y}] \end{aligned} \tag{5}$$

where $\mathcal{Y}$ denote the exceptional divisor of π.

Remark 3.1. The computation of coefficients n_i, m_i and p_i can be done by a topological argument based on the Morse theory and generic polar curves, see for instance [23]. In particular, in the isolated singularity case, $Sing X = \{p_0\}$, the coefficients at $T^*_{p_0}U$ are equal to $(-1)^{n-1}\mu_{n-1}$, $(-1)^{n-1}(\mu_n+\mu_{n-1})$, and μ_n respectively. Here μ_n denotes the Milnor number of f at p_0 and μ_{n-1} the Milnor number of the generic hyperplane section of f at p_0. One may show that $1-(-1)^{n-1}\mu_{n-1}$ equals the Euler characteristic of the complex link of X at p_0.

3.2. Global description of characteristic cycle

The singular scheme of X, that we denote by Y, is defined in local coordinates by $(f, \mathcal{J}_f) = \left(f, \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$. Since f belongs to the integral closure of $\mathcal{J}_f$, the normalizations of blow-ups of $\mathcal{J}_f$ and $(f, \mathcal{J}_f)$ are equal. Hence the formulas (5) hold true locally if we replace the blow-up of the former ideal by the blow-up of the latter one. We shall see that they hold true globally.

Let $B = \mathrm{Bl}_Y M \to M$ be the blow-up of M along Y. Let $\mathcal{X}$ and $\mathcal{Y}$ denote the total transform of X and the exceptional divisor in B respectively. To get a convenient description of B, we use the bundle $\mathcal{P}^1_M L$ of principal parts of L over M, as in [2], [31]. The differentials and the sections of L take values in $\mathcal{P}^1_M L$ and also $\mathcal{P}^1_M L$ fits in an exact sequence

$$0 \to T^*M \otimes L \to \mathcal{P}^1_M L \to L \to 0.$$

Thus f determines a section of $\mathcal{P}^1_M L$ that is written locally as $(df, f) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}, f\right)$. The closure of the image of the meromorphic map $M \dashrightarrow \mathbb{P}\mathcal{P}^1_M L$ induced by this section is the blow-up $B \to M$. Thus we may treat B as a subvariety of $\mathbb{P}\mathcal{P}^1_M L$. Clearly, the total transform $\mathcal{X}$ of X equals $B \cap \mathbb{P}(T^*M \otimes L)$, that we identify with a subvariety of $\mathbb{P}(T^*M \otimes L)$. Since $\mathbb{P}(T^*M \otimes L) = \mathbb{P}(T^*M)$ we see that the formulas (5) hold globally.

By an elementary computation on $\mathbb{P}(T^*M \otimes L)$, see [31], this gives

$$\begin{aligned} c_*(X) &= c\,(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}]-[\mathcal{Y}]}{1+\mathcal{X}-\mathcal{Y}}\right) \\ c_*(\chi) &= c\,(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}]}{1+\mathcal{X}-\mathcal{Y}}\right); \\ c_*(\mu) &= (-1)^{n-1} c\,(TM|_X) \cap \pi_* \left(\frac{[\mathcal{Y}]}{1+\mathcal{X}-\mathcal{Y}}\right). \end{aligned} \tag{6}$$

The first of the above formulas was obtained by Aluffi [2] by means of resolution of singularities and a detailed description of how the formula changes under a blowing-up. He has also got the following formula for the Chern-Mather class of X.

$$c_M(X) = c\,(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}']}{1+\mathcal{X}-\mathcal{Y}} \right) ,$$

where $\mathcal{X}'$ is the proper transform of X. In local coordinates $[\mathcal{X}'] = \mathbb{P}T_X U$ and hence this result follows from (4).

3.3. Aluffi's formulas. Milnor class of a hypersurface.

From (6) we derive the formulas obtained by Aluffi in [1]. First note that in our case

$$c^F(X) = c^{FJ}(X) = c(TM|_X - L|_X) \cap [X] .$$

By birational invariance of Segre classes [16], Chap.4:

$$\begin{aligned} c^F(X) = c(TM|_X) \cap s(X,M) \quad &= \quad c(TM|_X) \cap \pi_* s(\mathcal{X}, B) \\ &= \quad c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}]}{1+\mathcal{X}} \right) . \end{aligned}$$

In [1] Aluffi defines a "thickening" of X along its singular subscheme Y: X^k is the subscheme of M defined by the ideal $\mathcal{I}_X \mathcal{I}_Y^k$. He shows that the Fulton class of X^k is a polynomial in k with the CSM class being equal to $c^F(X^{-1})$. Indeed, as above,

$$c^F(X^k) = c(TM|_X) \cap s(X^k, M) = c(TM|_X) \cap \pi_* \left(\frac{[\mathcal{X}] + k[\mathcal{Y}]}{1+\mathcal{X}-k\mathcal{Y}} \right) . \tag{7}$$

This can be expressed in the following suggestive form, cf. [1],

$$c_*(X) = c^F(X^{-1}) = c(TM|_X) \cap s(X \setminus Y, M).$$

The *Milnor class* was first defined by Yokura [44] as

$$\mathcal{M}(X) := (-1)^{n-1} \big(c^{FJ}(X) - c_*(X) \big).$$

As follows from (6), (7)

$$\begin{aligned} \mathcal{M}(X) \qquad &= (-1)^{n-1} c\,(TM|_X) \cap \pi_* \left(\frac{[\mathcal{Y}]}{(1+\mathcal{X})(1+\mathcal{X}-\mathcal{Y})} \right) \\ &= (-1)^{n-1} c\,(TM|_X) \cap \pi_* \left(\frac{[\mathbb{P}\,\mathrm{Ch}(\mu)]}{(1+\mathcal{X})(1+\mathcal{X}-\mathcal{Y})} \right) \\ &= c(L|_X)^{-1} c_*(\mu), . \end{aligned} \tag{8}$$

Let $\mathcal{S} = \{S\}$ be any stratification of X such that μ is constant on the strata of $\mathcal{S}$. (One may take, for instance, any Whitney stratification of X.) Denote the value of μ on the stratum S by μ_S and let

$$\alpha(S) := \mu_S - \sum_{S' \neq S, S \subset \overline{S'}} \alpha(S')$$

be the numbers defined inductively on descending dimension of S. Then $\mu = \sum_{S \in \mathcal{S}} \alpha(S) \mathbb{1}_{\overline{S}}$ and

$$\mathrm{Ch}(\mu) = \sum_{S \in \mathcal{S}} \alpha(S) \, \mathrm{Ch}(\mathbb{1}_{\overline{S}}).$$

This gives the following formula on the Milnor class, see [29],

$$\mathcal{M}(X) = \sum_{S \in \mathcal{S}} \alpha(S) \, c(L|_X)^{-1} \cap (i_{\overline{S},X})_* \, c_*(\overline{S}), \tag{9}$$

where $i_{\overline{S},X} : \overline{S} \to X$ denotes the inclusion. This formula was first conjectured by Yokura in [44] and proven in [31]. If Y is smooth and μ is constant on Y, equal say μ_Y, then it reads

$$\mathcal{M}(X) = \mu_Y (c(L|_X)^{-1} \cap (i_{Y,X})_* \, c_*(Y)).$$

If the singular set of X is finite, then we get

$$\mathcal{M}(X) = \sum_{x \in Sing(X)} \mu(x)[x]. \tag{10}$$

This formula was obtained also by Suwa [39] in a more general set-up of isolated singularities of (global) complete intersections.

The Milnor class is related to the μ-class supported on Y, introduced by Aluffi in [1],

$$\mu_L(Y) = c(T^*M \otimes L) \cap s(Y, M).$$

As shown in [2]

$$\mathcal{M}(X) = (-1)^n c(L)^{n-1} (\mu_L(Y)^\vee \otimes L).$$

For the notation $\mu_L(Y)^\vee$ and the proof we refer the reader to [2]. We just note that the above formula can be derived from (8).

Following [3] we give another interpretation of (8). Let Y be a subvariety of M. Let $\pi : C_Y M \to Y$ denote the normal cone to Y in M and let C_i be the irreducible components of $C_Y M$. Denote by m_i their multiplicities and let $Y_i = \pi(C_i)$. Then the *weighted Mather class of* Y is defined by

$$c_{wM}(Y) := \sum (-1)^{\dim Y_i} m_i c_M(Y_i).$$

(here the factor $(-1)^{\dim Y}$ is removed from the original Aluffi's definition). Note that $c_{wM}(Y)$ depends on the scheme structure of Y, in particular it is sensitive to the presence of embedded components. An important property is that $c_{wM}(Y)$ is intrinsic to Y (independent of the ambient nonsingular variety). In the particular case when Y is the singular scheme of a hypersurface X

$$c_{wM}(Y) = c_*(\mu) = c(L|_Y) \cap \mathcal{M}(X).$$

Thus (8) takes the following form

$$c^{FJ}(X) - c_*(X) = (-1)^{n-1} c(L|_{Sing(X)})^{-1} \cap c_{wM}(Sing(X)) \tag{11}$$

with the right hand side depending only on the scheme structure of $Sing(X)$ and on $c(L|_{Sing(X)})$.

Remark 3.2. Many of the results presented above were motivated by their corresponding formulas for the Euler characteristic. The generalized Milnor number was first defined in [28] as

$$\mu(X) := (-1)^{n-1}(c(TM|_X - L|_X) \cap [X] - \chi(X))).$$

If X has only isolated singularities then the generalized Milnor number of X equals the sum of their local Milnor numbers. Yokura's Conjecture, i.e. formula (9), was motivated by a similar formula for the Euler characteristic established in [29].

3.4. Specialization

Suppose now that the line bundle L admits a section $g \in H^0(M, L)$ such that $X' = g^{-1}(0)$ is smooth and transverse to the strata of a Whitney stratification of X. This is for instance the case when L is very ample. The Milnor class, and so the Fulton class, of X equals the CSM class of a simple constructible function on X.

For $t \in \mathbb{C}$, denote $f_t = f - tg$. In this paragraph by $\mathcal{X}$ we denote

$$\mathcal{X} := \left\{(x, t) \in M \times \mathbb{C} \mid f_t(x) = 0\right\}.$$

Let $p : \mathcal{X} \to \mathbb{C}$ be the restriction to $\mathcal{X}$ of the projection onto the second factor of $M \times \mathbb{C}$. Then $p^{-1}(t) = \{x \in M \mid f_t(x) = 0\}$ for $t \in \mathbb{C}$. Denote by

$$\mathrm{sp} : F(\mathcal{X}) \to F(X)$$

the specialization by p. By Proposition 5.1 of [31]

$$\operatorname{sp} \mathbb{1}_{\mathcal{X}}(x) = \begin{cases} \chi(x) = 1 + (-1)^{n-1}\mu(x) & \text{for } x \notin X \cap X', \\ 1 & \text{for } x \in X \cap X'. \end{cases}$$

Then by the commutativity of the CSM class with specialization

$$c^F(X) = c_*(\operatorname{sp} \mathbb{1}_{\mathcal{X}}), \quad \mathcal{M}(X) = (-1)^{n-1} c_*(\operatorname{sp} \mathbb{1}_{\mathcal{X}} - \mathbb{1}_X),$$

see [31] for details. In particular the last equality gives the formula (8) for the Milnor class.

Remark 3.3. One may show easily that in the algebraic case $c^F(X)$ or $\mathcal{M}(X)$, or indeed any algebraic cycle, is of the form $c_*(\alpha)$ for a constructible function α. Note that the above formulas give such α's explicitely (under the assumption of ampleness of L).

§4. Milnor classes

One would like to extend the results described in the previous section to the local complete intersection case. To be more precise, let M be a nonsingular variety and let $i : X \hookrightarrow M$ be a regular embedding (cf. [16] Appendix B7). Then

$$c^F(X) = c^{FJ}(X) = c(\tau_X) \cap [X],$$

Let $N_X M$ denote its normal bundle. The question is whether there is a formula so that

$$\mathcal{M}(X) = c(N_X M)^{-1} \cap class(Sing X) \quad ?$$

and $class(SingX)$ is a characteristic class depending only on some data given by $Sing(X)$, see also [46] for similar questions.

By a result of Suwa [39] this is the case if X has only isolated singularities. Then

$$\mathcal{M}(X) = \sum_{x \in Sing(X)} \mu_x \, [x],$$

where μ_x denotes the Milnor number at x.

But the very first obstacle to extend the hypersurface case is the absence of a good candidate for constructible function μ. For $f : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0), k \geq 2$, the Milnor fibration (or the nearby cycles functor) is not well-defined in general unless f is "sans éclatement en codimension 0", see [18], that is the case, for instance, in the isolated singularity case.

Nevertheless there are some partial answers that we describe below.

4.1. Milnor class by the obstruction theory

In [8] the authors assume that M is a complex manifold and $X \subset M$ is a local complete intersection. We assume moreover that X is the zero set of a holomorphic section, generically transverse to the zero section, of a holomorphic vector bundle E on M (this case is sometimes called a global complete intersection). X is also assumed to be compact. Let $n = \dim X$. It is showed in [8] that $\mathcal{M}(X)$ can be localized at a connected component S of the singular set Y of X. For such a component S and the following data: a tubular neighbourhood U of S in X, a positive integer r, and an r frame $v^{(r)}$ of vectors tangent to X defined on $\partial U \cap D$, D being the $2(n-r+1)$-skeleton of a cellular decomposition of X, the authors define two classes: the localized Schwartz class $Sch(v^{(r)}, S)$ and the localized virtual class $Vir(v^{(r)}, S)$ both living in $\in H_{2(r-1)}(S)$. The former class contributes to $c_{r-1}(X)$ and the latter to the homology characteristic class of the virtual tangent bundle $\tau_X = TM|_X - N_X M$. Then, as shown in [8],

$$\mu_{r-1}(X,S) := (-1)^{n-1}((Sch(v^{(r)}, S) - Vir(v^{(r)}, S))$$

is independent on the choices and the total Milnor class is the sum over the connected components S_α

$$\mathcal{M}(X) = \sum_\alpha (i_\alpha)_* \mu_*(X, S_\alpha)$$

where $i_\alpha : S_\alpha \hookrightarrow X$ and $\mu_*(X, S_\alpha) = \sum \mu_i(X, S_\alpha)$.

4.2. On Verdier-type Riemann-Roch for CSM classes

Let $f : X \to Y$ be a local complete intersection morphism (an l.c.i. for short), that is the composition of a regular embedding i and a smooth morphism p, cf. [16] Appendix B. 7. Guided by the Riemann-Roch theorem and by the bivariant theory of Chern classes, Yokura [43] posed the question of commutativity (or rather of understanding the non-commutativity) of the following diagram

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & A_*(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle c(T_f)\cap f^*} \\ F(X) & \xrightarrow{c_*} & A_*(X) \end{array}$$

where T_f is the virtual tangent bundle, f^* on the left-hand side is the pull-back of constructible functions, f^* on the right-hand side is the Gysin homomorphism, i.e., the composition of the smooth pull-back p^*

and the Gysin i^* for regular embeddings. The non-commutativity of the above diagram is related to the Milnor class as follows, cf. [43] Example (3.1). Let $i : X \hookrightarrow M$ be a regular embedding as above and let $p : M \to pt$ be the projection to a point. Then $f = p \circ i : X \to pt$ is an l.c.i. morphism and applying the morphisms of the diagram to $1\!\!1_{pt}$ we get

$$c_*(f^* 1\!\!1_{pt}) = c_*(X),$$
$$c(T_f) \cap f^* c_*(1\!\!1_{pt}) = c(T_f) \cap [X] = c^{FJ}(X).$$

Thus, in this case, the non-commutativity of the diagram is measured exactly by the Milnor class.

Actually, only the regular embeddings contribute to the non-commutativity of the diagram. Yokura [43] shows that the diagram is commutative for smooth morphisms. Indeed, let us verify it for X and Y non-singular on $1\!\!1_Y \in F(Y)$:

$$\begin{aligned} c(T_f) \cap f^* c_*(1\!\!1_Y) &= c(T_f) \cap f^*(c(TY) \cap [Y]) \\ &= (c(T_f) \cup c(f^*TY)) \cap f^*[Y] \\ &= (c(T_f) \cup f^* c(TY)) \cap [X] \\ &= c(TX) \cap [X]. \end{aligned}$$

The general case can be reduced to the above one by the resolution of singularities.

4.3. Schürmann's formula

We present the main result of [35] that generalizes the results on the hypersurface case to the case of the regular embedding. Recall that for a general holomorphic map $f : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^k, 0)$, $k \geq 2$, the Milnor fibration and the nearby Euler characteristic are not well-defined. The main idea of Schürmann is to overcome this difficulty by replacing X by a hypersurface using the classical construction of the deformation to the normal cone, cf. [16] Ch. 5.

Let $i : X \hookrightarrow Y$ be a regular embedding, and we do not have to assume that Y is smooth. Let $C_X Y$ be the normal cone of X in Y and let $\pi : C_X Y \to X$ and $k : X \to C_X Y$ denote the projection and the embedding as the zero section respectively. Since i is a regular embedding, $C_X Y$ is equal to the normal bundle $N_X Y$. In paricular, π is smooth. Denote by $M_X Y \to \mathbb{C}$ the deformation of Y to the normal cone C_XY. We have the commutative diagram

$$\begin{array}{ccccc} C_XY & \hookrightarrow & M_XY & \hookleftarrow & Y\times\mathbb{C}^* \\ \downarrow & & \downarrow{\scriptstyle flat} & & \downarrow \\ \{0\} & \hookrightarrow & \mathbb{C} & \hookleftarrow & \mathbb{C}^* \end{array}$$

Denote by $\tilde{\pi} : Y\times\mathbb{C}^* \to Y$ the projection to the first factor. Schürmann defines the "constructible function version" of Verdier specialization as

$$\mathrm{sp} = \mathrm{sp}_{X\setminus Y} = \psi_h \circ \tilde{\pi}^* : F(Y) \to F_{mon}(C_XY),$$

whose image is in monodromic, i.e. conical, constructible functions on C_XY. By Verdier specialization, the CSM class commutes with the analogously defined specialization on homology $\mathrm{Sp} : H_*(Y) \to H_*(C_XY)$: $\mathrm{Sp}\circ c_* = c_* \circ \mathrm{sp}$. The "vanishing Euler characteristic" transformation associated to the embedding i is defined by $\Phi_i = \mathrm{sp} - \pi^* i^*$. Thus

$$(12)\quad c_*(\Phi_i(\cdot)) \quad = c_*(\mathrm{sp}(\cdot) - \pi^* i^*(\cdot)) = \mathrm{Sp}(c_*(\cdot)) - c_*(\pi^* i^*(\cdot)).$$

This formula holds in $H_*(C_XY)$. To go down to $H_*(X)$, we use the Gysin isomorphism $k^* = (\pi^*)^{-1} : H_*(C_XY) \to H_*(X)$ (recall that $k : X \to C_XY$ denotes the embedding on the zero section)

$$k^* c_*(\Phi_i(\alpha)) = k^* \,\mathrm{Sp}(c_*(\alpha)) - k^* c_*(\pi^* i^*(\alpha)).$$

The Gysin homomorphism i^* is defined by $i^* = k^* \circ \mathrm{Sp}$. Since π is smooth, by Yokura's theorem $c_*(\pi^*(\cdot)) = c(T_\pi) \cap \pi^*(c_*(\cdot))$ and hence

$$k^* c_*(\pi^*(\cdot)) = c(k^* T_\pi) \cap k^* \pi^*(c_*(\cdot)) = c(N_XY) \cap c_*(\cdot).$$

Consequently

$$k^* c_*(\Phi_i(\alpha)) = i^*(c_*(\alpha)) - c(N_XY) \cap c_*(i^*(\alpha)).$$

Thus applying $c(N_XY)^{-1} \cap k^*$ to both sides of (12)

$$(13)\qquad c(N_XY)^{-1} \cap k^* c_*(\Phi_i(\cdot)) = c(N_XY)^{-1} \cap i^* c_*(\cdot) - c_*(i^*(\cdot)).$$

This is the formula of Schürmann [35].

If Y is smooth, this formula applied to $1\!\!1_Y$ reads

$$(14)\qquad c(N_XY)^{-1} \cap k^* c_*(\Phi_i(1\!\!1_Y)) = c^F(X) - c_*(X).$$

If $i : X \hookrightarrow Y$ is a regular embedding of codimension 1, Y arbitrary, then the specialization in local coordinates defines the vanishing Euler characteristic functor $\mu : F(Y) \to F(X)$. It satisfies $\Phi_i = \pi^* \circ \mu - k_* \circ \mu$ and Schürmann's formula takes the following simple form

(15) $c(N_XY)^{-1} \cap c_*(\mu(\alpha)) = c(N_XY)^{-1} \cap i^*c_*(\alpha) - c_*(i^*(\alpha))$,

for $\alpha \in F(Y)$. If Y is smooth and $\alpha = 1\!\!1_Y$, we recover the formula of Yokura's Conjecture (8).

Suppose that Y is smooth. Since the geometric construction of deformation onto the normal cone can be localized, Schürmann's formula for the Milnor class can be also localized at each connected component of $Sing(X)$. For such a connected component S denote: $i_S : S \hookrightarrow X$ the inclusion, $n_S : N_XY|S \to N_XY$ the induced inclusion of normal cones, $\mu_S := n_S^*\Phi_i(1\!\!1_Y)$, and $k_S : S \to N_XY|S$ is the inclusion on the zero section. Then

$$c^{FJ}(X) - c_*(X) = \sum_S (i_S)_*(c(N_XY|S)^{-1} \cap k_S^* c_*(\mu_S)).$$

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On the classification of 7th degree real decomposable curves

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Abstract.

A survey of recent results in the problem of the topological classification of 7th degree decomposable curves in the real projective plane is given.

Let (x_0, x_1, x_2) be point coordinates in the real projective plane $\mathbb{R}P^2$. An algebraic curve of degree m is a homogeneous polynomial $F_m(x_0, x_1, x_2)$ over $\mathbb{R}$ of degree m considered up to a constant nonzero factor. The set

$$\mathbb{R}F_m = \{(x_0, x_1, x_2) \in \mathbb{R}P^2 | F_m(x_0, x_1, x_2) = 0\} \subset \mathbb{R}P^2$$

is called the set of real points of the curve. The algebraic curve F_m is called an M-curve if the set $\mathbb{R}F_m$ consists of $(m-1)(m-2)/2 + 1$ connected components.

The polynomial F_m is *decomposable* (in the product of two factors) if

$$F_m(x_0, x_1, x_2) = A_k(x_0, x_1, x_2) \cdot B_{m-k}(x_0, x_1, x_2),$$

where $k \leq [m/2]$, and the polynomials $A_k(x_0, x_1, x_2)$ of degree k and $B_{m-k}(x_0, x_1, x_2)$ of degree $m-k$ are irreducible over $\mathbb{R}$. Our problem is to obtain the topological classification of triples $(\mathbb{R}P^2, \mathbb{R}F_m, \mathbb{R}A_k)$, which satisfy the following conditions of maximality and general position:

(i) the curves A_k and B_{m-k} are M-curves ;

(ii) the set $\mathbb{R}A_k \bigcap \mathbb{R}B_{m-k}$ consists of $k(m-k)$ distinguish points;

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(iii) all points of the set $\mathbb{R}A_k \bigcap \mathbb{R}B_{m-k}$ are situated on the same connected component of $\mathbb{R}A_k$ and on the same connected component of $\mathbb{R}B_{m-k}$.

In case $m = 6$ this problem was solved by the author (under weaker conditions) – see [P1]–[P4]. In particular, the following theorem provides the classification for the case $m = 6, k = 1$.

Theorem 1. *Under conditions* (i)-(iii), *the classification of triples* $(\mathbb{R}P^2, \mathbb{R}F_6, \mathbb{R}A_1)$ *consists of 4 types shown in Figure 1.*

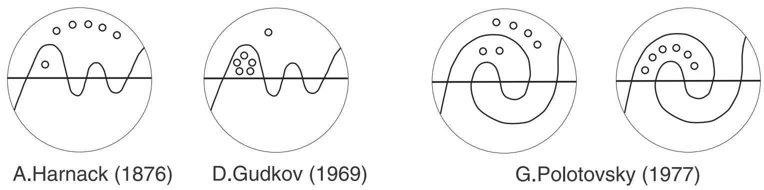

Fig. 1. Line and M-quintic (authors of the first constructions are marked).

Here and below we use the Poincaré disk (i.e. disk where every two diametrically opposite points of its boundary circle are identified) as model of the projective plane $\mathbb{R}P^2$.

The classification of arrangements of a quintic and a line in general position has important application: it gives classification of smoothings of generic five-fold point (singularity N_{16} in Arnold's notations). O.Viro showed (see [V1], [V2]) that from topological point of view, smoothing such a singular point is a result of gluing an affine quintic instead of a neighborhood of the point under condition of coincidence of asymptotic directions of the quintic with tangents to the branches at the singular point. E.Shustin proved [Sh1] that it is always possible to obtain this coincidence. We would like to point out that Theorem 1 provides also the classification of affine M-quintics: it is sufficient to consider the line $\mathbb{R}A_1$ as the line at infinity for the affine plane (in Figure 2 the line $\mathbb{R}A_1$ is shown as the boundary of the Poincaré disc).

In one's turn, smoothing five-fold points has been used by many authors for the constructions of nonsingular algebraic curves.

The classification of triples $(\mathbb{R}P^2, \mathbb{R}F_m, \mathbb{R}A_k)$ for $m = 6$ has many different applications, therefore it is naturally to consider the problem for $m = 7$. Below we give a survey of results in this direction.

The classification of affine M-sextics has been recently completed. One can find the proof in series of papers [O-Sh1], [K1], [K2], [Sh-K], [Sh2], [O1], [O2], [O-Sh2], [F-O]. The result is formulated in the following theorem.

Theorem 2. *Under conditions* (i)-(iii), *the classification of triples* $(\mathbb{R}P^2, \mathbb{R}F_7, \mathbb{R}A_1)$ *consists of 35 types shown in Figure 2.*

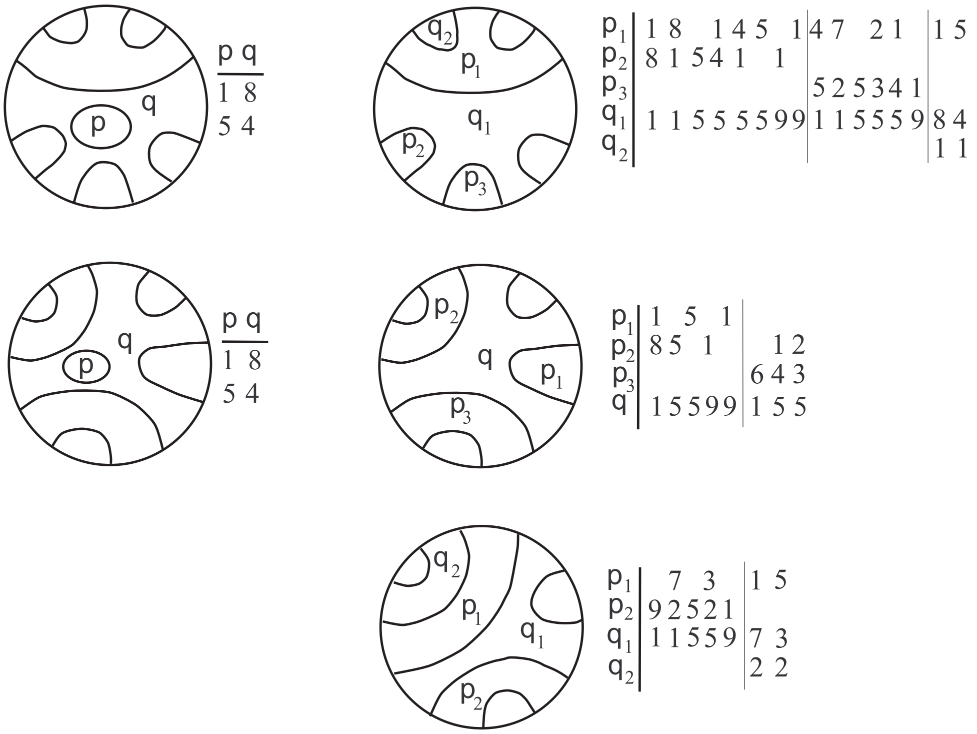

Fig. 2. Line and M-sextic (letters denote the number of ovals in domains of the same names).

Further it is natural to consider separately the cases when points of set $\mathbb{R}A_k \bigcap \mathbb{R}B_{7-k}$

a) are situated on ovals[1] of curves-factors and

b) lie on the odd branch of the factor of odd degree.

Note, that in case a) there always exists a pseudo-line which has no intersections with ovals[2]. Below for pictures we assume that this line is the boundary of the Poincaré disk and we do not draw it in the Figures. In case a) we also do not draw the odd branch.

One can find the proof of the classification for the case $k = 2$ under condition a) in papers [O3], [P5], [P6] and complete answer is:

[1]By definition, an oval and the odd branch are respectively a two-sided and one-sided circles embedded in $\mathbb{R}P^2$.

[2]In the opposite case there exists a pseudo-line consisting from arcs of ovals therefore the odd branch will intersect an oval, but it contradicts to the assumption a).

Theorem 3. *Under conditions* (i)-(iii) *in case* a) *the classification of triples* $(\mathbb{R}P^2, \mathbb{R}F_7, \mathbb{R}A_2)$ *consists of 42 types shown in Figure 3.*

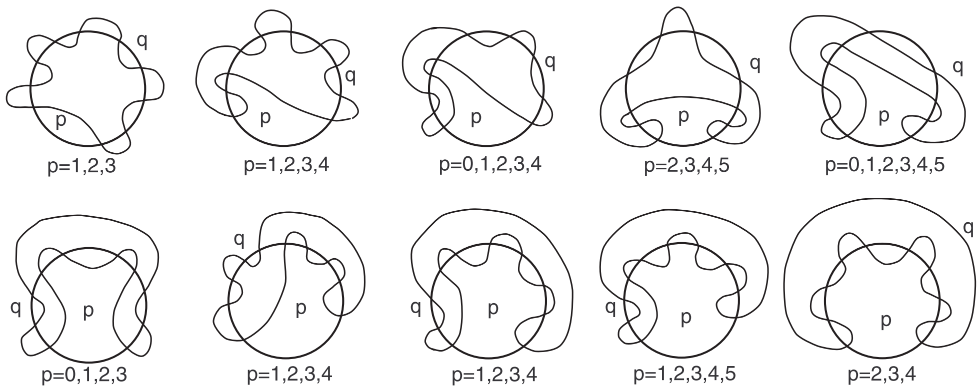

Fig. 3. Conic and M-quintic with common points on ovals; $p+q=5$.

In the case b) for $k=2$ the classification is in progress. In particular, at the present time about 60 types are constructed and for the same number of types the question about realizability is still open. Some details can be found in [P5], [G1], [G2]. In [K-P] was considered a problem intimately connected with case b) for $k=2$: the classification of arrangements of a M-quintic and pair of lines. Namely, in [K-P] the following theorem was proved.

Theorem 4. *Every arrangement of two lines and quintic in maximal general position, for which there are only two arcs of odd branch having ends in points of intersection lying on different lines, is homeomorphic to one of 20 model depicted in Figure 4.*

The most difficult case is case $k=3$ of mutual arrangements of a M-cubic and a M-quartic. In [O-P] the answer for case a) was obtained:

Theorem 5. *Under conditions* (i)-(iii) *in case* a) *the classification of triples* $(\mathbb{R}P^2, \mathbb{R}F_7, \mathbb{R}A_3)$ *consists of 31 types shown in Figure 5.*

In the case b) for $k=3$ the classification has not been completed. Below we describe some details for this case. Simultaneously it will give an illustration of a general approach to the classification which consists in the following.

We draw *topological models*, i.e., collections of smooth circles in $\mathbb{R}P^2$, which may pretend to represent a triple of the kind $(\mathbb{R}P^2, \mathbb{R}F_m, \mathbb{R}A_k)$ up to a homeomorphism, and for each such a model we try to find out, to which extent this pretention can be justified. In other words, our procedure consists of the following steps.

Step 1. *Enumeration* of all admissible models.

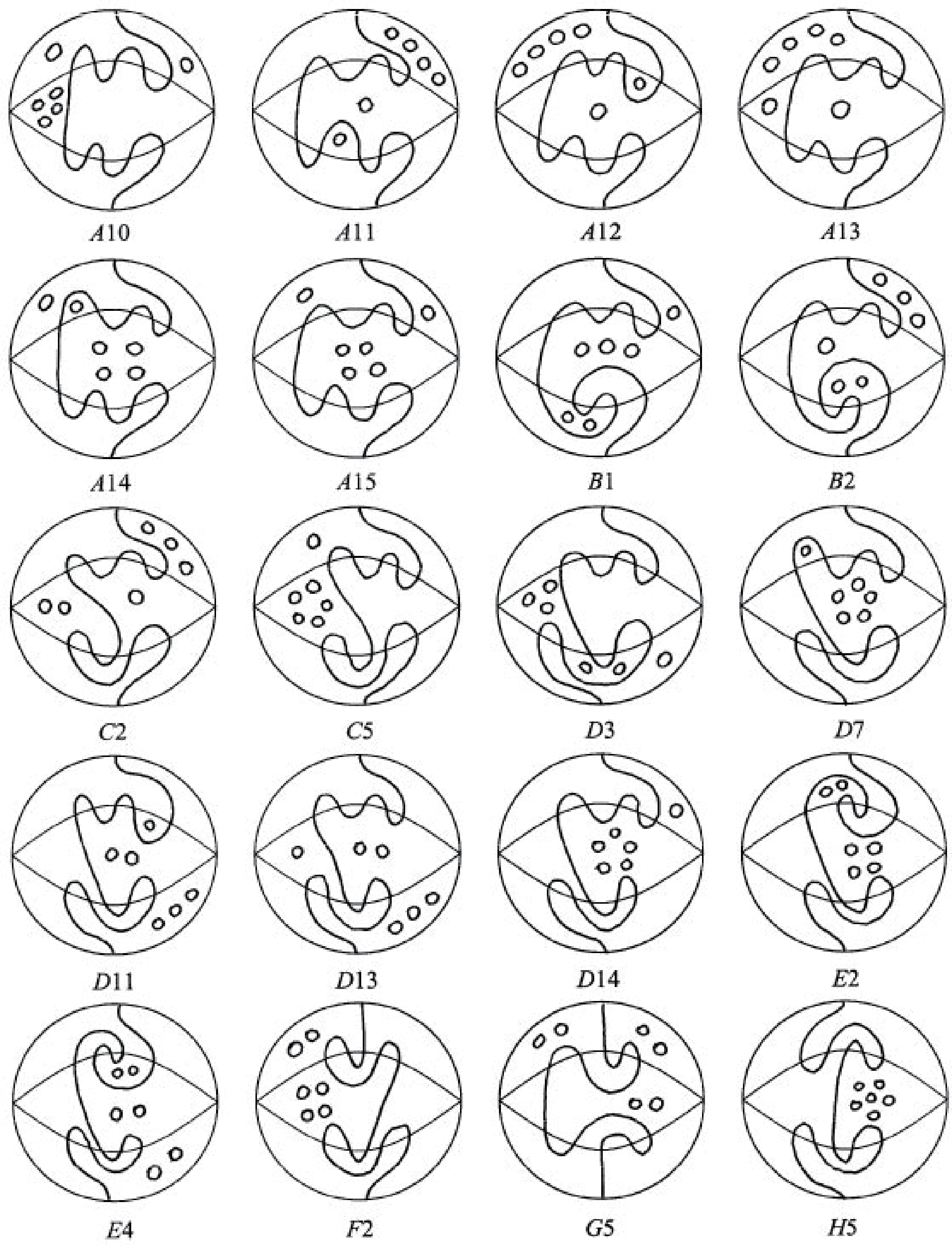

Fig. 4. M-quintic and pair of lines.

In essence, each time this is a special combinatorial problem, the algorithms for solution of the problems were described in [P6] (for $m = 6$ – in [P2], [P4]).

Step 2. *Constructions*, i.e., attempts to realize a given admissible model by a 7th degree curve.

Fig. 5. M-cubic and M-quartic with common points on ovals; $\alpha + \beta = 3$.

For the constructions, different variants of the small parameter methods (including Viro's technique of gluing of charts of polynomials [V1], [V2]) and quadratic transformations were applied.

Step 3. *Prohibitions*, i.e., attempts to prove that a given admissible model cannot be realized by a 7th degree curve.

The main methods of prohibitions are the Orevkov method [O2] based on the link theory, and the Hilbert-Rohn-Gudkov method (see [O-Sh2]) based on the bifurcation theory.

Now we return to the case b) for $k = 3$: M-cubic and a M-quartic with 12 common points on an oval O_4 of the quartic and the odd branch J_3 of the cubic.

Let the system of coordinates in $\mathbb{R}P^2$ be such that the straight line $x_2 = 0$ does not intersect ovals of the quartic (to get this, it is sufficient to assume that $x_2 = 0$ is the result of small shifting a double tangent line to the quartic). To reduce the space of our paper, we consider here only the case when there exists a pseudo-line S such that the odd branch of the cubic intersect this pseudo-line at one point only. Let us consider S as "the line at infinity" (i.e. the boundary of the Poincaré disk). There are 12 arcs on the odd branch J_3 and 12 arcs on the oval O_4, which appear under intersection of the odd branch with the oval O_4. We assume that the endpoints of the arc of the odd branch, which intersect the line S, coincide with two endpoints of the same arc of the oval O_4 (series "A" in [P6]).

The admissible models of $(\mathbb{R}P^2, O_4 \bigcup J_3)$ are enumerated by codes which are lexicographically ordered in the second column of Table 1. To obtain the model, which corresponds to a code, it is sufficient
(i) to draw a circle in the interior of the Poincaré disk, which displays as a model of the oval O_4,
(ii)to mark on this circle 12 points and denote consecutively these points by symbols $1, 2, \ldots, 9, a, b, c$ successively, and
(iii)to draw the model of J_3 in the order given by the code so that the arc $(c, 1)$ of J_3 (with the endpoints c and 1) intersects the line at infinity (in our case the boundary of the Poincarè disk) at one point.

For each of 83 models of Table 1 the set $(\mathbb{R}P^2 \setminus O_4 \bigcap J_3)$ consists of 13 connected domains: the closures of 12 of them are homeomorphic to a disk and the closure of one domain is homeomorphic to a Möbius band. In all cases we denote the last domain by β. The set $\mathbb{R}F_m \setminus (O_4 \bigcup J_3)$ consists of four ovals, which are called "free". The quartic provides three free ovals and the cubic provides one free oval denoted by O_3. The free ovals are located in these domains. Simple arguments (topological corollaries of the Bézout theorem and so on, see for details [P6]) show that some of the domains can not contain free ovals, and free ovals can not surround each other in the domain different from the domain β [3].

[3]Sometimes (for example, in case no.1) such situation for free ovals in β is possible.

Table 1.

no	code	♯ of cases for ovals	realized (no in [O4])	no	code	♯ of cases for ovals	realized (no in [O4])
1	123456789abc	13	1 – 5	43	12543a9678bc	3	–
2	12345678ba9c	12	13	44	12543a9876bc	3	–
3	1234567a98bc	12	–	45	1256789ab43c	12	134
4	123456987abc	12	–	46	125678ba943c	3	–
5	1234569ab87c	12	29,30	47	12569ab8743c	3	–
6	123456ba789c	9	31	48	1256ba98743c	3	–
7	123456ba987c	13	32 – 34	49	1276589ab43c	12	136
8	123458769abc	12	35	50	1278965ab43c	12	138
9	12345876ba9c	12	37	51	12789ab6345c	9	139
10	1234589a76bc	9	–	52	12789ab6543c	13	140
11	12345a9678bc	12	43	53	1278ba96345c	3	–
12	12345a9876bc	13	44,45	54	1278ba96543c	3	–
13	1234765a98bc	3	–	55	1298765ab43c	12	142
14	123478965abc	12	61,62	56	12987ab6345c	8	143
15	1234789ab65c	12	65,66	57	12987ab6543c	12	144
16	123478ba965c	3	–	58	129ab834567c	8	–
17	123498567abc	9	–	59	129ab854367c	8	–
18	123498765abc	13	68 – 70	60	129ab876345c	9	145
19	1234987a65bc	12	74,75	61	129ab876543c	13	146
20	12349ab8567c	9	76	62	12ba3456789c	9	–
21	12349ab8765c	13	77 – 80	63	12ba5436789c	8	–
22	1234ba56789c	8	–	64	12ba7634589c	8	–
23	1234ba76589c	8	–	65	12ba7654389c	8	–
24	1234ba98567c	9	81	66	12ba9834567c	9	–
25	1234ba98765c	13	82 – 85	67	12ba9854367c	8	–
26	123654987abc	12	–	68	12ba9876345c	9	147
27	1236549ab87c	12	87	69	12ba9876543c	13	148
28	123654ba789c	8	88	70	1432789ab65c	12	149
29	123654ba987c	12	89	71	1432987ab65c	12	151
30	12367854ba9c	8	–	72	14329ab8567c	8	152
31	1236789a54bc	8	–	73	14329ab8765c	12	153
32	12367a9854bc	8	–	74	1432ba56789c	8	–
33	12387456ba9c	12	95	75	1432ba76589c	8	–
34	12387654ba9c	12	99	76	1432ba98567c	8	154
35	12389a7456bc	9	–	77	1432ba98765c	8	155
36	12389a7654bc	9	–	78	1456329ab87c	8	156
37	123a945678bc	12	–	79	145632ba789c	8	–
38	123a945876bc	12	–	80	145632ba987c	8	157
39	123a965478bc	3	–	81	1652349ab87c	12	158
40	123a987456bc	13	–	82	165234ba987c	12	159
41	123a987654bc	13	–	83	165432ba987c	8	160
42	1254389a76bc	3	–		Total	784	63

Note that domain β is always admissible for the free ovals. The number of admissible distributions of ovals is shown in the third column of table 1 [4].

Some constructions of arrangements of M-cubic and a M-quartic having 12 points of intersection of the oval O_4 and the odd branch J_3 were described in [P6]. Recently using some new approach, S.Orevkov [O4] obtained a list of 237 distinguish arrangements of such sort (his list includes results of all previous constructions). In the fourth column of Table 1, we indicate the numbers of realized models from the Orevkov list[5].

Now we give a short explanation of the application of the Orevkov prohibition method [O2] taking as an example case no.3 from the Table 1 (many details of the method can be found in [O2],[O3],[K-P],[O-P]). The topological model for this case is shown to the left picture of Figure 6, where each Greek letter denotes the numbers of free ovals in the domain and the same time the name of this domain. The right picture represents the same model in the more realistic view.

Suppose that this model with some distribution of free ovals is realized by some curve C_7 of degree 7. The enumeration of admissible distribution of free ovals is very simple: the oval O_3 can be in one of the domains α, β, δ (the domain γ is free of free ovals by virtue of the complex orientations formulas); for free ovals of quartic we have $\beta + \delta = 3$ for every position of O_3. Thus, the total number of distributions of free ovals is 12 (compare with Table 1).

1. To apply the Orevkov method [O2] we need in a pencil L_P of lines in $\mathbb{R}P^2$ with center at a point $P \in \mathbb{R}P^2 \setminus \mathbb{R}C_7$, which has a maximal general position with respect to the curve $\mathbb{R}C_7$. Here the *maximal general position* means that (i) for every line $l \in L_P$ the set $l \bigcap \mathbb{R}C_7$ consists of at least 5 points and there exists some such line having 7 common points with $\mathbb{R}C_7$, (ii)the multiplicity of intersection of every line $l \in L_P$ and the curve $\mathbb{R}C_7$ at every point is no more than two, and (iii) for every line $l \in L_P$ the number of such points with multiplicity two is no more than one. The points of intersection of l and $\mathbb{R}C_7$ with multiplicity two are called *critical* of the pencil L_P. They can be either points of tangency of l and $\mathbb{R}C_7$ or double points of $\mathbb{R}C_7$. A line l having critical points is called *critical.*

[4]Here corollaries of the Rokhlin and Mishachev formulas of complex orientations are taken into account; applications of these formulas in such situations are described in [P6], [K-P], [O-P].

[5]Pictures in the Orevkov list in [O4] are not numbered. We enumerate them along rows of his figures.

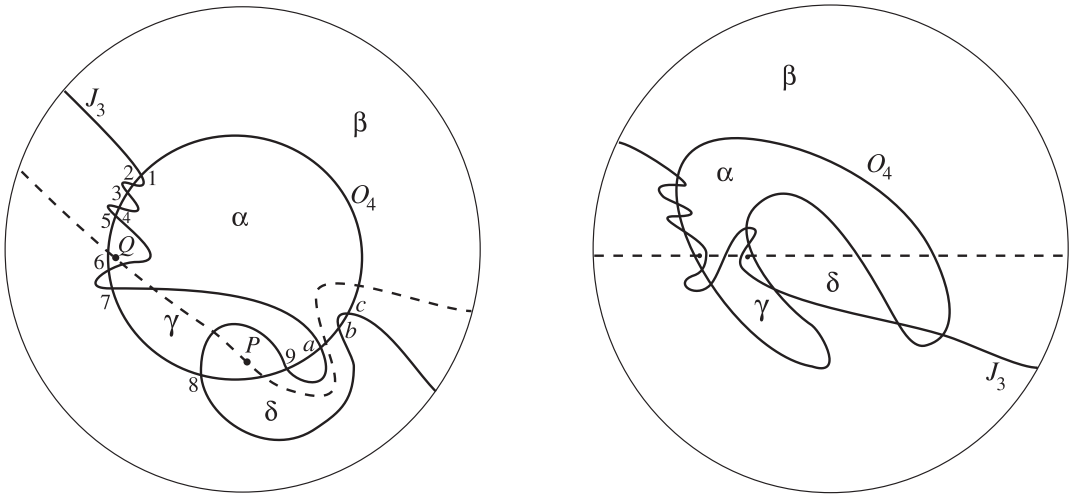

Fig. 6. Model for arrangement with code $(12345678ba9c)$ (no.3 from Table 1).

Let a center P of the pencil be chosen by an appropriate way for a given topological model of $J_3 \bigcup O_4$. After that we need to consider all different admissible possibilities for mutual arrangement of the model of the pencil with respect to the model of $J_3 \bigcup O_4$. The Bézout theorem admits several (usually two or three) such essentially different arrangements[6].

2. Let us choose point P in the interior of the digon with vertices 8, 9 (see Figure 6). Let Q be some interior point in the digon with vertices 5, 6. The dotted line in Figure 6 represents one of admissible positions of the line PQ. It is convenient to redraw the picture such that line PQ becomes the boundary circle of the Poincaré disk, see Figure 7. If we draw the corresponding affine plane, where the center P of the pencil L_P is located on the line at infinity, then the pencil L_P in this affine plane constitutes a set of parallel lines. Free ovals may be only in vertical zones bounded by critical lines and filled by lines of the pencil, each line of which has 5 real points of intersection with $J_3 \bigcup O_4$. We must consider all admissible distributions of free ovals in these zones taking into account their mutual order.

3. Consider complexification of our construction. Let

$$\mathbb{C}C_7 = \{(x_0, x_1, x_2) \in \mathbb{C}P^2 | C_7 = 0\}$$

[6]"Essentially different" means that corresponding braids, which will be constructed below, are nonconjugate in the braid group.

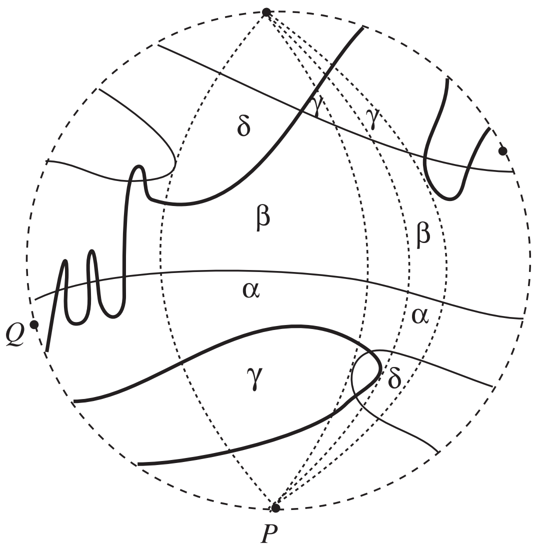

Fig. 7. Model no.3 and pencil L_P.

be the set of complex points of the curve C_7, $\mathbb{C}l$ be the set of complex points of the line l and $\mathbb{C}L_P = \bigcup \mathbb{C}l$ for $l \in L_P$. The intersection $\mathbb{C}C_7 \bigcap \mathbb{C}L_P$ can be described as a union of 7 circles. Every two circles either are disjoint or intersect at critical points of the pencil $\mathbb{C}L_P$; and the intersection of every three circles is empty.

Some standard perturbation (see details in [O2]) of the union of circles turns it into a link K of disjoint circles. Let b be a braid in the group B_7 of braids of 7 strings, whose closure $\bar{b}$ coincides with the link K. It is clear that the braid b is defined up to conjugation in the group B_7. The fact that the pencil L_P is in maximal general position with respect to $\mathbb{R}C_7$ implies that the braid b is uniquely defined (up to conjugation) by visible mutual arrangement of the model of $\mathbb{R}C_7$ and the pencil L_P in $\mathbb{R}P^2$.

The construction implies that the link $K = \bar{b}$ is the boudary link for a part of a surface $\mathbb{C}C_7 \in \mathbb{C}P^2$. It is well known (see, for example, [R]) that it is possible only if the braid b is a so called quasi-positive braid. As a necessary condition of quasipositivity, as in [O2],[K-P],[O-P], we apply the Murasugi-Tristram Inequality, which for our case can be written in the form

$$h = |\sigma(\bar{b})| + n - e(b) - \mathrm{null}(\bar{b}) \leq 0,$$

where $\sigma(\bar{b})$ is the signature, $\mathrm{null}(\bar{b})$ is the nullity of the link $\bar{b}$, and $e(b) = \sum k_i$ for $b = \prod \sigma_i^{k_i}$, where σ_i, $i = 1, 2, \ldots, 6$, are standard generators of the B_7.

4. One can check that for every position of the pencil with respect to the model no. 3 and for every distribution of free ovals, the value of h is always positive. Thus, the model no. 3 from Table 1 is unrealizable by an algebraic curve of degree 7.

For all of other considered cases, including all cases of the Table 1, we have obtain $h = 0$ only for the arrangements which were realized by S.Orevkov. This leads to the following

Conjecture. *Under conditions* (i)-(iii) *in the case* b) *every union of an M-cubic and M-quartic is homeomorphic to some disposition from the Orevkov's list* [O4] *of realized models.*

Remarks. 1. The most difficult step in the application of the Orevkov method is the choice of the point P and enumeration of admissible arrangements of the pencil L_P, i.e., items 1 and 2 above. These steps were made "by hand". All other steps were made on a computer by using of a number of programs written by M.Guschin. Other variant of programs was created by S.Orevkov.

2. The prohibitions for cases of the Table 1, which satisfy the assumption that the oval O_3 lies outside of ovals of the quartic and outside of β (for the example, in δ of the model No. 3 above), were independently considered by S.Orevkov (see Proposition 6.2 in [O4]). In these cases, one can choose the center P of the pencil L_P inside the oval O_3; and the disposition of L_P with respect to the model is easily determined.

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$\mathcal{A}$-topological triviality of map germs and Newton filtrations

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Abstract.

We apply the method of constructing controlled vector fields to give sufficient conditions for the $\mathcal{A}$-topological triviality of deformations of map germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of type $f_t(x) = f(x)+th(x)$, with $n \geq p$ or $n \leq 2p$. These conditions are given in terms of an appropriate choice of Newton filtrations for $\mathcal{O}_n$ and $\mathcal{O}_p$ and are for the $\mathcal{A}$-tangent space of the germ f.

For the case $n \geq p$, we follow the technique used by M. A. S. Ruas in her Ph.D. Thesis [7] and construct *control functions* in the target and in the source to obtain, via a partition of the unit, a unique control function. We use the control function of the target to give an estimate for the case $p \geq 2n$. Moreover, in this case we show that if the coordinates of the map germ satisfy a Newton non-degeneracy condition, deformations by terms of higher filtration are topologically trivial.

As an application we obtain for both cases, $n \geq p$ and $p \geq 2n$, the results of Damon in [3] for deformations of weighted homogeneous map germs.

§1. Introduction

The determinacy of topological triviality for families of map-germs is a fundamental subject in singularity theory. As we see in the articles of Damon, [4] and [3] for example, the method of constructing controlled vector fields is a very powerful tool to compute the topological triviality.

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M. A. S. Ruas in her PhD. Thesis gives an explicit order such that the $\mathcal{A}$-topological structure of a polynomial map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $n \geq p$, is preserved after higher order perturbations.

In this paper we apply this method to give sufficient conditions for the $\mathcal{A}$-topological triviality of deformations of map germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of type $f_t(x) = f(x) + th(x)$, with $n \geq p$ or $n \leq 2p$. These conditions are given in terms of an appropriate choice of Newton filtrations for $\mathcal{O}_n$ and $\mathcal{O}_p$ and are for the $\mathcal{A}$-tangent space of the germ f.

First we generalize the results of M. A. S. Ruas [7], by considering different Newton filtrations A_k for $\mathcal{O}_n$ and B_k for $\mathcal{O}_p$, these results are given for the case $n \geq p$. We construct *control functions* in the target and in the source to obtain, via a partition of the unit, a unique control function. We remark that in [7] these control functions are homogeneous, since they are associated to the usual filtration, given by the degree of monomials.

In the case $p \geq 2n$ we give an estimate in terms of the control function of the target. Moreover, if $p \geq 2n$, we apply the results of Gaffney in [6] to show that deformations by higher Newton filtration are $\mathcal{A}$-topologically trivial if the map germs satisfy a Newton non-degeneracy condition.

In both cases we also show that the results of Damon for the topological triviality of unfoldings of weighted homogenous map germs can be obtained from our results.

§2. Newton filtration and control functions

To construct controlled vector fields that guarantee the topological triviality we define a convenient *control function* in terms of a fixed Newton polyhedron. An analytic function $\rho : \mathbb{C}^n \to \mathbb{R}$ is a control if there exist constants C and α such that $\rho(x) \geq C|x|^\alpha$.

First we construct a control function in the target, denoted by ρ_{m} and a function in the source, denoted by ρ_{f}. When $n \geq p$, the control function ρ is defined from these, via a partition of the unity. For $p \geq 2n$, the control function is ρ_{m}.

Fix coordinate systems $\mathbf{x}$ in $(\mathbb{C}^n, 0)$, $\mathbf{y}$ in $(\mathbb{C}^p, 0)$ and denote by $\mathcal{O}_n$, $\mathcal{O}_p$, the sets of holomorphic germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}, 0)$ and from $(\mathbb{C}^p, 0)$ to $(\mathbb{C}, 0)$. We identify these sets with the rings of convergent power series $\mathbb{C}[[x]]$ and $\mathbb{C}[[y]]$ respectively.

To fix the notation we follow [1] and say that a subset $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a *Newton polyhedron* if there exist some $k_1, \ldots, k_r \in \mathbb{Q}^n_+$ such that Γ_+ is the convex hull in $\mathbb{R}^n_+$ of the set $\{k_i + v : v \in \mathbb{R}^n_+, i = 1, \ldots, r\}$

and Γ_+ intersects all the coordinate axis. Denote by Γ the union of the compact faces of Γ_+ and consider the *Newton filtration* of $\mathcal{O}_n = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots$, by the ideals $\mathcal{A}_q = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq \phi_\Gamma^{-1}(q + \mathbb{N})\}$, for all $q \in \mathbb{N}$, here ϕ_Γ is the Newton function of Γ.

We fix a Newton polyhedron Γ_+ in $\mathbb{R}^n_+$ with its associate Newton filtration, then for any germ of function $g = \sum_k a_k x^k$, denote $\mathrm{d}(g) = \max\{q : g \in \mathcal{A}_q\}$ and by $\mathrm{in}(g)$, the polynomial $\mathrm{in}(g) = \sum a_k x^k$ such that $\phi_\Gamma(k) = \mathrm{d}(g)$.

To define a Newton filtration in $\mathcal{O}_p$ we consider a fixed map germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $g = (g_1, \dots, g_p)$, call $D_i = \mathrm{d}(g_i)$ and say that $D_1 \leq D_2 \leq \dots \leq D_p$. In this case we call $\mathrm{d}(g) = (D_1, D_2, \dots, D_p)$.

Denote by M_I the determinant of the $p \times p$ minor of the matrix of the partial derivatives of g indexed by $I = \{i_1, \dots, i_p\} \subset \{1, \dots n\}$, with $i_1 < \dots < i_p$. We fix an order for these determinants calling $M_1, M_2, \dots, M_r$ in such a way that $\mathrm{d}(M_j) \leq \mathrm{d}(M_{j+1})$ and call $L_j = \mathrm{d}(M_j)$.

Now, call $D = \text{m.c.m.}\{D_1, D_2, \dots, D_p, L_1, \dots, L_q\}$ and define the weighted homogeneous control function in the target, $\rho_{\mathrm{m}} : \mathbb{C}^p \to \mathbb{R}$ by

$$\rho_{\mathrm{m}}(y) = |y_1|^{2r_1} + |y_2|^{2r_2} + \dots + |y_p|^{2r_p}, \text{ where } r_i = \frac{D}{D_i} \text{ for all } i = 1, \dots, p.$$

The Newton filtration of $\mathcal{O}_p = \mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \dots$ is associate to the control function $\rho_{\mathrm{m}}(y)$. Therefore any ideal $\mathcal{B}_k$ has a Newton polyhedron which only one compact face with normal vector $w = (w_1, ..., w_p)$, where $w_i = \frac{R}{r_i}$ and $R = \text{m.c.m.}\{r_1, \dots, r_p\}$, for all $i = 1, ..., p$.

For any monomial $y^\beta = y_1^{\beta_1} y_2^{\beta_2} \dots y_n^{\beta_n} \in \mathcal{O}_p$, denote $\mathrm{d}_w(y^\beta) = w_1\beta_1 + \dots + w_p\beta_p$, and for any $g \in \mathcal{O}_p, \mathrm{d}_w(g) = \min . \mathrm{d}_w(y^\beta)$ for all y^β with nonzero coefficient in the Taylor series of g, then $B_k = \{g \in \mathcal{O}_p; \mathrm{d}_w(g) \geq k\}$.

Here we have $\mathrm{d}_w(\rho_m) = 2R$ and as $\rho_{\mathrm{m}}(g(x)) = |g_1|^{2r_1} + |g_2|^{2r_2} + \dots + |g_p|^{2r_p}$, $\mathrm{d}(\rho_m \circ g) = \mathrm{d}\left(|g_1|^{2r_1} + |g_2|^{2r_2} + ... + |g_p|^{2r_p}\right) = 2D$.

Now define the control function in the source

$$\rho_v(x_1, \dots, x_n) = \sum_{j=1}^{r} x_1^{2v_1^j} \dots x_n^{2v_n^j},$$

with $v^j = (v_1^j, \dots, v_n^j)$, $j = 1, \dots, r$ being the vertices of the Newton polyhedron $\Gamma_+(A_D)$, therefore $\mathrm{d}(\rho_v) = 2D$.

We also define the function $\rho_{\mathrm{f}}(g) : \mathbb{C}^n \to \mathbb{R}$, $\rho_{\mathrm{f}}(g)(x) = \sum |M_j|^{2\alpha_j}$, where $\alpha_j = \frac{D}{L_j}$ for all $j = 1, \ldots, q$. We remember that $\rho_{\mathrm{f}}(g)$ is not a control function, however, under some conditions, it is important in the construction of the controlled vector fields. We remark that all these constructions are done to obtain $\mathrm{d}(\rho_{\mathrm{f}}(g)) = \mathrm{d}(\rho_m \circ g) = \mathrm{d}(\rho_v) = 2D$.

Example: Let $g(x, y) = (xy, x^4 + y^5 + xy^2)$ and fix the Newton polyhedron $\Gamma_+(g_2)$. Call Δ_1 the face with vertices $\{(0, 5), (1, 2)\}$ and Δ_2 the face with vertices $\{(4, 0), (1, 2)\}$, $C(\Delta_i)$ denotes the cone with vertex at 0 passing through Δ_i.

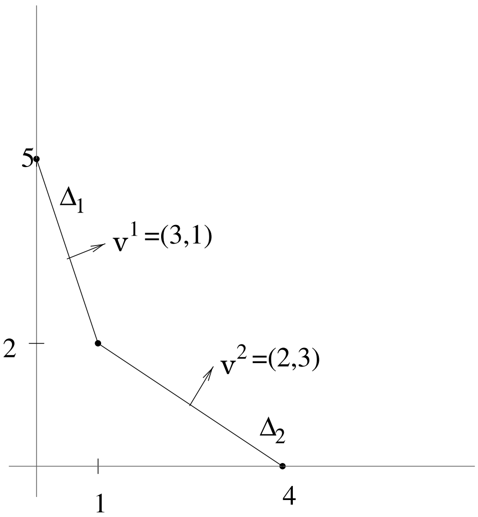

Fig. 1. The Newton polyhedron $\Gamma_+(g_2)$.

The Newton filtration $\varphi_{\Gamma_+(g_2)}$ is

$$\varphi(x^a y^b) = \begin{cases} 24a + 8b, & \text{if } (a, b) \in C(\Delta_1) \\ \\ 10a + 15b, & \text{if } (a, b) \in C(\Delta_2). \end{cases}$$

Then $\mathrm{d}(g_1) = 25$ and $\mathrm{d}(g_2) = 40$, therefore $D = 200 = \mathrm{m.c.m}\{25, 40\}$ and the control function in the target is $\rho_{\mathrm{m}}(y) = |y_1|^{16} + |y_2|^{10}$, and $\mathrm{d}_w(\rho_{\mathrm{m}}(y)) = 80 = R$.

Now let $M(x, y) = 5y^5 - 4x^4 + xy^2$, be the determinant of the Jacobian matrix of g, then $\mathrm{d}(M) = 40$ and $\rho_{\mathrm{f}}(g) = (5y^5 - 4x^4 + xy^2)^{10}$, with $\mathrm{d}(\rho_{\mathrm{f}}(g)) = 2D = 400$.

Since $\mathrm{d}(g_1^{16}) = \mathrm{d}(g_2^{10}) = \mathrm{d}(\rho_{\mathrm{f}}(g)) = 2D = 400$, the control function in the source ρ_v associate to $\Gamma_+(g_2^{10})$ is $\rho_v(x, y) = x^{40} + y^{32} + x^{20}y^{40}$ and $\mathrm{d}(\rho_v) = 2D = 400$.

§3. $\mathcal{A}$-topological triviality

Denote by

$$F : (\mathbb{C}^n \times \mathbb{C}, (0,0)) \to (\mathbb{C}^p \times \mathbb{C}, (0,0))\,, F(x,\lambda) = (f(x,\lambda), \lambda)$$

a one parameter unfolding of a finitely determined map germ

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$$

and call the family of map germs $f_\lambda(x) = f(x,\lambda)$ a deformation of the germ f.

An unfolding $F(x,\lambda)$ of f is $\mathcal{A}$-topologically trivial if, for small values of λ, there are germs of homeomorphisms $H : (\mathbb{C}^n \times \mathbb{C}, (0,0)) \to (\mathbb{C}^n \times \mathbb{C}, (0,0))$, of type $H(x,\lambda) = (h(x,\lambda),\lambda)$, with $h(0,\lambda) = 0$ and $K : (\mathbb{C}^p \times \mathbb{C}, (0,0)) \to (\mathbb{C}^p \times \mathbb{C}, (0,0))$ of type $K(x,\lambda) = (k(x,\lambda),\lambda)$ with $k(0,\lambda) = 0$ such that $K \circ F \circ H^{-1} = (f_0(x),\lambda)$.

In this case we say that the deformation $f_\lambda(x)$ is $\mathcal{A}$-topologically trivial, since for small values of λ, the families of homeomorphisms $h_\lambda : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, with $h_\lambda(x) = h(x,\lambda)$ and $k_\lambda : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ with $k_\lambda(x) = k(x,\lambda)$ give

$$k_\lambda \circ f_\lambda \circ h_\lambda^{-1} = f_0.$$

Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finitely determined map germ satisfying

$$A_{2D+D_1}\theta_g \subseteq tg(A_{2D}\theta_n) + wg(B_{2R+1}\theta_p). \tag{1}$$

From the above constructions we have the following:

Proposition 3.1.

(1) *If $n \geq p$, suppose that in a neighborhood V of 0 in $\mathbb{C}^n$, there exist constants α e β such that $\rho_{\mathrm{f}}(g(x)) \geq \beta\rho_v(x)$, for all $x \in V \cap \{x; \rho_{\mathrm{m}}(g(x)) < \alpha\rho_v(x)\}$.*

(2) *If $p \geq 2n$, suppose that $\rho_m(g(x)) \geq c\rho_v(x)$, $\forall x$ in a neighborhood V of 0.*

Then deformations $g_\lambda = g + \lambda h$ of g, with $\mathrm{d}(h_i) \geq D_i$, $\forall\, i = 1,\dots,p$, are $\mathcal{A}$-topologically trivial for small values of λ.

In the next Lemma, essential in the proof of this Proposition, we show that it is possible to extend the filtration condition of the equation (1) to the tangent space of an unfolding of the germ g.

We call m_1 the maximal ideal in $\mathcal{O}_1$ and $\tilde{A}_{2D+D_1}$ the ideal in $\mathcal{O}_{n+1}$ generated by the monomial λ and the ideal A_{2D+D_1}.

Lemma 3.2. *Let $G(x,\lambda) = (g_\lambda(x), \lambda)$, with $g_\lambda(x) = (g_{1\lambda}(x), \ldots g_{n\lambda}(x))$ be an unfolding of $g_0(x) = g(x)$, such that $g_{i\lambda} - g_{i0} \in m_1.A_{D_i}\theta_G$ for all $i = 1, \ldots, n$ and $|\lambda| < \epsilon$ for small values of ϵ. If the equation (1) holds, then*

$$tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) \supseteq A_{2D+D_1}\theta_G.$$

Proof: Since

$$\begin{array}{rcl} A_{2D+D_1}\theta_G & = & A_{2D+D_1}\theta_g + \lambda A_{2D+D_1}\theta_G \\ & \subseteq & tg(A_{2D}\theta_n) + wg(B_{2R+1}.\theta_p) + \lambda A_{2D+D_1}\theta_G, \end{array}$$

and

$$\begin{array}{l} tg(A_{2D}\theta_n) + wg(B_{2R+1}.\theta_p) \\ \qquad \subseteq tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + \lambda A_{2D+D_1}\theta_G \end{array}$$

it follows that

$$(2) \quad A_{2D+D_1}\theta_G \subseteq tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + \lambda A_{2D+D_1}\theta_G.$$

Let E be the finitely generated $\mathcal{O}_{n+1}$-modulo defined as

$$E = \frac{tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1})}.$$

We remark that E is a $G^*(\mathcal{O}_{p+1})$-modulo and $(\lambda).E = E$ since

$$\begin{array}{rl} (\lambda).E & = \frac{tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1})}{tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1})} \; + \\ & \quad + \; \frac{(\lambda)[tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + A_{2D+D_1}\theta_G]}{tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1})} \\ & = \frac{tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1})} = E. \end{array}$$

Therefore if we show that E is finitely generated as $G^*(\mathcal{O}_{p+1})$-modulo we apply the Nakayama's Lemma to obtain $E = 0$, or

$$A_{2D+D_1}\theta_G \subseteq tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}).$$

We show now that E is finitely generated as $G^*(\mathcal{O}_{p+1})$- module.

Let E' be the finitely generated $\mathcal{O}_{n+1}$-module

$$E' = \frac{tG(A_{2D}\theta_{n+1}) + A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1})}.$$

Then we need to show that E' is finitely generated as a $G^*(\mathcal{O}_{p+1})$-module.

From the Malgrange's Preparation Theorem, E' is a finitely generated $G^*(\mathcal{O}_{p+1})$- module if, and only if $\dim_{\mathbb{C}} \frac{E'}{G^*(m_{p+1})E'} < +\infty$.
Write

$$\begin{aligned} \frac{E'}{G^*(m_{p+1})E'} &= \frac{\frac{tG(A_{2D}\theta_{n+1}) + A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1})}}{\frac{G^*(m_{p+1})[tG(A_{2D}\theta_{n+1}) + A_{2D+D_1}\theta_G] + tG(A_{2D}\theta_{n+1})}{tG(A_{2D}\theta_{n+1})}} \\ &= \frac{tG(A_{2D}\theta_{n+1}) + A_{2D+D_1}\theta_G}{tG(A_{2D}\theta_{n+1}) + G^*(m_{p+1})[tG(A_{2D}\theta_{n+1}) + A_{2D+D_1}\theta_G]}, \end{aligned}$$

denote $S = A_{2D+D_1}\theta_G$ and $T = tG(A_{2D}\theta_{n+1}) + G^*(m_{p+1})A_{2D+D_1}\theta_G$.

Therefore by the isomorphism theorem we obtain $\dfrac{T+S}{T} \cong \dfrac{S}{T \cap S}$.

From

$$\begin{aligned} &tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}.\theta_{p+1}) + \lambda A_{2D+D_1}\theta_G \\ &\qquad \subseteq tG(A_{2D}\theta_{n+1}) + G^*(m_{p+1})\theta_G \end{aligned}$$

and by the equation (2) we conclude that

$$tG(A_{2D}\theta_{n+1}) + G^*(m_{p+1})\theta_G \supseteq A_{2D+D_1}\theta_G.$$

Multiplying by A_{2D+D_1} we obtain

$$tG(A_{4D+D_1}\theta_{n+1}) + G^*(m_{p+1})A_{2D+D_1}\theta_G \supseteq A_{4D+2D_1}\theta_G.$$

On the other hand,

$$tG(A_{4D+D_1}\theta_{n+1}) + G^*(m_{p+1})A_{2D+D_1}\theta_G \supseteq \lambda A_{2D+D_1}\theta_G.$$

Hence, $\dim_{\mathbb{C}} \frac{S}{T\cap S} \leq \dim_{\mathbb{C}} \frac{A_{2D+D_1}\theta_G}{\tilde{A}_{2D+D_1}A_{2D+D_1}\theta_G} < +\infty$. $\|$

Proof of the Proposition 3.1: Let $G(x,\lambda)=(g_\lambda(x),\lambda)$ be an unfolding of g, with $g_\lambda(x)=g(x)+\lambda h(x)$ and $h=(h_1,\dots,h_p)$ with each $h_i\in A_{D_i}$.

From the general hypotheses, since $A_{D_i}\subset A_{D_1}$ for all $i=1,\dots,n$ we obtain

$$h.\rho_{\mathrm{m}}(g)\in tg(A_{2D}\theta_n)+wg(B_{2R+1}\theta_p).$$

From the Lemma 3.2. we conclude that there exist analytic vector fields $\xi\in A_{2D}\theta_{n+1}$ and $\eta\in B_{2R+1}\theta_{p+1}$ such that the above inclusion holds for deformations, i.e.

$$h.\rho_{\mathrm{m}}(g_\lambda)=tG(\xi)+\eta\circ G. \tag{3}$$

From the equation (3) we construct the vector field controlled by ρ_{m}. Define ω in $(\mathbb{C}^p\times\mathbb{C},0\times 0)$ as:

$$\omega(y,\lambda)=\begin{cases}\dfrac{\eta(y,\lambda)}{\rho_{\mathrm{m}}(y)}, & \text{if } y\neq 0\\[2ex] 0, & \text{if } y=0.\end{cases}$$

Since $\mathrm{d}_w(\eta)\geq 2R+1>\mathrm{d}_w(\rho_{\mathrm{m}})=2R$ we apply Lemmas (1) e (2) of [9] to conclude that the vector field ω is integrable.

Proof of the case $n\geq p$. In order to define the vector field controlled by the function ρ_{f}, for each $I=\{i_1,i_2,\dots,i_p\}\subset\{1,2,\dots,n\}$ write $\dfrac{\partial g_\lambda}{\partial\lambda}M_{I_\lambda}=tG(\gamma_I)$, with $\gamma_I=\sum\gamma_i\frac{\partial}{\partial x_i}$ and each γ_i is defined as

$$\begin{cases}\gamma_i=0, & \text{if } i\notin I\\[2ex] \gamma_{i_m}=\sum N_{ji_m}\left(\frac{\partial g_\lambda}{\partial\lambda}\right)_j, & \text{if } i_m\in I.\end{cases} \tag{4}$$

where N_{ji_m} is the $(p-1)\times(p-1)$ cofactor of $\frac{\partial g_j}{\partial x_{i_m}}$.

Since $\rho_{\mathrm{f}}(g_\lambda)=\sum|M_{I_\lambda}|^{2\alpha_I}$, we obtain

$$h.\rho_{\mathrm{f}}(g_\lambda)=tG\left(\sum\gamma_I M_{I_\lambda}^{\alpha_I-1}\overline{M_{I_\lambda}}^{\alpha_I}\right),$$

therefore $h = tG(\psi)$, with $\psi = \dfrac{\sum \gamma_I M_{I_\lambda}^{\alpha_I - 1} \overline{M_{I_\lambda}}^{\alpha_I}}{\rho_{\mathrm{f}}(g_\lambda)}$.

Denote $\gamma_{\mathcal{R}} = \sum \gamma_I M_{I_\lambda}^{\alpha_I - 1} \overline{M_{I_\lambda}}^{\alpha_I}$, then $\mathrm{d}(\gamma_{\mathcal{R}}) = \mathrm{d}(\rho_{\mathrm{f}}(g_\lambda)) + r$, with $r = \min\limits_{i,k} \left\{ \frac{M}{\ell(v^k)} . v_i^k \right\}$.

The integrability of the vector field $\psi = \dfrac{\gamma_R}{\rho_{\mathrm{f}}(g_\lambda)}$, follows from the hypotheses of the following:

Lemma 3.3. There exist positive constants α_1, β and a neighborhood V of the origin in $\mathbb{C}^n$ such that

$$\rho_{\mathrm{f}}(g_\lambda(x)) \geq \alpha_1 \rho_v(x), \ \forall\, x \in \ V \cap \{\rho_{\mathrm{m}}(g_\lambda(x)) < \beta \rho_v(x)\}.$$

Proof: Since $g_\lambda = g + \lambda h$, and $\mathrm{d}(h_i) \geq \mathrm{d}(g_i)$ for all $i = 1, \ldots, p$ we obtain

$$\rho_{\mathrm{f}}(g_\lambda) \geq \rho_{\mathrm{f}}(g) - \lambda\theta(x, \lambda), \text{with} \quad \mathrm{d}(\theta) \geq \mathrm{d}(\rho_{\mathrm{f}}(g)).$$

By hypotheses $\rho_{\mathrm{f}}(g) \geq \alpha \rho_v(x)$ for $x \in V \cap \{x; \rho_{\mathrm{m}}(g(x)) < \beta \rho_v(x)\}$ hence there exists a constant $c > 0$ such that $\lambda\theta(x, \lambda) \leq c\rho_v(x)$. Since $\rho_{\mathrm{m}}(g_\lambda(x)) < \rho_{\mathrm{m}}(g(x))$, for each $x \in V \cap \{x; \rho_{\mathrm{m}}(g_\lambda(x)) < \rho_{\mathrm{m}}(g(x)) < \alpha \rho_v(x)\}$, we obtain

$$\begin{aligned} \rho_{\mathrm{f}}(g_\lambda(x)) &\geq \rho_{\mathrm{f}}(g(x)) - \lambda\theta(x, \lambda) \\ &\geq (\alpha - c)\rho_v(x) \\ &= \alpha_1 \rho_v(x). \end{aligned}$$

$\|$

To finish the proof of the Proposition 3.1., consider the following partition of the unity.

Let $H = (V \times I) - (0 \times I)$, with $I = (-\epsilon, \epsilon)$ and the following sets

$F_1 = (\{(x, \lambda); g_\lambda(x) = 0\} - (0 \times \mathbb{C})) \cap H$, $F_2 = \{(x, \lambda); \rho_{\mathrm{m}}(g_\lambda(x)) \geq \alpha \rho_v(x)\} \cap H$,

$E_1 = \{(x, \lambda); \rho_{\mathrm{m}}(g_\lambda(x)) < \alpha_1 \rho_v(x)\} \cap H$ and $E_2 = \{(x, \lambda); \rho_{\mathrm{m}}(g_\lambda(x)) < \alpha_2 \rho_v(x)\} \cap H$,

with $\alpha_1 < \alpha < \alpha_2$.

We remark that F_1 and F_2 are closed and disjoint from H.

Define $\zeta(x) = \zeta_1(x) + \zeta_2(x)$, a partition of the unity related to $\{E_2, (\overline{E_1})^c\}$.

$$\zeta_1(x,\lambda) = \begin{cases} 1, & \text{if } (x,\lambda) \in F_1 \\ 0, & \text{if } (x,\lambda) \in (E_2)^c \end{cases}$$

and

$$\zeta_2(x,\lambda) = \begin{cases} 1, & \text{if } (x,\lambda) \in F_2 \\ 0, & \text{if } (x,\lambda) \in \overline{E_1}. \end{cases}$$

Call $\nu_2(x,\lambda) = \begin{cases} \frac{\xi(x,\lambda)}{\rho_{\mathrm{m}}(g_\lambda(x))}, & \text{if } (x,\lambda) \in {F_1}^c \\ 0, & \text{if } (x,\lambda) \in F_1, \end{cases}$

where $\xi(x,\lambda)$ is given in equation (3), and define

$$\nu_1(x,\lambda) = \begin{cases} \frac{\gamma_{\mathcal{R}}}{\rho_{\mathrm{f}}(g_\lambda(x))}, & \text{if } (x,\lambda) \in {F_2}^c \\ 0, & \text{if } (x,\lambda) \in F_2. \end{cases}$$

Since all functions defined above can be extended in such a way that they are zero at $0 \times \lambda$, let ν be the vector field in $(\mathbb{C}^n \times \mathbb{C}, 0 \times 0)$ defined as

$$\nu(x,\lambda) = \begin{cases} \zeta_1(x,\lambda)\nu_1(x,\lambda) + \zeta_2(x,\lambda)\nu_2(x,\lambda), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Then the vector field ν is continuous, integrable and $h = tG(\nu(x,\lambda)) + w(G(x,\lambda))$.

From the integral curve solutions of ν e ω we construct the germs of homeomorphisms

$H : (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^n \times \mathbb{C}, 0 \times 0)$, $H(x,\lambda) = (h(x,\lambda),\lambda)$, $h(x,0) = x$,

and

$K : (\mathbb{C}^p \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^p \times \mathbb{C}, 0 \times 0)$, $K(y,\lambda) = (k(y,\lambda),\lambda)$, $k(y,0) = y$

to obtain $K \circ G \circ H^{-1} = (g, id_{\mathbb{C}})$. ||

Proof of the case $p \geq 2n$: From the equation (3) we have

$$h = tG\left(\frac{\xi}{\rho_{\mathrm{m}}(g_\lambda)}\right) + \frac{\eta \circ G}{\rho_{\mathrm{m}}(g_\lambda)}.$$

By the general hypotheses $\xi \in A_{2D}$ and $\eta \in A_{2R+1}$, from (2) we see that
$\rho\mathrm{m}(g_\lambda)(x) \geq c.\rho_v(x)$, therefore the vector field $\dfrac{\xi}{\rho\mathrm{m}(g_\lambda)}$ is integrable, and also the vector field $\dfrac{\eta}{\rho\mathrm{m}}$, then the homeomorphisms H and K are obtained as above. ||

§4. The non-degenerate case when $p \geq 2n$

In this section we are interested in the topological triviality of families $g_\lambda : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of type $g_\lambda = g + \lambda h$, with $p \geq 2n$ and $g = (g_1, g_2, \ldots, g_p)$ being an $\mathcal{A}$-finitely determined map germ.

We show the $\mathcal{A}$-topological triviality of the family g_λ in terms of the filtration of the map germ h, if the ideal I generated by the system $\{g_1, g_2, \ldots, g_p\}$ satisfies some non-degeneracy conditions with respect to its Newton polyhedron.

We recover the basic definitions needed for these non-degeneracy conditions.

Let $g = \sum_k a_k x^k$ in $\mathcal{O}_n$, denote $\operatorname{supp} g$ the set of points $k \in \mathbb{Z}^n$ with $a_k \neq 0$. If I is an ideal in $\mathcal{O}_n$, define $I = \cup_{g \in I} \operatorname{supp} g$.

Fix an ideal I, consider its Newton polyhedron $\Gamma_+(I)$, the convex hull in $\mathbb{R}^n_+$ of $\{k + v : v \in \mathbb{R}^n_+,\ k \in \operatorname{supp}(I)\}$ and its induced Newton filtration.

For each compact face Δ of $\Gamma(I)$, call $C(\Delta)$ the cone with vertex at the origin and passing through Δ and $\mathcal{A}_\Delta$ denotes the sub-ring with unity of $\mathcal{O}_n$, $\mathcal{A}_\Delta = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq C(\Delta)\}$. The Newton filtration of $\mathcal{O}_n$ induces a filtration on $\mathcal{A}_\Delta$ in a natural way.

For any germ $g \in \mathcal{O}_n$, denote $g_\Delta = \sum a_k x^k$ with $k \in \operatorname{supp} g \cap \Delta$, and $\mathrm{in}_\Delta(g)$, the polynomial

$$\mathrm{in}_\Delta(g) = \sum \big\{ a_k x^k : k \in \operatorname{supp} g \cap C(\Delta) \text{ and } \mathrm{d}(x^k) = \mathrm{d}(g) \big\}.$$

Definition 4.1. *The ideal I is Newton non-degenerate if there exists a system of generators $\{f_1, \ldots, f_s\}$ of I such that for each compact face $\Delta \subseteq \Gamma$, the ideal generated by the system $\{f_{1\Delta_1}, \ldots, f_{s\Delta_1}\}$ has finite colength in $\mathcal{A}_{\Delta_1}$, for all subfaces Δ_1 of Δ.*

Definition 4.2. *A system of generators $\{f_1, \ldots, f_s\}$ of an ideal I is non-degenerate on $\Gamma_+(I)$ if, for each compact face $\Delta \subseteq \Gamma$, the ideal of $\mathcal{A}_\Delta$ generated by $\mathrm{in}_\Delta(f_1), \ldots, \mathrm{in}_\Delta(f_s)$ has finite colength in $\mathcal{A}_\Delta$.*

Now we consider the ideal $I = \langle g_1, g_2, \ldots, g_p\rangle$, for each generator g_i of I, denote $\mathrm{d}(g_i) = D_i$ and consider $D_1 \leq D_2 \leq \ldots \leq D_p$.

In the case that the ideal I is non-degenerate on some Newton polyhedron Γ_+ we have the following:

Proposition 4.3. *Suppose that I is non-degenerate on some Newton polyhedron Γ_+. Then, deformations of g of type $g_\lambda = g + \lambda h$, with $\mathrm{d}(h_i) \geq D_p$, for all $i = 1, \ldots, p$ are $\mathcal{A}$-topologically trivial.*

When the ideal I is Newton non-degenerate we obtain the following:

Corollary 4.4. *Suppose that I is Newton non-degenerate. Then, deformations of g of type $g_\lambda = g + \lambda h$, with $\mathrm{d}(h_i) \geq D_i$, for all $i = 1, \ldots, p$ are $\mathcal{A}$-topologically trivial.*

Since $p \geq 2n$ any map germ $g = (g_1, \ldots, g_p)$ is $\mathcal{A}$-finitely determined if, and only if, g is $\mathcal{L}$- finitely determined, where $\mathcal{L}$ denotes the $\mathcal{L}$-group of Mather.

Let $G(x, \lambda) = (g_\lambda, \lambda)$ be the one parameter unfolding of g. Since g is $\mathcal{L}$-finitely determined we can choose an integer number s and a vector field $\eta \in m_p\theta_{p+1}$, such that

$$\frac{\partial g_\lambda}{\partial \lambda}\left(g_1^{2D/D_1} + g_2^{2D/D_2} + \ldots + g_p^{2D/D_p}\right)^s = \eta \circ G.$$

Consider the control function in the target $\rho : \mathbb{C}^p \to \mathbb{R}$, defined by

$$\rho(y) = \left(|y_1|^{2D/D_1} + |y_2|^{2D/D_2} + \ldots + |y_p|^{2D/D_p}\right)^{1/2}$$

To prove that the vector field $\dfrac{\eta \circ g}{(\rho(g))^{2s}}$ is integrable, it is sufficient to show that there exists a constant $C > 0$ such that $\left|\frac{\partial g_\lambda}{\partial \lambda}(\lambda, y_1, \ldots, y_p)\right| \leq C\rho(y)$, (see Gaffney in [6] p.482 and Fukui-Paunescu in [5], p.87).

We can compose the terms of this inequality with G to get an equivalent inequality on $\mathbb{C}^n$, $\left|\frac{\partial g_\lambda}{\partial \lambda}(\lambda, x_1, \ldots, x_n)\right| \leq C\rho(g_\lambda)$.

Proof of the Proposition 4.3:

From the proof of the Theorem 3.6 of [1] we see that the ideal I is non-degenerate on Γ_+ if, and only if, the ideal

$$J = \left\langle g_1^{D/D_1}, g_2^{D/D_2}, \dots g_p^{D/D_p} \right\rangle$$

is Newton non-degenerate. In this case, if a germ h satisfies $\mathrm{d}(h) \geq D_p$, then $\Gamma_+(h^{D/D_p}) \in \Gamma_+(J)$, since J is Newton non-degenerate we obtain $h^{D/D_p} \in \overline{J}$. Now we can use the valuative criterion for the integral closure (see [12] p. 288), to obtain that

$$\left| \frac{\partial g_\lambda}{\partial \lambda}(\lambda, x_1, \dots, x_n) \right| \leq C\rho(g_\lambda)$$

and the result follows.

Proof of the Corollary 4.4:

If the ideal I is Newton non-degenerate, we obtain from the Theorem 3.4 of [10] that any germ h with $\Gamma_+(h) \subset \Gamma_+(I)$ is in the integral closure of I. ¿From the condition $\mathrm{d}(h_i) \geq \mathrm{d}(g_i)$, since $\Gamma_+(g_i) \subset \Gamma_+(I)$ we obtain $\Gamma_+(h_i) \subset \Gamma_+(I)$. $\|$

4.1. An example in $\mathbb{C}^2 \to \mathbb{C}^4$

Let $g : \mathbb{C}^2 \to \mathbb{C}^4$ be the map germ $g = (g_1, g_2, g_3, g_4)$ with

$$\begin{aligned}
g_1(x,y) &= \alpha_1 x^5 + \alpha_2 y^5 + a_1 x^3 y + a_2 x y^3;\\
g_2(x,y) &= \beta_1 x^7 + \beta_2 y^7 + b_1 x^3 y^2 + b_2 x^2 y^3;\\
g_3(x,y) &= \theta_1 x^{11} + \theta_2 y^{11} + c_1 x^5 y^3 + c_2 x^3 y^5;\\
g_4(x,y) &= \gamma_1 x^{12} + \gamma_2 y^{12} + d x^4 y^4;
\end{aligned}$$

We see in the example 2.1 of [2] that g is $\mathcal{A}$-finitely determined for generic values of α_i, β_i, θ_i and γ_i, with a_i, b_i, c_i and d being all distinct prime numbers.

Here we fix the Newton polyhedron $\Gamma_+(g_4)$, with vertices $(12,0)$, $(0,12), (4,4)$ to obtain that I is non-degenerate on $\Gamma_+(g_4)$, therefore any deformation of type $g_\lambda = g + \lambda h$ with $\mathrm{d}(h_j) \geq \mathrm{d}(g_4)$ for $j = 1,2,3,4$ is topologically trivial.

§5. The weighted homogeneous case

Damon in [3] investigates the topological triviality of unfoldings of $\mathcal{A}$-finitely determined map germs which are weighted homogenous. His theorem 1. shows that polynomial unfoldings of non negative weights of these map germs are topologically trivial.

From the results shown above we obtain similar results for one parameter linear unfoldings of any weighted homogenous $\mathcal{A}$-finitely determined map germ. We remark that in the weighted homogenous case, the results of Damon are for any pair of dimensions (n, p) while the results shown here are only for $n \geq p$ and $p \geq 2n$

Definition 5.1. *Given $(w_1, \dots, w_n; d_1, \dots, d_p)$ with $w_i,\ d_j \in \mathbb{Q}_+$, a map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is weighted homogeneous of type $(w_1, \dots, w_n; d_1, \dots, d_p)$ if for all $\lambda \in K - \{0\}$*

$$f(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \dots, \lambda^{w_n} x_n) = (\lambda^{d_1} f_1(x), \lambda^{d_2} f_2(x), \dots, \lambda^{d_p} f_p(x)).$$

For a fixed set of weights $w = (w_1, \dots, w_n)$ consider the *Newton filtration* of $\mathcal{O}_n = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots$, by the ideals $\mathcal{A}_q = \{g \in \mathcal{O}_n : \mathrm{d}_w(g) \geq q\}$.

Proposition 5.2. *Let g be an $\mathcal{A}$-finitely determined map germ which is weighted homogenous of type $(w_1, \dots, w_n; d_1, \dots, d_p)$. Then deformations of $g_\lambda = g + \lambda h$ of g, with $\mathrm{d}_w(h_i) \geq d_i$, $\forall\ i = 1, \dots, p$, are $\mathcal{A}$-topologically trivial for small values of λ.*

Proof: To show this result we follow the proof of the Proposition 3.2.

We should prove that the germ g satisfies the equation 1, however the main purpose of this equation is to guarantee that

$$h\rho_{\mathrm{m}}(g) \in tg(A_{2D}\theta_n) + wg(B_{2R+1}\theta_p)$$

and them from the Lemma 3.2. we obtain that this condition also holds for deformations, i. e.,

$$h.\rho_{\mathrm{m}}(g_\lambda) \in tG(A_{2D}\theta_{n+1}) + wG(B_{2R+1}\theta_{p+1})$$

In the case of weighted homogenous map germs, we see in the item *ii* of the proposition 7.4 of [3] p.319, that it is possible to obtain vector fields η and ψ satisfying the condition

$$h.\rho_{\mathrm{m}}(g_\lambda) = tg_\lambda(\psi(x, \lambda)) + \eta(g(x, \lambda)).$$

and as g and ρ_m are weighted homogeneous, we may assume that the vector field ψ is in $A_{2D}\theta_{n+1}$ and η is in $B_{2R+1}\theta_{p+1}$.

In the case $p \geq 2n$ we use the fact that each germ g_j is weighted homogeneous of type $(w_1, \dots, w_n; d_j)$, then we consider the ideal I generated by the system $\{g_1^{2r_1}, \dots, g_p^{2r_p}\}$, where r_j are integers such that $r_j d_j = D$ for some D and each $g_j^{2r_j}$ is weighted homogenous of type $(w_1, \dots, w_n; 2D)$.

Since g is $\mathcal{A}$-finitely determined it is also $\mathcal{L}$-finitely determined and the ideal I is Newton non-degenerate.

Therefore we obtain the inequality $\rho_{\mathrm{m}}(g(x)) \geq c\rho_v(x)$, $\forall x$ in a neighborhood V of 0.

In the case $n \geq p$ we need to show that there exist a neighborhood V of 0 in $\mathbb{C}^n$, and constants α e β such that $\rho_{\mathrm{f}}(g(x)) \geq \beta\rho_v(x)$, for all $x \in V \cap \{x; \rho_{\mathrm{m}}(g(x)) < \alpha\rho_v(x)\}$, but in this case this condition follows from the lemma 7.7, p.319 of [3].

Therefore, we are ready to follow the final part of the proof of the Proposition 3.1. to obtain the result. $\|$

§6. Examples

Example 6.1. ([7], p. 102.) Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$, $f(x, y, z) = (x^2 + y^2 + x^3 + z^3, x^2 + y^3 + z^2)$.

We remark that it is not possible to apply the Proposition 5.2 for this case since the map germ f is not weighted homogenous.

The best filtrations to choose for $\mathcal{O}_n$ and $\mathcal{O}_p$ in this example are the usual filtrations given by the degree.

Here we have $tf(m_n^3\theta_n) + wf(m_p^2\theta_p) = m_n^4\theta_f$, moreover we see that $tf(m_n^3\theta_n) + wf(m_p^2\theta_p) = tf(m_n^3\theta_n) + f^*(m_p)m_n^{k-1}\theta_f = m_n^4\theta_f$, hence this germ is $(k-1) - C^0 - \mathcal{K}-$ determined. From this condition we show that this germ satisfies the conditions of the Proposition 3.2., therefore deformations by order higher than 2 are $\mathcal{A}$-topologically trivial.

Example 6.2. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $f(x, y) = (xy, g(x, y))$, with $g(x, y) = x^4 + xy^2 + y^5$.

This is a special case of a pre-weighted homogeneous map germ which is in the $\mathcal{K}$-orbit of the $\mathcal{A}$-finitely determined weighted homogenous map germ $k(x, y) = (xy, x^4 + y^5)$, therefore it is also $\mathcal{A}$-finitely determined. See [11] for more details about the $\mathcal{A}$-finite determinacy of pre-weighted homogenous map germs.

We fix the Newton polygon $\Gamma_+(g)$ and are interested in the topological triviality of families of type $g_\lambda(x,y) = (xy, x^4 + xy^2 + y^5 + \lambda h(x,y))$, with $\mathrm{d}(h) \geq \mathrm{d}(g)$.

The main difficulty with this type of example is to show that the equation 1 holds, or if we follow the proof for the weighted homogenous case, we need to show that it is possible to obtain vector fields η and ψ satisfying the condition

$$h.\rho_{\mathrm{m}}(g_\lambda) = tg_\lambda(\psi(x,\lambda)) + \eta(g(x,\lambda)).$$

in such a way that the vector field ψ is in $A_{2D}\theta_{n+1}$ and η is in $B_{2R+1}\theta_{p+1}$.

In fact, in this case we can show that for each germ h it is possible to find an specific vector field ψ, which depends of the cone $C(\Delta)$ that h belongs, such that ψ is not in A_{2D}, however it is in an appropriate level of filtration in such a way that we obtain the integrability of the vector field $\nu_1(x,\lambda)$ given in the proof of the Lemma 3.3.

Therefore we can follow the method of the proof of the Proposition 3.2. to show that any deformation of this type is topologically trivial for small values of λ.

This example is a particular case of the following:

Proposition 6.3. [8] Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $f(x,y) = (xy, g(x,y))$, with $g(x,y) = x^a + x^r y^s + y^b$, be a pre-weighted homogeneous map germ in the $\mathcal{K}$-orbit of an $\mathcal{A}$-finitely determined map germ $k(x,y) = (xy, x^a + y^b)$. Then deformations of type $f(x,y) = (xy, x^a + x^r y^s + y^b + \lambda h(x,y))$ with $\Gamma_+(h) \subset \Gamma_+(g)$ are topologically trivial for small values of λ.

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On the topology of symmetry sets of smooth submanifolds in $\mathbb{R}^k$

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Abstract.

We study topology of symmetry sets, conflict sets and medial axes in the case when they have only stable singularities of corank 1. Singularities of these sets satisfy various conditions of coexistence. For example, isolated singularities and singularities forming smooth non-closed curves define a graph. If this graph is finite, then there is the following incidence relation: the sum of the local degrees of vertices of the graph is equal to the doubled number of its edges (the local degree of a vertex is the number of edges that are incident to this vertex; loops are counted twice). We give many-dimensional generalizations of this relation for sets mentioned above. These generalizations follow from some general facts on coexistence of wave front singularities found recently by the author.

Let M be a C^∞-smooth closed (compact without boundary) submanifold in the k-dimensional Euclidean space $\mathbb{R}^k$. The manifold M can have several connected components, perhaps of different dimensions (including isolated points). We will assume everywhere below that M lies neither in any hyperplane nor in any hypersphere in $\mathbb{R}^k$.

Let us equip the space of all embeddings $M \to \mathbb{R}^k$ by the C^∞-topology. Submanifolds in $\mathbb{R}^k$ corresponding to embeddings from an open dense subset in this space are called *generic*.

DEFINITION. The *symmetry set* of the manifold M is the closure of the set of centers of hyperspheres S^{k-1} in $\mathbb{R}^k$ that are tangent to M at two different points.

The symmetry set is a complicated singular subset in the ambient space. This set and various of its subsets (conflict sets, medial axes, etc)

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found applications in computer vision and other applied fields (see, for example, [3],[5],[6],[15]).

The symmetry set can be noncompact. Moreover, some of its points can be centers of several hyperspheres (of different radii) every of which is tangent to the manifold M at two or more points. Therefore, we will consider more simple object.

Let $\mathcal{H}(\mathbb{R}^k)$ be the set of all hyperspheres and hyperplanes in $\mathbb{R}^k$. This is a smooth $(k+1)$-dimensional manifold. Hyperplanes in $\mathbb{R}^k$ form a smooth hypersurface $\mathcal{P}(\mathbb{R}^k)$ in $\mathcal{H}(\mathbb{R}^k)$. The complement to $\mathcal{P}(\mathbb{R}^k)$ is the total space of a smooth fiber bundle

$$\varrho : \mathcal{H}(\mathbb{R}^k) \setminus \mathcal{P}(\mathbb{R}^k) \to \mathbb{R}^k,$$

that takes each hypersphere to its center. The fiber of this (trivial) bundle is the set of positive real numbers (the radius of a hypersphere).

Definition. *Generalized symmetry set* $\Sigma(M)$ *of the manifold* M *is the closure in* $\mathcal{H}(\mathbb{R}^k)$ *of the set of hyperspheres in* $\mathbb{R}^k$ *which are tangent to* M *at two different points.*

The set $\Sigma(M)$ is a compact subset in $\mathcal{H}(\mathbb{R}^k)$. The image of the set $\Sigma(M)\setminus\mathcal{P}(\mathbb{R}^k)$ with respect to the projection ϱ is the symmetry set of the manifold M. Moreover, the intersection of the tangent cone to $\Sigma(M)$ at any point from $\Sigma(M) \setminus \mathcal{P}(\mathbb{R}^k)$ and the tangent space to the fiber of the bundle ϱ at this point is 0.

Denote by $\mathcal{F}(M)$ the set of hyperspheres and hyperplanes in $\mathbb{R}^k$ that are tangent to the submanifold M. This is the front of some Legendre mapping into $\mathcal{H}(\mathbb{R}^k)$ (see [9]). Its simplest singularities are *stable singularities of corank* 1. These singularities are classified by (nonzero) elements $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p}$ of the free additive Abelian semigroup $\mathbb{A}$ whose generators are the symbols $A_1, A_2, \dots, A_\mu, \dots$

Namely a generic front $\mathcal{F}$ in a smooth n-dimensional manifold V has a singularity of type A_μ at a given point $v \in V$ if its germ $(\mathcal{F}, v)$ at this point is diffeomorphic to a germ at zero of the hypersurface in $\mathbb{R}^n = \{(\lambda_0, \dots, \lambda_{n-1})\}$ formed by points, where the polynomial $t^{\mu+1} + \lambda_{\mu-1}t^{\mu-1} + \cdots + \lambda_1 t + \lambda_0$ has a multiple real root. The front $\mathcal{F}$ has a singularity of type $A_{\mu_1} + \cdots + A_{\mu_p}$ at a point v if the germ $(\mathcal{F}, v)$ consists of p irreducible components having (as germs of fronts) singularities of types $A_{\mu_1}, \dots, A_{\mu_p}$ at the point v, and if, moreover, germs of the manifolds of these singularities on the corresponding components intersect transversally at the point v. The numbers $\mu_1, \dots, \mu_p$ are called multiplicities of a singularity of type $A_{\mu_1} + \cdots + A_{\mu_p}$.

Let us suppose that the front $\mathcal{F}(M)$ has only stable singularities of corank 1. Then for a generic manifold M, the points of the front $\mathcal{F}(M)$, where it has singularities of type $\mathcal{A} \in \mathbb{A}$, are in one to one correspondence with tangent $\mathcal{A}$-spheres and $\mathcal{A}$-planes of the manifold M. The latter are defined by the following way.

Definition. A hypersphere (hyperplane) π in $\mathbb{R}^k$ is said to be *tangent* $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p}$*-sphere* (*plane*) of the manifold M if

1) it is tangent to M exactly at p points $x_1, \ldots, x_p$ that are the vertices of an $(p-1)$-dimensional simplex;

2) for any $i = 1, \ldots, p$ and for any smooth function in $\mathbb{R}^k$ equal to 0 on π and having noncritical value at x_i, a germ at x_i of the restriction of this function onto M is given by the formula

$$\pm t_1^{\mu_i+1} \pm t_2^2 \pm \cdots \pm t_{m_i}^2$$

in suitable local coordinates $t_1, \ldots, t_{m_i}$ on M.

Remark. By definition, any hypersphere (hyperplane) in $\mathbb{R}^k$ passing through a given point is a tangent A_1-sphere (plane) for this point.

Tangent $\mathcal{A}$-spheres (planes) of the manifold M are called its *tangent hyperspheres* (*hyperplanes*) *of corank* 1. The number $c(\mathcal{A}) = \mu_1 + \cdots + \mu_p$ is called the *codimension* of tangency of an $\mathcal{A}$-sphere (plane) with the manifold M. The number $d(\mathcal{A}) = c(\mathcal{A}) + p$ is called the *degree* of this tangency.

Let M be a generic manifold. Then $c(\mathcal{A}) \leq k+1$ for any its tangent $\mathcal{A}$-sphere and $c(\mathcal{A}) \leq k$ for any tangent $\mathcal{A}$-plane. Moreover, for any fixed $\mathcal{A} \in \mathbb{A}$ the set $\mathcal{A}_M$ consisting of all tangent $\mathcal{A}$-spheres and $\mathcal{A}$-planes of the manifold M is a smooth submanifold (generally speaking, nonclosed) of codimension $c(\mathcal{A})$ in $\mathcal{H}(\mathbb{R}^k)$. The restriction of the projection ϱ onto the manifold $\mathcal{A}_M \setminus \mathcal{P}(\mathbb{R}^k)$ is a smooth immersion.

Now, if $\mathcal{A} = 2A_1$ or $c(\mathcal{A}) > 2$, then the manifold $\mathcal{A}_M$ belongs to the set $\Sigma(M)$ and is called the manifold of singularities of type $\mathcal{A}$ of this set. The numbers $c(\mathcal{A})$ and $d(\mathcal{A})$ are called the codimension and the degree of these singularities, respectively.

Denote by $\chi(\mathcal{A}_M)$ the topological Euler characteristic of the manifold $\mathcal{A}_M$ (the alternated sum of the Betti numbers of the homology groups with compact supports). Sometimes, to simplify notations, we omit the subscript M and write $\chi(\mathcal{A})$.

Definition. The *index* $I_{\mathcal{A}}(X)$ of a singularity of type $X = A_{\nu_1} + \cdots + A_{\nu_q} \in \mathbb{A}$ with respect to a singularity of type $\mathcal{A} = A_{\mu_1} + \cdots + A_{\mu_p}$ is the nonnegative integer defined recursively by the following conditions:

1) if $\mu^* = \max\{\mu_1, \ldots, \mu_p\} > \nu^* = \max\{\nu_1, \ldots, \nu_q\}$, then $I_{\mathcal{A}}(X)$ is equal to 0;

2) if $\mu^* \le \nu^*$, then $I_{\mathcal{A}}(X)$ is equal to

$$\sum_{\nu_i=\mu^*,\mu^*+1} I_{\mathcal{A}-A_{\mu^*}}(X - A_{\nu_i}) + \sum_{\nu_i>\mu^*+1} I_{\mathcal{A}-A_{\mu^*}}(X - A_{\nu_i} + A_{\nu_i-\mu^*-1}),$$

where $I_{\emptyset}(Y) = 1$ for any Y (here $\emptyset$ is the zero of the semigroup $\mathbb{A}$).

Theorem 1. *Let M be a smooth closed submanifold in $\mathbb{R}^k$. Suppose that all tangent hyperspheres (hyperplanes) of M are tangent hyperspheres (hyperplanes) of corank 1. Then for a generic M the following statements are valid.*

1) *If the manifold $\mathcal{A}_M$ of singularities of type $\mathcal{A} \in \mathbb{A}$ of the generalized symmetry set $\Sigma(M)$ of the manifold M has an odd dimension, then its Euler characteristic $\chi(\mathcal{A}_M)$ is a linear combination*

$$\chi(\mathcal{A}_M) = \sum_X K_{\mathcal{A}}(X)\chi(X_M) \tag{1}$$

of the Euler characteristics $\chi(X_M)$ of even-dimensional manifolds X_M of singularities of types $X \in \mathbb{A}$ where $c(X) > c(\mathcal{A})$. This combination is universal in the sense that every its coefficient $K_{\mathcal{A}}(X)$ depends only on $\mathcal{A}$ and X (that is, it does not depend on the topology of the manifold M). Namely,

$$K_{\mathcal{A}}(X) = \sum_{i=0}^{[c(X)-c(\mathcal{A})-1]/2} (-1)^i P_i(\mathcal{A}, X),$$

where $P_i(\mathcal{A}, X)$ is equal to the sum of the products of the form

$$\prod_{j=0}^{i} \frac{I_{Y_j}(Y_{j+1})}{I_{Y_j}(Y_j)}$$

by all ordered sets $(Y_0, Y_1, \dots, Y_{i+1})$ of elements of the semigroup $\mathbb{A}$ such that $Y_0 = \mathcal{A}, Y_{i+1} = X$ and

$$c(\mathcal{A}) < c(Y_1) < \cdots < c(Y_i) < c(X),$$

$$c(Y_1) \equiv \cdots \equiv c(Y_i) \equiv c(\mathcal{A}) \pmod 2 .$$

2) *The list of the formulas (1) for singularities of codimension $c \le 6$ is given in Table 1 for an even k and in Table 2 for an odd k.*

3) *The Euler characteristic $\chi(\Sigma(M))$ of the set $\Sigma(M)$ is a universal linear combination*

$$\chi(\Sigma(M)) = \sum_X \mathcal{K}(X)\chi(X_M) \tag{2}$$

of the Euler characteristics $\chi(X_M)$ of even-dimensional manifolds X_M of singularities of types $X \in \mathbb{A} \setminus \{A_1, A_2\}$. The coefficients of this combination are calculated by the formula

$$\mathcal{K}(X) = 1 - \sum_{\mathcal{A}} K_{\mathcal{A}}(X),$$

where the summation is taken by all nonzero elements $\mathcal{A}$ of the semigroup $\mathbb{A}$ such that $\mathcal{A} \neq A_1$, $\mathcal{A} \neq A_2$, $c(\mathcal{A}) < c(X)$ and $c(\mathcal{A}) \equiv k \pmod 2$.

4) *The formula* (2) *for an even k has the form*

$$\begin{aligned}
\chi(\Sigma(M)) = {} & \tfrac{1}{2}[\chi(A_3) - 4\chi(3A_1)] \\
& + \tfrac{1}{2}[\chi(A_5) + 3\chi(A_4 + A_1) + 2\chi(A_3 + A_2) + 6\chi(A_3 + 2A_1) \\
& \qquad + 4\chi(2A_2 + A_1) + 12\chi(A_2 + 3A_1) + 32\chi(5A_1)] \\
& - \tfrac{1}{4}[27\chi(A_7) + 52\chi(A_6 + A_1) + 38\chi(A_5 + A_2) + 96\chi(A_5 + 2A_1) \\
& \qquad + 41\chi(A_4 + A_3) + 70\chi(A_4 + A_2 + A_1) + 168\chi(A_4 + 3A_1) \\
& \qquad + 74\chi(2A_3 + A_1) + 52\chi(A_3 + 2A_2) + 124\chi(A_3 + A_2 + 2A_1) \\
& \qquad + 288\chi(A_3 + 4A_1) + 88\chi(3A_2 + A_1) + 208\chi(2A_2 + 3A_1) \\
& \qquad + 480\chi(A_2 + 5A_1) + 1088\chi(7A_1)] + \dots
\end{aligned}$$

(*the dots denote a universal linear combination of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension* 9); *for an odd k*

$$\begin{aligned}
\chi(\Sigma(M)) = {} & \chi(2A_1) \\
& - \tfrac{1}{2}[2\chi(A_4) + 3\chi(A_3 + A_1) + 2\chi(2A_2) \\
& \qquad + 4\chi(A_2 + 2A_1) + 6\chi(4A_1)] \\
& + \tfrac{1}{2}[10\chi(A_6) + 14\chi(A_5 + A_1) + 11\chi(A_4 + A_2) \\
& \qquad + 20\chi(A_4 + 2A_1) + 10\chi(2A_3) + 15\chi(A_3 + A_2 + A_1) \\
& \qquad + 28\chi(A_3 + 3A_1) + 12\chi(3A_2) + 22\chi(2A_2 + 2A_1) \\
& \qquad + 40\chi(A_2 + 4A_1) + 70\chi(6A_1)] + \dots
\end{aligned}$$

(*the dots denote a universal linear combination of the Euler characteristics of manifolds of singularities of even codimensions starting from the codimension* 8).

Proof. Let a generic front $\mathcal{F}$ in a smooth manifold V have only stable singularities of corank 1. Assume that the closure of the submanifold in V formed by singularities of type $2A_1$ of this front is compact. Then there are universal linear relations between the Euler characteristics of the manifolds of singularities of $\mathcal{F}$ lying in the mentioned closure. These relations are given in [12]. Theorem 1 is a corollary of these results in the case $\mathcal{F} = \mathcal{F}(M)$.

REMARK. Theorem 1 is valid for any generic curve in $\mathbb{R}^k$. In the case $\dim M > 1$, the main condition of Theorem 1 (all tangent hyperspheres (hyperplanes) of M are tangent hyperspheres (hyperplanes) of corank 1) is valid only for very special manifold M. For example, a smooth closed connected generic surface with such a property in $\mathbb{R}^3$ has no umbilic points. Therefore it is diffeomorphic to a torus (see [4]).

EXAMPLE. Let M be a smooth closed curve (not necessary connected) in the plane $\mathbb{R}^2$. Then its generalized symmetry set $\Sigma(M)$ is the union of a graph and a smooth closed one-dimensional manifold. The edges of the graph are simply connected components of the manifold $(2A_1)_M$. The vertices are singularities $3A_1, A_2 + A_1, A_3$. The incidence relation in this graph has the form: $2\chi(2A_1) = 6\chi(3A_1) + 2\chi(A_2 + A_1) + \chi(A_3)$. This is the first formula of Table 1 (in the case $k = 2$).

Consider *supporting* hyperspheres and hyperplanes of the manifold M. They are tangent hyperspheres and hyperplanes such that M lies on one side of them. If a supporting hypersphere (hyperplane) is a tangent $A_{\mu_1} + \cdots + A_{\mu_p}$-sphere (plane), then $\mu_1, \ldots, \mu_p$ are odd numbers.

Let $\mathbb{A}_{odd} \subset \mathbb{A}$ be the free additive Abelian semigroup whose generators are the symbols $A_1, A_3, \ldots, A_{2l+1}, \ldots$ Then for $k \leq 6$ the set of supporting hyperspheres (hyperplanes) of a smooth closed generic submanifold M in $\mathbb{R}^k$ consists of supporting $\mathcal{A}$-spheres (planes) where $\mathcal{A} \in \mathbb{A}_{odd}$ and $c(\mathcal{A}) \leq k+1$ ($c(\mathcal{A}) \leq k$ for hyperplanes). If the dimension of each connected component of the manifold M is at most 1, then this is true for any k (see [1],[16]).

Supporting hyperspheres and hyperplanes of the manifold M that lie in its generalized symmetry set $\Sigma(M)$ are called *singular*. Singular supporting hyperspheres and hyperplanes of the manifold M form a compact subset $\Sigma_{sup}(M)$ in $\mathcal{H}(\mathbb{R}^k)$. This subset is the closure in $\mathcal{H}(\mathbb{R}^k)$ of the set of supporting hyperspheres that are tangent to M at two different points.

The set of singular supporting $\mathcal{A}$-spheres and $\mathcal{A}$-planes of a generic manifold M (in dimensions mentioned above) is a smooth submanifold of codimension $c(\mathcal{A})$ in $\mathcal{H}(\mathbb{R}^k)$ for any $\mathcal{A} \in \mathbb{A}_{odd}$. It is called the manifold of singularities of type $\mathcal{A}$ of the set $\Sigma = \Sigma_{sup}(M)$ and is denoted by $\mathcal{A}_\Sigma$. The numbers $c(\mathcal{A})$ and $d(\mathcal{A})$ are called the codimension and the degree of these singularities, respectively.

THEOREM 2. *Let M be a smooth closed submanifold in $\mathbb{R}^k$. Assume that $k \leq 6$ or the dimension of each connected component of the manifold*

M is at most 1. Then for a generic M the statements 1 – 5 below are valid for $\Sigma = \Sigma_{sup}(M)$ and $\chi_0 = (-1)^k[1 - \chi(M)] + 1$.

1) *If the manifold $\mathcal{A}_\Sigma$ of singularities of type $\mathcal{A} \in \mathbb{A}_{odd}$ of the set Σ has an odd dimension, then its topological Euler characteristic $\chi(\mathcal{A}_\Sigma)$ is a linear combination*

$$\chi(\mathcal{A}_\Sigma) = \sum_X K_{\mathcal{A}}^{odd}(X)\,\chi(X_\Sigma) \tag{3}$$

of the Euler characteristics $\chi(X_\Sigma)$ of even-dimensional manifolds X_Σ of singularities of types $X \in \mathbb{A}_{odd}$ where $c(X) > c(\mathcal{A})$. This combination is universal in the sense that every its coefficient $K_{\mathcal{A}}^{odd}(X)$ depends only on $\mathcal{A}$ and X (that is, it does not depend on the topology of the manifold M). Namely,

$$K_{\mathcal{A}}^{odd}(X) = \sum_{i=0}^{[c(X)-c(\mathcal{A})-1]/2} (-1)^i \widetilde{P}_i(\mathcal{A}, X),$$

where $\widetilde{P}_i(\mathcal{A}, X)$ is equal to the sum of the products of the form

$$\prod_{j=0}^{i} \left(\frac{1}{2}\right)^{\operatorname{sign}[d(Y_{j+1})-d(Y_j)]} \frac{I_{Y_j}(Y_{j+1})}{I_{Y_j}(Y_j)}$$

by all ordered sets $(Y_0, Y_1, \ldots, Y_{i+1})$ of elements of the semigroup $\mathbb{A}_{odd}$ such that $Y_0 = \mathcal{A}, Y_{i+1} = X$ and

$$c(\mathcal{A}) < c(Y_1) < \cdots < c(Y_i) < c(X),$$

$$c(Y_1) \equiv \cdots \equiv c(Y_i) \equiv c(\mathcal{A}) \pmod 2\,.$$

2) *The list of the formulas (3) for singularities of codimension $c \leq 8$ is given in Table 3 for an even k and in Table 4 for an odd k.*

3) *The Euler characteristic $\chi(\Sigma)$ of the set Σ is equal to χ_0. From the other side it is a universal linear combination*

$$\chi(\Sigma) = \sum_X \mathcal{K}^{odd}(X)\chi(X_\Sigma) \tag{4}$$

of the Euler characteristics $\chi(X_\Sigma)$ of even-dimensional manifolds X_Σ of singularities of types $X \in \mathbb{A}_{odd} \setminus \{A_1\}$. The coefficients of this combination are calculated by the formula

$$\mathcal{K}^{odd}(X) = 1 - \sum_{\mathcal{A}} K_{\mathcal{A}}^{odd}(X),$$

where the summation is taken by all nonzero elements $\mathcal{A}$ *of the semigroup* $\mathbb{A}_{odd}$ *such that* $\mathcal{A} \neq A_1$, $c(\mathcal{A}) < c(X)$ *and* $c(\mathcal{A}) \equiv k \pmod 2$.

4) *The formula* (4) *for an even* k *has the form*

$$\begin{aligned}\chi(\Sigma) = {} & \tfrac{1}{2}\,[\chi(A_3) - \chi(3A_1)] + \tfrac{1}{2}\,[\chi(A_5) + 2\chi(5A_1)] + \\ & + \tfrac{1}{4}[\chi(A_7) - 2\chi(A_5 + 2A_1) - \chi(2A_3 + A_1) \\ & \quad - 3\chi(A_3 + 4A_1) - 17\chi(7A_1)] \\ & + \tfrac{1}{4}[3\chi(A_9) + 4\chi(A_7 + 2A_1) + 3\chi(A_5 + A_3 + A_1) \\ & \quad + 13\chi(A_5 + 4A_1) + 2\chi(3A_3) + 8\chi(2A_3 + 3A_1) \\ & \quad + 30\chi(A_3 + 6A_1) + 124\chi(9A_1)] \\ & - \tfrac{1}{4}[7\chi(A_{11}) + 22\chi(A_9 + 2A_1) + 16\chi(A_7 + A_3 + A_1) \\ & \quad + 56\chi(A_7 + 4A_1) + 17\chi(2A_5 + A_1) + 12\chi(A_5 + 2A_3) \\ & \quad + 42\chi(A_5 + A_3 + 3A_1) + 152\chi(A_5 + 6A_1) \\ & \quad + 30\chi(3A_3 + 2A_1) + 106\chi(2A_3 + 5A_1) \\ & \quad + 378\chi(A_3 + 8A_1) + 1382\chi(11A_1)] + \dots\end{aligned}$$

(*the dots denote a universal linear combination of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension* 13)*; for an odd* k,

$$\begin{aligned}\chi(\Sigma) = {} & \chi(2A_1) - \chi(4A_1) \\ & + \tfrac{1}{2}[6\chi(6A_1) + \chi(A_3 + 3A_1) + \chi(A_5 + A_1)] \\ & - \tfrac{1}{2}[34\chi(8A_1) + 8\chi(A_3 + 5A_1) + 2\chi(2A_3 + 2A_1) \\ & \quad + 4\chi(A_5 + 3A_1) + \chi(A_5 + A_3) + \chi(A_7 + A_1)] \\ & + \tfrac{1}{4}[620\chi(10A_1) + 167\chi(A_3 + 7A_1) + 46\chi(2A_3 + 4A_1) \\ & \quad + 13\chi(3A_3 + A_1) + 71\chi(A_5 + 5A_1) + 19\chi(A_5 + A_3 + 2A_1) \\ & \quad + 8\chi(2A_5) + 26\chi(A_7 + 3A_1) + 7\chi(A_7 + A_3) \\ & \quad + 11\chi(A_9 + A_1)] + \dots\end{aligned}$$

(*the dots denote a universal linear combination of the Euler characteristics of manifolds of singularities of even codimensions starting from the codimension* 12)*;*

5) *Let* k *be even* ($k \leq 16$) *and* $\chi(\mathcal{A}_\Sigma) = 0$ *for any* $\mathcal{A} \in \mathbb{A}_{odd} \setminus \{A_1\}$ *such that* $d(\mathcal{A}) \leq k$. *Then*

$$\sum_{\mathcal{A}\in\mathbb{A}_{odd}\colon c(\mathcal{A})=k+1} w(\mathcal{A})\chi(\mathcal{A}_\Sigma) = (1 + k/2)\, w(A_{k+1})\, \chi_0, \tag{5}$$

where

$$w(A_{\mu_1} + \dots + A_{\mu_p}) = (-1)^{[p/2]} w(A_{\mu_1}) \dots w(A_{\mu_p}),$$

$$w(A_{2l+1}) = \frac{1}{2l+1}\binom{2l+1}{l},$$

and $[x]$ is the integral part of the number x. The number $w(A_{2l+1})$ is the l'th Catalan number. The formulas (5) *for $k \leq 12$ are given in Table* 5 *below.*

PROOF. The subset $\Sigma_{sup}(M) \subset \mathcal{H}(\mathbb{R}^k)$ is diffeomorphic to the set of singular points of the boundary Γ of a connected component of the complement to some generic front $\mathcal{F}$ in a $(k+1)$-dimensional space (see [10]; the closure of this component is a $(k+1)$-dimensional C^0-manifold and Γ is its boundary). For any $\mathcal{A} \in \mathbb{A}_{odd}$, the manifold $\mathcal{A}_\Sigma$ is diffeomorphic to the manifold $\mathcal{A}_\Gamma$ of singularities of type $\mathcal{A}$ of the hypersurface Γ (that is the set of singularities of type $\mathcal{A}$ of the front $\mathcal{F}$ at points of Γ). Therefore Theorem 2 follows from [11] (see also [14]) where we found universal linear relations between the Euler characteristics of the manifolds of singularities on the boundary of a connected component of the complement to a generic front under the condition that this boundary has only stable corank 1 singularities with odd multiplicities.

REMARK. The restriction $k \leq 16$ in the condition of the statement 5 of Theorem 2 is pure technical. I think that it can be omitted.

EXAMPLE. The formula (5) is valid for any smooth closed convex generic curve M in the even-dimensional space $\mathbb{R}^k$ (see [13]; recall that a curve in $\mathbb{R}^k$ is convex if it intersects any hyperplane at most at k points taking multiplicities into account).

REMARK. One can prove that the formula (5) is valid for any smooth closed generic curve M in $\mathbb{R}^k$ (k is even) such that for any k of its points (taking multiplicities into account) there is a hypersphere that passes through these points and does not intersect the curve at other points.

Consider the set of supporting hyperspheres of the manifold M. A supporting hypersphere is called *externally* (*internally*) supporting if M lies outside (inside) the ball bounded by this hypersphere.

Let $\Sigma_{ext}(M)$ ($\Sigma_{int}(M)$) be the set of singular externally-supporting (internally-supporting, respectively) hyperspheres of the manifold M. The sets $\Sigma_{ext}(M)$ and $\Sigma_{int}(M)$ are disjoint (recall, we assume that M does not lie in any hypersphere and in any hyperplane in $\mathbb{R}^k$). The union of these sets is equal to $\Sigma_{sup}(M) \setminus \mathcal{P}(\mathbb{R}^k)$.

DEFINITION. The *externally-supporting* (*internally-supporting*) *symmetry set* of the manifold M is the closure $\mathcal{C}_{ext}(M)$ ($\mathcal{C}_{int}(M)$, respectively) in $\mathbb{R}^k$ of the set of centers of externally-supporting (internally-supporting) hyperspheres that are tangent to M at two different points.

The supporting symmetry sets of the manifold M are subsets of its symmetry set. It is easy to see that $\mathcal{C}_{ext}(M) = \varrho(\Sigma_{ext}(M))$, $\mathcal{C}_{int}(M) = \varrho(\Sigma_{int}(M))$. Moreover, the projection ϱ defines a homeomorphism of the sets $\Sigma_{ext}(M)$ and $\Sigma_{int}(M)$ with the sets $\mathcal{C}_{ext}(M)$ and $\mathcal{C}_{int}(M)$, respectively.

Proposition. *Let M be a smooth closed submanifold in $\mathbb{R}^k$. Then the set $\mathcal{C}_{ext}(M)$ is compact if an only if one of the connected components of M is a strictly convex hypersurface and all other components lie inside the compact domain in $\mathbb{R}^k$ bounded by this hypersurface.*

The similar statement is valid for the set $\mathcal{C}_{int}(M)$ as well. Moreover, if $M = M_0 \cup (M \setminus M_0)$ where M_0 is a smooth closed strictly convex hypersurface and $M \setminus M_0$ lies inside the compact domain in $\mathbb{R}^k$ bounded by the hypersurface M_0, then $\mathcal{C}_{int}(M) = \mathcal{C}_{int}(M_0)$.

Proof. The set $\mathcal{C}_{ext}(M)$ (or $\mathcal{C}_{int}(M)$) is compact if and only if M has no singular supporting hyperplanes. The set of all supporting hyperplanes of such a manifold is diffeomorphic to S^{k-1}. The mapping $S^{k-1} \to \mathbb{R}^k$ that assigns to a supporting hyperplane the point of tangency with M is a smooth embedding ([8]). The image of this embedding is the desired strictly convex connected component of M. Proposition is proved.

Let M be a smooth closed submanifold in $\mathbb{R}^k$ where $k \leq 6$. Then the set of centers of singular externally-supporting (internally-supporting) $\mathcal{A}$-spheres of a generic M is a smooth submanifold of codimension $c(\mathcal{A})-1$ in $\mathbb{R}^k$ for any $\mathcal{A} \in \mathbb{A}_{odd}$. It is called the manifold of singularities of type $\mathcal{A}$ of the set $\Sigma = \mathcal{C}_{ext}(M)$ ($\Sigma = \mathcal{C}_{int}(M)$, respectively) and is denoted by $\mathcal{A}_\Sigma$. The numbers $c(\mathcal{A})$ and $d(\mathcal{A})$ are called the codimension and the degree of these singularities.

Theorem 3. *Let M be a smooth closed submanifold in $\mathbb{R}^k$ where $k \leq 6$. Suppose that one of the connected components of M is a strictly convex hypersurface and all other components lies inside the compact domain in $\mathbb{R}^k$ bounded by this hypersurface. Then for a generic M, the statements* 1 – 5 *from Theorem* 2 *are valid in every of the following two cases:*

1) $\Sigma = \mathcal{C}_{ext}(M)$ *and* $\chi_0 = (-1)^k[1 - \chi(M)]$;
2) $\Sigma = \mathcal{C}_{int}(M)$ *and* $\chi_0 = 1$.

Proof. Theorem 3 is a corollary of the main result of [13]. Indeed, the set $\mathcal{C}_{ext}(M)$ ($\mathcal{C}_{int}(M)$) is the Maxwell set of global minima (maxima, respectively) of the family of functions $F(x, \lambda)$ of $x \in M$ depending on the parameter $\lambda \in \mathbb{R}^k$, where $F(x, \lambda)$ is the square of the distance between λ and x (see [1]). The manifold $\mathcal{A}_\Sigma$ is the manifold of singularities of type $\mathcal{A}$ of the corresponding Maxwell set for any $\mathcal{A} \in \mathbb{A}_{odd}$.

Remark. If the sets $\mathcal{C}_{ext}(M)$ and $\mathcal{C}_{int}(M)$ are compact, then the Euler characteristic of the manifold of Morse global minima of the family

$F(x,\lambda)$ is equal to $\chi(M)$. The Euler characteristic of the manifold of Morse global maxima of the family $F(x,\lambda)$ is equal to $\chi(S^{k-1})$.

Now, take an arbitrary connected component U of the complement to a smooth closed generic submanifold M in $\mathbb{R}^k$. By M_U denote the union of the connected components of M lying strictly inside the closure $\overline{U}$ of the domain U.

Let $\Sigma_{sup}(U)$ be the set of singular supporting hyperspheres and hyperplanes of M lying in $\overline{U}$. The set $\Sigma_{sup}(U)$ is a connected component of the set $\Sigma_{sup}(M)$. Singularities of $\Sigma_{sup}(M)$ at points from $\Sigma_{sup}(U)$ are called singularities of $\Sigma_{sup}(U)$.

Definition. The *middle points set* of the domain U is the set $\mathcal{C}(U)$ of points in U being the centers of singular externally-supporting hyperspheres of the manifold M. Sometimes, depending on M and U, this set is called *medial axe* or *conflict set*.

The set $\mathcal{C}(U)$ is a subset of the externally-supporting symmetry set of the manifold M. If the domain U is bounded, then its middle points set is a compact connected component $\mathcal{C}(U) = \varrho(\Sigma_{sup}(U))$ of the set $\mathcal{C}_{ext}(M)$. Singularities of the set $\mathcal{C}_{ext}(M)$ at points from $\mathcal{C}(U)$ are called singularities of $\mathcal{C}(U)$.

Theorem 4. *Let M be a smooth closed submanifold in $\mathbb{R}^k$, where $k \le 6$. Assume that the complement $\mathbb{R}^k \setminus M$ is disconnected and U is one of its connected components. Then for a generic M, the statements* $1-5$ *from Theorem* 2 *are valid for*

1) $\Sigma = \mathcal{C}(U)$ *and* $\chi_0 = \chi(\overline{U}) - (-1)^k\chi(M_U)$ *if U is bounded;*

2) $\Sigma = \Sigma_{sup}(U)$ *and* $\chi_0 = \chi(\overline{U}) - (-1)^k\chi(M_U) + (-1)^k$ *if U is unbounded.*

Proof. Theorem 4 in the case of the bounded component U follows from [13] by analogy with Theorem 3. Namely, the set $\mathcal{C}(U)$ is the Maxwell set of global minima of the family of functions $F(x,\lambda)$ of $x \in M$ depending on the parameter $\lambda \in \overline{U} \setminus \partial\overline{U}$, where $F(x,\lambda)$ is the square of the distance between λ and x, and $\partial\overline{U}$ is the boundary of $\overline{U}$. The Euler characteristic of the manifold of Morse global minima of this family is equal to $\chi(\partial\overline{U}) + \chi(M_U)$.

The case of the unbounded component U is reduced to the previous one after the inversion of the space $\mathbb{R}^k$ with respect to a hypersphere of a small radius having the centre at a point from the complement $\mathbb{R}^k \setminus (\overline{U} \cup M)$.

Example. Let M be a smooth closed connected curve in the plane $\mathbb{R}^2$. Suppose that it has no self-intersections and is generic. Take a connected component U of the complement $\mathbb{R}^2 \setminus M$. Let $\chi(A_3)$ be the number of the curvature circles of M lying in $\overline{U}$ and $\chi(3A_1)$ be the number of circles in $\overline{U}$ that are tangent to M at three points. Then Theorem 4 implies,

$$\chi(A_3) - \chi(3A_1) = 2. \tag{6}$$

Remark. At the first time, the relation (6) was obtained in [2] for a convex curve M. In [7], it was proved in the non-convex case as well. Theorem 4 extends these results onto the case of non-connected curves and gives their many-dimensional generalizations.

Table 1 (k is even)

$c=2$	$2\chi(2A_1) = 6\chi(3A_1) + 2\chi(A_2+A_1) + \chi(A_3)$ $- 40\chi(5A_1) - 18\chi(A_2+3A_1) - 8\chi(2A_2+A_1)$ $- 11\chi(A_3+2A_1) - 5\chi(A_3+A_2)$ $- 7\chi(A_4+A_1) - 4\chi(A_5)$ $+ 672\chi(7A_1) + 320\chi(A_2+5A_1) + 152\chi(2A_2+3A_1)$ $+ 72\chi(3A_2+A_1) + 204\chi(A_3+4A_1)$ $+ 97\chi(A_3+A_2+2A_1) + 46\chi(A_3+2A_2)$ $+ 62\chi(2A_3+A_1) + 129\chi(A_4+3A_1)$ $+ 61\chi(A_4+A_2+A_1) + 39\chi(A_4+A_3)$ $+ 81\chi(A_5+2A_1) + 38\chi(A_5+A_2)$ $+ 50\chi(A_6+A_1) + 31\chi(A_7) + \dots$
$c=4$	$4\chi(4A_1) = 20\chi(5A_1) + 4\chi(A_2+3A_1) + 2\chi(A_3+2A_1)$ $- 280\chi(7A_1) - 100\chi(A_2+5A_1) - 32\chi(2A_2+3A_1)$ $- 8\chi(3A_2+A_1) - 58\chi(A_3+4A_1)$ $- 18\chi(A_3+A_2+2A_1) - 4\chi(A_3+2A_2)$ $- 10\chi(2A_3+A_1) - 26\chi(A_4+3A_1)$ $- 6\chi(A_4+A_2+A_1) - 3\chi(A_4+A_3) - 12\chi(A_5+2A_1)$ $- 2\chi(A_5+A_2) - 4\chi(A_6+A_1) - \chi(A_7) + \dots$
	$2\chi(A_2+2A_1) = 6\chi(A_2+3A_1) + 4\chi(2A_2+A_1)$ $+ 2\chi(A_3+2A_1) + \chi(A_3+A_2) + 2\chi(A_4+A_1)$ $- 40\chi(A_2+5A_1) - 36\chi(2A_2+3A_1) - 24\chi(3A_2+A_1)$ $- 24\chi(A_3+4A_1) - 21\chi(A_3+A_2+2A_1)$ $- 14\chi(A_3+2A_2) - 12\chi(2A_3+A_1) - 24\chi(A_4+3A_1)$ $- 17\chi(A_4+A_2+A_1) - 9\chi(A_4+A_3) - 16\chi(A_5+2A_1)$ $- 10\chi(A_5+A_2) - 11\chi(A_6+A_1) - 6\chi(A_7) + \dots$
	$2\chi(2A_2) = 2\chi(2A_2+A_1) + 2\chi(A_3+A_2) + \chi(A_5)$ $- 4\chi(2A_2+3A_1) - 6\chi(3A_2+A_1) - 4\chi(A_3+A_2+2A_1)$ $- 5\chi(A_3+2A_2) - 4\chi(2A_3+A_1) - 4\chi(A_4+A_2+A_1)$ $- 4\chi(A_4+A_3) - 2\chi(A_5+2A_1) - 5\chi(A_5+A_2)$ $- 3\chi(A_6+A_1) - 4\chi(A_7) + \dots$
	$\chi(A_3+A_1) = 2\chi(A_3+2A_1) + \chi(A_3+A_2)$ $+ \chi(A_4+A_1) + \chi(A_5)$ $- 8\chi(A_3+4A_1) - 4\chi(A_3+A_2+2A_1) - 2\chi(A_3+2A_2)$ $- 5\chi(2A_3+A_1) - 6\chi(A_4+3A_1) - 3\chi(A_4+A_2+A_1)$ $- 4\chi(A_4+A_3) - 6\chi(A_5+2A_1) - 3\chi(A_5+A_2)$ $- 5\chi(A_6+A_1) - 4\chi(A_7) + \dots$
	$2\chi(A_4) = 2\chi(A_4+A_1) + 2\chi(A_5)$ $- 4\chi(A_4+3A_1) - 2\chi(A_4+A_2+A_1) - \chi(A_4+A_3)$ $- 4\chi(A_5+2A_1) - 2\chi(A_5+A_2)$ $- 4\chi(A_6+A_1) - 4\chi(A_7) + \dots$

$c=6$	$2\chi(6A_1)=14\chi(7A_1)+2\chi(A_2+5A_1)+\chi(A_3+4A_1)+\dots$
	$2\chi(A_2+4A_1)=10\chi(A_2+5A_1)$ $+4\chi(2A_2+3A_1)+2\chi(A_3+4A_1)$ $+\chi(A_3+A_2+2A_1)+2\chi(A_4+3A_1)+\dots$
	$2\chi(2A_2+2A_1)=6\chi(2A_2+3A_1)+6\chi(3A_2+A_1)$ $+2\chi(A_3+A_2+2A_1)+\chi(A_3+2A_2)$ $+2\chi(A_4+A_2+A_1)+\chi(A_5+2A_1)+\dots$
	$2\chi(3A_2)=2\chi(3A_2+A_1)$ $+2\chi(A_3+2A_2)+\chi(A_5+A_2)+\dots$
	$\chi(A_3+3A_1)=4\chi(A_3+4A_1)+\chi(A_3+A_2+2A_1)$ $+\chi(2A_3+A_1)+\chi(A_4+3A_1)+\chi(A_5+2A_1)+\dots$
	$\chi(A_3+A_2+A_1)=2\chi(A_3+A_2+2A_1)+2\chi(A_3+2A_2)$ $+2\chi(2A_3+A_1)+\chi(A_4+A_2+A_1)$ $+\chi(A_4+A_3)+\chi(A_5+A_2)+\chi(A_6+A_1)+\dots$
	$2\chi(2A_3)=2\chi(2A_3+A_1)+2\chi(A_4+A_3)+\chi(A_7)+\dots$
	$2\chi(A_4+2A_1)=6\chi(A_4+3A_1)+2\chi(A_4+A_2+A_1)$ $+\chi(A_4+A_3)+2\chi(A_5+2A_1)+2\chi(A_6+A_1)+\dots$
	$\chi(A_4+A_2)=\chi(A_4+A_2+A_1)+\chi(A_4+A_3)$ $+\chi(A_5+A_2)+\chi(A_7)+\dots$
	$\chi(A_5+A_1)=2\chi(A_5+2A_1)+\chi(A_5+A_2)$ $+\chi(A_6+A_1)+\chi(A_7)+\dots$
	$\chi(A_6)=\chi(A_6+A_1)+\chi(A_7)+\dots$

The dots in formulas of Table 1 denote universal linear combinations of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension 9.

Table 2 (k is odd)

$c=3$	$2\chi(3A_1) = 8\chi(4A_1) + 2\chi(A_2+2A_1) + \chi(A_3+A_1)$ $- 80\chi(6A_1) - 32\chi(A_2+4A_1) - 12\chi(2A_2+2A_1)$ $- 4\chi(3A_2) - 19\chi(A_3+3A_1) - 7\chi(A_3+A_2+A_1)$ $- 4\chi(2A_3) - 10\chi(A_4+2A_1) - 3\chi(A_4+A_2)$ $- 5\chi(A_5+A_1) - 2\chi(A_6) + \dots$
	$2\chi(A_2+A_1) = 4\chi(A_2+2A_1) + 4\chi(2A_2)$ $+ 2\chi(A_3+A_1) + 2\chi(A_4)$ $- 16\chi(A_2+4A_1) - 16\chi(2A_2+2A_1)$ $- 12\chi(3A_2) - 12\chi(A_3+3A_1) - 11\chi(A_3+A_2+A_1)$ $- 8\chi(2A_3) - 12\chi(A_4+2A_1) - 10\chi(A_4+A_2)$ $- 10\chi(A_5+A_1) - 8\chi(A_6) + \dots$
	$\chi(A_3) = \chi(A_3+A_1) + \chi(A_4)$ $- 2\chi(A_3+3A_1) - \chi(A_3+A_2+A_1) - \chi(2A_3)$ $- 2\chi(A_4+2A_1) - \chi(A_4+A_2)$ $- 2\chi(A_5+A_1) - 2\chi(A_6) + \dots$
$c=5$	$2\chi(5A_1) = 12\chi(6A_1) + 2\chi(A_2+4A_1) + \chi(A_3+3A_1) + \dots$
	$2\chi(A_2+3A_1) = 8\chi(A_2+4A_1) + 4\chi(2A_2+2A_1)$ $+ 2\chi(A_3+3A_1) + \chi(A_3+A_2+A_1)$ $+ 2\chi(A_4+2A_1) + \dots$
	$2\chi(2A_2+A_1) = 4\chi(2A_2+2A_1) + 6\chi(3A_2)$ $+ 2\chi(A_3+A_2+A_1) + 2\chi(A_4+A_2)$ $+ \chi(A_5+A_1) + \dots$
	$\chi(A_3+2A_1) = 3\chi(A_3+3A_1) + \chi(A_3+A_2+A_1)$ $+ \chi(2A_3) + \chi(A_4+2A_1) + \chi(A_5+A_1) + \dots$
	$\chi(A_3+A_2) = \chi(A_3+A_2+A_1) + 2\chi(2A_3)$ $+ \chi(A_4+A_2) + \chi(A_6) + \dots$
	$\chi(A_4+A_1) = 2\chi(A_4+2A_1) + \chi(A_4+A_2)$ $+ \chi(A_5+A_1) + \chi(A_6) + \dots$
	$\chi(A_5) = \chi(A_5+A_1) + \chi(A_6) + \dots$

The dots in formulas of Table 2 denote universal linear combinations of the Euler characteristics of manifolds of singularities of even codimensions starting from the codimension 8.

Table 3 (k is even)

$c=2$	$2\chi(2A_1)=3\chi(3A_1)+\chi(A_3)$ $-5\chi(5A_1)-\chi(A_3+2A_1)-\chi(A_5)$ $+21\chi(7A_1)+5\chi(A_3+4A_1)+\chi(2A_3+A_1)$ $+3\chi(A_5+2A_1)+\chi(A_7)$ $-153\chi(9A_1)-41\chi(A_3+6A_1)-11\chi(2A_3+3A_1)$ $-3\chi(3A_3)-19\chi(A_5+4A_1)-5\chi(A_5+A_3+A_1)$ $-7\chi(A_7+2A_1)-3\chi(A_9)+\dots$
$c=4$	$4\chi(4A_1)=10\chi(5A_1)+2\chi(A_3+2A_1)$ $-35\chi(7A_1)-9\chi(A_3+4A_1)-3\chi(2A_3+A_1)$ $-4\chi(A_5+2A_1)-\chi(A_7)$ $+252\chi(9A_1)+70\chi(A_3+6A_1)+20\chi(2A_3+3A_1)$ $+6\chi(3A_3)+31\chi(A_5+4A_1)+9\chi(A_5+A_3+A_1)$ $+12\chi(A_7+2A_1)+5\chi(A_9)+\dots$
	$\chi(A_3+A_1)=\chi(A_3+2A_1)+\chi(A_5)$ $-\chi(A_3+4A_1)-\chi(A_5+2A_1)-\chi(A_7)$ $+3\chi(A_3+6A_1)+\chi(2A_3+3A_1)+3\chi(A_5+4A_1)$ $+\chi(A_5+A_3+A_1)+2\chi(A_7+2A_1)+\chi(A_9)+\dots$
$c=6$	$4\chi(6A_1)=14\chi(7A_1)+2\chi(A_3+4A_1)$ $-84\chi(9A_1)-20\chi(A_3+6A_1)-5\chi(2A_3+3A_1)$ $-\chi(3A_3)-6\chi(A_5+4A_1)-\chi(A_5+A_3+A_1)$ $-\chi(A_7+2A_1)+\dots$
	$2\chi(A_3+3A_1)=4\chi(A_3+4A_1)$ $+2\chi(2A_3+A_1)+2\chi(A_5+2A_1)$ $-10\chi(A_3+6A_1)-5\chi(2A_3+3A_1)-3\chi(3A_3)$ $-8\chi(A_5+4A_1)-4\chi(A_5+A_3+A_1)$ $-5\chi(A_7+2A_1)-2\chi(A_9)+\dots$
	$4\chi(2A_3)=2\chi(2A_3+A_1)+2\chi(A_7)$ $-\chi(2A_3+3A_1)+3\chi(3A_3)-\chi(A_5+A_3+A_1)$ $-\chi(A_7+2A_1)-2\chi(A_9)+\dots$
	$\chi(A_5+A_1)=\chi(A_5+2A_1)+\chi(A_7)$ $-\chi(A_5+4A_1)-\chi(A_7+2A_1)-\chi(A_9)+\dots$
$c=8$	$2\chi(8A_1)=9\chi(9A_1)+\chi(A_3+6A_1)+\dots$
	$\chi(A_3+5A_1)=3\chi(A_3+6A_1)$ $+\chi(2A_3+3A_1)+\chi(A_5+4A_1)+\dots$
	$2\chi(2A_3+2A_1)=3\chi(2A_3+3A_1)+3\chi(3A_3)+$ $+2\chi(A_5+A_3+A_1)+\chi(A_7+2A_1)+\dots$
	$2\chi(A_5+3A_1)=4\chi(A_5+4A_1)$ $+\chi(A_5+A_3+A_1)+2\chi(A_7+2A_1)+\dots$
	$2\chi(A_5+A_3)=\chi(A_5+A_3+A_1)+2\chi(A_9)+\dots$
	$\chi(A_7+A_1)=\chi(A_7+2A_1)+\chi(A_9)+\dots$

The dots in formulas of Table 3 denote universal linear combinations of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension 11.

Table 4 (k is odd)

$c=3$	$4\chi(3A_1) = 8\chi(4A_1) + 2\chi(A_3 + A_1)$ $- 20\chi(6A_1) - 5\chi(A_3 + 3A_1) - 2\chi(2A_3) - 3\chi(A_5 + A_1)$ $+ 112\chi(8A_1) + 30\chi(A_3 + 5A_1) + 8\chi(2A_3 + 2A_1)$ $+ 15\chi(A_5 + 3A_1) + 4\chi(A_5 + A_3) + 6\chi(A_7 + A_1) + \dots$
	$4\chi(A_3) = 2\chi(A_3 + A_1) - \chi(A_3 + 3A_1)$ $+ 2\chi(2A_3) - \chi(A_5 + A_1)$ $+ 2\chi(A_3 + 5A_1) + 2\chi(A_5 + 3A_1)$ $+ \chi(A_5 + A_3) + \chi(A_7 + A_1) + \dots$
$c=5$	$4\chi(5A_1) = 12\chi(6A_1) + 2\chi(A_3 + 3A_1)$ $- 56\chi(8A_1) - 14\chi(A_3 + 5A_1) - 4\chi(2A_3 + 2A_1)$ $- 5\chi(A_5 + 3A_1) - \chi(A_5 + A_3) - \chi(A_7 + A_1) + \dots$
	$4\chi(A_3 + 2A_1) = 6\chi(A_3 + 3A_1) + 4\chi(2A_3) + 4\chi(A_5 + A_1)$ $- 10\chi(A_3 + 5A_1) - 4\chi(2A_3 + 2A_1) - 9\chi(A_5 + 3A_1)$ $- 5\chi(A_5 + A_3) - 7\chi(A_7 + A_1) + \dots$
	$4\chi(A_5) = 2\chi(A_5 + A_1) - \chi(A_5 + 3A_1)$ $+ \chi(A_5 + A_3) - \chi(A_7 + A_1) + \dots$
$c=7$	$2\chi(7A_1) = 8\chi(8A_1) + \chi(A_3 + 5A_1) + \dots$
	$2\chi(A_3 + 4A_1) = 5\chi(A_3 + 5A_1)$ $+ 2\chi(2A_3 + 2A_1) + 2\chi(A_5 + 3A_1) + \dots$
	$2\chi(2A_3 + A_1) = 2\chi(2A_3 + 2A_1)$ $+ 2\chi(A_5 + A_3) + \chi(A_7 + A_1) + \dots$
	$2\chi(A_5 + 2A_1) = 3\chi(A_5 + 3A_1)$ $+ \chi(A_5 + A_3) + 2\chi(A_7 + A_1) + \dots$
	$2\chi(A_7) = \chi(A_7 + A_1) + \dots$

The dots in formulas of Table 4 denote universal linear combinations of the Euler characteristics of manifolds of singularities of odd codimensions starting from the codimension 10.

Table 5

$k=2$	$\chi(A_3)-\chi(3A_1)=2\chi_0$
$k=4$	$2\chi(A_5)-\chi(A_3+2A_1)+\chi(5A_1)=6\chi_0$
$k=6$	$5\chi(A_7)-2\chi(A_5+2A_1)-\chi(2A_3+A_1)$ $+\chi(A_3+4A_1)-\chi(7A_1)=20\chi_0$
$k=8$	$14\chi(A_9)-5\chi(A_7+2A_1)-2\chi(A_5+A_3+A_1)$ $+2\chi(A_5+4A_1)-\chi(3A_3)+\chi(2A_3+3A_1)$ $-\chi(A_3+6A_1)+\chi(9A_1)=70\chi_0$
$k=10$	$42\chi(A_{11})-14\chi(A_9+2A_1)-5\chi(A_7+A_3+A_1)$ $+5\chi(A_7+4A_1)-4\chi(2A_5+A_1)-2\chi(A_5+2A_3)$ $+2\chi(A_5+A_3+3A_1)-2\chi(A_5+6A_1)$ $+\chi(3A_3+2A_1)-\chi(2A_3+5A_1)$ $+\chi(A_3+8A_1)-\chi(11A_1)=252\chi_0$
$k=12$	$132\chi(A_{13})-42\chi(A_{11}+2A_1)-14\chi(A_9+A_3+A_1)$ $+14\chi(A_9+4A_1)-10\chi(A_7+A_5+A_1)-5\chi(A_7+2A_3)$ $+5\chi(A_7+A_3+3A_1)-5\chi(A_7+6A_1)-4\chi(2A_5+A_3)$ $+4\chi(2A_5+3A_1)+2\chi(A_5+2A_3+2A_1)$ $-2\chi(A_5+A_3+5A_1)+2\chi(A_5+8A_1)$ $+\chi(4A_3+A_1)-\chi(3A_3+4A_1)+\chi(2A_3+7A_1)$ $-\chi(2A_3+10A_1)+\chi(13A_1)=924\chi_0$

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An infinitesimal criterion for topological triviality of families of sections of analytic varieties

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Abstract.

We present sufficient conditions for the topological triviality of families of germs of functions defined on an analytic variety V. The main result is an infinitesimal criterion using the integral closure of a convenient ideal as the tangent space to a subset of the set of topologically trivial deformations of a given germ. Applications to the problem of equisingularity of families of sections of V are also discussed.

§1. Introduction

Let $V, 0$ be the germ of an analytic subvariety of k^n ($k = \mathbb{R}$ or $\mathbb{C}$) and let $\mathcal{R}_V$ (respectively C^0-$\mathcal{R}_V$) be the group of germs of diffeomorphisms (respectively homeomorphisms) preserving $V, 0$. In this paper we introduce a sufficient condition for the C^0-$\mathcal{R}_V$- triviality of families of map germs $h : k^n \times k, 0 \to k^p, 0$, based on the integral closure of $T\mathcal{R}_V(h)$, the tangent space to the orbit of h under the action of the group $\mathcal{R}_V$. Our main result establishes that if $\frac{\partial h}{\partial t} \in \overline{T\mathcal{R}_V(h)}$, then h is topologically $\mathcal{R}_V$-trivial.

We are specially concerned with the case $p = 1$, that is, with families $h : k^n \times k, 0 \to k, 0$. In this case $h^{-1}(0)$ defines a family of sections of the analytic variety $V, 0$.

As a corollary of the method, we obtain sharp results when the analytic variety is weighted homogeneous and the family of sections is a deformation of a weighted homogeneous map germ h_0 (consistent with V) by terms of filtration higher than or equal to the filtration of h_0.

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This result was previously proved by Damon in [8]. In the final section, we introduce a notion of V-equisingularity of the family of sections and we show that the hypothesis of the main theorem implies this geometric condition. A weighted approach for the topological triviality of families of sections of analytic varieties was presented in [16]. For other results related to the subject discussed in this paper, see for instance [2], [8], [19].

§2. Basic results

Let $\mathcal{O}_n$ be the ring of germs of analytic functions $h : k^n, 0 \to k$, $k = \mathbb{R}$ or $\mathbb{C}$.

A germ of a subset $V, 0 \subset k^n, 0$ is the germ of an analytic variety if there exist germs of analytic functions $f_1, ..., f_r$ such that $V = \{x : f_1(x) = \cdots = f_r(x) = 0\}$.

Our aim is to study map germs $h : k^n, 0 \to k^p, 0$ under the equivalence relation that preserves the analytic variety $V, 0$. We say that two germs h_1 and $h_2 : k^n, 0 \to k^p, 0$ are $\mathcal{R}_V$-equivalent (respectively C^0-$\mathcal{R}_V$-equivalent) if there exists a germ of a diffeomorphism (respectively homeomorphism) $\phi : k^n, 0 \to k^n, 0$ with $\phi(V) = V$ and $h_1 \circ \phi = h_2$. That is,

$$\mathcal{R}_V = \{\phi \in \mathcal{R} : \phi(V) = V\},$$

where $\mathcal{R}$ is the group of germs of diffeomorphisms of $k^n, 0$.

A one parameter deformation $h : k^n \times k, 0 \to k^p, 0$ of $h_0 : k^n, 0 \to k^p, 0$ is topologically $\mathcal{R}_V$-trivial (or C^0-$\mathcal{R}_V$-trivial) if there exists homeomorphism $H : k^n \times k, 0 \to k^n \times k, 0$, $H(x,t) = (\bar{h}(x,t), t)$, such that $h \circ H(x,t) = h_0(x)$ and $H(V \times k) = V \times k$.

We denote by θ_n the set of germs of tangent vector fields in $k^n, 0$; θ_n is a free $\mathcal{O}_n$ module of rank n. Let $I(V)$ be the ideal in $\mathcal{O}_n$ consisting of germs of analytic functions vanishing on V. We denote by $\Theta_V = \{\eta \in \theta_n : \eta(I(V)) \subseteq I(V)\}$, the submodule of germs of vector fields tangent to V (see [2] for more details).

The tangent space to the action of the group $\mathcal{R}_V$ is $T\mathcal{R}_V(h) = dh(\Theta_V^0)$, where Θ_V^0 is the submodule of Θ_V given by the vector fields that are zero at zero.

The group $\mathcal{R}_V$ is a geometric subgroup of the contact group, as defined by J.Damon [5], [6], hence the infinitesimal criterion for $\mathcal{R}_V$-determinacy holds (see [2] for a proof).

Theorem 2.1. *The germ h is $\mathcal{R}_V$-finitely determined if and only if there exists a positive integer k such that $T\mathcal{R}_V(h) \supset \mathcal{M}_n^k$.*

The following theorem is the geometric criterion for the $\mathcal{R}_V$-finite determinacy.

Theorem 2.2. ([2]) *Let $V, 0 \subseteq \mathbb{C}^n, 0$ be the germ of an analytic variety and let $h : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ be the germ of an analytic function. Let*

$$V(h) = \{x \in \mathbb{C}^n : \xi h(x) = 0 \text{ for all } \xi \in \Theta_V\}.$$

Then h is $\mathcal{R}_V$-finitely determined if and only if $V(h) = \{0\}$ or $\emptyset$.

As a consequence of this result, it follows that if h is $\mathcal{R}_V$-finitely determined, then $h^{-1}(c)$ is transverse to V away from 0, for sufficiently small values of c.

In the real case, the necessary condition remains true, that is, if h is $\mathcal{R}_V$-finitely determined then the set $\{x \in \mathbb{R}^n : \xi h(x) = 0 \text{ for all } \xi \in \Theta_V\}$ is $\{0\}$ or $\emptyset$.

§3. Basic facts on integral closure of ideals

Let I be an ideal in a ring A. An element $h \in A$ is said to be integral over I if it satisfies an integral dependence relation $h^n + a_1 h^{n-1} + ... + a_n = 0$ with $a_i \in I^i$. The set of such elements form an ideal in A, called the integral closure of I.

When $A = \mathcal{O}_{X,x_0}$, the local ring of a complex analytic set, Teissier gives in [18] various notions equivalent to the above concept.

Theorem 3.1. ([11], Proposition 1.2) *Let I be an ideal in $\mathcal{O}_{X,x_0}$ and $\overline{I}$ its integral closure, where X is a complex analytic space. The following statements are equivalent:*
(a) $h \in \overline{I}$.
(b) *For each choice of generators $\{g_i\}$ of I there exist a neighbourhood U of x_0 and a constant $C > 0$ such that for all $x \in U$:*

$$|h(x)| \leq C \sup_i |g_i(x)|.$$

(c) *For each analytic curve $\varphi : \mathbb{C}, 0 \to X, x_0$, $h \circ \varphi$ lies in $(\varphi^*(I))\mathcal{O}_1$.*
(d) *There exists a faithful $\mathcal{O}_{X,x_0}$ module L of finite type such that $h.L \subset I.L$.*

In the real case, the above algebraic definition of integral closure is not appropriate. But, one can use condition (c) above as a definition. More precisely,

Definition 3.2. *Let I be an ideal of the ring $\mathcal{O}_{X,x_0}$, where X is a real analytic set. The real integral closure $\overline{I}$ of I is the set of h such that for all analytic $\varphi : \mathbb{R}, 0 \to X, x_0$, we have $h \circ \varphi \in (\varphi^*(I))\mathcal{O}_1$.*

Gaffney ([11], p. 30) shows that $h \in \overline{I}$ if and only if for each choice of generators $\{g_i\}$ of I there exists a neighbourhood U of x_0 and a constant $C > 0$ such that for all $x \in U$:

$$|h(x)| \leq C\text{sup}_i|g_i(x)|.$$

§4. The main result

Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ of an analytic function and let $h : k^n \times k, 0 \to k, 0$ be an analytic deformation of h_0. In the sequel, we shall assume $h(0, t) = 0$. The property of being $\mathcal{R}_V$-finitely determined is open in the sense that the germ $\{x \in k^n : dh_t\xi(x) = 0, \forall\, \xi \in \Theta_V\}$ at 0 is $\{0\}$ or empty for sufficiently small values of the parameters ([2]). However, this does not guarantee the existence of a neighbourhood U of 0 in $k^n, 0$ and an open ε-ball, B_ε, centered at the origin in k such that the above condition holds $\forall\, x \in U$ and $\forall\, t \in B_\varepsilon$. We then need the following definition:

Definition 4.1. *Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ. We say that a deformation $h : k^n \times k, 0 \to k, 0$ of h_0 is a* good deformation *if $V(h) \subseteq \{0\} \times k, 0$, where $V(h) = \{(x, t) \in k^n \times k, 0\,;\, dh_t(x)\xi(x) = 0\, \forall\, \xi \in \Theta_V\}$.*

Example 4.2. *Let V be the x-axis in k^2; Θ_V is generated by $(1, 0)$ and $(0, y)$. The germ $h_0(x, y) = x^2 + y^3$ is $\mathcal{R}_V$-finitely determined. The deformation $h_t(x, y) = x^2 + y^3 + ty^2$ of h_0 has the property that h_t is $\mathcal{R}_V$-finitely determined for each fixed t, but we cannot find $\varepsilon > 0$ such that the above condition holds for all $t \in B_\varepsilon$.*

Our main result is the following theorem:

Theorem 4.3. *Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ and let $h : k^n \times k, 0 \to k, 0$ be a good deformation of h_0. If $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ for all $t \in k$ sufficiently near 0, then h is C^0-$\mathcal{R}_V$-trivial.*

The proof of the theorem is a consequence of the following results.

In what follows we can assume that $dh_t\xi(0) = 0, \forall\, \xi \in \Theta_V$. In fact, if $\xi \in \Theta_V$, then $dh_t\xi.\frac{\partial h}{\partial t} = dh_t(\frac{\partial h}{\partial t}.\xi)$. If $dh_t\xi_0(0) \neq 0$ for some ξ_0, then

$$\frac{\partial h}{\partial t} = dh_t\Big(\frac{\frac{\partial h}{\partial t}.\xi_0}{dh_t\xi_0}\Big)$$

and hence the deformation is C^ω-$\mathcal{R}_V$-trivial (i.e. analytically trivial). Observe that $\frac{\frac{\partial h}{\partial t}.\xi_0}{dh_t\xi_0} \in \Theta_V^0$.

Lemma 4.4. *Let I and J be ideals in $\mathcal{O}_n$ with $\mathcal{M}_n I \subseteq J \subseteq I$ and $V(I) = \{0\}$, where $V(I)$ is the variety of the ideal I. Then $V(J) = \{0\}$.*

Proof. From the hypothesis, $V(\mathcal{M}_n I) \supseteq V(J) \supseteq V(I)$. Since $V(M_n I) = V(\mathcal{M}_n) \cup V(I) = \{0\} \cup \{0\}$, we get $V(J) = \{0\}$. Q.E.D.

Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ and let $h : k^n \times k, 0 \to k, 0$ be a good deformation of h_0. Let $\{\xi_1, ..., \xi_r\}$ be generators of Θ_V and $I = \langle dh_t \xi_1, ..., dh_t \xi_r \rangle$ the ideal in $\mathcal{O}_{n+1}$ then $V(I) \subseteq \{0\} \times k$, since h is a good deformation of h_0. Let $\{\alpha_1, ..., \alpha_m\}$ be the generators of Θ_V^0, $dh_t \alpha_i = \rho_i$ and $J = \langle \rho_1, ..., \rho_m \rangle$. Since the α_i and hence the ρ_i vanish on $\{0\} \times k$, it follows that $V(J) \supseteq \{0\} \times k$. On the other hand, $M_n I \subset J \subset I$, and it follows from Lemma 4.4, that $V(J) \subseteq \{0\} \times k$.

Let $\rho(x,t) = \sum_{i=1}^m |\rho_i|^2$. The condition $V(J) = \{0\} \times k$ implies that $\rho \geq 0$, and $\rho_t(x) = 0$ is equivalent to $x = 0$. Then, the following result holds.

Lemma 4.5. *Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ and let $h : k^n \times k, 0 \to k, 0$ be a good deformation of h_0. If $\rho(x,t) = \sum_{i=1}^m |dh_t \alpha_i|^2$, then $V(\rho(x,t)) = \{0\} \times k$.*

Lemma 4.6. *Let $h : k^n \times k, 0 \to k, 0$ be a deformation of h_0. Suppose there is a continuous vector field $(W, 1) \in \Theta_{V \times k}$ such that:*
(i) $\rho \frac{\partial h}{\partial t} = dh_t(W)$, where ρ is a control function, that is, $\rho : k^n \times k, 0 \to \mathbb{R}$ with $\rho(x,t) \geq 0$ and $\rho(x,t) = 0$ if and only if $x = 0$.
(ii) $(-\frac{W}{\rho}, 1)$ is locally integrable.
Then h is topologically $\mathcal{R}_V$-trivial.

Proof. Let $\phi(x,t,\tau)$ be the flow of the on $k^n \times k, 0$ defined by $(-\frac{W}{\rho}, 1)$, so $\frac{\partial \phi}{\partial \tau} = (-\frac{W}{\rho}, 1) \circ \phi$, $\phi(x,t,0) = (x,t)$. When $k = \mathbb{R}$, we define

$$\varphi(x,t) = \phi(x,0,t) = (\overline{\varphi}(x,t), t).$$

Taking the derivative of $h(\varphi(x,t)) = h(\overline{\varphi}(x,t), t)$ with respect to t, we get

$$\begin{aligned}
\frac{\partial}{\partial t}(h(\varphi(x,t))) &= \sum_{i=1}^n \frac{\partial h}{\partial x_i}(\overline{\varphi}(x,t), t) \frac{\partial \overline{\varphi_i}}{\partial t}(x,t) + \frac{\partial h}{\partial t}(\overline{\varphi}(x,t), t) \\
&= - \sum_{i=1}^n \frac{\partial h}{\partial x_i}(\overline{\varphi}(x,t), t) \frac{W_i}{\rho}(\overline{\varphi}(x,t), t) + \frac{\partial h}{\partial t}(\overline{\varphi}(x,t), t) \\
&= (\frac{\partial h}{\partial t} - \sum_{i=1}^n \frac{W_i}{\rho} \frac{\partial h}{\partial x_i})(\overline{\varphi}(x,t), t) = 0
\end{aligned}$$

where W_i are the components of W. Hence, fixing x, it follows that $h(\varphi(x,t))$ is constant, that is, $h(\varphi(x,t)) = h(\varphi(x,0)) = h(x,0) = h_0(x)$

for all t and x. Therefore h is topologically $\mathcal{R}_V$-trivial. When $k = \mathbb{C}$, we consider the restriction

$$h^1 = h|\mathbb{C}^n \times \mathbb{R} \times \{0\} \to \mathbb{C}.$$

It is sufficient to show that h is a $\mathcal{R}_V$-topologically trivial deformation of h^1, which in turn is a $\mathcal{R}_V$-topologically trivial deformation of h_0.

Let $\phi(x,t,\tau)$ be such that $\frac{\partial \phi}{\partial \tau} = (-\frac{W}{\rho}, 1) \circ \phi$ and $\phi(x,t,0) = (x,t)$. We consider $\phi_1(x, u+iv) = \phi(x,u,v)$ and $\phi_2(x,u) = \phi(x,0,u)$. It follows that $h \circ \phi_1$ is constant with respect to v and hence $h(\phi_1(x,u+iv)) = h(\phi_1(x,u)) = h(\phi(x,u,0)) = h(x,u) = h^1(x,u)$. One can also show that $h^1 \circ \phi_2$ is constant with respect to u, therefore $h^1(\phi_2(x,u)) = h^1(\phi_2(x,0)) = h^1(x,0) = h_0$ and the result follows. Q.E.D.

Proof of the Theorem 4.3. With the above notations, it follows that

$$|\rho_i|^2 \frac{\partial h}{\partial t} = dh_t(\overline{\rho_i} \frac{\partial h}{\partial t} \alpha_i).$$

Since $\rho = \sum_{i=1}^m |\rho_i|^2$, it follows that

$$\rho \frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} (\overline{\rho_1}\alpha_1 + \dots + \overline{\rho_m}\alpha_m) \right)$$

hence

$$\frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} \frac{1}{\rho} (\overline{\rho_1}\alpha_1 + \dots + \overline{\rho_m}\alpha_m) \right).$$

From Lemma 4.5, $V(\rho(x,t)) = \{0\} \times k$. We define the vector field X in $k^n \times k, 0$,

$$X(x,t) = \begin{cases} \left(-\frac{\partial h}{\partial t} \frac{1}{\rho} (\overline{\rho_1}\alpha_1 + \dots + \overline{\rho_m}\alpha_m), 1 \right) & \text{if } x \neq 0 \\ (0,1) & \text{if } x = 0 \end{cases}$$

The vector field $X(x,t)$ is real analytic away from $\{0\} \times k$.

From the hypothesis, $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ and hence by item (b) of Theorem 3.1

$$|\frac{\partial h}{\partial t}| \leq c \sup\{|\rho_i|\}.$$

Then

$$\begin{aligned}|X(x,t)-X(0,t)| &= |\tfrac{\partial h}{\partial t}\tfrac{1}{\rho}(\overline{\rho_1}\alpha_1+...+\overline{\rho_m}\alpha_m)| \\ &\leq |\tfrac{\partial h}{\partial t}|\tfrac{1}{\rho}(|\overline{\rho_1}||\alpha_1|+...+|\overline{\rho_m}||\alpha_m|) \\ &\leq c\sup\{|\rho_i|\}\tfrac{1}{\rho}(|\rho_1||\alpha_1|+...+|\rho_m||\alpha_m|) \\ &\leq c(|\alpha_1|+...+|\alpha_m|)\leq C|x|.\end{aligned}$$

Thus, X satisfies the Lipschitz condition around the solution $(0,t)$, and it follows from [4] or [13] that $X(x,t)$ is locally integrable in a neighbourhood of $(0,0)\in k^n\times k$. Then, there exists a family of homeomorphisms $\phi(x,t,\tau)$, $\phi: k^n\times k\times\mathbb{R},0\to k^n\times k,0$ such that $\frac{\partial\phi}{\partial\tau}=-X\circ\phi$ and $\phi(x,t,0)=(x,t)$. The proof follows now from Lemma 4.6 (see Lemma 6.2, in [9]). Q.E.D.

§5. Weighted homogeneous germs and varieties

Definition 5.1. *(a) Given $(w_1,...,w_n:d_1,...,d_p)$, w_i, $d_j\in\mathbb{Q}^+$, a map germ $f:k^n,0\to k^p,0$ is weighted homogeneous of type $(w_1,...,w_n:d_1,...,d_p)$ if for all $\lambda\in k-\{0\}$:*

$$f(\lambda^{w_1}x_1,\lambda^{w_2}x_2,...,\lambda^{w_n}x_n)=(\lambda^{d_1}f_1(x),\lambda^{d_2}f_2(x),...,\lambda^{d_p}f_p(x)).$$

In this case, the value w_i is called weight of the variable x_i and the value d_i, is the filtration of f_i with respect to the weights $(w_1,...,w_n)$. We write: weight$(x_i)=w(x_i)=w_i$ and filtration$(f)=$ fil$(f)=(d_1,...,d_p)$.

(b) Given $(w_1,...,w_n)$, and any monomial $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}$, we define fil$(x^\alpha)=\sum_{i=1}^n\alpha_i w_i$.

(c) We define a filtration in the ring $\mathcal{O}_n$ via the function defined by fil$(f)=\inf_{|\alpha|}\{$fil$(x^\alpha):\frac{\partial^{|\alpha|}f}{\partial x^\alpha}(0)\neq 0\}$, $|\alpha|=\alpha_1+\ldots+\alpha_n$.

Definition 5.2. *A germ of an analytic variety $V,0\subseteq k^n,0$ is weighted homogeneous if it is defined by a weighted homogeneous map germ $f:k^n,0\to k^p,0$.*

Definition 5.3. *Let $V,0\subseteq k^n,0$ be the germ of a weighted homogeneous analytic variety. We say that a set $\{\alpha_1,...,\alpha_r\}$ of generators of Θ_V is weighted homogeneous of type $(w_1,...,w_n:d_1,...,d_r)$ if α_{ij} are weighted homogeneous polynomials of type $(w_1,...,w_n:d_i+w_j)$ whenever $\alpha_{ij}\neq 0$, where $\alpha_i=\sum_{j=1}^n\alpha_{ij}\frac{\partial}{\partial x_j}$, $i=1...r$.*

When V is a weighted homogeneous variety, we always can choose weighted homogeneous generators for Θ_V. A proof can be found in [10].

Following [7], we define:

Definition 5.4. *Let V be defined by weighted homogeneous polynomials. We say that h is weighted homogeneous consistent with V if h is weighted homogeneous with respect to the same set of weights assigned to V.*

Example 5.5. *Let $V = \phi^{-1}(0) \subset k^3$ where $\phi(x,y,z) = z^2 - x^2y$. We have that ϕ is weighted homogeneous of type $(1,2,2:4)$. Let $h(x,y,z) = x^3 + xy + xz$ and $f(x,y,z) = x^3 + xy + z^2$. Then h is consistent with V, f is weighted homogeneous but not consistent with V.*

The following result does not follow as a corollary of the Theorem 4.3, but the proof is similar. It was previously proved by J. Damon in [8], but we include it here for completeness. In [16], we discuss a weighted approach for the topological triviality of families of sections of analytic varieties, which also gives Theorem 5.6 as a corollary.

Theorem 5.6. *Let V be a weighted homogeneous subvariety of $k^n,0$ and let $h_0 : k^n,0 \to k,0$ be weighted homogeneous consistent with V and $\mathcal{R}_V$-finitely determined. Then any deformation h of h_0 by terms of filtration greater than or equal to the filtration of h_0, is C^0-$\mathcal{R}_V$-trivial.*

Proof. Under the above conditions, any such h is a good deformation of h_0 (see [15]).

We have $dh_0(\alpha_i)$ is weighted homogeneous, where $\{\alpha_1, ..., \alpha_m\}$ is a set of weighted homogeneous generators of Θ_V. Let r_i be the filtration of $dh_0(\alpha_i)$, $i = 1, ..., m$ and

$$\omega_0(x) = |dh_0(\alpha_1)(x)|^{2s_1} + ... + |dh_0(\alpha_m)(x)|^{2s_r}$$

with $s_i = k/r_i$, and $k = \text{l.c.m.}\{r_i\}$. Let $\rho_i = dh_t(\alpha_i)$ and $\omega = \sum_{i=1}^m |\rho_i|^{2s_i}$. Since

$$|\rho_i|^2 \frac{\partial h}{\partial t} = dh_t(\overline{\rho_i} \frac{\partial h}{\partial t} \alpha_i),$$

it follows that

$$\omega \frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} (\overline{\rho_1} |\rho_1|^{2s_1 - 2} \alpha_1 + ... + \overline{\rho_m} |\rho_m|^{2s_m - 2} \alpha_m) \right).$$

Then

$$\frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} \frac{1}{\omega} (\overline{\rho_1} |\rho_1|^{2s_1 - 2} \alpha_1 + ... + \overline{\rho_m} |\rho_m|^{2s_m - 2} \alpha_m) \right).$$

The proof now follows analogously to the proof of Theorem 4.3. Q.E.D.

Example 5.7. *Let $V, 0 \subset \mathbb{R}^3, 0$ (or $\mathbb{C}^3, 0$) be defined by $\varphi(x,y,z) = 2x^{k+1}y^2 + y^3 - z^2 + x^{2(k+1)}y = 0$. This is the implicit equation for the S_k-singularities classified by D. Mond* [14]. *The function-germ φ is weighted homogeneous of weights 2, $2k+2$ and $3k+3$ for x, y and z respectively. We have that $h(x,y,z) = y + a_{k+1}x^{k+1}$ is $\mathcal{R}_V$-finitely determined for $a_{k+1} \neq 0, 1$ and consistent with V. Therefore deformations of h by terms of order higher than or equal to fil(h) are C^0-$\mathcal{R}_V$-trivial. For k odd, $h_1(x,y,z) = z + ax^{3(k+1)/2}$ and $h_2(x,y,z) = z + bx^{(k+1)/2}y$ are consistent with V and $\mathcal{R}_V$-finite for all $a^2 \neq -4/27$ and $b \neq \pm 2$. Thus deformations of h_1 and h_2, respectively by terms of order higher than or equal to fil(h_1) and fil(h_2) are C^0-$\mathcal{R}_V$-trivial.*

§6. V-Equisingularity

Bernard Teissier developed in [18] an infinitesimal theory and a theory of geometrical invariants to study the equisingularity of families of complex analytic hypersurfaces X_t^d with isolated singularities. The integral closure of an ideal I is the right object to the infinitesimal part of that theory. T. Gaffney in [11] extended Teissier results, using the integral closure of a convenient module to obtain necessary and sufficient conditions for the equisingularity of families of complete intersections with isolated singularities.

Definition 6.1. *Suppose (X,x) is a complex analytic germ, $\mathcal{O}_{X,x}$ its local ring and M a submodule of $\mathcal{O}_{X,x}^p$. Then an element $h \in \mathcal{O}_{X,x}^p$ is in $\overline{M}$ if and only if for all $\phi : \mathbb{C}, 0 \to X, x$, $h \circ \phi$ is in $(\phi^*(M))\mathcal{O}_1$.*

Theorem 6.2. ([11], Theorem 2.5) *Let $F : \mathbb{C}^t \times \mathbb{C}^N \to \mathbb{C}^p, 0$, defining $X = F^{-1}(0)$ with reduced structure, $Y = \mathbb{C}^t \times 0$ and X_0 the smooth part of X. Then $\frac{\partial F}{\partial s} \in \overline{\left\langle z_i \frac{\partial F}{\partial z_j} \right\rangle}_{\mathcal{O}_X}$ for all tangent vectors $\frac{\partial}{\partial s}$ to $\mathbb{C}^t \times 0$ iff (X_0, Y) are Whitney regular.*

Our purpose in this section is to show that the infinitesimal condition in Theorem 4.3 gives a sufficient condition for equisingularity of families of sections of analytic varieties. We also show with an example that it is not a necessary condition.

Let $V \subset \mathbb{C}^n$ be an analytic variety. The family of sections of V is defined by $h(x,t) = 0$, where $h : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}, 0$, $h(0,t) = 0$, is a good deformation of a $\mathcal{R}_V$-finitely determined map germ $h_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$.

In order to define the notion of V-equisingularity, we will construct a stratified diagram of mappings which satisfies Thom's second isotopy lemma.

From now on, we assume that V admits a Whitney stratification $\mathcal{S}_V$ in a neighbourhood U of the origin, for which $\{0\}$ is a stratum. We can also extend this stratification to the neighbourhood U of the origin in a natural way, that is, the strata are the strata of $\mathcal{S}_V$ and the complement of V in U. We denote by $\tilde{V}$ the subvariety of $\mathbb{C}^n \times \mathbb{C}, 0$ defined by $\tilde{V} = V \times \mathbb{C}$. The product stratification is clearly Whitney regular. Since the germ $h : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}, 0$ is a good deformation, we can choose a representative, which we also denote by h, given by $h : U \times B_r, 0 \to \mathbb{C}, 0$, where B_r is an open ball in $\mathbb{C}$ centered at the origin with the property that $h^{-1}(0)$ is transversal to the strata of $\tilde{V}$ away from $0 \times B_r$.

We refine the stratification $\tilde{\mathcal{S}}$ of $U \times B_r$ as follows. Given a stratum S of $\mathcal{S}$, we define the new strata $\tilde{S}$ of $\tilde{\mathcal{S}}$ as one of the following types: $(S \times B_r) - h^{-1}(0)$ and $(S \times B_r) \cap h^{-1}(0)$. This refinement defines a new stratification $U \times B_r$, since h is transversal to $\tilde{V}$ away from zero. We denote this new stratification by the same notation $\tilde{\mathcal{S}}$.

Definition 6.3. *With the above notation, h is V-equisingular if there exists $\varepsilon > 0$ such that:*
(1) $(B_\varepsilon \times B_r, \tilde{\mathcal{S}})$ is Whitney regular;
(2) $B_\varepsilon \times B_r \xrightarrow{F} \mathbb{C} \times B_r \xrightarrow{\pi} B_r$ satisfies the second isotopy lemma, where B_ε is the closed ball in $\mathbb{C}^n$ with radius ε, B_r is the closed ball in $\mathbb{C}$ of radius r, and $F : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C} \times \mathbb{C}, 0$ is given by $F(x,t) = (h(x,t), t)$.

In the following theorem we show that $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ is a sufficient condition for V-equisingularity.

Theorem 6.4. *Let $V = \phi^{-1}(0)$, $\phi : \mathbb{C}^n, 0 \to \mathbb{C}, 0$, $h_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ $\mathcal{R}_V$-finitely determined and $h : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}, 0$ a good deformation of h_0. Let $h^{-1}(0) \cap \Sigma_\phi = \{0\} \times \mathbb{C}$, where Σ_ϕ is the singular set of ϕ. If $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$, then h is V-equisingular.*

J.W. Bruce in [1] considers an analogous question. He describes the topological type of generic families of sections of a semialgebraic stratification $\mathcal{T}$ of a neighbourhood of the origin in $\mathbb{R}^n$, with 0 being a stratum. Such families are *generalised transverse* (G.T) with respect to the stratification, that is, for every pair of strata S_1 and S_2, and a sequence of points $(x_i) \in S_1$ such that $\lim_{i\to\infty} x_i = x \in S_2$ and the limit of the tangent spaces $\lim_{i\to\infty} T_{x_i}S_1 = T$ then $dh(x) : T \to \mathbb{R}$ has maximal rank, that is, $h^{-1}(h(x))$ is transversal to T.

The following theorem is proved in [1]:

Theorem 6.5. ([1], Proposition 1.4) *Let $\mathcal{T}$ a Whitney stratification of an open neighbourhood U of the origin in $\mathbb{R}^n$, with 0 being a stratum. Let $h : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ be a family of submersions, with $h(0,t) = 0$ and $h_t(x) = h(x,t)$. If the family h is generalised transverse with respect to $\mathcal{T}$, for all $t \in [0,1]$, then there exists a germ of homeomorphism $G : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ preserving the strata of $\mathcal{T}$ such that $h_0 \circ G = h_1$.*

Good examples of families satisfying the G.T. condition are the families of sections of an analytic variety defined by generic families of hyperplanes in $\mathbb{R}^n$. In this work, we substitute the G.T. condition by the finite determinacy of h_0 and the integral closure condition. Under these hypothesis we are able to obtain the topological triviality of families that do not satisfy the G.T. condition.

Example 6.6. *Let $V, 0 \subseteq \mathbb{C}^3, 0$ be the swallowtail parametrized by $(x, -4y^3 - 2xy, -3y^4 - xy^2)$. The module Θ_V is generated by $\eta_1 = (2x, 3y, 4z)$, $\eta_2 = (6y, -2x^2 - 8z, xy)$ and $\eta_3 = (-4x^2 - 16z, -8xy, y^2)$. The $\mathcal{R}_V$ classification of germs $h : \mathbb{C}^3, 0 \to \mathbb{C}, 0$ given by Theorem 4.10 in* [3]*, gives the normal form $z + ax^n + tx^{n+1}$, $n \geq 2$ which is finitely determined for $a \neq 0, n \neq 2$, and $a \neq 0, a \neq 1/12, n = 2$. Let $h_0(x,y,z) = z + ax^n$ from Theorem 4.3 we have that the family $h_t(x,y,z) = z + ax^n + tx^{n+1}$ is topologically $\mathcal{R}_V$-trivial. However this family h_t is not G.T. at 0, since $dh_t(0,0,0) = (0,0,1)$ and the limit of tangent planes to the smooth part of V is the xy-plane.*

To prove Theorem 6.4, we first prove the following Lemma.

Lemma 6.7. *Let $\phi : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ and $V = \phi^{-1}(0)$. Given $h : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$, define $G : \mathbb{C}^{n+1}, 0 \to \mathbb{C}^2, 0$, by $G(x,t) = (h(x,t), \phi(x))$. If $g \in \overline{dh_t(\Theta_V^0)}_{\mathcal{O}_{n+1}}$ then $(g, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{G^{-1}(0)}}$.*

Proof. By hypothesis, for any analytic curve $\varphi : \mathbb{C}, 0 \to \mathbb{C}^{n+1}, 0$, it follows that $g \circ \varphi \in \langle dh_t(\alpha_i) \circ \varphi \rangle$ where α_i are generators of Θ_V^0 . Then for all $\varphi : \mathbb{C}, 0 \to V \times \mathbb{C}, 0$, we also have
$(g \circ \varphi, 0) \in \langle dh_t(\alpha_i) \circ \varphi, d\phi(\alpha_i) \circ \varphi \rangle$, since $d\phi(\alpha_i) \in \langle \phi \rangle$ and $\phi(V) = 0$. Therefore $(g \circ \varphi, 0) \in \left\langle (x_i \frac{\partial h}{\partial x_j}, x_i \frac{\partial \phi}{\partial x_j}) \circ \varphi \right\rangle$. Thus,
$(g, 0) \in \overline{\left\langle (x_i \frac{\partial h}{\partial x_j}, x_i \frac{\partial \phi}{\partial x_j}) \right\rangle}_{\mathcal{O}_{V\times\mathbb{C}}} = \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{V\times\mathbb{C}}}$. In particular, $(g, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{G^{-1}(0)}}$. Q.E.D.

Remark 6.8. *The above result remains true under the weaker hypothesis $g \in \overline{dh(\Theta_V^0)}_{\mathcal{O}_{V\times\mathbb{C}}}$.*

We now proceed to prove Theorem 6.4; our proof is analogous to the proof of Theorem 6.5 in [1]. As in [1], we divide the proof in steps:

Step 1. The stratification $\tilde{S}$ is Whitney regular.

Proof. The Whitney regularity of a pair of strata (S_1, S_2) follows easily, with exception of the regularity condition of the strata over $\{0\} \times \mathbb{C}$. Clearly the strata of type $(S \times \mathbb{C}) - h^{-1}(0)$ are regular with respect to $\{0\} \times B_r$, since the original stratification satisfies the Whitney conditions. Then we only have to verify that $(S \times B_r) \cap h^{-1}(0)$ is regular over $\{0\} \times B_r$. From hypothesis, $\frac{\partial h}{\partial t} \in \overline{dh_t \Theta_V^0}$ and from Lemma 6.7 it follows that $(\frac{\partial h}{\partial t}, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{G^{-1}(0)}}$. Now, from Theorem 6.2, $(G^{-1}(0) - \Sigma_{G^{-1}(0)}, \{0\} \times B_r) = (h^{-1}(0) \cap \tilde{V} - \{0\} \times B_r, \{0\} \times B_r)$ is Whitney regular. Q.E.D.

Step 2. For some $\varepsilon' > 0$ and all $0 < \varepsilon \leq \varepsilon'$ the product of the boundary of the ε-ball, ∂B_ε, by B_r meets the strata of $\tilde{\mathcal{S}}$ transversaly.

Proof. The argument is the same as in Theorem 6.5 in [1]. Let us suppose that the statement is false. Then we can find a sequence of points (x_i, t_i) in some stratum $\tilde{S}$ with $x_i \to 0$ and $T_{(x_i,t_i)}\tilde{S} \subset T_{(x_i,t_i)}(\partial B_{\varepsilon_i} \times B_r)$ where $\varepsilon_i = ||x_i||$. Then $(x_i, 0)$ is perpendicular to $T_{(x_i,t_i)}\tilde{S}$. This contradicts the Whitney condition B. Q.E.D.

We then have the first approximation to our stratified diagram, that is,

$$B_\varepsilon \times B_r \xrightarrow{F} \mathbb{C} \times B_r \xrightarrow{\pi} B_r$$

where B_ε is the closed ball in $\mathbb{C}^n$ of radius ε, $\varepsilon \leq \varepsilon'$, $F(x,t) = (h(x,t), t)$ and π is the projection to the second factor. We stratify $\mathbb{C} \times B_r$ by $(\mathbb{C} - \{0\}) \times B_r \cup \{0\} \times B_r$ and we refine the stratification of $B_\varepsilon \times B_r$, taking the intersection of the strata in $\tilde{\mathcal{S}}$ with $\partial B_\varepsilon \times B_r$ and $int B_\varepsilon \times B_r$. We would like to show that this stratification satisfies Thom's condition, but h_t might have critical points on ∂B_ε. To get around this difficulty we need the following.

Step 3. For some $\delta > 0$, $B_\delta - \{0\}$ in $\mathbb{C}$ consists only of regular values of h_t for every $t \in B_r$.

Proof. This follows from the fact that h is a good deformation of h_0. Q.E.D.

In the above diagram we change $\mathbb{C}$ by B_δ, where B_δ is the ball with radius δ, with the stratification $\partial B_\delta \cup \{0\} \cup int B_\delta - \{0\}$, and satisfying the conditions in Step 3. We then get a new stratification of $F^{-1}(B_\delta \times B_r)$ pulling back the strata. We consider now

$$F^{-1}(B_\delta \times B_r) \xrightarrow{F} B_\delta \times B_r \xrightarrow{\pi} B_r$$

Step 4. The above diagram is Thom stratified.

Proof. We have to show that the diagram satisfies the condition A_{h_t}. Given two strata $\tilde{S}_1$, $\tilde{S}_2$ with $(x_i, t_i) \in \tilde{S}_1$, and $(x_i, t_i) \to (x, t) \in \tilde{S}_2$, the restriction of the kernel of $dF(x_i, t_i)$ to $T_{(x_i,t_i)}\tilde{S}_1$, say K_i, is $T_{(x_i,t_i)}\tilde{S}_1 \cap (\ker dh_{t_i}(x_i) \times \{0\})$. The limit of this sequence of spaces is contained in $T \cap (\ker dh_t(x) \times \{0\})$ where $T = \lim_{i\to\infty} T_{(x_i,t_i)}\tilde{S}_1$. If $x \neq 0$ then $\ker dh_t(x) \times \{0\}$ is transversal to T, hence $\lim_{i\to\infty} K_i = T \cap (\ker dh_t(x) \times \{0\})$. Since $T \supset T_{(x,t)}\tilde{S}_2$ (Whitney condition A), then $\lim_{i\to\infty} K_i$ contains the restriction of the kernel of $dF(x,t)$ to $T_{(x,t)}\tilde{S}_2$. If $x = 0$ then $\tilde{S}_2 = \{0\} \times B_r$, and Thom condition follows trivially. Q.E.D.

Remark 6.9. *The V-equisingularity of a family h as above implies that h is topologically $\mathcal{R}_V$-trivial.*

In fact, from Thom's second isotopy lemma ([12], p.62), there exist homeomorphisms

$$H : F^{-1}(B_\delta \times \{0\}) \times B_r \to F^{-1}(B_\delta \times B_r)$$

$$H' : B_\delta \times \{0\} \times B_r \to B_\delta \times B_r,$$

preserving the stratifications, such that the following diagram commutes:

$$\begin{array}{ccccc} F^{-1}(B_\delta \times \{0\}) \times B_r & \xrightarrow{F \times id} & B_\delta \times \{0\} \times B_r & \xrightarrow{\pi_3} & B_r \\ \downarrow H & & \downarrow H' & & \downarrow id \\ F^{-1}(B_\delta \times B_r) & \xrightarrow{F} & B_\delta \times B_r & \xrightarrow{\pi_2} & B_r \end{array}$$

Then $H(x, 0, t) = (\overline{h}(x,t), t)$, $F(\overline{h}(x,t), t) = (h(x,0), t)$ and it follows that $h(\overline{h}(x,t), t) = h_0(x)$ for all t and x. Therefore h is topologically $\mathcal{R}_V$-trivial.

The example below shows that the condition $g \in \overline{dh(\Theta_V^0)}_{\mathcal{O}_{V\times\mathbb{C}}}$ is stronger than the condition $(g, 0) \in \overline{\left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle}_{\mathcal{O}_{G^{-1}(0)}}$.

Example 6.10. *Let $V,0 \subset k^3,0$ be defined by $\phi(x,y,z) = 2x^2y^2 + y^3 - z^2 + x^4y = 0$ and $h : \mathbb{C}^4,0 \to \mathbb{C},0$, $h(x,y,z,t) = y + (a+t)x^2$ and $G : \mathbb{C}^4,0 \to \mathbb{C}^2,0$ given by $G(x,y,z,t) = (y+(a+t)x^2, 2x^2y^2+y^3-z^2+x^4y)$. The module Θ_V is generated by $\eta_1 = (2x,4y,6z)$, $\eta_2 = (0,2z,x^4+4x^2y+3y^2)$, $\eta_3 = (x^2+3y,-4xy,0)$ and $\eta_4 = (z,0,2x^3y+2xy^2)$. The element $\frac{\partial h}{\partial t} = x^2$ is not in the integral closure of the ideal $dh_t(\Theta_V^0)$ (it also follows that $x^2 \notin \overline{\langle dh(\eta_i)\rangle}_{\mathcal{O}_{V\times\mathbb{C}}}$). In fact, given $\phi : k,0 \to k^4,0$, $\phi(s) = (s,-as^2,0,0)$, it follows that $\frac{\partial h}{\partial t} \circ \phi$ is not in $(\phi^*(dh_t(\Theta_V^0)))\mathcal{O}_1$, then by Theorem 3.1, $\frac{\partial h}{\partial t} = x^2 \notin \overline{dh_t(\Theta_V^0)}$. We can verify that $(x^2,0) \in \overline{\left\langle x_i\frac{\partial G}{\partial x_j}\right\rangle}_{\mathcal{O}_{G^{-1}(0)}}$. In fact, we will show that $(x^2,0) \in \overline{\left\langle x_i\frac{\partial G}{\partial x_j}, e_iG_j\right\rangle}_{\mathcal{O}_4}$ and the result will follow from this. We have*

$$\begin{aligned}
&(a)\ zG_z = (0,-2z^2)\\
&(b)\ e_1G_1 = (y+(a+t)x^2,0)\\
&(c)\ x^2G_y = (x^2, 4x^4y+3x^2y^2+x^6)\\
&(d)\ e_2G_2 + \tfrac{1}{2}zG_z = (0, 2x^2y^2+y^3+x^4y)
\end{aligned}$$

Let $\varphi : \mathbb{C},0 \to \mathbb{C}^4,0$ be given by $\varphi(u) = (\varphi_1(u),\varphi_2(u),\varphi_3(u),\varphi_4(u))$. We shall see that $(\varphi_1^2,0) \in \left\langle (x_i\frac{\partial G}{\partial x_j}, e_iG_j)\circ\varphi\right\rangle_{\mathcal{O}_1}$. Let $r = ord(\varphi_1)$ and $s = ord(\varphi_2)$, if $s \le r$ or $2s = r$ then it follows from (b) that $(\varphi_1^2,0) \in \left\langle (x_i\frac{\partial G}{\partial x_j}, e_iG_j)\circ\varphi\right\rangle_{\mathcal{O}_1}$.

If $s > r$ then it follows from (c) that

$$x^2G_y\circ\varphi = (\varphi_1^2, 4\varphi_1^4\varphi_2+3\varphi_1^2\varphi_2^2+\varphi_1^6) = (\varphi_1^2,0)+(0,4\varphi_1^4\varphi_2+3\varphi_1^2\varphi_2^2+\varphi_1^6)$$

and from (d) we get that $(0,4\varphi_1^4\varphi_2+3\varphi_1^2\varphi_2^2+\varphi_1^6) \in \left\langle (x_i\frac{\partial G}{\partial x_j}, e_iG_j)\circ\varphi\right\rangle_{\mathcal{O}_1}$, hence,
$(\varphi_1^2,0) \in \left\langle (x_i\frac{\partial G}{\partial x_j}, e_iG_j)\circ\varphi\right\rangle_{\mathcal{O}_1}$ or $(x^2,0) \in \overline{\left\langle x_i\frac{\partial G}{\partial x_j}, e_iG_j\right\rangle}_{\mathcal{O}_4}$.

Remark 6.11. *When the variety V reduces to 0, Tessier in [17] proved that the set*

$$\{t \in \mathbb{C},0 \,:\, \frac{\partial h}{\partial t} \in \overline{\left\langle x_i\frac{\partial h}{\partial x_j}\right\rangle}\}$$

is open and dense. In the relative case, we can obtain a similar result as a consequence of Gaffney in [11], that is :

$$\{t \in \mathbb{C},0 \,:\, (\frac{\partial h}{\partial t},0) \in \overline{\left\langle x_i\frac{\partial G}{\partial x_j}\right\rangle}_{\mathcal{O}_{G^{-1}(0)}}\}$$

is open and dense. However, the corresponding statement does not hold for $\overline{dh_t(\Theta_V^0)}$. In fact, with a slight modification of the arguments in the above example, we see that the set:

$$\{t \in \mathbb{C}, 0 \, : \, \frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}\}$$

is empty.

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Lines of principal curvature near singular end points of surfaces in $\mathbb{R}^3$

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Abstract.

In this paper are studied the nets of principal curvature lines on surfaces embedded in Euclidean 3−space near their end points, at which the surfaces tend to infinity.

This is a natural complement and extension to smooth surfaces of the work of Garcia and Sotomayor (1996), devoted to the study of principal curvature nets which are structurally stable –do not change topologically– under small perturbations on the coefficients of the equations defining algebraic surfaces.

This paper goes one step further and classifies the patterns of the most common and stable behaviors at the ends, present also in generic families of surfaces depending on one-parameter.

§1. Introduction

A *surface* of *smoothness class* C^k in Euclidean (x, y, z)-space $\mathbb{R}^3$ is defined by the variety $A(\alpha)$ of zeros of a real function α of class C^k in $\mathbb{R}^3$. The exponent k ranges among the positive integers as well as on the symbols ∞, ω (for analytic) and $a(n)$ (for algebraic of degree n).

In the class $C^{a(n)}$ of algebraic surfaces of degree n, we have $\alpha = \sum \alpha_h$, $h = 0, 1, 2, ..., n$, where α_h is a homogeneous polynomial of degree h with real coefficients: $\alpha_h = \sum a_{ijk}x^iy^jz^k$, $i + j + k = h$.

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The space $\mathbb{R}^3$ will be endowed with the Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$ also denoted by $<,>$, and with the positive orientation induced by the volume form $\Omega = dx \wedge dy \wedge dz$.

An *end point* or *point at infinity* of $A(\alpha)$ is a point in the unit sphere $\mathbb{S}^2$, which is the limit of a sequence of the form $p_n/|p_n|$, for p_n tending to infinity in $A(\alpha)$.

The *end locus*, $E(\alpha)$, of $A(\alpha)$ is the collection of its end points. This set is a geometric measure of the non-compactness of the surface and describes how it tends to infinity.

A surface $A(\alpha)$ is said to be *regular at* $p \in E(\alpha)$ if in a neighborhood of p, $E(\alpha)$ is a regular smooth curve in $\mathbb{S}^2$. Otherwise, p is said to be a *critical end point* of $A(\alpha)$.

For the class $a(n)$, $E(\alpha)$ is contained in the algebraic curve $E_n(\alpha) = \{p \in \mathbb{S}^2;\ \alpha_n(p) = 0\}$. The regularity of $E(\alpha)$ is equivalent to that of $E_n(\alpha)$.

The gradient vector field of α, will be denoted by $\nabla\alpha = \alpha_x \partial/\partial x + \alpha_y \partial/\partial y + \alpha_z \partial/\partial z$, where $\alpha_x = \partial\alpha/\partial x$, etc.

The zeros of this vector field are called *critical points* of α; they determine the set $C(\alpha)$. The regular part of $A(\alpha)$ is the smooth surface $S(\alpha) = A(\alpha) \setminus C(\alpha)$. When $C(\alpha)$ is disjoint from $A(\alpha)$, the surface $S(\alpha) = A(\alpha)$ is called *regular*. The *orientation* on $S(\alpha)$ will be defined by taking the gradient $\nabla\alpha$ to be the *positive normal*. Thus $A(-\alpha)$ defines the same surface as $A(\alpha)$ but endowed with the opposite orientation on $S(-\alpha)$.

The Gaussian normal map N, of $S(\alpha)$ into the sphere $\mathbb{S}^2$, is defined by the unit vector in the direction of the gradient: $N_\alpha = \nabla\alpha/|\nabla\alpha|$. The eigenvalues $-k^1_\alpha(p)$ and $-k^2_\alpha(p)$ of the operator $DN_\alpha(p)$, restricted to $T_pS(\alpha)$, the tangent space to the surface at p, define the principal curvatures, $k^1_\alpha(p)$ and $k^2_\alpha(p)$ of the surface at the point p. It will be always assumed that $k^1_\alpha(p) \leq k^2_\alpha(p)$.

The points on $S(\alpha)$ at which the principal curvatures coincide, define the set $U(\alpha)$ of *umbilic points* of the surface $A(\alpha)$. On $S(\alpha) \setminus U(\alpha)$, the eigenspaces of DN_α, associated to $-k^1_\alpha$ and $-k^2_\alpha$ define line fields $L_1(\alpha)$ and $L_2(\alpha)$, mutually orthogonal, called respectively *minimal* and *maximal principal line fields* of the surface $A(\alpha)$. The smoothness class of these line fields is C^{k-2}, where $k-2 = k$ for $k = \infty$, ω and $a(n)-2 = \omega$.

The maximal integral curves of the line fields $L_1(\alpha)$ and $L_2(\alpha)$ are called respectively the lines of minimal and maximal principal curvature, or simply the *principal lines* of $A(\alpha)$.

What was said above concerning the definition of these lines is equivalent to require that they are non trivial solutions of Rodrigues' differential equations:

$$(1) \qquad DN_\alpha(p)dp + k^i_\alpha(p)dp = 0, \qquad < N(p), dp >= 0, \;\; i = 1, 2.$$

where $p = (x, y, z)$, $\alpha(p) = 0$, $dp = dx\partial/\partial x \;+ dy\partial/\partial y \;+ dz\partial/\partial z$. See [22, 23].

After elimination of k^i_α, $i = 1, 2$, the first two equations in (1) can be written as the following single implicit quadratic equation:

$$(2) \qquad < DN_\alpha(p)dp \wedge N(p), dp >= [DN_\alpha(p)dp, N_\alpha(p), dp] = 0.$$

The left (and mid term) member of this equation is the *geodesic torsion* in the direction of dp. In terms of a local parametrization $\overline{\alpha}$ introducing coordinates (u, v) on the surface, the equation of lines of curvature in terms of the coefficients (E, F, G) of the first and (e, f, g) of the second fundamental forms is, see [22, 23],

$$(3) \qquad [Fg - Gf]dv^2 + [Eg - Ge]dudv + [Ef - Fe]du^2 = 0.$$

The net $F(\alpha) = (F_1(\alpha), F_2(\alpha))$ of orthogonal curves on $S(\alpha) \setminus U(\alpha)$, defined by the integral foliations $F_1(\alpha)$ and $F_2(\alpha)$ of the line fields $L_1(\alpha)$ and $L_2(\alpha)$, will be called *the principal net* on $A(\alpha)$.

The study of families of principal curves and their umbilic singularities on immersed surfaces was initiated by Euler, Monge, Dupin and Darboux, to mention only a few. See [2, 18] and [14, 22, 23] for references.

Recently this classic subject acquired new vigor by the introduction of ideas coming from Dynamical Systems and the Qualitative Theory of Differential Equations. See the works [14], [5], [7], [13] of Gutierrez, Garcia and Sotomayor on the structural stability, bifurcations and genericity of principal curvature lines and their umbilic and critical singularities on compact surfaces.

The scope of the subject was broadened by the extension of the works on structural stability to other families of curves of classical geometry. See [12], for the asymptotic lines and [8, 9, 10, 11] respectively for the arithmetic, geometric, harmonic and general mean curvature lines. Other pertinent directions of research involving implicit differential equations arise from Control and Singularity Theories, see Davydov [3] and Davydov, Ishikawa, Izumiya and Sun [4].

In [6] the authors studied the behavior of the lines of curvature on algebraic surfaces, i.e. those of $C^{a(n)}$, focusing particularly their generic and stable patterns at end points. Essential for this study was the operation of compactification of algebraic surfaces and their equations (2) and (3) in $\mathbb{R}^3$ to obtain compact ones in $\mathbb{S}^3$. This step is reminiscent of the Poincaré compactification of polynomial differential equations [19].

In this paper the study in [6] will be extended to the broader and more flexible case of C^k-smooth surfaces.

As mentioned above, in the case of algebraic surfaces studied in [6], the ends are the algebraic curves defined by the zeros, in the Equatorial Sphere $\mathbb{S}^2$ of $\mathbb{S}^3$, of the highest degree homogeneous part α_n of the polynomial α. Here, to make the study of the principal nets at ends of smooth surfaces tractable by methods of Differential Analysis, we follow an inverse procedure, going from compact smooth surfaces in $\mathbb{S}^3$ to surfaces in $\mathbb{R}^3$. This restriction on the class of surfaces studied in this paper is explained in Subsection 1.1.

The new results of this paper on the patterns of principal nets at end points are established in Sections 2 and 3. Their meaning for the Structural Stability and Bifurcation Theories of Principal Nets is discussed in Section 4, where a pertinent problem is proposed. The essay [21] presents a historic overview of the subject and reviews other problems left open.

1.1. Preliminaries

Consider the space $\mathcal{A}_c^k$ of real valued functions α^c which are C^k-smooth in the three dimensional sphere $\mathbb{S}^3 = \{|p|^2+|w|^2 = 1\}$ in $\mathbb{R}^4$, with coordinates $p = (x, y, z)$ and w. The meaning of the exponent k is the same as above and $a(n)$ means polynomials of degree n in four variables of the form $\alpha^c = \sum \alpha_h w^h$, $h = 0, 1, 2, ..., n$, with α_h homogeneous of degree h in (x, y, z).

The *equatorial* sphere in $\mathbb{S}^3$ will be $\mathbb{S}^2 = \{(p, w) : \ |p| = 1, \ w = 0\}$ in $\mathbb{R}^3$. It will be endowed with the positive orientation defined by the outward normal. The northern hemisphere of $\mathbb{S}^3$ is defined by $\mathbb{H}^+ = \{(p, w) \in \mathbb{S}^3 : \ w > 0\}$.

The surfaces $A(\alpha)$ considered in this work will be defined in terms of functions $\alpha^c \in \mathcal{A}_c^k$ as $\alpha = \alpha^c \circ \mathbb{P}$, where $\mathbb{P}$ is the *central projection* of $\mathbb{R}^3$, identified with the tangent plane at the north pole $\mathbb{T}_w^3$, onto $\mathbb{H}^+$, defined by:

$$\mathbb{P}(p) = (p/(|p|^2 + 1)^{1/2}, \ \ 1/(|p|^2 + 1)^{1/2}).$$

For future reference, denote by $\mathbb{T}_y^3$ the tangent plane to $\mathbb{S}^3$ at the point $(0, 1, 0, 0)$, identified with $\mathbb{R}^3$ with orthonormal coordinates (u, v, w), with w along the vector $\omega = (0, 0, 0, 1)$. The central projection $\mathbb{Q}$ of $\mathbb{T}_y^3$ to $\mathbb{S}^3$ is such that $\mathbb{P}^{-1} \circ \mathbb{Q} : \mathbb{T}_y^3 \to \mathbb{T}_w^3$ has the coordinate expression $(u, v, w) \to (u/w, v/w, 1/w)$.

For $m \leq k$, the following expression defines uniquely, the functions involved:

$$(4) \qquad \alpha^c(p,w) = \sum w^j \alpha_j^c(p) \ + \ o(|w|^m),\ j = 0,\ 1,\ 2, ..., m.$$

In the algebraic case ($k = a(n)$) studied in [6], $\alpha^c = \sum w^{n-h}\alpha_h$, $h = 0,\ 1,\ 2, .., n$, where the obvious correspondence $\alpha_h = \alpha^c_{n-h}$ holds.

The end points of $A(\alpha)$, $E(\alpha)$, are contained in $E(\alpha^c) = \{\alpha_0^c(p) = 0\}$.

At a *regular end point* p of $E(\alpha)$ it will be required that α_0^c has a regular zero, i.e. one with non-vanishing derivative i.e. $\nabla\alpha_0^c(p) \neq 0$. At regular end points, the end locus is oriented by the positive unit normal $\nu(\alpha) = \nabla\alpha_0^c/|\nabla\alpha_0^c|$. This defines the positive unit tangent vector along $E(\alpha)$, given at p by $\tau(\alpha)(p) = p \wedge \nu(\alpha)(p)$. An end point is called *critical* if it is not regular.

A regular point $p \in E(\alpha)$ is called a *biregular end point* of $A(\alpha)$ if the geodesic curvature, k_g, of the curve $E(\alpha)$ at p, considered as a spherical curve, is different from zero; it is called an *inflexion end point* if k_g is equal to zero.

When the surface $A(\alpha)$ is regular at infinity, clearly $E(\alpha) = E(\alpha^c) = \{\alpha_0^c(p) = 0\}$.

The analysis in sections 2 and 3 will prove that there is a natural *extension* $F_c(\alpha) = (F_{c1}(\alpha), F_{c2}(\alpha))$ of the net $(\mathbb{P}(F_1(\alpha)), \mathbb{P}(F_2(\alpha))$ to $A_c(\alpha) = \{\alpha^c = 0\}$, as a net of class C^{k-2}, whose singularities in $E(\alpha)$ are located at the *inflexion* and *critical* end points of $A(\alpha)$. This is done by means of special charts used to extend the quadratic differential equations that define $(\mathbb{P}(F_1(\alpha)), \mathbb{P}(F_2(\alpha))$ to a full neighborhood in $A_c(\alpha)$ of the arcs of biregular ends. The differential equations are then extended to a full neighborhood of the singularities. See Lemma 2, for regular ends, and Lemma 4, for critical ends.

The main contribution of this paper consists in the resolution of singularities of the extended differential equations, under suitable genericity hypotheses on α^c. This is done in sections 2 and 3. It leads to eight patterns of principal nets at end points. Two of them – elliptic and hyperbolic inflexions– have also been studied in the case of algebraic surfaces [6].

§2. Principal Nets at Regular End Points

Lemma 1. *Let p be a regular end point of $A(\alpha)$, $\alpha = \alpha^c \circ \mathbb{P}$. Then there is a mapping $\overline{\alpha}$ of the form*

$\overline{\alpha}(u,w) = (x(u,w), y(u,w), z(u,w))$, $w > 0$, defined by

$$x(u,w) = \frac{u}{w}, \qquad y(u,w) = \frac{h(u,w)}{w}, \qquad z(u,w) = \frac{1}{w}. \tag{5}$$

which parametrizes the surface $A(\alpha)$ near p, with

$$\begin{aligned} h(u,w) =& k_0 w + \frac{1}{2}au^2 + buw + \frac{1}{2}cw^2 \\ &+\frac{1}{6}(a_{30}u^3 + 3a_{21}u^2w + 3a_{12}uw^2 + a_{03}w^3) \\ &+\frac{1}{24}(a_{40}u^4 + 4a_{31}u^3w + 6a_{22}u^2w^2 \\ &+4a_{13}uw^3 + a_{04}w^4) + h.o.t \end{aligned} \tag{6}$$

Proof. With no lost of generality, assume that the regular end point p is located at $(0,1,0,0)$, the unit tangent vector to the regular end curve is $\tau = (1,0,0,0)$ and the positive normal vector is $\nu = (0,0,1,0)$. Take orthonormal coordinates u, v, w along $\tau, \nu, \omega = (0,0,0,1)$ on the tangent space, $\mathbb{T}^3_y$ to $\mathbb{S}^3$ at p. Then the composition $\mathbb{P}^{-1} \circ \mathbb{Q}$ writes as $x = u/w,\ y = v/w,\ z = 1/w$.

Clearly the surface $A(\alpha^c)$ near p can be parametrized by the central projection into $\mathbb{S}^3$ of the graph of a C^k function of the form $v = h(u,w)$ in $\mathbb{T}^3_y$, with $h(0,0) = 0$ and $h_u(0,0) = 0$. This means that the surface $A(\alpha)$, with $\alpha = \alpha^c \circ \mathbb{P}^{-1}$ can be parametrized in the form (5) with h as in (6).

Q.E.D.

Lemma 2. *The differential equation (3) in the chart $\overline{\alpha}$ of Lemma 1, multiplied by $w^8\sqrt{EG - F^2}$, extends to a full domain of the chart (u,w) to one given by*

$$\begin{aligned} &Ldw^2 + Mdudw + Ndu^2 = 0, \\ L =& - b - a_{21}u - a_{12}w - (c + a_{22})uw \\ &-(b + \frac{1}{2}a_{31})u^2 - \frac{1}{2}a_{13}w^2 + h.o.t. \\ M =& - a - a_{30}u - a_{21}w - \frac{1}{2}(2a + a_{40})u^2 \\ &-a_{31}uw + \frac{1}{2}(2c - a_{22})w^2 + h.o.t. \\ N =& w[au + bw + a_{30}u^2 + 2a_{21}uw + a_{12}w^2 + h.o.t.] \end{aligned} \tag{7}$$

where the coefficients are of class C^{k-2}.

Proof. The coefficients of first fundamental form of $\overline{\alpha}$ in (5) and (6) are given by:

$$\begin{aligned} E(u,w) =& \frac{1+h_u^2}{w^2} \\ F(u,w) =& \frac{h_u(wh_w - h) - u}{w^3} \\ G(u,w) =& \frac{1+u^2+(wh_w-h)^2}{w^4} \end{aligned}$$

The coefficients of the second fundamental form of $\overline{\alpha}$ are:

$$\begin{aligned} e(u,w) =& \frac{h_{uu}}{w^4\sqrt{EG-F^2}}, \\ f(u,w) =& \frac{h_{uw}}{w^4\sqrt{EG-F^2}}, \\ g(u,w) =& \frac{h_{ww}}{w^4\sqrt{EG-F^2}} \end{aligned}$$

where $e = [\overline{\alpha}_{uu}, \overline{\alpha}_u, \overline{\alpha}_w]/|\overline{\alpha}_u \wedge \overline{\alpha}_w|$, $f = [\overline{\alpha}_{uw}, \overline{\alpha}_u, \overline{\alpha}_w]/|\overline{\alpha}_u \wedge \overline{\alpha}_w|$ and $g = [\overline{\alpha}_{ww}, \overline{\alpha}_u, \overline{\alpha}_w]/|\overline{\alpha}_u \wedge \overline{\alpha}_w|$.

The differential equation of curvature lines (3) is given by $Ldw^2 + Mdudw + Ndu^2 = 0$, where $L = Fg - Gf$, $M = Eg - Ge$ and $N = Ef - Fe$.

These coefficients, after multiplication by $w^8\sqrt{EG-F^2}$, keeping the same notation, give the expressions in (7). Q.E.D.

The differential equation (7) is non-singular, i.e., defines a regular net of transversal curves if $a \neq 0$. This will be seen in item a) of next proposition. Calculation expresses a as a non-trivial factor of k_g.

The singularities of equation (7) arise when $a = 0$; they will be resolved in item b), under the genericity hypothesis $a_{30}b \neq 0$.

Proposition 1. *Let $\overline{\alpha}$ be as in Lemma 1. Then the end locus is parametrized by the regular curve $v = h(u,w)$, $w = 0$.*

a) *At a biregular end point, i.e., regular and non inflexion, $a \neq 0$, the principal net is as illustrated in Fig. 1, left.*

b) *If p is an inflexion, bitransversal end point, i.e., $\beta(p) = a_{30}b \neq 0$, the principal net is as illustrated in Fig. 1, hyperbolic $\beta < 0$, center, and elliptic $\beta > 0$, right.*

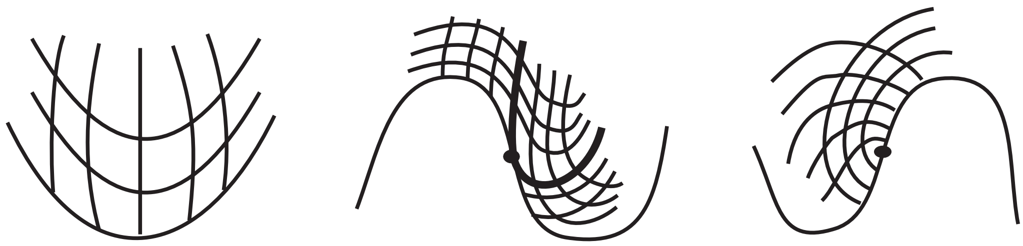

Fig. 1. Curvature lines near regular end points: biregular (left) and inflexions: hyperbolic (center) and elliptic (right).

Proof. Consider the implicit differential equation

$$\begin{aligned}\mathcal{F}(u,w,p) =& -(b+a_{21}u+a_{12}w+h.o.t.)p^2\\ &-(a+a_{30}u+a_{21}w+h.o.t.)p\\ &+w(au+bw+h.o.t.)=0.\end{aligned} \tag{8}$$

The Lie-Cartan line field tangent to the surface $\mathcal{F}^{-1}(0)$ is defined by $X = (\mathcal{F}_p, p\mathcal{F}_p, -(\mathcal{F}_u + p\mathcal{F}_w))$ in the chart $p = dw/du$ and $Y = (q\mathcal{F}_q, \mathcal{F}_q, -(q\mathcal{F}_u + \mathcal{F}_w))$ in the chart $q = du/dw$. Recall that the integral curves of this line field projects to the solutions of the implicit differential equation (8).

If $a \neq 0$, $\mathcal{F}^{-1}(0)$ is a regular surface, $X(0) = (-a,0,0) \neq 0$ and $Y(0) = (b,a,0)$. So by the Flow Box theorem the two principal foliations are regular and transversal near 0. This ends the proof of item a).

If $a = 0$, $\mathcal{F}^{-1}(0)$ is a quadratic cone and $X(0) = 0$. Direct calculation shows that

$$DX(0) = \begin{pmatrix} -a_{30} & -a_{21} & -2b \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{pmatrix}$$

Therefore 0 is a saddle point with non zero eigenvalues $-a_{30}$ and a_{30} and the associated eigenvectors are $e_1 = (1,0,0)$ and $e_2 = (b,0,-a_{30})$.

The saddle separatrix tangent to e_1 is parametrized by $w = 0$ and has the following parametrization $(s,0,0)$. The saddle separatrix tangent to e_2 has the following parametrization:

$$u(s) = s + O(s^3),\ w(s) = -\frac{a_{30}}{b}\frac{s^2}{2} + O(s^3),\ p(s) = -\frac{a_{30}}{b}s + O(s^3).$$

If $a_{30}b < 0$ the projection $(u(s), w(s))$ is contained in the semiplane $w \geq 0$. As the saddle separatrix is transversal to the plane $\{p = 0\}$ the phase portrait of X is as shown in the Fig. 2 below. The projections of the integral curves in the plane (u, w) shows the configurations of the principal lines near the inflexion point.

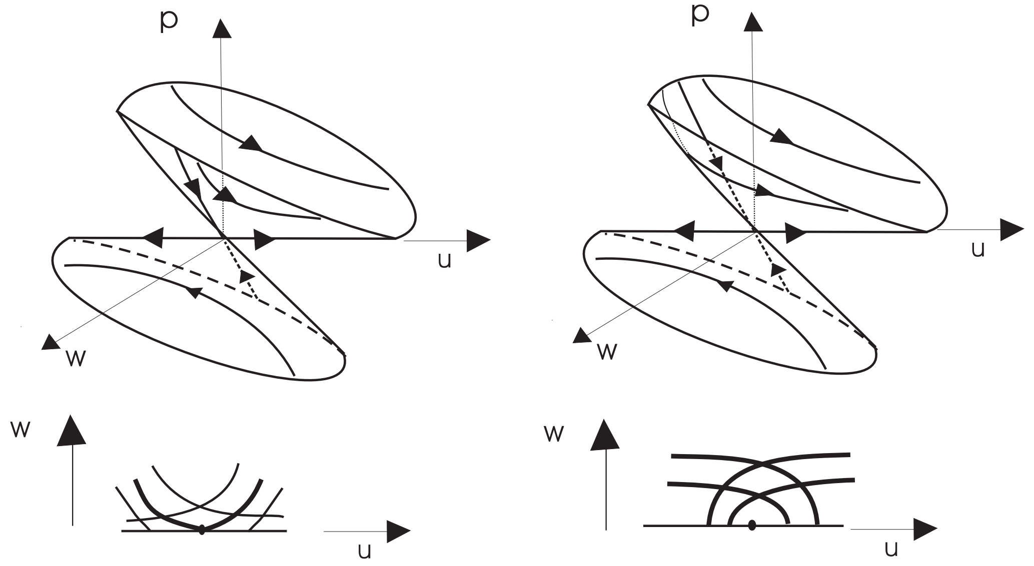

Fig. 2. Phase portrait of X near the singular point of saddle type

Q.E.D.

Proposition 2. *Let $\overline{\alpha}$ be as in Lemma 1. Suppose that, contrary to the hypothesis of Proposition 1, $a = 0$, $a_{30} = 0$, but $ba_{40} \neq 0$ holds.*

The differential equation (7) of the principal lines in this case has the coefficients given by:

$$
\begin{aligned}
L(u,w) =& -[b + a_{21}u + a_{12}w + \frac{1}{2}(2b + a_{31})u^2 \\
&+(c + a_{22})uw + \frac{1}{2}a_{13}w^2 + h.o.t.] \\
M(u,w) =& -[a_{21}w + \frac{1}{2}a_{40}u^2 \\
&+a_{31}uw + \frac{1}{2}(a_{22} - 2c)w^2 + h.o.t.] \\
N(u,w) =& w^2(b + 2a_{21}u + a_{12}w + h.o.t.)
\end{aligned}
\tag{9}
$$

The principal net is as illustrated in Fig. 3.

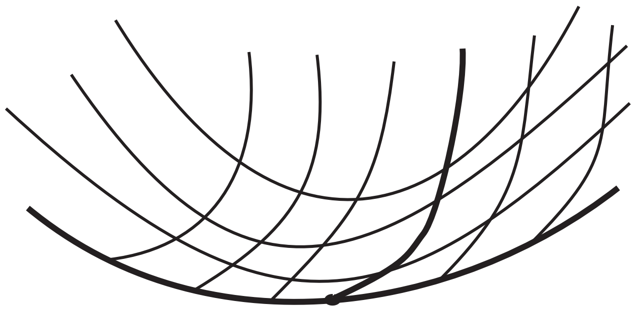

Fig. 3. Curvature lines near a hyperbolic-elliptic inflexion end point

Proof. From equation (7) it follows the expression of equation (9) is as stated. In a neighborhood of 0 this differential equation factors in to the product of two differential forms $X_+(u,w) = A(u,w)dv - B_+(u,w)du$ and $X_-(u,w) = A(u,w)dv - B_-(u,w)du$, where $A(u,w) = 2L(u,w)$ and $B_\pm(u,w) = M(u,w) \pm \sqrt{(M^2 - 4LN)(u,w)}$. The function A is of class C^{k-2} and the functions $B_\pm$ are Lipschitz. Assuming $a_{40} > 0$, it follows that $A(0) = -2b \neq 0$, $B_-(u,0) = 0$ and $B_+(u,0) = a_{40}u^2 + h.o.t.$ In the case $a_{40} < 0$ the analysis is similar, exchanging B_- with B_+.

Therefore, outside the point 0, the integral leaves of X_+ and X_- are transversal. Further calculation shows that the integral curve of X_+ which pass through 0 is parametrized by $(u, -\frac{a_{40}}{6b}u^3 + h.o.t.)$.

This shows that the principal foliations are extended to regular foliations which however fail to be a net a single point of cubic contact. This is illustrated in Fig. 3 in the case $a_{40}/b < 0$. The case $a_{40}/b > 0$ is the mirror image of Fig. 3. Q.E.D.

Proposition 3. *Let $\overline{\alpha}$ be as in Lemma 1. Suppose that, contrary to the hypothesis of Proposition 1, $a = 0$, $b = 0$, but $a_{30} \neq 0$ holds.*

The differential equation of the principal lines in this chart is given by:

$$
\begin{aligned}
&-[a_{21}u + a_{12}w + \frac{1}{2}a_{31}u^2 + (c + a_{22})uw + \frac{1}{2}a_{13}w^2 + h.o.t.]dw^2 \\
(10)\quad &-[a_{30}u + a_{21}w + \frac{1}{2}a_{40}u^2 + a_{31}uw + \frac{1}{2}(a_{22} - 2c)w^2 + h.o.t.]dudw \\
&+w[(a_{30}u^2 + 2a_{21}uw + a_{12}w^2) + h.o.t.]du^2 = 0.
\end{aligned}
$$

a) *If $(a_{21}^2 - a_{12}a_{30}) < 0$ the principal net is as illustrated in Fig. 4 (left).*

b) *If* $(a_{21}^2 - a_{12}a_{30}) > 0$ *the principal net is as illustrated in Fig. 4(right).*

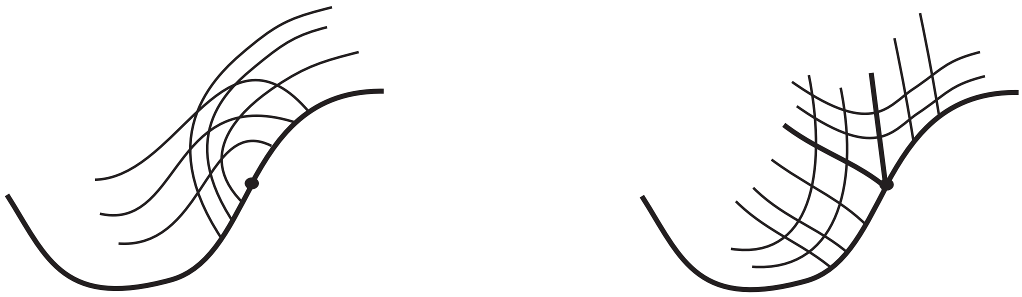

Fig. 4. Curvature lines near an umbilic-inflexion end point

Proof. Consider the Lie-Cartan line field defined by

$$X = (\mathcal{F}_p, p\mathcal{F}_p, -(\mathcal{F}_u + p\mathcal{F}_w))$$

on the singular surface $\mathcal{F}^{-1}(0)$, where

$$\begin{aligned}\mathcal{F}(u,w,p) = &- [a_{21}u + a_{12}w + h.o.t.]p^2 - [a_{30}u + a_{21}w + h.o.t.]p \\ &+ w[(a_{30}u^2 + 2a_{21}uw + a_{12}w^2) + h.o.t.] = 0.\end{aligned}$$

The singularities of X along the projective line (axis p) are given by the polynomial equation $p(a_{30} + 2a_{21}p + a_{12}p^2) = 0$. So X has one, respectively, three singularities, according to $a_{21}^2 - a_{12}a_{30}$ is negative, respectively positive. In both cases all the singular points of X are hyperbolic saddles and so, topologically, in a full neighborhood of 0 the implicit differential equation (10) is equivalent to a Darbouxian umbilic point D_1 or to a Darbouxian umbilic point of type D_3. See Fig. 5 and [14, 17].

In fact,

$$DX(0,0,p) = \begin{pmatrix} -2a_{21}p - a_{30} & -2a_{12}p - a_{21} & 0 \\ -p(2a_{21}p + a_{30}) & -p(2a_{12}p + a_{21}) & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where, $A_{31} = p((c+a_{22})p^2+2a_{31}p+a_{40})$, $A_{32} = p[a_{13}p^2+(2a_{22}-c)p+a_{31}]$ and $A_{33} = 4a_{21}p + a_{30} + 3a_{12}p^2$.

The eigenvalues of $DX(0,0,p)$ are $\lambda_1(p) = -(a_{30} + 3a_{21}p + 2a_{12}p^2)$, $\lambda_2(p) = 4a_{21}p + a_{30} + 3a_{12}p^2$ and $\lambda_3 = 0$.

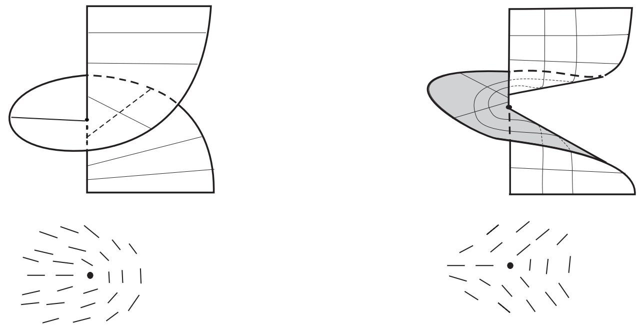

Fig. 5. Resolution of a singular point by a Lie-Cartan line field

Let p_1 and p_2 be the roots of $r(p) = a_{30} + 2a_{21}p + a_{12}p^2 = 0$.
Therefore, $\lambda_1(p_i) = a_{21}p_i + a_{30}$ and $\lambda_2(p_i) = -2(a_{21}p_i + a_{30})$.
As

$$r(-\frac{a_{30}}{a_{21}}) = \frac{a_{30}(a_{12}a_{30} - a_{21}^2)}{a_{21}^2} \neq 0,$$

it follows that $\lambda_1(p_i)\lambda_2(p_i) < 0$ and $\lambda_1(0)\lambda_2(0) = -a_{30}^2 < 0$. So the singularities of X are all hyperbolic saddles. If $a_{21}^2 - a_{12}a_{30} < 0$, X has only one singular point $(0,0,0)$. If $a_{21}^2 - a_{12}a_{30} > 0$, X has three singular points $(0,0,0)$, $(0,0,p_1)$ and $(0,0,p_2)$.

In the first case in a full neighborhood of $(0,0)$ the principal foliations have the topological type of a D_1 Darbouxian umbilic point. In the region $w > 0$ the behavior is as shown in Fig. 4 (left). In the second case the principal foliations have the topological type of a D_3 Darbouxian umbilic point and so the behavior in the finite region $w > 0$ is as shown in Fig. 4 (right). Q.E.D.

§3. Principal Nets at Critical End Points

Let p be a critical end point of the surface $A(\alpha)$, $\alpha = \alpha^c \circ \mathbb{P}$. Without lost of generality assume that the point p is located at $(0,1,0,0)$ and that the surface $\alpha^c = 0$ is given by the graph of a function $w = h(u,v)$, where h vanishes together with its first partial derivatives at $(0,0)$ and the u and v are the principal axes of the quadratic part of its second order jet.

Through the central projection $\mathbb{Q}$, the coordinates (u,v,w) can be thought to be orthonormal in the tangent space $\mathbb{T}_p^3$ to $\mathbb{S}^3$ at p, with w along $\omega = (0,0,0,1)$, u along $(1,0,0,0)$ and v along $(0,1,0,0)$.

Lemma 3. *Let p be a critical end point of the surface $A(\alpha)$, $\alpha = \alpha^c \circ \mathbb{P}$. Then there is a mapping $\overline{\alpha}$ of the form*

$$\overline{\alpha}(u,v) = (x(u,v), y(u,v), z(u,v))$$

defined by

$$x(u,v) = \frac{u}{h(u,v)}, \quad y(u,v) = \frac{v}{h(u,v)}, \quad z(u,v) = \frac{1}{h(u,v)} \tag{11}$$

which parametrizes the surface $A(\alpha)$ near p. The function h is as follows.

i) *If p is a definite critical point of h, then*

$$\begin{aligned} h(u,v) = &(a^2u^2 + b^2v^2) + \frac{1}{6}(a_{30}u^3 + 3a_{21}u^2v + 3a_{12}uv^2 + a_{03}v^3) \\ &+ \frac{1}{24}(a_{40}u^4 + 6a_{31}u^3v + 4a_{22}u^2v^2 + 6a_{13}uv^3 + a_{04}v^4) + h.o.t. \end{aligned} \tag{12}$$

ii) *If p is a saddle critical point of h, then*

$$\begin{aligned} h(u,v) = &(-au + v)v + \frac{1}{6}(a_{30}u^3 + 3a_{21}u^2v + 3a_{12}uv^2 + a_{03}v^3) \\ &+ \frac{1}{24}(a_{40}u^4 + 6a_{31}u^3v + 4a_{22}u^2v^2 + 6a_{13}uv^3 + a_{04}v^4) + h.o.t. \end{aligned} \tag{13}$$

Proof. The map $x = u/w$, $y = v/w$, $z = 1/w$ from $\mathbb{T}^3_y$ to $\mathbb{T}^3_w$, expresses the composition $\mathbb{P}^{-1} \circ \mathbb{Q}$.

Therefore the surface $A(\alpha)$, with $\alpha = \alpha^c \circ (\mathbb{P})^{-1}$ can be parametrized with the functions x, y, z as is stated in equation (11).

The function h takes the form given in equation (12) if it is definite positive. If it is a non degenerate saddle, after a rotation of principal axes, h can be written in the form given in equation (13). Q.E.D.

Lemma 4. *The differential equation (3) in the chart $\overline{\alpha}$ of Lemma 3, multiplied by $h^8\sqrt{EG - F^2}$, extends to a full domain of the chart (u,v) to one given by*

$$\begin{aligned} &Ldw^2 + Mdudw + Ndu^2 = 0, \\ L =& h^8\sqrt{EG - F^2}(Fg - Gf), \\ M =& h^8\sqrt{EG - F^2}(Eg - Ge), \\ N =& h^8\sqrt{EG - F^2}(Ef - Fe). \end{aligned} \tag{14}$$

where the coefficients are of class C^{k-2}. Here (E,F,G) and (e,f,g) are the coefficients of the first and second fundamental forms of the surface in the chart $\overline{\alpha}$.

Proof. The first fundamental form of the surface parametrized by $\overline{\alpha}$, equation (11), in Lemma 3 is given by:

$$\begin{aligned}E(u,v) =&\frac{(h-uh_u)^2+(v^2+1)h_u^2}{h^4}\\ F(u,v) =&\frac{-h(uh_u+vh_v)+(u^2+v^2+1)h_uh_v}{h^4}\\ G(u,v) =&\frac{(h-vh_v)^2+(u^2+1)h_v^2}{h^4}\end{aligned}$$

The coefficients of the second fundamental form are given by :

$$\begin{aligned}e(u,v) =& -\frac{h_{uu}}{h^4\sqrt{EG-F^2}}\\ f(u,v) =& -\frac{h_{uv}}{h^4\sqrt{EG-F^2}}\\ g(u,v) =& -\frac{h_{vv}}{h^4\sqrt{EG-F^2}}\end{aligned}$$

where $e=[\overline{\alpha}_{uu},\overline{\alpha}_u,\overline{\alpha}_v]/|\overline{\alpha}_u\wedge\overline{\alpha}_v|$, $f=[\overline{\alpha}_{uv},\overline{\alpha}_u,\overline{\alpha}_v]/|\overline{\alpha}_u\wedge\overline{\alpha}_v|$ and $g=[\overline{\alpha}_{vv},\overline{\alpha}_u,\overline{\alpha}_v]/|\overline{\alpha}_u\wedge\overline{\alpha}_v|$.

Therefore the differential equation of curvature lines, after multiplication by $h^8|\overline{\alpha}_u\wedge\overline{\alpha}_v|$ is as stated. Q.E.D.

3.1. Differential Equation of Principal Lines around a Definite Critical End Point

Proposition 4. *Suppose that* 0 *is a critical point of* h *given by equation (12), with* $a>0,\ b>0$ *(local minimum).*

In polar coordinates $u=br\cos\theta,\ v=ar\sin\theta$ *the differential equation (14) is given by* $Ldr^2+Mdrd\theta+Nd\theta^2=0$, *where:*

$$\begin{aligned}L =&l_0+l_1r+h.o.t,\\ M =&m_0+m_1r+h.o.t,\\ N =&r^2(\frac{1}{2}n_0+\frac{1}{6}n_1r+\frac{1}{24}n_2r^2+h.o.t.)\end{aligned}\tag{15}$$

with $m_0=M(\theta,0)=-8a^7b^7\neq 0$ *and the coefficients* $(l_0,l_1,m_1,\ n_0,\ n_1,n_2)$ *are trigonometric polynomials with coefficients depending on the fourth order jet of* h *at* $(0,0)$, *expressed in equations (16) to (19).*

Proof. Introducing polar coordinates $u = br\cos\theta$, $v = ar\sin\theta$ in the equation (14), where h is given by equation (12), it follows that the differential equation of curvature lines near the critical end point 0, is given by $Ldr^2 + Mdrd\theta + Nd\theta^2 = 0$, where: $m_0 = M(\theta, 0) = -8a^7b^7$, $N(\theta, 0) = 0$, and $\frac{\partial N}{\partial r}(\theta, 0) = 0$.

The Taylor expansions of L, M and N are as follows:

$$\begin{aligned} L =& l_0 + l_1 r + h.o.t, \\ M =& m_0 + m_1 r + h.o.t, \\ N =& r^2(n_0/2 + n_1 r/6 + n_2 r^2/24 + h.o.t.) \end{aligned}$$

After a long calculation, corroborated by computer algebra, it follows that:

$$\begin{aligned} l_0 = &= 2a^5b^5[a_{30}b^3\cos^2\theta\sin\theta + a_{21}ab^2(2\cos\theta - 3\cos^3\theta) \\ &+ a_{12}a^2b(\sin\theta - 3\cos^2\theta\sin\theta) + a_{03}a^3(\cos^3\theta - \cos\theta)] \end{aligned} \tag{16}$$

$$\begin{aligned} n_0 =& 4b^5a^5[(-3a_{21}b^2a + a_{03}a^3)\cos^3\theta \\ &+ (a_{30}b^3 - 3a_{12}ba^2)\sin\theta\cos^2\theta \\ &+ (-a_{03}a^3 + 2a_{21}b^2a)\cos\theta + 4ba^2a_{12}\sin\theta] \end{aligned} \tag{17}$$

$$m_1 = -4b^5a^5[(a_{30}b^3 + a_{12}ba^2)\cos\theta + (a_{21}b^2a + a_{03}a^3)\sin\theta] \tag{18}$$

(19)

$$
\begin{aligned}
n_2 = &- 2a^3b^3[(a^3b^4(24a_{30}a_{13} + 6a_{03}a_{40} + 108a_{21}a_{22} + 72a_{12}a_{31})\\
&+6a^7a_{03}a_{04} - 18b^2a^5(a_{21}a_{04} + 4a_{12}a_{13} + 2a_{03}a_22) - 6ab^6(3a_{21}a_{40}\\
&+4a_{30}a_{31})) \cos^7\theta + (a^4b^3(6a_{04}a_{30} + 72a_{21}a_{13} + 108a_{12}a_{22}\\
&+24a_{03}a_{31}) + 6a_{40}b^7a_{30} - 18b^5a^2(a_{40}a_{12} + 4a_{21}a_{31} + 2a_{30}a_{22})\\
&-6a^6b(4a_{03}a_{13} + 3a_{12}a_{04})) \sin\theta \cos^6\theta + (24a^7b^2a_{03} + 72b^6a^3a_{21}\\
&-b^4a^5(24a_{03} + 72a_{21}) + ab^6(50a_{30}a_{31} + 36a_{21}a_{40})\\
&-a^3b^4(10a_{03}a_{40} + 198a_{21}a_{22} + 46a_{30}a_{13} + 126a_{12}a_{31})\\
&+b^2a^3(18a_{12}^2a_{21} - 6a_{21}^2a_{03} - 12a_{30}a_{12}a_{03}) + b^4a(3a_{30}^2a_{03}\\
&+6a_{30}a_{12}a_{21} - 9a_{21}^3) + a^5b^2(54a_{03}a_22 + 114a_{12}a_{13} + 30a_{21}a_{04})\\
&+a^5(3a_{03}^2a_{21} - 3a_{12}^2a_{03}) - 8a_{03}a^7a_{04}) \cos^5\theta\\
&+(a^6b(22a_{03}a_{13} + 18a_{12}a_{04}) + a^4b^5(72a_{12} + 24a_{30}) - 10a_{40}b^7a_{30}\\
&-a^4b^3(90a_{21}a_{13} + 126a_{12}a_{22} + 8a_{04}a_{30} + 26a_{03}a_{31})\\
&+a^2b^5(54a_{30}a_22 + 24a_{40}a_{12} + 102a_{21}a_{31}) + a^2b^3(12a_{30}a_{03}a_{21}\\
&+6a_{12}^2a_{30} - 18a_{21}^2a_{12}) - 24b^7a^2a_{30} + 3b^5(a_{21}^2a_{30} - a_{30}^2a_{12})\\
&-a^4b(3a_{03}^2a_{30} + 6a_{21}a_{12}a_{03} - 9a_{12}^3) - 72a^6b^3a_{12}) \sin\theta \cos^4\theta\\
&+(-2a_{03}a^7a_{04} - 24a^7b^2a_{03} + 6a^5(a_{12}^2a_{03} - a_{03}^2a_{21}) - ab^6(16a_{21}a_{40}\\
&+24a_{30}a_{31}) - a^5b^2(12a_{21}a_{04} + 18a_{03}a_22 + 48a_{12}a_{13})\\
&+a^5b^4(24a_{03} + 96a_{21}) + a^3b^2(15a_{30}a_{12}a_{03} + 12a_{21}^2a_{03} - 27a_{12}^2a_{21})\\
&+a^3b^4(4a_{03}a_{40} + 120a_{21}a_22 + 70a_{12}a_{31} + 26a_{30}a_{13})\\
&+ab^4(6a_{21}^3 - 3a_{30}^2a_{03} - 3a_{30}a_{12})a_{21} - 96b^6a^3a_{21})) \cos^3\theta + (48a^6a_{21}b^3\\
&+(4a_{13}a_{03} + 2a_{21}a_{04})a^6b + (9a_{21}a_{03}a_{21} + 3a_{03}^2a_{30} - 12a_{21}^3)a^4b\\
&-(48a_{21} + 24a_{30})a^4b^5 + (6a_{03}a_{31} + 34a_{13}a_{21} + 48a_{21}a_{22}\\
&+2a_{04}a_{30})a^4b^3 + 24b^7a^2a_{30} - (36a_{31}a_{21} + 18a_{22}a_{30} + 6a_{40}a_{21})a^2b^5\\
&+(9a_{21}^2a_{21} - 9a_{30}a_{03}a_{21})a^2b^3) \sin\theta \cos^2\theta\\
&+(4a_{03}a^7a_{04} + (-3a_{12}^2a_{03} + 3a_{03}^2a_{21})a^5\\
&+6a_{13}b^2a^5a_{12} - 36a^5b^4a_{21} + 3(3a_{12}^2a_{21} - 2a_{21}^2a_{03} - a_{12}a_{03}a_{30})b^2a^3\\
&-4(6a_{21}a_{22} + a_{13}a_{30} + 3a_{12}a_{31})b^4a^3 + 36b^6a^3a_{21}) \cos\theta\\
&+(12b^5a^4a_{12} - 12a^6b^3a_{12} - (2a_{12}a_{04} + 2a_{13}a_{03})a^6b\\
&-(4a_{13}a_{21} - 6a_{12}a_{22})a^4b^3 + (3a_{12}^3 - 3a_{12}a_{03}a_{21})a^4b) \sin\theta].
\end{aligned}
$$

$$
\begin{aligned}
l_1 =& -\frac{1}{6}a^3b^3[(18b^5aa_{21}a_{30}+18ba^5a_{12}a_{03}\\
&-6b^3a^3(a_{30}a_{03}+9a_{21}a_{12}))\cos^6\theta\\
&+(3a^6a_{03}^2-9(2a_{21}a_{03}+3a_{12}^2)b^2a^4\\
&+9(2a_{30}a_{12}+3a_{21}^2)b^4a^2-3b^6a_{30}^2)\sin\theta\cos^5\theta\\
&+(-15b^5aa_{21}a_{30}+9b^3a^3(9a_{21}a_{12}+a_{30}a_{03})\\
&-39ba^5a_{12}a_{03}-32b^3a^5a_{13}+32b^5a^3a_{31})\cos^4\theta\\
&+(48b^4a^4a_{22}-8b^6a^2a_{40}-6b^4a^2(3a_{21}^2+2a_{12}a_{30})-8b^2a^6a_{04}\\
&-6a^6a_{03}^2+12b^2a^4(2a_{21}a_{03}+3a_{12}^2))\sin\theta\cos^3\theta\\
&+(-24b^5a^3a_{31}+40b^3a^5a_{13}+24ba^5a_{12}a_{03}\\
&-3b^3a^3(9a_{21}a_{12}+a_{30}a_{03}))\cos^2\theta\\
&+(-12b^6a^4-3b^2a^4(3a_{12}^2+2a_{21}a_{03})+8b^2a^6a_{04}\\
&+12b^4a^6+3a^6a_{03}^2-24b^4a^4a_{22})\sin\theta\cos\theta\\
&-(3ba^5a_{12}a_{03}+8b^3a^5a_{13})]
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
n_1 =& a^3b^3[(18b^5aa_{21}a_{30}-6b^3a^3(9a_{21}a_{12}+a_{30}a_{03})\\
&+18ba^5a_{12}a_{03})\cos^6\theta\\
&+(9b^4a^2(2a_{12}a_{30}+3a_{21}^2)-9b^2a^4(2a_{21}a_{03}+3a_{12}^2)\\
&-3b^6a_{30}^2+3a^6a_{03}^2)\sin\theta\cos^5\theta\\
&+(16b^3a^5a_{13}+9b^3a^3(a_{30}a_{03}+9a_{21}a_{12})\\
&-16b^5a^3a_{31}-39b^5aa_{21}a_{30}-15ba^5a_{12}a_{03})\cos^4\theta\\
&+(-24b^4a^4a_{22}+6b^6a_{30}^2-12b^4a^2(3a_{21}^2+2a_{12}a_{30})+4b^2a^6a_{04}\\
&+4b^6a^2a_{40}+6b^2a^4(2a_{21}a_{03}+3a_{12}^2))\sin\theta\cos^3\theta\\
&+(-20b^3a^5a_{13}-6ba^5a_{12}a_{03}+12b^5a^3a_{31}\\
&+18b^5aa_{21}a_{30}-3b^3a^3(13a_{21}a_{12}+a_{30}a_{03}))\cos^2\theta\\
&+(-12b^6a^4-3a_{12}^2b^2a^4+12b^4a^4a_{22}+12b^4a^6\\
&+6b^4a^2(2a_{21}^2+a_{12}a_{30})-3a^6a_{03}^2-4b^2a^6a_{04})\sin\theta\cos\theta\\
&+(4b^3a^5a_{13}+6b^3a^3a_{21}a_{12}+3ba^5a_{12}a_{03})].
\end{aligned}
\tag{21}
$$

Q.E.D.

3.2. Principal Nets around a Definite Critical End Points

Proposition 5. *Suppose that p is an end critical point and consider the chart defined in Lemma 3 such that h is given by equation (12), with $a > 0$, $b > 0$ (local minimum). Then the behavior of curvature lines near p is the following.*

i) *One principal foliation is radial.*

ii) *The other principal foliation surrounds p and the associated return map Π is such that $\Pi(0) = 0$, $\Pi'(0) = 1$, $\Pi''(0) = 0$, $\Pi'''(0) = 0$ and $\Pi''''(0) = \dfrac{\pi}{2^{10}a^5b^5}\Delta$, where*

$$\begin{aligned}\Delta =&12(a_{30}a_{21} + 3a_{03}a_{30} - 5a_{12}a_{21})b^6a^4\\ &+12(5a_{12}a_{21} - a_{12}a_{03} - 3a_{03}a_{30})a^6b^4\\ &+4(3a_{04}a_{30}a_{21} + a_{13}a_{21}^2 + 10a_{31}a_{03}a_{21})a^4b^4\\ &-4(10a_{13}a_{12}a_{30} + 3a_{40}a_{12}a_{03} + a_{31}a_{12}^2)a^4b^4\\ &+4(a_{13}a_{03}^2 - a_{04}a_{12}a_{03})a^8 + 4(a_{40}a_{30}a_{21} - a_{31}a_{30}^2)b^8\\ &+3(a_{30}^3a_{03} + 2a_{30}a_{21}^3 - 3a_{30}^2a_{21}a_{12})b^6\\ &+3(3a_{12}a_{03}^2a_{21} - 2a_{12}^3a_{03} - a_{30}a_{03}^3)a^6\\ &+4[a_{03}(2a_{13}a_{21} - 3a_{04}a_{30} - 3a_{31}a_{03} + 12a_{22}a_{12})\\ &+5a_{04}a_{12}a_{21} - 13a_{13}a_{12}^2]a^6b^2\\ &+4[a_{30}(+3a_{13}a_{30} - 2a_{31}a_{12} + 3a_{40}a_{03} - 12a_{22}a_{21})\\ &-5a_{40}a_{21}a_{12} + 13a_{31}a_{21}^2]a^2b^6\\ &+9(a_{30}a_{21}^2a_{03} - 2a_{30}a_{21}a_{12}^2 + a_{12}a_{21}^3)a^2b^4\\ &+9(-a_{12}^3a_{21} - a_{30}a_{03}a_{12}^2 + 2a_{12}a_{21}^2a_{03})a^4b^2\\ &+12b^2a^8a_{12}a_{03} - 12b^8a^2a_{30}a_{21}\end{aligned}$$

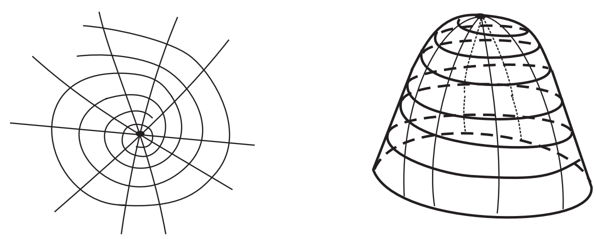

Fig. 6. Curvature lines near a definite focal critical end point, $\Delta > 0$.

Proof. Consider the implicit differential equation (15)

$$(l_0 + rl_1 + h.o.t)dr^2 + (m_0 + m_1 r + h.o.t)drd\theta$$
$$+r^2(n_0/2 + n_1 r/6 + n_2 r^2/24 + h.o.t.)d\theta^2 = 0.$$

As $m_0 = -8a^7b^7 \neq 0$ this equation factors in the product of two equations in the standard form, as follows.

$$\begin{aligned}\frac{dr}{d\theta} =& -\frac{1}{2}\frac{n_0}{m_0}r^2 + \frac{1}{6}\frac{3m_1 - n_0 + m_0 n_1}{m_0^2}r^3\\ &-\frac{1}{24}\frac{(12m_1^2 n_0 + m_0^2 n_2 + 6l_0 n_0^2 - 6m_0 m_2 n_0 - 4m_0 m_1 n_1)}{m_0^3}r^4\\ &+h.o.t.\\ =&\frac{1}{2}d_2(\theta)r^2 + \frac{1}{6}d_3(\theta)r^3 + \frac{1}{24}d_4(\theta)r^4 + h.o.t.\end{aligned} \tag{22}$$

$$\frac{d\theta}{dr} = -\frac{l_0}{m_0} + h.o.t. \tag{23}$$

The solutions of the nonsingular differential equation (23) defines the radial foliation.

Writing $r(\theta, h) := h + q_1(\theta)h + q_2(\theta)h^2/2 + q_3(\theta)h^3/6 + q_4(\theta)h^4/24 + h.o.t.$ as the solution of differential equation (22) it follows that:

$$\begin{aligned}q_1'(\theta) =&0\\ q_2'(\theta) =&d_2(\theta) = -\frac{n_0}{m_0}\\ q_3'(\theta) =&3d_2(\theta)q_2(\theta) + d_3(\theta)\\ q_4'(\theta) =&3d_2(\theta)q_2(\theta)^2 + 4d_2(\theta)q_3(\theta) + 6d_3(\theta)q_2(\theta) + d_4(\theta)\end{aligned} \tag{24}$$

As $q_1(0) = 0$ it follows that $q_1(\theta) = 0$. Also $q_i(0) = 0,\ i = 2,\ 3,\ 4.$ So it follows that $q_2(\theta) = -\int_0^\theta \frac{n_0}{m_0}d\theta$. From the expression of n_0, an odd polynomial in the variables $c = \cos\theta$ and $s = \sin\theta$, it follows that $q_2(2\pi) = 0$ and therefore $\Pi'(0) = 1,\ \ \Pi''(0) = 1$.

Now, $q_3(\theta) = \int_0^\theta [q_2(\theta)q_2'(\theta) + d_3(\theta)]d\theta$.

Therefore $q_3(\theta) = \frac{1}{2}q_2^2(\theta) + \int_0^\theta d_3(\theta)d\theta$.

So,

$$\Pi'''(0) = q_3(2\pi) = \int_0^{2\pi} d_3(\theta)d\theta.$$

A long calculation, confirmed by algebraic computation, shows that $q_3(2\pi) = 0$.

Integrating the last linear equation in (24), it follows that:

$$q_4(\theta) = 3q_2(\theta)^3 + 4q_2(\theta)\int_0^\theta d_3(\theta)d\theta + 2\int_0^\theta q_2(\theta)d_3(\theta)d\theta + \int_0^\theta d_4(\theta)d\theta.$$

Therefore,

$$\Pi''''(0) = q_4(2\pi) = 2\int_0^{2\pi} q_2(\theta)d_3(\theta)d\theta + \int_0^{2\pi} d_4(\theta)d\theta.$$

Integration of the right hand member, corroborated by algebraic computation, gives $\Pi''''(0) = \frac{\pi}{2^{10}a^5b^5}\Delta$. This ends the proof. Q.E.D.

Remark 1. *When $\Delta \neq 0$ the foliation studied above spirals around p. The point is then called a focal definite critical end point.*

3.3. Principal Nets at Saddle Critical End Points

Let p be a saddle critical point of h as in equation (13) with the finite region defined by $h(u,v) > 0$.

Then the differential equation (14) is given by

(25)

$$\begin{aligned}
&Ldv^2 + Mdudv + Ndu^2 = 0,\\
L(u,v) =& -a^3u^2 + 2a^2uv - 3aa_{12}uv^2 + (2a^2a_{12} - 3aa_{21} - 2a_{30})u^2v\\
&+(aa_{30} + 2a^2a_{21})u^3 + (2a_{12} + aa_{03})v^3 + h.o.t.\\
M(u,v) =& -2a^2v^2 + (4a_{30} - a^2a_{12})uv^2 + (a^2a_{21} - 2aa_{30})u^2v\\
&+a^2a_{30}u^3 + (2aa_{12} - a^2a_{03} + 4a_{21})v^3 + h.o.t.\\
N(u,v) =& av^2[a^2 - 2(aa_{21} + a_{30})u - 2(aa_{12}u + a_{21})v] + h.o.t.
\end{aligned}$$

Proposition 6. *Suppose that p is a saddle critical point of the surface represented by $w = h(u,v)$ as in Lemma 3. Then the behavior of the extended principal foliations in the region ($h(u,v) \geq 0$), near p, is the following.*

i) *If $aa_{30}(a_{03}a^3 + 3aa_{21} + 3a^2a_{12} + a_{30}) > 0$ then the curvatures of both branches of $h^{-1}(0)$ at p have the same sign and the behavior is as in Fig. 7, left -even case.*

ii) *If $aa_{30}(a_{03}a^3 + 3aa_{21} + 3a^2a_{12} + a_{30}) < 0$ then the curvatures of both branches of $h^{-1}(0)$ at p have opposite signs and the behavior is as in the Fig. 7, right - odd case.*

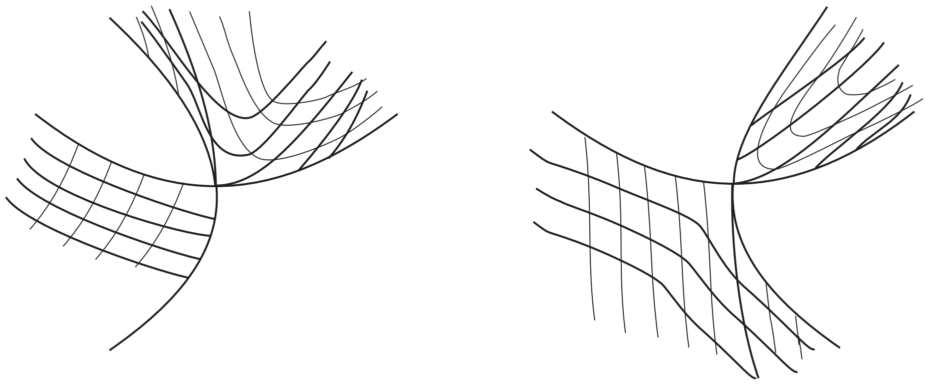

Fig. 7. Curvature lines near saddle critical end point: even case, left, and odd case, right.

Proof. In order to analyze the behavior of the principal lines near the branch of $h^{-1}(0)$ tangent to $v = 0$ consider the projective blowing-up $u = u, v = uw$.

The differential equation $Ldv^2 + Mdudv + Ndu^2 = 0$ defined by equation (25) is, after some simplification, given by:

$$(26) \qquad \begin{aligned}&(-\frac{1}{4}a_{30}^2u + aa_{30}w + O(2))du^2 + (aa_{30}u - 2a^2w + O(2))dwdu\\ &-(a^2u + O(2))dw^2 = 0.\end{aligned}$$

To proceed consider the resolution of the singularity $(0,0)$ of equation (26) by the Lie-Cartan line field $X = (q\mathcal{G}_q, \mathcal{G}_q, -(q\mathcal{G}_u + \mathcal{G}_w)), \quad q = \frac{du}{dw}$. Here $\mathcal{G}$ is

$$\mathcal{G} = (-\frac{1}{4}a_{30}^2u + aa_{30}w + O(2))q^2 + (aa_{30}u - 2a^2w + O(2))q - (a^2u + O(2)).$$

The singularities of X, contained in axis q (projective line), are the solutions of the equation $q(2a - a_{30}q)(6a - a_{30}q) = 0$.

Also,

$$DX(0,0,q) = \begin{pmatrix} \frac{1}{2}(qa_{30}(2a - a_{30}q)) & -2qa(a - a_{30}q) & 0\\ \frac{1}{2}(a_{30}(2a - a_{30}q)) & -2a(a - a_{30}q) & 0\\ 0 & 0 & A_{33}\end{pmatrix}$$

where $A_{33} = 3a^2 - 4aa_{30}q + \dfrac{3}{4}a_{30}^2q^2$.

The eigenvalues of $DX(0,0,q)$ are $\lambda_1(q) = -2a^2 + 3aa_{30}q - \dfrac{1}{2}a_{30}^2q^2$, $\lambda_2(q) = 3a^2 - 4aa_{30}q + \dfrac{3}{4}a_{30}^2q^2$ and $\lambda_3(q) = 0$.

Therefore the non zero eigenvalues of $DX(0)$ are $-2a^2$ and $3a^2$. At $q_1 = \frac{2a}{a_{30}}$ the eigenvalues of $DX(0,0,q_1)$ are $2a^2$ and $-2a^2$. Finally at $q_2 = \frac{6a}{a_{30}}$ the eigenvalues of $DX(0,0,q_2)$ are $6a^2$ and $-2a^2$.

As a conclusion of this analysis we assert that the net of integral curves of equation (26) near $(0,0)$ is the same as one of the generic singularities of quadratic differential equations, well known as the Darbouxian D_3 or a tripod, [14, 17]. See Fig. (5).

Now observe that the curvature at 0 of the branch of $h^{-1}(0)$ tangent to $v=0$ is precisely $k_1 = \dfrac{a_{30}}{a}$ and that $h^{-1}(0)\backslash\{0\}$ is solution of equation (25). So, after the blowing down, only one branch of the invariant curve $v = \dfrac{a_{30}}{6a}u^2 + O(3)$ is contained in the finite region $\{(u,v) : h(u,v) > 0\}$.

Analogously, the analysis of the behavior of the principal lines near the branch of $h^{-1}(0)$ tangent to $v = au$ can be reduced to the above case. To see this perform a rotation of angle $\tan\theta = a$ and take new orthogonal coordinates $\bar{u}$ and $\bar{v}$ such that the axis $\bar{u}$ coincides with the line $v = au$.

The curvature at 0 of the branch of $h^{-1}(0)$ tangent to $v = au$ is

$$k_2 = -\frac{a_{03}a^3 + 3aa_{21} + 3a^2a_{12} + a_{30}}{3a}.$$

Performing the blowing-up $v = v, \quad u = sv$ in the differential equation (25) we conclude that it factors in two transversal regular foliations.

Gluing the phase portraits studied so far and doing their blowing down, the net explained below is obtained.

The finite region $(h(u,v) > 0)$ is formed by two sectorial regions R_1, with $\partial R_1 = C_1 \cup C_2$ and R_2 with $\partial R_2 = L_1 \cup L_2$. The two regular branches of $h^{-1}(0)$ are given by $C_1 \cup L_1$ and $C_2 \cup L_2$.

If $k_1k_2 < 0$ – odd case – then one region, say R_1, is convex and ∂R_1 is invariant for one extended principal foliation and ∂R_2 is invariant for the other one. In each region, each foliation has an invariant separatrix tangent to the branches of $h^{-1}(0)$. See Fig. 7, right.

If $k_1k_2 > 0$ – even case – then in a region, say R_1, the extended principal foliations are equivalent to a trivial ones, i.e., to $dudv = 0$, with C_1 being a leaf of one foliation and C_2 a leaf of the other one. In the region R_2 each extended principal foliation has a hyperbolic sector, with separatrices tangent to the branches of $h^{-1}(0)$ as shown in Fig. 7, left. Here $C_2 \cup L_1$ are leaves of one principal foliation and $C_1 \cap L_2$ are leaves of the other one. Q.E.D.

§4. Concluding Comments and Related Problems

We have studied here the simplest patterns of principal curvature lines at end points, as the supporting smooth surfaces tend to infinity in $\mathbb{R}^3$, following the paradigm established in [6] to describe the structurally stable patterns for principal curvature lines escaping to infinity on algebraic surfaces.

We have recovered here –see Proposition 1 – the main results of the structurally stable inflexion ends established in [6] for algebraic surfaces: namely the hyperbolic and elliptic cases.

In the present context a surface $A(\alpha^c)$ with $\alpha^c \in \mathcal{A}_c^k$ is said to be *structurally stable* at a singular end point p if the C^s, topology with $s \leq k$ if the following holds. For any sequence of functions $\alpha_n^c \in \mathcal{A}_c^k$ converging to α^c in the C^s topology, there is a sequence p_n of end points of $A(\alpha_n^c)$ converging to p such that the extended principal nets of $\alpha_n = \alpha_n^c \circ \mathbb{P}$, at these points, are topologically equivalent to extended principal net of $\alpha = \alpha^c \circ \mathbb{P}$, at p.

Recall (see [6]) that two nets $N_i\; i = 1, 2$ at singular points $p_i\; i = 1, 2$ are topologically equivalent provided there is a homeomorphism of a neighborhood of p_1 to a neighborhood of p_2 mapping the respective points and leaves of the respective foliations to each other.

The analysis in Proposition 1 makes clear that the hyperbolic and elliptic inflexion end points are also structurally stable in the C^3 topology for defining α^c functions in the space $\mathcal{A}_c^k$, $k \geq 4$.

We have studied also six new cases –see Propositions 2 to 6 – which represent the simplest patterns where the structural stability conditions fail.

The lower smoothness class C^k for the validity of the analysis in the proofs of these propositions is as follows. In Propositions 2 and 6 we must assume $k \geq 4$. In Proposition 5, clearly $k \geq 5$ must hold.

In each of these cases it is not difficult to describe partial aspects of possible topological changes –bifurcation phenomena– under small perturbations of the defining functions α^c.

However it involves considerably technical work to provide the full analysis of bifurcation diagrams of singular end points and their global effects in the principal nets.

We recall here that the study of the bifurcations of principal nets away from end points, i.e., in compact regions was carried out in [13], focusing the umbilic singular points. There was also established the connection between umbilic codimension one singularities and their counterparts in critical points of functions and the singularities of vector fields, following the paradigm of first order structural stability in the sense of

Andronov and Leontovich [1], generalized and extended by Sotomayor [20]. Grosso modo this paradigm aims to characterize the structurally stable singularities under small perturbations inside the space of non-structurally stable ones.

To advance an idea of the bifurcations at end points, below we will suggest pictorially the local bifurcation diagrams in the three regular cases studied so far.

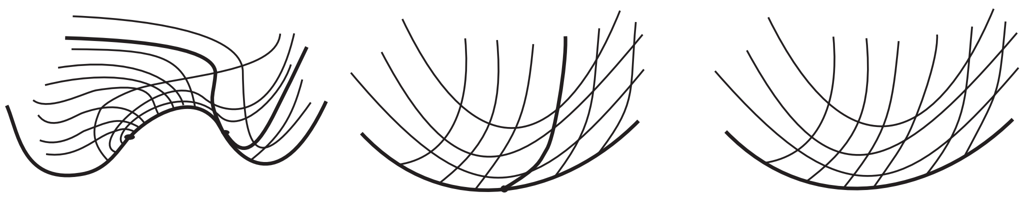

Fig. 8. Bifurcation Diagram of Curvature lines near regular end points: elimination of hyperbolic and elliptic inflexion points

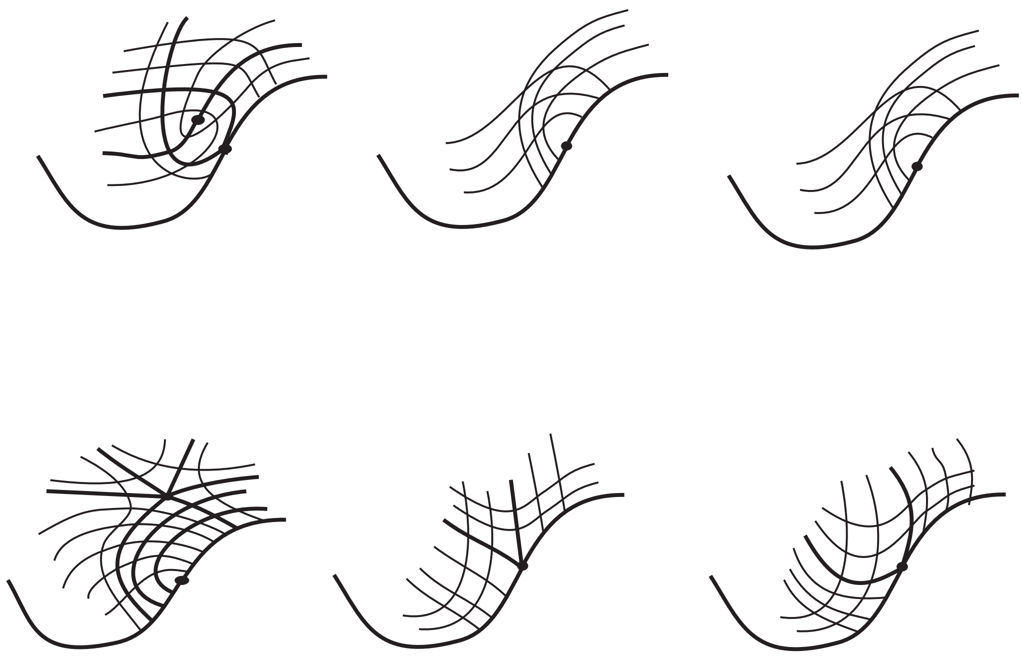

Fig. 9. Bifurcation Diagram of Curvature lines near umbilic-inflexion end points. Upper row: D_1 umbilic - hyperbolic inflexion. Lower row: D_3 umbilic - elliptic inflexion.

The description of the bifurcations in the critical cases, however, is much more intricate and will not be discussed here.

The full analysis of the non-compact bifurcations as well as their connection with first order structural stability will be postponed to a future paper.

Concerning the study of end points, see also [16], where Gutierrez and Sotomayor studied the behavior of principal nets on constant mean curvature surfaces, with special analysis of their periodic leaves, umbilic and end points. However, the patterns of behavior for this class of surfaces is non-generic in the sense of the present work.

We conclude proposing the following problem.

Problem 1. Concerning the case of the focal critical end point, we propose to the reader to provide a conceptual analysis and a proof of Proposition 5, avoiding long calculations and the use of Computer Algebra.

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(r) does not imply (n) or (npf) for definable sets in non polynomially bounded o-minimal structures.

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Abstract.

It is known that if two subanalytic strata satisfy Kuo's ratio test, then the normal cone of the larger stratum Y along the smaller X satisfies two nice properties: the fiber of the normal cone at any point of X is the tangent cone to the fiber of Y over that point; the projection of the normal cone to X is open ("normal pseudo-flatness"). We present an example with X a line and Y a surface which is definable in any non polynomially bounded o-minimal structure such that the pair satisfies Kuo's ratio test, but neither of the above properties hold for the normal cone.

In **[OT2]** P. Orro and the first author defined a regularity condition (r^e) for C^2 stratifications which provides a way of quantifying Kuo's ratio test (r) **[K]**, because for subanalytic stratifications, Whitney's condition (a) and (r^e) hold, for some e, $0 < e < 1$, if and only if Kuo's ratio test (r) is satisfied. They further showed that if $0 \leq e < 1$, $(a + r^e)$ implies rather good behaviour of the normal cone along strata: the special fibre of the normal cone at a point x in a stratum X is equal to the tangent cone to the normal slice to X through x (this property is denoted by (n) in **[OT2]**), and the stratification is normally pseudo-flat (abbreviated to (npf)). Thus for subanalytic stratifications, (r) implies both (n) and (npf).

In the example below, which is not subanalytic, (r) holds, but neither (n) nor (npf) hold, and one can check that (r^e) fails for all $0 \leq e < 1$, so that in particular Verdier's condition (w) fails ((w) is equivalent to $(a + r^0)$). Example 4.2 of **[OT2]** provides a different non-subanalytic example without (n) or (npf), called a Kuo Escargot (cf. **[OT1]**), which was (b)-regular and not (r)-regular, but this example was not definable

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in any o-minimal structure, due to spiralling. The example below is log-analytic, so is definable in the o-minimal structure $R_{exp,an}$, but it is not definable in *any* polynomially bounded o-minimal structure, by Miller's dichotomy **[M]** stating that an o-minimal structure is not polynomially bounded if and only if it possesses the exponential function as a definable function. By the same dichotomy, our example is definable in *every* o-minimal structure which is not polynomially bounded.

It is straightforward to show that (r) implies (r^e) for some $e, 0 \leq e < 1$, for stratified sets whose strata are definable in a polynomially bounded o-minimal structure, as the proof of the implication in **[OT2]** uses only curve selection and the Łojasiewicz inequality (see **[DM]** or **[V]**).

One can check easily that $(r_{\mathrm{cod}\,1})$ fails for our example showing that (r) does not imply (r^*) for definable sets in non polynomially bounded o-minimal structures. The proof in **[NT]** that (r) implies (r^*) for subanalytic strata presumably works for polynomially bounded o-minimal structures (but it would be good to have a complete proof of this).

One can also check that (b) holds for the example, showing that (b) does not imply (b^*) along a stratum X for definable sets in non polynomially bounded o-minimal structures, even when $\dim X = 1$. Recall from **[NT]** that (b) implies (b^*) for subanalytic strata if $\dim X = 1$ because then (r) and (b) are equivalent, by **[K]**.

Presumably, for definable sets in polynomially bounded o-minimal structures, (r) implies (b), and (b) implies (r) if $\dim X = 1$, so that then (b) would imply (b^*) if $\dim X = 1$.

In the example below the density is actually constant along the small stratum, so in particular it is continuous. In 2000, G. Comte **[C]** has shown continuity of the density along strata of any (r)-regular subanalytic stratification (hence along 1-dimensional strata of any (b)-regular subanalytic stratification). In 2003 G. Valette found a different proof of this result **[V]** with a strengthened conclusion and has very recently (2003) announced an extension to any (b)-regular subanalytic stratification.

Are these results about the density true for definable sets in any o-minimal structure ?

Definitions. Below k will denote an integer greater than or equal to 2. Let S be a closed stratified subset of $\mathbb{R}^n$, whose strata are differentiable submanifolds of class C^k. For each stratum X of S denote by $C_X S$ the normal cone of S along X, that is the restriction to X of the closure of the set $\{(x, \mu(x\pi(x))) : x \in S - X\} \subset \mathbb{R}^n \times S^{n-1}$, where π is the local canonical projection onto X, $\mu(x)$ is the unit vector $\frac{x}{\|x\|}$, and here and throughout the paper pq denotes the vector $q-p$. In fact C_XS is the union of the normal cones C_XY_i, where $\{Y_i\}$ are the strata of S whose closures contain X.

Condition (n): The fibre $(C_XS)_x$ of C_XS at a point x of X is the tangent cone $C_x(S_x)$ to the fibre $S_x = S \cap \pi^{-1}(x)$ of S at x, for every stratum X of S.

Normal pseudo-flatness (npf): The projection $p : C_XS \to X$ is open for every stratum X of S.

When a stratification satisfies two conditions, for example Whitney (a)-regularity and (n)-regularity, we say it is $(a+n)$-regular. Subanalytic stratifications satisfying $(a+n)$ or (npf) have a normal cone with good behaviour from the point of view of the dimension of its fibres. In fact they satisfy the condition

$$\dim(C_XS)_x \leq \dim S - \dim X - 1. \qquad (*)$$

This is obvious for $(a+n)$, while for (npf) it follows from $(5.1.ii')$ of [**OT2**]. For differentiable stratifications one first needs to be able to define the dimension.

Despite this limitation, the tangent cone $C_x(S_x)$ to the fibre $S_x = S \cap \pi^{-1}(x)$ (hence the fibre $(C_XS)_x$ of the normal cone, assuming (n)) can be quite arbitrary: recent work of Ferrarotti, Fortuna and Wilson show that every closed semi-algebraic cone of codimension ≥ 1 is realised as the tangent cone at a point of a certain real algebraic variety [**FFW**], while Kwieciński and Trotman showed that *every* closed cone is realised as the tangent cone at an isolated singularity of a certain $C^\infty(b)$-regular stratified espace [**KT**].

Hironaka showed in [**H**] that a Whitney stratification (i.e. (b)-regular) of an analytic set (real or complex) is normally pseudo-flat along each stratum. J.-P. Henry et M. Merle [**HM2**] obtained (n) with S replaced by $X \cup Y$ when X and Y are two adjacent strata of a subanalytic Whitney stratification of $X \cup Y$.

Every C^2 (w)-regular stratification satisfies automatically (a) and (r^e), i.e. $(a+r^e)$. For subanalytic strata the combination $(a+r^e)$ is equivalent to the ratio test (r) introduced by T.-C. Kuo in 1971, which implies Whitney's condition (b) [**K**]; since [**T**] we know that (r) is strictly weaker than (w) in the semialgebraic case, and there even exist real algebraic examples [**BT**]. The equivalence of (b), (r) and (w) for complex analytic stratifications was completed by Teissier in 1982 ([**Te2**], [**HM1**]).

In **[OT2]** it is proved that every $(a + r^e)$-regular stratification is normally pseudoflat and satisfies condition (n). Hence for (r)-regular stratifications which are definable in a polynomially bounded o-minimal structure, (n) and (npf) hold.

We recall the definitions of the conditions (a) and (b) of Whitney, (r) of Kuo **[K]**, and (w) of Kuo-Verdier **[Ve]**.

Let X and Y be two submanifolds of $\mathbb{R}^n$ such that $X \subset \overline{Y}$, and let π be the local projection onto X. Following Hironaka **[H]**, denote by $\alpha_{Y,X}(y)$ the distance of T_yY to $T_{\pi(y)}X$, which is

$$\alpha_{Y,X}(y) = \max\{< \mu(u), \mu(v) >: u \in N_yY - \{0\}, v \in T_{\pi(y)}X\},$$

and by $\beta_{Y,X}(y)$ the distance of $y\pi(y)$ to T_yY expressed as

$$\beta_{Y,X}(y) = \max\{< \mu(u), \mu(y\pi(y))\} >: u \in N_yY - \{0\}\},$$

where $<,>$ is the scalar product on $\mathbb{R}^n$.

For $v \in \mathbb{R}^n$, the distance of the vector v to a plane B is

$$\eta(v, B) = \sup\{< v, n >: n \in B^{\perp}, \|n\| = 1\}.$$

Set

$$d(A, B) = \sup\{\eta(v, B) : v \in A, \|v\| = 1\},$$

so that in particular

$$\alpha_{Y,X}(y) = d(T_{\pi(y)}X, T_yY).$$

Set also

$$R_{Y,X}(y) = \frac{\|y\|\alpha_{Y,X}(y)}{\|y\pi(y)\|} \quad \text{and} \quad W_{Y,X}(y, x) = \frac{d(T_xX, T_yY)}{\|yx\|}.$$

Definition. The pair of strata (X, Y) satisfies, at $0 \in X$: condition (a) if, for y in Y,

$$\lim_{y \to 0} \alpha_{Y,X}(y) = 0,$$

condition (b^{π}) if, for y in Y,

$$\lim_{y \to 0} \beta_{Y,X}(y) = 0,$$

condition (b) if, for y in Y,

$$\lim_{y \to 0} \alpha_{Y,X}(y) = \lim_{y \to 0} \beta_{Y,X}(y) = 0,$$

condition (r) if, for y in Y,

$$\lim_{y \to 0} R_{Y,X}(y) = 0,$$

condition (w) if, for y in Y and x in X, $W_{Y,X}(y, x)$ is bounded near 0.

In **[OT2]** P. Orro and the first author introduced the following condition of Kuo-Verdier type.

Definition. Let $e \in [0, 1)$. One says that (X, Y) satisfies condition (r^e) at $0 \in X$ if, for $y \in Y$, the quantity $R_e(y) = \frac{\|\pi(y)\|^e \alpha_{Y,X}(y)}{\|y\pi(y)\|}$ is bounded near 0.

This condition is a C^2 diffeomorphism invariant. It is Verdier's condition (w) when $e = 0$, hence (w) implies (r^e) for all $e \in [0, 1)$. But, unlike (w), condition (r^e) when $e > 0$ does not imply condition (a) : a counter-example which is a semi-algebraic surface can be obtained by pinching a half-plane $\{z \geq 0, y = 0\}$ of $\mathbb{R}^3$, with boundary the axis $0x = X$, in a cuspidal region $\Gamma = \{x^2 + y^2 \leq z^p\}$, where p is an odd integer such that $p > \frac{2}{e}$, such that in Γ there are sequences tending to 0 for which condition (a) fails. Such an example will be (r^e)-regular.

Theorem[OT2]. *Every $(a + r^e)$-regular stratification is normally pseudo-flat and satisfies condition (n).*

Corollary. *For (r)-regular stratifications which are definable in a polynomially bounded o-minimal structure, (n) and (npf) hold.*

Now we recall the definition of E^*-regularity for E an equisingularity condition, as in **[OT1]**. This notion came from the discussion of B. Teissier in his 1974 Arcata lectures **[Te1]**. Teissier stated that one requirement for an equisingularity condition to be "good" is that it be preserved after intersection with generic linear spaces containing a given linear stratum. Various equisingularity conditions have been shown to have this property, notably Whitney (b)-regularity for complex analytic stratifications ([**Te2**], [**HM1**]), and Kuo's ratio test (r) and Verdier's condition (w) for subanalytic stratifications [**NT**].

Definition. Let M be a C^2-manifold. Let X be a C^2-submanifold of M and $x \in X$. Let Y be a C^2-submanifold of M such that $x \in \overline{Y}$, and $X \cap Y = \emptyset$. Let E denote an equisingularity condition (examples: Whitney (b), (r), (w)). Then (X, Y) is said to be $E_{\text{cod } k}$-regular at x $(0 \leq k < \text{cod } X)$ if there is an open dense subset U^k of the Grassmann manifold of codimension k subspaces of T_xM containing T_xX such that if W is a C^2 submanifold of M with $X \subset W$ near x, and $T_xW \in U^k$, then W is transverse to Y near x, and $(X, Y \cap W)$ is E-regular at x.

Definition. (X, Y) is said to be E^*-regular at x if (X, Y) is $E_{\text{cod } k}$-regular for all k, $0 \leq k < \text{cod } X$.

Theorem[NT]. *For subanalytic stratifications,* (r) *implies* (r^*) *and* (w) *implies* (w^*).

Corollary. *For subanalytic* (b)*-regular stratifications,* (b^*) *holds over every* 1*-dimensional stratum.*

In the log-analytic example below, (r) and (b) hold, but (r^*) and (b^*) fail.

Example.

In $\mathbf{R}^3$ consider the graph Y of the function $f(x, z)$, for $z > 0$, and x and z small, where

$$y = f(x, z) = z - \frac{z}{\ln z} \ln(x + \sqrt{x^2 + z^2}).$$

Note that $\lim_{z \to 0} f(x, z) = 0$.

Then let X be the x-axis, so that $X \subset \overline{Y}$, and X and Y are disjoint C^∞ submanifolds of $\mathbf{R}^3$. We consider the closed stratified set S with just 2 strata (X, Y).

Remark 1. $f(x, z) = -f(-x, z)$, *i.e.* f *is an odd function of* x.

Proof.

$$\begin{aligned} f(x, z) + f(-x, z) &= 2z - \frac{z}{\ln z}[\ln(x + \sqrt{x^2 + z^2}) + \ln(-x + \sqrt{x^2 + z^2})] \\ &= 2z - \frac{z}{\ln z}[\ln(x^2 + z^2 - x^2)] = 2z - \frac{z.2 \ln z}{\ln z} = 0. \end{aligned}$$

Q.E.D.

Remark 2. $X \subset \overline{Y}$, *because* $\lim_{z\to 0} f(x,z) = 0$.

Proof. Obviously

$$\lim_{z\to 0} z = 0, \quad \text{and} \quad \lim_{z\to 0} \frac{1}{\ln z} = 0.$$

If $x > c > 0$, then $|\ln(x + \sqrt{x^2+z^2})| < |\ln(2c)|$, so that

$$\lim_{z\to 0} z \ln(x + \sqrt{x^2+z^2}) = 0.$$

By remark 1 we do not need to study the case of $x < 0$.
If both x and z tend to 0, consider the cases :
(i) $\frac{|z|}{|x|} \to 0$. Then

$$|z \ln(x + \sqrt{x^2+z^2})| < |z \ln(2x)| < |x \ln(2x)| \to 0 \quad \text{as} \quad x \to 0.$$

(ii) $\frac{|x|}{|z|}$ is bounded. Then

$$|z \ln(x + \sqrt{x^2+z^2})| \equiv |z \ln z| \to 0 \quad \text{as} \quad x \to 0.$$

Q.E.D.

We prove below that the following five properties hold :

(1) (n) **and** (npf) **fail at** $(0,0,0)$**.**
(2) (r) **holds.**
(3) (b) **holds.**
(4) (b^*) **and** (r^*) **fail at** $(0,0,0)$**.**
(5) The density of S **is constant along** X**.**

Property 1. (n) *and* (npf) *fail at* $(0,0,0)$.

Proof. We will show that the limits of secants from $(x,0,0)$ to $(x, f(x,z), z)$ as (x,z) tends to $(x_0, 0)$ are the straight lines which in the (y,z)-plane have equations

$$\begin{aligned} y &= z & &\text{if} \quad x_0 > 0 \\ y = \sigma z \text{ for all } \sigma &\in [-1,1] & &\text{if} \quad x_0 = 0 \\ y &= -z & &\text{if} \quad x_0 < 0. \end{aligned} \tag{1.1}$$

However, for the secants from $(0,0,0)$ to $(0, f(0,z), z)$ as z tends to 0, the limiting secant is $y = 0$. Hence (n) fails (the tangent cone to

$C_0(S_0)$ does not equal the fibre at 0 of the normal cone). Moreover (npf) fails since for $x_0 \neq 0$ the fibre at x_0 of the normal cone is 0-dimensional, while the fibre at 0 is 1-dimensional.

Proof of (1.1). First observe that, for all $0 < z < 1$, the secant from $(0,0,0)$ to $(0, f(0,z), z)$ has slope

$$\frac{f(0,z)}{z} = 1 - \frac{\ln z}{\ln z} = 0.$$

Take $x_0 > 0$ and let (x,z) tend to $(x_0, 0)$. The slope of the secant from $(x,0,0)$ to $(x, f(x,z), z)$ is

$$\frac{f(x,z)}{z} = 1 - \frac{\ln(x + \sqrt{x^2+z^2})}{\ln z}$$

which tends to 1 as z tends to 0 and x tends to x_0.

By symmetry (Remark 1), when $x_0 < 0$ the limiting slope is -1.

Now suppose (x,z) tends to $(0,0)$.

By symmetry (Remark 1 again) it will be enough to study the case $x > 0$ and to show that all the values $\sigma \in [0, +1]$ are realised. So we must show that the limits of

$$\frac{\ln(x + \sqrt{x^2+z^2})}{\ln z}$$

take all values in $[0,1]$ as x and z tend to 0 when $x > 0$.

First notice that if $x < Cz$ for some positive constant C, then

$$\lim_{x \to 0, z \to 0} \frac{\ln(x + \sqrt{x^2+z^2})}{\ln z} = 1,$$

because

$$\ln(x + \sqrt{x^2+z^2}) = \ln z + \ln\left(\frac{x}{z} + \sqrt{\left(\frac{x}{z}\right)^2 + 1}\right),$$

and the second term is bounded and non-negative.

So it remains to check that $\ln(x + \sqrt{x^2+z^2})$ takes all values in $[0,1]$, when $z = o(x)$. Write

$$\frac{\ln(x + \sqrt{x^2+z^2})}{\ln z} = \frac{\ln x}{\ln z} + \frac{\ln\left(1 + \sqrt{1 + \left(\frac{z}{x}\right)^2}\right)}{\ln z}.$$

The second term on the right has a bounded numerator so goes to 0 as (x, z) goes to $(0, 0)$. Because $0 < z < x < 1$, the first term on the right belongs to $(0, 1)$.

Let $\alpha \in (0, 1)$. On the curve $x = z^\alpha$,

$$\frac{\ln x}{\ln z} = \alpha,$$

so that

$$\lim \frac{\ln(x + \sqrt{x^2 + z^2})}{\ln z} = \alpha.$$

On the curve $x|\ln z| = +1$,

$$\frac{\ln x}{\ln z} = x|\ln x|,$$

with limit 0 as x tends to 0.

This completes the proof of (1.1), and hence the proof of Property 1. Q.E.D.

Property 2. (r) *holds for the pair of strata* (X, Y) *at* $(0, 0)$.

Proof. Recall that Kuo's ratio test (r) holds when

$$\frac{|(x, y, z)|.d(T_{(x,0,0)}X, T_{(x,y,z)}Y)}{|(y, z)|} \to 0$$

as (x, y, z) tends to $(0, 0, 0)$ on Y.

Now,

$$d(T_{(x,0,0)}X, T_{(x,y,z)}Y) = \frac{|\frac{\partial f}{\partial x}|}{|(\frac{\partial f}{\partial x}, -1, \frac{\partial f}{\partial z})|} < |\frac{\partial f}{\partial x}|.$$

And

$$\frac{|\frac{\partial f}{\partial x}|.|(x, y, z)|}{|(y, z)|} \approx \frac{|\frac{\partial f}{\partial x}|}{|z|}.\sqrt{x^2 + z^2}$$

$$= \frac{z}{|\ln z|}.\frac{1}{x + \sqrt{x^2 + z^2}}.(1 + \frac{x}{\sqrt{x^2 + z^2}}).\frac{\sqrt{x^2 + z^2}}{z}$$

$$= \frac{1}{|\ln z|}$$

which tends to 0 as z tends to 0.

We check directly that (a) holds. As above, $d(T_{(x,0,0)}X, T_{(x,y,z)}Y) < |\frac{\partial f}{\partial x}|$.

But $|\frac{\partial f}{\partial x}| = \frac{|z|}{|\ln z|}.\frac{1}{\sqrt{x^2+z^2}} < \frac{1}{|\ln z|}$, which tends to 0 as z tends to 0, as required. Q.E.D.

Note that although (r) holds, this argument does not show that (r^e) (of **[OT2]**) holds. In fact we know already, by the main theorem of **[OT2]**, that (r^e) must fail, because (a) holds, while (n) and (npf) fail.

Property 3. *(b) holds for (X, Y) at $(0,0,0)$.*

Proof. We have just seen that (a) holds. Thus we need only prove that (b^π) holds.

Again by remark 1, we need only treat the case $x \geq 0$.

Suppose $0 < z < 1$, and $0 \leq x$, for x small.

Then $x + \sqrt{x^2+z^2} \geq z$, so that

$$0 < \frac{\ln(x+\sqrt{x^2+z^2})}{\ln z} \leq 1. \qquad (*)$$

Also

$$0 < \frac{z^2}{x\sqrt{x^2+z^2}+x^2+z^2} \leq 1. \qquad (**)$$

The zy slope of the secant line from $(x,0,0)$ to $(x, f(x,z), z)$ is

$$\frac{f(x,z)}{z} = 1 - \frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}.$$

The zy slope of the tangent in the z direction on Y is

$$\frac{\partial f}{\partial z}(x,z) = 1 - \frac{\ln(x+\sqrt{x^2+z^2})}{\ln z} - z\frac{\partial}{\partial z}\left(\frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}\right).$$

To prove that Whitney (b^π) holds at $(0,0,0)$ we must show that

$$\lim_{(x,z)\to(0,0)}\left(z\frac{\partial}{\partial z}\left(\frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}\right)\right) = 0.$$

But

$$z\frac{\partial}{\partial z}\left(\frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}\right) = \frac{1}{\ln z}\left(\frac{z^2}{x\sqrt{x^2+z^2}+x^2+z^2} - \frac{\ln(x+\sqrt{x^2+z^2})}{\ln z}\right),$$

which tends to 0 as z tends to 0 for x small, by $(*)$ and $(**)$. This implies that (b^π) holds, and hence that (b) holds for (Y, X) on a neighbourhood of $(0,0,0)$ in X. Q.E.D.

Property 4. *(b^*) and (r^*) fail at* $(0,0,0)$.

Proof. We intersect Y with planes $\{y = az, -1 < a < 1\}$ to obtain

$$(1-a)z = \frac{z}{\ln z}\ln(x + \sqrt{x^2+z^2}),$$

which becomes

$$z^{1-a} = x + \sqrt{x^2+z^2}.$$

Squaring, we get $x^2 + z^2 = z^{2-2a} - 2xz^{1-a} + x^2$ which simplifies to

$$x = \frac{z^{1-a} - z^{1+a}}{2}.$$

This curve in the xz-plane passes through $(0,0)$ if $-1 < a < 1$, showing that (b^*) and (r^*) fail to hold, since $Y \cap \{y = az\}$ contains curves passing through $(0,0,0)$, and so $(X, Y \cap \{y = az\})$ cannot be (b)-regular, because (a) fails for $(X, Y \cap \{y = az\})$, and hence (X, Y) is not ($b_{\mathrm{cod}\ 1}$)-regular and (b^*) fails, implying that (r^*) fails also. Q.E.D.

Property 5. *The density of S is constant along X.*

Proof. We show first that the tangent cone to S at $(0,0,0)$ is the half-plane $\{y = 0, z \geq 0\}$.

Each definable curve on Y which passes through $(0,0,0)$ and which is not tangent to X has a projection to the (x,z)-plane tangent to some line $x = cz$, where c is a nonzero constant. On such a curve,

$$y = z\left(1 - \frac{\ln(cz + o(z) + \sqrt{c^2z^2 + 2cz.o(z) + o(z^2) + z^2})}{\ln z}\right)$$

$$= \frac{z\ln(c + o(1) + \sqrt{c^2 + 2c.o(1) + 1})}{\ln z} = o(z).$$

Hence such a curve on Y is tangent to $\{y = 0\}$.

Now consider a curve whose projection to the (x,z)-plane is tangent to the x-axis, so of the form $(x, y(x,z), z(x))$ where $z = o(x)$. Then

$$y = z\left(1 - \frac{\ln(x + \sqrt{x^2+z^2})}{\ln z}\right)$$

$$= O(z) = o(x),$$

so that again the curve itself is tangent to the x-axis. It follows that the tangent cone to S at $(0,0,0)$ is $\{y = 0, z \geq 0\}$.

It is easy to see that the tangent cone at $(x_0, 0, 0)$ equals $\{y = z\}$ if $x_0 > 0$ and equals $\{y = -z\}$ if $x_0 < 0$.

It follows that the density of S at points of the x-axis has the constant value 1/2. Q.E.D.

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Valuations and local uniformization

Michel Vaquié

Abstract.

We give the principal notions in valuation theory, the value group and the residue field of a valuation, its rank, the compositions of valuations, and we give some classical examples. Then we introduce the Riemann-Zariski variety of a field, with the topology defined by Zariski. In the last part we recall the result of Zariski on local uniformization and give a sketch of the proof in the case of an algebraic surface.

§ Introduction

In these notes, we are going to give an idea of the proof of the resolution of singularities of an algebraic surface by O. Zariski. This proof is based on the theory of the valuations of algebraic function fields and could be seen as one of the most important applications of this theory in algebraic geometry.
In the first part of the paper we give the principal definitions and properties of valuations that we need for resolution. We don't speak about the problems of extension of valuations in a field extension, neither the problems of ramification.
In the second part we define the Riemann-Zariski variety of a field, what is called "abstract Riemann surface" or "Riemann manifold" by Zariski, and we give the principal property of this space.
In the last part we give a sketch of the proof of local uniformization in the case of an algebraic surface over an algebraically closed field of characteristic zero, and how we can deduce the resolution.

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All the results on valuations of this paper are classical, we give a proof of some of them, otherwise we send back the reader to the books of Bourbaki ([Bo]), Endler ([En]), Ribenboim ([Ri]) or Zariski and Samuel ([Za-Sa]), or to the articles of Zariski and of the author ([Va]).

§1. Valuations

1.1. Valuation rings and valuations

A commutative ring A is called a local ring if the non-units form an ideal, this ideal is the unique maximal ideal $\mathfrak{m}$ of A and we note $(A, \mathfrak{m})$ the local ring. The quotient field $k = A/\mathfrak{m}$ is called the residue field of $(A, \mathfrak{m})$. We don't assume that the ring A is noetherian, in Zariski's terminology such a ring is called a "quasi-local ring".
Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be two local rings, we say that B *dominates* A, and we note $A \preceq B$, if $A \subset B$ and $\mathfrak{m} = A \cap \mathfrak{n}$. If we assume $A \subset B$, then B dominates A if and only if $\mathfrak{m} \subset \mathfrak{n}$.
We deduce from the definition that if A is dominated by B, we have an inclusion between the residue fields: $A/\mathfrak{m} \subset B/\mathfrak{n}$.

Example 1. If $(A, \mathfrak{m})$ is a noetherian local ring, the completion $\hat{A}$ of A for the $\mathfrak{m}$-topology dominates A.
Let A and B be two integral domains with $A \subset B$, then for any prime ideal $\mathfrak{q}$ in B, the local ring $B_{\mathfrak{q}}$ dominates the local ring $A_{\mathfrak{p}}$ where $\mathfrak{p}$ is the prime ideal of A defined by $\mathfrak{p} = A \cap \mathfrak{q}$.
Let $f\colon X \longrightarrow Y$ be a morphism between algebraic varieties, or schemes, for any point x in X, the local ring $\mathcal{O}_{Y,y}$ dominates the local ring $\mathcal{O}_{X,x}$, where $y = f(x)$.

Let K be a field, the relation B *dominates* A, or A *is dominated by* B, defines a partial ordering on the set of the local rings contained in K. Then we can give the following definition.

Definition. Let V be an integral domain; then V is a *valuation ring* of K if K is the fraction field of V and if V is a maximal element of the set of local rings contained in K ordered by the relation of domination. If V is an integral domain, we say that V is a valuation ring if V is a valuation ring of its fraction field.

With this definition, it is easy to prove the existence of valuation rings, more precisely, we have the following result.

Proposition 1.1. ([Bo], Chap. 6, §1, n°2, Théorème 2, page 87.) *Let A be a subring of a field K and $h\colon A \longrightarrow L$ be a morphism of A in L an algebraically closed field, then there exists a valuation ring V of K*

with $A \subset V$ and a morphism $f\colon V \longrightarrow L$ which extends h and such that $max(V) = f^{-1}(0)$.

Proof. We consider the set $\mathcal{H} = \{(B, f) \,/\, B \subset K \,\text{and}\, f\colon B \to L\}$; we order $\mathcal{H}$ by $(B, f) \preceq (C, g)$ if $B \subset C$ and g extends f. Any totally ordered subset $((B_\alpha, f_\alpha))$ of $\mathcal{H}$ has an upper bound (B, f) in $\mathcal{H}$, with $B = \cup_\alpha B_\alpha$, then by Zorn's lemme the set $\mathcal{H}$ contains a maximal element (W, g) and if we note $\mathfrak{p}$ the kernel of $g\colon W \longrightarrow L$, the local ring $W_\mathfrak{p}$ is the valuation ring satisfying the required condition.

Corollary. *Any local subring A of a field K is dominated by at least one valuation ring V of K.*

Remark 1.1. In the cases we shall consider, we have a ground field k and we have the following result. Let A be a k-subalgebra of K and $h\colon A \longrightarrow L$ a k-morphism from A in an algebraically closed field L, then there exists a valuation ring V of K wich is a k-algebra, with $A \subset V$ and a k-morphism $f\colon V \longrightarrow L$ which extends h and such that $max(V) = f^{-1}(0)$. In particular we get that the ground field k is included in $V \setminus max(V)$.

We are going to give now the principal characteristic properties of valuation rings.

Theorem 1.2. ([Bo], Chap. 6, §2, n°2, Théorème 1, page 85.) *Let V be an integral domain, contained in a field K, then the following conditions are equivalent:*

a) V is a valuation ring of K;

b) let $x \in K$, then $x \notin V \Longrightarrow x^{-1} \in V$;

c) K is the fraction field of V and the set of ideals of V is totally ordered by inclusion;

c') K is the fraction field of V and the set of principal ideals of V is totally ordered by inclusion.

Remark 1.2. From the condition b), we deduce that any valuation ring is integrally closed. In fact, we have the following result:

let A be an integral domain and K a field containing A, then the intersection of all the valuation rings V of K with $A \subset V$ is the integral closure of A in K.

From the condition c), we deduce that any finitely generated ideal of a valuation ring is principal.

Let Γ be an additive abelian totally ordered group. We add to Γ an element $+\infty$ such that $\alpha < +\infty$ for every α in Γ, and we extend the law on $\Gamma_\infty = \Gamma \cup \{+\infty\}$ by $(+\infty) + \alpha = (+\infty) + (+\infty) = +\infty$.

Definition. Let A be a ring, a *valuation* of A with values in Γ is a mapping ν of A in Γ_∞ such that the following conditions are satisfied:

1) $\nu(x.y) = \nu(x) + \nu(y)$ for every $x, y \in A$,
2) $\nu(x + y) \geq \min\bigl(\nu(x), \nu(y)\bigr)$ for every $x, y \in A$,
3) $\nu(x) = +\infty \Longleftrightarrow x = 0$.

Remark 1.3. The condition 1) means that the valuation ν is a homomorphism of $A \setminus \{0\}$ with the multiplicative law in the group Γ, hence we have $\nu(1) = 0$ and more generally, for any root of unity z, i. e. $z^n = 1$ for some $n > 0$, we have also $\nu(z) = 0$ because Γ has no torsion.

From the conditions 1) and 3) it follows that if there is a valuation ν on A, then A is an integral domain. More generally, if we have a mapping $\nu \colon A \longrightarrow \Gamma_\infty$ with the conditions 1), 2) and with $\nu(0) = +\infty$, but if we don't assume that ν takes the value $+\infty$ only for 0, the set $\mathcal{P} = \nu^{-1}\{+\infty\}$ is a prime ideal of A and ν induces a valuation on the integral domain $A/\mathcal{P}$.
If A is an integral domain, any valuation ν on A with values in Γ extends in a unique way in a valuation of the fraction field K of A with values in Γ.
The set of elements of Γ vhich are values of elements of $A \setminus \{0\}$ generates a subgroup Γ' of Γ and we have $\Gamma' = \nu(K^*)$.

The valuation ν defined by $\nu(x) = 0$ for any x in $A \setminus \{0\}$ is called *the trivial valuation.*

Proposition 1.3. ([Bo], Chap. 6, §3, n°1, Proposition 1, page 97.) *Let ν be a valuation of A, then for any family $\{x_1, \dots, x_n\bigr)$ in A we have the inequality:*

$$\nu\Bigl(\sum_{i=1}^{n} x_i\Bigr) \geq \min\bigl\{\nu(x_1) \dots, \nu(x_n)\bigr\}.$$

More over, if the minimum is reached by only one of the $\nu(x_i)$ we have the equality:

$$\nu\Bigl(\sum_{i=1}^{n} x_i\Bigr) = \min\bigl\{\nu(x_1) \dots, \nu(x_n)\bigr\}.$$

Proposition 1.4. *Let ν be a valuation of a field K with values in a group Γ, then the set A of elements x of K with $\nu(x) \geq 0$ is a valuation ring of K and the maximal ideal $max(A)$ is the set of elements x of K with $\nu(x) > 0$.*
Conversely, we can associate to any valuation ring V of K a valuation ν of K with values in a group Γ such that V is the inverse image $\nu^{-1}\bigl(\{\alpha \in \Gamma \mid \alpha \geq 0\}\bigr)$.

Proof. We deduce from the conditions 1) and 2) of the definition of a valuation that the set $A = \{x \in K \mid \nu(x) \geq 0\}$ is a subring of K, and by property b) of theorem 1.2 we get that A is a valuation ring.

To get the converse, we are going to construct the group Γ and the mapping ν from the ring V. More generally, if C is an integral domain with fraction field K, the set $U(C)$ of invertible elements of C is a subgroup of the multiplicative group K^* and we call Γ_C the quotient group. The divisibility relation on C defines a partial order on Γ_C, compatible with the group structure, and we deduce from the remark following the theorem 1.2 that Γ_C is totally ordered if and only if C is a valuation ring. Then the canonical mapping $K^* \longrightarrow \Gamma_C = K^*/U(C)$ induces a valuation ν on K with $C = \{x \mid \nu(x) \geq 0\}$ and with values in the group Γ_C.

Definition. The valuation ring V associated to the valuation ν of K is called the valuation ring of ν and we note it $V = R_\nu$, and the field $\kappa(V) = V/max(V)$ is called *the residue field* of ν and we denote it κ_ν. The subgroup $\Gamma' = \nu(K^*)$ of Γ is called *the value group* of ν and we note it $\Gamma' = \Gamma_\nu$. We deduce from the proof of the proposition that the value group Γ_ν is isomorphic to $\Gamma_V = K^*/U(V)$. In general we shall assume that Γ is the value group, i. e. that ν is surjective from K^* into Γ.

We say that two valuations ν and ν' of a field K are *equivalent* if they have the same valuation rings, i. e. $R_\nu = R_{\nu'}$.

Proposition 1.5. ([Bo], Chap. 6, §3, n°2, Proposition 3, page 99.) *Two valuations ν and ν' of K are equivalent if and only if there exists an order preserving isomorphism φ of Γ_ν onto $\Gamma_{\nu'}$ such that $\nu' = \varphi \circ \nu$.*

We make no distinction between equivalent valuations and we identify them.

We often consider a fixed field k; all the fields K are extensions of k and we say that a valuation ν of K is a valuation of K/k if the restriction of ν to k is trivial, i.e. if for all elements x in k^* we have $\nu(x) = 0$. If V is the valuation ring of K associated to the valuation ν, this is equivalent to demand to V to be a k-algebra. The natural map $k \longrightarrow V$ has its image included in $V \setminus max(V)$, then we get an inclusion $k \subset \kappa_\nu$, i.e. the residue field of the valuation is also an extension of k.

1.2. Rank of a valuation and composite valuation

Definition. A subset Δ of a totally ordered group Γ is called a *segment* if Δ is non-empty and if for any element α of Γ which belongs to Δ, all the elements β of Γ which lie between α and $-\alpha$, i. e. such that $-\alpha \leq \beta \leq \alpha$ or $\alpha \leq \beta \leq -\alpha$, also belong to Δ.

A subgroup Δ of Γ is called an *isolated subgroup* if Δ is a segment of Γ.

Proposition 1.6. ([Bo], Chap. 6, §4, n°2, Proposition 3, page 108.) *The kernel of an order preserving homorphism of totally ordered groups of* Γ *in* Γ' *is an isolated subgroup of* Γ.
Conversely, if Δ *is an isolated subgroup of a totally ordered group* Γ*, the quotient group* Γ/Δ *has a structure of totally ordered group and the canonical morphism* $\Gamma \longrightarrow \Gamma/\Delta$ *is ordered preserving.*

The set of all the segments Δ of Γ is totally ordered by the relation of inclusion, then we can give the following definition.

Definition. The ordinal type of the totally ordered set of proper isolated subgroups Δ of Γ is called *the rank* of the group Γ.

Let ν be a valuation of a field K, with value group Γ, and let V be the valuation ring associated to ν. For any part A of V containing 0 we denote by Δ_A the set of all the elements Γ_∞ in the complementary of $\big(\nu(A)\big) \cup \big(-\nu(A)\big)$.

Theorem 1.7. ([Za-Sa], Chap.VI, §10, Theorem 14, page 40.) *If* $\mathcal{I}$ *is an ideal of* V*,* $\mathcal{I} \neq V$*, then* $\Delta_{\mathcal{I}}$ *is a segment in* Γ*. The mapping* $\mathcal{I} \longrightarrow \Delta_{\mathcal{I}}$ *is a bijection from the set of all proper ideals of* V *onto the set of all segments of* Γ*, which is order-reversing for the relation of inclusion.*
Moreover, the segment $\Delta_{\mathcal{I}}$ *is an isolated subgroup of* Γ *if and only if* $\mathcal{I}$ *is a prime ideal of* V*.*

The maximal ideal $max(V)$ is the prime ideal corresponding to the isolated subgroup $\Delta = \{0\}$, and the ideal (0) is the prime ideal corresponding to the isolated subgroup $\Delta = \Gamma$.

Definition. We define *the rank* of a valuation ν as the rank of its value group.

Remark 1.4. From the theorem, we see that the set of the prime ideals of the valuation ring V associated to the valuation ν is totally ordered by inclusion and the rank of the valuation ν is by definition the ordinal type of the set of prime ideals of V.

When the ordinal type of the set of prime ideals of the valuation ring V is finite, we say that the valuation ν is of finite rank and we denote $\text{rank}(\nu) = n$ with $n \in \mathbb{N}$. Otherwise we say that the valuation is of infinite rank.

Corollary. *The rank of the valuation* ν *is equal to the Krull dimension of the valuation ring* V *associated to* ν*.*

For any element α in the group Γ we can define the ideals $\mathcal{P}_\alpha(V)$ and $\mathcal{P}_{\alpha+}(V)$ of the valuation ring V by:

$$\mathcal{P}_\alpha(V) = \{x \in V \,/\, \nu(x) \geq \alpha\} \quad \text{and} \quad \mathcal{P}_{\alpha+}(V) = \{x \in V \,/\, \nu(x) > \alpha\}.$$

We can also define the Rees-like algebras introduced in [Te], 2.1, associated to this family of ideals:

$$\mathcal{A}_\nu(R_\nu) = \bigoplus_{\alpha\in\Gamma} \mathcal{P}_\alpha(R_\nu) v^{-\alpha} \subset R_\nu[v^\Gamma] \quad \text{and} \quad \mathbf{gr}_\nu(V) = \bigoplus_{\alpha\in\Gamma} \mathcal{P}_\alpha(V)/\mathcal{P}_{\alpha+}(V).$$

Remark 1.5. If the value group Γ is not isomorphic to the group of integers $\mathbb{Z}$, then there may exist ideals $\mathcal{I}$ of V which are different from ideals $\mathcal{P}_\alpha$ or $\mathcal{P}_{\alpha+}$ for all α in Γ.
If the value group Γ of ν is equal to $\mathbb{Q}$, for any real number $\beta > 0$ in $\mathbb{R} \setminus \mathbb{Q}$, the set $\mathcal{I} = \{x \in V \,/\, \nu(x) \geq \beta\}$, which is also equal to $\{x \in V \,/\, \nu(x) > \beta\}$, is an ideal of V, but there is no α in Γ such that $\mathcal{I}$ is equal to $\mathcal{P}_\alpha$ or $\mathcal{P}_{\alpha+}$.
If the value group Γ of the valuation ν is of rank bigger than one, and if $\mathcal{P}$ is a prime ideal of the valuation ring V different from (0) and from the maximal ideal $max(V)$, there doesn't exist α in Γ such that $\mathcal{P}$ is equal to $\mathcal{P}_\alpha$ or to $\mathcal{P}_{\alpha+}$.

Proposition 1.8. ([Bo], Chap.6, §4, n°1, Proposition 1, page 110; [Va], Proposition 3.3, page 547.) *Let V be a valuation ring of a field K.*
a) Any local ring R with $V \subset R \subset K$ is a valuation ring of K, and the maximal ideal $max(R)$ of R is contained in V and is a prime ideal of V.
b) The mapping $\mathcal{P} \mapsto V_{\mathcal{P}}$ is a bijection from the set of prime ideals of V onto the set of local rings R with $V \subset R \subset K$, which is order-reversing for the relation of inclusion. The inverse map is defined by $R \mapsto max(R)$.

Proof. From the condition b) of theorem 1.2 we see that the ring R is a valuation ring and that $max(R)$ is an ideal of V. Since $max(R)$ is a prime ideal of R, it is also a prime ideal of V.
For any prime ideal $\mathcal{P}$ of V the local ring $V_{\mathcal{P}}$ is such that $V \subset V_{\mathcal{P}} \subset K$, and if $\mathcal{P} \subset \mathcal{Q}$ we have $V_{\mathcal{Q}} \subset V_{\mathcal{P}}$. We can verify that the maximal ideal $\mathcal{P}V_{\mathcal{P}}$ of the local ring $V_{\mathcal{P}}$ is equal to $\mathcal{P}$.

Let V be a valuation ring of a field K associated to a valuation ν of value group Γ, and we assume that ν is of finite rank r. We denote respectively $\mathcal{P}_i$, Δ_i and V_i, $0 \leq i \leq r$, the prime ideals of V, the isolated subgroups of Γ and the local subrings of K containing V, with the relations $\mathcal{P}_i = max(V_i)$, $V_i = V_{\mathcal{P}_i}$ and $\Delta_i = \Delta_{\mathcal{P}_i}$. We have the inclusions:

$$\begin{array}{l}(0) = \mathcal{P}_0 \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_{r-1} \subset \mathcal{P}_r = \max(V) \\ V = V_r \subset V_{r-1} \subset \ldots \subset V_1 \subset V_0 = K \\ (0) = \Delta_r \subset \Delta_{r-1} \subset \ldots \subset \Delta_1 \subset \Delta_0 = \Gamma.\end{array}$$

We shall study later the relations between the valuations ν_i associated to the valuation rings V_i and the valuation ν, more precisely the relations between their value groups Γ_i and the isolated subgroups Δ_i of Γ (cf. the proposition 1.11).

Example 2. The trivial valuation of K, i.e. the valuation ν defined by $\nu(x) = 0$ for all the non-zero elements x of K, is the unique valuation of rank 0.

Example 3. The valuation ν of K is of rank one if and only if the value group Γ of ν is isomorphic to a subgroup of $(\mathbb{R}, +)$. It is equivalent to say that the group Γ is *archimedean*, i.e. Γ satisfies the following condition: if α and β are any two elements of Γ with $\alpha > 0$, then there exists an integer n such that $n\alpha > \beta$. The valuation ring V associated to ν is of dimension 1 and we deduce from the proposition that V is a maximal subring of K for the relation of inclusion.

Definition. We say that a totally ordered group Γ is a *discrete* group if it is of finite rank r and if all the quotient groups Δ_{i+1}/Δ_i, where the Δ_i are the isolated subgroups of Γ, are isomorphic to $\mathbb{Z}$. It is equivalent to say that the ordered group Γ is isomorphic to a subgroup of $(\mathbb{Z}^n, +)$ with the lexicographic order. We say that a valuation ν is *discrete* if its value group Γ is a discrete group.

If the value group of ν is discrete of rank one, i.e. if ν is discrete valuation of rank one, we can assume that the value group is $\mathbb{Z}$.

Proposition 1.9. ([Bo], Chap. 6, §3, n°6, Proposition 9, page 105.) *Let A be a local integral domain, then the following conditions are equivalent:*

a) A is a discrete valuation ring of dimension 1*;*

b) A is principal;

c) the maximal ideal $max(A)$ is principal and A is noetherian;

d) A is a noetherian valuation ring.

We see that in that case, if we assume that the value group Γ of the valuation ν associated to the ring A is the ring $\mathbb{Z}$, the maximal ideal $\mathfrak{m} = max(A)$ is generated by any element x in A such that $\nu(x) = 1$. Then any element y of the fraction field K of A can be written $y = ux^n$,

with $u \in A \smallsetminus \mathfrak{m}$ and with $n \in \mathbb{Z}$, and we have $\nu(y) = n$. We say that the valuation ν is the $\mathfrak{m}$-*adic* valuation, i.e. the valuation defined by the relation: $\nu(y) \geq n$ if and only if $y \in \mathfrak{m}^n$.
The only ideals of A are the ideals $\mathcal{P}_n(A) = \{x \in A \,/\, \nu(x) \geq n\}$,(cf. remark 1.5), and $\mathcal{P}_n(A)$ is the principal ideal generated by x^n.

Definition. Let Γ be a commutative group, then the maximum number of rationally independent elements of Γ is called *the rational rank* of the group Γ. We define *the rational rank* of a valuation ν as the rational rank of its value group Γ.

The rational rank is an element of $\mathbb{N} \cup \{+\infty\}$, we denote it rat.rank$(\Gamma)$, and we have rat.rank$(\Gamma) = \dim_{\mathbb{Q}}\big(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}\big)$.
The rational rank of a group is zero if and only if Γ is a torsion group. If Γ is a value group of a valuation, Γ is totally ordered, then its rational rank is zero if and only if $\Gamma = \{0\}$, i.e. if and only if the valuation is the trivial valuation.

Proposition 1.10. ([Bo], Chap. 6, §10, n°2, Proposition 3, page 159.) *Let Γ be a commutative group and Γ' a subgroup of Γ. Then we have the equality:*

$$\textit{rat.rank}(\Gamma) = \textit{rat.rank}(\Gamma') + \textit{rat.rank}\big(\Gamma/\Gamma'\big)\,.$$

If Γ is a totally ordered group we have the inequality:

$$\textit{rank}(\Gamma) \leq \textit{rank}(\Gamma') + \textit{rat.rank}\big(\Gamma/\Gamma'\big)\,.$$

Corollary. *The rank of a valuation ν is never greater than its rational rank:*

$$\textit{rank}(\nu) \leq \textit{rat.rank}(\nu)\,.$$

Let ν be a valuation of a field K, with value group Γ and valuation ring V, and we denote $\mathfrak{m}$ the maximal ideal of V. If the rank of ν is bigger than one, there exists a proper isolated subgroup Δ of Γ, $\Delta \neq (0)$, and let $\mathfrak{m}'$ the prime ideal of V associated to Δ by the theorem 1.7. We know by proposition 1.8 that $\mathfrak{m}'$ is a prime ideal of V and that the local ring $V' = V_{\mathfrak{m}'}$ is a valuation ring of K with $\mathfrak{m}'$ as maximal ideal and such that $V \subset V'$. We denote ν' the valuation of K associated to V' and Γ' the value group of ν'.

Proposition 1.11. ([Za-Sa], Chap.VI, §10, Theorem 17, page 43.) *a) The value group Γ' is isomorphic to the quotient group Γ/Δ, and the valuation $\nu'\colon K^* \longrightarrow \Gamma'$ is the composition of $\nu\colon K^* \longrightarrow \Gamma$ and $\lambda\colon \Gamma \longrightarrow \Gamma/\Delta$.*

b) The quotient ring $\bar{V} = V/\mathfrak{m}'$ is a valuation ring of the residue field $\kappa_{\nu'} = V'/\mathfrak{m}'$ of the valuation ν', and the value group of the valuation $\bar{\nu}$ associated to $\bar{V}$ is isomorphic to Δ.

Proof. a) The valuations ν and ν' are defined as the natural applications $\nu\colon K^* \longrightarrow \Gamma = K^*/U(V)$ and $\nu'\colon K^* \longrightarrow \Gamma' = K^*/U(V')$. Then we deduce from $V \subset V'$ and $\mathfrak{m} \subset \mathfrak{m}'$ that $U(V)$ is included in $U(V')$ and that ν' is equal to $\lambda \circ \nu$. We have to show that the kernel of λ is isomorphic to the isolated subgroup Δ, which is a consequence of the relation between the prime ideal $\mathfrak{m}'$ and $\Delta = \Delta_{\mathfrak{m}'}$.
b) Since V is a valuation ring of K, the quotient ring $\bar{V}$ is also a valuation ring of the fraction field $\bar{K}$ of $\bar{V}$ and we have $\bar{K} = V'/\mathfrak{m}'$. To show that the value group of the valuation $\bar{\nu}$ is equal to the group Δ it is enough to remark that we have the exact sequence: $0 \longrightarrow \bar{K}^*/U(\bar{V}) \longrightarrow K^*/U(V) \longrightarrow K^*/U(V') \longrightarrow 0$.

Definition. ([Za-Sa], Chap.VI, §10.) The valuation ν is called *the composite valuation with the valuations ν' and $\bar{\nu}$* and we write $\nu = \nu' \circ \bar{\nu}$.

Corollary. *If ν is the composite valuation $\nu' \circ \bar{\nu}$ we have the equalities:*

$$rank(\nu) = rank(\nu') \ + \ rank(\bar{\nu})$$

$$rat.rank(\nu) = rat.rank(\nu') \ + \ rat.rank(\bar{\nu})\,.$$

Conversely, if we have a valuation ν' of a field K and a valuation $\bar{\nu}$ of the residue field $\bar{K} = \kappa_{\nu'}$, we can define the composite valuation $\nu = \nu' \circ \bar{\nu}$.

Proposition 1.12. ([Va], Proposition 4.2, page 552.) *Let ν' be a valuation of K with valuation ring V' and residue field $\kappa_{\nu'} = \bar{K}$ and $\bar{\nu}$ be a valuation of $\bar{K}$, then the composite valuation $\nu = \nu' \circ \bar{\nu}$ is the valuation of the field K associated to the valuation ring V defined by $V = \big\{x \in V' \,/\, \bar{\nu}(\bar{x}) \geq 0\big\}$.*

We notice that the residue field of the composite valuation ν is equal to the residue field $\kappa_{\bar{\nu}}$ of the valuation $\bar{\nu}$.

Remark 1.6. If we have the valuations ν' of K and $\bar{\nu}$ of $\kappa_{\nu'}$, the composite valuation $\nu = \nu' \circ \bar{\nu}$ defines an extension of the value group Γ' of ν' by the value group $\bar{\Gamma}$, i.e. an exact sequence of totally ordered groups: $0 \longrightarrow \bar{\Gamma} \longrightarrow \Gamma \longrightarrow \Gamma' \longrightarrow 0$.
If this exact sequence splits, the value group Γ is isomorphic to the group $(\Gamma' \times \bar{\Gamma})$ with the lexicographic order. If the valuation ν' is a discrete valuation of rank one, i.e. for $\Gamma' \simeq \mathbb{Z}$, the exact sequence always splits and we can describe the composite valuation $\nu = \nu' \circ \bar{\nu}$ in the

following way. The maximal ideal of the valuation ring V' associated to ν' is generated by an element u and we can associate to any non zero element x in K the non zero element $\bar{y}$ in the residue field $\kappa_{\nu'}$ which is the class of $y = x.u^{-\nu'(x)}$. The composite valuation ν is then defined by $\nu(x) = \big(\nu'(x), \bar{\nu}(\bar{y})\big)$.

Remark 1.7. If ν_1 is a valuation of a field K and if ν_2 is a valuation of the residue field κ_1 of ν_1, we have defined the composite valuation ν of K, $\nu = \nu_1 \circ \nu_2$. By induction we may define in the same way the composite valuation $\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$, where each valuation ν_i is a valuation of the residue field κ_{i-1} of the valuation ν_{i-1}, $1 \le i \le r$, with $\nu_0 = \nu$. For any $1 \le t \le r$, we decompose the valuation ν as $\nu = \nu'_{(t)} \circ \bar{\nu}_{(t)}$, where $\nu'_{(t)} = \nu_1 \circ \ldots \circ \nu_t$ is a valuation of K and $\bar{\nu}_{(t)} = \nu_{t+1} \circ \ldots \circ \nu_r$ is a valuation of the residue field $\kappa_{\nu'_{(t)}}$ of $\nu'_{(t)}$, with $\kappa_{\nu'_{(t)}} = \kappa_t$. If we denote $V_{(t)}$ the valuation ring of K associated to $\nu'_{(t)}$, the family of valuations $\big(\nu'_{(1)}, \ldots, \nu'_{(r)} = \nu\big)$ corresponds to the sequence $V = V_{(r)} \subset \ldots \subset V_{(1)} \subset K$. We call the valuation $\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$, *the composite valuation with the family* $\big(\nu_1, \nu_2, \ldots, \nu_r\big)$.

Let ν_1 and ν_2 be two valuations of a field K and let $(V_1, \mathfrak{m}_1)$ and $(V_2, \mathfrak{m}_2)$ be the valuation rings respectively associated to ν_1 and ν_2. We assume that there exists a valuation ring V of K, $V \neq K$, which contains the rings V_1 and V_2, then there exists a non trivial valuation ν of K such that the valuations ν_1 and ν_2 are composite with ν. More precisely, there exist two valuations $\bar{\nu}_1$ and $\bar{\nu}_2$ of the residue field κ_ν with $\nu_1 = \nu \circ \bar{\nu}_1$ and $\nu_2 = \nu \circ \bar{\nu}_2$. This is also equivalent to say that there exists an non zero subset $\mathfrak{m}$ of $V_1 \cap V_2$ which is a prime ideal of the two rings V_1 and V_2.

Definition. ([Za-Sa], Chap.VI, §10, page 47.) Two valuations ν_1 and ν_2 of a field K are said *independent* if they are not composite with a same non trivial valuation ν.
A family $\big\{\nu_1, \nu_2, \ldots, \nu_k\big\}$ of valuations of a field K is called a *family of independent valuations* if any two of them are independent.

In fact we can define a partial order on the set of all the valuations of a field K by $\nu_1 \preceq \nu_2$ if and only if $V_2 \subset V_1$, where V_i is the valuation ring associated to ν_i, $i = 1, 2$. This equivalent to say that ν_2 is composite with ν_1, i.e. that there exists a valuation $\bar{\nu}$ of the residue field κ_{ν_1} such that $\nu_2 = \nu_1 \circ \bar{\nu}$. If ν_1 and ν_2 are two valuations of K, we can define the valuation $\nu = \nu_1 \wedge \nu_2$ as the "biggest" valuation ν such that $\nu \preceq \nu_1$ and $\nu \preceq \nu_2$. This valuation $\nu = \nu_1 \wedge \nu_2$ is the valuation associated to smallest valuation ring V of K which contains the valuation rings V_1

and V_2 associated to ν_1 and ν_2. Then two valuations ν_1 and ν_2 of K are independent if and only if $\nu_1 \wedge \nu_2$ is the trivial valuation.
If ν is a valuation of rank one, the valuation ring V associated to ν is maximal among the valuation rings of K, i.e. the valuation ν is minimal among the non trivial valuations of K. Then for any valuation ν' of K, ν and ν' are not independent if and only if $\nu \preceq \nu'$, i.e. if and only if ν' is composite with ν. If ν and ν' are two distinct valuations of K of rank one, then ν and ν' are independent.

The notion of independence of valuations is important because of the following result which is called *the approximation theorem.*

Theorem 1.13. ([Za-Sa], Chap.VI, §10, Theorem 18, page 47.) *Let $\{\nu_1, \nu_2, \ldots, \nu_k\}$ be a family of independent valuations of a field K; given k arbitrary elements $x_1, \ldots, x_k$ of K and k arbitrary elements $\alpha_1, \ldots, \alpha_k$ of the value groups $\Gamma_1, \ldots, \Gamma_k$ of the valuations $\nu_1, \ldots, \nu_k$ respectively, then there exists an element x of K such that*

$$\nu_i(x - x_i) = \alpha_i\,, \quad i = 1, 2, \ldots, k\,.$$

1.3. Extension of a valuation

Let K be a field and let L be an overfield of K. If μ is a valuation of L, the restriction of μ to K is a valuation of K, the value group Γ_ν of ν is a subgroup of the value group Γ_μ and the valuation ring R_ν associated to ν is equal to $R_\mu \cap K$ where R_μ is the valuation ring associated to μ.

Definition. We say that the valuation μ is *an extension of the valuation ν to L.*

Remark 1.8. In fact the valuation ring R_ν is dominated by the valuation ring R_μ. More generally if V and W are valuation rings of K and L respectively, we have W dominates V if and only if $V = W \cap K$.
Since the valuation ring R_μ dominates the valuation ring R_ν, we have an inclusion of the residue fields $\kappa_\nu \subset \kappa_\mu$.

Proposition 1.14. *For any valuation ν of a field K and for any overfield L of K, there exists at least one valuation μ of L wich is an extension of ν.*

Proof. By the corollary at the proposition 1.1 there exists at least one valuation ring W of the field L which dominates the valuation ring V associated to the valuation ν. Then the valuation μ associated to W is an extension of ν.

Let ν be a valuation of a field K and let μ be any extension of ν to an overfield L of K. We want to study the extensions Γ_μ and κ_μ of respectively the value group Γ_ν and the residue field κ_ν of ν.

Definition. The ramification index of μ relative to ν is the index of the subgroup Γ_ν in Γ_μ:

$$e(\mu/\nu) \;=\; \left[\Gamma_\mu : \Gamma_\nu\right].$$

The residue degree of μ relative to ν is the degree of the extension of the residue fields:

$$f(\mu/\nu) \;=\; \left[\kappa_\mu : \kappa_\nu\right].$$

The ramification index and the residue degree are elements of $\bar{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$.

Remark 1.9. If μ' is an extension of μ to an overfield L' of L, then μ' is an extension of ν and we have the equalities:

$$e(\mu'/\nu) = e(\mu'/\mu)e(\mu/\nu) \quad \text{and} \quad f(\mu'/\nu) = f(\mu'/\mu)f(\mu/\nu)\,.$$

Proposition 1.15. *Let ν be a valuation of a field K and let μ be an extension of ν to an overfield L; if the field extension $L|K$ is finite of degree n, then we have the inequality*

$$e(\mu/\nu)f(\mu/\nu) \leq n\,.$$

We deduce that the ramification index $e(\mu/\nu) = \left[\Gamma_\mu : \Gamma_\nu\right]$ and the residue degree $f(\mu/\nu) = \left[\kappa_\mu : \kappa_\nu\right]$ are finite.

Proof. Let r and s be two integers with $r \leq e(\mu/\nu)$ and $s \leq f(\mu/\nu)$, and we want to show $rs \leq n$. There exist r elements $x_1, x_2, \ldots, x_r$ of L such that for any (i,j) with $i \neq j$, $\mu(x_i) \not\equiv \mu(x_j) \bmod \Gamma_\nu$, and there exist s elements $y_1, y_2, \ldots, y_s$ in the valuation ring R_μ such that their images $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_s$ in the residue field κ_μ are linearly independent over κ_ν. It is enough to show that the rs elements $x_i y_j$, $1 \leq i \leq r$ and $1 \leq j \leq s$, of L are linearly independent over K.
We assume that there exists a non trivial relation

$$(*) \quad \sum a_{i,k} x_i y_k \;=\; 0\,, \quad \text{with } a_{i,k} \in K\,.$$

We choose an (j,m) such that for all (i,k) we have $\mu(a_{j,m}x_j y_m) \leq \mu(a_{i,k}x_i y_k)$. Since $\mu(y_k) = 0$ for all the y_k and since $\mu(x_j) - \mu(x_i) \notin \Gamma_\nu$ for all $i \neq j$, we have $\mu(a_{j,m}x_j y_m) \neq \mu(a_{i,k}x_i y_k)$ for $i \neq j$. If we multiply the relation $(*)$ by $(a_{j,m}x_j)^{-1}$ we get a relation $\sum b_k y_k + z = 0$, with $b_k = a_{j,k}/a_{j,m} \in R_\mu \cap K$ and $z \in max(R_\mu)$, then we get in the residue field κ_μ the relation $\sum \bar{b}_k \bar{y}_k = 0$ with $\bar{b}_m = 1$. This is a non trivial relation of linear dependence of the $\bar{y}_k$ over κ_ν, which is impossible by hypothesis.

Proposition 1.16. *If L is an algebraic extension of K the quotient group Γ_μ/Γ_ν is a torsion group, i.e. every element has finite order, and the residue field κ_μ is an algebraic extension of κ_ν.*

Proof. We can write $L = \lim_{\rightarrow} L_\alpha$, where the fields L_α are finite extensions of K. Then the value group is equal to $\bigcup_\alpha \Gamma_\alpha$ where $\big(\Gamma_\alpha = \mu(L_\alpha^*)\big)$ is a filtered family of groups with $\big[\Gamma_\alpha ; \Gamma_\nu\big] < +\infty$ for all the α. If we denote κ_α the residue field of the valuation $\mu_{|L_\alpha}$, then the residue field κ_μ is equal to $\lim_{\rightarrow} \kappa_\alpha$ where the κ_α are finite extensions of κ_ν.

Remark 1.10. If K is an algebraic extension of k, the unique extension to K of the trivial valuation of k is also the trivial valuation of K. Then if K is algebraic over k the unique valuation of K/k, i.e. the unique valuation of K which is trivial on k, is the trivial valuation. More generally, if K is any extension of k, a valuation of K/k is also a valuation of $K/\bar{k}$, where $\bar{k}$ is the algebraic closure of k in K.

Let ν be a valuation of a field K with value group Γ and residue field κ, then for any algebraic extension L of K and for any extension μ of ν to L we can consider that the value group Γ_μ is contained in the divisible closure Γ^* of Γ and that the residue field κ_μ is contained in the algebraic closue $\bar{\kappa}$ of κ. The divisible closure Γ^* of Γ is the quotient of the group $\Gamma \times (\mathbb{N} \setminus \{0\})$ by the equivalence relation $\sim$ defined by $(\alpha, p) \sim (\beta, q) \iff p\beta = q\alpha$, endowed with the natural addition and ordering.

Corollary. *If L is an algebraic extension of K then the rank and the rational rank of the valuation μ are equal respectively to the rank and the rational rank of the valuation ν. Moreover if L is a finite extension of K, μ is a discrete valuation if and only if ν is a discrete valuation.*

Remark 1.11. If the extension $L|K$ is algebraic but not finite, we can find an extension μ of a discrete valuation ν of K which is not a discrete valuation.

Let ν be a valuation on a field K and let $L|K$ be an extension, then we want to study the set of all the extensions μ of ν to L. We know that the valuation ring V associated to ν is an integrally closed ring and we consider the integral closure $\bar{V}$ of V in the extension L. In general the ring $\bar{V}$ is not a local ring but we always have $\bar{V} = \bigcap W$ where the rings W are the valuation rings associated to all the extensions of ν to L. In the case of an algebraic extension we have the following result.

Theorem 1.17. ([Bo], Chap. 06, §8, n°6, Proposition 6, page 147.) *Let $L|K$ be an algebraic extension and let ν be a valuation of K with valuation ring V, then there is a bijection from the set of the maximal* *ideals of the integral closure* $\bar{V}$ *of* V *in* L *onto the set of the extensions of* ν *to* L*, which is defined in the following way:*

for any maximal ideal $\mathfrak{p}$ *of* $\bar{V}$ *the local ring* $\bar{V}_{\mathfrak{p}}$ *is a valuation ring of* L *which dominates* V*, and we associate to* $\mathfrak{p}$ *the valuation* μ *of* L *associated to* $\bar{V}_{\mathfrak{p}}$*.*

We can deduce from the theorem that if W and W' are valuation rings of L which dominate V, they are not comparable with respect to the inclusion, then if μ and μ' are two extensions of ν to L, they are not comparable with respect to relation $\preceq$. If the valuation ν is of finite order, it's a consequence of the corollary of the proposition 1.16 and of the corollary of the proposition 1.11.

If L is a finite extension of K, for any valuation ν of K the set $\mathcal{V} = \mathcal{V}_L(\nu)$ of valuations μ of L which are extensions of ν to L is finite. In fact we have the inequality $\mathrm{card}(\mathcal{V}) \leq [L:K]_{sep}$, where $[L:K]_{sep}$ is the separable degree of the extension $L|K$. If the extension $L|K$ is purely inseparable, there exists only one extension μ of ν: for any element x in L there exists an integer $n \geq 0$ such that x^{p^n} belongs to K, where p is the characteristic of K, then the valuation μ is defined by $\mu(x) = p^{-n}\nu\big(x^{p^n}\big)$.

For a finite extension $L|K$ we have the following important result:

Theorem 1.18. ([Za-Sa], Chap.VI, §11, Theorem 19, page 55 and Theorem 20, page 60.) *Let* L *be a finite extension of* K *of degree* n*, let* ν *be a valuation of* K *and let* $\mu_1, \mu_2, \ldots, \mu_g$ *be the extensions of* ν *to* L*. If* e_i *and* f_i *are respectively the ramification index and the residue degree of* μ_i *relative to* ν*, then:*

$$e_1 f_1 + e_2 f_2 + \ldots + e_g f_g \leq n\,.$$

If we assume that the integral closure $\bar{V}$ *in* L *of the valuation ring* V *associated to* ν *is a finite* V*-module, then we have equality:*

$$e_1 f_1 + e_2 f_2 + \ldots + e_g f_g = n\,.$$

Remark 1.12. If the extension $L|K$ is separable and if ν is discrete valuation of rank one, which is equivalent to say that the valuation ring V is noetherian, the integral closure $\bar{V}$ is a finite V-module and we always have equality. But it is possible to find inseparable extension $L|K$ and a discrete rank one valuation ν of K, or a separable extension $L|K$ and a non-discrete valuation ν of K, such that the equality fails.

Let ν be a valuation of a field K and let $L|K$ be a normal algebraic extension. Then the Galois group $G = Gal(L/K)$ acts transitively on the set $\mathcal{V}$ of the extensions of ν to L. To any extension μ of ν to L, we can associate subgroups of G called the *decomposition group* $G^d(\mu) = G_Z$, the *inertia group* $G^i(\mu) = G_T$ and the *ramification group* $G^r(\mu) = G_V$. The ramification theory of valuations, which is the study of the properties of these groups, is very important part of the theory of valuations. Since we don't need it in the following, we are not going to develop this theory here.

Let ν be a valuation of a field K and let ν' be an extension of ν to an extension K' of K of positive transcendence degree. We called V and V', Γ and Γ' and κ and κ' respectively the valuation rings, the residue fileds and the value groups of the valuations ν and ν'. We want to study the relations between the transcendence degree of the extensions K'/K and κ'/κ and the rational rank and the rank of the quotient group Γ'/Γ. We first consider the case $K' = K(x)$, whith x transcendental over K.

Proposition 1.19. ([Bo], Chap. 6, §10, n°1, Proposition 1 and Proposition 2, page 157.) *Let ν be a valuation of a field K with value group Γ and residue field κ.*
a) If Γ" is a totally ordered group which contains Γ and if ξ is an element of Γ" satisfying the condition $n.\xi \in \Gamma \Longrightarrow n = 0$, there exists a valuation ν' and only one which is an extension of ν to $K' = K(x)$, with values in the group Γ" and such that $\nu'(x) = \xi$. Then the value group of ν' is equal to $\Gamma' = \Gamma + \mathbb{Z}.\xi$ and the residue field $\kappa_{\nu'}$ of ν' is equal to κ.
b) There exists a valuation ν' and only one which is an extension of ν to the field $K' = K(x)$ such that $\nu'(x) = 0$ and such that the image t of x in the residue field $\kappa_{\nu'}$ is transcendental over κ. Then the value group Γ' of ν' is equal to Γ and the residue field $\kappa_{\nu'}$ is equal to $\kappa(t)$.

Remark 1.13. There may exist valuations ν' which are extension of ν to the extension $K' = K(x)$ with value group Γ' such that the quotient Γ'/Γ is a nontrivial torsion group, or with residue field κ' a non trivial algebraic extension of κ.

Proof. For any element ξ of a totally ordered group Γ" containing Γ, the map ν' from the polynomial ring $K[x]$ to Γ" by $\nu'\big(\sum a_j x^j\big) = \min\big(\nu(a_j) + j.\xi\big)$, is a valuation of $K[x]$ which extends to a valuation ν' of the field $K' = K(x)$ and ν' is an extension of ν.

In the first case, we see that if μ is any extension of ν with $\nu(x) = \xi$ we have $\mu(a_i x^i) = \nu(a_j) + i.\xi$ and as for $i \neq j$, $\nu(a_i) + i.\xi \neq \nu(a_j) + j.\xi$, we must have $\mu\big(\sum a_j x^j\big) = \min\big(\nu(a_j) + j.\xi\big)$ (cf. proposition 1.3). Then there exists only one extension of ν which is the valuation ν' that we have defined, and the value group is obviously the group $\Gamma + \mathbb{Z}.\xi$. Any element y of $K' = K(x)$ may be written $y = x^n b(1 + u)$, with $n \in \mathbb{Z}$, $b \in K^*$ and

$u \in K'$, $\nu'(u) > 0$, then if $\nu'(y) = 0$, i.e. if y is in $V' \setminus max(V')$ where V' is the valuation ring associated to ν', $n = 0$ and the residue class $\bar{y}$ of y in $\kappa_{\nu'}$ is equal to the residue class of b in κ, and then $\kappa_{\nu'} = \kappa$.

In the second case we want to show that there is only one extension μ of ν, and that this valuation μ is again defined by $\mu\big(\sum a_j x^j\big) = \min\big(\nu(a_j) + j.\xi\big) = \min\big(\nu(a_j)\big)$. Let y be in K', and we may assume that $y = \sum a_j x^j$ with $a_j \in V$ and with $\nu(a_l) = 0$ for one index l. Then by the proposition 1.3 we have $\mu(y) \geq \min\big(\nu(a_j)\big) = 0$. The image $\bar{y}$ of y in the residue field κ_μ is equal to $\sum \bar{a}_j t^j$, and since t is transcendental over κ, we have $\bar{y} \neq 0$ which is equivalent to $\mu(y) = 0$.

We consider now the general case, K' is an extension of K with transcendental degree tr.deg.K'/K and let ν' be an extension of a valuation ν of K to K'. We denote V, κ, Γ and V', κ', Γ' respectively the valuation ring, the residue field, the value group of ν and ν'.

Theorem 1.20. ([Bo], Chap. 6, §10, n°3, Théorème 1, page 161.) *Let $x_1, \dots, x_s$ be elements of the valuation ring V' such that their images $\bar{x}_1, \dots, \bar{x}_s$ in the κ' are algebraically independent over κ, and let $y_1, \dots, y_r$ be elements of K' such that the images of $\nu'(y_1), \dots, \nu'(y_r)$ in the quotient group Γ'/Γ are linearly independent over $\mathbb{Z}$. Then the $r + s$ elements $x_1, \dots, x_s, y_1, \dots, y_r$ of K' are algebraically independent over K. If we denote ν" the restriction of the valuation ν' to the field K" $= K(x_1, \dots, x_s, y_1, \dots, y_r)$, the value group Γ" of ν" is equal to $\Gamma + \mathbb{Z}.\nu'(y_1) + \ldots + \mathbb{Z}.\nu'(y_r)$ and the residue field κ" is equal to $\kappa(\bar{x}_1, \dots, \bar{x}_s)$.*

For a polynomial $f = \sum a_{(\underline{\beta},\underline{\gamma})} x^{\underline{\beta}} y^{\underline{\gamma}}$ in $K[x_1, \dots, x_s, y_1, \dots, y_r]$, the valuation of f is defined by:

$$\nu''(f) = \min_{(\underline{\beta},\underline{\gamma})} \nu''\Big(a_{(\underline{\beta},\underline{\gamma})} x^{\underline{\beta}} y^{\underline{\gamma}}\Big) = \min_{(\underline{\beta},\underline{\gamma})} \Big(\nu\big(a_{(\underline{\beta},\underline{\gamma})}\big) + \sum_{1 \leq j \leq r} \gamma_j \nu'\big(y_j\big)\Big) .$$

Proof. We make a proof by induction on $r + s$, and it is enough to consider the two cases $r = 1$ and $s = 0$ or $r = 0$ and $s = 1$. Then the result is a consequence of the proposition 1.19.

Corollary. *a) We have the inequality:*

$$rat.rank\big(\Gamma'/\Gamma\big) \; + \; tr.deg.\kappa'/\kappa \; \leq \; tr.deg.K'/K \, .$$

Moreover if we have equality and if we assume that K' is a finitely generated extension of K, the group Γ'/Γ is a finitely generated $\mathbb{Z}$-module and the residue field κ' is a finitely generated extension of κ.
b) We have the inequality:

$$rank(\nu') \; + \; tr.deg.\kappa'/\kappa \; \leq \; rank(\nu) \; + \; tr.deg.K'/K \, .$$

Moreover if we have equality, and if we assume that K' is a finitely generated extension of K and that Γ is discrete, i.e. $\Gamma \simeq (\mathbb{Z}^h, +)_{\mathrm{lex}}$, then the residue field κ' is a finitely generated extension of κ and Γ' is discrete.

Example 4. Let K be a field and let ν be a valuation of K, then from the theorem 1.20 there exists a unique extension ν' of ν to the purely transcendental extension $K' = K(x_1, \ldots, x_s)$ of K such that $\nu'(x_i) = 0$ for all i and such that the images $\bar{x}_1, \ldots, \bar{x}_s$ in the residue field $\kappa_{\nu'}$ are algebraically independent over κ_ν. The valuation ν' is defined by

$$\nu'\Big(\sum a_{\underline{\beta}} x^{\underline{\beta}}\Big) = \min_{\underline{\beta}}\Big(\nu\big(a_{\underline{\beta}}\big)\Big) .$$

The valuation ν' is called *the Gauss valuation.*
Let L/K be an extension of transcendence degree s, let ν be a valuation of K and μ be an extension μ of ν to L such that tr.deg.$\kappa_\mu/\kappa_\nu =$ tr.deg.$L/K = s$. Then there exist s elements $x_1, \ldots, x_s$ of L, algebraically independent over K such that L is an algebraic extension of $K' = K(x_1, \ldots, x_s)$ and such that μ is the extension of a Gauss valuation ν' of K'.

Let k be a field, K be an extension of k and we consider a valuation ν of K/k, i.e. that ν is a valuation of K which induces the trivial valuation on k, and let κ be the residue field of ν. We define the dimension of the valuation ν by the following.

Definition. The *dimension* of the valuation ν is the transcendence degree of the residue field κ of ν over the field k: $\dim(\nu) = $ tr.deg.κ/k.

Remark 1.14. Let k be a field, K be an extension of k and let ν be a valuation of K/k, with residue filed κ. We can apply the corollary in the case of a valuation ν of K/k, where K is an extension of k and we find the inequalities:

$$\mathrm{rank}(\nu) \ + \ \dim(\nu) \ \leq \ \mathrm{rat.rank}(\nu) + \ \dim(\nu) \ \ \leq \ \mathrm{tr.deg}.K/k\,.$$

If we assume that K is *a function field over* k, i.e. that K is a finitely generated extension of k, and if we have the equality rat.rank.$(\nu)+\dim(\nu) =$ tr.deg.K/k then the value group Γ is a finitely generated $\mathbb{Z}$-module and the residue field κ is finitely generated over k, moreover if we have the equality rank$(\nu)+\dim(\nu) =$ tr.deg.K/k, the valuation ν is discrete.

Remark 1.15. Let k be a field, K be an extension of k and let ν be a valuation of K/k which is composite $\nu = \nu' \circ \bar{\nu}$, where ν' is a valuation of K/k with residue field κ' and $\bar{\nu}$ is a valuation of κ'/k. We deduce from the proposition 1.11 that the valuations ν and $\bar{\nu}$ have the same residue field κ, and that we have the equalities rank(ν) = rank(ν')+ rank$(\bar{\nu})$. Hence, if we are in the case of equality for the rank for the valuation ν:

$$\mathrm{rank}(\nu) \ + \ \dim(\nu) \ = \ \mathrm{tr.deg.}K/k\,,$$

we are also in the case of equality for the rank for the valuations ν' and $\bar{\nu}$:

$$\begin{aligned} \mathrm{rank}(\nu') \ + \ \dim(\nu') \ &= \ \mathrm{tr.deg.}K/k\ , \\ \mathrm{rank}(\bar{\nu}) \ + \ \dim(\bar{\nu}) \ &= \ \mathrm{tr.deg.}\kappa'/k\,. \end{aligned}$$

We have the same result for the rational rank of the valuations ν, ν' and $\bar{\nu}$.

Definition. Let K be a field and let ν a valuation on K with value group Γ and residue field κ. Let K^* be an extension of K and let ν^* be an extension of ν to K^*, with value group Γ^* and residue field κ^*, then we say that the valued field (K^*,ν^*) is *an immediate extension* of the valued field (K,ν) if the group Γ is canonically isomorph to Γ^* and the field κ is canonically isomorph to κ^*.

This equivalent to the following condition:

$$\forall x^* \in K^* \quad \exists x \in K \quad \text{such that} \quad \nu^*(x^*-x) > \nu^*(x^*)\ .$$

A valued field (K,ν) is called *a maximal valued field* if there doesn't exist any immediate extension. It is possible to prove that for any valued field (K,ν) there exists an immediate extension (K^*,ν^*) which is a maximal valued field, but in general this extension is not unique ([Ku]).

1.4. Examples

Example 5. Prime divisor ([Za-Sa], Chap.VI, §14, page 88.)
Let K be a function field over a field k, of transcendence degree d, then *a prime divisor* of K over k is a valuation ν of K/k which have dimension $d-1$, i.e. such that tr.deg.$\kappa/k = d-1$ where κ is the residue field of ν. Since the valuation ν is non trivial, we have rank$(\nu) \geq 1$, and we deduce from the corollary of the theorem 1.20 that we have rank$(\nu) = 1$, hence we are in the case of equality for the rank for the valuation ν: rank(ν)+ dim(ν) = tr.deg.K/k, the valuation ν is discrete of rank one, i.e. its value group is isomorphic to $\mathbb{Z}$, and its residue field κ is finitely generated over k. We deduce from the proposition 1.9 that the valuation ring V associated to ν is a noetherian ring.
Furthermore, we can always find a normal integral domain R, finitely generated over k, having K as fraction field, and a prime ideal $\mathfrak{p}$ of height one of R, such that the valuation ring V is equal to the local ring

$R_{\mathfrak{p}}$. The valuation ν is the "$\mathfrak{p}$-adic valuation", i.e. the valuation defined by $\nu(g) = \max\{n \in \mathbb{N} \ / \ g \in \mathfrak{p}^n\}$, for any g in R. If we consider the affine algebraic variety X associated to R, $X = \mathrm{Spec}R$, the prime ideal $\mathfrak{p}$ defines a *prime Weil divisor* D on X, i.e. a reduced irreducible closed subscheme of codimension one, and the valuation ν is the valuation defined by the order of vanishing along the divisor D. Moreover the valuation ring associated to ν is equal to the local ring $\mathcal{O}_{X,D}$ of the generic point of D in X. Conversely, any prime Weil divisor D on a normal algebraic variety X defines a prime divisor ν of the function field $K = F(X)$ of the variety X, and the valuation ring associated to ν is the local ring $\mathcal{O}_{X,D}$.

Example 6. Composition of prime divisors
Let K be a function field over a field k, of transcendence degree d, and let ν be a valuation of K/k of rank r and of dimension $d - r$. Then we are in the case of equality for the rank in the corollary of the theorem 1.20, and we know that the value group Γ of ν is isomorphic to $(\mathbb{Z}^r, +)_{\mathrm{lex}}$ and that the residue field κ of ν is also finitely generated over k. We deduce from the remark 1.15 that if we write ν as a composite valuation $\nu = \nu' \circ \bar{\nu}$, with $\mathrm{rank}(\nu') = 1$, then ν' is a prime divisor of K and $\bar{\nu}$ is a valuation of the residue field κ' of ν' which satisfies also the equality for the rank. By induction we can write the valuation ν as the composite of a family of valuations $\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$ (cf remark 1.7). All the valuations ν_i, $1 \leq i \leq r$, are discrete valuations of rank one, the residue field κ_i of the valuation ν_i is a function field over k and the valuation ν_{i+1} is a prime divisor of κ_i.

Example 7. Field of generalized power series ([Za-Sa], Chap. VI, §15, Example 2, page 101.)
We want to construct a valuation with a preassigned value group. More precisely, let Γ be a totally ordered group and let k be a ground field, and we want to find a field K, extension of k, and a valuation ν of K/k whose value group is isomorphic to the group Γ.
We define the ring R of *generalized power series* of a variable x with coefficients in the field k and with exponents in the group Γ by the following: R is the set of the expressions ξ of the form $\xi = \sum_{\gamma \in \Gamma} c_\gamma x^\gamma$ whose support $\mathrm{supp}(\xi) = \{\gamma \in \Gamma \ / \ c_\gamma \neq 0\}$ is a well ordered subset of Γ.
We recall that an ordered set A is well ordered if any non empty subset B of A has a minimal element. Then we can define an addition and a multiplication on R in the usual way: for $\xi = \sum_\gamma c_\gamma x^\gamma$ and $\zeta = \sum_\gamma d_\gamma x^\gamma$

in R, we put:

$$\xi + \zeta = \sum_\gamma s_\gamma x^\gamma \qquad \text{with} \quad s_\gamma = c_\gamma + d_\gamma \ ,$$

$$\xi.\zeta = \sum_\gamma m_\gamma x^\gamma \qquad \text{with} \quad m_\gamma = \sum_{\alpha+\beta=\gamma} c_\alpha d_\beta \ ,$$

this last sum is well defined because since the supports of ξ and ζ are well ordered sets, for any $\gamma \in \Gamma$, there exists only a finite number of couples (α, β) in $\text{supp}(\xi) \times \text{supp}(\zeta)$ with $\alpha + \beta = \gamma$.
Hence we have a ring, in fact a k-algebra, and we denote it by $R = k[[x^\Gamma]]$. The ring R is integral, we call its fraction field $K = Fr(R)$ *the field of generalized power series* and we denote it by $K = k((x^\Gamma))$.
We can define a valuation ν on $K = k((x^\Gamma))$. Let ξ be an element of the ring $k[[x^\Gamma]]$, $\xi = \sum_{\gamma\in\Gamma} c_\gamma x^\gamma$, $\xi \neq 0$, then we put $\nu(\xi) = \min(\text{supp}(\xi))$, this is well defined because since the support of ξ is a non empty well ordered subset of Γ, it has a minimal element; and for $\xi = 0$ we put $\nu(0) = +\infty$. It is easy to prove that the valuation ν is a valuation of $K = k((x^\Gamma))$, which is trivial on k and such that its residue field κ is equal to k and its value group is Γ.
Moreover, the valued field $(k((x^\Gamma)), \nu)$ is maximal, i.e. there exist no immediate extension. ([Ri], Chap. D, Corollaire au Théorème 2, page 103.)

Example 8. Valuations of *k(x,y)/k*
We are going to give two examples of valuations of K/k, where K is the pure transcendental extension $K = k(x, y)$ of k of degree 2. We construct these valuations by their restrictions to the polynomial ring $R = k[x, y]$.
i) The first one is a valuation ν on $R = k[x, y]$ whose value group is the group of rational numbers $\mathbb{Q}$.
We put:

$$\nu(y) = 1 \quad \text{and} \quad \nu(x) = 1 + \frac{1}{2} = \frac{3}{2} \ .$$

The "first" element z_2 of R, i.e. a polynomial in x, y of minimal degree, such that the value $\nu(z)$ is not uniquely determined by the values in $z_0 = y$ and in $z_1 = x$ is the element $z_2 = x^2 + y^3$. We must have $\nu(z_2) \geq 3$ and we put:

$$\nu(z_2) = 3 + \frac{1}{3} = \frac{10}{3} \ .$$

We can define a sequence (z_n) of elements of R such that for any integer $n \geq 2$, the value $\nu(z_n)$ is not determined by the values on ν in the z_r for $r < n$. For any n the value $\nu(z_n)$ is a rational number $\gamma_n = \dfrac{p_n}{n+1}$, with p_n a positive integer and $(p_n, n+1) = 1$. The sequence (z_n) is constructed by induction in the following fashion, we assume we have found the elements z_r, $0 \le r \le n$ and the values $\nu(z_r) = \gamma_r$, then we put:

$$z_{n+1} = z_n + y^{p_n} \ ,$$

with the value:

$$\nu(z_{n+1}) = \gamma_{n+1} = p_n + \frac{1}{n+2} = \frac{p_{n+1}}{n+2} \quad \text{with } p_{n+1} = p_n(n+2) + 1 \ .$$

We have constructed a valuation ν of R, hence a valuation of the field $K = k(x,y)$, whose value group is $\mathbb{Q}$ and it is easy to see that the residue filed κ of ν is the ground field k.

There is another construction of such a valuation in [Za 1], I §6, page 648.

More generally, for any preassigned value group Γ of $\mathbb{R}$, we can construct a valuation with value group Γ. ([Za-Sa], Chap.VI, §15, Example 3, page 102 or [ML-Sc].)

ii) The second example is what we call *an analytic arc* on the surface $\mathbf{A}^2_k = \mathrm{Spec}R$. ([Za 1], I §5, page 647.)
Let $\hat{R}$ be the completion of the ring R, $\hat{R}$ is the ring of power series $k[[x,y]]$, and let $\hat{K}$ its fraction field $\hat{K} = k((x,y))$. We consider an element t of $\hat{R}$:

$$t = x + \sum_{i=1}^{+\infty} c_i y^i \ , \quad c_i \in k^* \quad \text{for all} \quad i \ge 1 \ ,$$

which is not algebraic over the field K.
We define the valuation $\hat{\nu}$ of $\hat{K}/k$ with values in the group $\hat{\Gamma} = (\mathbb{Z}^2, +)_{\mathrm{lex}}$, by:

$$\hat{\nu}(y) = (0,1) \quad \text{and} \quad \hat{\nu}(t) = (1,0) \ .$$

For any element ξ of $\hat{R}$, with $\xi = \sum_{i=1}^{+\infty} d_i y^i$ and $d_1 \neq 0$, we have $\hat{\nu}(\xi) = (0,1)$, hence since the coefficients c_i are non zero, for any $N \ge 1$ we have:

$$\hat{\nu}\Big(\sum_{i+n}^{+\infty} c_i y^i\Big) \ = \ \hat{\nu}\big(y^N\big) \ = \ (0,N) \ ,$$

then for any $N \ge 1$ we get:

$$\hat{\nu}\big(x + \sum_{i=1}^{N-1} c_i y^i\big) \ = \ (0,N) \ .$$

We define the valuation ν on $K = k(x,y)$ as the restriction to K of the valuation $\hat{\nu}$ of $\hat{K}$, then the value group Γ is equal to $\mathbb{Z}$, i.e. the valuation ν is a discrete valuation of rank one. But this valuation is not a prime divisor, its residue field is equal to k.

We may describe the valuation ν on R in a different way. We have an injective k-morphism φ of $R = k[x,y]$ to the power series ring $k[[y]]$ defined by

$$\varphi(x) = x - t = -\sum_{i=1}^{+\infty} c_i y^i \quad \text{and} \quad \varphi(y) = y \ .$$

Then the valuation ν is the restriction to R of the y-adic valuation on $k[[y]]$.

Example 9. A non finitely generated residue field ([Za 2], Chap.3 II , footnote 12, page 864.)
Let K be a function field over a field k and let ν be a valuation of K/k, then if we don't assume that we have the equality rat.rank$(\nu)+$ dim$(\nu) =$ tr.deg.K/k, it may happen that the residue field κ of ν is not finitely generated over k.

Let k be a field and $K = k(x,y,z)$ be an extension of k with x, y and z algebraically independent elements over k and we consider the valuation ν *defined by the formal power serie in y:*

$$z = x^{1/2}.y + x^{1/4}.y^2 + x^{1/8}y^3 \ldots = \sum_{n\geq 1} x^{1/2^n}.y^n \ .$$

We can give the following description of the valuation ν.
Let $A = k[x,y,z]$ be the polynomial ring and let $R = \cup_{n\geq 1} k\big[\big[x^{1/2^n}, y, z\big]\big]$ be the ring of formal power series in y, z and the $x^{1/2^n}$, $n \geq 1$. Let $f = z - \sum_{n\geq 1} x^{1/2^n}.y^n$ be in R and $\bar{R}$ be the quotient ring $R/(f)$. Then the map $A \longrightarrow \bar{R}$ induced by $A \subset R$ is an injection. We consider on $\bar{R}$ the y-adic valuation μ, i.e. the valuation defined by the order in y. If we denote $\bar{k}$ the field $\bar{k} = \cup_{n\geq 1} k((x^{1/2^n}))$ and L the fraction field of $\bar{R}$, L is an extension of $\bar{k}$ and μ is a discrete rank one valuation of $L/\bar{k}$, then the valuation ring W associated to μ is a noetherian $\bar{k}$-algebra and the residue field κ_μ is an extension of $\bar{k}$. The valuation ν is the restriction of μ to A, we have for instance $\nu(x) = 0$, $\nu(y) = \nu(z) = 1$.
We shall see that the residue field κ of ν is not finitely generated over k, in fact we have $\bar{k} \subset \kappa$, i.e. all the elements $x^{1/2^n}$, for $n \geq 1$, belong to κ. If we denote by $[u]$ the image in $\bar{R}$ of any element $u \in R$, we have $\mu([z/y - x^{1/2}]) = 1$ and the residue class of $[z/y]$ in κ_μ is equal to $x^{1/2}$. Then we have also that $x^{1/2}$ is the residue class of z/y in κ. In

the same way we can write $[z^2/y^2 - x] = [(z/y - x^{1/2})(z/y + x^{1/2})] = [2x^{1/2}x^{1/4}y + \ldots]$, and we find that the residue class of $(z^2 - xy^2)/y^3$ in κ is equal to $2x^{3/4}$, and we can continue in this fashion to show that all the elements $x^{1/2^n}$, for $n \geq 1$, belong to κ.
Since the valuation ν is the restriction of discrete valuation μ, ν is also a discrete valuation of rank one, i.e. its value group Γ is equal to $\mathbb{Z}$, hence rank.(ν) = rat.rank.$(\nu) = 1$, and we have seen that the transcendence degree of the residue field κ over k is equal to one, i.e. $\dim(\nu) = \text{tr.deg.}\kappa/k = 1$. Then for this valuation we have $\text{rank}(\nu) + \dim(\nu) = 2 < \text{tr.deg.}K/k = 3$.

§2. Riemann variety

2.1. Center of a valuation

Let K be a field and ν be a valuation of K, we denote V the valuation ring associated to ν and $\mathfrak{m}$ its maximal ideal.

Definition. Let A be a subring of K with $A \subset V$, i.e. such that $\nu(x)$ is non negative for all the elements x of A, then the *center* of the valuation ν on A is the ideal $\mathfrak{p}$ of A defined by $\mathfrak{p} = A \cap \mathfrak{m}$.

Remark 2.1. The center $\mathfrak{p}$ of the valuation ν on A is the unique prime ideal $\mathfrak{q}$ of A such that the valuation ring V dominates the local ring $A_{\mathfrak{q}}$. If A is a local ring, the center of ν on A is the maximal ideal of A if and only if V dominates A.

Let X be *an algebraic variety over a field* k, i.e. an irreducible reduced scheme of finite type over k, and let $K = F(X)$ be the function field of X, then K is a finitely generated extension of k and the dimension of the variety X is equal to the transcendence degree of K over k. We want to define the center of a valuation ν of K/k, or more generally of a valuation ν of L/k where L is an extension of K, on the variety X.

We consider first that X is an affine variety, $X = \text{Spec}A$, where A is an integral k-algebra of finite type, with $A \subset L$. If A is contained in the valuation ring V associated to ν, i.e. if the valuation ν is non negative for all the elements $x \in A$, the center of ν on X is the point ξ of X corresponding to the prime ideal $\mathfrak{p}$ where $\mathfrak{p}$ is the center of the valuation ν on A, i.e. $\mathfrak{p} = A \cap \mathfrak{m}$. Since the center $\mathfrak{p}$ is a prime ideal of A, the closed subscheme Z of X defined by $\mathfrak{p}$ is an integral subscheme, i.e. an irreducible reduced subscheme of X, and ξ is the generic point of Z. We say also that the closed subscheme Z is the center of the valuation ν on the affine variety X. If A is not contained in the valuation ring V, then we say that the valuation ν has no center on X or that the center Z of the valuation ν on X is the empty set.

We want to generalise this definition for any algebraic variety X over k, with function field K, and say that the center of a valuation ν of a field L, with $K \subset L$, on the variety X is a point ξ of X such that the local ring $\mathcal{O}_{X,\xi}$ is dominated by the valuation ring V associated to ν. This is equivalent to say that we have a morphism of $T = \mathrm{Spec} V$ to X such that the image of the closed point t of T, corresponding to the maximal ideal of V, is the point ξ, and that this morphism induces the inclusion $K \subset L$, i.e. that the image of the generic point of T is the generic point of X.

Before defining the center of a valuation on any algebraic variety, we shall recall the valuation criterions of separatedness and of properness. ([EGA], Proposition 7.23 and Théorème 7.3.8, or [Ha], Chap.II, Theorem 4.3 and Theorem 4.7.)

Valuative criterion of separatedness. *let X and Y be noetherian schemes, let $f\colon X \longrightarrow Y$ be a morphism of finite type, then f is separated if and only if for every field L, for every valuation ring V of L and for every morphism $g\colon U = \mathrm{Spec} L \longrightarrow X$ and $h\colon T = \mathrm{Spec} V \longrightarrow Y$ forming a commutative diagram*

$$\begin{array}{ccc} U = \mathrm{Spec}\, L & \xrightarrow{g} & X \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle f} \\ T = \mathrm{Spec}\, V & \xrightarrow{h} & Y \end{array}$$

there exists at most one morphism $\bar{h}\colon T \longrightarrow X$ making the whole diagram commutative.

Valuative criterion of properness. *let X and Y be noetherian schemes, let $f\colon X \longrightarrow Y$ be a morphism of finite type, then f is proper if and only if for every field L, for every valuation ring V of L and for every morphism $g\colon U = \mathrm{Spec} L \longrightarrow X$ and $h\colon T = \mathrm{Spec} V \longrightarrow Y$ forming a commutative diagram*

$$\begin{array}{ccc} U = \mathrm{Spec}\, L & \xrightarrow{g} & X \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle f} \\ T = \mathrm{Spec}\, V & \xrightarrow{h} & Y \end{array}$$

there exists a unique morphism $\bar{h}\colon T \longrightarrow X$ making the whole diagram commutative.

The valuative criterion of separatedness will give the unicity of the center of a valuation on an algebraic variety and the valuative criterion of properness will give a condition on a variety for any valuation to have a center on X.

Proposition 2.1. *Let X be an algebraic variety over k and let ν be a valuation of a field L, extension of the function field $K = F(X)$ of X, then there exists at most one point ξ of X such that the local ring $\mathcal{O}_{X,\xi}$ is dominated by the valuation ring V associated to ν.*
Moreover the irreducible closed subvariety Z of X defined by $Z = \overline{\{\xi\}}$ is the subset of the points $x \in X$ whose the local ring $\mathcal{O}_{X,x}$ is contained in the valuation ring V associated to ν.

Proof. Since X is an algebraic variety, the morphism $f \colon X \longrightarrow \mathrm{Spec}k$ is separated, and the unicity of the point ξ is a consequence of the valuative criterion of separatedness, where the morphism $g \colon U = \mathrm{Spec}L \longrightarrow X$ is defined by the inclusion $F(X) \subset L$ and where the morphism $h \colon T = \mathrm{Spec}V \longrightarrow \mathrm{Spec}k$ is defined because the valuation ν is trivial on k. Then there exists at most one morphism $\bar{h} \colon T = \mathrm{Spec}V \longrightarrow X$, i.e. at most one point ξ on the variety X such that its local ring $\mathcal{O}_{X,\xi}$ is dominated by the valuation ring V.
To show that the set $Z = \overline{\{\xi\}}$ is equal to $\big\{x \in X \,/\, \mathcal{O}_{X,x} \subset V\big\}$, we can assume that X is an affine variety $X = \mathrm{Spec}A$ and that Z is exactly the closed subscheme of X defined by the center $\mathfrak{p}$ of the valuation ν on A. Then it is enough to see that for any prime ideal $\mathfrak{q}$ of A we have $\mathfrak{p} \subset \mathfrak{q}$ if and only if $A_{\mathfrak{q}} \subset V$.

Definition. The *center of the valuation ν on the variety X* is the point ξ, when it exists, defined in the proposition. We say also that the center of the valuation ν on the variety X is the subvariety $Z = \overline{\{\xi\}}$. If there doesn't exist ξ we say that the valuation ν has no center on the variety X or that the center Z is empty.

The valuation ν may have no center on the variety X, for instance if X is an affine variety $X = \mathrm{Spec}A$, with A non contained in V. But, if X is a projective variety, any valuation ν has a center on X. In fact we have the following result. We recall that an algebraic variety X over a field k is complete if the morphism $X \longrightarrow \mathrm{Spec}k$ is proper.

Theorem 2.2. *If X is a complete variety over a field k, any valuation ν of L/k, L an extension of the function field $K = F(X)$ of X, has a center on X.*
Conversely, the variety X is complete over k if all the valuations ν of K/k have a center on X.

Proof. If X is a complete variety, the morphism $f \colon X \longrightarrow \mathrm{Spec}k$ is proper, and we can apply the criterion of properness where the morphism $g \colon U = \mathrm{Spec}L \longrightarrow X$ is defined by the inclusion $F(X) \subset L$ and where the morphism $h \colon T = \mathrm{Spec}V \longrightarrow \mathrm{Spec}k$ is defined because the valuation

ν is trivial on k. Then we obtain a morphism $\bar{h}\colon T = \mathrm{Spec}V \longrightarrow X$ and the image of the closed point of T is the center of the valuation ν on X.

If X is an algebraic variety over a field k, then the transcendence degree of the function field $K = F(X)$ of X over k is equal to the dimension of X. Let ν be a valuation of the function field K, then we are going to show that the dimension of ν, i.e. the transcendence degree of the residue field κ of ν over k, is always bigger or equal to the dimension of its center Z on X.

Proposition 2.3. *Let X be an algebraic variety over a field k with function field $K = F(X)$, and let ν be a valuation of K/k with residue field κ. Then if the center Z of ν on X is non empty whe have $dimZ \leq dim(\nu)$. Moreover if we have a strict inequality, there exists a proper birational morphism $Y \longrightarrow X$ such that the dimension of the center of ν on Y is equal to $dim(\nu)$.*

Proof. Let Z be the center of the valuation ν on X and let ξ be the generic point of Z. We denote by A the local ring $\mathcal{O}_{X,\xi}$ of X in ξ and by $\mathfrak{p}$ its maximal ideal, then the valuation ring V associated to ν dominates A and we have $A/\mathfrak{p} \subset V/\mathfrak{m}$, i.e. an inclusion of the function field $F(Z)$ of Z in the residue field κ. Since Z is an algebraic variety over k we have $\mathrm{dim}Z = \mathrm{tr.deg.}F(Z)/k$, then $\mathrm{dim}Z \leq \mathrm{tr.deg.}\kappa/k$.

If the inequality is strict, let $x_1, \dots, x_r$ be elements of V such that their images $\bar{x}_1, \dots, \bar{x}_r$ in κ is a transcendental basis of κ over $F(Z)$, and we can write $x_i = p_i/q$ with p_i and q in A, $i = 1, \dots, r$. We consider an ideal $\mathcal{I}$ of $\mathcal{O}_X$ which is locally generated by $q, p_1, \dots, p_r$ and Y the blowing up of $\mathcal{I}$ in X. Then the center Z' of ν on Y satisfies $F(Z)(\bar{x}_1, \dots, \bar{x}_r) \subset F(Z')$ and we obtain $\mathrm{dim}Z' = \mathrm{tr.deg.}\kappa/k$.

Remark 2.2. Let ν be a prime divisor of the function field K of an algebraic variety X over k (cf exemple 5), let Z be the center of ν on X and we assume that Z is non empty. Then we have $\mathrm{codim}Z \geq 1$ and we deduce from the proposition 2.3 that there exists a proper birational morphism $\pi\colon Y \longrightarrow X$ such that the center D of ν on Y is a prime Weil divisor, moreover if we choose Y normal, the valuation ring V associated to ν is equal to the local ring $\mathcal{O}_{Y,D}$ of D in Y.
Conversely, if we consider a prime Weil divisor Z on an algebraic variety X over k, then we deduce from the proposition 2.3 that any non trivial valuation ν of the function field K of X with center Z is a prime divisor. It is possible to show that the set of prime divisors ν of the function field K of X which have center Z on X is finite and non empty. Moreover if the variety X is normal there exists only one prime divisor ν with

center Z, this the valuation ν associated to the local ring $\mathcal{O}_{X,Z}$, which is a noetherian valuation ring ([Za-Sa]).

Let X be an algebraic variety over a field k with function field $K = F(X)$ and let ν be a valuation of K/k with residue field κ. We assume that the transcendence degree of κ over k is positive, then there exists non trivial valuations $\bar{\nu}$ of κ/k and we can define the composite valuation $\nu' = \nu \circ \bar{\nu}$ which is also a valuation of K/k. If the center Z of ν on X is non empty the function field $\bar{K} = F(Z)$ of Z is contained in the residue field κ and we can consider the center on Z of a valuation $\bar{\nu}$ of κ/k.

Proposition 2.4. *The center on Z of the valuation $\bar{\nu}$ is equal to the center on X of the composite valuation $\nu' = \nu \circ \bar{\nu}$.*

Proof. We may assume that X is an affine variety $X = \mathrm{Spec}A$. We denote respectively V, V', $\bar{V}$ and $\mathfrak{m}$, $\mathfrak{m}'$, $\bar{\mathfrak{m}}$ the valuation rings associated to ν, ν', $\bar{\nu}$ and their maximal ideals, then we have $(0) \subset \mathfrak{m} \subset \mathfrak{m}' \subset V' \subset V \subset K$ and $\bar{\mathfrak{m}} \subset \bar{V} = V'/\mathfrak{m} \subset \bar{K} = V/\mathfrak{m}$. The centers of the valuations ν and ν' on X are defined by the prime ideals $\mathfrak{p} = A \cap \mathfrak{m}$ and $\mathfrak{p}' = A \cap \mathfrak{m}'$ of A, and the center of the valuation $\bar{\nu}$ on $Z = \mathrm{Spec}\bar{A}$, with $\bar{A} = A/\mathfrak{p}$ is defined by the prime ideal $\bar{\mathfrak{p}} = \bar{A} \cap \bar{\mathfrak{m}}$ of $\bar{A}$. Then the proposition is a consequence of the equality $\bar{\mathfrak{p}} = \mathfrak{p}'/\mathfrak{p}$.

We have seen that if ν' is a composite valuation $\nu' = \nu \circ \bar{\nu}$ of K/k, where K is the function field of an algebraic variety X over k, the center Z' of ν' is contained in the center Z of ν. This a consequence of the proposition 2.4 if the center Z of ν is non empty. If the center Z is empty, no local ring $\mathcal{O}_{X,x}$ for $x \in X$ is contained in the valuation ring V associated to ν, then none is contained in the valuation ring V' associated to ν' since we have $V' \subset V$. More generally if the valuation ν is composite with the family $\big(\nu_1, \nu_2, \ldots, \nu_r\big)$, and if we denote $\nu'_{(t)}$ the valuation of K defined by $\nu'_{(t)} = \nu_1 \circ \ldots \circ \nu_t$, $0 \leq t \leq r$, and ξ_t the center of $\nu'_{(t)}$ on the variety X, we obtain a family $\big(\xi_1, \xi_2, \ldots, \xi_r\big)$ of points of X such that ξ_t is a specialization of ξ_{t+1}, i.e. $\xi_t \in \overline{\{\xi_{t+1}\}}$, for $1 \leq t \leq r-1$.

Conversely, we have the following result.

Theorem 2.5. ([Za-Sa], Chap.VI, §16, Theorem 37, page 106.) *Let X be an algebraic variety over a field k of dimension d, let r be an integer such that $r \leq d$ and let $\big(\xi_1, \xi_2, \ldots, \xi_r\big)$ be a family of points of X such that ξ_t is a specialization of ξ_{t+1}, $1 \leq t \leq r-1$. Then there exists a valuation ν composite with a family $\big(\nu_1, \nu_2, \ldots, \nu_r\big)$, such that the center of the composite valuation $\nu'_{(t)} = \nu_1 \circ \ldots \circ \nu_t$ is the point ξ_t, for $t = 1, \ldots, r$.*

Remark 2.3. Let ν be a valuation of K/k of rank r, where K is the function field of an algebraic variety X over k of dimension d. We assume that the center Z of ν on X is non empty and that we have $\dim Z = d-r$. Then we are in the case of equality of the corollary of the theorem 1.20 for the rank, and we know that we can write the valuation ν as $\nu = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_r$ (cf example 6), where each valuation ν_i is a prime divisor. In that case the center ξ_t of the valuation $\nu'_{(t)} = \nu_1 \circ \nu_2 \circ \ldots \circ \nu_t$ defines a divisor in $\overline{\{\xi_{t+1}\}}$ and the valuation ν is *the composition of the orders of vanishing along these divisors* (cf remark 1.6).

2.2. Riemann variety

Let k be a field, we want to study the set of all the valuations ν of K/k where K is an extension of k, i.e. the set of all the valuations ν of K which are trivial on k.

Definition. ([Za-Sa], Chap.VI, §17, page 110.) *The Riemann variety* or *the Rieman manifold* or *the abstract Riemann surface* of K relative to k is the set of all the valuations ν of K which are trivial on k. We denote this set by $S = S(K/k)$.

More generally we can define the Riemann variety of K/k when k is a subring of K, not necessarily a field. In that case the Riemann variety is the set of all the valuations of K which are not negative on k, i.e. the valuations of K such that the valuation ring V associated to ν contains k.

Remark 2.4. We deduce from the remark 1.10 that the Riemann variety $S(K/k)$ and $S(K/\bar{k})$ are isomorphic, where $\bar{k}$ is the integral closure of k in K, and if K is an algebraic extension of k, the Riemann variety contains one unique element, the trivial valuation.

We give sometimes another definition of the Riemann variety, we consider only the non trivial valuations ν of K which are trivial on k, and we denote this set $S^*(K/k)$, i.e. $S(K/k) = S^*(K/k) \cup \{\nu_0\}$ where ν_0 is the trivial valuation of K. With this definition, if K is an algebraic extension of k, the Riemann variety $S^*(K/k)$ is empty.

We introduce a topology in the Riemann variety $S = S(K/k)$, by defining a basis of open sets.

Definition. Let A be a subring of K containing k, then we denote $E(A)$ the set of all the valuations ν of K/k which are non negative on A, i.e. the set defined by $E(A) = \left\{\nu \in S(K/k) \,/\, A \subseteq V_\nu\right\}$, where V_ν is the valuation ring associated to ν. We define the topology in S by taking as basis of open sets the family of all the sets $E(A)$ where A range over the family of all k-subalgebras of K which are finitely generated over k. We call this topology *the Zariski topology.*

If A and A' are two finitely generated k-subalgebras, we denote $[A, A']$ the subalgebra of K generated by A and A'. This algebra is finitely generated over k and we notice that the intersection $E(A) \cap E(A')$ is equal to $E([A, A'])$. Therefore the intersection of two basic open subsets is again a basic open subset, and hence we have indeed defined a topology in S. Any finitely generated k-subalgebra A of K is of the form $A = k[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are elements of K. Then we can write the basic open set $E(A) = E(k[x_1]) \cap \ldots \cap E(k[x_n])$, and hence the topology in S is generated by the open sets $E(k[x]) = \{\nu \in S(K/k) \,/\, \nu(x) \geq 0\}$, where x range K^*. We may also notice that if A and A' are two k-subalgebras of K with $A \subset A'$, then we have $E(A') \subseteq E(A)$.

Theorem 2.6. *Let ν be a valuation of K/k, then the closure of the set $\{\nu\}$ consisting of the single element ν in S is the set of all the valuations ν' of K/k which are composite with ν:*

$$\overline{\{\nu\}} = \left\{\nu' \in S \ / \ \nu' \textit{ is composite with } \nu\right\}.$$

More precisely the closure $\overline{\{\nu\}}$ is isomorphic to the Riemann variety $S(\kappa/k)$ of the residue field κ of the valuation ν.

Proof. Let ν and ν' be two valuations of K, then ν' is composite with ν if and only if the valuation ring V' associated to ν' is contained in the valuation ring V associated to ν. If ν' is in the closure of $\{\nu\}$, for any finitely generated k-subalgebra A of K we have $\nu' \in E(A) \Longrightarrow \nu \in E(A)$, i.e. $A \subseteq V' \Longrightarrow A \subseteq V$, hence V' is contained in V. Conversely, il ν' is not in the closure of $\{\nu\}$, there exists a finitely generated k-algebra A with $A \subseteq V'$ and $A \not\subseteq V$, hence V' is not contained in V.

We deduce from the proposition 1.12 that the map ϕ of $\overline{\{\nu\}}$ to the Riemann variety $S(\kappa/k)$ of the residue field κ of ν, which sends a composite valuation $\nu' = \nu \circ \bar{\nu}$ to the valuation $\bar{\nu}$ of κ, is a bijection. By definition of the valuation ring $\bar{V}$ associated to $\bar{\nu}$, we see that for all the elements x in the valuation ring V, we have $\nu'(x) \geq 0$ if and only if $\bar{\nu}(\bar{x}) \geq 0$, where $\bar{x}$ is the image of x in κ, hence the map ϕ is an homeomorphism.

Remark 2.5. Let ν_0 be the trivial valuation of K, all the valuations of K are composite with ν_0. The valuation ring associated to ν_0 is the field K, hence the valuation ν_0 belongs to all the non empty open sets $E(A)$, and ν_0 is a generic point of the Riemann variety $S(K/k)$.

Even if we consider the variety $S^*(K/k) = S(K/k) \smallsetminus \{\nu_0\}$, we see that this space is never a Hausdorff space, in the case where k is a field.

Theorem 2.7. ([Za-Sa], Chap.VI, §17, Theorem 40, page 113.) *The Riemann variety $S = S(K/k)$ is quasi-compact, i.e. every open covering of S contains a finite subcovering.*

Proof. We give a sketch of the Chevalley's proof which is exposed with more details in [Za-Sa] or in [Va].
Any valuation ν of K is uniquely determined by its valuation ring, hence to know a valuation it is enough to know the sets of the elements x of K where ν is positive, equal to zero or negative and we can consider the Riemann variety $S = S(K/k)$ as a subset of the set Z^K of the applications of K to $Z = \{+, 0, -\}$.
We define a topology in Z by taking as open sets $\emptyset$, $\{0, +\}$ and Z and we introduce the product topology on Z^K. Then the induced topology in S has for basis of open sets the sets E defined as follows: $E = \{\nu \in S \,/\, \nu(x_i) \geq 0, i = 1, 2, \ldots, r\}$ where $\{x_1, x_2, \ldots, x_r\}$ is a finite subset of K. This definition agree with the preceding definition, hence we can consider the Riemann variety as a subset of the topological space Z^K.
We shall modify temporarily the topology on Z^K, we introduce the discrete topology on Z, then Z is compact and by Tychonoff's theorem the product space Z^K is also compact. With this new topology S becomes closed in Z^K, hence is compact. Since this topology is stronger that the preceding one, we deduce that the Riemann variety is quasi-compact with the Zariski topology.

We shall show that the Riemann variety $S(K/k)$ may be regarded as the projective limit of an inverse system of integral schemes: $S = \varprojlim X_\alpha$. More precisely, if k is a field anf K a function field over k, i.e. a finitely generated extension of k, we define a *model* M of K (over k) as an algebraic variety M over k such that K is the function field of M. We say that M is a complete, resp. projective, model of K if M is a complete, resp. projective, algebraic variety over k.

We call L the set of local k-subalgebras P of K, and for any P we denote $m(P)$ its maximal ideal: $L = \big\{P \,\text{local}\, k\text{-algebra} \,/\, k \subset P \subset K\big\}$. For any k-subalgebra A of K, non necessarily local, we call $L(A)$ the subset of L of the local k-algebras P containing A: $L(A) = \big\{P \in L \,/\, A \subset P\big\}$. Then we define a topology in L such that the set of the $L(A)$, for A ranging the finitely generated k-algebras, is a basis of open sets.
Let A be a finitely generated k-subalgebra of K and let Spec$A$ be the affine scheme associated to A, then we can define a map $f_A\colon L(A) \to$

Spec A by $f_A(P) = m(P) \cap A = \mathfrak{p}$. We have a topology in Spec$A$, the Zariski topology, such that the closed subsets are the sets $V(\mathcal{I}) = \{\mathfrak{p} \in \operatorname{Spec} A \,/\, \mathcal{I} \subset \mathfrak{p}\}$, where $\mathcal{I}$ range the ideals of A. Moreover the closed subset $V(\mathcal{I})$ is isomorphic to the affine scheme Spec$A/\mathcal{I}$.

Proposition 2.8. *The map f_A is continuous from $L(A)$ to* Spec$A$, *and induces an homeomorphism of $V(A)$ into* Spec$A$, *where $V(A)$ is the subset of $L(A)$ defined by $V(A) = \{A_{\mathfrak{p}} \,/\, \mathfrak{p} \in \operatorname{Spec} A\}$.*

Proof. To show that the map f_A is continuous, we have to show that the inverse image of any open subset O of Spec$A$ is open in $L(A)$, and we may consider only the open sets $O = D(x) = \{\mathfrak{p} \in \operatorname{Spec} A \,/\, x \notin \mathfrak{p}\}$, and we recall that $D(x)$ is isomorphic to the affine scheme SpecA_x. We shall see that the inverse image $f_A^{-1}\big(D(x)\big)$ is equal to the open set $L(A_x)$. A local ring P of $L(A)$ belongs to $f_A^{-1}\big(D(x)\big)$ if and only if the prime ideal $\mathfrak{p} = m(P) \cap A$ doesn't contain x, i.e. x doesn't belong to the maximal ideal $m(P)$, and as x belongs to $A \subset P$ and P is local this equivalent to demand to x^{-1} to belong to P, hence to demand to A_x to be contained in P.
By definition the map f_A induces a bijection of the subset $V(A)$ into Spec$A$ and we have to show that f_A identify the topology in $V(A)$ induced by the topology of L to the Zariski topology in Spec$A$. Any open set in $V(A)$ is a finite intersection of sets $O(x)$ of the following type $O(x) = \{P \in V(A) \,/\, x \in P\}$, for x a non zero element of the fraction field of A, and it is enough that the set $f_A\big(O(x)\big)$ is open in Spec$A$. In fact we see that this set is the complementary in Spec$A$ of the closed subset $V(\mathcal{I})$ where $\mathcal{I}$ is the ideal $\mathcal{I} = (A : x) = \{c \in A \,/\, cx \in A\}$. An element x of the fraction field of A belongs to the local ring $A_{\mathfrak{p}}$ if and only if we have $x = a/b$ with $a \in A$ and $b \in A \setminus \mathfrak{p}$, i.e. if and only if there exists b with $b \in \mathcal{I}$ and $b \notin \mathfrak{p}$.

Remark 2.6. If M is a model of the field K, i.e. if M is an algebraic variety over k with function field K, then we can associate to any point x of M the local ring $\mathcal{O}_{M,x}$ in L. By the preceding proposition, the map f defined by $f(x) = \mathcal{O}_{M,x}$ is a homeomorphism of M, with the Zariski topology, into a subset of L.
In the same way, if we associate to any valuation ν of the Riemann variety $S = S(K/k)$ the valuation ring $V = V_\nu$, we see that S is a subset of L. And by definition we see that the topology in L induces the Zariski topology in S.

Let A be an integral k-algebra, finitely generated over k and with fraction field K, then the set of valuations ν of the Riemann variety $S(K/k)$ which have a non empty center on the affine scheme $X = \operatorname{Spec} A$

is equal to the open set $E(A)$. We can define a map g_A of $E(A)$ to X by $g_A(\nu) = x$ where $x = \mathfrak{p}$ is the center of ν on X, and as the center $\mathfrak{p}$ is by definition equal to $A \cap m(V)$, where $m(V)$ is the maximal ideal of the valuation ring V, this map is the restriction of the map f_A, hence is continuous. More generaly, we have the following result.

Proposition 2.9. *Let X be an algebraic variety over k with function field K. The set of valuations ν of K/k which have a non empty center on X is an open set $U(X)$ of the Riemann variety $S(K/k)$ and the map g_X which associate to any valuation ν of $U(X)$ its center x_ν on X is continuous. Moreover, the variety X is complete if and only if the open set $U(X)$ is equal to the whole Riemann variety, and we get a continuous map $g_X \colon S(K/k) \longrightarrow X$.*

Proof. We deduce from the proposition 2.1 that the center of a valuation ν of K/k on X is well defined, and from the theorem 2.2 that all the valuations ν of K/k have a center on X if and only if the variety X is complete over k.
We can write the algebraic variety X as the union of a finite number of affine open sets $X = \bigcup_{i=1}^n X_i$, where $X_i = \mathrm{Spec} A_i$ and A_i is a finitely generated k-algebra with fraction field K. Then the subset $U(X)$ of the valuations ν in $S(K/k)$ which have a center on the variety X is the union of the open subsets $E\big(A_i\big)$, hence $U(X)$ is open in $S(K/k)$. The restriction of the map g_X on each subset $E\big(A_i\big)$ is the continuous map f_{A_i}, hence the map g_X is also continuous of $U(X)$ to X.

Let X and X' be two algebraic varieties over k, with the same function field K, and let $h \colon X' \longrightarrow X$ be a birational morphism of X' to X. Let ν be a valuation of K/k belonging to the open set $U(X')$, then there exits a birational morphism f' of $T = \mathrm{Spec} V$ to X', where V is the valuation ring associated to ν, and the image ξ' of the closed point t of T is the center of ν on X'. Then the composite morphism $f = h \circ f'$ is a birational morphism of $T = \mathrm{Spec} V$ to X, hence the valuation ν has also a center on X, i.e. the valuation ν belongs to the open set $U(X)$, and this center $\xi = f(t)$ is equal to the image $h(\xi')$ of the center of ν on X'.
We can also notice that for any point x' of X', its image $x = h(x')$ is the point of X such that the local ring $\mathcal{O}_{X,x}$ is dominated by the local ring $\mathcal{O}_{X',x'}$. Then if ξ' is the center of the valuation ν on X', we have the local ring $\mathcal{O}_{X'.\xi'}$ which is dominated by V, and since the relation of domination is transitive, the local ring $\mathcal{O}_{X,\xi}$ is dominated by V, where $\xi = h(\xi')$, i.e. ξ is the center of ν on X.

We have shown that the open set $U(X')$ is contained in the open set $U(X)$ and that the restriction of the map g_X to $U(X')$ is equal to $h \circ g_{X'}$.

Moreover, if the morphism $h\colon X' \longrightarrow X$ is proper, for any valuation ν of K/k having a center ξ on X, we can apply the valuative criterion of properness to the following commutative diagram:

$$\begin{array}{ccc} U = \operatorname{Spec} K & \xrightarrow{j} & X' \\ \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle h} \\ T = \operatorname{Spec} V & \xrightarrow{f} & X \end{array}$$

where the morphism $f\colon T \longrightarrow X$ is defined by the existence of the center of the valuation on X. Then we deduce the existence of a morphism $f'\colon V \longrightarrow X'$ and the image ξ' of the closed point t of T is the center of the valuation ν on X' and $h(\xi')$ is equal to ξ. Hence, if the birational morphism $h\colon X' \longrightarrow X$ is proper, any valuation ν having a center on X has also a center on X', i.e. the open sets $U(X)$ and $U(X')$ are equal.

We have proven the following result.

Proposition 2.10. *Let X and X' be two algebraic varieties over a field k, with the same function field K. If there exists a birational morphism $h\colon X' \longrightarrow X$, then we have the inclusion $U(X') \subset U(X)$ in the Riemann variety $S(K/k)$.*
Moreover, the morphism $h\colon X' \longrightarrow X$ is proper if and only if we have equality $U(X') = U(X)$.

Let X be an algebraic variety over k, with function field K, and let $U = U(X)$ the open subset of the Riemann variety $S = S(K/k)$ of the valuations ν having a center on X. For any algebraic variety Y such that there exists a proper birational morphism $h_Y : Y \longrightarrow X$, the open subset $U(Y)$ of S is equal to U and the continuous map $g_Y : U \longrightarrow Y$ satisfies $g_X = h_Y \circ g_Y$. Let Y and Y' be two algebraic varieties with proper birational morphisms $h_Y : Y \longrightarrow X$ and $h_{Y'} : Y' \longrightarrow X$, we denote $Y \prec Y'$ if there exists a morphism $h_{Y',Y}$ of Y' to Y such that $h_{Y'} = h_Y \circ h_{Y',Y}$, in that case the morphism $h_{Y',Y}$ is also proper birational. We call $\mathcal{D}$ the inverse system of the (Y, h_Y) with the relation $\prec$ and we may define the projective limit

$$\mathcal{X} = \varprojlim_{\mathcal{D}} Y\,.$$

This projective limit is the subset of the product space $\prod_{\mathcal{D}} Y$ of the elements $\bar{x} = \big(x_Y\big)$ such that $h_{Y',Y}\big(x_{Y'}\big) = x_Y$ for any couple (Y, Y') with $Y \prec Y'$. We introduce in $\mathcal{X}$ the topology induced by the product topology on $\prod_{\mathcal{D}} Y$, and the natural maps $t_Y : \mathcal{X} \longrightarrow Y$ are continuous.

Theorem 2.11. ([Za-Sa], Chap.VI, §17, Theorem 41, page 122.) *There exists a natural homeomorphism $g\colon U \longrightarrow \mathcal{X}$ of the open subset*

$U = U(X)$ of the Riemann variety $S(K/k)$ to the projective limit $\mathcal{X} = \varprojlim_{\mathcal{D}} Y$. Hence the Riemann variety $S(K/k)$ may be identified with the projective limit of the inverse system $\mathcal{C}$ of the complete algebraic varieties Z over k, with function field K: $S = \varprojlim_{\mathcal{C}} Z$.

Remark 2.7. The algebraic varieties Y of the inverse system $\mathcal{D}$ are models of the function field K, and if we have two algebraic varieties Y and Y' in $\mathcal{D}$ with $Y \prec Y'$, we say that Y' dominates Y. To calculate the projective limit $\mathcal{X} = \varprojlim_{\mathcal{D}} Y$ of the inverse system $\mathcal{D}$, we can consider a cofinal subset $\mathcal{D}'$ of $\mathcal{D}$, i.e. a subfamily $\mathcal{D}'$ of $\mathcal{D}$ such that for any Y in $\mathcal{D}$ there exists an element Z in $\mathcal{D}'$ with $Y \prec Z$.
The algebraic varieties of the system $\mathcal{C}$ are the complete models of K, and the Chow lemma says that the inverse system of projective models $\mathcal{P}$ is cofinal in the inverse system of complete models, hence we have also the equality $S(K/k) = \varprojlim_{\mathcal{P}} P$.

Remark 2.8. The complete algebraic varieties are quasi-compact topological spaces, hence we could deduce the quasi-compacity of the Riemann variety as projective limit of quasi-compact spaces, but to prove the theorem 2.11 we use the quasi-compacity of $S(K/k)$.

Proof. For every couple (Y, Y') of the inverse system $\mathcal{D}$ with $Y \prec Y'$, the maps $g_Y : U \to Y$ and $g_{Y'} : U \to Y'$ are continuous and satisfy $g_Y = h_{Y',Y} \circ g_{Y'}$. Hence we obtain a continuous map g of the open subset U of S in the projective limit $\mathcal{X}$, such that $t_Y \circ g = g_Y$ on U.
We shall show that this map $g : U \longrightarrow \mathcal{X}$ is onto. Let $\bar{x} = (x_Y)$ be a point in $\mathcal{X}$ and let R_Y the local ring of the point $x_Y = t_Y(\bar{x})$, $R_Y = \mathcal{O}_{Y,x_Y}$. Since for $Y \prec Y'$ the local ring R_Y is dominated by $R_{Y'}$, the ring $R = \bigcup_{Y \in \mathcal{D}} R_Y$ is a local ring, contained in K, with maximal ideal $max(R) = \bigcup_{Y \in \mathcal{D}} max(R_Y)$. There exists a valuation ring V, associated to a valuation ν of K/k, which dominates the local ring R. For all the Y in $\mathcal{D}$ the valuation ring dominates also the local rings $R_Y = \mathcal{O}_{Y,x_Y}$, then the center of the valuation ν on Y is x_Y, i.e. ν belongs to the open subset U and its image by g_y is x_Y, hence the image of ν by the map g is the point $\bar{x}$.
To show that the map $g : U \longrightarrow \mathcal{X}$ is injective, we shall show that for any point $\bar{x} = (x_Y)$ of $\mathcal{X}$, the local ring $R = \bigcup_{Y \in \mathcal{D}} R_Y$ defined by $R_Y = \mathcal{O}_{Y,x_Y}$, is a valuation ring of K. Let w be an element of K, and we have to show that either w, either w^{-1} belongs to the ring R. We can write $w = u/v$ with u and v in R, and there exist Y' and Y'' in $\mathcal{D}$ such that $u \in R_{Y'}$ and $v \in R_{Y''}$, and since there exists Y in $\mathcal{D}$ with $Y' \prec Y$ and $Y'' \prec Y$, we may assume that u and v belong to the same local ring R_Y. Let $\mathcal{I}$ be a sheaf of ideals on the variety Y such that $\mathcal{I}_{Y,x_Y}$ is equal

to the ideal (u, v) of the local ring $R_Y = \mathcal{O}_{Y,x_Y}$ and let $r\colon Z \longrightarrow Y$ be the blowing up of center $\mathcal{I}$ in Y. Then Z belongs to the inverse system $\mathcal{D}$ and let x_Z be the point in Z with $x_Z = t_Z(\bar{x})$. By definition the ideal $\mathcal{I}R_Z$ is principal, i.e. is generated by one of the elements u or v. If $\mathcal{I}R_Z$ is generated by u, then $w^{-1} = v/u$ belongs to R_Z anf if $\mathcal{I}R_Z$ is generated by v, $w = u/v$ belongs to R_Z, hence we deduces that w or w^{-1} belongs to R.
We have to prove that the map $g\colon U \longrightarrow \mathcal{X}$ is closed. We deduce from the proposition 2.8 that the maps $g_Y : U \longrightarrow Y$ are closed, then the map $g\colon U \longrightarrow \mathcal{X}$ is also closed because for any closed subset F of U we have $g(F) = \mathcal{X} \cap \left(\prod_{Y\in\mathcal{D}} g_Y(F)\right)$.

Proposition 2.12. *A valuation ν of K/k is a closed point of the Riemann variety $S(K/k)$ if and only if the residue field κ of ν is an algebraic extension of k, i.e. if and only if the valuation ν is zero-dimensional.*

Proof. The valuation ν is a closed point of $S(K/k)$ if and only if $\overline{\{\nu\}}$ is reduced to one point, hence from the theorem 2.6, if and only if the Riemann variety $S(\kappa/k)$ of the valuations of the residue field κ of ν which are trivial on k, contains one element, and we deduce from the remark 2.4 that this is equivalent to demand to κ to be an algebraic extension of the field k. By definition of the dimension of a valuation ν of K/k, this also equivalent to say that the dimension of ν is zero.

§3. Uniformization and resolution of singularities

3.1. The general problem

Let X be a scheme, a point x of X is said *non-singular*, or *simple*, if the local ring $\mathcal{O}_{X,x}$ is a regular ring. If we assume that X is an excellent scheme, for instance if X is a scheme of finite type over a field K, and if X is reduced, the set of all the non-singular points of X is a dense open subset X_{reg} of X. We say that the scheme X is *non-singular* if all the points x of X are non-singular, i.e. if $X = X_{reg}$. Hence all the connected components of X are irreducible. By definition *a resolution of singularities* of a reduced scheme X is a proper birational morphism $\pi\colon \tilde{X} \longrightarrow X$ of a non-singular scheme $\tilde{X}$ onto X, which induces an isomorphism over the non-singular open subset X_{reg} of X. We may also demand more conditions on the morphism π, for instance that the exceptional locus, i.e. the closed subset E in $\tilde{X}$ where π is not an isomorphism, $E = \tilde{X} \smallsetminus \pi^{-1}(X_{reg})$, is a normal crossings divisor in $\tilde{X}$, or that π is a composition of blowups in regular centers.
If we assume that X is an algebraic variety over a field k, any resolution

of singularities $\pi\colon \tilde{X} \longrightarrow X$ of X will give a non-singular model $\tilde{X}$ of the function field $K = F(X)$ of X, i.e. a non-singular algebraic variety $\tilde{X}$ over k with function field $F(\tilde{X})$ equal to K. Moreover, if the variety X is complete, the variety $\tilde{X}$ is also complete because the morphism π is proper, then we get a complete non-singular model $\tilde{X}$ of the function field K. Then we may also define a problem, which is weaker than the resolution of singularities, by the following:

let K be a function field over a field k, then does there exist a complete non-singular model Y of K?

The stategy of Zariski to solve the problem of the resolution of singularities of an algebraic variety X over a field k, with function field $F(X) = K$, is to study all the valuations of K/k, which belong to the open subset $U(X)$ of the Riemann variety $S(K/k)$, and to try to find for each valuation ν of $U(X)$ a model Y of K such that the center ξ of ν on Y is a non-singular point, i.e. such that the local ring $\mathcal{O}_{Y,\xi}$ is a regular ring. This is this problem we call *the local uniformization of a valuation ν.*

We may notice that there are also two ways to define the problem of the uniformization, one we call the abstract form, or the invariantive form in Zariski's terminology, and one we call the strong form, or the projective form in Zariski's terminology ([Za 2]).

Uniformization problem in the abstract form. *Let K be a function field over a field k and let ν be a valuation of K/k, then does there exist a complete model V of K over k on wich the center ξ of the valuation ν is a non-singular point?*

Uniformization problem in the strong form. *Let X be an algebraic variety over a field k, with function field K, and let ν be a valuation of K/k which belongs to the open subset $U(X)$ of the Riemann variety $S(K/k)$, then does there exist a proper birational morphism $\pi\colon \tilde{X} \longrightarrow X$ of an algebraic variety $\tilde{X}$ onto X, such that the center $\tilde{\xi}$ of the valuation ν on $\tilde{X}$ is a non-singular point?*

Remark 3.1. Zariski gives a different definition of the uniformization problem in the strong form. He considers a model X of the field K and a valuation ν of K/k with center ξ on X and he wants to find a new model $\tilde{X}$ of K such that the center $\tilde{\xi}$ of ν on $\tilde{X}$ is a non-singular point and such that the local ring $\mathcal{O}_{X,\xi}$ is contained in the local ring $\mathcal{O}_{\tilde{X},\tilde{\xi}}$. Since the two local rings $\mathcal{O}_{X,\xi}$ and $\mathcal{O}_{\tilde{X},\tilde{\xi}}$ are dominated by the valuation ring V associated to the valuation ν, we have $\mathcal{O}_{X,\xi}$ contained in $\mathcal{O}_{\tilde{X},\tilde{\xi}}$ if and only if $\mathcal{O}_{X,\xi}$ is dominated by $\mathcal{O}_{\tilde{X},\tilde{\xi}}$. Hence if there exists a proper birational morphism $\pi\colon \tilde{X} \longrightarrow X$ the local ring $\mathcal{O}_{X,\xi}$ is contained in the local ring $\mathcal{O}_{\tilde{X},\tilde{\xi}}$, and conversely if the local ring $\mathcal{O}_{X,\xi}$ is contained in the local ring $\mathcal{O}_{\tilde{X},\tilde{\xi}}$, there exists locally in a neighbourhood of $\tilde{\xi}$ a birational morphism π of $\tilde{X}$ to X with $\pi(\tilde{\xi}) = \xi$.

To see how the uniformization problem is a step to get the resolution of singularities, we need the following result.

Proposition 3.1. (cf. [Za 2], Chap.II, §5, , page 855.) *Let X be an algebraic variety over a field k, with function field K, then the set of valuations ν of K/k wich have a center ξ on X which is a non-singular point of X is an open subset of the Riemann variety $S(K/k)$.*

Proof. The set V of valuations ν of K/k wich have a non-singular center ξ on X is a subset of the open subset $U = U(X)$ of valuation which have a center on X, and to prove that V is open we have to show that V is stable under generalization, i.e. that for any valuation ν in V and for any valuation μ in U with $\nu \in \overline{\{\mu\}}$, we have μ which belongs to V. If ξ and ζ are the centers on X respectively of the valuations ν and μ, then ζ is again a generalization of ξ, because the map $g_X : U \longrightarrow X$ is continuous. Since the subset of non-singular points of an algebraic variety is an open subset, we see that if ν has a non-singular center ξ on X, then μ has also a non-singular center ζ on X.

Corollary. *The uniformization theorem for zero-dimensional valuations implies the uniformization theorem for all the valuations of K/k.*

Proof. By the proposition it is enough to show that for any valuation μ of K/k, there exists a zero-dimensional valuation ν such that $\nu \in \overline{\{\mu\}}$. If the valuation μ is of dimension $d > 0$, then by definition deg.tr.κ/k is positive and there exists a zero-dimensional valuation $\bar{\nu}$ of κ/k. Hence the composite valuation $\nu = \mu \circ \bar{\nu}$ is a zero-dimensional valuation of K/k which belongs to $\overline{\{\mu\}}$.

However, to prove the uniformization theorem we do not prove the result for zero-dimensional valuations and then use the corollary to get the result for all the valuations of K/k, i.e. we don't uniformize a zero-dimensional valuation $\nu = \mu \circ \bar{\nu}$ to get the uniformization of the valuation μ. We do the converse, we first uniformize the valuation μ and we then use this result to get the uniformization theorem for the valuations ν which are composite with μ. The reason for this is that we get the proof by induction on the rank of the valuations and we have seen that for $\nu = \mu \circ \bar{\nu}$ we have rank$(\nu) =$ rank$(\mu)+$ rank$(\bar{\nu})$ by proposition 1.10 ([Za 2], Chap.III, §7, , page 857).

We may enonce the local version of the uniformization theorem in the strong form in the following way. Let R be an integral finitely generated k-algebra, $R = k[x_1, x_2, \ldots, x_n]$ and let ν be a valuation of K, where K is the fraction field of R, $K = Fr(R)$, with $\nu(x) \geq 0$ for all the elements x in R, i.e. we assume that the valuation ν has a center on R. We denote V the valuation ring associated to ν, $\mathfrak{m}$ its maximal ideal and $\mathfrak{p} = R \cap \mathfrak{m}$ the center of ν on R. Then there exists a finitely generated k-algebra S, with fraction field $Fr(S) = K$, i.e. we have $S = R[u_1, u_2, \ldots, u_t]$ with $u_i \in K$ for $i = 1, 2, \ldots t$, such that the center $\mathfrak{q} = S \cap \mathfrak{m}$ of ν on S is regular, i.e, such that the local ring $S_{\mathfrak{q}}$ is regular, and such that the local ring $R_{\mathfrak{p}}$ is contained in the local ring $S_{\mathfrak{q}}$. Since the local rings $R_{\mathfrak{p}}$ and $S_{\mathfrak{q}}$ are dominated by the valuation ring V, we have also $R_{\mathfrak{p}}$ dominated by $S_{\mathfrak{q}}$ and the inclusion $R_{\mathfrak{p}} \subset S_{\mathfrak{q}}$ induces a birational correspondence $\pi \colon \mathrm{SpecS} \dashrightarrow \mathrm{Spec}R$ which is defined in a neighbourhood of $\mathfrak{q}$ and with $\pi(\mathfrak{q}) = \mathfrak{p}$. We may replace the ring S by $S^* = S[v_1, v_2, \ldots, v_s]$ in such a way that the ring S^* is a regular ring and R is contained in S^*. This ring S^* corresponds to a non-singular affine open subvariety $U = \mathrm{Spec}S^*$ of the affine variety $\mathrm{Spec}S$, which contains the non-singular point $\mathfrak{q}$, a such subvariety U exists because the set of non-singular points of $\mathrm{Spec}S$ is open. Then we get a birational morphism $\pi^* \colon \mathrm{Spec}S^* \longrightarrow \mathrm{Spec}R$ with $\mathrm{Spec}S^*$ a non-singular affine algebraic variety and such that the valuation ν has a center on $\mathrm{Spec}S^*$. Let d be the dimension of the ring R, then d is equal to the transcendence degree of the fraction field K over k. We can find d elements $\xi_1, \xi_2, \ldots, \xi_d$ of K algebraically independent over k, and the field K is an algebraic extension of $k(\xi_1, \xi_2, \ldots, \xi_d)$. Let X be the affine algebraic variety associated to the k-algebra R. If we write $R = k[X_1, X_2, \ldots, X_n]/I$, where $X_1, X_2, \ldots, X_n$ are algebraically independent over k, then X is the closed subvariety of the n-dimensional affine space $\mathbf{A}_k^n$ defined by the ideal I of the polynomial ring $k[X_1, X_2, \ldots, X_n]$. We say that the k-algebra R is *an hypersurface ring* if we may write $R = k[x_1, x_2, \ldots, x_{d+1}]$ with $d = \dim R$, i.e. if the affine variety X associated to R is an hypersurface in the affine space $\mathbf{A}_k^{d+1}$. In that case the ideal I is generated by only one element, $I = (f)$.

Proposition 3.2. (cf. [Za 2], Chap.IV, §9, , page 858.) *Let k be a field of characteristic zero and let K be a function field over k. If the uniformization problem is resolved for all the k-algebras R with fraction field K which are hypersurface rings, then it is resolved for any k-algebra with fraction field K.*

Proof. Let $R = k[x_1, x_2, \ldots, x_n]$ be an integral k-algebra with fraction field K and let ν be a valuation of K/k with valuation ring V, we assume that ν is non negative on R, i.e. $R \subset V$ and let $\mathfrak{p} = R \cap \mathfrak{m}$ be the center of ν on R. By the Emmy Noether normalization theorem there exists d elements $y_1, y_2, \dots y_d$, with $d = \dim R = \text{tr.deg.}K/k$, such that R is integral over the k-algebra $k[y_1, y_2, \dots, y_d]$. Then K is a finite extension of $L = k(y_1, y_2, \dots, y_d)$, and since the characteristic of k is zero, there exists z in K with $K = L(z)$ and we may assume $z \in R$. Let R^* be the k-algebra $R^* = k[y_1, y_2, \dots, y_d, z]$, then R^* satisfies $R^* \subset R$, $Fr(R^*) = Fr(R) = K$ and R is integral over R^*. Moreover, by construction R^* is an hypersurface ring. Let $\mathfrak{p}^*$ be the center of the valuation ν on R^*, $\mathfrak{p}^* = R^* \cap \mathfrak{m}$, and by hypothesis there exists an uniformization of ν over R^*, i.e. a k-algebra S with $R^* \subset S$, $Fr(S) = Fr(R^*) = K$, and such that the center $\mathfrak{q} = S \cap \mathfrak{m}$ of ν on S is non-singular. Since the ring $S_\mathfrak{q}$ is regular, $S_\mathfrak{q}$ is integrally closed in its fraction field K, then we get also $R_\mathfrak{p} \subset S_\mathfrak{q}$ and S is a uniformization of ν over R.

The most important result on the uniformization problem is the theorem of Zariski for algebraic varieties over a field of characteristic zero. For varieties over a field of positive characteristic, we have the theorem for the dimensions $d \leq 3$, and there are also results for some special valuations ([Kn-Ku]).

Uniformization theorem. ([Za 2]) *Let X be an algebraic variety over an arbitrary ground field k of characteristic zero, with function field K, and let ν be a valuation of K/k which belongs to the open subset $U(X)$ of the Riemann variety $S(K/k)$, then there exists a proper birational morphism $\pi\colon \tilde{X} \longrightarrow X$ of an algebraic variety $\tilde{X}$ onto X, such that the center $\tilde{\xi}$ of the valuation ν on $\tilde{X}$ is a non-singular point.*

Now, if we assume that we have the uniformization theorem we shall see that the resolution is a consequence of *a gluing problem of a finite number of local uniformizations.* More precisely, let X be a complete algebraic variety over a field k, with function field K, and we assume that for any valuation ν of K/k, there exists a proper birational morphism $\pi(\nu)\colon \tilde{X}(\nu) \longrightarrow X$ such that the center $\xi(\nu)$ of ν on $\tilde{X}(\nu)$ is a non-singular point. By the proposition 3.1, there exists an open subset $V(\nu)$ of the Riemann variety $S(K/k)$, such that the center of any valuation μ in $V(\nu)$ is also a non-singular point of $\tilde{X}(\nu)$. By the quasi-compacity of the Riemann variety (theorem 2.7), there exists a finite number of valuations $\nu_1, \nu_2, \dots, \nu_t$ of K/k such that the family $V(\nu_1), V(\nu_2), \dots, V(\nu_t)$ is a covering of the Riemann variety $S(K/k)$. We have obtained a finite family of proper birational morphisms $\pi_i\colon \tilde{X}_i \longrightarrow X$, $i = 1, 2, \dots, t$, such that for any valuation ν of K/k there exists i such that ν has a non-singular center on $\tilde{X}_i$. Hence, the problem of the resolution of singularities is reduced to the following gluing problem. Let X be an algebraic variety, and let $\pi_1 : \tilde{X}_i \longrightarrow X$ and $\pi_2 : \tilde{X}_2 \longrightarrow X$ be two proper birational morphisms. Then, does there exist a variety Y and two proper birational morphisms $\rho_1 : Y \longrightarrow \tilde{X}_1$ and $\rho_2 : Y \longrightarrow \tilde{X}_2$, with $\pi_1 \circ \rho_1 = \pi_2 \circ \rho_2$, and such that the open subset Y_{reg} of non-singular points of Y satisfies ${\rho_1}^{-1}\big(\tilde{X}_{1reg}\big) \cup {\rho_2}^{-1}\big(\tilde{X}_{2reg}\big) \subset Y_{reg}$.

The Zariski proof of resolution of singularities of surfaces over an algebraically closed field k of characteristic zero is the first algebraic or arithmetic proof. He used the uniformization theorem and the theory of integrally closed ideals to show that we can obtain the resolution of singularities by a finite sequence of normal blowing ups ([Za 1]. Zariski give a new proof of the resolution of singularities for surfaces and later a proof of resolution of singularities for three dimensional varieties over an arbitrary ground field of characteristic zero by using the method of the uniformization of valuations and by showing that it is possible to solve the gluing problem ([Za 3], [Za 4]).

3.2. The case of algebraic surfaces

In this section, we give an idea of the Zariski proof of the uniformization theorem for surfaces over an algebraically closed field of characteristic zero ([Za 1]).

Let k be an algebraically closed field of characteristic zero, let K be a function field over k with transcendence degree $d = 2$. First of all we shall study all the valuations ν of K which are trivial on the ground field k. Let ν be a non trivial valuation of K/k, we recall that we have the inequalities (remark 1.14):

$$\mathrm{rank}(\nu) \;+\; \dim(\nu) \;\leq\; \mathrm{rat.rank}(\nu) + \;\dim(\nu) \;\;\leq\; \mathrm{tr.deg.}K/k \;= 2\,,$$

where the dimension of the valuation is the transcendence degree of the residue field κ of ν over the ground field k. Since the valuation ν is non trivial its rank is positive, then we have the four following possibilities:

$$\begin{array}{rlllll}
i) & \mathrm{rank}(\nu) \;= 1 & = & \mathrm{rat.rank}(\nu) \;=\; 1 & \text{and} & \dim(\nu) \;=\; 1\;. \\
ii) & \mathrm{rank}(\nu) \;= 1 & = & \mathrm{rat.rank}(\nu) \;=\; 1 & \text{and} & \dim(\nu) \;=\; 0\;; \\
iii) & \mathrm{rank}(\nu) \;= 1 & < & \mathrm{rat.rank}(\nu) \;=\; 2 & \text{and} & \dim(\nu) \;=\; 0\;; \\
iv) & \mathrm{rank}(\nu) \;= 2 & = & \mathrm{rat.rank}(\nu) \;=\; 2 & \text{and} & \dim(\nu) \;=\; 0\;;
\end{array}$$

We are going to give a description of the valuation ν in the four cases.

Remark 3.2. Since K is finitely generated over the ground field k, we deduce from the corollary of the theorem 1.20 that in the cases i),

iii) and *iv*) the value group Γ of the valuation ν is finitely generated over $\mathbb{Z}$ and the residue field κ is finitely generated over k, and moreover in the cases *i*) and *iv*) the value group Γ is discrete, i.e. is isomorphic to $(\mathbb{Z}, +)$ or to $(\mathbb{Z}^2, +)_{\text{lex}}$.
Moreover since we have assumed that the field k is algebraically closed, in the cases *ii*), *iii*) and *iv*) the residue field κ, which is algebraic over k, is equal to k.

i) $\mathbf{rank}(\nu) = \mathbf{rat.rank}(\nu) = 1,\ \mathbf{dim}(\nu) = 1$

The valuation ν is a prime divisor of the function field K (cf example 5). If X is a model of the field K such that the valuation ν has a center ξ on X, the dimension of the center $Z = \overline{\{\xi\}}$ is either zero, i.e. ξ is a closed point of X, either a curve, which we may call an *algebraic arc* on X.
Let $U = \text{Spec}R$ be an affine open neighbourhood of ξ in X, then R is contained in the valuation ring V associated to ν, and let $\mathfrak{p}$ the center of ν in R, i.e. the prime ideal of R corresponding to ξ. There exists an affine normal model $Y = \text{Spec}S$ of K, i.e. S is a finitely generated k-algebra, integrally closed in its fraction field K, and there exists a prime ideal $\mathfrak{q}$ of S with height one, such that $R \subset S$ and $S_{\mathfrak{q}} = V$. Hence the valuation ν is the $\mathfrak{q}$-adic valuation, and we may write $\nu(f) = \text{order}_{\mathfrak{q}}(f)$ for any element f in S.
The residue field κ of the valuation ν is a function field of transcendence degree one over the ground field k. Let C be the center of the valuation ν on $\text{Spec}S$, i.e. let C be the affine algebraic curve defined by the prime ideal $\mathfrak{q}$, then the residue field of the valuation is the function field of C: $\kappa = F(C)$.

ii) $\mathbf{rank}(\nu) = \mathbf{rat.rank}(\nu) = 1,\ \mathbf{dim}(\nu) = 0$

We consider first the case where the valuation ν is discrete, i.e. that its value group Γ is isomorphic to $\mathbb{Z}$, and we may assume $\Gamma = \mathbb{Z}$. Let u be an element of the field K such that $\nu(u) = 1$. Then for any element x in K, $x \neq 0$, we have $\nu(x) = n_0$ with $n_0 \in \mathbb{Z}$, hence $\nu(x/u^{n_0}) = 0$. Since the residue field κ of the valuation ν is equal to k, there exists a uniquely determined element c_0 of k such that x/u^{n_0} and c_0 have the same image in κ, i.e. such that $\nu(x/u^{n_0} - c_0) > 0$. Hence we may define x_1 in K such that:

$$x = c_0 u^{n_0} + x_1, \quad \text{with } \nu(x_1) = n_1 > n_0 \ .$$

By induction we may construct uniquely determined sequences (c_i) in k, (n_i) in $\mathbb{Z}$ and x_i in K by:

$$x = c_0 u^{n_0} + c_1 u^{n_1} + c_2 u^{n_2} + \ldots + c_{i-1} u^{n_{i-1}} + x_i \text{ with } \nu(x_i) = n_i > n_{i-1} \ .$$

Let $\xi(x) = \sum_{i\geq 0} c_i u^{n_i}$ be the power series expension for x, hence the map $x \mapsto \xi(x)$ defines an injective morphism of K to the field $k((u))$ of integral power series of u with coefficients in k. The restriction to K of the u-adic valuation of $k((u))$, i.e. the valuation by the order in u, is the valuation ν.
If we choose any model X of K such that the valuation ν has a center on X, for instance if we choose a complete model of K, then the center of the valuation is a closed point p of X because the dimension of the center is always non greater than the dimension of the valuation. In a neighbourhood of the center p, we may choose coordinates $(x_1, x_2, \dots, x_n)$, i.e. we choose an affine neighbourhood $U = \text{Spec } k[x_1, x_2, \dots, x_n]$, with $\nu(x_i) \geq 0$ for $i = 1, 2, \dots, n$, and we may write the power series expensions for these coordinates:

$$x_i = c_{0,i}u^{n_{0,i}} + c_{1,i}u^{n_{1,i}} + c_{2,i}u^{n_{2,i}} + \dots \quad 1 \leq i \leq n \ .$$

These expensions represent an *analytic arc* on U which is not algebraic, i.e. which is not supported by an algebraic curve in U.

We consider now the case where the value group Γ is not discrete, by hypothesis we may assume $\mathbb{Z} \subset \Gamma \subset \mathbb{Q}$. There exists a family of prime numbers $P = \{p_i\}$, which may be finite or infinite, and for any prime number p_i in P a number n_i, $1 \leq n_i \leq \infty$ such that the value group Γ consists of all the rational numbers whose denominators are of the form $p_1^{a_1} p_2^{a_2} p_3^{n_3} \dots$ with $0 \leq a_i \leq n_i$ for any i.
Since Γ is non discrete, the denominators of the elements of Γ are not bounded, hence if all the numbers n_i are finite, the family P must be infinite.

iii) $\mathbf{rank}(\nu) = 1,\ \mathbf{rat.rank}(\nu) = 2,\ \mathbf{dim}(\nu) = 0$

Since the field K is finitely generated over k, we deduce from the corollary of the theorem 1.20 that the value group Γ is finitely generated subgroup of $\mathbb{R}$ with $\dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}) = 2$. Then the group Γ is generated by two elements linearly independent over $\mathbb{Q}$ and we may assume $\Gamma = \mathbb{Z} \oplus \beta\mathbb{Z}$, with $\beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\beta \geq 0$.
Let x and y be elements of K with $\nu(x) = 1$ and $\nu(y) = \beta$, then x, y are algebraically independent over k and K is a finite extension of the field $K^* = k(x, y)$. We denote R^* the polynomial ring $R^* = k[x, y]$ and ν^* the restriction of the valuation ν to K^*, since $\beta \notin \mathbb{Q}$ the value $\nu^*(x^i y^j) = i + j\beta$ is equal to $\nu^*(x^{i'} y^{j'}) = i' + j'\beta$ if and only if $(i, j) = (i', j')$, then the valuation ν^* is defined on the polynomial ring R^* by the following: for any polynomial $f = \sum_{i,j} a_{i,j} x^i y^j$ in R^*, we have $\nu^*(f) = \min\big\{i + j\beta \ / \ a_{i,j} \neq 0\big\}$.

The valuation ν^* on K^* is obtained by putting formally $y = x^\beta$, i.e. we may define an injective morphism of K^* in the field $k((x^\Gamma))$ of power series of x with exponents in the group Γ (cf example 7), and the valuation ν^* is the restriction of the natural valuation μ defined on $k((x^\Gamma))$ to the field K^*. The equation $y = x^\beta$ represents formally an arc on the affine plane $\mathbf{A}^2_k = \mathrm{Spec}R^*$, which we call a *transcendental branch* on $\mathbf{A}^2_k$.

iv) $\mathbf{rank}(\nu) = \mathbf{rat.rank}(\nu) = 2,\ \mathbf{dim}(\nu) = 0$

The valuation ν is composite of prime divisors: $\nu = \nu' \circ \bar{\nu}$, the valuation ν' is a prime divisor of K and the valuation $\bar{\nu}$ is a prime divisor of the residue field κ' of ν', which a function field of transcendence degree one over k (cf example 6).
For the valuation ν' we are in the case i), and we can consider an affine normal model $Y = \mathrm{Spec}S$, with S integrally closed in its fraction field K and such that the center $\mathfrak{q}$ of ν' in S is a height one prime ideal. Then the residue field κ' is equal to the function field of the algebraic curve C and the valuation $\bar{\nu}$ is a valuation of $F(C)$ whose center ξ is also the center of the valuation ν. The local ring $S_\mathfrak{q}$ is the valuation ring associated to the discrete valuation ν' of rank one, and let u be a generator of its maximal ideal $\mathfrak{q}S_\mathfrak{q}$. Then we can write the composite valuation $\nu = \nu' \circ \bar{\nu}$ in the following form:

$$\nu(f) \ = \ \big(\nu'(f), \bar{\nu}(\overline{f})\big) \ = \ \big(\mathrm{order}_\mathfrak{q}(f), \bar{\nu}(\overline{f})\big) \ ,$$

where we denote $\overline{f}$ the image of $fu^{-\nu'(f)}$ in the residue field κ'.
Moreover, if the center ξ of the valuation $\bar{\nu}$ is a regular point of the curve C, the local ring $\mathcal{O}_{C,\xi}$ is the discrete rank one valuation ring associated to $\bar{\nu}$ and we can write

$$\bar{\nu}(\overline{f}) = \mathrm{order}_\mathfrak{m}(\overline{f}) \ ,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{C,\xi}$.

Uniformization

We shall give an idea of Zariski's proof of the uniformization theorem in the abstract form for a valuation of rank one and of rational rank two, i.e. in the case *iii*), the proof in the other cases are simpler, cases *i*) and *iv*), ore are quite similar, case *ii*) ([Za 1]).

Let Γ be the value group of the valuation ν, we may write $\Gamma = \mathbb{Z} \oplus \tau\mathbb{Z}$, with $\tau \in \mathbb{R} \setminus \mathbb{Q}$ and $\tau > 0$, and we choose two elements x and y in the function field K such that $\nu(x) = 1$ and $\nu(y) = \tau$. Then the elements x and y are algebraically independent over k and since the characteristic of k is zero, there exists a primitive element z for the algebraic extension

$K/k(x,y)$, i.e. $K = k(x,y,z)$, and we choose z with $\nu(z)$ positive.
Hence we consider the affine model $M = \mathrm{Spec}R$ of the function field K where R is an hypersurface ring $R = k[x,y,z]$, i.e. M is a surface defined in the affine space $\mathbb{A}^3_k = \mathrm{Spec}k[X,Y,Z]$ by a polynomial $f(X,Y,Z) = \sum a_{r,s,t}X^rY^sZ^t$ which satisfies the equality:

$$(1) \qquad f(x,y,z) \ = \ \sum a_{r,s,t}x^ry^sz^t \ = \ 0 \ .$$

Since we have $\nu(x)$, $\nu(y)$ and $\nu(z)$ positive, the center ξ of the valuation ν is the point $\xi = (0,0,0)$, then $a_{0,0,0} = 0$. Moreover, since $f(x,y,z) = 0$ there must exist at least two monomials $a_{r,s,t}x^ry^sz^t$ in $f(x,y,z)$ for which the minimal value $\nu\big(x^ry^sz^t\big)$ is reached. If $x^{r_1}y^{s_1}z^{t_1}$ and $x^{r_2}y^{s_2}z^{t_2}$ are two distinct terms of f which have the same value, then we have $t_1 \neq t_2$. In fact if $t_1 = t_2$, we get $\nu(x^{r_1}y^{s_1}) = \nu(x^{r_2}y^{s_2})$, then $r_1 + s_1\tau = r_2 + s_2\tau$ and since $\tau \notin \mathbb{Q}$ this implies $r_1 = r_2$ and $s_1 = s_2$.
We may write the polynomial f in the form:

$$(2) \qquad f(x,y,z) \ = \ \sum_{i=1}^{d} a_{r_i,s_i,t_i}x^{r_i}y^{s_i}z^{t_i} \ + \ \sum_{j=d+1}^{D} a_{r_j,s_j,t_j}x^{r_j}y^{s_j}z^{t_j} \ ,$$

such that the monomials $x^{r_i}y^{s_i}z^{t_i}$, $1 \leq i \leq d$, have the same value $\nu(x^{r_i}y^{s_i}z^{t_i}) = \gamma$ and such that the monomials $x^{r_j}y^{s_j}z^{t_j}$, $d+1 \leq j \leq D$, have a value $\nu(x^{r_j}y^{s_j}z^{t_j}) > \gamma$, and we may assume $t_1 < t_2, \ldots < t_d$. If $\nu(z) = u + v\tau$, with u and v in $\mathbb{Z}$, and if $\gamma = M + N\tau$, with M and N in $\mathbb{Z}$, we get the equalities:

$$(3) \qquad M = r_1 + ut_1 = r_2 + ut_2 = \ldots = r_d + ut_d \ ,$$

$$(4) \qquad N = s_1 + vt_1 = s_2 + vt_2 = \ldots = s_d + vt_d \ ,$$

and for any $j > d$ we have:

$$(5) \qquad (r_j + ut_j) + (s_j + vt_j)\tau \ > \ M + N\tau \ .$$

We may assume that $f(0,0,z)$ is not identically zero, i.e. that the z-axis Z does not lie on the model $M \subset \mathbf{A}^3_k$; it is enough if necessary to replace z by $z + c'x + c''y$ for sufficiently general c' and c'' in k. We may write $f(0,0,z) = \sum_{t=m}^{T} a_{0,0,t}z^t$, with $a_{0,0,m} \neq 0$, i.e. $\xi = (0,0,0)$ is an m-fold point of the zero-dimensional subvariety $Z \cap M$. If ξ is a regular point of $Z \cap M$, then ξ is also a regular point of the variety M, i.e. the center of the valuation ν on the model M is a non-singular point. Hence we may assume that ξ is a singular point of $Z \cap M$, which is equivalent to $m > 1$, and the polynomial $f(x,y,z)$ contains the term az^m with

$a = a_{0,0,m} \in k^*$. Moreover, since we have $\nu(z^m) \geq \gamma = \nu(x^{r_d} y^{s_d} z^{t_d})$, we must have

$$t_d \ \leq \ m \ . \tag{6}$$

We consider the expansion of τ in continued fraction:

$$\tau \ = \ h_1 + \cfrac{1}{h_2 + \cfrac{1}{h_3 + \dots}} \quad .$$

We denote (f_i/g_i) the sequence of convergent fractions of τ:

$$\frac{f_i}{g_i} \ = \ h_1 + \cfrac{1}{h_2 + \cfrac{1}{\dots \ + \cfrac{1}{h_i}}} \quad , \quad \text{with} \quad \big(f_i, g_i\big) = 1 \ ,$$

and moreover for all $i \geq 2$ we have:

$$f_{i-1} g_i - f_i g_{i-1} \ = \ (-1)^{i-1} \quad \text{and} \quad (-1)^{i-1}\big(\tau - f_i/g_i\big) \ > \ 0 \ .$$

Since we have $\lim_{q \to \infty} f_q/g_q = \tau$ and since we have a finite number of terms $x^{r_j} y^{s_j} z^{t_j}$ with $\nu(x^{r_j} y^{s_j} z^{t_j}) > \gamma$, we can find an integer p, sufficiently high, such that for any $q \geq p-1$, we have also the inequality

$$(r_j + u t_j) + (s_j + v t_j) \frac{f_q}{g_q} \ > \ M + N \frac{f_q}{g_q} \ , \tag{7}$$

for all these terms $x^{r_j} y^{s_j} z^{t_j}$, $d+1 \leq j \leq D$.

We want to construct a new ring $R_1 = k[x_1, y_1, z_1]$, with $R \subset R_1 \subset K$, or in other words we want to construct a birational morphism $M_1 \longrightarrow M$, such that the situation is better in some sense for the singularity of the center ξ_1 of ν on M_1. We pass from the elements x, y, z to the new elements x_1, y_1, z_1 of the function field K, by doing the following *Cremona transformation*:

$$x \ = \ x_1^{g_p} y_1^{g_{p-1}}, \quad y \ = \ x_1^{f_p} y_1^{f_{p-1}}, \quad z \ = \ x^u y^v (z_1 + c) \ , \tag{8}$$

where c is the element of k^* determined as follows: since $\nu(z) = u + v\tau = \nu(x^u y^v)$, and since the residue field of the valuation ν is equal to the ground field k, there exists a unique element c in k^* such that $\nu(z - c x^u y^v) > \nu(z)$. Hence we have

$$\nu(z_1) \ = \ \nu\Big(\frac{z}{x^u y^v} - c\Big) \ > \ 0 \ . \tag{9}$$

If we denote $\varepsilon = (-1)^{p-1}$, we deduce from the equality $f_{p-1}g_p - f_p g_{p-1} = \varepsilon$ and from the inequalities $\varepsilon(-\tau + f_{p-1}/g_{p-1}) > 0$ and $\varepsilon(\tau - f_p/g_p) > 0$, that we have:

$$(10) \qquad x_1 \ = \ \left(\frac{x^{f_{p-1}}}{y^{g_{p-1}}}\right)^{\varepsilon} \quad \text{and} \quad y_1 \ = \ \left(\frac{y^{g_p}}{x^{f_p}}\right)^{\varepsilon} ,$$

whence:
(11)

$$\nu(x_1) \ = \ \varepsilon g_{p-1}\left(\frac{f_{p-1}}{g_{p-1}} - \tau\right) \ > \ 0 \quad \text{and} \quad \nu(y_1) \ = \ \varepsilon g_p\left(\tau - \frac{f_p}{g_p}\right) \ > \ 0 \ .$$

Then the center of the valuation ν on R_1 is the maximal ideal (x_1, y_1, z_1), i.e. the center of ν on the model $M_1 = \mathrm{Spec}R_1$ is the point $\xi_1 = (0,0,0)$.

The ring R_1 is again an hypersurface ring, i.e. R_1 is isomorphic to the quotient ring $k[X_1, Y_1, Z_1]/(f_1)$ where f_1 is the polynomial defined as follows. Every monomial $x^r y^s z^t$ is transformed by the Cremona transformation into the polynomial $x_1^{r'} y_1^{s'} (z_1 + c)^t$ with $r' = (r+tu)g_p + (s+tv)f_p$ and $s' = (r+tu)g_{p-1} + (s+tv)f_{p-1}$. Hence, we deduce from the equalities (3) and (4) and from the inequality (7) that all the terms of f are divisible by the monomial $x_1^{Mg_p+Nf_p} y_1^{Mg_{p-1}+Nf_{p-1}}$, and that this monomial is the biggest factor of all the terms. We may write:

$$(12) \qquad f(x,y,z) \ = \ x_1^{Mg_p+Nf_p} y_1^{Mg_{p-1}+Nf_{p-1}} . f_1(x_1, y_1, z_1) \ ,$$

with f_1 irreducible.
More precisely we notice that all the terms $x^r y^s z^t$ of f with minimal value have exactly $x_1^{Mg_p+Nf_p} y_1^{Mg_{p-1}+Nf_{p-1}}$ as factor, and that the other terms acquire a factor $x_1^{r'} y_1^{s'}$ with $r' > Mg_p + Nf_p$ and $s' > Mg_{p-1} + Nf_{p-1}$. Then we deduce from (2) that the polynomial $f_1(x_1, y_1, z_1)$ has the form:
(13)

$$f_1(x_1, y_1, z_1) \ = \ (z_1 + c)^{t_1}\left(\sum_{i=1}^{d} a_{r_i, s_i, t_i} (z_1 + c)^{t_i - t_1}\right) \ + \ x_1 y_1 g(x_1, y_1, z_1) \ .$$

We put $a_{r_i, s_i, t_i} = a_i$ for $i = 1, 2, \ldots, d$ and we notice $h(u)$ the polynomial defined by $h(u) = a_1 + a_2 u^{t_2 - t_1} + \ldots + a_d u^{t_d - t_1} = \sum_{j=0}^{t_d - t_1} h_j u^j$, hence we have

$$f_1(x_1, y_1, z_1) = (z_1 + c)^{t_1} h(z_1 + c) + x_1 y_1 g(x_1, y_1, z_1) \ .$$

Since the center ξ_1 of the valuation on M_1 is the point $(0,0,0)$ of $\mathbf{A}_k^3$, we have $(0,0,0) \in M_1$, i.e. we must have $f_1(0,0,0) = 0$, and since $c \neq 0$,

$u = c$ is a root of the polynomial $h(u)$. If c is an m_1-fold root of $h(u)$, i.e. if we have $h(u) = (u-c)^{m_1}h'(u)$ with $h'(c) \neq 0$, then ξ_1 is also a m_1-root of $Z_1 \cap M_1$, where Z_1 is the z_1-axis. Since $\deg h(u) = t_d - t_1$ and from the inequality (6) we deduce:

$$m_1 \ \leq \ t_d - t_1 \ \leq \ m \ . \tag{14}$$

To prove the theorem of uniformization, we have to show that with this process, we can get a point ξ_1 which is *better* than the point ξ. If we have $m_1 < m_0 = m$, then we have succeeded, we can make an induction on m_i and for $m_N = 1$ we have a non-singular point.
We assume that we have $m_1 = m$, then from (14) we deduce that we have:

$$t_1 \ = \ 0 \ , \quad t_d \ = \ m \quad \text{and} \quad h(u) \ = \ a_d(u-c)^m \ .$$

Then the sum of the terms of minimal value of $f(x,y,z)$ is equal to:

$$\begin{aligned}\sum_{i=1}^{d} a_i x^{r_i} y^{s_i} z^{t_i} \quad &= \quad x_1^{Mg_p+Nf_p} y_1^{Mg_{p-1}+Nf_{p-1}} z_1^m \\ &= \quad h_0 x^{b_0} y^{c_0} + h_1 x^{b_1} y^{c_1} z^1 + \ldots + h_m x^{b_m} y^{c_m} z^m \ ,\end{aligned}$$

with $h_j \neq 0$ and $\nu(x^{b_j}y^{c_j}z^j) = \gamma$ for all j, and we may deduce $\nu(z) = b_{m-1} + c_{m-1}\tau$.
Then we have the following result: if by the transformation (8) the multiplicity m does not decrease, the value $\nu(z)$ is of the form $\nu(z) = u + v\tau$ with $u, v \geq 0$.

We assume that we have choosen the element z with $\nu(z) = u + v\tau$, with $u, v \geq 0$. Since the residue field of the valuation ν is equal to the ground field k, there exists a unique $c \in k^*$ such that $\nu(z - cx^u y^v) > \nu(z)$. Let

$$\begin{aligned} z^{[1]} \quad &= \quad z - cx^u y^v \ , \qquad \nu(z^{[1]}) > \nu(z) \\ f^{[1]}(x,y,z^{[1]}) \quad &= \quad f(x,y,cx^u y^v + z^{[1]}) \ = \ f(x,y,z) \ .\end{aligned}$$

We have found a new presentation of the model M as closed subvariety of $\mathbf{A}_k^3$, now defined by the polynomial $f^{[1]}$. The center ξ of the valuation ν on M belongs to the $z^{[1]}$-axis $Z^{[1]}$, and the multiplicity of ξ on $Z^{[1]} \cap M$ is also equal to m, i.e. $z^{[1]} = 0$ is also a m-fold root of $f^{[1]}(0,0,z^{[1]}) = 0$. We can apply a Cremona transformation such as (8), and we get a new model M_1 with a center ξ_1 of "multiplicity" m_1. If we have $m_1 < m$, then we have got a better variety.
We assume that we have again the equality $m_1 = m$, then we deduce

from the previous result that the value of $z^{[1]}$ is also of the form

$$\nu(z^{[1]}) \;=\; u_1 + v_1\tau \;, \quad \text{with } u_1, v_1 \geq 0 \;.$$

Then we put again

$$z^{[2]} \;=\; z^{[1]} - c_1 x^{u_1} y^{v_1} \;,$$

where the constant $c_1 \in k^*$ is choosen such that $\nu(z^{[2]}) > \nu(z^{[1]})$, and we make the same construction as before.

Hence we may assume that we have found by induction a sequence $(z^{[i]})$, $o \leq i \leq p$, of elements of K, with $z^{[0]} = z$, such that

$$(15)\;\; z^{[i+1]} \;=\; z^{[i]} - c_i x^{u_i} y^{v_i} \quad \text{and} \quad \nu(z^{[i+1]}) \;>\; \nu(z^{[i]}) \;=\; u_i + v_i\tau \;.$$

Each $z^{[i]}$ defines a polynomial $f^{[i]}(x, y, z^{[i]})$ and an embedding of the model M in $\mathbf{A}^3_k$, or equivalently, if we have fixed the first embedding defined by $f(x, y, z)$, each element $z^{[i]}$ defines a curve $C^{[i]}$ on $\mathbf{A}^3_k$, $C^{[i]}$ corresponds to the $z^{[i]}$-axis on $\mathbf{A}^3_k$. The center ξ of the valuation ν on M belongs to all the curves $C^{[i]}$ and the multiplicity $m = \mathrm{mult}_\xi(C^{[i]} \cap M)$ does not depend from i.
Such as (8), we can associate to each $z^{[i]}$, $0 \leq j \leq p-1$ a Cremona transform $M_1^{[i]} \longrightarrow M$, and the center $\xi_1^{[i]}$ of the valuation ν on this model $M_1^{[i]}$ satisfies

$$m_1^{[i]} \;=\; \mathrm{mult}_{\xi_1^{[i]}}(C_1^{[i]} \cap M_1^{[i]}) \;=\; m \;.$$

If for $j = p$ we have an inequality, i.e. $m_1^{[p]} < m$, then we consider the new model $M_1 = M_1^{[p]}$ of K for which the situation is better.

If for $j = p$ we have again an equality $m_1^{[p]} = m$, then we have shown that we can find a new element $z^{[p+1]}$ of K with $z^{[p+1]} = z^{[p]} - c_p x^{u_p} y^{v_p}$ and $\nu(z^{[p+1]}) > \nu(z^{[p]})$.
Hence it is enough to show that it is impossible to find an infinite sequence $(z^{[i]})$ of elements of K which satisfies the property (15).
By definition, for any i, $i \geq 0$, we have

$$f^{[i+1]}(x, y, z^{[i+1]}) \;=\; f^{[i]}(x, y, c_i x^{u_i} y^{v_i} + z^{[i+1]}) \;,$$

hence we deduce

$$\frac{\partial f^{[i+1]}}{\partial z^{[i+1]}} \;=\; \frac{\partial f^{[i]}}{\partial z^{[i]}} \;.$$

Since in each polynomial $f^{[i]}$ the term in $(z^{[i]})^m$ must be among the minimum value terms, it follows that

$$\nu\left(\frac{\partial f^{[i]}}{\partial z^{[i]}}\right) \;\geq\; (m-1)\nu(z^{[i]}) \;\geq \nu(z^{[i]}) \;,$$

hence we have for all i the inequality

$$\nu(z^{[i]}) = u_i + v_i\tau \ \leq \ \nu\left(\frac{\partial f}{\partial z}\right) . \tag{16}$$

If we have an infinite sequence $\left(z^{[i]}\right)$, we find an infinite sequence $\left((u_i, v_i)\right)$ in $\mathbb{N}^2$ such that the sequence of real numbers $\alpha_i = u_i + v_i\tau$ is increasing, but in that case the sequence (α_i) is non bounded, which contradicts the inequality (16).

In fact we have proven that the sequence is finite, i.e. there exists a curve $C = C^{[p]}$ which is *better* than the other ones. In particular we have shown that the Cremona transformation associated to this curve C will give a new situation which is better than the initial one because the multiplicity m_1 is srictly smaller than the multiplicity m. This is the curve which corresponds to the *maximal contact*.

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Arc spaces, motivic integration and stringy invariants

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Abstract.

The concept of *motivic integration* was invented by Kontsevich to show that birationally equivalent Calabi-Yau manifolds have the same Hodge numbers. He constructed a certain measure on the *arc space* of an algebraic variety, the motivic measure, with the subtle and crucial property that it takes values not in $\mathbb{R}$, but in the Grothendieck ring of algebraic varieties. A whole theory on this subject was then developed by Denef and Loeser in various papers, with several applications.

Batyrev introduced with motivic integration techniques new singularity invariants, the *stringy invariants*, for algebraic varieties with mild singularities, more precisely log terminal singularities. He used them for instance to formulate a topological Mirror Symmetry test for pairs of singular Calabi-Yau varieties. We generalized these invariants to almost arbitrary singular varieties, assuming Mori's Minimal Model Program.

The aim of these notes is to provide a gentle introduction to these concepts. There exist already good surveys by Denef-Loeser [DL8] and Looijenga [Loo], and a nice elementary introduction by Craw [Cr]. Here we merely want to explain the basic concepts and first results, including the p-adic number theoretic pre-history of the theory, and to provide concrete examples.

The text is a slightly adapted version of the 'extended abstract' of the author's talks at the 12th MSJ-IRI "Singularity Theory and Its Applications" (2003) in Sapporo. At the end we included a list of various recent results.

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Table of contents

1 Pre-history

1.1. Let $f \in \mathbb{Z}[x_1, \cdots, x_m]$ and $r \in \mathbb{Z}_{>0}$. A very general problem in number theory is to compute the number of solutions of the congruence $f(x_1, \cdots, x_m) = 0 \bmod r$ (in $(\mathbb{Z}/r\mathbb{Z})^m$). Thanks to the Chinese remainder theorem it is enough to consider the case where r is a power of a prime.

So we fix a prime number p and we investigate congruences modulo varying powers of p. We denote by F_n the number of solutions of $f(x_1, \cdots, x_m) = 0 \bmod p^{n+1}$.

1.2. *Examples.*

1. $f_1 = y - x^2$. It should be clear that $F_n = p^{n+1}$.

2. $f_2 = x \cdot y$. Exercise : $F_n = (n+2)p^{n+1} - (n+1)p^n$.

3. $f_3 = y^2 - x^3$. We list F_n for small n : $F_0 = p$,

$$
\begin{array}{lll}
F_1 = p(2p-1) & F_5 = p^5(p^2+p-1) & F_7 = p^7(2p^2-1) \\
F_2 = p^2(2p-1) & F_6 = p^6(p^2+p-1) & F_8 = p^8(2p^2-1) \\
F_3 = p^3(2p-1) & & F_9 = p^9(2p^2-1) \\
F_4 = p^4(2p-1) & & F_{10} = p^{10}(2p^2-1) \\
& F_{11} = p^{11}(p^3+p^2-1) & \\
& F_{12} = p^{12}(p^3+p^2-1). &
\end{array}
$$

Note that the plane curve $\{f_1 = 0\}$ is nonsingular, $\{f_2 = 0\}$ has the easiest curve singularity, an ordinary node, and $\{f_3 = 0\}$ has a slightly more complicated singularity, an ordinary cusp. It is in fact this cusp which is responsible for the at first sight not so nice behavior of the F_n for f_3.

More generally, the problem of the behavior of the F_n turns out to be non-obvious precisely when $\{f = 0\}$ has singularities.

1.3. We now know that, for any $f \in \mathbb{Z}[x_1, \cdots, x_m]$, the F_n do satisfy the following 'regular' behavior.

Conjecture [Borewicz, Shafarevich] = **Theorem** [Igusa]. *The generating formal series $J_p(T) := J_p(f, T) = \sum_{n \geq 0} F_n T^n$ is a rational function in T. (In particular the F_n are determined by a* finite *number of them.)*

Igusa showed this in 1975 [Ig1] using

(1) a 'translation' of $J_p(T)$ into a p-adic integral (more precisely into $\int_{\mathbb{Z}_p^m} |f|_p^s |dx|$, which is now called *Igusa's local zeta function*, and which is the ancestor of the motivic zeta function of section 6),

(2) an embedded resolution of singularities for $\{f = 0\}$,

(3) the change of variables formula for integrals.

(We will see later an analogue of this strategy in the theory of motivic integration.)

1.4. *Examples* (continuing 1.2).

1. $J_p(f_1; T) = \frac{p}{1-pT}$ (easy).
2. Exercise : $J_p(f_2; T) = \frac{2p-1-p^2T}{(1-pT)^2}$.
3. Claim : $J_p(f_3; T) = p\frac{1+(p-1)T+(p^6-p^5)T^5-p^7T^6}{(1-p^7T^6)(1-pT)}$.

1.5. We already want to mention another connection with singularity theory; the famous (still open) *monodromy conjecture* of Igusa relates the poles of $J_p(T)$ with eigenvalues of local monodromy of f considered as a map $f : \mathbb{C}^n \to \mathbb{C}$, see (6.8).

1.6. Before introducing arc spaces and motivic integration in the next sections, we present a hopefully motivating analogy between this number theoretic setting and the geometric arc setting.

$f \in \mathbb{Z}[x_1, \cdots, x_m]$	$f \in \mathbb{C}[x_1, \cdots, x_m]$
solution of $f = 0$ over the ring $\mathbb{Z}/p^{n+1}\mathbb{Z} \cong \mathbb{Z}_p/p^{n+1}\mathbb{Z}_p$, i.e. an m-tuple with coordinates of the form $a_0 + a_1 p + ... + a_n p^n$, $a_i \in \{0, 1, ..., p-1\}$	solution of $f = 0$ over the ring $\mathbb{C}[t]/(t^{n+1}) \cong \mathbb{C}[[t]]/(t^{n+1})$, i.e. an m-tuple with coordinates of the form $a_0 + a_1 t + ... + a_n t^n$, $a_i \in \mathbb{C}$ ("n-jet" of $\{f = 0\}$)
solution of $f = 0$ over $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^{n+1}\mathbb{Z}$, i.e. with coordinates of the form $\sum_{n=0}^{\infty} a_i p^i$	solution of $f = 0$ over $\mathbb{C}[[t]] = \varprojlim \mathbb{C}[t]/(t^{n+1})$, i.e. with coordinates of the form $\sum_{n=0}^{\infty} a_i t^i$ ("arc" of $\{f = 0\}$)
integrate over $\mathbb{Z}_p^m$	integrate over $\mathcal{L}(\mathbb{C}^m) := \{$ arcs of $\mathbb{C}^m\}$

Warning. Here and further on we sometimes use other (better ?) normalizations than in the original papers.

2 Arc spaces

Let X be an algebraic variety over $\mathbb{C}$. (The theory of arc spaces and motivic integration can be generalized to any field of characteristic zero, see e.g. [DL8].)

2.1. The *space of arcs modulo* t^{n+1} or *space of* n*-jets on* X is an algebraic variety $\mathcal{L}_n(X)$ over $\mathbb{C}$ such that

$$\begin{aligned}&\{\text{points of } \mathcal{L}_n(X) \text{ with coordinates in } \mathbb{C}\}\\ =&\{\text{points of } X \text{ with coordinates in } \frac{\mathbb{C}[t]}{(t^{n+1})}\}.\end{aligned}$$

For all n there are obvious 'truncation maps' $\pi_n^{n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$, obtained by reducing $(n+1)$-jets modulo t^{n+1}, and more generally $\pi_n^m : \mathcal{L}_m(X) \to \mathcal{L}_n(X)$ for $m \geq n$. This description is somewhat informal, but is essentially what is needed. We now first provide examples and give the 'exact' definition later.

2.2. *Example.* Let $X = \mathbb{C}^d$. Then

$$\begin{aligned}\mathcal{L}_n(X) =& \{(a_0^{(1)} + a_1^{(1)}t + \cdots + a_n^{(1)}t^n, \cdots, a_0^{(d)} + a_1^{(d)}t + \cdots + a_n^{(d)}t^n), \\ & \quad \text{with all } a_i^{(j)} \in \mathbb{C}\} \\ \cong& \mathbb{C}^{(n+1)d}.\end{aligned}$$

2.3. *Example.* Let $X = \{y^2 - x^3 = 0\}$.

(0) $\mathcal{L}_0(X) = \{(a_0, b_0) \in \mathbb{C}^2 | b_0^2 = a_0^3\} = X$.

(1) $\mathcal{L}_1(X)$

$$\begin{aligned}&= \{(a_0 + a_1t, b_0 + b_1t) \in (\mathbb{C}[t]/(t^2))^2 \mid (b_0 + b_1t)^2 = (a_0 + a_1t)^3 \text{mod } t^2\} \\ &= \{(a_0 + a_1t, b_0 + b_1t) \in (\mathbb{C}[t]/(t^2))^2 \mid b_0^2 = a_0^3 \text{ and } 2b_0b_1 = 3a_0^2a_1\}.\end{aligned}$$

So we can consider $\mathcal{L}_1(X)$ as the (two-dimensional) algebraic variety in $\mathbb{C}^4$ with equations $b_0^2 = a_0^3$ and $2b_0b_1 = 3a_0^2a_1$ in the coordinates a_0, a_1, b_0, b_1. The map $\pi_0^1 : \mathcal{L}_1(X) \to \mathcal{L}_0(X) = X$ is induced by the projection $\mathbb{C}^4 \to \mathbb{C}^2 : (a_0, a_1, b_0, b_1) \mapsto (a_0, b_0)$.

The fibre of π_0^1 above $(0, 0)$ is $\{(0, a_1, 0, b_1)\} \cong \mathbb{C}^2$; this corresponds to the fact that the tangent space to X at $(0, 0)$ is the whole $\mathbb{C}^2$. The fibre above $(a_0, b_0) \neq (0, 0)$ is the line in the (a_1, b_1)-plane with equation $2b_0b_1 = 3a_0^2a_1$, which corresponds to the tangent line at X in (a_0, b_0). In other words : $\mathcal{L}_1(X)$ is the tangent bundle TX, and π_0^1 is the natural projection $TX \to X$.

(2) $\mathcal{L}_2(X) = \{(a_0 + a_1t + a_2t^2, b_0 + b_1t + b_2t^2) \in (\mathbb{C}[t]/(t^3))^2 \mid (b_0 + b_1t + b_2t^2)^2 = (a_0 + a_1t + a_2t^2)^3 \text{mod } t^3\}$ is given in $\mathbb{C}^6$ by the equations

$$\begin{cases} b_0^2 = a_0^3 \\ 2b_0b_1 = 3a_0^2a_1 \\ b_1^2 + 2b_0b_2 = 3a_0a_1^2 + 3a_0^2a_2. \end{cases}$$

Exercise. a) Verify the description of $\mathcal{L}_2(X)$ and note that the map $\pi_1^2 : \mathcal{L}_2(X) \to \mathcal{L}_1(X)$ is not surjective. More precisely, the fibre of π_0^2 above $(0, 0)$ is $\{(0, a_1, a_2, 0, 0, b_2)\} \cong \mathbb{C}^3$, but its image by π_1^2 is not the whole (a_1, b_1)-plane; it is just the line $\{b_1 = 0\}$.

b) Compute $\mathcal{L}_3(X)$ and note that also $\pi_2^3 : \mathcal{L}_3(X) \to \mathcal{L}_2(X)$ is not surjective.

c) However, above the nonsingular part of $X = \mathcal{L}_0(X)$ all considered maps $\pi_n^{n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ are fibrations with fibre $\mathbb{C}$.

2.4. Some observations in the examples are easily seen to be satisfied in general.
(1) $\mathcal{L}_0(X) = X, \quad \mathcal{L}_1(X) = TX.$
(2) If X is smooth of dimension d, then all π_n^{n+1} are locally trivial fibrations (w.r.t. the Zariski topology) with fibre $\mathbb{C}^d$.

2.5. The *space of arcs on* X is an 'algebraic variety of infinite dimension' $\mathcal{L}(X)$ over $\mathbb{C}$ such that

$$\begin{aligned}&\{\text{points of } \mathcal{L}(X) \text{ with coordinates in } \mathbb{C}\}\\ =&\{\text{points of } X \text{ with coordinates in } \mathbb{C}[[t]]\}.\end{aligned}$$

We provide the 'exact' definition after continuing the examples. Now we have for all n truncation maps $\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$, obtained by reducing arcs modulo t^{n+1}.

2.6. *Example.* Let $X = \mathbb{C}^d$. Then

$$\mathcal{L}(X) = \{(\sum_{n=0}^{\infty} a_n^{(1)} t^n, \cdots, \sum_{n=0}^{\infty} a_n^{(d)} t^n), \text{ with all } a_n^{(j)} \in \mathbb{C}\},$$

which can be considered as an infinite dimensional affine space.

2.7. *Example.* Let $X = \{y^2 - x^3 = 0\}$. Then $\mathcal{L}(X)$ is given in the infinite dimensional affine space with coordinates

$$\begin{cases} a_0, a_1, a_2, \cdots, a_n, \cdots \\ b_0, b_1, b_2, \cdots, b_n, \cdots \end{cases}$$

by the infinite number of equations

$$\begin{cases} b_0^2 = a_0^3 \\ 2b_0 b_1 = 3a_0^2 a_1 \\ b_1^2 + 2b_0 b_2 = 3a_0 a_1^2 + 3a_0^2 a_2 \\ \cdots \end{cases}$$

2.8. *More precise definitions.*

(i) The 'base extension operation' $Y \to Y \times_{\mathbb{C}} \mathbb{C}[t]/(t^{n+1})$ is a covariant functor on the category of complex algebraic varieties, and it has a right adjoint $X \to \mathcal{L}_n(X)$. (Even more precisely we should say that we consider the *reduced* scheme $\mathcal{L}_n(X)$ associated to this right adjoint scheme.) This says that, for any $\mathbb{C}$-algebra R, the set of R-valued points of $\mathcal{L}_n(X)$ is in natural bijection with the set of $R[t]/(t^{n+1})$-valued points of X. In particular, as we said in (2.1), the $\mathbb{C}$-valued points of $\mathcal{L}_n(X)$ can be naturally identified with the $\mathbb{C}[t]/(t^{n+1})$-valued points of X.

(ii) Then $\mathcal{L}(X)$ is the inverse limit $\varprojlim \mathcal{L}_n(X)$. (Technically, it is important here that the truncation morphisms $\pi_n^{n+1} : \mathcal{L}_{n+1}(X) \to \mathcal{L}_n(X)$ are affine.) The K-valued points of $\mathcal{L}(X)$, for any field $K \supset \mathbb{C}$, are in natural bijection with the $K[[t]]$-valued points of X. We mention the following result, attributed to Kolchin : if X is irreducible, then $\mathcal{L}(X)$ is irreducible.

See [DL3] for more information.

2.9. When X is an affine variety, i.e. given by a finite number of polynomial equations, one can describe equations for the $\mathcal{L}_n(X)$ and for $\mathcal{L}(X)$ as in Examples 2.3 and 2.7.

2.10. Some first natural and fundamental questions are how the $\mathcal{L}_n(X)$ and $\pi_n(\mathcal{L}(X))$ change with n. (For $\pi_n(\mathcal{L}(X))$ this was already considered by Nash [Na].) Note that $\mathcal{L}_n(X)$ describes by definition the n-jets on X, and $\pi_n(\mathcal{L}(X))$ those n-jets that can be lifted to arcs on X.

This can be compared with the number theoretical setting of the previous section : there the question was how the solutions over $\mathbb{Z}/p^{n+1}\mathbb{Z}$ changed with n, and we could consider the same question for those solutions over $\mathbb{Z}/p^{n+1}\mathbb{Z}$ that can be lifted to solutions over $\mathbb{Z}_p$.

2.11. We now introduce the Grothendieck ring of algebraic varieties, which is the 'best' framework to answer these questions, and which is moreover (essentially) the value ring for motivic integration, to be explained in the next section.

Recall first two fundamental properties of the topological Euler characteristic $\chi(\cdot) \in \mathbb{Z}$ on complex algebraic varieties :
(1) $\chi(V) = \chi(Z) + \chi(V \setminus Z)$ if Z is (Zariski-)closed in V,
(2) $\chi(V \times W) = \chi(V) \cdot \chi(W)$.

A finer invariant satisfying these properties is the Hodge-Deligne polynomial $H(\cdot) = H(\cdot; u, v) \in \mathbb{Z}[u, v]$, given for an algebraic variety V of dimension d by

$$H(V;u,v) := \sum_{p,q=0}^{d} (\sum_{i=0}^{2d} (-1)^i h^{p,q}(H_c^i(V,\mathbb{C})))u^p v^q,$$

where $h^{p,q}(\cdot)$ denotes the dimension of the (p,q)-component of the mixed Hodge structure. (When we would work over an arbitrary field of characteristic zero, we use an embedding into $\mathbb{C}$ of the field of definition of V. The $u^p v^q$–coefficients of $H(V;u,v)$ do not depend on the chosen embedding, since for a smooth projective V they are equal to $(-1)^{p+q} \dim H^q(V,\Omega_V^p)$.)

Note that $H(V;1,1) = \chi(V)$.

The Grothendieck ring is the value ring of the 'universal Euler characteristic' on algebraic varieties.

Definition. (i) The Grothendieck group of (complex) algebraic varieties is the abelian group $K_0(Var_{\mathbb{C}})$ generated by symbols $[V]$, where V is an algebraic variety, with the relations $[V] = [W]$ if V and W are isomorphic, and $[V] = [Z] + [V \setminus Z]$ if Z is (Zariski-) closed in V.

(ii) there is a natural ring structure on $K_0(Var_{\mathbb{C}})$ given by $[V] \cdot [W] := [V \times W]$.

— So by construction the map $\{\text{Varieties over } \mathbb{C}\} \to K_0(Var_{\mathbb{C}}) : V \mapsto [V]$ is indeed universal with respect to the two properties above. Of course we still loose some information by this operation. For example $X = \{y^2 - x^3 = 0\} \subset \mathbb{A}^2$ satisfies $[X] = [\mathbb{A}^1]$. Also, when $V \to B$ is a locally trivial fibration with fibre F, then $[V] = [B] \cdot [F]$. —

(iii) Let C be a constructible subset of some variety V, i.e. a disjoint union of (finitely many) locally closed subvarieties A_i of V, then $[C] \in K_0(Var_{\mathbb{C}})$ is well defined as $[C] := \sum_i [A_i]$.

(iv) We denote 1:= [point], $\mathbb{L} := [\mathbb{A}^1]$ and $\mathcal{M}_{\mathbb{C}} := K_0(Var_{\mathbb{C}})_{\mathbb{L}}$ the ring obtained from $K_0(Var_{\mathbb{C}})$ by inverting $\mathbb{L}$.

The rings $K_0(Var_{\mathbb{C}})$ and $\mathcal{M}_{\mathbb{C}}$ are quite mysterious. For instance, it was shown only recently that $K_0(Var_{\mathbb{C}})$ is not a domain [Po], and it is still not known whether $\mathcal{M}_{\mathbb{C}}$ is a domain or not, or whether the natural map $K_0(Var_{\mathbb{C}}) \to \mathcal{M}_{\mathbb{C}}$ is injective.

Remark. There is an interesting alternative description of $K_0(Var_{\mathbb{C}})$ as the abelian group, generated by isomorphism classes $[V]$ of *nonsingular* *projective* varieties V, with the relations $[\emptyset]=0$ and $[\tilde{V}]-[E]=[V]-[Z]$, where $\tilde{V}\to V$ is the blowing-up with centre Z and exceptional variety E [Bi1].

2.12. We now answer the questions in (2.10). We will consider $[\mathcal{L}_n(X)]$ and $[\pi_n(\mathcal{L}(X))]$ in $\mathcal{M}_{\mathbb{C}}$. For the latter we use a theorem of Greenberg [Gr], saying that $\pi_n(\mathcal{L}(X))$ is a constructible subset of $\mathcal{L}_n(X)$.

Theorem [DL3][DL8]**.** *The generating formal series*

$$J(T) := \sum_{n\geq 0}[\mathcal{L}_n(X)]T^n \text{ and } P(T) := \sum_{n\geq 0}[\pi_n(\mathcal{L}(X))]T^n$$

in $\mathcal{M}_{\mathbb{C}}[[T]]$ *are rational, with moreover as denominators products of polynomials of the form* $1-\mathbb{L}^a T^b$*, where* $a\in\mathbb{Z}$ *and* $b\in\mathbb{Z}_{>0}$*.*

The proof uses motivic integration, which 'explains' why $\mathcal{M}_{\mathbb{C}}$ is needed instead of $K_0(Var_{\mathbb{C}})$; see section 3.

This result specializes to the analogous statement, replacing $[\cdot]$ by $\chi(\cdot)$ or $H(\cdot)$. Note for this that $\chi : K_0(Var_{\mathbb{C}})\to\mathbb{Z}$ and $H : K_0(Var_{\mathbb{C}})\to\mathbb{Z}[u,v]$ obviously extend to $\chi:\mathcal{M}_{\mathbb{C}}\to\mathbb{Z}$ and $H:\mathcal{M}_{\mathbb{C}}\to\mathbb{Z}[u,v][\frac{1}{uv}]$. When $X=\{f=0\}$ for some polynomial f, the statement for $J(T)$ should be compared with Theorem 1.3 for $J_p(T)$! In this case, we will outline a proof for $J(T)$ later. We just mention that the proof for $P(T)$ uses techniques from logic, more precisely quantifier elimination.

2.13. *Example.* When X is smooth of dimension d, all $\mathcal{L}_n(X)=\pi_n(\mathcal{L}(X))$ are locally trivial over X with fibre $\mathbb{C}^{nd}$. Hence

$$J(T)=P(T)=\sum_{n\geq 0}[X]\mathbb{L}^{nd}T^n=\frac{[X]}{1-\mathbb{L}^d T}.$$

2.14. *Example.* Let $X=\{y^2-x^3=0\}$. The descriptions in Example 2.3 yield $[\mathcal{L}_0(X)]=[X]=\mathbb{L},[\mathcal{L}_1(X)]=\mathbb{L}^2+(\mathbb{L}-1)\mathbb{L}=2\mathbb{L}^2-\mathbb{L},[\mathcal{L}_2(X)]=\mathbb{L}^3+(\mathbb{L}-1)\mathbb{L}^2=2\mathbb{L}^3-\mathbb{L}^2$. We claim that

$$J(T)=\mathbb{L}\frac{1+(\mathbb{L}-1)T+(\mathbb{L}^6-\mathbb{L}^5)T^5-\mathbb{L}^7T^6}{(1-\mathbb{L}^7T^6)(1-\mathbb{L}T)},$$

see section 6. (Compare with 1.4(3)!) The formula in [DL5, Proposition 10.2.1] yields

$$P(T) = \frac{\mathbb{L} + (1-\mathbb{L})T - \mathbb{L}T^2}{(1-\mathbb{L}T)(1-T^2)}.$$

2.15. *Example.* Let $X = \{xy = 0\}$. EXERCISE :
(i) $[\mathcal{L}_n(X)] = (n+2)\mathbb{L}^{n+1} - (n+1)\mathbb{L}^n$. Then

$$J(T) = \frac{2\mathbb{L} - 1 - \mathbb{L}^2 T}{(1-\mathbb{L}T)^2}.$$

(Compare again with Examples 1.2 and 1.4.)
(ii) $[\pi_n(\mathcal{L}(X))] = 2\mathbb{L}^{n+1} - 1$. Then

$$P(T) = \frac{2\mathbb{L} - 1 - \mathbb{L}T}{(1-\mathbb{L}T)(1-T)}.$$

2.16. [Mu1] To conclude this section, we relate some properties of the spaces of n-jets on X to properties of X. Let X be irreducible of dimension d.

(i) The closure in $\mathcal{L}_n(X)$ of $(\pi_0^n)^{-1}(X_{\mathrm{reg}})$ is an irreducible component of $\mathcal{L}_n(X)$ of dimension $d(n+1)$.

(ii) Suppose that X is locally a complete intersection. Then
(1) $\mathcal{L}_n(X)$ is pure dimensional if and only if $\dim \mathcal{L}_n(X) \leq d(n+1)$.
(2) $\mathcal{L}_n(X)$ is irreducible if and only if $\dim(\pi_0^n)^{-1}(X_{\mathrm{sing}}) < d(n+1)$.
(3) If $\mathcal{L}_{n+1}(X)$ is pure dimensional or irreducible, then so is $\mathcal{L}_n(X)$.
(4) If $\mathcal{L}_n(X)$ is irreducible for some $n > 0$, then X is normal.
(5) $\mathcal{L}_n(X)$ is irreducible for all $n > 0$ if and only if X has rational singularities.

(iii) When $d = 1$ we have for any $n > 0$ that $\mathcal{L}_n(X)$ is irreducible if and only if X is nonsingular.

3 Motivic integration

This notion is due to Kontsevich [Ko] on nonsingular varieties. It has been further developed by Batyrev [Ba2][Ba3], and especially by Denef and Loeser [DL3][DL4][DL6][DL8], with some improvements by Looijenga [Loo]. Probably the best way to view and understand it, is as being an analogue of p-adic integration.

Let in this section X be any algebraic variety of pure dimension d.

3.1. A subset A of $\mathcal{L}(X)$ is called *constructible* or *cylindric* or *a cylinder* if $A = \pi_m^{-1} C$ for some m and some constructible subset C of $\mathcal{L}_m(X)$. These can be considered as 'reasonably nice' subsets of the arc space $\mathcal{L}(X)$, being precisely all arcs obtained by lifting a nice subset of a jet space.

3.2. Suppose that X is nonsingular. Then such a constructible subset $A = \pi_m^{-1} C$ satisfies the property

$$[\pi_n(A)] = \mathbb{L}^{(n-m)d}[C] \quad \text{for all } n \geq m,$$

since $\pi_m^n : \mathcal{L}_n(X) = \pi_n(\mathcal{L}(X)) \to \mathcal{L}_m(X) = \pi_m(\mathcal{L}(X))$ is a locally trivial fibration with fibre $\mathbb{C}^{(n-m)d}$. We have in particular that the

$$\frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$$

are all equal in $\mathcal{M}_{\mathbb{C}}$ for $n \geq m$.

For general X, a constructible set $A \subset \mathcal{L}(X)$ which is disjoint with $\mathcal{L}(X_{sing})$ still satisfies the property that the $\frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$ stabilize for n big enough [DL3, Lemma 4.1]. More precisely we have the following.

Definition. We call a set $A \subset \mathcal{L}(X)$ *stable* if for some $m \in \mathbb{N}$ we have

(i) $\pi_m(A)$ is constructible and $A = \pi_m^{-1}(\pi_m(A))$, and

(ii) for all $n \geq m$ the projection $\pi_{n+1}(A) \to \pi_n(A)$ is a piecewise trivial fibration with fiber $\mathbb{C}^d$.

(So in particular A is constructible.)

Lemma [DL3]. *If $A \subset \mathcal{L}(X)$ is constructible and $A \cap \mathcal{L}(X_{\text{sing}}) = \emptyset$, then A is stable.*

Hence for such A it makes sense to consider $\lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{nd}} \in \mathcal{M}_{\mathbb{C}}$ as an invariant of A; it is called its *naive motivic measure.* Note that for nonsingular X the measure of $\mathcal{L}(X)$ is just $[X]$.

3.3. For arbitrary constructible $A \subset \mathcal{L}(X)$ the sequence $\frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$ will not stabilize.

Example. Let $X = \{xy = 0\}$. From Example 2.15 we see that

$$\frac{[\pi_n(\mathcal{L}(X))]}{\mathbb{L}^{nd}} = \frac{2\mathbb{L}^{n+1} - 1}{\mathbb{L}^n} = 2\mathbb{L} - \frac{1}{\mathbb{L}^n}.$$

This sequence 'almost' stabilizes (the singular point of X of course causes the trouble), and it would be nice to be able to consider $2\mathbb{L}$ as the limit of this sequence.

This will indeed work in Kontsevich's *completed Grothendieck ring* $\hat{\mathcal{M}}_{\mathbb{C}}$. This is by definition the completion of $\mathcal{M}_{\mathbb{C}}$ with respect to the decreasing filtration $F^m, m \in \mathbb{Z}$, of $\mathcal{M}_{\mathbb{C}}$, where F^m is the subgroup of $\mathcal{M}_{\mathbb{C}}$ generated by the elements $\frac{[S]}{\mathbb{L}^i}$ with S an algebraic variety and $\dim S - i \leq -m$. Note that this is indeed a ring filtration : $F^m \cdot F^n \subset F^{m+n}$. So $\hat{\mathcal{M}}_{\mathbb{C}} = \varprojlim_{m} \frac{\mathcal{M}_{\mathbb{C}}}{F^m}$.

Continuing the example. Indeed in $\hat{\mathcal{M}}_{\mathbb{C}}$ we have

$$\lim_{n\to\infty} \frac{[\pi_n(\mathcal{L}(X))]}{\mathbb{L}^{nd}} = 2\mathbb{L} - \lim_{n\to\infty} \frac{1}{\mathbb{L}^n} = 2\mathbb{L}.$$

Theorem [DL3]. *Let A be a constructible subset of $\mathcal{L}(X)$. Then the limit*

$$\mu(A) := \lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$$

exists in $\hat{\mathcal{M}}_{\mathbb{C}}$.

We call $\mu(A)$ the *motivic measure* of A. This yields a σ-additive measure μ on the Boolean algebra of constructible subsets of $\mathcal{L}(X)$. Thus, given any sequence $A_i, i \in \mathbb{Z}_{>0}$, of disjoint constructible subsets in $\mathcal{L}(X)$ such that $\lim_{i\to\infty} \mu(A_i) = 0$, we have that $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ in $\hat{\mathcal{M}}_{\mathbb{C}}$.

Note. It is not known whether the natural map $\mathcal{M}_{\mathbb{C}} \to \hat{\mathcal{M}}_{\mathbb{C}}$ is injective; its kernel is $\cap_{m\in\mathbb{Z}} F^m$. However, e.g. the topological Euler characteristic $\chi(\cdot)$ and the Hodge-Deligne polynomial $H(\cdot)$ factor through the image of $\mathcal{M}_{\mathbb{C}}$ in $\hat{\mathcal{M}}_{\mathbb{C}}$.

Remark. Let $S \subsetneq X$ be a closed subvariety; it is not difficult to see that $\mathcal{L}(S)$ is *not* a constructible subset of $\mathcal{L}(X)$. It is possible to introduce more generally *measurable* subsets of $\mathcal{L}(X)$, and to associate analogously a motivic measure (in $\hat{\mathcal{M}}_{\mathbb{C}}$) to those subsets [Ba2][DL6]; we then have that such $\mathcal{L}(S)$ are measurable of measure zero.

3.4. We briefly compare with the p-adic case. Let M be a d-dimensional submanifold of $\mathbb{Z}_p^m$, defined algebraically. We denote by $|\pi_n(M)|$ the cardinality of the image of M under the natural truncation map $\pi_n : (\mathbb{Z}_p)^m \to (\mathbb{Z}_p/p^{n+1}\mathbb{Z}_p)^m = (\mathbb{Z}/p^{n+1}\mathbb{Z})^m$. Then $\frac{|\pi_n(M)|}{p^{(n+1)d}} \in \mathbb{Z}[\frac{1}{p}]$ is constant for n big enough and is called the volume $\mu_p(M)$ of M.

For a singular d-dimensional subvariety Z of $\mathbb{Z}_p^m$ one defines its *volume* as $\mu_p(Z) := \lim_{\epsilon \to 0} \mu_p(Z \setminus T_\epsilon(Z_{\text{sing}})) \in \mathbb{R}$, where T_ϵ denotes a small tubular neighbourhood 'of radius ϵ'. Then by a Theorem of Oesterlé [Oe] we have, with analogous notation $|\pi_n(Z)|$,

$$\mu_p(Z) = \lim_{n \to \infty} \frac{|\pi_n(Z)|}{p^{(n+1)d}} .$$

Note the analogy

	p-adic	motivic
integrate over	$\mathbb{Z}_p^m$	$(\mathbb{C}[[t]])^m$
value rings	$\mathbb{Z}$	$K_0(Var_{\mathbb{C}})$
	$\mathbb{Z}[\frac{1}{p}]$	$\mathcal{M}_{\mathbb{C}}$
	$\mathbb{R}$	$\hat{\mathcal{M}}_{\mathbb{C}}$

The brilliant idea of Kontsevich was to use $\hat{\mathcal{M}}_{\mathbb{C}}$ instead of $\mathbb{R}$ as a value ring for integration.

3.5. We can now consider in a natural way *motivic integration.* We do not treat the most general setting; the following suffices in practice. Let $A \subset \mathcal{L}(X)$ be constructible and $\alpha : A \to \mathbb{Z} \cup \{+\infty\}$ a function with constructible fibres $\alpha^{-1}\{n\}, n \in \mathbb{Z}$. Then

$$\int_A \mathbb{L}^{-\alpha} d\mu := \sum_{n \in \mathbb{Z}} \mu(\alpha^{-1}\{n\})\mathbb{L}^{-n}$$

in $\hat{\mathcal{M}}_{\mathbb{C}}$, whenever the right hand side converges in $\hat{\mathcal{M}}_{\mathbb{C}}$. Then we say that $\mathbb{L}^{-\alpha}$ is *integrable* on A. (This will always be the case if α is bounded from below.)

3.6. An important example of an integrable function is induced by an effective Cartier divisor D on X, i.e. D is an (eventually non-reduced) subvariety of X which is locally given by one equation. Define $ord_t D : \mathcal{L}(X) \to \mathbb{N} \cup \{+\infty\} : \gamma \mapsto ord_t f_D(\gamma)$, where f_D is a local equation of D in a neighbourhood of the origin $\pi_0(\gamma)$ of γ. Note e.g. that $(ord_t D)(\gamma) = +\infty$ if and only if $\gamma \in \mathcal{L}(D_{\text{red}})$ and $(ord_t D)(\gamma) = 0$ if and only if $\pi_0(\gamma) \notin D_{\text{red}}$. One easily verifies that $\mathbb{L}^{-ord_t D}$ is integrable on $\mathcal{L}(X)$.

We note that $(ord_t D)^{-1}(+\infty) = \mathcal{L}(D_{\text{red}})$ is *not* constructible; it is however measurable with measure zero.

Example. Take $X = \mathbb{A}^1$ and D the divisor associated to the function x^N, i.e. the 'origin with multiplicity N'.

Exercise. (i) $N | (ord_t D)(\gamma)$ for all $\gamma \in \mathcal{L}(\mathbb{A}^1)$ and

$$\mu(\{\gamma \in \mathcal{L}(\mathbb{A}^1) \mid (ord_t D)(\gamma) = iN\}) = \frac{\mathbb{L}-1}{\mathbb{L}^i} \text{ for all } i \in \mathbb{N}.$$

(ii) $\int_{\mathcal{L}(\mathbb{A}^1)} \mathbb{L}^{-ord_t D} d\mu = \frac{(\mathbb{L}-1)\mathbb{L}^{N+1}}{\mathbb{L}^{N+1}-1} = (\mathbb{L}-1) + \frac{\mathbb{L}-1}{\mathbb{L}^{1+N}-1}$.

This example is the easiest case of the following very useful formula.

Proposition [Ba3][Cr]**.** *Let X be nonsingular and take a normal crossings divisor $D = \sum_{i \in S} N_i D_i$ on X, i.e. all D_i are nonsingular hypersurfaces intersecting transversely (and occurring with multiplicity N_i). Denote $D_I^\circ := (\cap_{i\in I} D_i) \setminus (\cup_{\ell \notin I} D_\ell)$ for $I \subset S$; the $D_I^\circ, I \subset S$, form a natural locally closed stratification of X (note that $D_\emptyset^\circ = X \setminus (\cup_{\ell \in S} D_\ell)$). Then*

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-ord_t D} d\mu = \sum_{I \subset S} [D_I^\circ] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{1+N_i}-1}.$$

3.7. The construction in (3.6) can be generalized as follows. Let $\mathcal{I}$ be a sheaf of ideals on X. Then we define

$$ord_t \mathcal{I} : \mathcal{L}(X) \to \mathbb{N} \cup \{+\infty\} : \gamma \mapsto \min_g ord_t g(\gamma),$$

where the minimum is taken over $g \in \mathcal{I}$ in a neighbourhood of $\pi_0(\gamma)$. Of course, when $\mathcal{I}$ is the ideal sheaf of an effective Cartier divisor D, then $ord_t\mathcal{I} = ord_t D$.

3.8. The most crucial ingredient in the theory of motivic integration is the *change of variables formula* or *transformation rule* for motivic integrals under a birational morphism.

Theorem [DL3]. *(i) Let $h : Y \to X$ be a proper birational morphism between algebraic varieties X and Y, where Y is nonsingular. Let $A \subset \mathcal{L}(X)$ be constructible and $\alpha : A \to \mathbb{Z} \cup \{+\infty\}$ such that $\mathbb{L}^{-\alpha}$ is integrable on A. Then*

$$\int_A \mathbb{L}^{-\alpha} d\mu = \int_{h^{-1}A} \mathbb{L}^{-(\alpha \circ h) - ord_t(Jac_h)} d\mu.$$

Here the ideal sheaf Jac_h is defined as follows. When also X is nonsingular, it is locally generated by the 'ordinary' Jacobian determinant with respect to local coordinates on X and Y. For general X, the sheaf of regular differential d-forms $h^*(\Omega^d_X)$ is still a submodule of Ω^d_Y; but now $h^*(\Omega^d_X)$ is not necessarily locally generated by one element. Taking (locally) a generator ω_Y of Ω^d_Y, each $h^*(\omega)$ for $\omega \in \Omega^d_X$ can be written as $h^*(\omega) = g_\omega \omega_Y$, and Jac_h is defined as the ideal sheaf which is (locally) generated by these g_ω.

(ii) When also X is nonsingular and $\alpha = ord_t D$ for some effective divisor D on X, we can rewrite the formula as follows :

$$\int_A \mathbb{L}^{-ord_t D} d\mu = \int_{h^{-1}A} \mathbb{L}^{-ord_t(h^*D + K_{Y|X})} d\mu.$$

Here h^*D is the pullback of D, i.e. locally given by the equation $f \circ h$, if D is given by the equation f. And $K_{Y|X}$ is the relative canonical divisor, which is precisely the effective divisor with equation the Jacobian determinant. Alternatively, $K_{Y|X} = K_Y - h^*K_X$ where $K_{\boldsymbol{.}}$ denotes the (ordinary) canonical divisor, i.e. the divisor of zeros and poles of a differential d-form.

Note. The birational morphism h above must be proper in order to induce a bijection from $\mathcal{L}(Y)$ to $\mathcal{L}(X)$ outside subsets of measure zero. More precisely, denoting by Exc the exceptional locus of h, we have a bijection from $\mathcal{L}(Y) \setminus \mathcal{L}(Exc)$ to $\mathcal{L}(X) \setminus \mathcal{L}(h(Exc))$. This is an easy consequence of the valuative criterion of properness [Har, Theorem II.4.7].

EXERCISE. Check the change of variables formula in the following special case : h is the blowing-up of a nonsingular X in a nonsingular centre, $A = \mathcal{L}(X)$ and α is the zero function.

4 First applications

4.1. Here we mean by a *Calabi-Yau manifold* M of dimension d a nonsingular complete (=compact) algebraic variety, which admits a nowhere vanishing regular differential d-form ω_M. Alternative formulations of this last condition are that the first Chern class of the tangent bundle of M is zero, or that the canonical divisor K_M of M is zero.

Theorem [Ko]**.** *Let X and Y be birationally equivalent Calabi-Yau manifolds. Then $[X] = [Y]$ in $\hat{\mathcal{M}}_{\mathbb{C}}$.*

Proof. Since X and Y are birationally equivalent there exist a nonsingular complete algebraic variety Z and birational morphisms $h_X : Z \to X$ and $h_Y : Z \to Y$. By the definition of the motivic measure and the change of variables formula we have in $\hat{\mathcal{M}}_{\mathbb{C}}$:

$$[X] = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(X)} 1 d\mu = \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t K_{Z|X}} d\mu = \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t K_Z} d\mu$$

and of course $[Y]$ is given by the same right hand side. Q.E.D.

This implies that birationally equivalent Calabi-Yau manifolds have the same Hodge-Deligne polynomial, meaning that they have the same Hodge numbers. This result was Kontsevich's motivation to invent motivic integration !

The same proof gives the following more general result. Two nonsingular complete algebraic varieties are called *K-equivalent* if there exists a nonsingular complete algebraic variety Z and birational morphisms $h_X : Z \to X$ and $h_Y : Z \to Y$ such that $h_X^* K_X = h_Y^* K_Y$. This is an important notion in birational geometry.

Theorem. *Let X and Y be K-equivalent varieties. Then $[X] = [Y]$ in $\hat{\mathcal{M}}_{\mathbb{C}}$.*

4.2. Let $h : Y \to X$ be a proper birational morphism between nonsingular algebraic varieties. We assume that the exceptional locus *Exc* of h, i.e. the subvariety of Y where h is not an isomorphism, is a normal crossings divisor. Let $E_i, i \in S$, be the irreducible components of *Exc*. The relative canonical divisor $K_{Y|X}$ is supported on *Exc*; let $\nu_i - 1$ be the multiplicity of E_i in this divisor, so $K_{Y|X} = \sum_{i \in S}(\nu_i - 1)E_i$. Denoting $E_I^\circ := (\cap_{i \in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$ for $I \subset S$, we have

$$[X] = \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_i} - 1} = \sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{1}{[\mathbb{P}^{\nu_i - 1}]}$$

in $\hat{\mathcal{M}}_{\mathbb{C}}$. Indeed, by the change of variables formula we have again that

$$[X] = \mu(\mathcal{L}(X)) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t K_{Y|X}} d\mu,$$

and then Proposition 3.6 yields the stated formula. Specializing to the topological Euler characteristic yields the remarkable formula

$$\chi(X) = \sum_{I \subset S} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i},$$

which was first surprisingly obtained in [DL1], using p-adic integration and the Grothendieck-Lefschetz trace formula.

5 Motivic volume

Here X is again any algebraic variety of pure dimension d.

5.1. Definition. The *motivic volume* of X is $\mu(\mathcal{L}(X)) \in \hat{\mathcal{M}}_{\mathbb{C}}$, thus the motivic measure of the whole arc space of X. Recall that $\mu(\mathcal{L}(X)) = \lim_{n \to \infty} \frac{[\pi_n(\mathcal{L}(X))]}{\mathbb{L}^{nd}}$, and that it equals $[X]$ when X is nonsingular.

We computed in (3.3) the motivic volume of $X = \{xy = 0\}$ as $\mu(\mathcal{L}(X)) = 2\mathbb{L}$ by the defining limit procedure. For more complicated

X, the following formula in terms of a suitable resolution of singularities is very useful.

5.2. Theorem [DL3]. *Let $h : Y \to X$ be a log resolution of X; i.e. h is a proper birational morphism from a nonsingular Y such that the exceptional locus Exc of h is a normal crossings divisor. Assume also that the image of $h^*(\Omega^d_X)$ in Ω^d_Y is locally principal, i.e. locally generated by one element.*

Denote by $E_i, i \in S$, the irreducible components of Exc, and let $\rho_i - 1$ be the multiplicity along E_i of the divisor associated to $h^*(\Omega^d_X)$, i.e. the (effective) divisor locally given by the zeroes of a generator of $h^*(\Omega^d_X)$. Finally, set $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$ for $I \subset S$. Then

$$\mu(\mathcal{L}(X)) = \sum_{I\subset S} [E_I^\circ] \prod_{i\in I} \frac{\mathbb{L}-1}{\mathbb{L}^{\rho_i}-1} = \sum_{I\subset S} [E_I^\circ] \prod_{i\in I} \frac{1}{[\mathbb{P}^{\rho_i-1}]}$$

in $\hat{\mathcal{M}}_{\mathbb{C}}$; in particular $\mu(\mathcal{L}(X))$ belongs to the subring of $\hat{\mathcal{M}}_{\mathbb{C}}$, obtained from (the image of) $\mathcal{M}_{\mathbb{C}}$ by inverting the elements $1+\mathbb{L}+\cdots+\mathbb{L}^j = [\mathbb{P}^j]$.

We will denote this subring by $\mathcal{M}_{loc}$.

5.3. *Example.* Let $X = \{y^2 - x^3 = 0\}$ in $\mathbb{A}^2$. We take $\mathbb{A}^1 \to X : u \mapsto (u^2, u^3)$ as a log resolution. Since Ω^1_X is generated by dx and dy (subject to the relation $2ydy = 3x^2dx$), one easily verifies that $h^*\Omega^1_X$ is generated by udu. Hence the image of $h^*\Omega^1_X$ in Ω^1_Y is principal and we can apply Theorem 5.2.

Note that $Exc = E_1 = \{0\}$, occurring with multiplicity 1 in the divisor of udu. So $\rho_1 = 2$ and

$$\mu(\mathcal{L}(X)) = \mathbb{L} - 1 + \frac{1}{[\mathbb{P}^1]} = \frac{\mathbb{L}^2}{\mathbb{L}+1}.$$

(Recall that $[X] = \mathbb{L}$.)

5.4. *Example.* Let $X = \{z^2 = xy\}$ in $\mathbb{A}^3$.

Exercise. (i) Verify that $\mu(\mathcal{L}(X)) = \mathbb{L}^2$. (The 'obvious' log resolution satisfies the assumption of Theorem 5.2, and the unique component E_1 of the exceptional locus has $\rho_1 = 2$.)

(ii) Note that also $[X] = \mathbb{L}^2$; this could be interpreted as the singularity of X being 'very mild'.

5.5. Exercise. Compute again that the motivic volume of $X = \{xy = 0\}$ is $2\mathbb{L}$; now using Theorem 5.2. (Note here that $[X] = 2\mathbb{L} - 1$; one could say that the motivic volume counts the double point twice.)

5.6. Recall that for nonsingular X its universal Euler characteristic $[X] \in K_0(Var_{\mathbb{C}})$ specializes to its Hodge-Deligne polynomial $H(X) \in \mathbb{Z}[u, v]$ and further to $\chi(X) \in \mathbb{Z}$.

Since $\chi(\cdot)$ and $H(\cdot)$ factor through the image of $\mathcal{M}_{\mathbb{C}}$ in $\hat{\mathcal{M}}_{\mathbb{C}}$, they induce natural maps $\chi : \mathcal{M}_{loc} \to \mathbb{Q}$ and $H : \mathcal{M}_{loc} \to \mathbb{Z}[[u, v]]$. Applying these specialization maps to the motivic measure of X yields new (numerical) singularity invariants, which generalize the usual $\chi(X)$ and $H(X)$ for nonsingular X. Denef and Loeser call $\chi(\mu(\mathcal{L}(X)))$ the *arc-Euler characteristic* of X.

For example the arc-Euler characteristic of $\{y^2 - x^3 = 0\}$ is $\frac{1}{2}$ and the one of $\{xy = 0\}$ is 2.

6 Motivic zeta functions

In this section M is a nonsingular irreducible algebraic variety of dimension m, and $f : M \to \mathbb{C}$ is a non-constant regular function.

6.1. For each $n \in \mathbb{N}$ the morphism $f : M \to \mathbb{A}^1 = \mathbb{C}$ induces a morphism $f_n : \mathcal{L}_n(M) \to \mathcal{L}_n(\mathbb{A}^1)$. A point $\alpha \in \mathcal{L}_n(\mathbb{A}^1)$ corresponds to an element $\alpha(t) \in \mathbb{C}[t]/(t^{n+1})$; we denote as usual the largest e such that t^e divides $\alpha(t)$ by $ord_t\alpha \in \{0, 1, \cdots, n, +\infty\}$. We set

$$\mathcal{X}_n := \{\gamma \in \mathcal{L}_n(M) \mid ord_t f_n(\gamma) = n\} \quad \text{for } n \in \mathbb{N};$$

it is a locally closed subvariety of $\mathcal{L}_n(M)$.

Exercise. Denote $X := \{f = 0\}$. Then $[\mathcal{X}_n] = \mathbb{L}^m[\mathcal{L}_{n-1}(X)] - [\mathcal{L}_n(X)]$ for $n \geq 1$, and $[\mathcal{X}_0] = [M] - [X]$.

Definition. The *motivic zeta function* of $f : M \to \mathbb{C}$ is the formal power series

$$Z(T) := \sum_{n \geq 0} [\mathcal{X}_n](\mathbb{L}^{-m}T)^n$$

in $\mathcal{M}_{\mathbb{C}}[[T]]$.

6.2. Considering the exercise above, it is not a surprise that for $X := \{f = 0\}$ the series $J(T) = \sum_{n\geq 0}[\mathcal{L}_n(X)]T^n$ and $Z(T)$ determine each other. Indeed, one easily verifies that

$$J(T) = \frac{Z(\mathbb{L}^m T) - [M]}{\mathbb{L}^m T - 1}\,.$$

6.3. The definition of $Z(T)$ is inspired by the p-adic Igusa zeta function, associated to a polynomial $f \in \mathbb{Z}_p[x_1, \cdots, x_m]$, which is defined as

$$Z_p(s) := \int_{\mathbb{Z}_p^m} |f(x)|_p^s |dx|$$

for $s \in \mathbb{C}$ with $\Re(s) > 0$. Recall that each $z \in \mathbb{Z}_p \setminus \{0\}$ can be expressed as $z = p^\ell u$ with $\ell \in \mathbb{Z}_{\geq 0}$ and $u \in \mathbb{Z}_p^\times$. One denotes $\mathrm{ord}_p(z) := \ell$ and $|z|_p := p^{-\mathrm{ord}_p z} = p^{-\ell}$. To compare with 6.1, note that $Z_p(s)$ can be rewritten as

$$\begin{aligned} Z_p(s) &= \sum_{n\geq 0} volume\{x \in \mathbb{Z}_p^m \mid ord_p f(x) = n\} p^{-ns} \\ &= \frac{1}{p^m} \sum_{n\geq 0} \#\{x \in (\mathbb{Z}/p^{n+1}\mathbb{Z})^m \mid ord_p f(x) = n\} (p^{-m}p^{-s})^n. \end{aligned}$$

6.4. Exercise. Write D for the (effective) divisor of zeros of f, i.e. D is "$\{f = 0\}$ with multiplicities". Then

$$\int_{\mathcal{L}(M)} \mathbb{L}^{-ord_t D} d\mu = Z(\mathbb{L}^{-1})$$

in $\hat{\mathcal{M}}_{\mathbb{C}}$, meaning in particular that the substitution in the right hand side yields a well-defined element of $\hat{\mathcal{M}}_{\mathbb{C}}$.

6.5. As for the motivic volume, there is an important (similar) formula for $Z(T)$ in terms of a resolution.

Theorem [DL2]. *Let $h : Y \to M$ be an embedded resolution of $\{f = 0\}$; i.e. h is a proper birational morphism from a nonsingular Y such that h is an isomorphism on $Y \setminus h^{-1}\{f = 0\}$ and $h^{-1}\{f = 0\}$ is a normal crossings divisor. Let $E_i, i \in S$, be the irreducible components* *of $h^{-1}\{f = 0\}$. For $i \in S$ we denote by N_i the multiplicity of E_i in the divisor of $f \circ h$ on Y, and by $\nu_i - 1$ the multiplicity of E_i in the divisor of $h^*\omega$, where ω is a local generator of Ω^m_M. (Equivalently: $div(f \circ h) = \sum_{i\in S} N_i E_i$ and $K_{Y|M} = \sum_{i\in S}(\nu_i - 1)E_i$.) Set finally $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{\ell\notin I} E_\ell)$ for $I \subset S$. Then*

$$Z(T) = \sum_{I\subset S}[E_I^\circ]\prod_{i\in I}\frac{(\mathbb{L}-1)T^{N_i}}{\mathbb{L}^{\nu_i} - T^{N_i}};$$

in particular $Z(T)$ is rational and belongs more precisely to the subring of $\mathcal{M}_{\mathbb{C}}[[T]]$ generated by $\mathcal{M}_{\mathbb{C}}$ and the elements $\frac{T^N}{L^\nu - T^N}$, where $\nu, N \in \mathbb{Z}_{>0}$.

6.6. *Corollaries.*

(i) In the special case that $X = \{f = 0\}$ is a hypersurface this yields the stated rationality of $J(T)$ in (2.12).

(ii) Let $M = \mathbb{A}^m$ and $f \in \mathbb{Z}[x_1, \cdots, x_m]$. Then by a similar formula of Denef [De2] for the p-adic Igusa zeta functions $Z_p(s)$, Theorem 6.5 yields that $Z(T)$ specializes to the $Z_p(s)$ for all p except a finite number. See [DL2] for a precise statement. Similarly $J(T)$ specializes to $J_p(T)$ for all p except a finite number [DL8, Theorem 6.1].

(iii) For any $f : M \to \mathbb{C}$ we now explain how $Z(T)$ specializes to the *topological zeta function* of f. Using Theorem 6.5 and the notation there, we evaluate $Z(T)$ at $T = \mathbb{L}^{-s}$ for any $s \in \mathbb{N}$; this yields the well-defined elements

$$\sum_{I\subset S}[E_I^\circ]\prod_{i\in I}\frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i+sN_i}-1} = \sum_{I\subset S}[E_I^\circ]\prod_{i\in I}\frac{1}{[\mathbb{P}^{\nu_i+sN_i-1}]}$$

in (the image in $\hat{\mathcal{M}}_{\mathbb{C}}$ of) the localization of $\mathcal{M}_{\mathbb{C}}$ with respect to the elements $[\mathbb{P}^j]$. Applying the Euler characteristic specialization map $\chi(\cdot)$ yields the rational numbers

$$\sum_{I\subset S}\chi(E_I^\circ)\prod_{i\in I}\frac{1}{\nu_i + sN_i}$$

for $s \in \mathbb{N}$. The *topological zeta function* $Z_{top}(s)$ of f is the unique rational function in one variable s admitting the values above for $s \in \mathbb{N}$.

Without the specialization argument above it is not at all clear that $Z_{top}(s)$ does not depend on the chosen resolution $h : Y \to M$. In fact $Z_{top}(s)$ was first introduced in [DL1], in terms of a resolution, and p-adic

Igusa zeta functions and the Grothendieck-Lefschetz trace formula were needed to prove independence of the chosen resolution.

6.7. We just mention that there is an important generalization of the motivic zeta function, working over a relative and equivariant Grothendieck ring; it specializes by a limit procedure to objects in (an equivariant version of) $\mathcal{M}_{\mathbb{C}}$, which are shown to be a good *virtual motivic incarnation* of the Milnor fibres of f at the points of $\{f = 0\}$. It is quite remarkable that a definitely non-algebraic notion as the Milnor fibre has such an algebraic incarnation. See [DL2][DL7].

Moreover these objects satisfy a *motivic Thom-Sebastiani Theorem*, generalizing the known results of Varchenko and Saito. See [DL4].

6.8. *Monodromy Conjecture.*

There is an intriguing conjectural relation between the poles of the topological zeta function and the eigenvalues of the local monodromy of f.

Monodromy conjecture. *If s_0 is a pole of $Z_{top}(s)$, then $e^{2\pi i s_0}$ is an eigenvalue of the local monodromy action on the cohomology of the Milnor fibre of f at some point of $\{f = 0\}$.*

One can also state the analogous conjecture for the motivic zeta function, but then one has to be careful with the notion of pole, see [RV2]. Alternatively, we can formulate this monodromy conjecture for $Z(T)$ as follows, without mentioning poles [DL2] :

$Z(T)$ belongs to the ring generated by $\mathcal{M}_{\mathbb{C}}$ and the elements $\frac{T^N}{L^\nu - T^N}$, where $\nu, N \in \mathbb{Z}_{>0}$ and $e^{2\pi i \frac{\nu}{N}}$ is an eigenvalue of the local monodromy as above.

Actually, it was originally stated for the p-adic Igusa zeta function, being even more remarkable, for then it relates number theoretical invariants of $f \in \mathbb{Z}[x_1, \cdots, x_m]$ to differential topological invariants of f, considered as function $\mathbb{C}^n \to \mathbb{C}$.

The conjecture was shown by Loeser for $M = \mathbb{A}^2$ [Loe1]; a shorter proof in dimension 2 is in [Ro]. In dimension 3 there is a lot of 'experimental evidence' [Ve1], and by now various special cases are proved [ACLM1][ACLM2][Loe2][RV1].

Example. Let $M = \mathbb{A}^2$ and $f = y^2 - x^3$.
EXERCISE. Compute, using Theorem 6.5,

$$Z(T) = \mathbb{L}^2(\mathbb{L}-1)\frac{\mathbb{L}^5 - \mathbb{L}^3T + \mathbb{L}^3T^2 - T^5}{(\mathbb{L}^5 - T^6)(\mathbb{L}-T)}$$

and

$$Z_{top}(s) = \frac{5+4s}{(5+6s)(1+s)}.$$

(This is how we computed $J(T)$ in Example 2.14.) In particular, the poles of $Z_{top}(s)$ are -1 and $-5/6$. On the other hand, it is well known that the monodromy eigenvalues of f are $1, e^{\frac{\pi i}{3}}$, and $e^{-\frac{\pi i}{3}}$. Hence the monodromy conjecture is indeed satisfied here.

Note. The previous example was too simple to exhibit the 'typical' situation. Each irreducible component E_i in Theorem 6.5 induces a candidate-pole $-\frac{\nu_i}{N_i}$, and quite miraculously, for a generic example with a lot of components E_i, 'most' of these candidates cancel. This experimental fact is compatible with the monodromy conjecture, see [Ve1].

7 Batyrev's stringy invariants

Using motivic integration, Batyrev [Ba1][Ba2] introduced new singularity invariants for algebraic varieties with 'mild' singularities, more precisely with at worst log terminal singularities. He used them for instance to formulate a topological mirror symmetry test for singular Calabi-Yau varieties, to give a conjectural definition for stringy Hodge numbers, and to prove a version of the McKay correspondence.

We first explain log terminal and related singularities; for this we need the Gorenstein notion.

7.1. Let X be a *normal* algebraic variety of dimension d. In particular X is irreducible, X_{sing} has codimension at least 2 in X, and X has a well defined canonical divisor K_X (up to linear equivalence). One can view (a representative of) K_X as the divisor of zeroes and poles of a *rational* differential d-form on X; it is also the Zariski-closure of the usual canonical divisor on X_{reg}.

When X is nonsingular, K_X is a Cartier divisor, i.e. locally given by one equation. This is not true in general.

Definition. A normal variety X is *Gorenstein* if K_X is a Cartier divisor. Alternatively : X is Gorenstein if the rational differential d-forms on X, which are regular on X_{reg}, are locally generated by one element.

Example. Let $X = \{z^2 = xy\}$; then those differential 2-forms are generated by $\frac{dx\wedge dy}{2z} = \frac{dx\wedge dz}{x} = -\frac{dy\wedge dz}{y}$ (which is indeed regular on X_{reg}).

This notion is quite general; for instance all (normal) hypersurfaces and even complete intersections are Gorenstein.

Note. In the literature one often uses the term *Gorenstein* alternatively for varieties X for which all local rings $\mathcal{O}_{X,x} (x \in X)$ are Gorenstein rings, and then the property that K_X is a Cartier divisor is called *1-Gorenstein.*

7.2. We now introduce a certain 'badness' for singularities, in terms of numerical invariants of a resolution.

Let X be Gorenstein of dimension d. Take a log resolution $\pi : Y \to X$ of X and denote by $E_i, i \in S$, the irreducible components of the exceptional locus Exc of h. We associate as follows an integer a_i to each E_i.

(1) *Description with divisors.* Since K_X is Cartier, the pullback $\pi^* K_X$ makes sense and one can consider the relative canonical divisor $K_{Y|X} = K_Y - \pi^* K_X$, which is supported on Exc. Then $a_i - 1$ is the multiplicity of E_i in $K_{Y|X}$, i.e. $K_{Y|X} = \sum_{i\in S}(a_i - 1)E_i$.

(2) *Description with differential forms.* Take a general point Q_i of E_i and local coordinates $y_1, y_2, \cdots, y_d$ around Q_i such that the local equation of E_i is $y_1 = 0$. Let ω_i be a local generator around $\pi(Q_i)$ of the d-forms on X, which are regular on X_{reg}. (Such an ω_i exists by the Gorenstein property.) Then around Q_i one can write $\pi^*\omega_i$ as

$$\pi^*\omega_i = u y_1^{a_i - 1} dy_1 \wedge dy_2 \wedge \cdots \wedge dy_d,$$

where u is regular and nonzero around Q_i.

In general the $a_i \in \mathbb{Z}$, and when X is nonsingular they satisfy $a_i \geq 2$.

Terminology. One calls a_i the *log discrepancy* of E_i with respect to X (and $a_i - 1$ the *discrepancy*).

Example. The standard log resolution of $X = \{z^2 = xy\}$ has one exceptional curve $E \cong \mathbb{P}^1$ with log discrepancy $a = 1$.

7.3. We also have to consider a technical generalization : the normal variety X is called $\mathbb{Q}$-*Gorenstein* if rK_X is Cartier for some $r \in \mathbb{Z}_{>0}$. In this case the log discrepancies are defined analogously by $K_{Y|X} = \sum_{i \in S}(a_i - 1)E_i$, which should be considered as an abbreviation of $rK_{Y|X} = rK_Y - rK_X = \sum_{i \in S} r(a_i - 1)E_i$. Now the $r(a_i - 1) \in \mathbb{Z}$, and hence $a_i \in \frac{1}{r}\mathbb{Z}$.

Example. Let X be the quotient of $\mathbb{A}^2$ by the action of $\mu_3 = \{z \in \mathbb{C} \mid z^3 = 1\}$ given by $(x, y) \mapsto (\epsilon x, \epsilon y)$ for $\epsilon \in \mu_3$. Concretely, X is given in $\mathbb{A}^4$ by the equations

$$\{u_1 u_3 - u_2^2 = u_2 u_4 - u_3^2 = u_1 u_4 - u_2 u_3 = 0\},$$

in particular it is *not* a complete intersection. Here K_X is not Cartier; a representative of K_X is for example $\{u_1 = u_2 = u_3 = 0\}$. However, $3K_X$ is Cartier; a representative is $\{u_1 = 0\}$.

The standard log resolution of X has one exceptional curve $E \cong \mathbb{P}^1$ with log discrepancy $a = \frac{2}{3}$.

A nice introduction to these notions is in [Re1].

7.4. Definition. (i) Let X be a $\mathbb{Q}$-Gorenstein variety. Take a log resolution $\pi : Y \to X$ of X; let $E_i, i \in S$, be the irreducible components of the exceptional locus of π with log discrepancies a_i. Then X is called *terminal, canonical, log terminal* and *log canonical* if $a_i > 1, a_i \geq 1$, $a_i > 0$ and $a_i \geq 0$, respectively, for all $i \in S$.

One can show that these conditions do not depend on the chosen resolution.

(ii) We say that X is *strictly log canonical* if it is log canonical but not log terminal.

We should note that 0 is indeed the relevant 'border value' here; if some $a_i < 0$ on some log resolution, then one can easily construct log resolutions with arbitrarily negative a_i.

The log terminal singularities should be considered 'mild', the singularities which are not log canonical 'general', and the strictly log canonical ones as a special 'border' class.

7.5. *Example.* (1) When X is a surface ($d = 2$) terminal is equivalent to non-singular, the canonical singularities are precisely the so-called ADE singularities or rational double points, and the log terminal singularities are precisely the Hirzebruch-Jung or quotient singularities.

(2) Let $X = \{x_1^k + x_2^k + \cdots + x_{d+1}^k = 0\}$ in $\mathbb{A}^{d+1}$. The origin is the only singular point of X, and the blowing-up with the origin as centre yields a log resolution $\pi : Y \to X$ of X with exceptional locus consisting of one irreducible component E, which is isomorphic to $\{x_1^k + x_2^k + \cdots + x_{d+1}^k = 0\} \subset \mathbb{P}^d$.

Exercise. (i) The log discrepancy of E with respect to X is $d + 1 - k$.

(ii) X is log terminal, strictly log canonical, and not log canonical when $k < d + 1, k = d + 1$, and $k > d + 1$, respectively.

7.6. There are nice results of Ein, Mustaţă and Yasuda, relating the previous notions with jet spaces.

Theorem [Mu1][EMY][EM]**.** *Let X be a normal variety, which is locally a complete intersection. Then X is terminal, canonical, and log canonical if and only if $\mathcal{L}_n(X)$ is normal, irreducible, and equidimensional, respectively, for every n.*

7.7. Definition. Let X be a log terminal algebraic variety. Take a log resolution $\pi : Y \to X$ of X. Let $E_i, i \in S$, be the irreducible components of the exceptional locus of π with log discrepancies a_i ($\in \mathbb{Q}_{>0}$). Denote also $E_I^\circ := (\cap_{i\in I} E_i) \setminus (\cup_{\ell \notin I} E_\ell)$ for $I \subset S$.

(i) The *stringy Euler number* of X is

$$e_{st}(X) := \sum_{I\subset S} \chi(E_I^\circ) \prod_{i\in I} \frac{1}{a_i}.$$

(ii) The *stringy E-function* of X is

$$E_{st}(X) := \sum_{I\subset S} H(E_I^\circ) \prod_{i\in I} \frac{uv - 1}{(uv)^{a_i} - 1}.$$

(iii) The *stringy $\mathcal{E}$-invariant* of X is

$$\mathcal{E}_{st}(X) := \sum_{I\subset S} [E_I^\circ] \prod_{i\in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1}.$$

Remarks. (1) Clearly $e_{st}(X) \in \mathbb{Q}; E_{st}(X)$ is a rational function in u, v (with 'fractional powers'), and $\mathcal{E}_{st}(X)$ lives in a finite extension of $\hat{\mathcal{M}}_{\mathbb{C}}$. We have specialization maps $\mathcal{E}_{st}(X) \mapsto E_{st}(X) \mapsto e_{st}(X)$.

(2) Strictly speaking, Batyrev defined and used only the levels (i) and (ii) [Ba2][Ba3].

When X is nonsingular, $\mathcal{E}_{st}(X) = [X]$ (this is 4.2), and of course $E_{st}(X) = H(X)$ and $e_{st}(X) = \chi(X)$. So also these invariants are new singularity invariants, generalizing $[\cdot], H(\cdot)$ and $\chi(\cdot)$, respectively, for nonsingular X. (Just as the motivic volume and its specializations. We give a comparing example in 7.11.)

7.8. The crucial point is that the defining expressions above do not depend on the chosen resolution. We indicate three different arguments, supposing for simplicity that X is Gorenstein, i.e. the $a_i \in \mathbb{Z}_{>0}$.

(1) Let $\pi : Y \to X$ and $\pi' : Y' \to X$ be two log resolutions of X. By the formula of Proposition 3.6 we have in fact

$$\sum_{I \subset S} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_i} - 1} = \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t K_{Y|X}} d\mu.$$

So we must show that $\int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t K_{Y|X}} d\mu = \int_{\mathcal{L}(Y')} \mathbb{L}^{-ord_t K_{Y'|X}} d\mu$. To this end we take a log resolution $\rho : Z \to X$, dominating π and π'; i.e. we have $\rho : Z \xrightarrow{\sigma} Y \xrightarrow{\pi} X$ and $\rho : Z \xrightarrow{\sigma'} Y' \xrightarrow{\pi'} X$. By the change of variables formula in (3.8) we have

$$\begin{aligned} \int_{\mathcal{L}(Y)} \mathbb{L}^{-ord_t K_{Y|X}} d\mu &= \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t (\sigma^* K_{Y|X} + K_{Z|Y})} d\mu \\ &= \int_{\mathcal{L}(Z)} \mathbb{L}^{-ord_t (K_{Z|X})} d\mu \,, \end{aligned}$$

and of course the same is true for the integral over $\mathcal{L}(Y')$.

This is essentially Batyrev's proof.

(2) We can define $\mathcal{E}_{st}(X)$ intrinsically, using motivic integration on X [Ya1][DL6]. There is an ideal sheaf $\mathcal{I}_X$ on X such that

$$\mathcal{E}_{st}(X) = \int_{\mathcal{L}(X)} \mathbb{L}^{ord_t \mathcal{I}_X} d\mu,$$

using the setting of (3.5) and (3.7). More precisely, denoting by ω_X the sheaf of differential d-forms on X which are regular on X_{reg}, we have a natural map $\Omega_X^d \to \omega_X$ whose image is $\mathcal{I}_X \omega_X$. See [Ya1, Lemma 1.16].

(3) Using the Weak Factorization Theorem, see below, one essentially has to show that the defining expressions in (7.7) do not change after blowing-up Y in a nonsingular centre which intersects $\cup_{i\in S} E_i$ transversely. This is straightforward.

7.9. Weak Factorization Theorem [AKMW][Wł].

(1) Let $\phi : Y \dashrightarrow Y'$ be a proper birational map between nonsingular irreducible varieties, and let $U \subset Y$ be an open set where ϕ is an isomorphism. Then ϕ can be factored as follows into a sequence of blow–ups and blow–downs with smooth centres disjoint from U.

There exist nonsingular irreducible varieties $Y_1, \dots, Y_{\ell-1}$ and a sequence of birational maps

$$Y = Y_0 \overset{\phi_1}{\dashrightarrow} Y_1 \overset{\phi_2}{\dashrightarrow} \cdots \overset{\phi_{i-1}}{\dashrightarrow} Y_{i-1} \overset{\phi_i}{\dashrightarrow} Y_i \overset{\phi_{i+1}}{\dashrightarrow} \cdots \overset{\phi_{\ell-1}}{\dashrightarrow} Y_{\ell-1} \overset{\phi_\ell}{\dashrightarrow} Y_\ell = Y'$$

where $\phi = \phi_\ell \circ \phi_{\ell-1} \circ \cdots \circ \phi_2 \circ \phi_1$, such that each ϕ_i is an isomorphism over U (we identify U with an open in the Y_i), and for $i = 1, \dots, \ell$ either $\phi_i : Y_{i-1} \dashrightarrow Y_i$ or $\phi_i^{-1} : Y_i \dashrightarrow Y_{i-1}$ is the blowing–up at a nonsingular centre disjoint from U, and is thus a morphism.

(1′) There is an index i_0 such that for all $i \le i_0$ the map $Y_i \to Y$ is a morphism, and for $i \ge i_0$ the map $Y_i \to Y'$ is a morphism.

(2) If $Y \setminus U$ and $Y' \setminus U$ are normal crossings divisors, then the factorization above can be chosen such that the inverse images of these divisors under $Y_i \to Y$ or $Y_i \to Y'$ are also normal crossings divisors, and such that the centres of blowing–up of the ϕ_i or ϕ_i^{-1} intersect these divisors transversely.

Remark. (i) In [AKMW] and [Wł] the theorem is stated for a birational map ϕ between complete Y and Y'; the generalization to proper birational maps between not necessarily complete Y and Y' is mentioned by Bonavero [Bo].

(ii) In [AKMW, Theorem 0.3.1] the first claim of (2) is not explicitly stated, but can be read off from the proof (see [AKMW, 5.9 and 5.10]).

7.10. *Important Intermezzo.* Using weak factorization instead of motivic integration, we can define $\mathcal{E}_{st}(X)$ in a localization of (a finite extension of) $\mathcal{M}_{\mathbb{C}}$, which is a priori finer than in (a finite extension of) $\hat{\mathcal{M}}_{\mathbb{C}}$, since we do not know whether the natural map $\mathcal{M}_{\mathbb{C}} \to \hat{\mathcal{M}}_{\mathbb{C}}$ is injective.

This remark also applies e.g. to (4.1), yielding $[X] = [Y]$ in the localization of $\mathcal{M}_{\mathbb{C}}$ with respect to the $[\mathbb{P}^j]$ instead of merely in $\hat{\mathcal{M}}_{\mathbb{C}}$.

7.11. *Example.* Let $X = \{x_1^k + x_2^k + \cdots + x_{d+1}^k = 0\} \subset \mathbb{A}^{d+1}$.
Exercise. We use the notation E of Example 7.5.
(i) $\mathcal{E}_{st}(X) = (\mathbb{L}-1)[E] + [E]\frac{\mathbb{L}-1}{\mathbb{L}^{d+1-k}-1}$,
(ii) $\mu(\mathcal{L}(X)) = (\mathbb{L}-1)[E] + [E]\frac{\mathbb{L}-1}{\mathbb{L}^{d}-1}$,
(iii) $[X] = (\mathbb{L}-1)[E] + 1$.

(Note also that (ii) and (iii) are consistent with Example 5.4.)

7.12. *Applications.*
(i) Topological mirror symmetry test for singular Calabi-Yau mirror pairs [Ba2].
(ii) A conjectural definition of stringy Hodge numbers for certain canonical Gorenstein varieties [Ba2].
(iii) A proof of a version of the McKay correspondence [Ba3][DL6][Ya1].
(iv) A new birational invariant for varieties of nonnegative Kodaira dimension, assuming the Minimal Model Program [Ve2, (2.8)].

8 Stringy invariants for general singularities

In this section X is a $\mathbb{Q}$-Gorenstein variety.

8.1. For a log resolution $\pi : Y \to X$ of X, we use the notation E_i and $a_i, i \in S$, and $E_I^\circ, I \subset S$, as before. There are (at least) two natural questions concerning a possible generalization of Batyrev's stringy invariants beyond the log terminal case.

Question I. Suppose there exists at least one log resolution $\pi : Y \to X$ of X for which *all* log discrepancies $a_i \neq 0$. Is (e.g.)

$$\sum_{I \subset S} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{a_i}$$

independent of a chosen such resolution ?

This question is still open (a positive answer would yield a generalized stringy invariant for those X admitting such a log resolution). Note that, when using the weak factorization theorem to connect two such log resolutions by chains of blowing-ups, log discrepancies on 'intermediate varieties' could be zero, obstructing an obvious attempt of proof.

QUESTION II. Do there exist any kind of invariants, associated to all or 'most' $\mathbb{Q}$-Gorenstein varieties, which coincide with Batyrev's stringy invariants if the variety is log terminal ?

Concerning this question, we obtained the following result [Ve4]. We associated invariants to 'almost all' $\mathbb{Q}$-Gorenstein varieties, more precisely to all $\mathbb{Q}$-Gorenstein varieties without strictly log canonical singularities, which do generalize Batyrev's invariants for log terminal varieties. (Note that in particular log discrepancies can be zero in a log resolution of a non log canonical variety !)

- To construct these invariants we have to assume Mori's Minimal Model Program (in fact the relative and log version).
- As in the previous section, we can work on any level : $\chi(\cdot), H(\cdot)$, and $[\cdot]$. For simplicity we treat here just the roughest level $\chi(\cdot)$; the other levels are analogous.

8.2. We associate to any $\mathbb{Q}$-Gorenstein X without strictly log canonical singularities a rational function $z_{st}(X;s)$ in one variable s, the *stringy zeta function* of X. It will turn out that for log terminal X, this rational function is in fact a constant and equal to $e_{st}(X)$.

We just present the main idea of our construction. The 'pragmatic' idea is to split the log discrepancies a_i of a log resolution $\pi : Y \to X$ as $a_i = \nu_i + N_i$ such that $(\nu_i, N_i) \neq (0,0)$ for all i, and to define $z_{st}(X;s)$ as

$$\sum_{I \subset S} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i + sN_i} \in \mathbb{Q}(s).$$

This is done in a geometrically meaningful way via factoring π through a certain 'partial resolution' $p : X^m \to X$ of X, which is called a *relative log minimal model of* X. This is a natural object in the (relative, log) Minimal Model Program; important here is that it is not unique and that X^m can have certain mild singularities. (Its existence is the key point in this Program and this is for the moment proved only in dimensions 2 and 3.)

For the specialists : p is a proper birational morphism, X^m is $\mathbb{Q}$-factorial, the pair (X^m, E^m) is divisorial log terminal, and $K_{X^m}+E^m$ is p-nef, where E^m denotes the reduced exceptional divisor of p. References for these notions are e.g. in [KM][KMM][Ma].

We consider the factorization $\pi : Y \xrightarrow{h} X^m \xrightarrow{p} X$. In general h is only a birational map (maybe not everywhere defined), but we suppose for the moment that it is a morphism. We justify this later. Denoting as usual by $E_i, i \in S$, the irreducible components of the exceptional divisor of π, we let $E_i^m, i \in S^m$, be the images in X^m of those E_i which 'survive' in X^m, i.e. which are not contracted by h to varieties of smaller dimension. Then

$$\begin{aligned}\sum_{i\in S} a_i E_i =& K_Y + \sum_{i\in S} E_i - \pi^* K_X \\ =& \underbrace{K_Y + \sum_{i\in S} E_i - h^*(K_{X^m} + \sum_{i\in S^m} E_i^m)}_{(1)} \\ & \underbrace{+ h^*(K_{X^m} + \sum_{i\in S^m} E_i^m) - h^* p^* K_X}_{(2)} .\end{aligned}$$

Both (1) and (2) are divisors on Y, supported on $\cup_{i\in S} E_i$. We write (1) as $\sum_{i\in S} \nu_i E_i$; all $\nu_i \geq 0$ because the pair $(X^m, \sum_{i\in S^m} E_i^m)$ has only mild singularities (more precisely, because it is divisorial log terminal). We can rewrite (2) as

$$h^*(K_{X^m} + \sum_{i\in S^m} E_i^m - p^* K_X) = h^*(\sum_{i\in S^m} a_i E_i^m);$$

and it is well known that all $a_i, i \in S^m$, are non-positive (more precisely, this follows since $K_{X^m} + \sum_{i\in S^m} E_i^m$ is p-nef). So we can write (2) as $\sum_{i\in S} N_i E_i$ where all $N_i \leq 0$.

With these definitions of ν_i and N_i we indeed have $a_i = \nu_i + N_i$ for $i \in S$, with moreover $\nu_i \geq 0$ and $N_i \leq 0$. One can show that, if X has no strictly log canonical singularities, the situation $\nu_i = N_i = 0$ *cannot* occur.

When X is log terminal, the morphism $p : X^m \to X$ has *no* exceptional divisors, so $S^m = \emptyset$, all $N_i = 0$ and $\nu_i = a_i$, and as promised $z_{st}(X; s) = e_{st}(X)$.

In fact we FIRST choose a relative log minimal model $p : X^m \to X$ of X, we secondly choose a log resolution $h : Y \to X^m$ of the pair (X^m, E^m), where E^m is the reduced exceptional divisor of p, and then we put $\pi := p \circ h$.

The point is again that $z_{st}(X; s)$ is independent of both choices, for which a crucial ingredient is the Weak Factorization Theorem.

8.3. Theorem [Ve4]. *Let X be any* surface *without strictly log canonical singularities. Then*

$$\lim_{s\to 1} z_{st}(X; s) \in \mathbb{Q}.$$

(Recall that this is non-obvious since some a_i can be zero. The clue is that if $a_i = 0$, then E_i must be rational and must intersect exactly once or twice other components; this then easily implies the cancelling of $\nu_i + sN_i$ in the denominator of $z_{st}(X; s)$.) So we can define in dimension 2 a generalized stringy Euler *number* $e_{st}(X)$ as the limit above for any such surface X. In fact we constructed this generalized $e_{st}(X)$ in [Ve3] by a 'direct' approach.

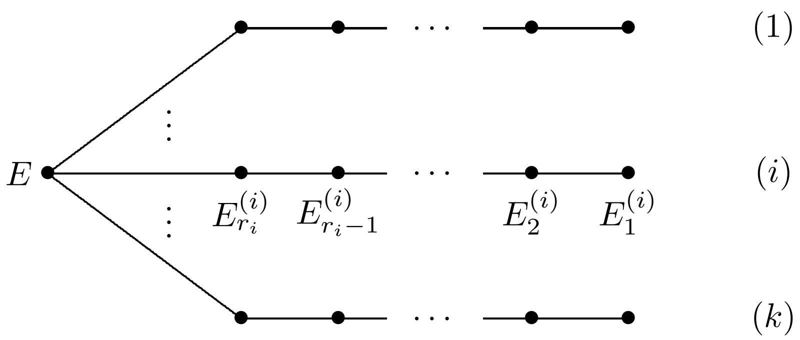

FIGURE 1

8.4. *Example* [Ve3]. Let $P \in X$ be a normal surface singularity with dual graph of its minimal log resolution $\pi : X \to S$ as in Figure 1. There is a central curve E with genus g and self-intersection number $-\kappa$, and all other curves are rational. Each attached chain $E_1^{(i)} - \cdots - E_{r_i}^{(i)}$ is determined by two co-prime numbers n_i and q_i, which are the absolute value of the determinant of the intersection matrix of $E_1^{(i)}, \ldots, E_{r_i}^{(i)}$ and $E_1^{(i)}, \ldots, E_{r_i-1}^{(i)}$, respectively. Finally, we denote by d the absolute value of the determinant of the total intersection matrix of $\pi^{-1}P$. This is a quite large class of singularities; it includes all weighted homogeneous isolated complete intersection singularities, for which the numbers $\{g;\kappa;(n_1,q_1),\cdots,(n_k,q_k)\}$ are called the *Seifert invariants* of the singularity.

If $P \in X$ is not strictly log canonical, then

$$e_{st}(X) = \lim_{s\to 1} z_{st}(X;s) = \frac{1}{a}(2-2g-k+\sum_{i=1}^{k} n_i) + \chi(X \setminus \{P\}) \,,$$

where

$$a = \frac{2-2g-k+\sum_{i=1}^{k}\frac{1}{n_i}}{\kappa - \sum_{i=1}^{k}\frac{q_i}{n_i}} = \frac{\prod_{i=1}^{k} n_i}{d}(2-2g-k+\sum_{i=1}^{k}\frac{1}{n_i})$$

is the log discrepancy of E.

We note that some other log discrepancies might be zero. A particular example is the so-called *triangle singularity*, given by $g=0, \kappa=1$, $k=3$ and $r_1=r_2=r_3=1$. So, concretely, there is a central rational curve with self-intersection -1 to which three other rational curves are attached. Then $a=-1$ and the three other log discrepancies are zero, and $e_{st}(X) = 1-(n_1+n_2+n_3)+\chi(X\setminus\{P\})$.

When such $P \in X$ is a weighted homogeneous isolated *hypersurface* singularity, this generalized stringy Euler number appears in some Taylor expansion associated to it, studied by Némethi and Nicolaescu [NN].

8.5. *Example.* [Ve4] Here we mention a concrete example of a threefold singularity $P \in X$, which has an exceptional surface with log discrepancy zero in a log resolution, and such that nevertheless $\lim_{s\to 1} z_{st}(X;s) \in \mathbb{Q}$, i.e. such that the evaluation $z_{st}(X;1)$ makes sense.

Let X be the hypersurface $\{x^4+y^4+z^4+t^5=0\}$ in $\mathbb{A}^4$; its only singular point is $P=(0,0,0,0)$. We sketch the following constructions in Figure 2; we denote varieties and their strict transforms by the same symbol.

The blowing-up $\pi_1 : Y_1 \to X$ with centre P is already a resolution of X (Y_1 is smooth). Its exceptional surface E_1 is the affine cone over the smooth projective plane curve $C=\{x^4+y^4+z^4=0\}$. Let $\pi_2 : Y_2 \to Y_1$ be the blowing-up with centre the vertex Q of this cone, and exceptional surface $E_2 \cong \mathbb{P}^2$. Then $E_1 \subset Y_2$ is a ruled surface over C which intersects

E_2 in a curve isomorphic to C. The composition $\pi = \pi_1 \circ \pi_2$ is a log resolution of $P \in X$, and one easily verifies that the log discrepancies are $a_1 = 0$ and $a_2 = -1$; in particular $P \in X$ is not log canonical.

Now $E_1 \subset Y_2$ can be contracted (more precisely one can check that the numerical equivalence class of the fibre of the ruled surface E_1 is an extremal ray). Let $h : Y_2 \to X^m$ denote this contraction, and let $\pi = p \circ h$. As the notation suggests, one can verify that $K_{X^m} + E_2$ is p-nef, implying that (X^m, E_2) is a relative log minimal model of $P \in X$.

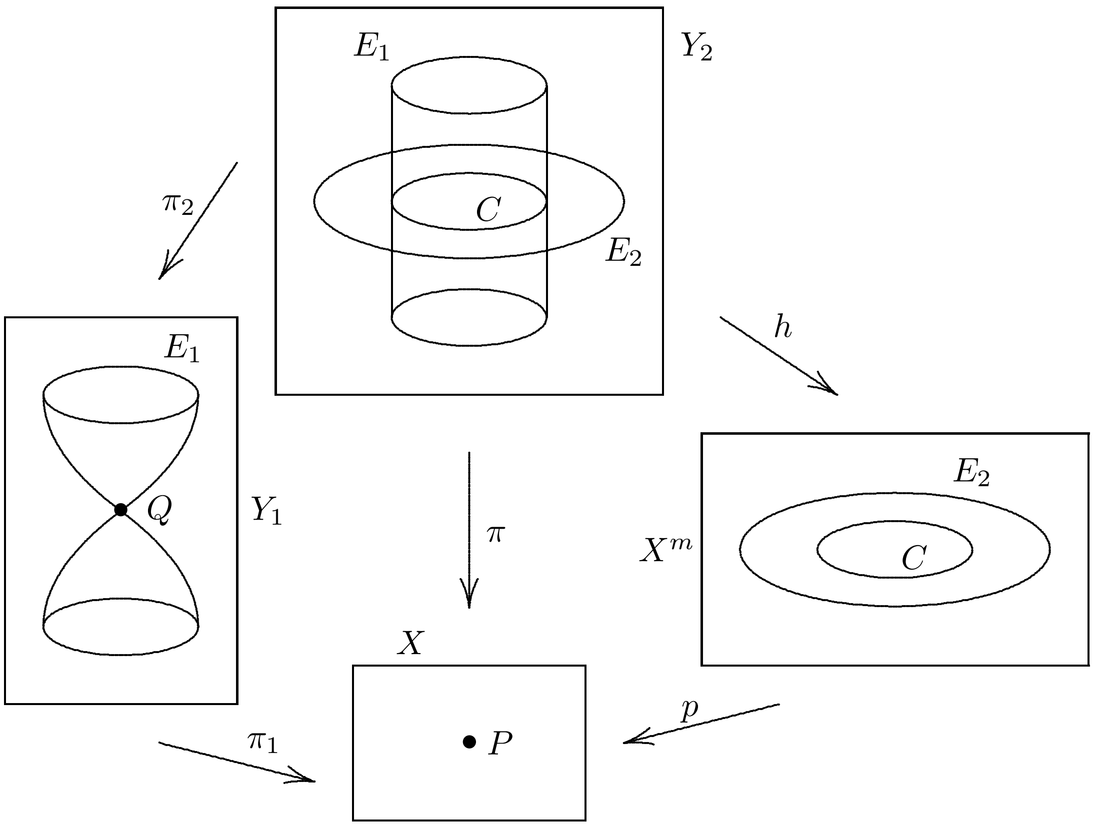

FIGURE 2

Denoting as usual

$$K_{Y_2} = h^*(K_{X^m} + E_2) + (\nu_1 - 1)E_1 + (\nu_2 - 1)E_2$$

and

$$h^*(a_2 E_2) = N_1 E_1 + N_2 E_2$$

we have clearly that $\nu_2 = 0$ and $N_2 = -1$, and one computes that $\nu_1 = \frac{1}{5}$ and $N_1 = -\frac{1}{5}$. So

$$\begin{aligned} &z_{st}(X;s) \\ &= \frac{\chi(C)}{(\nu_1 + sN_1)(\nu_2 + sN_2)} + \frac{\chi(E_1 \setminus C)}{\nu_1 + sN_1} + \frac{\chi(E_2 \setminus C)}{\nu_2 + sN_2} + \chi(X \setminus \{P\}) \\ &= \frac{-4}{(\frac{1}{5} - \frac{1}{5}s)(-s)} + \frac{-4}{\frac{1}{5} - \frac{1}{5}s} + \frac{7}{-s} + \chi(X \setminus \{P\}) = \frac{13}{s} + \chi(X \setminus \{P\}), \end{aligned}$$

yielding $\lim_{s \to 1} z_{st}(X;s) = z_{st}(X;1) = 13 + \chi(X \setminus \{P\})$.

8.6. *Question.* Let X be a $\mathbb{Q}$-Gorenstein variety of arbitrary dimension without strictly log canonical singularities. When is

$$\lim_{s \to 1} z_{st}(X;s) \in \mathbb{Q}\ ?$$

9 Miscellaneous recent results

Here we gather a collection of various results, which were obtained after the redaction of the survey paper [DL8].

• Aluffi noticed in [Al1] that the Euler characteristic formula in (4.2) implies interesting similar statements about Chern-Schwartz-MacPherson classes. Then in [Al2] he studies the birational behavior of Chern classes with respect to the 'motivic integration philosophy'. There he also introduces stringy Chern classes of log terminal varieties, which was done simultaneously by de Fernex, Lupercio, Nevins and Uribe in [dFLNU].

• Bittner [Bi2] calculated the relative dual of the motivic nearby fibre and constructed a nearby cycle morphism on the level of the Grothendieck group of varieties.

• More exotic motivic measures are introduced by Bondal, Larsen and Lunts [BLL] and Drinfeld [Dr].

• Using arc spaces and motivic integration, Budur [Bu] relates the Hodge spectrum of a hypersurface singularity to its *jumping numbers* (which come from multiplier ideals).

• Campillo, Delgado and Gusein-Zade [CDG1][CDG2][CDG3], and Ebeling and Gusein-Zade [EG1][EG2] studied filtrations on the ring of germs of functions on a germ of a complex variety, defined by arcs on the singularity. An important technique is integration with respect to the Euler characteristic over the projectivization of the space of function germs; this notion is similar to (and inspired by) motivic integration.

• Cluckers and Loeser [CL1][CL2][CL3] built a more general theory for relative motivic integrals, avoiding moreover the completion of Grothendieck rings. These integrals specialize to both 'classical' and arithmetic motivic integrals.

More 'relative theory' is in [Ni3].

• Dais and Roczen obtained formulas for the stringy Euler number and stringy E-function for some special classes of singularities [Da][DR].

• Now available are the ICM 2002 survey [DL9] and the recent expository paper of Hales [Hal3] on the theory of arithmetic motivic measure of Denef and Loeser [DL5]. Related work is in [DL10] and [Ni3].

• In [dSL] du Sautoy and Loeser associate motivic zeta functions to a large class of infinite dimensional Lie algebras.

• Ein, Lazarsfeld, Mustaţă and Yasuda have various other papers about spaces of jets, relating them for instance to singularities of pairs, in particular to the log canonical threshold, and to multiplier ideals [ELM][Mu2][Ya2].

• Koike and Parusiński [KP] associated motivic zeta functions to real analytic function germs and showed that these are invariants of blow-analytic equivalence. Fichou [Fi] obtained similar results in the context of Nash funcion germs. Both constructions are useful for classification issues.

• Gordon [Go] introduced a motivic analogue of the Haar measure for the (non locally compact) groups $G(k((t)))$, where G is a reductive algebraic groups, defined over an algebraically closed field k of characteristic zero.

• Guibert [Gui] computed the motivic zeta function associated to irreducible plane curve germs, yielding a new proof of the formula expressing the spectrum in terms of the Puiseux data. Here he studied also a motivic zeta function for a family of functions and related it with the Alexander invariants of the family; this is used to obtain a formula for the Alexander polynomial of a plane curve.

• Guibert, Loeser and Merle [GLM1] introduced iterated motivic vanishing cycles and proved a motivic version of a conjecture of Steenbrink concerning the spectrum of hypersurface singularities.

• Gusein-Zade, Luengo and Melle Hernández [GLM2] treat integration over spaces of non-parametrized arcs and introduce motivic versions of the classical monodromy zeta function. They indicate a formula connecting the motivic zeta function with this monodromy zeta function.

• Arithmetic motivic integration in the context of p-adic orbital integrals and transfer factors is considered by Gordon and Hales in [GH] and [Hal2]. An introduction to this theory is [Hal1].

• Ishii and Kollár [IK] found counter examples in dimensions at least 4 to the Nash problem, which relates irreducible components of the space of arcs through a singularity to exceptional components of a resolution. (And they proved it in general for toric singularities.) Reguera [Reg] showed in any dimension that the Nash problem is equivalent to the so-called wedge problem.

For a toric variety, Ishii [Is] described precisely the relation between arc families and valuations, and obtained the answer to the embedded version of the Nash problem.

• Ito produced an alternative proof that birational smooth minimal models have equal Hodge numbers [It1], and that Batyrev's stringy E-function is well defined [It2], using p-adic Hodge theory.

• Kapranov [Ka] introduced another motivic zeta function as the generating series for motivic measures of varying n-fold symmetric products of a fixed variety. Larsen and Lunts [LL1][LL2] determined for which surfaces this is a rational function over $K_0(Var_{\mathbb{C}})$. It is not known whether it is always a rational function over $\mathcal{M}_{\mathbb{C}}$. See also [DL10, §7] and [BDN].

• For toric surfaces, Lejeune-Jalabert and Reguera [LR] and Nicaise [Ni1] computed an explicit formula for the series $P(T)$ and $J(T)$, respectively. This last paper also contains a sufficient condition for the equality of $P(T)$ and the arithmetic Poincaré series of a toric singularity, which is always satisfied in the surface case. A counter example for this equality in dimension 3 is given.

In [Ni2] Nicaise provides a concrete formula for $P(T)$ if the variety has an embedded resolution of a simple form; this yields a short proof of the formula for toric surfaces.

• Loeser [Loe3] studied the behavior of motivic zeta functions of prehomogeneous vector spaces under castling transformations; he deduced in particular how the motivic Milnor fibre and the Hodge spectrum at the origin behave under such transformations.

• In [NS] Nicaise and Sebag establish the motivic zeta function as a Weil zeta function of the rigid Milnor fibre.

• Sebag [Se1][Se2] studied motivic integration and motivic zeta functions in the context of formal schemes. Loeser and Sebag [LS] developed a theory of motivic integration for smooth rigid varieties, obtained a motivic Serre invariant, and provided new geometric birational invariants of degenerations of algebraic varieties.

• The author introduces motivic principal value integrals and investigates their birational behavior in [Ve5].

• Vojta provides in [Vo] a general reference for jet spaces and jet differentials (at the level of EGA), using Hasse-Schmidt higher differentials.

• Yasuda [Ya1][Ya3] introduced so-called *twisted* jets and arcs over Deligne-Mumford stacks and studied then motivic integration over them. As applications he obtained a McKay correspondence for general orbifolds (see also [LP]), and a common generalization of the stringy E-function and the orbifold cohomology.

• Yokura [Yo] constructs Chern-Schwartz-MacPherson classes on pro-algebraic varieties and relates this to the motivic measure.

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Finite Dehn surgery along A'Campo's divide knots

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Abstract.

We give two geometric methods of constructing plane curves giving cable knots of torus knots via A'Campo's divide knot theory, related to both singularity theory and knot theory. We point out a relationship between "area"of the plane curves and the coefficients of *finite Dehn surgery*, which is Dehn surgery yielding three-dimensional manifolds with finite fundamental group.

§1. Introduction

The *divide* is a relative, generic immersion of a finite number of copies of an arc and a circle in the unit disk D in $\mathbf{R}^2$. N. A'Campo formulated the following definition to associate to each divide P a link $L(P)$ in the 3-dimensional sphere S^3 ([1, 2, 3, 4]):

$$L(P) = \{(u, v) \in D \times T_u D \,|\, u \in P,\, v \in T_u P,\, |u|^2 + |v|^2 = 1\} \subset S^3,$$

where $T_u P$ is the subset consisting of vectors tangent to P in the tangent space $T_u D$ of D at u. The number of components of $L(P)$ is $\sharp$arc + 2$\sharp$circle, where $\sharp$arc (and $\sharp$circle, respectively) is the number of immersed components of arcs (and circles) in P. In this paper, we will study the case where P consists of one immersed arc, thus $L(P)$ is a knot, and we say "a curve P gives a link L" if $L(P) = L$.

The class of links of divides properly contains the class of the links arising from isolated singularities of complex plane curves, for example,

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each torus link $T(a, b)$ of type (a, b) with $a, b > 0$ appears as the link of the singularity of the curve $z^a - w^b = 0$ in $\mathbf{C}^2$ at the origin. In particular, if a and b are coprime, then $T(a, b)$ is a torus knot. A *cable knot* of a (non-trivial) knot K is a knot in the boundary T_K of a regular neighborhood of K. A cable knot is called (p, q)-cable of K and denoted by $C(K; p, q)$ if it is homologous to $pl_K + qm_K$ in T_K, where $\{m_K, l_K\}$ is a meridian-longitude system on T_K. For the torus knot $T(a, b)$, if the pair (p, q) of coefficients satisfies the inequarity $q > abp$ then the cable knot $C(T(a, b); p, q)$ also appears as the link of the singularity in $\mathbf{C}^2$, see [11, p.51]. Note that it is well-known that the link of a singularity is a torus knot, a cable knot or a knot obtained by iteration of cablings of them, called an "iterated torus knot".

In this paper, we give two geometric methods of constructing divides that give some cable knots of torus knots. They are different from A'Campo's original method [4]. The first method, in the next section, is a generalization of [14], in which the author and co-authors showed that a billiard curve in a rectangle $a \times b$ gives a torus knot $T(a, b)$ from the view point of knot theory. The second one, in Section 3, is a modification of A'Campo's, but we will use *fold-maps* of rectangles instead of immersions. In Section 4, we point out a relationship between such divide representation of knots and *finite Dehn surgery*, i.e., Dehn surgery yielding a 3-manifold whose fundamental group is finite. The reason why we show such alternative methods is that ours seems more convenient in 3-dimensional topology.

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§2. Method 1. Billiard curve

Let X be the infinite 45° lattice defined by

$$X := \{(x, y) \in \mathbf{R}^2 | \cos \pi x = \cos \pi y\}$$

in the real xy-plane. For a pair (a, b) of positive integers and $(m, n) \in \mathbf{Z}^2$, by $R(a \times b)_{(m,n)}$ we denote the rectangle at (m, n) of size $a \times b$ in the following sense:

$$R(a \times b)_{(m,n)} := \{(x, y) \in \mathbf{R}^2 | \, m \leq x \leq m + a \text{ and } n \leq y \leq n + b\}.$$

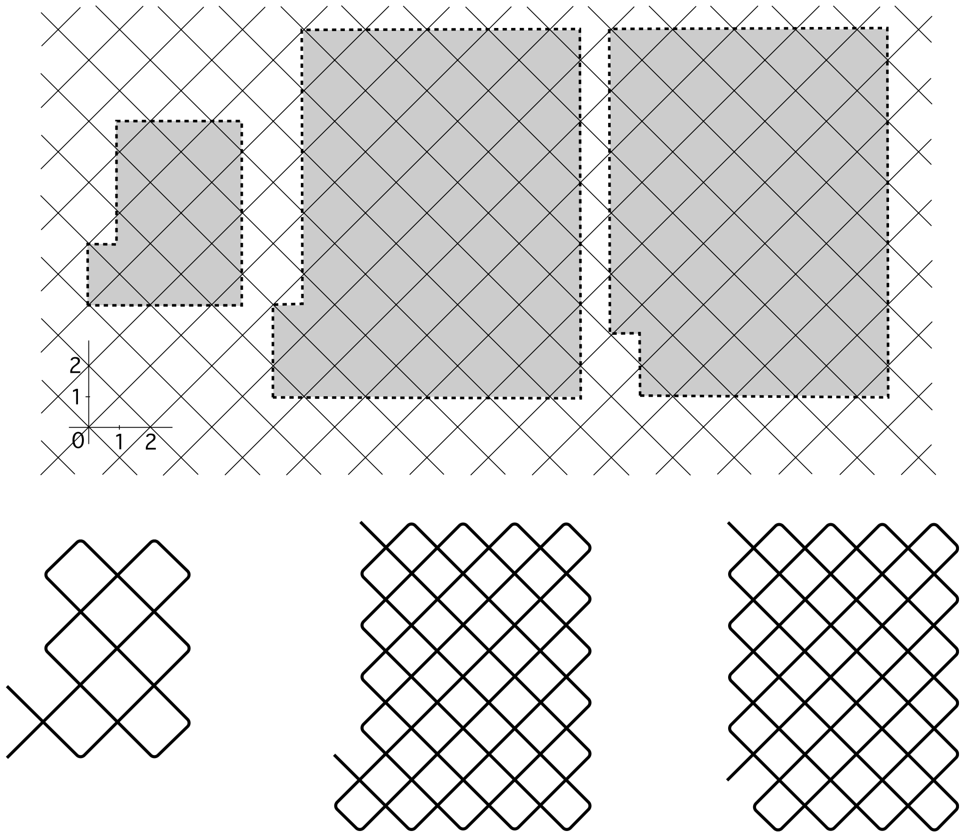

Fig. 1. $P(2,3;2,13)$, $P(3,4;3,37)$ and $P(3,4;3,35)$

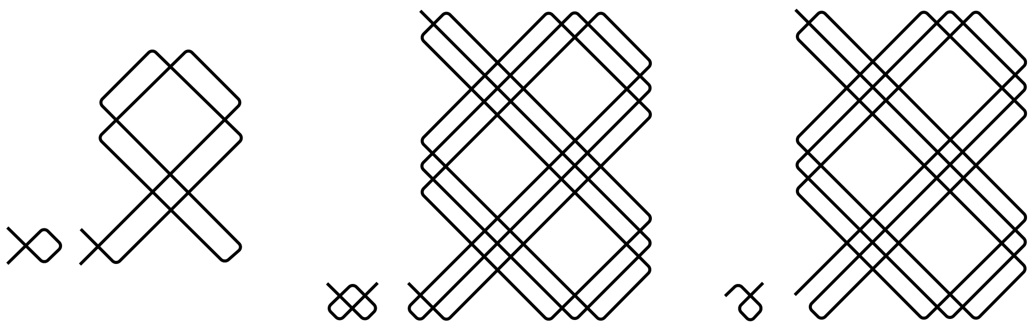

Fig. 2. $B(2,3)\langle B_0^+(2)\rangle$ and $B(3,4)\langle B_0^\pm(3)\rangle$

For such a rectangle or a *union* $\mathcal{R}$ of such rectangles, we regard $X \cap \mathcal{R}$ as a piecewise linear curve (shortly, a PL curve), where we regard each point in $X \cap \partial\mathcal{R}$ as a break point if it is on the edges of $\mathcal{R}$, or a endpoint if it is on a corner of $\partial\mathcal{R}$. From such a PL curve, we get a divide by rounding the break points smoothly and setting it in the unit disk in $\mathbf{R}^2$

by an isotopy. By $X \cap \mathcal{R}$, we also denote such a smooth divide derived from the PL curve.

In [14], the author and co-authors proved the following proposition and found a nice diagram of $T(a,b)$ and Murasugi-sum structure of their fiber surfaces from the view point of knot theory. (Proposition 2.1 itself has been shown in [17], [5], or can be shown by more recent works [10], [18].)

Proposition 2.1. ([14]) *Let (a,b) be a pair of positive integers and (m,n) any pair of integers. Then a divide $X \cap R(a \times b)_{(m,n)}$ gives a torus link $T(a,b)$.*

Now, for a pair (a,b) of positive coprime integers and a positive integer p, we define a region $\mathcal{R}(a,b;p,pab+1)$ in $\mathbf{R}^2$ and a divide $P(a,b;p,pab+1)$ as:

$$\begin{array}{rrcl} (+) & \mathcal{R}(a,b;p,pab+1) & := & R(pa \times pb)_{(0,0)} \cup R(1 \times p)_{(-1,0)}, \\ & P(a,b;p,pab+1) & := & X \cap (\mathcal{R}(a,b;p,pab+1) + \vec{\delta}), \end{array}$$

and also define a region $\mathcal{R}(a,b;p,pab-1)$ in $\mathbf{R}^2$ and a divide $P(a,b;p,pab-1)$ as:

$$\begin{array}{rrcl} (-) & \mathcal{R}(a,b;p,pab-1) & := & cl\left(R(pa \times pb)_{(0,0)} \backslash R(1 \times (p-1))_{(0,0)}\right), \\ & P(a,b;p,pab-1) & := & X \cap (\mathcal{R}(a,b;p,pab-1) + \vec{\delta}), \end{array}$$

where cl means the closure of the region, $\vec{\delta} = (\delta_1, \delta_2) \in \mathbf{Z}^2$ with $\delta_1 + \delta_2 \equiv p+1 \mod 2$ and $+\vec{\delta}$ means the parallel transformation by $\vec{\delta}$ in $\mathbf{R}^2$, see Figure 1.

Theorem 2.2. *For a pair (a,b) of positive coprime integers and a positive integer p, the divide $P(a,b;p,pab \pm 1)$ gives a $(p, pab \pm 1)$-cable of the torus knot $T(a,b)$, i.e.,*

$$L(P(a,b;p,pab \pm 1)) = C(T(a,b);p,pab \pm 1).$$

Proof. First, from the pair $(a_0,b_0) := (a,b)$, we construct a word $w_1 w_2 \cdots w_n$ of two letters L (left) and R (right) by Euclidean algorithm, see Figure 3:

If $a_i > b_i$, then $w_{i+1} := L$ and $(a_{i+1}, b_{i+1}) := (a_i - b_i, b_i)$.

If $a_i < b_i$, then $w_{i+1} := R$ and $(a_{i+1}, b_{i+1}) := (a_i, b_i - a_i)$.

By coprime-ness of (a,b), after some n steps, the pair (a_n, b_n) becomes $(1,1)$. Then this step is over.

Second, we regard the word as a rule of constructing the $a \times b$ rectangle. In fact, $R(a,b)_{(0,0)}$ is obtained from $R(1 \times 1)_{(0,0)}$ ruled by the word in inversed order as follows $(j = 1, 2, ..., n)$ (see also [19]):

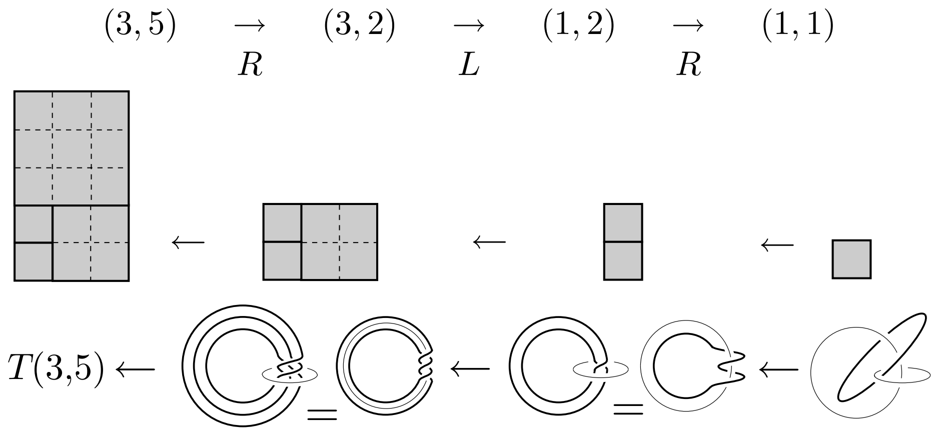

Fig. 3. Euclidean Algorithm

If $w_{n+1-j} = L$, then we add a square from the right.

If $w_{n+1-j} = R$, then we add a square from the top, see Figure 3. This process corresponds to a blowing-up sequence of the singularity $z^a - w^b = 0$, thus also to a twisting sequence of torus knots. According to growing up of the rectangle from $R(1 \times 1)$ to $R(a \times b)$, the corresponding knot K changes from the unknot $K_0 := T(1,1)$ to $K_n := T(a,b)$ by Proposition 2.1.

Finally, we starting with the region $R((p+1) \times p)_{(-1,0)} + \vec{\delta}$ or $cl\left(R(p \times p)_{(0,0)} \backslash R(1 \times (p-1))_{(0,0)}\right) + \vec{\delta}$ according to the sign at $\pm$, which gives $C(T(1,1); p, p \pm 1) = T(p \pm 1, p)$ regarded as a curve in T_{K_0}. We add $p \times p$ extended squares to the starting rectangle from the right or the top according to the word w_{n+1-j} is L or R ($j = 1, 2, ..., n$) as same as in the last step. Then we have $\mathcal{R}(a, b; p, pab \pm 1)$ and the divide $P(a, b; p, pab \pm 1)$.

Generally, if a knot K' is obtained from K by a positive twisting along a disk d, the homology class m_K in T_K becomes to $m_{K'}$ in $T_{K'}$ and the class l_K in T_K becomes to $l_{K'} + lk(K, \partial d)^2 m_{K'}$ in $T_{K'}$, where $lk(K, \partial d)$ is the linking number of K and the boundary of d, see [20, p.11]. In the divide theory, the intersection number between two divides equals to the linking number of the corresponding components of the link.

In our j-th process in the case of $w_{n+1-j} = R$, the boundary of the disk d_j corresponds to the right edge whose length is b_{n+1-j}, which equals to the linking number $lk(K_{j-1}, \partial d_j)$.

Thus $pl_{K_{j-1}} + (pa_{n+1-j}b_{n+1-j} \pm 1)m_{K_{j-1}}$ becomes to

$$\begin{aligned} p(l_{K_j} + b^2_{n+1-j}m_{K_j}) + (pa_{n+1-j}b_{n+1-j} \pm 1)m_{K_j} \\ = pl_{K_j} + \{p(a_{n+1-j} + b_{n+1-j})b_{n+1-j} \pm 1\}m_{K_j} \\ = pl_{K_j} + (pa_{n-j}b_{n-j} \pm 1)m_{K_j}. \end{aligned}$$

The case of $w_{n+1-j} = L$ is similar (a and b are changed). After the final n-th step, we have the cable knot $C(T(a,b); p, pab \pm 1)$. The proof is completed. Q.E.D.

§3. Method 2. Fold-immersion

Let P be a PL curve obtained by cutting out $X \cap \mathcal{R}$ from the lattice X as in the last section.

Definition 3.1. For such a PL curve P, by b we denote the number of break points of P on the edges of $\mathcal{R}$. We say that a map $f : [0,1]^2 \to \mathbf{R}^2$ is a *fold-immersion of* $[0,1]^2$ *along* P if it satisfies the following condition:

(1) $f([0,1] \times \{\frac{1}{2}\}) = P$,

(2) There exists a sequence $0 < t_1 < t_2 < \cdots < t_b < 1$ such that

 (i) f is an immersion over $([0,1] \backslash \{t_1, t_2, \cdots, t_b\}) \times [0,1]$ and

 (ii) Near each $\{t_i\} \times [0,1]$, f is locally given as shown in Figure 4. For example, in the case of Figure 4, f near $\{t_i\} \times [0,1]$ is determined by the map

$$\begin{array}{rccc} \varphi : & (t_i - \epsilon, t_i + \epsilon) \times [0,1] & \to & \mathbf{R}^2 \\ & (t,s) & \mapsto & (t+s, |t|). \end{array}$$

 If the break point is not on the bottom edge, then f is locally given by the $\pi/2, \pi$ or $3\pi/2$ rotation of Figure 4 or its reflection.

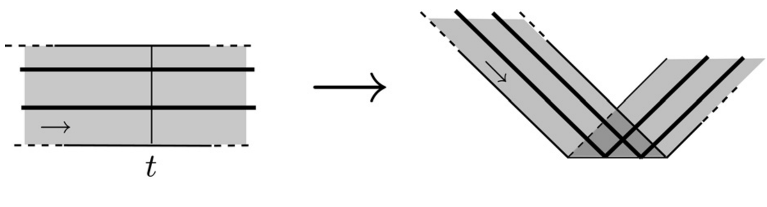

Fig. 4. Fold-map

We remark that triangle moves on divides shown in Figure 5 do not change the ambient isotopy type of the links of the divides. Let $B_0^{\pm}(p)$

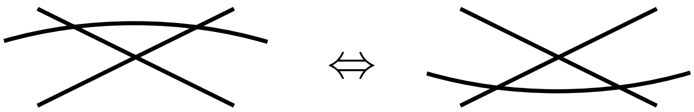

Fig. 5. Triangle move

and $B(a,b)$ be the billiard curves defined by

$$\begin{array}{rcl} B_0^+(p) & := & X \cap R((p+1)\times p)_{(0,-p)}, \\ B_0^-(p) & := & X \cap cl\left(R(p\times p)_{(0,-p)} \backslash R(1\times(p-1))_{(0,-p)}\right) \text{ and} \\ B(a,b) & := & X \cap R(a\times b)_{(0,0)}. \end{array}$$

By scaling smaller, we regard $B_0^{\pm}(p)$ in the rectangle as a curve in $[0,1]^2$. Then, the image of $B_0^{\pm}(p)$ under a (generic) fold-immersion along $B(a,b)$ is well-defined up to triangle moves. We denote such a curve by $B(a,b)\langle B_0^{\pm}(p)\rangle$, see Figure 2, placed near Figure 1 for convenience.

Theorem 3.2. *For a pair (a,b) of positive coprime integers and a positive integer p, the divide $B(a,b)\langle B_0^{\pm}(p)\rangle$ gives a $(p, pab\pm 1)$-cable of $T(a,b)$, i.e.,*

$$L(B(a,b)\langle B_0^{\pm}(p)\rangle) = C(T(a,b); p, pab\pm 1).$$

Proof. It is easy to see that, for a nice choice of fold-immersion, or in other words, by some triangle moves, the curve $B(a,b)\langle B_0^{\pm}(p)\rangle$ is isotopic to $P(a,b;p,pab\pm 1)$. Q.E.D.

We remark that A'Campo constructed in [3] the divide $B(2,3)\langle B_0^+(2)\rangle$ in our notation as the image of $B(2,9)$ under an immersion along $B(2,3)$ and denoted by $P_{2,9} * P_{2,3}$, see Figure 6.

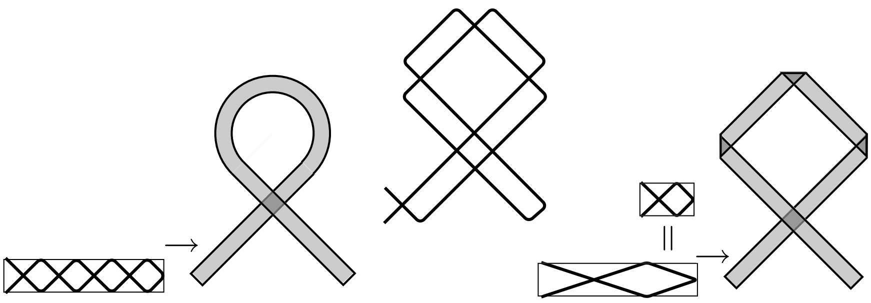

Fig. 6. Comparison

§4. Area of divide and Dehn surgery

Let K be a knot in S^3 and n an integer. By $M(K,n)$ we denote the 3-manifold obtained by Dehn surgery along K with coefficient n, i.e., removing a solid torus V_K along K and regluing it back such that the meridian comes to a curve homologous to $l_K + n\, m_K$, where $\{m_K, l_K\}$ is a meridian-longitude system on the boundary of V_K. Here we are concerned with $M(K,n)$ whose fundamental group $\pi_1(M(K,n))$ is finite. Such research is called "finite Dehn surgery" ([6, 9, 13]). Note that $H_1(M(K,n); \mathbf{Z}) \cong \mathbf{Z}/n\mathbf{Z}$.

What we would like to point out in this section is that in some examples of integral finite surgery $M(K,n)$, the knot K is given by a plane curve as $X \cap \mathcal{R}$ via A'Campo's divide theory, and that in such a case the coefficient n is near to the area $A(\mathcal{R})$ of the region $\mathcal{R}$ in the plane.

The links which can be obtained from the billiard curves by the method in section 2 and 3 are only cable knots of torus knots. In [13] and [6], it is proved that an iterated torus knot other than a torus knot or a cable knot of a torus knot has no finite surgery. The cable knots of torus knots with finite surgery, completely listed in [6], can be obtained as the knots of billiard curves except for the following four cases: $C(T(2,3);p,q)$ with $(p,q) = (2,9), (2,15), (3,16)$ and $(3,20)$. We can state the following:

Theorem 4.1.

(i) *Let $\mathcal{R}$ be a rectangle $R(a \times b)_{(m,n)}$ with a pair (a,b) of any coprime integers, K the knot of the billiard curve obtained from $\mathcal{R}$, and n a coefficient of finite surgery of S^3 along K. Then the inequality $|n - A(\mathcal{R})| \leq 1$ holds.*

(ii) *Let $\mathcal{R}$ be a region defined by $(+)$, K the knot of the billiard curve obtained from $\mathcal{R}$, and n a coefficient of finite surgery of S^3 along K. Then the inequality $|n - A(\mathcal{R})| \leq 1$ holds.*

(iii) *Let $\mathcal{R}$ be a region defined by $(-)$, K the knot of the billiard curve obtained from $\mathcal{R}$, and n a coefficient of finite surgery of S^3 along K. Then the inequality $|n - A(\mathcal{R})| \leq 2$ holds.*

Proof. We start with families of finite surgery along torus knots and cable knots of torus knots.

Example 4.2. *Each of the followings is finite surgery:*
(1) ([21]) *$M(\,T(a,b), n\,)$ with $n = ab \pm 1$.*

(2) ([8, 13]) $M(C(T(a,b);2,2ab\pm 1),\, n)$ *with* $n = 4ab \pm 1$,
(3) ([13]) $M(C(T(2,b);3,6b\pm 1),\, n)$ *with* $n = 18b \pm 2$.

In (1) and (2) (or (3), respectively), resulting 3-manifolds are lens spaces (or prism manifolds). The case (i) in the theorem is shown by (1) above: K is the torus knot $T(a,b)$ by Proposition 2.1 and the area of $R(a\times b)_{(m,n)}$ is ab. For (2) and (3), we have that $A(\mathcal{R}) = p^2ab+p$ for $\mathcal{R} = \mathcal{R}(a,b;p,pab+1)$ and that $A(\mathcal{R}) = p^2ab-p+1$ for $\mathcal{R} = \mathcal{R}(a,b;p,pab-1)$. Thus, in these examples, the inequality $|n - A(\mathcal{R})| \le 1$ holds.

Next, we recall more "exceptional" examples from the list in [6]. In the left-hand side of Table 1, we picked up all examples of integral finite surgery along knots of type $C(T(a,b);p,pab\pm 1)$ from Table 1 in [6], which is the complete list of 37 examples by [13]. Four examples marked by $*$ are included in Example 4.2 (3). In the right-hand side, we represent each knot by a divide of type $B(a,b)\langle B_0^{\pm}(p)\rangle$ (Method 2), which can be deformed as $X \cap \mathcal{R}$ (Method 1) and write its area $A(\mathcal{R})$.

$(+)$ $C(T(a,b);p,pab+1)$	n		$B(a,b)\langle B_0^+(p)\rangle$	$A(\mathcal{R})$
$C(T(2,3);2,13)$	27		$B(2,3)\langle B_0^+(2)\rangle$	26
$C(T(2,3);3,19)$	56	$*$	$B(2,3)\langle B_0^+(3)\rangle$	57
$C(T(2,3);4,25)$	99		$B(2,3)\langle B_0^+(4)\rangle$	100
$C(T(2,3);4,25)$	101		$B(2,3)\langle B_0^+(4)\rangle$	100
$C(T(2,3);5,31)$	154		$B(2,3)\langle B_0^+(5)\rangle$	155
$C(T(2,3);6,37)$	221		$B(2,3)\langle B_0^+(6)\rangle$	222
$C(T(2,5);2,21)$	43		$B(2,5)\langle B_0^+(2)\rangle$	42
$C(T(2,5);3,31)$	92	$*$	$B(2,5)\langle B_0^+(3)\rangle$	93
$C(T(2,5);4,41)$	163		$B(2,5)\langle B_0^+(4)\rangle$	164
$C(T(3,4);3,37)$	110		$B(3,4)\langle B_0^+(3)\rangle$	111
$C(T(3,5);3,46)$	137		$B(3,5)\langle B_0^+(3)\rangle$	138

$(-)$ $C(T(a,b);p,pab-1)$	n		$B(a,b)\langle B_0^-(p)\rangle$	$A(\mathcal{R})$
$C(T(2,3);2,11)$	21		$B(2,3)\langle B_0^-(2)\rangle$	23
$C(T(2,3);3,17)$	50		$B(2,3)\langle B_0^-(3)\rangle$	52
$C(T(2,3);3,17)$	52	$*$	$B(2,3)\langle B_0^-(3)\rangle$	52
$C(T(2,3);4,23)$	91		$B(2,3)\langle B_0^-(4)\rangle$	93
$C(T(2,3);4,23)$	93		$B(2,3)\langle B_0^-(4)\rangle$	93
$C(T(2,3);5,29)$	146		$B(2,3)\langle B_0^-(5)\rangle$	146
$C(T(2,3);6,35)$	211		$B(2,3)\langle B_0^-(6)\rangle$	211
$C(T(2,5);2,19)$	37		$B(2,5)\langle B_0^-(2)\rangle$	39
$C(T(2,5);3,29)$	88	$*$	$B(2,5)\langle B_0^-(3)\rangle$	88
$C(T(2,5);4,39)$	157		$B(2,5)\langle B_0^-(4)\rangle$	157
$C(T(3,4);3,35)$	106		$B(3,4)\langle B_0^-(3)\rangle$	106
$C(T(3,5);3,44)$	133		$B(3,5)\langle B_0^-(3)\rangle$	133

Table 1: Integral finite surgeries along cables

We have the theorem. Q.E.D.

Recent researchers' interest seems to be the finite surgery along hyperbolic knots. In the recent work [22], the author pointed out that every knot in a certain subfamily of Berge's knots ([7]) yielding lens spaces (It contains 19-surgery along the Pretzel knot of type $(-2, 3, 7)$, which was discovered in [12]), is a divide knot and given by a divide of $X \cap \mathcal{R}$ type. For them, it holds that $n = A(\mathcal{R})$.

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