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# The Bergman kernel and pluripotential theory

**Zbigniew Błocki**

**Abstract.**

We survey recent developments relating the notions of the Bergman kernel and pluripotential theory and indicate some open problems.

## §1. Introduction

We will discuss recent results relating the Bergman kernel and pluripotential theory. For $n = 1$ that there is such a relation is perhaps not surprising, since then the Bergman kernel can be expressed in terms of the Green function

$$K_\Omega = -\frac{2}{\pi}\frac{\partial^2 g_\Omega}{\partial z \partial \overline{w}}.$$

No counterpart of this is known for $n \geq 2$. Nevertheless, the pluricomplex Green function in several variables turned out to be a very useful tool in the theory of the Bergman kernel and Bergman metric. We will concentrate on the results that directly relate these two notions.

First we collect basic definitions, notations and assumptions. Good general references are for example [19], [25], [20] (for the Bergman kernel) and [23] (for pluripotential theoretic notions). Throughout $\Omega$ will always denote a bounded pseudoconvex domain in $\mathbb{C}^n$ (if $n = 1$ then every domain is pseudoconvex). The Bergman kernel $K_\Omega(z, w)$, $z, w \in \Omega$, is determined by

$$f(w) = \int_\Omega f(z)\overline{K_\Omega(z,w)}d\lambda(z), \quad w \in \Omega,\ f \in H^2(\Omega),$$

where $H^2(\Omega)$ is the (Hilbert) space of all holomorphic functions in $\Omega$ that belong to $L^2(\Omega)$. By $k_\Omega$ we will denote the Bergman kernel on the

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diagonal

$$(1.1)\quad k_\Omega(z) = K_\Omega(z,z) = \sup\left\{\frac{|f(z)|^2}{||f||^2} : f \in H^2(\Omega)\setminus\{0\}\right\}, \quad z \in \Omega,$$

($||f||$ is the $L^2$-norm). Then $\log k_\Omega$ is a smooth strongly plurisubharmonic function in $\Omega$ and the Bergman metric $B_\Omega$ is the Kähler metric given by the potential $\log k_\Omega$, that is

$$B_\Omega^2(z;X) = \sum_{j,k=1}^n \frac{\partial^2 \log k_\Omega}{\partial z_j \partial \overline{z}_k}(z) X_j \overline{X}_k, \quad z \in \Omega,\ X \in \mathbb{C}^n.$$

The Bergman metric defines the Bergman distance in $\Omega$ which will be denoted by $dist_\Omega$. We will call $\Omega$ *Bergman complete* if it is complete w.r.t. $dist_\Omega$, and *Bergman exhaustive* if $\lim_{z\to\partial\Omega} k_\Omega(z) = \infty$.

For a fixed $w \in \Omega$ the *pluricomplex Green function* with pole at $w$ is defined by $g_w := g_\Omega(z,w) = \sup \mathcal{B}_w$, where

$$\mathcal{B}_w = \{u \in PSH(\Omega) : u < 0, \limsup_{z\to w}\big(u(z) - \log|z-w|\big) < \infty\}.$$

Then $g_w \in \mathcal{B}_w$ and

$$c_\Omega(w) = \exp \limsup_{z\to w}\big(g_w(z) - \log|z-w|\big)$$

is the logarithmic capacity of $\Omega$ w.r.t. $w$. One of the main differences between one and several complex variables is the symmetry of $g_\Omega$: of course it is always symmetric if $n = 1$ and usually not true for $n \geq 2$ (the first counterexample was found by Bedford-Demailly [1]).

The domain $\Omega$ is called *hyperconvex* if it admits a bounded plurisubharmonic exhaustion function, that is there exists $u \in PSH(\Omega)$ such that $u < 0$ in $\Omega$ and $\lim_{z\to\partial\Omega} u(z) = 0$ (of course, if $n = 1$ then hyperconvexity is equivalent to the regularity of $\Omega$). It was shown by Demailly [12] that if $\Omega$ is hyperconvex then $g_\Omega$ is continuous on $\overline{\Omega}\times\Omega$ (off the diagonal, vanishing on the boundary) but it is still an open problem if it is continuous on $\Omega \times \partial\Omega$ (for partial results see [8], [7], [17] and [6]).

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## §2. Bergman completeness, Bergman exhaustivity and hyperconvexity

In this section we will concentrate on the relations between these three notions. We start with the following two results.

**Theorem 2.1.** (Ohsawa [26], [27]) *If $\Omega$ is hyperconvex then it is Bergman exhaustive.*

**Theorem 2.2.** (Herbort [16], Błocki-Pflug [7]) *If $\Omega$ is hyperconvex then it is Bergman complete.*

Theorem (2.2) was proved independently in [16] and [7] ([7] heavily relied on [9], where Theorem 2.2 was proved in particular for $n = 1$, whereas [16] was written independently of both [7] and [9]).

We are now going to sketch the main ideas behind the proof of Theorem 2.2. As a byproduct, the method also gives Theorem 2.1 (the original Ohsawa proofs from [26] and [27] were different, we will discuss the one from [27] later). First, we use the theory of the complex Monge-Ampère operator to estimate the volume of the sublevel sets $\{g_w < -1\}$ for $w$ near the boundary. In [5] it was shown that for hyperconvex $\Omega$ there exists a unique $u_\Omega \in PSH(\Omega) \cap C(\overline{\Omega})$ such that $u_\Omega = 0$ on $\partial\Omega$ and $(dd^c u_\Omega)^n = d\lambda$. Then integrating by parts (see [4])

$$\begin{aligned} vol(\{g_w < -1\}) &\leq \int_\Omega |g_w|^n (dd^c u_\Omega)^n \\ &\leq n!\, ||u_\Omega||^{n-1}_{L^\infty(\Omega)} \int_\Omega |u_\Omega| (dd^c g_w)^n \\ &\leq C(n, diam\,\Omega)\, |u_\Omega(w)|. \end{aligned} \tag{2.1}$$

In particular,

$$\Omega \text{ is hyperconvex } \Rightarrow \lim_{w \to \partial\Omega} vol(\{g_w < -1\}) = 0. \tag{2.2}$$

The above proof of (2.2) is taken from [7]. It was also independently shown in [16] (the argument there was due to Coman), where a result from [8] was used.

Before proceeding further, let us comment on the implication (2.2). As noticed in [32] (see p. 53), the reverse implication is true if $n = 1$. The following example from [16]

$$\{(z, w) \in \mathbb{C}^2 : |w| < e^{-1/|z|} < e^{-1}\}$$

shows that it is no longer true for $n \geq 2$ (see the review of [16] in Mathematical Reviews). (2.1) also shows that $g_w \to 0$ in $L^n(\Omega)$ as

$w \to \partial\Omega$ from which one can easily get that $g_w \to 0$ in $L^p(\Omega)$ for every $p < \infty$. The open problem of continuity of $g_\Omega$ on $\Omega\times\partial\Omega$ (for hyperconvex $\Omega$) is equivalent to locally uniform convergence $g_w \to 0$ in $\Omega$ as $w \to \partial\Omega$.

To finish the proof of Theorem 2.2 we use the following estimate from [16] (it is proved using Hörmander's $L^2$-estimate for the $\overline{\partial}$ operator [18]; see also [9] and [6])

$$\frac{|f(w)|^2}{k_\Omega(w)} \le c_n \int_{\{g_w<-1\}} |f|^2 d\lambda, \quad f \in H^2(\Omega),\ w \in \Omega. \tag{2.3}$$

Combining (2.2) with (2.3) we get, if $\Omega$ is hyperconvex,

$$\lim_{w\to\partial\Omega} \frac{|f(w)|^2}{k_\Omega(w)} = 0, \quad f \in H^2(\Omega). \tag{2.4}$$

This is precisely the criterion of Kobayashi [24] and we conclude that $\Omega$ is Bergman complete. In addition, if we use (2.3) with $f \equiv 1$ and (2.1) we obtain the following quantitative version of Theorem 2.1, which also gives a comparison between the Bergman kernel and the solution to the complex Monge-Ampère equation

$$k_\Omega \ge \frac{1}{C(n, diam\,\Omega)\,|u_\Omega|}. \tag{2.5}$$

The reverse implications in Theorems 2.1 and 2.2 are false even for $n = 1$. Ohsawa [26] considered Zalcman-type domains

$$\Delta(0,1) \setminus \overline{\bigcup_{k=1}^{\infty} \Delta(2^{-k}, r_k)}, \tag{2.6}$$

where $\Delta(z,r)$ denotes the disk centered ar $z$ with radius $r$ and $r_k$ is a sequence decreasing to 0 such that $r_k < 2^{-k}$ and $\overline{\Delta}(2^{-k}, r_k)\cap\overline{\Delta}(2^{-j}, r_j) = \emptyset$ for $k \ne j$. From Wiener's criterion it then follows that (2.6) is hyperconvex if and only if

$$\sum_{k=1}^{\infty} \frac{k}{-\log r_k} = \infty.$$

On the other hand, Ohsawa [26] showed that if for example $r_k = 2^{-k^3}$ (for $k \ge 2$) then (2.6) is Bergman exhaustive. Chen [9] proved that then (2.6) is also Bergman complete, we thus get a counterexample to reverse implications in Theorems 2.1 and 2.2.

The relation between Bergman exhaustivity and Bergman completeness is also of interest. The problem is related to the Kobayashi criterion

(2.4). For if (2.4) was equivalent to Bergman completeness (this problem was posed by Kobayashi) then Bergman completeness would imply Bergman exhaustiveness (putting $f \equiv 1$ in (2.4)). Let us first look at (2.4). By (1.1) we have

$$\frac{|f(z)|}{\sqrt{k_\Omega(z)}} \leq \frac{|h(z)|}{\sqrt{k_\Omega(z)}} + ||f-h||, \quad f, h \in H^2(\Omega),\ z \in \Omega,$$

and we easily see that to verify (2.4) it is enough to check it, for a given sequence $\Omega \ni w_j \to w_0 \in \partial\Omega$, for $f$ belonging to a dense subspace of $H^2(\Omega)$. Therefore, if $\Omega$ is Bergman exhaustive and $H^\infty(\Omega, w_0)$, the space of holomorphic functions in $\Omega$ that are bounded near $w_0$, is dense in $H^2(\Omega)$ for every $w_0 \in \partial\Omega$ then $\Omega$ satisfies (2.4) and is thus also Bergman complete. We use the following.

**Theorem 2.3.** (Hedberg [15], Chen [10]) *If $n = 1$ then $H^\infty(\Omega, w_0)$ is dense in $H^2(\Omega)$ for every $w_0 \in \partial\Omega$.*

**Corollary 2.1.** (Chen [10]) *If $n = 1$ then Bergman exhaustiveness implies Bergman completeness.*

The above results are false for $n \geq 2$ and the counterexample is the Hartogs triangle $\{(z, w) \in \mathbb{C}^2 : |w| < |z| < 1\}$. They hold however if one in addition assumes that for every $w_0 \in \partial\Omega$ there exists a neighborhood basis $U_j$ of $w_0$ such that $\Omega \cup U_j$ is pseudoconvex for every $j$ (in the case of Hartogs triangle this is not true at the origin) - see [6].

The remaining problem is therefore whether Bergman completeness implies Bergman exhaustiveness. It was settled in the negative by Zwonek [33] who showed that the following domain

$$\Delta(0,1) \setminus \overline{\bigcup_{k=2}^{\infty} \bigcup_{j=0}^{k^5-1} \Delta\big(k^{-5} e^{2\pi i j/k^5}, e^{-k^{19}}\big)}, \tag{2.7}$$

is Bergman complete but not Bergman exhaustive (see also [22]). Note that any such an example, by Theorem 2.3, does not satisfy (2.4) which shows that the Kobayashi criterion is not necessary for Bergman completeness.

It is possible to characterize Bergman exhaustive domains in terms of potential theory in dimension 1.

**Theorem 2.4.** (Zwonek [34]) *Assume $n = 1$. Then $\Omega$ is Bergman exhaustive if and only if*

$$\lim_{\Omega \ni z \to \partial\Omega} \int_0^{1/2} \frac{dt}{-t^3 \log cap\,(\overline{\Delta}(z,t) \setminus \Omega)} = \infty.$$

From Theorem 2.4 it follows in particular that (2.6) is Bergman exhaustive if and only if

$$\sum_{k=1}^{\infty} \frac{4^k}{-\log r_k} = \infty.$$

No characterization of Bergman completeness in terms of potential theory is known. Jucha [21] however showed that (2.6) is Bergman complete if and only if

$$\sum_{k=1}^{\infty} \frac{2^k}{\sqrt{-\log r_k}} = \infty.$$

As a consequence, one can simplifiy the Zwonek example (2.7): it is sufficient to take (2.6) with $r_k = 2^{-k^2 4^k}$.

From the definition it easily follows that Bergman completeness is a biholomorphically invariant notion, whereas Bergman exhaustiveness is not: the Hartogs triangle is biholomorphic to $\Delta \times \Delta_*$, which is not Bergman exhaustive. To author's knowledge, no such example is known for $n = 1$ (it would of course also show that the Kobayashi criterion is not necessary for Bergman completeness).

In [6] it was shown that the Kobayashi criterion (2.4) for Bergman completeness can be replaced with the following

$$\limsup_{w \to \partial\Omega} \frac{|f(w)|^2}{k_\Omega(w)} < ||f||^2, \quad f \in H^2(\Omega) \setminus \{0\}.$$

It remains an open problem if this condition is necessary for Bergman completeness.

## §3. Other results

Diederich-Ohsawa [14] proved a quantitative estimate for the Bergman distance in smooth pseudoconvex domains. Pluripotential theory turned out to be one of the main tools in establishing this result. The estimate from [14] was improved in [6] with help of the following theorem.

**Theorem 3.1.** ([6]) *Assume that $\Omega$ is psudoconvex and $z, w \in \Omega$ are such that $\{g_z < -1\} \cap \{g_w < -1\} = \emptyset$. Then $dist_\Omega(z, w) \geq c_n > 0$.*

On the other hand, the following estimate was used in [13] (see also [11]) to show a quantitative bound for the Bergman metric in smooth pseudoconvex domains.

**Theorem 3.2.** (Diederich-Herbort [13]) *There exists a positive constant $C$, depending only on $n$ and the diameter of $\Omega$, such that for any psudoconvex $\Omega$*

$$\frac{1}{C}B_{\{g_w<-1\}}(w;X) \leq B_\Omega(w;X) \leq CB_{\{g_w<-1\}}(w;X), \quad w \in \Omega,\ X \in \mathbb{C}^n.$$

No counterpart of Zwonek's Theorem 2.4, characterizing the domains where $\lim\limits_{z\to\partial\Omega} k_\Omega(z) = \infty$ in terms of potential theory, is known for $n \geq 2$. However, the domains with $\limsup\limits_{z\to\partial\Omega} k_\Omega(z) = \infty$ are characterized completely.

**Theorem 3.3.** (Pflug-Zwonek [29]) *The following are equivalent*

(1) *$\Omega$ is an $L^2$-domain of holomorphy (that is $\Omega$ is a domain of existence of a function from $H^2(\Omega)$);*
(2) *$\partial\Omega$ has no pluripolar part (that is if $U$ is open then $U \cap \partial\Omega$ is either empty or non-pluripolar);*
(3) $\limsup\limits_{z\to w} k_\Omega(z) = \infty,\ \ w \in \partial\Omega.$

The proof of Theorem 2.1 in [27] relied on the following quantitative estimate.

**Theorem 3.4.** (Ohsawa [27]) *Assume $n = 1$. There exists a positive numerical constant $C$ such that for any $\Omega$*

$$C\sqrt{k_\Omega(w)} \geq c_\Omega(w), \quad w \in \Omega.$$

The above result of course gives Theorem 2.1 for $n = 1$ and also provides another quantitative bound for the Bergman kernel from below in terms of potential theory, alternative to (2.5). Theorem 2.1 for arbitrary $n$ then follows easily from the Ohsawa-Takegoshi extension theorem [28].

Ohsawa [27] obtained $C = \sqrt{750\pi}$ in Theorem 3.4. Berndtsson [3] proved this estimate with $C = \sqrt{6\pi}$. The Suita conjecture [30] asserts that the estimate holds with $C = \sqrt{\pi}$. This constant would be then optimal - it is attained for the disk.

In fact, one can easily generalize Theorem 3.4 to higher dimensions. Without loss of generality we may assume that $\Omega$ is hyperconvex (the general case can be obtained by approximation). For a fixed $w \in \Omega$ by [31] one can find $\zeta \in \mathbb{C}^n$, $|\zeta| = 1$, such that

$$c_\Omega(w) = \exp \lim_{\lambda\to 0}(g_w(w + \lambda\zeta) - \log|\lambda|).$$

By $D$ denote the one dimensional slice $\{\lambda \in \mathbb{C} : w + \lambda\zeta \in \Omega\}$ and by $g$ the Green function for $D$ with pole at 0. Then $g(\lambda) \geq g_w(w + \lambda\zeta)$

and thus $c_D(0) \geq c_\Omega(w)$. By Theorem 3.4 and the Ohsawa-Takegoshi extension theorem

$$c_\Omega(w) \leq c_D(0) \leq C_S \sqrt{k_D(0)} \leq C_S C_{OT} \sqrt{k_\Omega(w)},$$

where $C_S$ is the constant from Theorem 3.4 and $C_{OT}$ the constant from the Ohsawa-Takegoshi extension theorem (Berndtsson [2] showed that if $\Omega \subset \{|z_1| \leq 1\}$ then one can take $C_{OT} = 4\pi$).

We do not know if $\lim\limits_{w \to \partial\Omega} c_\Omega(w) = \infty$ for hyperconvex $\Omega$ (and $n \geq 2$). If this was the case then the above estimate would give another quantitative version of Theorem 2.1.

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# Neumann eigenfunctions and Brownian couplings

**Krzysztof Burdzy**

**Abstract.**

This is a review of research on geometric properties of Neumann eigenfunctions related to the "hot spots" conjecture of Jeff Rauch. The paper also presents, in an informal way, some probabilistic techniques used in the proofs.

## §1. Introduction

In 1974 Jeff Rauch stated a problem at a conference, since then referred to as the "hot spots conjecture" (the conjecture was not published in print until 1985, in a book by Kawohl [K]). Informally speaking, the conjecture says that the second Neumann eigenfunction for the Laplacian in a Euclidean domain attains its maximum and minimum on the boundary. There was hardly any progress on the conjecture for 25 years but a number of papers have been published in recent years, on the conjecture itself and on problems related to or inspired by the conjecture. This article will review some of this body of research and techniques used in it, with focus on author's own research and probabilistic methods used in proofs of analytic results.

The paper is organized as follows. First, we will state and explain the conjecture. Then we will review the main results on the conjecture and related problems. Finally, we will review some techniques used in the proofs.

In order to explain the intuitive contents of the hot spots conjecture we will start with the heat equation. Suppose that $D$ is an open connected bounded subset of $\mathbb{R}^d$, $d \geq 1$. Let $u(t,x), t \geq 0, x \in D$, be the solution of the heat equation $\partial u/\partial t = \Delta_x u$ in $D$ with the Neumann

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boundary conditions and the initial condition $u(0,x) = u_0(x)$. That is, $u(t,x)$ is a solution to the following initial-boundary value problem,

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta_x u(t,x), & x \in D,\ t > 0, \\ \dfrac{\partial u}{\partial \mathbf{n}}(t,x) = 0, & x \in \partial D,\ t > 0, \\ u(0,x) = u_0(x), & x \in D, \end{cases} \tag{1.1}$$

where $\mathbf{n}(x)$ denotes the inward normal vector at $x \in \partial D$. The long time behavior of a "generic" solution (i.e., the solution corresponding to a "typical" initial condition) can be derived from the properties of the second eigenfunction using the following eigenfunction expansion. Under suitable conditions on the domain, such as convexity or Lipschitz boundary, and for a "typical" initial condition $u_0(x)$, we have

$$u(t,x) = c_1 + c_2 \varphi_2(x) e^{-\mu_2 t} + R(t,x), \tag{1.2}$$

where $c_1 \in \mathbb{R}$ and $c_2 \neq 0$ are constants depending on the initial condition, $\mu_2 > 0$ is the second eigenvalue for the Neumann problem in $D$, $\varphi_2(x)$ is a corresponding eigenfunction, and $R(t,x)$ goes to 0 faster than $e^{-\mu_2 t}$, as $t \to \infty$. Note that the first eigenvalue is equal to 0 and the first eigenfunction is constant. Suppose that $\varphi_2(x)$ attains its maximum at the boundary of $D$. Under this assumption, for "most" initial conditions $u_0(x)$, if $z_t$ is a point at which the function $x \to u(t,x)$ attains its maximum, then the distance from $z_t$ to the boundary of $D$ tends to zero as $t$ tends to $\infty$. In other words, the "hot spots" move towards the boundary.

**Hot Spots Conjecture** (Rauch (1974)). *The second eigenfunction for the Laplacian with Neumann boundary conditions in a bounded Euclidean domain attains its maximum at the boundary.*

The above version of the hot spots conjecture is somewhat ambiguous as it does not specify whether the maximum has to be strict, i.e., whether the eigenfunction can attain the same maximal value somewhere in the interior of the domain; it does not address the question of what might happen when the second eigenvalue is not simple, i.e., whether all eigenfunctions corresponding to the second eigenvalue have to satisfy the conjecture (in some domains, for example, the square, there are infinitely many eigenfunctions corresponding to the second eigenvalue). As we will see, it turns out that a precise statement of the conjecture is not needed because the results do not depend in a subtle way on its formulation.

The hot spots conjecture can be justified by appealing to our physical intuition and by examples amenable to explicit analysis. Intuitively, the "heat" and "cold" are substances that annihilate each other so it is easy to believe that the hottest and coldest spots lie as far as possible from each other, hence on the boundary of the domain. One can find explicit formulas for the eigenfunctions in some simple domains, for example, in a rectangle $[0, a] \times [0, b]$ with $a > b > 0$, we have $\varphi_2(x_1, x_2) = \cos(\pi x_1/a)$. All such explicit examples support the hot spots conjecture, i.e., the second eigenfunction attains the maximum on $\partial D$ in simple domains such as rectangles, discs and balls.

## §2. Main theorems on the "hot spots" problem

For 25 years, from 1974 to 1999, almost nothing was known about the "hot spots" conjecture. A notable exception was a result by Kawohl that appeared in his book [K] in 1985. Kawohl proved that if a set $D \subset \mathbb{R}^d$ is a cylindrical domain, i.e., if $d > 1$, and $D$ can be represented as $D = D_1 \times [0, 1]$ for some $D_1 \subset \mathbb{R}^{d-1}$, then the hot spots conjecture holds for $D$. This result has a simple proof based on the factorization of eigenfunctions in cylindrical domains. Kawohl's most lasting contributions are the realization that one should restrict attention to some classes of domains, and the statement of the currently most significant open problem in the area—Kawohl suggested that the hot spot conjecture might not be true in general but it should be true for convex domains.

The next paper on the hot spots conjecture, [BB1], appeared in 1999. The paper contained the proof of the hot spots conjecture for two classes of planar domains: domains with a line of symmetry and "lip" domains, to be described shortly. The results were not complete, in the sense that the authors imposed some extra "technical" assumptions on domains in each family. Those extra assumptions were removed for symmetric domains by Pascu [P] and for "lip" domains in [AB2].

Recall that a function $f$ is called Lipschitz with constant $c$ if $|f(x) - f(y)| \leq c|x - y|$ for all $x$ and $y$. A "lip" domain is a bounded planar domain such that its boundary consists of two graphs of Lipschitz functions with the Lipschitz constant equal to 1. For example, any obtuse triangle (i.e., a triangle with an angle greater than $\pi$) is a lip domain if it is properly oriented. In Fig. 2.1, $D_1$, $D_2$ and the interior of $\overline{D_1 \cup D_2}$ are lip domains.

**Theorem 2.1**. *The hot spots conjecture holds for $D \subset \mathbb{R}^2$ if*

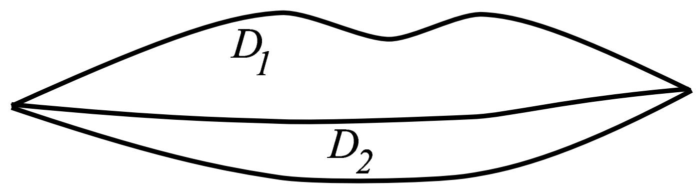


Figure 2.1.

(i) ([BB1], [P]) $D$ is convex and has a line of symmetry, or
(ii) ([BB1], [AB2]) $D$ is a lip domain.

The methods and techniques developed in [BB1] to prove the hot spots conjecture for some classes of domains turned out to be useful also in deriving negative results. The first of such results, [BW], appeared in 1999. The authors showed that there exists a planar domain where the second eigenvalue is simple and the eigenfunction corresponding to the second eigenvalue attains its maximum in the interior of the domain. This result was strengthened in [BB2], where it was shown that in some other planar domain, the second eigenvalue is simple and the second eigenfunction attains both its minimum and maximum in the interior of the domain. The domain constructed in [BB2] had many holes and the one constructed in [BW] had 2 holes. The intuitive idea behind the examples constructed in [BW] and [BB2] suggested that every counterexample to the hot spots conjecture in the plane must have at least two holes, and every counterexample in $\mathbb{R}^d$, $d \geq 3$, must have at least $d$ handles. This turned out not to be true—a new counterexample ([B2]) shows that there exists a planar domain with one hole and simple second eigenvalue, and such that the second eigenfunction attains both its maximum and minimum in the interior of the domain. The domain is depicted in Fig. 2.2. Its shape is much simpler than that of examples in [BW] and [BB2]. The maximum and minimum of the second eigenfunction are attained at the points marked on the figure.

**Theorem 2.2**. ([BW], [BB2], [B2]) *The hot spots conjecture fails for some domains $D \subset \mathbb{R}^2$.*

Before we discuss results related to the hot spots conjecture in various ways, we will state the most intriguing open problems in this area. The first one was proposed by Kawohl in [K], and the second one is known among the researchers interested in the subject.

**Open problems**. (i) ([K]) *Does the hot spots conjecture hold for bounded convex domains $D \subset \mathbb{R}^d$ for all $d \geq 1$?*

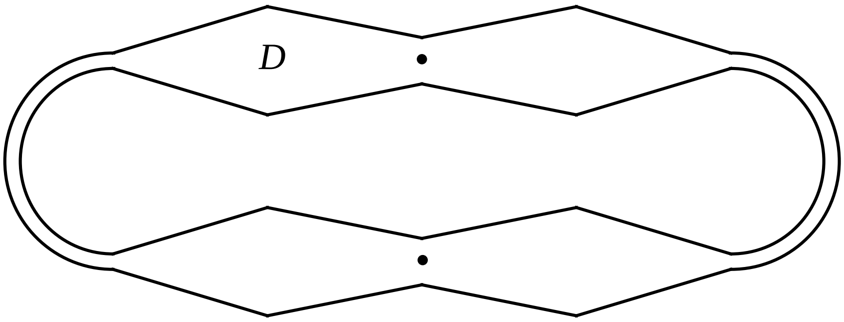


Figure 2.2.

(ii)*Does the hot spots conjecture hold for bounded simply connected planar domains?*

## §3. Results related to the "hot spots" problem

The hot spots conjecture inspired a number of papers on the properties of Neumann eigenfunctions. We will review those that seem to be the closest in spirit to the original conjecture. For a review of research in related areas, see [NTJ].

First of all, we mention a paper by Hempel, Seco and Simon [HSS], which appeared in 1991, long time before the current interest in the hot spots conjecture. The authors studied the spectrum of the Neumann Laplacian in bounded Euclidean domains with non-smooth boundaries. Roughly speaking, their results show that the spectrum does not need to be discrete, and in a sense, it can be completely arbitrary. For this reason, the hot spots conjecture must be limited to domains where the spectrum is discrete, such as domains with Lipschitz boundaries.

Athreya [A2] showed that some monotonicity properties of Neumann eigenfunctions hold also for solutions of some semi-linear partial differential equations related to a class of stochastic processes known as "superprocesses." He adapted the probabilistic techniques used in the research on the hot spots conjecture to the new setting.

Jerison [J] found the location (in an asymptotic sense) of the nodal line (i.e., the line where the eigenfunction vanishes) of the second Neumann eigenfunction in long and thin domains. Strictly speaking, this result is not directly related to the hot spots conjecture. However, the information about the location of the nodal line can be effectively used in the research on the hot spots conjecture. This was first done in [BB1], where the nodal line was identified with the line of symmetry in domains possessing a line of symmetry. The knowledge of the nodal line can be used to transform the Neumann problem to a problem with mixed Neumann and Dirichlet conditions—a problem much easier than the original one. Jerison and Nadirashvili considered in [JN] convex planar domains with two perpendicular lines of symmetry, and showed that under these strong assumptions one can provide some accurate information about the second eigenfunction. The location of the nodal line for the second eigenfunction is treated as a problem of its own interest in [AB1], where probabilistic techniques are used to give some results in this direction.

Atar investigated in [A1] a class of multidimensional domains. Techniques used in other papers on the hot spots problem seem to work only in planar domains so [A1] is the only paper (except for an early result in [K]) that contains results on the multidimensional version of the problem.

It was known for a long time, as a "folk law" among the experts in the field, that the hot spots conjecture does not hold for manifolds, see, e.g., remarks to this effect in [BB1] or [BB2]. However, the first rigorous paper studying the hot spots problem for manifolds was published by Freitas [F].

Although a paper by Ishige and Mizoguchi [IM] is not devoted to the hot spots problem in the sense of this article, it is related because it studies geometric properties of the heat equation solutions.

Two recent papers by Bañuelos and Pang, one of them joint with Pascu ([BP] and [BPP]) are devoted to variations of the hot spots problem. The purpose of [BP] is to prove an inequality for the distribution of integrals of potentials in the unit disk composed with Brownian motion which, with the help of Lévy's conformal invariance, gives another proof of Pascu's result [P]. The paper [BPP] investigates the "hot spots" property for the survival time probability of Brownian motion with killing and reflection in planar convex domains whose boundary consists of two curves, one of which is an arc of a circle, intersecting at acute angles. This leads to the "hot spots" property for the mixed Dirichlet-Neumann eigenvalue problem in the domain with Neumann conditions on one of the curves and Dirichlet conditions on the other.

## §4. Review of selected probabilistic techniques

The following review of techniques used in proofs of results related to the hot spots conjecture is highly subjective in its choices, dealing mostly with methods used by the author of this article in his own research. The review will mainly focus on "essential probabilistic techniques," i.e., those techniques that involve stochastic processes and cannot be easily translated into the language of analysis. A good way to illustrate this idea is to look at an example of a probabilistic concept that is *not* essential. The hitting distribution of Brownian motion on the boundary of a set can be identified with the harmonic measure—the two concepts are equivalent but knowing this equivalence does not immediately lead to any new results. We will focus on a probabilistic technique called "couplings." The technique was invented by Doeblin in 1930's and one can find a general review of this method in books by Lindvall [L] and Mu-Fa Chen [C]. The most frequent application of the coupling technique consists of a construction of two processes on the same probability space, run with the same clock. Often, the processes meet at a certain time, called the coupling time. Typically, the processes are *not* independent. One usually tries to find a coupling with as small coupling time as possible. A distinguishing feature of applications of couplings in the context of the hot spots conjecture is that the properties of the coupling time usually do not matter, and in a somewhat perverse way, the coupling time is infinite for some of the couplings. Couplings were used for the first time to study the hot spots conjecture in [BB1] but that paper owes a lot to an earlier project, [BK], devoted to a seemingly unrelated problem.

Many proofs of results on the hot spots conjecture are based on the eigenfunction expansion (1.2). First, a geometric property is proved for the heat equation and then it is translated into a statement about the second eigenfunction using (1.2), as $t \to \infty$.

For an introductory presentation of probabilistic concepts used below, such as Brownian motion, and their relationship to analysis, see a book by Bass [B1].

Let $X_t$ and $Y_t$ be reflected Brownian motions in $D$ starting from $x \in D$ and $y \in D$, resp. Then we can represent the solution $u(t,x)$ of the heat equation (1.1) as $u(t,x) = Eu_0(X_t)$, and similarly $u(t,y) = Eu_0(Y_t)$. We have by (1.2),

$$\begin{aligned} \varphi_2(x) - \varphi_2(y) &= c_3 e^{\mu_2 t}(u(t,x) - u(t,y)) + R_1(t,x,y) \\ &= c_3 e^{\mu_2 t}(Eu_0(X_t) - Eu_0(Y_t)) + R_1(t,x,y), \end{aligned} \tag{4.1}$$

where $R_1(t,x,y)$ goes to 0 as $t \to \infty$. Without loss of generality we will assume that $c_3 > 0$. Suppose that we can prove for some initial condition $u_0$ that for all $t > 0$,

$$Eu_0(X_t) - Eu_0(Y_t) \le 0. \tag{4.2}$$

This and (4.1) will then show that $\varphi_2(x) \le \varphi_2(y)$. If the last inequality can be proved for an appropriate family of pairs $(x, y)$, the hot spots conjecture will follow. We will next present a technique of proving (4.2).

For $x, y \in \mathbb{R}^2$, write $x \preceq y$ if the angle between $y - x$ and the positive horizontal half-line is within $[-\pi/4, \pi/4]$. Suppose that $D$ is a lip domain (defined in Section 2) and $x, y \in D$, $x \preceq y$. Suppose that $X_t$ and $Y_t$ are reflected Brownian motions in $D$, driven by the same Brownian motion, and starting from $x$ and $y$, resp. In other words,

$$\begin{aligned} X_t &= x + B_t + \int_0^t \mathbf{n}(X_s) dL_s^X, \\ Y_t &= y + B_t + \int_0^t \mathbf{n}(Y_s) dL_s^Y, \end{aligned} \tag{4.3}$$

where $\mathbf{n}(z)$ is the unit inward normal vector at $z \in \partial D$ and $L_s^X$ is the local time of $X$ on the boundary of $D$, i.e., $L^X$ is a non-decreasing process that does not increase when $X$ is inside $D$. In other words,

$$\int_0^\infty \mathbf{1}_D(X_s) dL_s^X = 0.$$

Similar remarks apply to the formula for $Y_t$. For domains which are piecewise $C^2$-smooth, the existence of processes satisfying (4.3) follows from results of Lions and Sznitman [LS]. For lip domains, one can use a recent result from [BBC]. The existence of a strong unique solution to an equation analogous to (4.3) but in a multidimensional Lipschitz domain remains an open problem at this time.

We have assumed that the domain $D$ is a lip domain so if the normal vector $\mathbf{n}(z)$ is well defined at $z \in \partial D$ (this is the case for almost all boundary points), it has to form an angle less than $\pi/4$ with the vertical. Then easy geometry shows that the "local time push" in (4.3), i.e., the term represented by the integral, is such that if $x \preceq y$ then

$$X_t \preceq Y_t \qquad \text{for all } t \ge 0. \tag{4.4}$$

Now consider a set $A \subset D$, such that both $A$ and $D \setminus A$ have a non-empty interior and $\partial A \cap \partial(D \setminus A)$ is a vertical line segment. Suppose that $A$ lies to the right of $D \setminus A$ and let the initial condition be $u_0(z) = \mathbf{1}_A(z)$. If (4.4) is satisfied, then for any fixed time $t \ge 0$, we may have $X_t, Y_t \in A$, or $X_t, Y_t \in D \setminus A$, or $X_t \in D \setminus A, Y_t \in A$, but we will never have $X_t \in A, Y_t \in D \setminus A$. This and the definition of $u_0$ imply (4.2). We combine this with (4.1) to conclude that $\varphi_2(x) \le \varphi_2(y)$ for $x \preceq y$. Any lip domain has the "leftmost" and "rightmost" points in the sense of the partial order "$\leq$" (see Fig. 2.1) so our argument has shown that the maximum and the minimum of the second eigenfunction are attained at these two points. Hence, the hot spots conjecture holds in lip domains.

Planar domains with a line of symmetry have to be approached in a different manner. Suppose that $D \subset \mathbb{R}^2$ is symmetric with respect to a vertical line $K$ and let $D_1$ be the part of $D$ lying to the right of $K$. Under some extra assumptions, the second eigenfunction $\varphi_2$ in $D$ with the Neumann boundary conditions is antisymmetric with respect to $K$ (this follows from a simple symmetrization argument). Therefore, $\varphi_2$ must vanish on $K$ and we see that $\varphi_2$ is the first eigenfunction for the Laplacian in $D_1$ with the Neumann boundary conditions on $\partial D_1 \setminus K$ and Dirichlet boundary conditions on $K$. Such boundary conditions correspond to the Brownian motion in $D_1$ that is reflected on $\partial D_1 \setminus K$ and killed on $K$.

We will choose the initial condition $u_0$ to be identically equal to 1 in $D_1$. Let $T_K^X$ be the hitting time of $K$ by $X$ and let $T_K^Y$ have the analogous meaning for $Y$. The strategy now is to construct Brownian motions $X_t$ and $Y_t$ in $D_1$, reflected on $\partial D_1 \setminus K$, killed on $K$, starting from $x$ and $y$, and such that (4.2) holds not for a fixed time $t$ but for an appropriate stopping time $T$. Let $T = T_K^X$. If we can show that $X$ must hit $K$ before $Y$ does, then (4.2) follows and we have $\varphi_2(x) \leq \varphi_2(y)$ for this particular pair $(x, y)$. We will not go into details of how it is best to choose $x$ and $y$ and what assumptions one must make about the geometry of $D$ to carry out the argument outlined above. Instead, we will describe a coupling of reflected Brownian motions (the "mirror" coupling) that keeps the two Brownian particles in a relative position that ensures that $T_K^X \leq T_K^Y$.

Let us start by defining the mirror coupling for free Brownian motions in $\mathbb{R}^2$. Suppose that $x, y \in \mathbb{R}^2$, $x \neq y$, and that $x$ and $y$ are symmetric with respect to a line $M$. Let $X_t$ be a Brownian motion starting from $x$ and let $\tau$ be the first time $t$ with $X_t \in M$. Then we let $Y_t$ be the mirror image of $X_t$ with respect to $M$ for $t \leq \tau$, and we let $Y_t = X_t$ for $t > \tau$. The process $Y_t$ is a Brownian motion starting from $y$. The pair $(X_t, Y_t)$ is a "mirror coupling" of Brownian motions in $\mathbb{R}^2$.

Next we turn to the mirror coupling of reflected Brownian motions in a half-plane $\mathcal{H}$, starting from $x, y \in \mathcal{H}$. One can construct reflected Brownian motions $X_t$ and $Y_t$ in $\mathcal{H}$, starting from $x$ and $y$, so that they have the following properties. The processes $X_t$ and $Y_t$ behave like free Brownian motions coupled by the mirror coupling as long as they are both strictly inside $\mathcal{H}$. When one of the processes hits the boundary, the two particles cannot behave as a "free" mirror coupling in the whole plane. We will describe their motion by specifying constraints on the particles—otherwise they can move in an arbitrary way. Let $M$ be the line of symmetry for $x$ and $y$ and $H = M \cap \partial\mathcal{H}$. Then for every $t$, the distance from $X_t$ to $H$ is the same as for $Y_t$. Let $M_t$ be the line of symmetry for $X_t$ and $Y_t$. The "mirror" $M_t$ may move, but only in a continuous way, while the point $M_t \cap \partial\mathcal{H} = H$ will never move. The absolute value of the angle between the mirror and the normal vector to $\partial\mathcal{H}$ at $H$ can only decrease. These properties are illustrated in Fig. 4.1. The processes stay together after the first time they meet. The most important property of the mirror coupling is that the two processes $X_t$ and $Y_t$ remain at the same distance from a fixed point, the "hinge" $H$.

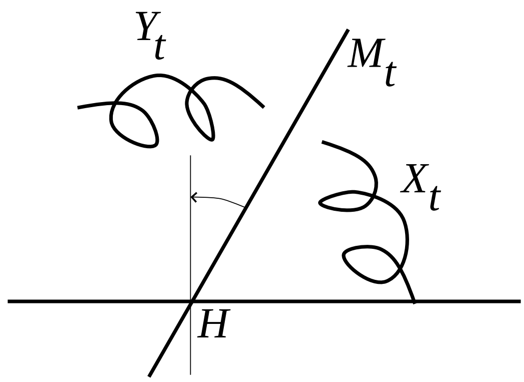


Figure 4.1.

When $D$ is a polygonal domain, the processes $X_t$ and $Y_t$ will reflect on different sides of $\partial D$ at different times. Since the reflecting particle cannot sense the global shape of the domain, the above description of the mirror coupling in a half-plane can be applied to describe the possible motions of the mirror (the line of symmetry between the processes) whenever only one of the processes is on the boundary. This simple recipe breaks down when the two processes hit the boundary at the same time. It is not obvious that two processes forming a mirror coupling can indeed hit the boundary at the same time but we conjecture that it is indeed true. The construction of the mirror coupling following the time when the two processes are simultaneously on the boundary has not been properly addressed in [BK] and [BB1]. In an earlier paper of Wang [W], mirror couplings were used without any proof of their existence. This unsatisfactory situation has been remedied recently as the full proof of the existence of mirror couplings in piecewise smooth domains has been given in [AB2], and the motion of the mirror following the time when both particles are on the boundary has been analyzed in [B2].

We will not present a detailed analysis of the motion of two particles related by a mirror coupling in a planar domain. The arguments involve no more than high school geometry.

The last coupling to be presented here is a "scaling coupling" introduced by Pascu [P]. This coupling is the most complex of the three couplings so we will only sketch the main ideas of this technique. The main objective of any coupling technique is to construct two processes whose relative motion is highly restricted, although each of the processes by itself is a reflected Brownian motion. This can lead to a condition such as (4.4) that can be in turn translated into an analytic statement using a formula such as (4.2).

Pascu's idea was to start with a planar Brownian motion $X_t$ and let $Y_t = X_{at}/\sqrt{a}$, for some fixed $a > 0$. It is well known that $Y$ is also a planar Brownian motion. The novelty of this coupling lies in the fact that although the shape of the trajectory of $Y$ is a scaled image of the shape of the trajectory of $X$, the corresponding pieces of the trajectory occur at different times. In other words, the two processes run with different clocks. This rules out straightforward reasoning such as that in (4.1)-(4.4) but nevertheless Pascu managed to translate the information about possible geometric positions of the two processes into an analytic statement.

Two further technical aspects of scaling couplings should be mentioned here. The hot spots problem needs a construction of a pair of reflected Brownian motions in a domain $D$, not free Brownian motions in the whole plane. Hence, the simple scaling idea has to be modified in a way somewhat reminiscent of the way the mirror coupling in the plane is modified to handle reflected Brownian motions, because if $X$ is a reflected Brownian motion in $D$ then $Y_t = X_{at}/\sqrt{a}$ is not. Second, Pascu combined scaling couplings with conformal mappings in order to be able to handle arbitrary convex domains with a line of symmetry (the first step was to do the construction in a semi-disc). Conformal mappings preserve reflected Brownian motions but they require a time change. It was a very non-trivial observation of Pascu that the time change involved in his argument had the properties needed to finish the argument when the domain was convex.

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# Brownian motion and harmonic measure in conic sections

**Tom Carroll**

**Abstract.**

This is a survey of results on the exit time and the exit position of Brownian motion from cones and parabola-shaped regions in Euclidean space. The paper begins with a section on harmonic measure.

## §1. Harmonic measure

### 1.1. The Dirichlet Problem and harmonic measure

Harmonic measure has long been a central theme of Potential Theory: that this is as true today as it was in the past is confirmed by the recent publication of the major book *Harmonic measure* by Garnett and Marshall [13].

The Dirichlet Problem is the boundary value problem for the Laplace equation: given a region $D$ in $\mathbf{R}^n$ and a bounded continuous function $f$ on the boundary of $D$, one is to find a function $u$ on the closure $\overline{D}$ of the region with the properties that

(i) $u$ is continuous on $\overline{D}$,
(ii) $u$ is harmonic in $D$, that is $\Delta u \equiv 0$ in $D$,
(iii) $u|_{\partial D} = f$

This boundary value problem arises in a number of physical contexts, for example that of determining the steady state temperature inside a region when the temperature on the boundary of the region is specified. This physical interpretation also sheds light on a defining characteristic of harmonic functions: they are the functions that satisfy the mean-value property, in that the average value of a harmonic function over

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a sphere is its value at the centre of the sphere [9, Theorem 1.9][1]. If the average steady state temperature on the sphere was, say, greater than the temperature at its centre there would then be a net flow of heat from the hotter sphere to its cooler centre. The temperature at the centre would then increase and this would not, after all, be a steady state temperature distribution.

The Dirichlet Problem as stated does not always have a solution: and even when it does it may not be unique, depending on how one treats points at infinity. Regions for which a solution exists, no matter what the specified boundary function may be, are called *regular* for the Dirichlet Problem. In the classical approach to characterizing such domains, a boundary point $\zeta$ is said to be *regular* if there is a barrier at that point, this being a function that is superharmonic and positive in $D$ near $\zeta$ and that tends to zero on approach to the boundary at $\zeta$. A domain is then proved to be regular if all of its boundary points are regular [15, Theorem 2.10] [1, Chapter 6]. The approach taken in Hayman and Kennedy's book is to deal first with bounded regular domains, and to consider unbounded possibly irregular domains later when the extra machinery needed, in particular that of polar sets, is in place [15, Section 5.7.1]. This is a dichotomy that may profitably be kept in mind.

A solution of the Dirichlet Problem corresponding to the continuous boundary function $f$ is called a harmonic extension of $f$. Such a harmonic extension may be considered for more general boundary data (with a suitable reformulation of condition (i)) [15, Theorems 2.10 and 2.17]. In particular, corresponding to a Borel measurable subset $E$ of the boundary of $D$, the boundary data $f = 1_E$ (so that $f$ takes the value 1 on the part $E$ of the boundary and 0 on the remaining boundary) has a harmonic extension into $D$. This solution is called the *harmonic measure* of $E$ and is denoted by $\omega(x, E; D)$. As a function of $x$, therefore, it is harmonic. If $x$ is held fixed and $E$ varies, then $\omega(x, E; D)$ is a measure on the boundary of $D$ [15, Theorem 3.10]. The harmonic extension of a general Borel measurable function can then be constructed by integration with respect to harmonic measure on the boundary. Intuitively, it helps to think of the harmonic measure of $E$ as the solution of the Dirichlet Problem with boundary data $f = 1_E$. From a technical point of view, it is best to construct harmonic measure as the measure

[1] This is but one of many possible references where a precise statement of this theorem may be found. I have chosen not to give perfectly precise statements of results in this article, with the excuse that specialists will know these results and will not need to read this article, while the rest of us are now forewarned not to take statements too literally and to consult at a minimum the cited references.

$\omega(x, E; D)$ on the boundary of $D$ for which

$$\int_{\partial D} f(\zeta)\, d\omega(x, \zeta; D)$$

returns the value at $x$ of the harmonic extension of $f$ into $D$ for any continuous bounded function $f$ on the boundary of $D$.

The study of harmonic measure in planar domains is facilitated by conformal mapping, as harmonic measure is *conformally invariant*. By conformal invariance of harmonic measure we mean that, if $f$ is analytic and one-to-one in the planar domain $D$, then

$$\omega(z, E; D) = \omega\left(f(z), f(E); f(D)\right).$$

To see why this holds true, we write $h$ for the function $\omega\left(\cdot, f(E); f(D)\right)$. Then $h \circ f$ satisfies

$$\Delta(h \circ f)(z) = (\Delta h)(f(z))\, |f'(z)|^2, \quad z \in D.$$

Thus $h \circ f$ is harmonic in $D$, since $h$ is harmonic in $f(D)$, and its boundary values are $1_E$. Hence $(h \circ f)(z) = \omega(z, E; D)$.

For example, to compute the harmonic measure $\omega(z, E; D)$ for a simply connected planar domain $D$, one maps $D$ conformally onto the unit disk $U$ by a map $f$ for which $f(z) = 0$. Then

$$\omega(z, E; D) = \omega(0, F; U), \quad F = f(E),$$

the latter being the normalized angular measure of $F$ on the unit circle. While this 'solves' the problem in principle, in practice relatively few explicit conformal mappings are known.

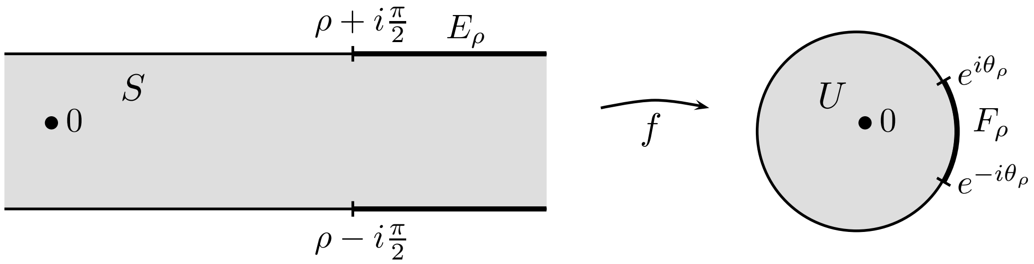
As a simple example that is relevant to the subject matter of this paper, we will compute the rate of decay of harmonic measure in the infinite strip $S = \{z\colon |\mathrm{Im}\, z| < \pi/2\}$. We set $E_\rho = \{z\colon |\mathrm{Im}\, z| = \pi/2 \text{ and } \mathrm{Re}\, z > \rho\}$ and set $\omega(\rho) = w(0, E_\rho; S)$.

$$f(z) = \frac{e^z - 1}{e^z + 1}$$

is a conformal map of $S$ onto the unit disk with $f(0) = 0$. For $\rho > 0$, the image $F_\rho$ of $E_\rho$ under $f$ is the shorter arc of the unit circle lying between $e^{-i\theta_\rho} = f(\rho - i\pi/2)$ and $e^{i\theta_\rho} = f(\rho + i\pi/2)$. The harmonic measure of this arc at 0 is its normalized angular measure, which is $\theta_\rho/\pi$.

Since

$$f\left(\rho + i\frac{\pi}{2}\right) = \frac{e^{\rho + i\pi/2} - 1}{e^{\rho + i\pi/2} + 1} = \frac{e^{2\rho} - 1 + 2ie^\rho}{1 + e^{2\rho}},$$




we see that

$$\theta_\rho = \arg\left[e^{2\rho} - 1 + 2ie^\rho\right] = \arctan\left(\frac{2e^\rho}{e^{2\rho}-1}\right).$$

Thus, in summary,

$$\omega(\rho) = w(0, E_\rho; S) = w(0, F_\rho; U) = \frac{1}{\pi}\arctan\left(\frac{2e^\rho}{e^{2\rho}-1}\right),$$

from which it follows that

$$\omega(\rho) = \frac{2}{\pi}e^{-\rho} + \mathrm{O}\left(e^{-3\rho}\right) \text{ as } \rho \to \infty.$$

### 1.2. Brownian motion and harmonic measure

Brownian motion in $\mathbf{R}^n$ is a mathematical model of the position of a particle that is subject to random buffeting with no preferred direction and whose intensity is independent of position. The possible paths of the particle are the continuous functions $\omega : [0,\infty) \to \mathbf{R}^n$. Brownian motion may be viewed as a measure, known as Wiener measure, on this space of continuous paths. We write $B_t(\omega) = \omega(t)$ for the position of the particle at time $t$ if it follows the path $\omega$. Then each $B_t$ is a random variable on path space. Wiener measure on path space is constructed so that (i) the net displacements in disjoint time intervals are independent and (ii) the net displacement $B_t - B_s$ between time $s$ and $t$ (with $s < t$) is normally distributed with mean zero and covariance matrix $t-s$ times the identity matrix. As is customary, we write $P_x$ to denote the probability (Wiener measure) of events (measurable sets of paths) that gives full measure to the paths with initial point $x$, and we write $E_x$ for the expectation (integral) with respect to $P_x$.

For a region $D$, we write

$$\tau_D = \inf\{t > 0\colon B_t \notin D\}.$$

This is the first exit time of Brownian motion from the region $D$, and plays a key role in this story. The exit time is a random variable, as its value depends on the particular path. The first exit position of Brownian motion from $D$ is then $B_{\tau_D}$.

The connection between Brownian motion and harmonic measure, first elucidated by Kakutani, is simply this: harmonic measure at $x$ in $D$ is the exit distribution from $D$ of Brownian motion with initial point $x$ [9, Section 1.4]. That is,

$$\omega(x, E; D) \;=\; P_x(\tau_D < \infty, B_{\tau_D} \in E),$$

for each Borel measurable subset $E$ of the boundary of $D$. In fact, for any domain $D$ and any bounded function $f$ on the boundary of $D$

$$h(x) \;=\; E_x\left[f(B_{\tau_D})\, 1_{\{\tau_D<\infty\}}\right], \quad x \in D,$$

is harmonic. This becomes 'obvious' once we remember that harmonic functions are those with the mean-value property. To see why, let's consider a sphere $S_x$ in $D$ with centre $x$, and consider those paths that first hit the sphere at a point $y$ on the sphere, before then going on to exit $D$. When we delete the initial sections between $x$ and $y$, the new paths constitute a new Brownian motion starting from $y$, and this doesn't change the exit position. Therefore the paths from $x$ that exit the sphere $S_x$ at $y$ contribute $h(y)$ times the probability of first exiting the sphere at $y$, and

$$h(x) \;=\; E_x\left[f(B_{\tau_D})\, 1_{\{\tau_D<\infty\}}\right] \;=\; \int_{S_x} h(y)\, d\sigma(y),$$

where $d\sigma$ is the distribution of the first hitting position on the sphere of Brownian motion with initial point at its centre. But $\sigma$ just has to be the uniform distribution on the sphere, which gives the mean-value property for $h$.

The probabilistic characterization of the regularity of a boundary point for the Dirichlet Problem is more intuitive than that involving the barrier function. A boundary point $\zeta$ of a region $D$ is regular if $P_\zeta(\tau_D = 0) = 1$ [20, Section 9.2]: there must be enough boundary near the boundary point $\zeta$ so that a Brownian motion with initial point $\zeta$ will immediately hit the boundary with probability one. If $\zeta$ is a regular boundary point, it is then not too hard to see that a Brownian motion, whose initial point $x$ is in $D$ and is near $\zeta$, will exit $D$ near $\zeta$ with high probability. If $f$ is continuous at $\zeta$, it will follow that $f(B_{\tau_D})$ will be close to $f(\zeta)$ with high probability. In effect, the function $h$ will be continuous at $\zeta$. This is the probabilistic solution of the Dirichlet problem [20, Section 9.2], [9, Section 1.6]. The case $f = 1_E$ is the assertion that harmonic measure and hitting probabilities of Brownian motion are one and the same.

### 1.3. Distortion Theorems

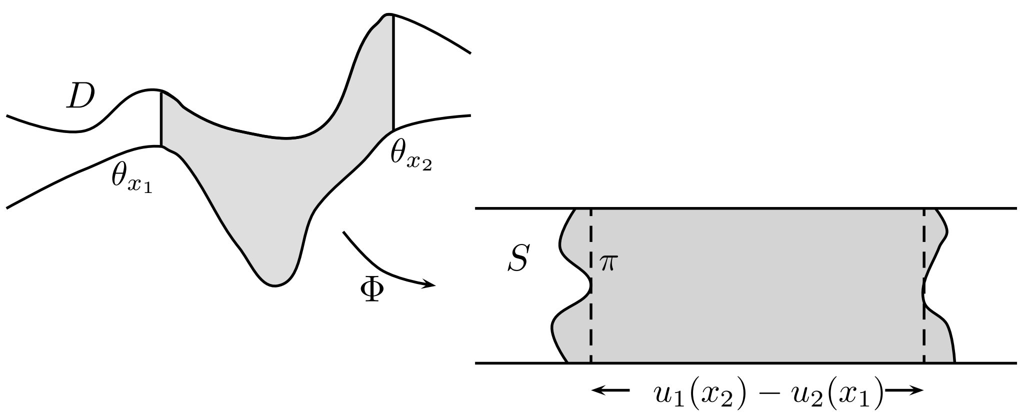
In his 1930 thesis, Ahlfors proved a distortion theorem for conformal mappings with which he settled the Denjoy conjecture and which has since proved to be useful and influential. We follow the eminently readable account in [2].

A domain $D$ is said to be *strip like* if it is simply connected and contains a curve $\beta(t)$, $0 < t < 1$, with $\operatorname{Re}\beta(t) \to -\infty$ as $t \to 0$ and $\operatorname{Re}\beta(t) \to \infty$ as $t \to 1$. Thus the curve determines two prime ends, that we call $-\infty$ and $\infty$. The domain $D$ is then mapped conformally onto the strip $S$ by a map $\Phi$ with $\operatorname{Re}\Phi(\beta(t)) \to -\infty$ as $t \to 0$ and $\operatorname{Re}\Phi(\beta(t)) \to \infty$ as $t \to 1$. For each $x$, the intersection of the vertical line $\operatorname{Re} z = x$ with $D$ consists of open line segments, one of which separates the two prime ends determined by the curve $\beta$. This crosscut of $D$ is traditionally labelled $\theta_x$ and its length is written as $\theta(x)$. Finally, we write

$$u_2(x) = \sup_{z\in\theta_x} \operatorname{Re}\Phi(z) \quad \text{and} \quad u_1(x) = \inf_{z\in\theta_x} \operatorname{Re}\Phi(z).$$

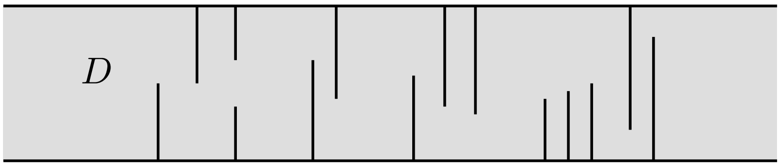
**Ahlfors' Distortion Theorem** *If* $\displaystyle\int_{x_1}^{x_2} \frac{dx}{\theta(x)} > 2\pi$, *then*

$$\frac{1}{\pi}\int_{x_1}^{x_2} \frac{dx}{\theta(x)} \le u_1(x_2) - u_2(x_1) + 4.$$

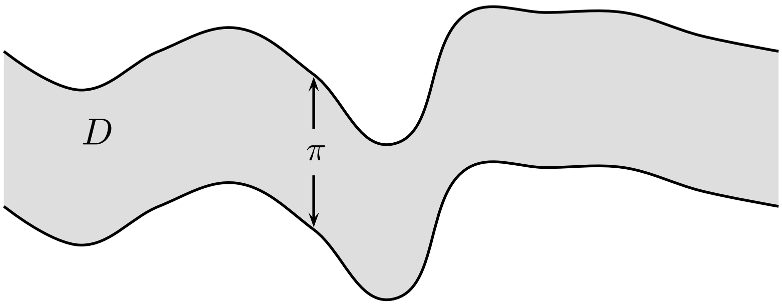



In geometric terms, this theorem is a lower bound on the area $\pi\,[u_1(x_2) - u_2(x_1)]$ of the largest rectangle contained in the image under $\Phi$ of that part of $D$ between the crosscuts $\theta_{x_1}$ and $\theta_{x_2}$. The distortion theorem shows that if the strip like domain is narrow, the conformal map $\Phi$ must stretch the distance, in the image strip, between the images of crosscuts in $D$.

There are essentially two situations in which Ahlfors' Distortion Theorem underestimates the rate of growth of the mapping $\Phi$. If, say, vertical slits are removed from $D$ then the integral $\int_{x_1}^{x_2} dx/\theta(x)$ will not detect them, yet one knows that the mapping $\Phi$ will then grow much faster. For example, in the case of the strip $S$ with as many slits removed as one may wish, the distortion theorem treats $\Phi$ as if it was the identity map. The second situation in which $\Phi$ grows faster than




predicted by the Ahlfors Distortion Theorem is typified by taking the strip $S$ and bending it into the shape of a snake or the crenellations on a castle, while leaving the lengths of the crosscuts unchanged.




In 1942, Warschawski [24] proved an upper bound on the area $\pi\,[u_2(x_2) - u_1(x_1)]$ of the smallest rectangle that contains the image under $\Phi$ of that part of $\Omega$ between the crosscuts $\theta_{x_1}$ and $\theta_{x_2}$. His theorems involve extra terms that measure the oscillation of the central line of the strip like domain and the oscillation of the width of the domain. We will state a special and slightly weaker case of his results that will be sufficient for our purposes.

**Distortion Theorem for certain symmetric domains** *Suppose that $D$ is a strip like domain that takes the form*

$$D = \left\{ z \colon |\mathrm{Im} z| < \frac{1}{2}\theta(\mathrm{Re} z) \right\},$$

*where $\theta$ is a non negative function on the real line with*

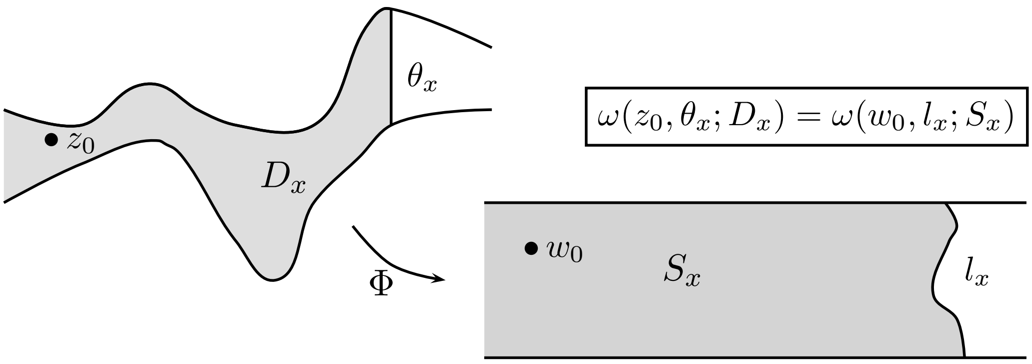
$$\int_{-\infty}^{\infty} \frac{\theta'(x)^2}{\theta(x)}\, dx \;<\; \infty.$$

*Then, for $z = x + iy \in D$ and fixed $x_0$,*

$$\operatorname{Re}\Phi(z) \;=\; \pi \int_{x_0}^{x} \frac{ds}{\theta(s)} + \mathrm{O}(1) \;\text{ as } x \to \infty.$$

### 1.4. Higher dimensional distortion theorems

The distortion theorems of Ahlfors and Warschawski may be viewed as harmonic measure estimates. Suppose that we wish to estimate the harmonic measure of the crosscut $\theta_x$, with respect to that part $D_x$ of $D$ to the left of $\theta_x$, at some fixed point $z_0$. Having mapped $D$ onto the strip $S$ by the map $\Phi$, we need the harmonic measure of the curve $l_x = \Phi(\theta_x)$ with respect to that part $S_x$ of the strip to the left of $l_x$, evaluated at the fixed point $w_0 = \Phi(z_0)$.




We infer the position of $l_x$ from the distortion theorems, in which we fix $x_1$ and take $x_2$ to be a varying $x$. Ahlfors lower bound on $u_1(x_2) - u_2(x_1)$ becomes, in effect, a lower bound on $u_1(x)$, and implies that $l_x$ must lie at least a certain distance to the right in the strip $S$. This gives an upper bound on the harmonic measure of $l_x$, as harmonic measure of a vertical cross cut in $S$ decreases as the cross cut is moved to the right. Warschawski's upper bound on $u_2(x)$ shows that $l_x$ cannot be too far to the right in the strip $S$, and therefore leads to a lower bound on the harmonic measure of $l_x$. We can then estimate the harmonic measure, by using the example at the end of Section 1.1. One will ideally end up with an estimate involving

$$\exp\left[-\pi \int_{x_0}^{x} \frac{ds}{\theta(s)}\right].$$

There are harmonic measure versions of these distortion theorems, particularly upper bounds, that work in any finite dimension, and for non simply connected domains in the plane. Tsuji, building on work of Carleman, proved just such an estimate from above involving a term analogous to that arising in the simply connected planar case. For the many developments in this area, the reader need go no further than the books by Tsuji [23] and Ohtsuka [19], forgetting neither Baernstein's account [2], nor Haliste's complete and careful exposition [14], nor Section 8.1.7 in Hayman [16].

## §2. Cones

### 2.1. Burkholder's 1977 paper on exit times of Brownian motion

In [7], Burkholder studies Brownian motion in $\mathbf{R}^n$ with starting point $x \in \mathbf{R}^n$ and an accompanying stopping time $\tau$. With

$$B_\tau^* \;=\; \sup_t |B_{t\wedge\tau}|,$$

he proves that if one of the random variables $\sqrt{n\tau + |x|^2}$, $|B_\tau|$ or $B_\tau^*$ is $p^{\text{th}}$-power integrable, with $p \in (0,\infty)$, then so are they all. To deduce that $B_\tau^*$ is $p^{\text{th}}$-power integrable if $|B_\tau|$ is $p^{\text{th}}$-power integrable, it is assumed that $\mathrm{E}_x \log\tau < \infty$ if the dimension is 2 and that $\mathrm{P}_x(\tau < \infty) = 1$ in higher dimensions. The norms of these three random variables are then comparable, with constants that depend only on $p$ and $n$.

These results are applicable when $\tau$ is the first exit time from a domain $D$ in $\mathbf{R}^n$, in which case he proves the additional result that $\tau^{1/2}$ is $p^{\text{th}}$-power integrable if and only if the function $|x|^p$ has a harmonic majorant in $D$ (this being a function $u$ that is harmonic in $D$, and for which $|x|^p \le u(x)$ for all $x$ in $D$). In Section 4 of the paper, Burkholder specializes to the case when $D$ is the image of a conformal map $F$ of the unit disk in two dimensions, and brings $H^p$ spaces into play.

### 2.2. Integrability of exit time and exit place for a cone

As a 'simple' application of his results, Burkholder works out everything for a right circular cone

$$\Gamma_\alpha = \{x \in \mathbf{R}^n \setminus \{0\}, 0 \le \theta(x) < \alpha\}$$

where $\theta(x)$ is the angle between $x$ and $(1,0,\dots,0)$. Let us write $T_\alpha$ for the exit time from $\Gamma_\alpha$, and go through the argument in two dimensions. If $p\alpha < \pi/2$ the function

$$u(x) = |x|^p \cos(p\theta)/\cos(p\alpha)$$

is harmonic in $\Gamma_\alpha$, and $|x|^p \le u(x)$ there. In this case, $|x|^p$ has a harmonic majorant and $T_\alpha^{1/2}$ is in $L^p$. In the case $p\alpha = \pi/2$,

$$u(x) = |x|^p \cos(p\theta)$$

is a harmonic function in $\Gamma_\alpha$, vanishes on the boundary of $\Gamma_\alpha$, and satisfies $0 < u(x) \le |x|^p$ in $\Gamma_\alpha$. From this Burkholder deduces that $T_\alpha^{1/2}$ is not in $L^p$. In fact, fixing any $x$ in the cone, he chooses a sequence of bounded domains $R_j$, each containing $x$, with $\overline{R_j} \subset R_{j+1}$ and whose union is the whole cone. He writes $T_j$ for the exit time from $R_j$. If $T_\alpha^{1/2}$ was $p^{\text{th}}$-power integrable then so would $B^*_{T_\alpha}$. Moreover, for each $j$,

$$\left|u\left(B_{T_j}\right)\right| \le |B_{T_j}|^p \le B^*_{T_\alpha} \quad (\text{since } T_j \le T_\alpha).$$

Since $u$ is harmonic and bounded in $R_{j+1}$, $\{u(B_{T_j \wedge t})\}_{t\ge 0}$ is a martingale and so, by optional stopping,

$$u(x) = E_x u(B_{T_j}).$$

Hence, by dominated convergence,

$$0 < u(x) = \lim_{j\to\infty} E_x u(B_{T_j}) = E_x\left[\lim_{j\to\infty} u(B_{T_j})\right] = E_x u(B_{T_\alpha}) = 0,$$

a contradiction. Thus $T_\alpha^{1/2} \notin L^p$ if $p\alpha = \pi/2$. Hence

$$T_\alpha^{1/2} \in L^p \iff \alpha < \frac{\pi}{2p}.$$

The same method works for higher dimensional cones, with the role of $|x|^p\cos(p\theta)$ being played by $|x|^p h(\theta)$ where $h$ is a certain hypergeometric function. This function has a smallest positive zero $\theta_{p,n}$ with $\theta_{p,n} < \pi$, and then

$$T_\alpha^{1/2} \in L^p \iff \alpha < \theta_{p,n}.$$

### 2.3. Some further developments

Burkholder's results, and the approach he took, gave rise to significant further research, for example that of Essén and Haliste [12]. Sakai [21] proved an interesting isoperimetric inequality in this area: if $u$ is the least harmonic majorant of $|x|^p$ in a bounded domain $D$ that contains 0, then

$$u(0)^{1/p} \le c\, r(D),$$

for some finite $c$ depending only on $p$ and the dimension. Here $r(D)$ is the volume radius of $D$, the radius of a ball in $\mathbf{R}^n$ with the same volume as $D$. He furthermore proved several interesting results about the best constant $c(p,n)$, bringing into play, as did Burkholder, exit times of Brownian motion, Hardy spaces and estimates of solutions of Poisson equations.

Expressions for the distribution function of the exit time from a cone have been obtained by Spitzer [22] in the planar case, from which the integrability result $T_\alpha^{1/2} \in L^p$ if and only if $\alpha < \pi/(2p)$ follows, by DeBlassie [10] in higher dimensions, and also in Bañuelos and Smits [5].

## §3. Paraboloids

### 3.1. Exit time

Relatively recently, Bañuelos, DeBlassie and Smits [4] set themselves the task of uncovering the tail distribution of the exit time of Brownian motion from another conic section, the parabola

$$\mathcal{P} \;=\; \left\{ z\colon \operatorname{Re} z > 0 \text{ and } |\operatorname{Im} z| < \sqrt{\operatorname{Re} z} \right\}$$

in the plane. The exit time from any bounded domain is exponentially integrable, while that from a cone is only power integrable. The authors' goal was to find a domain for which the integrability of its exit time was intermediate between these two extremes. The parabola can be fitted inside a cone whose aperture is as small as one may wish, simply by putting the vertex of the cone far out on the negative real axis and the axis of the cone in the direction of the positive real axis. The exit time from the parabola is less than that from the larger cone, and the latter will be in $L^p$ for large $p$ because the aperture of the cone is small. Consequently, the exit time $\tau_{\mathcal{P}}$ from the parabola is $p^{\text{th}}$-power integrable for every finite $p$. On the other hand, $E_x\left[\exp(c\tau_{\mathcal{P}})\right]$ cannot be finite for any positive $c$, since $\mathcal{P}$ contains disks of arbitrarily large radius. For a disk $B$ of radius $r$ and centre $x$ contained in $\mathcal{P}$, one has that $E_x\left[\exp(c\tau_{\mathcal{P}})\right] \geq E_x\left[\exp(c\tau_B)\right]$, and the latter is infinite when $c > a/r^2$, for an absolute constant $a$.

The estimate obtained by Bañuelos, DeBlassie and Smits for the distribution function of the exit time is

$$A_1 \leq \liminf_{t\to\infty} t^{-1/3} \log \frac{1}{P_z(\tau_{\mathcal{P}} > t)} \;\leq\; \limsup_{t\to\infty} t^{-1/3} \log \frac{1}{P_z(\tau_{\mathcal{P}} > t)} \leq A_2$$

for some positive constants $A_1$ and $A_2$. This indicates that $P_z(\tau_\alpha > t)$ may be about the size of $\exp\left[-A\, t^{1/3}\right]$.

This estimate was extended to higher dimensions and to more general paraboloids by Wembo Li [17], but it was Lifshits and Shi [18] who solved the problem completely. They showed that, for the exit time $\tau_\alpha$ from the parabola-shaped region

$$\mathcal{P}_\alpha \;=\; \{(x,Y) \in \mathbf{R} \times \mathbf{R}^{n-1} \colon x > 0,\, |Y| < Ax^\alpha\}$$

(where $A > 0$ and $0 < \alpha < 1$),

$$\lim_{t\to\infty} t^{\frac{\alpha-1}{\alpha+1}} \log \frac{1}{P_z(\tau_\alpha > t)}$$

exists, and they determined the finite, positive limit explicitly. In particular, they proved that

$$\lim_{t\to\infty} t^{-\frac{1}{3}} \log \frac{1}{P_z(\tau_{1/2} > t)} = \frac{3\pi^2}{8}.$$

The distribution function of the exit time from a related very general class of unbounded domains is investigated in [11]. Using the results of Lifshits and Shi, van den Berg [6] studied the behaviour of the heat kernel in parabola-shaped regions.

### 3.2. Exit place

In [3], Bañuelos and the author investigated the rate of decay of harmonic measure in the parabola-shaped regions $\mathcal{P}_\alpha$ in $\mathbf{R}^n$ or, equivalently, the distribution function of the exit position of Brownian motion from such domains. Setting $E_\rho$ to be that part of the boundary of $\mathcal{P}_\alpha$ lying outside the ball of centre 0 and radius $\rho$, we wished to estimate

$$\omega(\rho) = \omega(z_0, E_\rho; \mathcal{P}_\alpha) = P_{z_0}\left(|B_{\tau_\alpha}| > \rho\right),$$

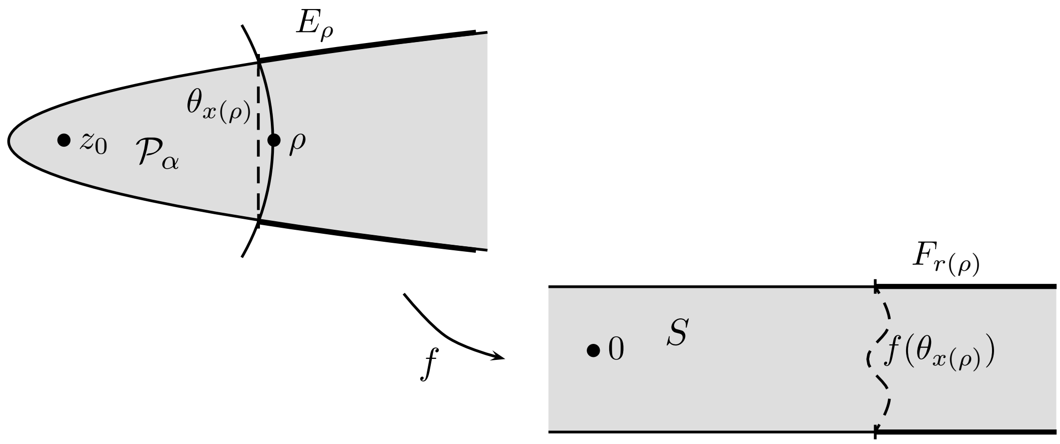
for some fixed $z_0$, say $z_0 = (1, 0, \ldots, 0)$, as accurately as possible.

This problem is easy to solve in the case of two dimensions using the techniques described earlier in this article. We map $\mathcal{P}_\alpha$ onto the strip $S$ by a symmetric conformal mapping $f$ with $f(z_0) = 0$. The part $E_\rho$ of the boundary of $\mathcal{P}_\alpha$ starts from the cross cut at $x = x(\rho)$, where $\rho - \rho^{2\alpha-1} < x(\rho) < \rho$. The length of the cross cut of $\mathcal{P}_\alpha$ at $x$ is $\theta(x) = 2x^\alpha$, which satisfies

$$\int^\infty \frac{\theta'(x)^2}{\theta(x)}\, dx \;<\; \infty.$$

Thus the Distortion Theorem for Certain Symmetric Domains of Section 1.3 is applicable and yields that the image of the cross cut $\theta_{x(\rho)}$ is within a bounded distance of

$$\frac{\pi}{2(1-\alpha)}\, x(\rho)^{1-\alpha}.$$




Writing $F_r$ for that part of the boundary of the strip $S$ where the real part is greater than $r$, the image of $E_\rho$ under $f$ is $F_{r(\rho)}$ where

$$r(\rho) = \frac{\pi}{2(1-\alpha)}\, x(\rho)^{1-\alpha} + \mathrm{O}(1).$$

At the end of Section 1.1, we worked out the rate of decay of harmonic measure in the strip, and found that

$$\omega(0, F_r; S) = \frac{2}{\pi}\, e^{-r} + \mathrm{o}(1) \quad \text{as } r \to \infty.$$

Thus

$$\omega(\rho) = \omega(1, E_\rho; \mathcal{P}_\alpha) = \omega(0, F_{r(\rho)}; S) \sim \exp\left[-\frac{\pi}{2(1-\alpha)}\, x(\rho)^{1-\alpha}\right].$$

A consequence of this is the following sharp integrability result,

$$E_1\left[\exp\left(b|B_{\tau_\alpha}|^{1-\alpha}\right)\right] < \infty \iff b < \frac{\pi}{2(1-\alpha)},$$

which is analogous to that of Burkholder for cones.

It does not seem to be so straightforward, however, to prove such precise results in higher dimensions. At the beginning of my talk at IWPT 2004, I asked the audience whether $P_{z_0}(|B_\tau| > \rho)$ is larger (for large $\rho$) for the exit time $\tau$ from

(i) the parabola $\{(x,y) : x > 0 \text{ and } |y| < \sqrt{x}\}$ in the plane

or from

(ii) the paraboloid $\{(x,Y) : x > 0 \text{ and } |Y| < \sqrt{x}\}$ in three dimensions.

The answer to this question follows from the explicit asymptotics that we derive in [3]. Chris Burdzy was kind enough to explain to me how he answered the question without knowing the exact asymptotics.

Brownian motion $\{B_t\}$ in the parabola-shaped region $\mathcal{P}_\alpha$ may be thought of as a one-dimensional Brownian motion $\{B_t^1\}$ in the $x_1$-direction and an independent Bessel process $\{X_t\}$ of order $n-1$ in the orthogonal direction. He showed that, for fixed $\alpha$ and $\rho$, $P_{z_0}\left(B^{1,*}_{\tau_\alpha} > \rho\right)$, where

$$B^{1,*}_{\tau_\alpha} = \sup_t B^1_{t\wedge\tau_\alpha},$$

decreases as the dimension increases. [This probability is the harmonic measure at $z_0$ of the cross section $\theta_\rho$ of $\mathcal{P}_\alpha$ at $x_1 = \rho$ with respect to that part of $\mathcal{P}_\alpha$ to the left of this cross section.] The probability that $B^{1,*}_{\tau_\alpha} > \rho$ is the probability that the one-dimensional Brownian motion $B_t^1$ hits level $\rho$ before the Bessel process $X_s$ exceeds $(B_s^1)^\alpha$. The Bessel process of order $n-1$ satisfies the stochastic differential equation

$$dX_t = dZ_t + \frac{n-2}{2}\,\frac{1}{X_t}\,dt$$

while the Bessel process of order $n$ corresponding to $\mathcal{P}_\alpha$, but one dimension higher, satisfies

$$dY_t = dZ_t + \frac{n-1}{2}\,\frac{1}{Y_t}\,dt,$$

where we may take $\{Z_t\}$ to be the *same* one dimensional Brownian motion in each case and to be independent of $\{B_t^1\}$. Since $X_0 = Y_0$ and since the drift coefficient $(n-1)/2$ for $Y_t$ is always greater than that for $X_t$, which is $(n-2)/2$, it follows from a general comparison result that, for any time $t$, $Y_t \ge X_t$ a.s. Thus $Y_s - (B_s^1)^\alpha$ will become non negative before $X_s - (B_s^1)^\alpha$ becomes non negative.

In [3], it is shown that the rate of decay of harmonic measure in $\mathcal{P}_\alpha$ satisfies, for each positive $\epsilon$,

$$\exp\left[-\frac{\sqrt{\lambda_1}}{1-\alpha}\,(1+\epsilon)\,\rho^{1-\alpha}\right] \;\le\; \omega(\rho) \le\; \exp\left[-\frac{\sqrt{\lambda_1}}{1-\alpha}\,(1-\epsilon)\,\rho^{1-\alpha}\right]$$

for all sufficiently large $\rho$, where $\lambda_1$ is the smallest eigenvalue for the Dirichlet Laplacian in the unit ball in $\mathbf{R}^{n-1}$. The upper bound comes from the Carleman estimate, which belongs to the family of upper bounds for harmonic measure described in Section 1.4. Lower bounds for harmonic measure, in situations such as that under consideration here, are harder to obtain. In his lower bound on harmonic measure mentioned in Section 1.3, Warschawski needed to take into account the oscillation of the width and the oscillation of the central line of the domain. The central lines of our parabola-shaped domains don't oscillate, while the width is increasing but not too quickly in that $\int \theta'(x)^2/\theta(x)\,dx$ is finite. It is natural to expect that the upper bound for harmonic measure given by the Carleman method would be achieved in this case. The problem, however, was how to prove this.

The harmonic measure $\omega(x, E_\rho; \mathcal{P}_\alpha)$ has symmetry that it inherits from the symmetry of the domain and that of its boundary values: at $(x, Y)$ in $\mathcal{P}_\alpha$ (where $x \in \mathbf{R}$, $Y \in \mathbf{R}^{n-1}$),

$$\omega((x, Y), E_\rho; \mathcal{P}_\alpha) = u(x, |Y|)$$

for some function $u(x, y)$ that is defined on the upper half of the domain $\mathcal{P}_\alpha$ in the plane. Whereas the harmonic measure satisfies Laplace's equation, the function $u$ satisfies

$$\Delta u + (n-2)\frac{u_y}{y} = 0.$$

We would like to transform from the parabola to the strip, as this worked so well in two dimensions. However, unlike Laplace's equation, this Bessel type operator is not conformally invariant. Transformation to the strip results in what is, at first sight, a messy expression that involves both the mapping $g$ from the strip to the parabola and its derivative. The asymptotics of the mapping $g$ and of $g'$ can be deduced relatively easily from Warschawski's work. However, the asymptotic estimates for the derivative are restricted to sub strips $S_\rho = \{z\colon |\mathrm{Im}\, z| < \rho\}$ where $\rho < \pi/2$. To transform the partial differential equation $\Delta u + (n-2)u_y/y = 0$ successfully from the parabola $\mathcal{P}_\alpha$ to the strip $S$ we needed *uniform* estimates on the derivative of the conformal mapping. These were obtained as part of the results in [8] and lead directly to the following result:

*Suppose that $g$ is a symmetric conformal mapping of the infinite strip $S$ onto $\mathcal{P}_\alpha$, with $g(x) \to \infty$ as $x \to \infty$. There is a function $\epsilon(w)$ in the strip $S$ with $\epsilon(w) \to 0$ as $\mathrm{Re}\, w \to \infty$, uniformly in the imaginary part of $w$. Moreover, whenever $u(x, y)$ satisfies the p.d.e. $\Delta u + (n-2)u_y/y = 0$ in $\mathcal{P}_\alpha^+ = \{(x, y) : 0 < y < x^\alpha\}$ then $v = u \circ g$ satisfies the p.d.e.*

$$\Delta v + (n + \epsilon(w) - 2)\frac{v_y}{y} = 0$$

*in the upper half of the strip.*

The Bessel operator is, in some sense, asymptotically conformally invariant. In our situation, the function $v$ arises from the harmonic measure of the exterior of a ball of radius $\rho$ in $\mathcal{P}_\alpha$ in $\mathbf{R}^n$. We know the rate of growth of the mapping from the planar parabola-shaped domain onto the strip, from which we can deduce the boundary values for $v$.

With a little more careful analysis involving the maximum principle, the lower bound for harmonic measure follows.

These bounds on harmonic measure lead directly to integrability results for the exit position, valid in each finite dimension:

$$E_1\left[\exp\left(b|B_{\tau_\alpha}|^{1-\alpha}\right)\right]$$

is finite if $b < \sqrt{\lambda_1}/(1-\alpha)$ and is infinite if $b > \sqrt{\lambda_1}/(1-\alpha)$. Just as in Burkholder's work on cones, we proved that $B^*_{\tau_\alpha}$ has the same integrability properties.

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# Radial limits of harmonic functions

**Stephen J. Gardiner**

**Abstract.**

A classical result of Alice Roth characterizes those functions on the unit circle that can arise from taking radial limits of entire functions. This paper describes recent progress on the characterization of radial limit functions of harmonic functions defined either in the unit ball or the whole of space. Some related open problems are posed.

## §1. Introduction

Let $\mathbb{T}$ denote the unit circle. The starting point for this article is the following question: which functions $f : \mathbb{T} \to \mathbb{C}$ can be expressed as

$$f(e^{i\theta}) = \lim_{r\to\infty} g(re^{i\theta}) \qquad (0 \le \theta < 2\pi) \tag{1.1}$$

for some entire function $g$? Such a function $f$ must, of course, be a Baire-one function, that is, the pointwise limit of a sequence from $C(\mathbb{T})$. One would expect, however, that only a restricted class of Baire-one functions on $\mathbb{T}$ can arise in this manner. The answer to the above question is found in the following classical result of Alice Roth [15], [16] (or see Chapter IV, §5 of the book by Gaier [9]).

**Theorem A.** *Let $f : \mathbb{T} \to \mathbb{C}$. The following statements are equivalent:*

*(a) there is an entire function $g$ such that (1.1) holds;*
*(b) $f$ is Baire-one, and is constant on each component of some relatively open dense subset $J$ of $\mathbb{T}$.*

*Further, if (b) holds, then (a) holds with the additional property that the convergence in (1.1) is locally uniform on $J$.*

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To see that (a) implies (b) in this result, suppose that (a) holds, let

$$K_k = \left\{ e^{i\theta} : \left| g(re^{i\theta}) \right| \le k \ \text{ for all } \ r > 0 \right\} \qquad (k \in \mathbb{N}),$$

and let $J_k$ denote the interior of $K_k$ relative to $\mathbb{T}$. Then $\cup_k K_k = \mathbb{T}$, so the set $J = \cup_k J_k$, which is relatively open in $\mathbb{T}$, must also be dense in $\mathbb{T}$ by a Baire category argument. Since $g$ is bounded on the set $\left\{ re^{i\theta} : r > 0, e^{i\theta} \in J_k \right\}$, the radial limit function $f$ must, by Montel's theorem, be constant on each component arc of $J_k$. Thus (b) follows (and, indeed, the convergence in (1.1) is locally uniform in $J$). The more difficult, and hence more interesting, part of the result is the converse. The proof of this involved adapting ideas from Runge's theorem on rational approximation to deal with approximation on non-compact sets, and foreshadowed much later celebrated work of Arakeljan [1], [2].

More recently, Boivin and Paramonov [6] obtained an analogue of Roth's result for radial limits of solutions of homogeneous elliptic partial differential equations of order two with constant complex coefficients in $\mathbb{R}^2$. In the particular case of harmonic functions the radial limit functions are characterized as those Baire-one functions on $\mathbb{T}$ that are first-degree polynomials of $\theta$ on each component arc of some relatively open dense subset of $\mathbb{T}$. The arguments used do not apply in higher dimensions.

In Section 2 below we will remain in the context of the plane and consider the nature of radial limit functions of harmonic functions that are defined in the unit disc. The corresponding problem in higher dimensions is still open. Then, in Section 3, we will move to higher dimensions and see a characterization of radial limit functions of entire harmonic functions. New features, and deeper arguments, apply in this setting.

## §2. Radial limits of harmonic functions in the disc

We now consider the question: which functions $f : \mathbb{T} \to \mathbb{R}$ can be expressed as

$$f(e^{i\theta}) = \lim_{r \to 1-} h(re^{i\theta}) \qquad (0 \le \theta < 2\pi) \tag{2.1}$$

for some harmonic function $h$ on the unit disc $\mathbb{D}$? To see what form the result should take we can follow the pattern of the argument outlined in Section 1. Let

$$K_k = \left\{ e^{i\theta} : \left| h(re^{i\theta}) \right| \le k \ \text{ whenever } \ 0 \le r < 1 \right\} \qquad (k \in \mathbb{N}),$$

and $J = \cup_k J_k$, where $J_k$ denotes the interior of $K_k$ relative to $\mathbb{T}$. It is not difficult to deduce from (2.1) that

$$H^{\mathbb{D}}_{f\chi_{J_k}}(re^{i\theta}) \to f(e^{i\theta}) \text{ as } r \to 1- \qquad (e^{i\theta} \in J_k),$$

where $H^U_g$ denotes the (generalised) solution to the Dirichlet problem on an open set $U$ with boundary data $g$ (the solution is given by a Poisson integral in the present context) and $\chi_A$ denotes the characteristic function of a set $A$. It follows from a converse of Fatou's theorem, due to Loomis [13] and valid for bounded boundary functions such as $f\chi_{J_k}$, that

$$\frac{1}{2t}\int_{(-t,t)} f(e^{i(\theta+\phi)})d\phi \to f(e^{i\theta}) \qquad \text{as } t \to 0+ \tag{2.2}$$

when $e^{i\theta} \in J_k$. We will say that $f$ is *asymptotically mean-valued* at $e^{i\theta}$ if (2.2) holds. One implication of the following result, taken from [11], has now been established.

**Theorem 1.** *Let $f : \mathbb{T} \to \mathbb{R}$. The following statements are equivalent:*

*(a) there is a harmonic function $h$ on $\mathbb{D}$ such that (2.1) holds for all $\theta$;*

*(b) $f$ is Baire-one, and there is a relatively open dense subset $J$ of $\mathbb{T}$ on which $f$ is locally bounded and asymptotically mean-valued.*

*Further, if (b) holds, then (a) holds with the additional property that the mapping $w \longmapsto \sup_{0<r<1} |h(rw)|$ is locally bounded on $J$.*

It remains to see why (b) implies (a). Suppose that condition (b) holds, let $\{J_j\}$ be the component arcs of $J$ and let $\{U_j\}$ be the corresponding sectors of $\mathbb{D}$. We write $J_j$ as $\left\{e^{i\theta} : |\theta - \theta_j| < a_j\right\}$. A naïve approach would now be to solve the Dirichlet problem in each sector $U_j$ with boundary data $f$ on $J_j$ and $f(e^{i(\theta_j \pm a_j)})$ on the boundary radii. The asymptotic mean value property of $f$ could then be used in conjunction with Fatou's theorem to deduce that the resulting Dirichlet solution had the desired radial limits at points of $\overline{J}_j$. There would remain, of course, the problem of how to "stitch together" the various Dirichlet solutions from different sectors to obtain a function that is harmonic on all of the disc. However, before we even get that far with this approach, an additional obstacle is that $f\big|_{J_j}$ need not be integrable with respect to harmonic measure for the sector $U_j$.

These difficulties can be overcome by refining our approach. Firstly, we modify the region in which we solve the Dirichlet problem by the removal of radial slits from the sector $U_j$. More precisely, let

$$S_{j,k}(\rho) = \left\{ re^{i\theta} : 0 \le r \le \rho, |\theta - \theta_j| = a_j \left(1 - 2^{-k}\right) \right\} \quad (0 \le \rho < 1; k \ge 1).$$

Then it is possible to choose a sequence $(\rho_{j,k})_{k\ge 1}$ in $(0,1)$, with limit 1, such that the function $f$ (interpreted as 0 off $\mathbb{T}$) is integrable with respect to harmonic measure for the set

$$V_j = U_j \backslash \left( \bigcup_{k\ge 1} S_{j,k}(\rho_{j,k}) \right).$$

If we denote the resultant harmonic function on $V_j$ by $h_j$, then it can be deduced, as above, from the asymptotic mean value property that

$$h_j(rw) \to f(w) \qquad \text{as } r \to 1- \qquad (w \in J_j).$$

Secondly, we must find some way of constructing a harmonic function on $\mathbb{D}$ that imitates the boundary behaviour of $h_j$ in $V_j$, for each $j$, and also has the right behaviour along radii ending in the closed set $\mathbb{T}\backslash J$. To do this, let $\rho_j : J_j \to (1 - j^{-1}, 1)$ be a continuous function such that $\rho_j(e^{i\theta}) \to 1$ as $\theta \to \theta_j \pm a_j$ and such that the set

$$E_j = \{rw : w \in J_j \ \text{ and } \ \rho_j(w) \le r < 1\}$$

does not intersect any of the radial slits $\{S_{j,k}(\rho_{j,k}) : k \ge 1\}$. Then $E_j$ is a relatively closed subset of $\mathbb{D}$ such that $E_j \subset V_j$. The set

$$E_0 = \left\{ rw : w \in \mathbb{T}\backslash J \ \text{ and } \ \frac{1}{2} \le r < 1 \right\}$$

is also closed relative to $\mathbb{D}$ and is, in addition, nowhere dense. Since $f$ is Baire-one, we can choose a continuous function $h_0$ on $E_0$ such that

$$h_0(rw) \to f(w) \qquad \text{as } r \to 1- \qquad (w \in \mathbb{T}\backslash J).$$

The disjoint union $E = \cup_{j\ge 0} E_j$ is a relatively closed subset of $\mathbb{D}$. Further, if we define $v$ on $E$ by setting it equal to $h_j$ on $E_j$ $(j \ge 0)$, then $v$ has radial limit function $f$. The theorem will therefore be established if we can approximate $v$ on $E$ by a harmonic function on $\mathbb{D}$ in such a way that the error of approximation tends to 0 at $\mathbb{T}$. Since $v$ is continuous on $E$ and harmonic on $E^\circ$, Corollary 3.21 of [10] (based on work of Armitage and Goldstein [4]) tells us that this can be done provided $\mathbb{D}^*\backslash E$ is connected and locally connected, where $\mathbb{D}^*$ denotes the Alexandroff (or one-point) compactification of $\mathbb{D}$. Our construction of $E$ evidently guarantees these connectivity hypotheses (see §3.2 of [10] for a discussion of local connectedness in this context), so the proof is complete.

Since our motivation came originally from classical function theory, it is natural to pose the following question.

**Problem 1.** *Which functions $f : \mathbb{T} \to \mathbb{C}$ can be expressed as*

$$f(e^{i\theta}) = \lim_{r\to 1-} g(re^{i\theta}) \qquad (0 \le \theta < 2\pi)$$

*for some holomorphic function $g$ on $\mathbb{D}$?*

Another obvious question concerns higher dimensions. Let $\mathbb{S}$ denote the unit sphere in $\mathbb{R}^n$.

**Problem 2.** *Which functions $f : \mathbb{S} \to \mathbb{R}$ can be expressed as*

$$f(w) = \lim_{r\to 1-} h(rw) \qquad (w \in \mathbb{S})$$

*for some harmonic function $h$ on the open unit ball of $\mathbb{R}^n$?*

We will see in the next section that moving from the context of the plane to higher dimensions is not routine.

## §3. Radial limits of harmonic functions in space

Now we develop the discussion of Section 1 in another direction by asking the question: which functions $f : \mathbb{S} \to \mathbb{R}$ can be expressed as

$$f(w) = \lim_{r\to \infty} h(rw) \qquad (w \in \mathbb{S}) \tag{3.1}$$

for some harmonic function $h$ on $\mathbb{R}^n$? Let $\delta$ denote the Laplace-Beltrami operator on $\mathbb{S}$; thus the Laplacian on $\mathbb{R}^n$ can be expressed in polar coordinates as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\delta.$$

If $I$ is a relatively open subset of $\mathbb{S}$ and the entire harmonic function $h$ in (3.1) is bounded on the conical set $\{rw : r > 0, w \in I\}$, then a simple dilation argument shows that $\delta f = 0$ on $I$. This observation, together with a Baire category argument, shows that a function $f$ of the form (3.1) must, in addition to being Baire-one, satisfy the Laplace-Beltrami equation $\delta f = 0$ on a relatively open dense subset of $\mathbb{S}$. In the case of two dimensions the latter equation reduces to $\partial^2 f/\partial\theta^2 = 0$ and so we arrive at the answer to the above question obtained by Boivin and Paramonov (see Section 1). However, this Baire category argument can be applied also in relation to the $\delta$-fine topology on $\mathbb{S}$, that is, the coarsest topology that makes all supersolutions of the Laplace-Beltrami equation continuous. This allows us to conclude that any function $f$ of the form (3.1) must be a $\delta$-fine solution of the Laplace-Beltrami equation on a $\delta$-finely open $\delta$-finely dense subset of $\mathbb{S}$. (We refer to Fuglede [8] for these notions from fine potential theory.) The important point here is that, when $n \geq 3$, there exist compact subsets of $\mathbb{S}$ that are nowhere dense in $\mathbb{S}$ and yet have non-empty $\delta$-fine interior. This shows that the answer to our question in higher dimensions will be more delicate.

In order to proceed we need some additional notation and terminology. Given a compact subset $J$ of $\mathbb{S}$ we write $u \in \mathcal{L}(J)$ if $u$ is a function on a relatively open subset $I$ of $\mathbb{S}$ such that $J \subset I$ and $\delta u = 0$ on $I$. Further, given $z \in J$, we denote by $\mathcal{N}_z(J)$ the collection of all $\mathcal{L}(J)$-representing measures for $z$, that is, probability measures $\mu$ on $J$ satisfying

$$u(z) = \int_J u \, d\mu \qquad \text{for every } u \in \mathcal{L}(J).$$

A bounded Borel function $f$ on $J$ will be called *$\mathcal{L}$-affine* on $J$ if

$$f(z) = \int f \, d\mu \qquad \text{whenever } z \in J \text{ and } \mu \in \mathcal{N}_z(J).$$

Clearly the collection of $\mathcal{L}$-affine functions on $J$ contains $\mathcal{L}(J)$.

We are now in a position to formulate the answer to our question in the following result, which is taken from [12].

**Theorem 2.** *Let $f : \mathbb{S} \to \mathbb{R}$. The following statements are equivalent:*

*(a) there is a harmonic function $h$ on $\mathbb{R}^n$ such that (3.1) holds;*

*(b) there is a sequence of compacts $J_k \uparrow \mathbb{S}$ such that, for each $k$, the restriction $f|_{J_k}$ is bounded, Baire-one and $\mathcal{L}$-affine on $J_k$.*

We will briefly outline below the main ideas of the proof and refer to [12] for full details. Although some things have to be verified, the implication $(a) \implies (b)$ is not difficult. As usual, the main interest lies in the proof of the converse. A key ingredient here is the result stated below. It follows from an abstract result of Lukeš *et al.* [14] that deals with approximation of bounded Baire-one functions in the context of simplicial function spaces. It can be applied in the present situation because of work of Bliedtner and Hansen [5] concerning simpliciality in potential theory.

**Theorem B.** *Let $J$ be a compact subset of $\mathbb{S}$ and let $f : J \to \mathbb{R}$ be a bounded Baire-one function. If $f$ is $\mathcal{L}$-affine on $J$, then there is a* *bounded sequence* $(u_m)$ *in* $C(J)$ *such that each function* $u_m$ *is* $\mathcal{L}$*-affine on* $J$ *and* $u_m \to f$ *pointwise on* $J$.

Now let $f : \mathbb{S} \to \mathbb{R}$ and suppose that condition (b) of Theorem 2 holds. We fix $k$ temporarily. By Theorem B there is a sequence $(u_{k,m})_{m\geq 1}$ in $C(J_k)$, and a positive constant $c_k$, such that

- $u_{k,m}$ is $\mathcal{L}$-affine on $J_k$ for each $m$,
- $|u_{k,m}| \leq c_k$ on $J_k$ for each $m$, and
- the sequence $(u_{k,m})_{m\geq 1}$ converges to $f$ pointwise on $J_k$.

Further, by an approximation result of Debiard and Gaveau [7], we may assume that each function $u_{k,m}$ satisfies $\delta u_{k,m} = 0$ on a neighbourhood of $\overline{I}_{k,m}$, where $I_{k,m}$ is some relatively open neighbourhood of $J_k$ in $\mathbb{S}$. (We may also assume that the sequence $(I_{k,m})_{m\geq 1}$ is decreasing.) Let $\omega_k$ denote the open set defined by

$$\omega_k = \bigcup_{m\geq 1} \left\{rz : z \in I_{k,m+1} \quad \text{and} \quad ((m-1)!)^4 < r < ((m+1)!)^4\right\}$$

and let $h_k$ denote the solution to the Dirichlet problem on $\omega_k$ with boundary data $g_k$ where, for each $m \geq 1$, the function $g_k$ is defined on the boundary subset

$$\left\{x \in \partial\omega_k : ((m-1)!)^4 m^2 \leq \|x\| < (m!)^4 (m+1)^2\right\}$$

by

$$g_k(x) = \frac{1}{m}\sum_{l=1}^{m} u_{k,l}\left(\frac{x}{\|x\|}\right).$$

Careful estimation of harmonic measure can be used to show that

$$h_k(rz) \to f(z) \quad \text{as} \quad r \to \infty \qquad (z \in J_k). \tag{3.2}$$

We now consider general values of $k$ and define $E = \cup_k E_k$, where

$$E_k = \begin{cases} \{rz : z \in J_1 \text{ and } r \geq 1\} & (k=1) \\ \{rz : z \in J_k, \operatorname{dist}(z, J_{k-1}) \geq \frac{1}{r} \text{ and } r \geq k\} & (k \geq 2). \end{cases}$$

Clearly the set $E$ is closed. We can obtain a harmonic function $v$ on a neighbourhood of $E$ by defining $v = h_k$ on an appropriate neighbourhood of $E_k$ for each $k$. Further, it is readily seen that the set $(\mathbb{R}^n \cup \{\infty\}) \backslash E$ is connected and locally connected, where $\infty$ denotes the point at infinity for $\mathbb{R}^n$. Under these circumstances we can appeal to another result from the theory of harmonic approximation (see [3], or Corollary 5.10 of [10]) to conclude that there is an entire harmonic function $h$ satisfying

$$|v(x) - h(x)| < \frac{1}{\|x\|} \qquad (x \in E).$$

Now let $z \in \mathbb{S}$ and $k_0 = \min\{k : z \in J_k\}$. For all sufficiently large values of $r$ we have $rz \in E_{k_0}$ and so

$$\begin{aligned} |f(z) - h(rz)| &\leq |f(z) - v(rz)| + \frac{1}{r} \\ &= |f(z) - h_{k_0}(rz)| + \frac{1}{r} \\ &\to 0 \quad \text{as} \quad r \to \infty \end{aligned}$$

by (3.2), as required.

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# Renewal theorems, products of random matrices, and toral endomorphisms

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***Dedicated to Martine Babillot***

**Abstract.**

We consider a subsemigroup $T$ of the linear group $G$ of the $d$-dimensional Euclidean space $V$, which is "sufficiently large". We study the orbit closures of $T$ in $V$ and we apply the results to semigroups of endomorphisms of the $d$-dimensional torus. The method uses the knowledge of the potential kernel of the Markov chain on $V$ defined by a probability measure supported on $T$. The condition of being "large" is satisfied for example by a subsemigroup of $SL(V)$, Zariski-dense in $SL(V)$.

## §1. Introduction

We denote by $G$ the linear group of the Euclidean space $V = \mathbb{R}^d$, by $\mathbb{T}^d$ the $d$-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, where $\mathbb{Z}^d$ is the lattice of integer points in $\mathbb{R}^d$. We denote by $M_{inv}(d, \mathbb{Z})$ the subsemigroup of elements $g$ in $G$ such that $g\mathbb{Z}^d \subset \mathbb{Z}^d$. These elements are $d \times d$ matrices with integer coefficients with non zero determinant. We observe that such a matrix defines a surjective endomorphism of $\mathbb{T}^d$.

Let us consider, to begin, the simplest situation $d = 1$. A basic fact of Diophantine approximation, is that, for given irrational $\alpha \in [0, 1[$ and $\varepsilon \in ]0, 1[$, there exists relatively prime integers $p, q$ such that: $q|\alpha - p/q| = \{q\alpha\} < \varepsilon$. Also, the set $\{\{q\alpha\}; q \in \mathbb{N}\}$ is dense in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $\alpha$ is rational then $\{\{q\alpha\}; q \in \mathbb{N}\}$ is finite. In [19], Hardy and Littlewood have considered such properties when $q$ belongs to a proper subset $Q$ of $\mathbb{N}$. In particular they have shown that $Q = \{n^2; n \in \mathbb{N}\}$ is such a set. We observe that $\{n^2; n \in \mathbb{N}\}$ is a multiplicative subsemigroup of $\mathbb{N}$. Hence one can ask for the validity of such properties for various subsets of $\mathbb{N}$. In [7], H. Furstenberg has proved that any "non lacunary" subsemigroup

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$T$ of $\mathbb{N}$ satisfies the required properties. In general, non lacunarity of $T$ means that $T$ is not contained in a multiplicative semigroup of the form $q^{\mathbb{N}} = \{q^n; n \in \mathbb{N}\}$ where $q$ is an integer. The simplest example of such a semigroup is $Q = \{2^m 3^n; m, n \in \mathbb{N}\}$. In this case, non lacunarity follows from the irrationality of $Log2/Log3$. This type of property has been considered for $d > 1$ and $T \subset M_{inv}(d, \mathbb{Z})$ with $T$ a commutative semigroup, by D. Berend in particular [5]. Also, it can be considered in the larger setting of multiparameter hyperbolic actions of groups or semigroups on manifolds (see for example [21]). In particular, G.A. Margulis has asked in [22] for necessary and sufficient conditions on a semigroup $T \subset M_{inv}(d, \mathbb{Z})$ for the dichotomy of density or finiteness of the $T$-orbits in $\mathbb{T}^d$.

Here, we give a brief exposition of some recent results on this problem; we concentrate mostly on the case where $T$ is non commutative and "large". The condition of $T$ being "large" can be made precise in terms of Zariski closure of $T$. We recall that the Zariski closure of a set $X$ in $G$ is the set of zeros of the polynomials on $G$ which vanish on $X$. The Zariski closure of $T$ is a group with a finite number of connected components. As this time two approaches on this problem have been developed. A direct one, by A. Muchnik [23], extends the arguments of D. Berend to the non necessarily commutative situation. A different approach uses properties of potential kernels of random walks on semi-simple groups ([17], [18]); it can be sketched as follows.

We denote by $\mu$ a finitely supported probability measure on $M_{inv}(d, \mathbb{Z})$ and we assume that its support generates a "large" semigroup of $G$. In this context we can obtain informations on orbit closures of $T$ in $\mathbb{R}^d$ from properties of $\mu$-invariant measures and potential measures on $\mathbb{R}^d$. Then, using projection on $\mathbb{T}^d$ and topological dynamics arguments, we can describe the orbit closures of $T$ on $\mathbb{T}^d$. Let us consider in more detail the case $d = 1$. Let $T$ be the multiplicative semigroup of $\mathbb{N}$ generated by $a = 2, b = 3$, and let $\mu = \frac{1}{2}(\delta_a + \delta_b)$. Then a basic fact, due to the irrationality of $Log2/Log3$, is that the semigroup $\{mLoga + nLogb; m, n \in \mathbb{N}\}$ is "more and more dense " in $\mathbb{R}$ at $+\infty$. This fact can be quantified by considering the potential measure $\sum_0^\infty \overline{\mu}^k$ of the additive random walk on $\mathbb{R}$ defined by $\overline{\mu}$, the image of $\mu$ by the map $x \to \mathrm{Log}x$. Then, according to the renewal theorem, this measure looks like Lebesgue measure at $+\infty$. Then, projecting on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and using arguments of topological dynamics, one can prove that, for every irrational $\alpha \in \mathbb{T}$, the orbit $T\alpha$ accumulates at zero, hence is dense in $\mathbb{T}$. This line of reasoning remains valid for $d > 1$; one has to use matricial analogues of the renewal theorem; useful results in this direction can be found in [2], [12], [14], [15], [17]. A noteworthy fact, in this context is that the non lacunarity condition, is automatically satisfied , if $T$ is non commutative and "large" (see Proposition 2 and Corollary 5). In order to describe a typical result in case $d > 1$, in a simplified situation, we introduce some notations.

**Definition 1.** *An element $g$ in $G$ is said to be proximal if we can write*

$$V = \mathbb{R}v_g \oplus V_g^<,$$

*where $gv_g = \lambda_g v_g, \lambda_g \in \mathbb{R}$, $gV_g^< = V_g^<$ and the spectral radius of $g$ in $V_g^<$ is strictly less than $|\lambda_g|$.*

**Definition 2.** *An element $g$ in $G$ is said to be quasi-expanding if its spectral radius is strictly greater than 1.*

**Definition 3.** *A semigroup $T \subseteq G$ is said to be strongly irreducible if there do not exist a finite union of proper subspaces which is $T$-invariant.*

For short, we will say that $T$ satisfies condition $(i-p)$ if $T$ is strongly irreducible and contains a proximal element. We will say that $T$ satisfies $(i-p-e)$ if $T$ satisfies $(i-p)$ and moreover, $T$ contains a quasi-expanding element. The condition $(i-p)$ is satisfied if the Zariski closure of $T$ contains $SL(V)$. The set of proximal elements in $T$ will be denoted by $T^{prox}$. In this paper we sketch a proof of the following

**Theorem 1** ([18])**.** *Assume $T$ is a subsemigroup of $M_{inv}(d, \mathbb{Z})$, which satisfies $(i-p-e)$. Then the $T$-orbits on $\mathbb{T}^d$ are finite or dense.*

**Example 1.** *Let $d = 2$, $a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and let $T$ be the subsemigroup of $SL(2, \mathbb{Z})$ generated by $a, b$. Then the conditions of the theorem are clearly satisfied. Hence the $T$-orbits in $\mathbb{T}^2$ are finite or dense.*

**Remark 1.** *a) The theorem gives a partial answer to the question of G.A. Margulis [22]. For the general case, see section 3 below.*

*b) The theorem will follow from a description of the $T$-orbits in $V$, where $T$ is a general subsemigroup of $G$ which satisfies $(i-p-e)$. Compactness of $\mathbb{T}^d$ allows us to restrict the study to $T$-orbits which accumulate at zero. The expansion property allows us to conclude that such an orbit is "large" according to a well known principle in hyperbolic dynamics.*

We need to consider the action of the semi-group $T$ on $\mathbb{P}(V)$, the projective space of $V$, as well as on the factor space $V/_{\{\pm Id\}}$. We denote by $\pi$ the projection of $V$ onto $V/_{\{\pm Id\}}$ and by $\pi(V)$ the image of $V \setminus \{0\}$. The projection of $v \in V \setminus \{0\}$ on $\mathbb{P}(V)$ will be denoted $\overline{v}$. The set of accumulation points of a subset $X$ of a topological space will be denoted by $X^{ac}$.

**Definition 4.** *Let $T$ be a subsemigroup of $G$ which satisfies condition $(i-p)$. We denote by $L_T$ the subset of $\mathbb{P}(V)$ defined by $L_T =$ closure $\{\overline{v}_g \in \mathbb{P}(V); g \in T^{prox}\}$. We denote by $\widetilde{L}_T$ the inverse image of $L_T$ in $V \setminus \{0\}$.*

It is easy to show (see for example [15]) that $L_T$ is the unique $T$-minimal subset of $\mathbb{P}(V)$. Then theorem 1 will be a consequence of the

**Theorem 2** ([18])**.** *Assume $T$ is a subsemigroup of $G$ which satisfies $(i-p-e)$, $\Phi$ is a closed $T$-invariant subset of $V \setminus \{0\}$ such that $0 \in \Phi^{ac}$. Then $\pi(\Phi) \supset \pi(\widetilde{L}_T)$.*

Let $\mu$ be a probability measure on $G$, $T_\mu$ the closed subsemigroup of $G$ generated by its support $S_\mu$. We observe that, if $X$ is a $G$-space, the action of $G$ on $X$ can be extended to probability measures as follows:

$$\mu * \rho = \int \delta_{gx} d\mu(g) d\rho(x).$$

We will consider in particular the Markov operators $P$ and $\widetilde{P}$ on $\mathbb{P}(V)$ and $\pi(V)$ respectively defined by

$$P(\overline{v}, .) = \mu * \delta_{\overline{v}}, \quad \widetilde{P}(v, .) = \mu * \delta_{\pi(v)}.$$

The space $\pi(V)$ can be written, using polar coordinates, in the form:

$$\pi(V) = \mathbb{P}(V) \times \mathbb{R}_+^*.$$

Furthermore, the group $\mathbb{R}_+^*$ acts naturally on the right on $\pi(V)$ by dilations. We denote by $\ell$ the Lebesgue measure on the group $\mathbb{R}$ or $\mathbb{R}_+^*$ (which is isomorphic to $\mathbb{R}$). We know (see [15]) that, if $T_\mu$ satisfies $(i-p)$, there exists on $\mathbb{P}(V)$ a unique $\mu$-stationary measure $\nu(\nu = \mu * \nu)$. The support of $\nu$ is $L_{T_\mu}$. If $S_\mu$ is compact, the limit of the sequence $\frac{1}{n} \int Log\|g\| d\mu^n(g)$ is finite and it will be denoted $\gamma(\mu)$. With these notations, theorem 2 will be a consequence of:

**Theorem 3.** *Assume $\mu$ is a probability measure on $G$ such that $S_\mu$ is compact, $T_\mu$ satisfies $(i-p-e)$, $\gamma(\mu) > 0$. Then, for any $v \in V \setminus \{0\}$,* *we have the following weak convergence on $\pi(V)$:*

$$\lim_{t\to 0} \sum_0^\infty \mu^k * \delta_{t\pi(v)} = \frac{1}{\gamma(\mu)} \nu \otimes \ell.$$

If $d = 1$, this statement is called the renewal theorem (see [6], p 300-309). It leads to Theorem 2 and to the following corollary, once a convenient measure $\mu$ on $T$ has been chosen. It has the following purely topological corollary;

**Corollary 1.** *Assume $T$ is a subsemigroup of $G$ which satisfies $(i - p - e)$. Then, for any $v \in V \setminus \{0\}$, we have the convergence:*

$$\liminf_{t\to 0} \quad t\pi(\overline{Tv}) \supset \pi(\widetilde{L}_T).$$

The convergence above is taken as the Hausdorff convergence on compact subsets of $\pi(V)$. The corollary says that $T$-orbits on $V$ are "large at the infinity".

## §2. Some extensions of the renewal theorem

We describe here a basic tool of the proof of Theorem 3.
The following is the classical renewal theorem;

**Theorem 4.** *Let $\mu$ be a probability measure on $\mathbb{R}$ such that the closed subgroup generated by its support is $\mathbb{R}$. Assume that $\mu$ has finite mean $\gamma(\mu) > 0$. Then we have the following weak convergence:*

$$\lim_{t\to -\infty} \sum_0^\infty \mu^k * \delta_t = \frac{1}{\gamma(\mu)} \ell,$$

*where $\ell$ is Lebesgue measure.*

In order to illustrate its meaning in dynamics, we consider the following special flow $(X, \widehat{\theta}^t)$. Assume $A = S_\mu \subset \mathbb{R}_+$ is finite and let $\Omega = A^{\mathbb{N}}, \rho = \mu^{\otimes \mathbb{N}}$. We consider the subset $X$ of $\Omega \times \mathbb{R}$:

$$X = \{(\omega, x) \;\; 0 \le x \le \omega_0\},$$

and endow $X$ with the probability measure $\widehat{\rho}$ which is the restriction of $\frac{1}{\gamma(\mu)} \rho \otimes \ell$ to $X$. We denote:

$$S_k(\omega) = \omega_0 + \cdots + \omega_{k-1},$$

and we observe that, for any $x, t \in \mathbb{R}_+$, there exists a unique $k \in \mathbb{N}$ such that $S_k(\omega) \leq x + t < S_{k+1}(\omega)$, we denote by $\theta$ the shift transformation on $\Omega$, and the flow $\widehat{\theta}^t$ on $X$ is defined by

$$\widehat{\theta}^t(\omega, x) = (\theta^k \omega, x + t - S_k(\omega)),$$

where $k$ is as above.

Then $\widehat{\theta}^t$ is the so called special flow under the function $\omega \to \omega_0$; the measure $\widehat{\rho}$ is $\widehat{\theta}^t$-invariant. The renewal theorem, for $\mu$ as above, is closely related to the following

**Proposition 1.** *The flow $\widehat{\theta}^t$ on $(X, \widehat{\rho})$ is mixing.*

*Proof.* We denote by $T$ the transformation on $X \times \mathbb{R}$ defined by

$$T(\omega, x) = (\theta\omega, x - \omega_0).$$

We observe that its adjoint with respect to $\rho \otimes \ell$ is the Markov kernel $\widetilde{P}$ defined by

$$\widetilde{P}\varphi(\omega, x) = \sum_{a \in A} \varphi(a\omega, x + a)\mu(a).$$

Furthermore $t \in \mathbb{R}$ acts by translation on $X \times \mathbb{R}$. In particular, for a function $\psi \in C(X \times \mathbb{R})$, we have: $(\delta_t * \psi)(\omega, x) = \psi(\omega, x - t)$.

On the other hand, for any $\varphi \in C(X)$ we have, with $k$ as above:

$$\varphi(\widehat{\theta}^t(\omega, x)) = \varphi o T^k(\omega, x + t),$$

$$\varphi(\widehat{\theta}^t(\omega, x)) = \sum_0^\infty \varphi o T^n(\omega, x + t).$$

Then, for any $\psi \in C(X)$, we have, using duality:

$$< \varphi \circ \widehat{\theta}^t, \psi >_{\widehat{\rho}} = < \varphi \circ \widehat{\theta}^t, \psi >_{\rho \otimes \ell} = \sum_0^\infty < \varphi, (\widetilde{P}^n \delta_{-t})(\psi) >_{\rho \otimes \ell} .$$

If $\psi$ is of the form $\psi' \otimes u$ with $\psi' \in C(\Omega)$ depending only of the first $r$ coordinates and $u$ is continuous with compact support on $\mathbb{R}$, we have for $n \geq r$:

$$(\widetilde{P}^n \delta_{-t})(\psi) = (\mu^{n-r} * \delta_{-t})(\widetilde{P}^r \psi),$$

where $\widetilde{P}^r \psi$ depends only of the $x$-coordinate. Also:

$$\sum_0^\infty (\widetilde{P}^n \delta_{-t})(\psi) = \sum_0^{r-1} (\widetilde{P}^n \delta_{-t})(\psi) + \sum_0^\infty (\mu^k * \delta_{-t})(\widetilde{P}^r \psi).$$

From the renewal theorem:

$$\lim_{t\to+\infty}\sum_0^\infty(\widetilde{P}^n\delta_{-t})(\psi)=\frac{1}{\gamma(\mu)}(\rho\otimes\ell)(\psi).$$

Hence, we have

$$\lim_{t\to+\infty}<\varphi\circ\widehat{\theta}^t,\psi>_{\widehat{\rho}}=\widehat{\rho}(\varphi)\widehat{\rho}(\psi),$$

for any $\varphi\in C(X)$, $\psi\in C(X)$ depending of finitely many coordinates. A density argument in $\mathbb{L}^2(X)$ allows to conclude. $\square$

The renewal theorem admits a natural extension to "Gibbsian random walks" defined in terms of a mixing subshift of finite type $(\Omega,\theta)$ endowed with a Gibbs measure $\rho$ and a Hölder function $f$ on $\Omega$ (See [11] [12] [27]).

**Definition 5.** *We say that the Hölder function $f$ on the subshift $(\Omega,\theta)$ is arithmetic if there exists a Hölder function $u$ on $\Omega$ such that $f+u\circ\theta-u$ takes its values in $c\mathbb{Z}$ for some $c\in\mathbb{R}$.*

We consider only a unilateral subshift $(\Omega,\theta)$ where $\Omega\subset A^{\mathbb{N}}$ and $A$ is finite. We denote by $p(\omega,a)$ the conditional probability of $\omega_{-1}=a$, given $\omega=(\omega_0,\omega_1,\cdots)$. The Markov kernel $\widetilde{P}$ on $\Omega\times\mathbb{R}$ is defined by:

$$\widetilde{P}\varphi(\omega,x)=\sum_{a\in A}\varphi(a\omega,x+f(\omega))p(\omega,a).$$

Then we have the following extension of the renewal theorem (See [12]):

**Theorem 5.** *With the above notations, assume $\gamma(f)=\int f(\omega)d\rho(\omega)>0$ and $f$ is non arithmetic. Then for any function $\varphi$ on $X\times\mathbb{R}$, continuous with compact support we have the following convergence:*

$$\lim_{t\to+\infty}\sum_0^\infty\widetilde{P}^k\varphi*\delta_t=\frac{1}{\gamma(f)}(\rho\otimes\ell)(\varphi).$$

This theorem is also closely related to the mixing of special flows over subshifts of finite type endowed with Gibbs measures.

Coming back to the matricial setting described in the introduction, we give below some important tools of the proof of Theorem 3.

**Proposition 2.** *Let $T\subset G$ be a subsemigroup which satisfies condition $(i-p)$ . We denote*

$$\textstyle\sum_T=\{Log|\lambda_g|\ \ g\in T^{prox}\}.$$

*Then* $\sum_T$ *generates a dense subgroup of* $\mathbb{R}$*.*

This result is proved in [18]. It is an analogue of the arithmetical condition assumed in the above theorem.

Keeping with the analogy of the above theorem, we consider the action of $G$ on $\mathbb{P}(V)$ and we denote $g \cdot b$ the result of the projective action of $g$ on $b \in \mathbb{P}(V)$. Then we write the action of $G$ on $\pi(V)$ as

$$g(b, x) = (g \cdot b\ ,\ \|gb\|x)$$

The Markov operator on $\mathbb{P}(V)$ (resp $\pi(V)$) associated with $\mu$ then can be written as:

$$P\varphi(b) = \int \varphi(g \cdot b)d\mu(g), \quad \text{resp.} \quad \widetilde{P}\varphi(b, x) = \int \varphi(g.b,\ x\|gb\|)d\mu(g).$$

It is convenient to "decompose" the operator $\widetilde{P}$, using the Fourier operators $P^s$ $(s \in \mathbb{R})$ which are defined on $\mathbb{P}(V)$ by:

$$P^s\varphi(b) = \int \varphi(g.b)\|gb\|^{is}\ d\mu(g).$$

For $\varepsilon \in ]0, 1]$, denote by $H_\varepsilon(\mathbb{P}(V))$ the space of $\varepsilon$-Hölder functions $\varphi$ on $\mathbb{P}(V)$ defined by the condition:

$$[\varphi]_\varepsilon = \underset{b,b' \in \mathbb{P}(V)}{Sup} \frac{|\varphi(b) - \varphi(b')|}{d^\varepsilon(b, b')} < +\infty,$$

where $d(b, b')$ is the distance on $\mathbb{P}(V)$ defined by

$$d(b, b') = \|b \wedge b'\| = |sin(b, b')|.$$

This space is a Banach space with respect to the norm defined by:

$$\|\varphi\|_\varepsilon = [\varphi]_\varepsilon + |\varphi|_\infty.$$

The operators $P^s$ have nice spectral properties, due to the following:

**Proposition 3.** *Assume that the probability measure* $\mu$ *has compact support and* $T_\mu$ *satisfies condition* $(i - p)$*. Then, for* $\varepsilon$ *small, there exists* $c_0 \in [0, 1[$ *and* $c(\varepsilon) \in \mathbb{R}_+$ *such that*

$$[P\varphi]_\varepsilon \leq c_0[\varphi]_\varepsilon,$$

$$[P^s\varphi]_\varepsilon \leq c_0[\varphi]_\varepsilon + c(\varepsilon)|\varphi|_\infty.$$

This allows, using [20], to define a principal eigenvalue $k(s) \in \mathbb{C}$ which plays the role of the Fourier transform in classical Probability Theory. Theorem 3 can be proved along the lines of the classical analytic proof of the renewal theorem (see [12]). On the other hand, Proposition 3 is a consequence of the properties of random walks on $G$, namely of the fact that the top Lyapunov exponent of $\mu$ is is simple in the Lyapunov spectrum of $\mu$ (see [9], [15]). In order to prove the corollary 1 , we make use of two facts. If $T$ is compactly generated, we can find $\mu$ such that $T_\mu = T$ and $\gamma(\mu) > 0$. Then the corollary follows from the fact that the support of $\nu \otimes \ell$ is $\pi(\widetilde{L}_T)$. If $T$ is not compactly generated, we proceed by exhaustion.

## §3. Orbit closures of semigroups of toral endomorphisms

We consider here natural extensions of Theorem 1 and we describe some of the tools required in their proofs. We refer to [16] [17] [23] for detailed proofs.

**Theorem 6** ([23]). *Assume $T$ is a subsemigroup of $M_{inv}(d, \mathbb{Z})$ which satisfy the conditions:*

*1) There is no $T$-invariant finite union of rational proper subspaces in $V$.*

*2) There is no relatively compact $T$-orbit in $V \setminus \{0\}$.*

*3) The group generated by $T$ is not a finite extension of a cyclic group.*

*Then the $T$-orbits in $\mathbb{T}^d$ are finite or dense.*

This result answers the question of Margulis in case of $\mathbb{T}^d (d \geq 1)$, since it is not difficult to show that the conditions of the theorem are necessary for the dichotomy of density or finiteness of $T$-orbits in $\mathbb{T}^d$. It can also be proved following the lines developed in section 2. We observe that condition (1) implies that the Zariski closure of $T$ in $G$ is reductive. An important component of the proof sketched here is the study of $T$-orbits on $V$, for a general subsemigroup of $G$ such that its Zariski closure, denoted $Zc(T)$, is semi-simple (see [17]). We sketch below some of the corresponding tools and simplify some technical aspects of [17], using the recent results of [16].

Let $S$ be an $\mathbb{R}$-algebraic and connected semi-simple group, $\mu$ a probability measure on $S$ such that the semigroup $T_\mu$ generated by its support is Zariski dense in $S$. We consider an Iwasawa decomposition of $S : S = KAN$; we denote by $M$ the centraliser of $A$ in $K$, by [M,M] its commutator subgroup. We choose a Weyl chamber $\mathcal{A}^+$ in the Lie algebra $\mathcal{A}$ of $A$ and we denote $A^+ = \exp \mathcal{A}^+ \subset A$. Then we have the polar decomposition $S = K\overline{A}^+K$ and we denote by $a(g)$ the $\overline{A}^+$-component of $g$. If $\mu$ has compact support, we can define (see [15]) the Lyapunov vector of $\mu$ by

$$\lambda(\mu) = \lim \frac{1}{n} \int Log\ a(g) d\mu^n(g) \in \overline{\mathcal{A}}^+.$$

We denote by $F = K/M$ the so-called Furstenberg boundary of $S$, and we observe that the $S$-homogeneous space $S/MN$ can be written as a product $(K/M) \times A$. Also the $S$-homogeneous space $S/N$ can be written as $K \times A$. The group $A$ acts on the right on these spaces. The result of the action of $a \in A$ on $v \in S/MN$ (resp $x \in S/N$) will be simply denoted $va$ (resp $xa$). We denote by $\ell$ the Lebesgue measure of $A$ and write $r = dimA$. It is known that there exists on $F$ a unique $\mu$-stationary measure $\nu$, and its support $\Lambda_{T_\mu}$ is the unique $T_\mu$-minimal subset of $F$. Also we have $\lambda(\mu) \in \mathcal{A}^+$ (see [15]) and [9]. The following is proved in [17], using results of [2].

**Theorem 7.** *With the above notations, for any $v \in S/MN$, we have the following weak convergence:*

$$\lim_{t \to +\infty} t^{\frac{r-1}{2}} \sum_0^\infty \mu^k * \delta_{ve^{-t\lambda(\mu)}} = c_\mu \nu \otimes \ell,$$

*where $c_\mu$ is a positive constant. Furthermore the measure $\nu \otimes \ell$ is $\mu$-invariant extremal .*

We need also to consider the potential kernel of $\mu$ on $S/N = K \times A$ and the $\mu$-stationary measures on $K = S/AN$.

**Theorem 8** ([16]). *With the above notations, the action of $\mu$ on $K = S/AN$ has only a finite number $p \leq 2^r$ of extremal $\mu$-stationary probabilities $\sigma_i (1 \leq i \leq p)$. Each of them is invariant under the right action of the connected component of $M$ and has projection $\nu$ on $K/M$. The $T_\mu$-minimal subsets of $K$ are the supports $\widehat{\Lambda}^i_T$ of the measures $\sigma_i$ $(1 \leq i \leq p)$.*

In case $S = SO(n,1)$ and $T$ is a Zariski dense sub semigroup of $S$, the theorem implies that the action of $T$ on $SO(n) = K$ has a unique minimal subset which is the inverse image in $K$ of the unique $T$-minimal subset $\Lambda_T$ of the boundary $SO(n)/SO(n-1)$ of $SO(n,1)$.

**Corollary 2.** *Each measure $\sigma_i \otimes \ell$ on $S/N$ $(1 \leq i \leq p)$ is $\mu$-invariant and extremal. For any $x \in S/N$, the cluster values, when*

$t \to +\infty$, *of the family of potentials* $t^{\frac{r-1}{2}} \sum_0^\infty \mu^k * \delta_{xe^{-t\lambda(\mu)}}$ *are positive linear combinations of the measures* $\sigma_i \otimes \ell$ $(1 \leq i \leq p)$.

Let $T$ be a Zariski dense semigroup of $S$. Then it is known (see for example [9], [24]) that $T$ contains $\mathbb{R}$-regular elements *i.e.* elements $g$ conjugate to elements of $MA^+$. We denote by $T^{prox}$ the set of $\mathbb{R}$-regular elements in $T$. For such a $g \in T^{prox}$ we denote

$$\lambda(g) = \lambda(\delta_g) \in \mathcal{A}^+,$$

and by $C_T$ we denote the closed subcone of $\overline{\mathcal{A}^+}$ generated by such elements $\lambda(g), g \in T^{prox}$. Also it is known that $C_T$ is convex and has non empty interior $C_T^\circ$ (see [3]). If $T$ is compactly generated, it is possible to find $\mu$, as above, with $T_\mu = T$ and $\lambda(\mu)$ proportional to $\lambda(g)$. Furthermore, for $g \in T^{prox}$, the projection modulo [M,M] of a conjugate of $g$ in $MA^+$ is uniquely defined. We denote this projection in $MA/[M,M]$ by $\sigma(g)$ and we consider the closed subgroup $\Delta_T \subset MA/[M,M]$ generated by the elements $\sigma(g)(g \in T^{prox})$. Then we have the

**Theorem 9** ([16]). *Let $T$ be a Zariski dense subsemigroup of the semi-simple group $S$. Then, the closed subgroup $\Delta_T \subset MA/[M,M]$ generated by the elements $\sigma(g)(g \in T^{prox})$ has finite index in $MA/[M,M]$.*

This result is a natural extension of Proposition 2. It extends also an analogous, result of [4], where $\sigma(g)$ is replaced by $\lambda(g) \in \mathcal{A}$. Furthermore, we observe that a deep study of analogous properties of Zariski dense subgroups of semi-simple groups has been developed by G. Prasad and A.S Rapinchuk (see for example [25], Theorem 2). This approach, based on embedding in linear groups over p-adic fields gives also, as a by-product, the statement of Theorem 9, in the case where, for some embedding $j$ in $GL(n, \mathbb{R})$, the coefficients of $j(T)$ belong to a number field (G. Prasad, personal communication). Using Theorem 9, we get the following corollaries.

**Corollary 3.** *Let $T$ be a Zariski dense subsemigroup of the semi-simple group $S$, $g$ an $\mathbb{R}$-regular element of $T$ such that $\lambda(g) \in C_T^\circ$. Let $\Phi \subset G/N$ be a closed $T$-invariant subset, and $x \in \widehat{\Lambda}_T^i \subset K$ such that*

$$g \in xMA^+x^{-1}, \quad g^{-\mathbb{N}}x \subset \Phi.$$

*Then $\Phi$ contains $\widehat{\Lambda}_T^i A$.*

This corollary leads to a description of the closed $T$-invariant subsets $\Phi$ of $V(dimV > 1)$, such that $0 \in \Phi^{ac}$, without assuming irreducibility of $T$, as in Theorem 2.

**Corollary 4.** *Let $T$ be a subsemigroup of $GL(V)$ such that its Zariski closure is semi-simple, $\Phi$ a closed $T$-invariant subset of $V$ such that $0 \in \Phi^{ac}$. Then there exists a non zero vector $u \in V$, a conjugate $A_u$ of $A$ in $S$, and an index $i \in [1,p]$ such that:*

$$0 \in \overline{A_u u} \subset \Phi,$$

$$\Phi \supset \widehat{\Lambda}^i_T A_u u.$$

From this we deduce the

**Corollary 5** ([17]). *Let $T$ be a subsemigroup of $M_{inv}(d,\mathbb{Z})$ such that its Zariski closure in $GL(V)$ is semi-simple. Assume that $T$ does not preserve a finite union of rational subspaces. Then every $T$-orbit in $\mathbb{T}^d$ is finite or dense.*

It can be shown (see [17]) that any infinite closed $T$-invariant subset $X$ of $\mathbb{T}^d$ contains 0 as an accumulation point. Hence, lifting $X$ to $\mathbb{R}^d$, Corollary 5 follows from Corollary 4.

As already said before, a proof of Theorem 6 can also be obtained along these lines. In this more general case, the Zariski closure of $T$ is reductive, hence one needs to show a renewal theorem for reductive groups.

Conditions (2) and (3) in Theorem 6 need to be added to the hypothesis of Theorem 8, in view of the presence of a non trivial center in this reductive group.

## §4. Final comments

Let $G$ be a connected Lie group, $G/H$ a non compact homogeneous space of $G$, $\mu$ a probability measure on $G$ such that the semi-group $T_\mu$ generated by the support of $\mu$ is "large". For example if $\mu$ has a density which is positive and continuous at $e$, one has $T_\mu = G$. If $G$ is algebraic, one can require $T_\mu$ to be Zariski dense in $G$.

It would be interesting to describe, in some special cases, the Radon measures on $G/H$ which are extremal solutions of the equation $\mu * \lambda = \lambda$. This is closely related to the description of the Martin boundary of the Markov chain on $G/H$ defined by $\mu$, *i.e* to the limits of the normalized potentials associated with $\sum_0^\infty \mu^k * \delta_x (x \in G/H)$ when $x$ goes to the infinity. It should be possible to obtain from these informations properties of the $T$-orbit closures in $G/H$ at infinity. In the situation considered in this paper one has $G = GL(V)$, $G/H = V \setminus \{0\}$. The case where $G$ is the affine group of $V$ and $G/H = V$ is also of interest for geometrical reasons. For a study of recurrence relations with random coefficients in such a setting, see [10].

From the analytic point of view, if $\mu$ has a density, very little seems to be known on the Martin boundary of $(G/H, \mu)$ even if $H = \{e\}$. If $G/H$ is a symmetric space and $\mu$ is defined by the heat kernel at time one, the Martin boundary has been calculated in ([13]). For $G/H = G$, the extremal solutions of the equation $\mu * \lambda = \lambda$ have been calculated in some cases (see [1], [2], [7], [11], [13], [26]). The knowledge of such measures for $G/H \neq G$, should also be useful for the study of orbit closures $\overline{Tx}$ $(x \in G/H)$ at the infinity, since it corresponds to a simpler situation.

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# Densities and harmonic measure

**Joaquim Ortega-Cerdà**

**Abstract.**

Several notions of densities related to zero sequences, interpolating sequences and sampling sequences of holomorphic functions are presented. Some ties with harmonic measure estimates are shown.

## §1. Introduction

In this survey we will present several notions of densities and its relation to some classical problems in function theory. We show how some of these densities can be computed through precise estimates of the harmonic measure on conveniently crafted domains. This new interpretation of the densities may be useful in the extension of the classical function theory in the disk to other domains or Riemann surfaces.

The results that we present here are not new, and we will point to the sources along the exposition. There exists a nice book [16] with the state of the art on the problems of interpolation and sampling sequences. If one is interested in an (elementary) survey on motivation of these problems and its conection to signal analysis see for instance [3] and the references therein.

## §2. Different densities

Given a sequence of points $\Lambda$ in $\mathbb{R}$ or $\mathbb{C}$ we will define different quantities $\mathcal{D}(\Lambda)$ that try to provide a mathematical definition to the intuitive concept of the density of the sequence. There are several possibilities as we will see, and each of these appeared in the literature to deal with

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different problems of function theory. The heuristic principle behind all the results that we present is that the density of the zero sequence of an holomorphic function is controlled by the growth of such function. Of course this is what lies behind Nenvanlinna theory but to begin with we would like to recall a classical result, the Beurling-Malliavin theorem:

### 2.1. The Beurling-Malliavin density

The Paley-Wiener $PW_\tau$ space consists of entire functions of exponential type lower or equal than $\tau$ ($|f(z)| \leq Ce^{\tau|\Im z|}$) and $f \in L^2(\mathbb{R})$.

Let us first discuss uniqueness sets $\Lambda \subset \mathbb{R}$ for $PW_\tau$, that is, sets for which $f \in PW_\tau$ and $f(\lambda) = 0\ \forall \lambda \in \Lambda$ implies $f \equiv 0$. Since every $f \in PW_\tau$ is entire, it is clear that every set $\Lambda$ with a finite accumulation point is a uniqueness set; it is also transparent that a finite set cannot be a uniqueness set, so we assume from now on that $\Lambda$ is an infinite sequence without acumulation points. It is intuitively clear that $\Lambda$ must be dense in some sense, so that $f_{|\Lambda} = 0$ implies $f = 0$. Now, if $f \in PW_\tau$ and $f(\alpha) = 0$, then the function $g(z) = f(z)\frac{(z-\beta)}{(z-\alpha)}$ is again in $PW_\tau$ and $g(\beta) = 0$; this means that we can move arbitrarily any finite number of points of $\Lambda$ without changing the problem. Consequently, the control on the density of the sequences $\Lambda$ should be asymptotic, depending just on how $\Lambda$ behaves "at infinity". In a series of deep and very celebrated papers, Beurling and Malliavin (see for instance [6]) proved some results giving an almost complete description of uniqueness sets for $PW_\tau$. They introduced a density $\mathcal{D}_{BM}(\Lambda)$, called now the *Beurling-Malliavin density*, and proved the following

**Theorem 2.1.** *If a real sequence $\Lambda$ satisfies $\mathcal{D}_{BM}(\Lambda) > 2\tau$ then $\Lambda$ is a uniqueness set for $PW_\tau$. Conversely if $\Lambda$ is a uniqueness set for $PW_\tau$ then $\mathcal{D}_{BM}(\Lambda) \geq 2\tau$.*

The definition of $\mathcal{D}_{BM}(\Lambda)$ is complicated, but geometric in nature. It is called a density because the number $\mathcal{D}_{BM}(\Lambda)$ depends on how many points does $\Lambda$ have in big intervals. It is closely related to the classical density

$$\overline{\mathcal{D}}\,(\Lambda) = \overline{\lim_{r\to 0}}\, \frac{n_\Lambda(r)}{2r},$$

where $n_\Lambda(r)$ indicates the number of points of $\Lambda$ in $[-r, r]$. In particular, $\mathcal{D}_{BM}(\Lambda) \geq \overline{\mathcal{D}}(\Lambda)$. The precise definition is the following: Let $\Lambda$ be a sequence of real numbers contained in $(0, +\infty)$. We fix $A > 0$ and we let

$$S_A(\Lambda) = \left\{ t > 0;\ \frac{n_\Lambda(\tau) - n_\Lambda(t)}{\tau - t} > A \text{ for some } \tau > t \right\}.$$

The set $S_A(\Lambda)$ is of the form $\bigcup_k (a_k, b_k)$ and we define

$$\|S_A(\Lambda)\| = \sum_k \frac{(b_k - a_k)^2}{a_k^2}.$$

Finally the density $\mathcal{D}_{BM}(\Lambda)$ is the infimum of all $A$ such that $\|S_A(\Lambda)\| < \infty$. If the sequence $\Lambda$ is real but not strictly positive we define $\Lambda_+ = \Lambda \cap (0, +\infty)$, $\Lambda_- = -(\Lambda \cap (-\infty, 0))$ and

$$\mathcal{D}_{BM}(\Lambda) = \max(\mathcal{D}_{BM}(\Lambda_+), \mathcal{D}_{BM}(\Lambda_-)).$$

The exact description of the uniqueness sets for $PW_\tau$ remains however unsolved.

### 2.2. The Beurling-Nyquist density

If one is interested instead in a uniqueness problem with stability, then the problem becomes the following. Describe the sequences $\Lambda \subset \mathbb{R}$ such that

$$\sum_\Lambda |f(\lambda)|^2 \lesssim \int_{\mathbb{R}} |f|^2 \lesssim \sum_\Lambda |f(\lambda)|^2, \tag{2.1}$$

for all functions $f \in PW_\tau$. For simplicity we will assume that $\Lambda$ is *separated*, i.e. $\inf_{\lambda \neq \lambda'} |\lambda - \lambda'| > 0$. The separated sequences that satisfy (2.1) are called sampling sequences for the Paley-Wiener space and they are very important in signal analysis because these are the sequences that allow a stable discretization of band-limited and finite energy signals. Their description can almost be achieved with a density very much like in the case of uniqueness sequences with the Beurling-Malliavin theorem. The following result was proved by Beurling (see [1]):

**Theorem 2.2.** *If $\Lambda$ is a uniformly separated real sequence and $\mathcal{D}^-_{BN}(\Lambda) > \tau$ then $\Lambda$ is a sampling sequence for $PW_\tau$. Conversely if $\Lambda$ is a sampling sequence for $PW_\tau$ then $\mathcal{D}^-_{BN}(\Lambda) \geq \tau$.*

The lower Beurling-Nyquist density $\mathcal{D}^-_{BN}(\Lambda)$ is defined as

$$\mathcal{D}^-_{BN}(\Lambda) = \lim_{r\to\infty} \min_{x\in\mathbb{R}} \frac{\#(\Lambda \cap (x, r+x))}{r}.$$

The corresponding upper Beurling-Nyquist density $\mathcal{D}^+_{BN}(\Lambda)$ is defined as

$$\mathcal{D}^+_{BN}(\Lambda) = \lim_{r\to\infty} \max_{x\in\mathbb{R}} \frac{\#(\Lambda \cap (x, r+x))}{r},$$

which is related to the following interpolation problem. For which separated $\Lambda \subset \mathbb{R}$, the restriction $PW_\tau \to \ell^2$, $f \to \{f(\lambda)\}$ is onto? This sequences are called interpolating sequences and they are relevant in one wants to codify a discrete signal over a continuous band-limited signal. The corresponding theorem by Beurling states

**Theorem 2.3.** *If $\Lambda$ is a separated real sequence such that $\mathcal{D}^+_{BN}(\Lambda) < \tau$ then $\Lambda$ is an interpolating sequence for $PW_\tau$. Conversely if $\Lambda$ is an interpolating sequence then $\mathcal{D}^+_{BN}(\Lambda) \le \tau$.*

Again the critical case where $\mathcal{D}^+_{BM}(\Lambda) = \mathcal{D}^-_{BM}(\Lambda) = \tau$ is not covered by the theorems. Sampling and interpolating sequences for the Paley-Wiener space have been recently described in [10] and [16] but the notions involved are more delicate.

### 2.3. The Bergman space, Seip's density

There are similar notions of sampling and interpolation for the Bergman space. The weighted Bergman space $B_\tau$ is defined as the holomorphic functions in the disk such that

$$\|f\|^2_{B_\tau} := \int_{\mathbb{D}} |f|^2(1-|z|)^{2\tau-1} < +\infty.$$

A sequence $\Lambda \subset \mathbb{D}$ is *separated* in this context if $\inf_{\lambda\neq\lambda'} \rho(\lambda,\lambda') > 0$, where $\rho(z,w) = |z-w|/|1-\bar{w}z|$ is the pseudohyperbolic distance.

The sampling sequences for the Bergman space are those sequences $\Lambda$ such that

$$\|f\|^2_{B_\tau} \simeq \sum |f(\lambda)|^2(1-|\lambda|)^{2\tau+1}.$$

and $\Lambda$ is an interpolating sequence for the Bergman space whenever for any sequence of values $\{v_\lambda\}$ such that $\sum |v_\lambda|^2(1-|\lambda|)^{2\tau+1} < +\infty$ there is a function in $B_\tau$ such that $f(\lambda) = v_\lambda$. Again there is a corresponding notion of density that describes the uniformly separated sampling (or interpolating) sequences in the Bergman space. This density was introduced by Seip in [13] and it is defined as

$$\mathcal{D}^+_S(\Lambda) = \limsup_{r\to 1} \sup_{z\in\mathbb{D}} \frac{\sum_{\rho(\lambda,z)<r} 1-\rho(z,\lambda)}{\log 1/(1-r)},$$
$$\mathcal{D}^-_S(\Lambda) = \liminf_{r\to 1} \inf_{z\in\mathbb{D}} \frac{\sum_{\rho(\lambda,z)<r} 1-\rho(z,\lambda)}{\log 1/(1-r)}.$$

The corresponding theorem is

**Theorem 2.4.** *A separated sequence $\Lambda$ is interpolating for $B_\tau$ if and only if $\mathcal{D}^+_S(\Lambda) < \tau$ and it is sampling if and only if $\mathcal{D}^-_S(\Lambda) > \tau$.*

### 2.4. Korenblum density

We introduce now a new density that almost describes the zeros in the Bergman space in the same sense that the Beurling-Malliavin almost describes the zeros of the Paley-Wiener space. This density was introduced by Korenblum in [7].

Given a finite set $E$ of points in $\mathbb{T}$ we define the Beurling-Carleson entropy of $E$ as

$$\hat{\kappa}(E) = \sum_k \frac{|I_k|}{2\pi}\left(\log \frac{2}{\pi}|I_k| + 1\right),$$

where $I_k$ are the arcs complementary to $E$ in $\mathbb{T}$. To each set $E$ we associate to it the Korenblum flower $F(E)$ as the union of Stolz regions with vertex on $E$, i.e. $\{z \in \mathbb{D}; d(\frac{z}{|z|}, E) \leq 1 - |z|\}$. Finally let $\sigma(\Lambda, E)$ be the Blashke sum of the points of $\Lambda$ that are inside the Korenblum flower, i.e.

$$\sigma(\Lambda, E) = \sum_{\lambda\in\Lambda\cap F(E)} \log 1/|\lambda|.$$

The density $\mathcal{D}_K(\Lambda)$ is defined as the infimum of all $A > 0$ such that

$$\sup_{E\subset\mathbb{T}} (\sigma(\Lambda, E) - A\hat{\kappa}(E)) < +\infty.$$

The following theorem is a refinement of Seip [15] and [14] of a previous work by Korenblum [7]

**Theorem 2.5.** *If $\Lambda$ is a sequence in the unit disk such that $\mathcal{D}_K(\Lambda) > \tau$ then $\Lambda$ is a uniqueness set for $B_\tau$ and conversely if $\Lambda$ is a uniqueness set for $B_\tau$ then $\mathcal{D}_K(\Lambda) \geq \tau$.*

In this context it should be noted that the original paper by Korenblum studied the zeros of functions in $\mathcal{A}^{-\infty} = \cup_{\tau>0} B_\tau$. The zeros, in view of Theorem 2.5 are the sequences such that $\mathcal{D}_K(\Lambda) < \infty$. The necessity of the density condition in the work of Korenblum was proved with a delicate study of the distortion of certain conformal mappings. There is a more elementary proof due to Bruna and Massaneda [2] that uses some estimates of the harmonic measure and that allows them to work in higher dimensions.

We will sketch this potential theoretic proof. Suppose that $f \in B_\tau$ with $f(0) = 1$ and $Z(f) = \Lambda$. Denote by $u = \log|f|$, $u$ is a subharmonic function in the disk with the growth $u^+ \leq C \log 1/(1 - |z|^2)$. Take any of the star shaped regions of Korenblum $F(E)$. Then

$$0 = u(0) = \int_{\partial F(E)} u(\zeta) d\omega(0, \zeta) - \int_{F(E)} g(0, \zeta, F(E))\Delta u(\zeta), \tag{2.2}$$

where $g(0,\zeta,F(E))$ is the Green function of $F(E)$ with pole at 0 and $\omega(0,\zeta)$ the hamonic measure evaluated at the origin. With a careful estimate of the harmonic measure it follows that

$$\int_{\partial F(E)} \log \frac{1}{1-|z|^2} d\omega(0,\zeta) \leq c\hat{\kappa}(E),$$

and more easily $g(0,y,F(E)) \geq c(1-|y|)$ for $|y| > 1/2$. Thus the necessary condition of Korenblum follows from (2.2) because $\Delta u = C\sum_{\lambda\in\Lambda}\delta_\lambda$.

This is part of a more general scheme, where the study of the zeros sequences of holomorphic functions is seen to be equivalent to the study of the Poisson equation $\Delta u = \mu$, where $\mu$ is a positive measure. We are interested in finding solutions $u$ to the equation without any boundary restriction but with some growth estimates. The conection is clear since for any holomorphic function $f$, $u = \log|f|$ is a subharmonic function and viceversa for any solution $u$ of $\Delta u = \sum_\Lambda \delta_\lambda$ there is an holomorphic function $f$ such that $u = \log|f|$ and $f$ vanishes on $\Lambda$. This connection has been exploited in many situations, see for instance [4].

### 2.5. Weighted densities

The study of these densities suggests the following pattern: The functions in the Paley-Wiener space are charactrized by the growth $e^{\tau|\Im z|}$ and the functions in the Bergman-space by $e^{\tau \log 1/(1-|z|^2)}$. In all cases the growth is of the type $e^{\phi(z)}$ where $\phi$ is a subharmonic function. This is very natural since $\log|f|$ is subharmonic whenever $f$ is holomorphic, but the striking point is that the densities in both cases are related to $\Delta\phi$, in the case of the Paley-Wiener case this corresponds to $\tau$ times the Lebesgue meaure on the real line and in the weighted Bergman space to $\tau$ times the invariant measure on the disk. This is no coincidence, in general the density of the sampling, interpolating and zero sequences must be measured in the geometry of the manifold endowed with a metric related to the Laplacian of the weight.

Consider for instance the following situation. Take $\phi$ a subharmonic function in $\mathbb{C}$ with some mild regularity (doubling Laplacian, i.e. there is a $C > 0$ such that for all disks $D$, $\mu(2D) \leq C\mu(D)$ where $\mu$ denotes the positive measure $\mu = \Delta\phi$. Let $\rho(z)$ be the radius such that $\mu(D(z,\rho(z))) = 1$ (one has to think of $\rho^2$ as a sort of regularized $\Delta\phi$). Let $\mathcal{F}_\phi$ be the space of entire functions $f$ such that $fe^{-\phi} \in L^\infty(\mathbb{C})$. The problem of describing interpolating and sampling sequences is the natural one in this setting. To solve it one has to introduce some densities tied to the metric in $\mathbb{C}$ induced by $\rho$.

**Definition 2.1.** A sequence $\Lambda$ is *$\rho$-separated* if there exists $\delta > 0$ such that

$$|\lambda - \lambda'| \geq \delta \max(\rho(\lambda), \rho(\lambda')) \qquad \lambda \neq \lambda'.$$

**Definition 2.2.** Assume that $\Lambda$ is a $\rho$-separated sequence and recall that we denote $\mu = \Delta\phi$.

The *upper uniform density of* $\Lambda$ with respect to $\Delta\phi$ is

$$\mathcal{D}^+_{\Delta\phi}(\Lambda) = \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{\#\big(\Lambda \cap \overline{D(z, r\rho(z))}\big)}{\mu(D(z, r\rho(z)))}.$$

The *lower uniform density of* $\Lambda$ with respect to $\Delta\phi$ is

$$\mathcal{D}^-_{\Delta\phi}(\Lambda) = \liminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{\#\big(\Lambda \cap \overline{D(z, r\rho(z))}\big)}{\mu(D(z, r\rho(z)))}.$$

The following theorem proved in [8] is

**Theorem 2.6.** *Let $\phi$ be a subharmonic function with a doubling Laplacian.*

(i) *A sequence $\Lambda$ is sampling for $\mathcal{F}_\phi$ , if and only if $\Lambda$ contains a $\rho$-separated subsequence $\Lambda'$ such that $\mathcal{D}^-_{\Delta\phi}(\Lambda') > 1/2\pi$.*

(ii) *A sequence $\Lambda$ is interpolating for $\mathcal{F}_\phi$, if and only if $\Lambda$ is $\rho$-separated and $\mathcal{D}^+_{\Delta\phi}(\Lambda) < 1/2\pi$.*

## §3. Riemann surfaces

This section is more especulative. All these results concern the study of function spaces defined on the whole $\mathbb{C}$ or in a disk with different growths. It is also possible to study the same problems in Riemann surfaces or in several complex variables. We will not deal with the multidimensional situation, although there has been some recent progress (see [9]), but we will rather concentrate on the Riemann surfaces. There are two (at least) possible approaches to define the right density that governs the interpolating or sampling sequences for holomorphic $L^2$ functions in the surface. Both use some potential theory to define them. In the first one as developed in [12] they compute the density of a sequence using instead of disks, the sublevel sets of the Green function. With these densities they have obtained some sufficient conditions (although not necessary) for a sequence to be sampling or interpolating in the Riemann surface. When restricted to the disk one reobtains Seip's characterization for the Bergman space.

The second approach consists in using some harmonic measure estimates to provide an alternative definition of the densities. We will present the result in the disk. By its invariant nature this new definition can be transported to any Riemann surface. We are inspired by a following result [5] due to Garnett Gehring and Jones. We need some notation. For a $z \in \mathbb{D}$, let $D(z,r)$ be the pseudohyperbolic disk $D(z,r) = \{w \in \mathbb{D}; \rho(z,w) < r\}$. As usual if $A$ is a portion of the boundary of an open set $\Omega$ and $z \in \Omega$, then $\omega(z, A, \Omega)$ denotes the harmonic measure of $A$ from the point $z$.

**Theorem 3.1.** *A separated sequence $\Lambda$ is an interpolating sequence for $H^\infty$ if and only if*

$$\inf_{\lambda \in \Lambda} \omega(\lambda, \partial\mathbb{D}, \mathbb{D} \setminus \bigcup_{\lambda' \neq \lambda} D(\lambda', c)) > 0$$

*for some $0 < c < 1$.*

To obtain a counterpart of this result, we define the following densities. Set

$$\Omega(z,r) = \Omega(\Lambda; z, r) = \mathbb{D} \setminus \bigcup_{1/2 < \rho(\lambda, z) < r} D(\lambda, 1-r),$$

which is a finitely connected domain. We see that the uniform pseudohyperbolic radius of the little disks tends to 0 as $r \to 1$. This decay is tuned with the growth of $r$ in such a way that the numbers

$$\mathcal{D}_h^-(\Lambda) = \liminf_{r \to 1^-} \inf_{z \in \mathbb{D}} \log \frac{1}{\omega(z, \partial\mathbb{D}, \Omega(z,r))}$$

and

$$\mathcal{D}_h^+(\Lambda) = \limsup_{r \to 1^-} \sup_{\lambda \in \Lambda} \log \frac{1}{\omega(\lambda, \partial\mathbb{D}, \Omega(\lambda,r))}$$

are positive when $\Lambda$ is uniformly dense. In fact, we have the following precise characterization that is proved in [11]

**Theorem 3.2.** *For a separated sequence $\Lambda$ in $\mathbb{D}$ we have*

$$\mathcal{D}_S^-(\Lambda) = \mathcal{D}_h^-(\Lambda) \quad \textit{and} \quad \mathcal{D}_S^+(\Lambda) = \mathcal{D}_h^+(\Lambda).$$

This theorem is proved with a direct proof that the harmonic measure density is comparable to the "geometric" density. It will be interesting to prove that whenever $\mathcal{D}_h^+(\Lambda) < \tau$ then $\Lambda$ is interpolating for $B_\tau$, directly without using Seip's characterization of interpolating sequences. This will possibly allow the generalization of such notions to Riemann surfaces.

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# Sobolev type spaces on metric measure spaces

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***Dedicated to Professor C. David Minda on his 61st birthday***

**Abstract.**

The aim of this note is to summarise some of the approaches to extending Sobolev space theory to metric measure spaces. In particular, we will give a brief survey of Hajłasz-Sobolev, Newton-Sobolev, and the Korevaar-Schoen type Sobolev spaces on metric measure spaces.

## §1. Introduction

Many developments in the study of quasiconformal mappings and quasiregular mappings between domains in manifolds were aided by Sobolev space theory. The groundbreaking paper [19] by Heinonen and Koskela already had indications that an analog of Sobolev space theory for metric measure spaces is desirable in the study of quasiconformal mappings. Meanwhile, certain degenerate elliptic partial differential equations were reformulated in terms of elliptic partial differential equations on Carnot groups such as the Heisenberg groups and were then studied using modifications of standard techniques of elliptic PDE theory; see for example [16], [15], [11], [7], and the references therein. It was therefore clear that a viable Sobolev space theory on metric measure spaces would aid in further development of the study of quasiconformal mappings between metric measure spaces and of the study of a wide class of partial differential equations.

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Using a characterization of Sobolev functions on Euclidean spaces, Hajłasz formulated a theory of Sobolev type function spaces on metric measure spaces in [12]; this theory was developed further in [13] and [9]. Following the definition of upper gradients given by Heinonen and Koskela in [19], the author and Cheeger independently proposed a theory of Sobolev type spaces on metric measure spaces; see [37] and [8]. Concurrently, using the theory of strongly local Dirichlet forms as a model, Korevaar and Schoen developed a theory of Sobolev mappings from Riemannian domains into metric space targets in [29], and they used this theory to study harmonic mappings in [30]. Their approach was modified by Koskela and MacManus in [33] to obtain another version of Sobolev type space of functions on metric measure spaces; see also [34] for a discussion connecting this Korevaar-Schoen type Sobolev space theory with the theory of Dirichlet forms on metric measure spaces.

In this note we will describe the above-mentioned function spaces and the connections between them without proofs. This note is arranged as follows. The next section will summarise the notations used throughout this paper. The third and fourth sections will discuss the Hajłasz and Newtonian approaches to defining Sobolev type spaces on metric measure spaces. The final section will describe a Korevaar-Schoen approach to constructing Sobolev type spaces on metric measure spaces and discuss the relationships between the three Sobolev type spaces under certain conditions. While no proofs are provided in this note, references to articles where the proofs can be found are given. However, the references given are not exhaustive, and many good references are left out for brevity of exposition.

## §2. Notations

In this note $X = (X, d, \mu)$ denotes a metric measure space with metric $d$ and measure $\mu$. Given $r > 0$ and $x \in X$, the (open) metric ball centered at $x$ with radius $r$ is denoted $B(x, r)$. We will assume throughout that $\mu$ is a Borel regular measure such that bounded sets have finite measure and non-empty open sets have positive measure. The Lebesgue measure of sets $A \subset \mathbb{R}^n$ is denoted $|A|$.

We fix an index $1 \le p < \infty$. Measurable functions $f : X \to \mathbb{R}$ are said to be in the class $L^p(X)$ if the integral $\|f\|^p_{L^p(X)} := \int_X |f|^p \, d\mu$ is finite. We say that $f \in L^p_{loc}(X)$ if $f \in L^p(Y)$ for every bounded subset $Y \subset X$. The integral average of a function $f \in L^1_{loc}(X)$ on a measurable set $A \subset X$ with $\mu(A) > 0$ is denoted $f_A$:

$$f_A := \frac{1}{\mu(A)} \int_A f(y)\, d\mu(y).$$

Given functions $f \in L^1_{loc}(X)$, we define the Hardy-Littlewood maximal function $Mf$ on $X$ by

$$Mf(x) = \sup_{r>0} |f|_{B(x,r)}.$$

The measure $\mu$ is said to be *doubling* if there is a constant $C \geq 1$ such that for every $x \in X$ and $r > 0$ we have $\mu(B(x, 2r)) \leq C\, \mu(B(x, r))$. The standard Lebesgue measure on $\mathbb{R}^n$ for example is a doubling measure. It is known that if $\mu$ is a doubling measure, then the Hardy-Littlewood maximal function operator $M : L^p(X) \to L^p(X)$ is a bounded sublinear operator for all $p > 1$; see for example [17]. Furthermore, $M : L^1(X) \to \text{wk}-L^1(X)$ boundedly. Here $\text{wk}-L^1(X)$ is the collection of all functions $f$ on $X$ for which there is a constant $C_f > 0$ such that for all $t > 0$,

$$\mu(\{x \in X\ :\ |f(x)| \geq t\}) \leq \frac{C_f}{t}.$$

The norm on $\text{wk} - L^1(X)$ is obtained by associating to each function $f$ in this class the infimum/minimum of all such numbers $C_f$.

A metric space is said to be proper if closed and bounded subsets of the space are compact in the metric topology. An easy topological argument shows that if $X$ is a complete metric space and $\mu$ is a doubling measure on $X$ then $X$ is proper.

## §3. The Hajłasz-Sobolev spaces

The following theorem was proven by Hajłasz in [12]. Recall that a domain $\Omega$ is a $p$-extension domain if and only if there is a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ with $Ef = f$ on $\Omega$.

**Theorem 3.1** (Hajłasz). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain or $\Omega = \mathbb{R}^n$, and $1 < p < \infty$. Suppose in addition that $\Omega$ is a $p$-extension domain. Then a function $f : \Omega \to \mathbb{R}$ is in the class $W^{1,p}(\Omega)$ if and only if there is a non-negative function $g \in L^p(\mathbb{R}^n)$ and a set $Z \subset \Omega$ with $|Z| = 0$ such that whenever $x, y \in \Omega \setminus Z$,*

$$|f(x) - f(y)| \leq |x - y|\ (g(x) + g(y))\,. \tag{3.1}$$

The function $g$ is called a *Hajłasz gradient* of $f$. For functions $f \in W^{1,p}(\mathbb{R}^n)$ it can be shown via a "telescoping sequence of balls" argument that $M|\nabla f|$ is a Hajłasz gradient of $f$.

It is clear that equation (3.1) can be used to extend the notion of Sobolev spaces to metric measure spaces. This is done in [12] as follows.

**Definition 3.1.** Given a function $f : X \to \mathbb{R}$, we say that a non-negative function $g : X \to \mathbb{R}$ is a *Hajłasz gradient* for $f$ if there exists a set $Z \subset X$ with $\mu(Z) = 0$ so that whenever $x, y \in X \setminus Z$,

$$|f(x) - f(y)| \le d(x, y)\ (g(x) + g(y))\,. \tag{3.2}$$

For functions $f \in L^p(X)$ we define the *Hajłasz-Sobolev norm* of $f$ by

$$\|f\|_{M^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all Hajłasz gradients $g$ of $f$. We denote by $M^{1,p}(X)$ the collection of all (equivalence classes of) functions $f \in L^p(X)$ for which the norm $\|f\|_{M^{1,p}(X)}$ is finite.

Lipschitz functions in $M^{1,p}(X)$ form a dense subclass of $M^{1,p}(X)$. If the measure $\mu$ is doubling, then functions in the class $M^{1,p}(X)$ always satisfy a weak $(1,p)$-Poincaré inequality: there are constants $C > 0$ and $\lambda \ge 1$ such that for all $f \in M^{1,p}(X)$ and all Hajłasz gradients $g \in L^p(X)$ of $f$,

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \le C\, r \left( \frac{1}{\mu(B(x,\lambda r))} \int_{B(x,\lambda r)} g^p\, d\mu \right)^{1/p}$$

whenever $B(x, r) \subset X$ is a ball in $X$ with radius $r$. Such Poincaré inequalities play a crucial role in potential theory. For example, using such inequalities it can be shown that solutions to the Dirichlet problem with boundary data from $M^{1,p}(X)$ always exist. Poincaré inequalities are also useful in the study of $p$-extension domains; see [14] for an elegant discussion of extension and trace theorems.

It should be noted however that by definition, if $f \in M^{1,p}(X)$ and $F \in L^p(X)$ such that $F = f$ $\mu$-a.e. in $X$, then $F \in M^{1,p}(X)$. This is one of the differences between the Hajłasz-Sobolev space and the Newton-Sobolev space discussed in the next section. Another crucial difference is as follows. If $U \subset X$ is a non-empty open set and $f \in M^{1,p}(X)$ is constant on $U$, it is not clear that we can choose a Hjłasz gradient $g$ of $f$ in $L^p(X)$ so that $g = 0$ $\mu$-a.e. in $U$. Such a truncation property for gradients is crucial in the current techniques used in the study of PDEs; for example, the truncation property is essential in the Nash-Moser proof and in the DeGiorgi proof of Harnack inequalities for energy minimizers and harmonic functions, and the lack of this truncation property in the Hajłasz-Sobolev space makes the related potential theory difficult.

## §4. Newtonian spaces

In [19], Heinonen and Koskela propose an alternative to distributional derivatives in the setting of metric measure spaces. Recall that if $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^1$-function, then by the fundamental theorem of calculus, for every pair of points $x, y \in \mathbb{R}^n$ and every rectifiable curve $\gamma$ joining $x$ and $y$ in $\mathbb{R}^n$,

$$|f(x) - f(y)| \le \int_\gamma |\nabla f|\, ds. \tag{4.1}$$

However, if $\Omega \subset \mathbb{R}^n$ is a domain and $f \in W^{1,p}(\Omega)$, then the collection of non-constant compact rectifiable curves $\gamma$ in $\Omega$ for which (4.1) fails is a *zero $p$-modulus collection of curves*; that is, there is a non-negative Borel measurable function $\rho_0 \in L^p(\Omega)$ such that for every such curve $\gamma$ we have $\int_\gamma \rho_0\, ds = \infty$. Using this fact as a motivation, Heinonen and Koskela proposed the following alternative to distributional derivatives for functions on metric measure spaces.

**Definition 4.1.** A family $\Gamma$ of non-constant compact rectifiable curves in a metric measure space $X$ is said to be a zero $p$-modulus family if there exists a non-negative Borel measurable function $\rho_0 \in L^p(X)$ such that for all curves $\gamma \in \Gamma$ the path integral $\int_\gamma \rho_0\, ds = \infty$. Given a function $f : X \to \mathbb{R}$, we say that a non-negative Borel measurable function $\rho$ on $X$ is an *upper gradient* of $f$ if for all non-constant compact rectifiable curves $\gamma$ in $X$,

$$|f(x) - f(y)| \le \int_\gamma |\nabla f|\, ds. \tag{4.2}$$

Here $x$ and $y$ denote the endpoints of $\gamma$. If (4.2) fails only for a zero $p$-modulus family of curves, then we say $\rho$ is a *$p$-weak upper gradient* of $f$.

It can be shown that if $f$ is a Lipschitz function on $X$, then the local Lipschitz constant function $\rho$ given by

$$\rho(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y,x)}$$

is an upper gradient of $f$; see [17]. On the other hand, if $(\rho_n)_n$ is a sequence of upper gradients of a function $f$ on $X$ such that each $\rho_n \in L^p(X)$ and $\rho_n \to \rho$ in $L^p(X)$, then the Borel function $\rho$, though may not be an upper gradient of $f$, is necessarily a $p$-weak upper gradient of $f$; see for example the discussion in [23].

Using the above definition of $p$-weak upper gradients given in [19], the author proposed in [37] the following version of Sobolev spaces, called Newtonian spaces.

**Definition 4.2.** Given a function $f : X \to \mathbb{R}$ such that $f$ belongs to an equivalence class in $L^p(X)$, the *Newtonian norm* of $f$ is given by

$$\|f\|_{N^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_{\rho} \|\rho\|_{L^p(X)},$$

where the infimum is taken over all upper gradients (or equivalently, all $p$-weak upper gradients) of $f$. We say that two functions $f_1, f_2$ on $X$ are equivalent, denoted $f_1 \sim f_2$, if $\|f_1 - f_2\|_{N^{1,p}(X)} = 0$. It is easy to see that $\sim$ defines an equivalence class on the collection of all functions $f$ on $X$ for which $\|f\|_{N^{1,p}(X)}$ is finite. The *Newton-Sobolev space* $N^{1,p}(X)$ is the collection of all such equivalence classes of functions.

It can be shown that $N^{1,p}(X)$, equipped with the above norm, is indeed a lattice and a normed vector space that is also a Banach space; see [37]. It should be noted that perturbations of functions in the Hajłasz space $M^{1,p}(X)$ on sets of $\mu$-measure zero are again in the Hajłasz space, and hence it is easy to see that $M^{1,p}(X)$ is a Banach space. However, perturbations of functions from $N^{1,p}(X)$ on sets of $\mu$-measure zero usually does not yield a function in $N^{1,p}(X)$; therefore the proof that $N^{1,p}(X)$ is a Banach space is more involved. On the other hand, if two functions $f_1$ and $f_2$ are in $N^{1,p}(X)$ and $f_1 = f_2$ $\mu$-a.e. we can see that $f_1 \sim f_2$ and hence they belong to the same equivalence class in $N^{1,p}(X)$.

Using the techniques found in the book [36] by Ohtsuka, it can be shown that whenever $X$ is a domain in $\mathbb{R}^n$, equipped with the Euclidean metric and the standard Lebesgue measure, $N^{1,p}(X) = W^{1,p}(X)$ both isometrically and isomorphically; see [37].

Given $f \in N^{1,p}(X)$, there are infinitely many $p$-weak upper gradients for $f$ in $L^p(X)$. Indeed, if $\rho$ is a $p$-weak upper gradient of $f$ and $g \in L^p(X)$ is a non-negative Borel measurable function, then $\rho + g$ is also a $p$-weak upper gradient of $f$. The following lemma is very useful in associating to each $f \in N^{1,p}(X)$ a unique $p$-weak upper gradient.

**Lemma 4.1.** *Let $f \in N^{1,p}(X)$. Then the collection of all $p$-weak upper gradients of $f$ in $L^p(X)$ forms a convex subset of $L^p(X)$. If $1 < p < \infty$, then there is a unique $p$-weak upper gradient $\rho_f \in L^p(X)$ of $f$*

*such that whenever $\rho \in L^p(X)$ is another $p$-weak upper gradient of $f$, we have $\rho_f \le \rho$ $\mu$-a.e.*

Such a $p$-weak upper gradient is called the *minimal* $p$-weak upper gradient of $f$.

The following lemma shows that the truncation property holds true for the class $N^{1,p}(X)$.

**Lemma 4.2.** *If $U \subset X$ is a closed or an open set and $f \in N^{1,p}(X)$ such that $f$ is constant $\mu$-a.e. in $U$, then $\rho_f = 0$ $\mu$-a.e. in $U$.*

On the other hand, unlike the Hajłasz-Sobolev class, functions in $N^{1,p}(X)$ need not satisfy a Poincaré inequality. We say that $N^{1,p}(X)$ satisfies a weak $(1,p)$-Poincaré inequality on $X$ if there exist constants $C > 0$ and $\lambda \ge 1$ such that whenever $f \in N^{1,p}(X)$ and $B(x,r)$ is a ball in $X$,

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \le C\, r \left(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)} \rho_f^p\, d\mu\right)^{1/p}.$$

Clearly, if $X$ has no non-constant compact rectifiable curve, then 0 would be an upper gradient for every function on $X$; in this case, $N^{1,p}(X) = L^p(X)$, and for every $f \in N^{1,p}(X)$ we have $\rho_f = 0$. In this event the above inequality can not be satisfied. Examples of such metric spaces include the so-called snow-flaked Euclidean space $X = \mathbb{R}^n$ with metric $d(x,y) = |x-y|^\epsilon$ for some fixed $0 < \epsilon < 1$. Other examples of metric spaces where Poincaré inequalities do not hold for $N^{1,p}(X)$ include certain fractal sets such as the Sierpinski gasket. However, there are many examples of non-Euclidean metric measure spaces supporting a Poincaré inequality; see [35], [6], [32], [31], and the references therein. Given that functions from $N^{1,p}(X)$ and their upper gradients satisfy the truncation property of Lemma 4.2, whenever the measure on $X$ is doubling and $X$ supports a weak $(1,p)$-Poincaré inequality, many of the classical methods of analysing harmonic functions in Euclidean domains can be modified to study energy minimizers and $p$-harmonic functions on domains in $X$; see for example [27], [26], [25], [28], [5], [4], [38], [3], [21], and [22].

While in general the Banach space $N^{1,p}(X)$ may not be reflexive, the following weak closure result from [23] demonstrates that one can almost apply Mazur's lemma to bounded sequences in $N^{1,p}(X)$.

**Lemma 4.3.** *Let $1 < p < \infty$. If $X$ is complete and $(f_j)_j$ is a sequence of functions in $L^p(X)$ with upper gradients $(g_j)_j$ in $L^p(X)$, such that $f_j$ weakly converges to $f$ and $g_j$ weakly converges to $g$ in $L^p(X)$,* *then $g$ is a weak upper gradient of $f$ after modifying $f$ on a set of measure zero, and there is a convex combination sequence $\tilde{f}_j = \sum_{k=j}^{N(j)} \lambda_{k,j} f_k$ and $\tilde{g}_j = \sum_{k=j}^{N(j)} \lambda_{k,j} g_k$ with $\sum_{k=j}^{N(j)} \lambda_{k,j} = 1$, $\lambda_{k,j} \geq 0$, so that $\tilde{f}_j$ converges to $f$ and $\tilde{g}_j$ converges to $g$ in $L^p(X)$.*

In [8], Cheeger independently developed a theory of Sobolev type spaces on metric spaces using the notion of upper gradients, and he used this theory to prove a Rademacher-type differentiability theorem for Lipschitz functions on metric measure spaces whose measure is doubling and supports a weak $(1, p)$-Poincaré inequality. The approach in [8] is as follows.

**Definition 4.3.** Given $f \in L^p(X)$, we say that $f \in H^{1,p}(X)$ if the following norm is finite:

$$\|f\|_{H^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_{(f_j,\rho_j)_j} \liminf_{j\to\infty} \|\rho_j\|_{L^p(X)},$$

where the infimum is taken over all sequences of function-upper gradient pairs $(f_j, \rho_j)_j$ with $f_j \to f$ in $L^p(X)$.

Using the uniform convexity of $L^p(X)$ and Lemma 4.3, it is clear that whenever $1 < p < \infty$ and $X$ is complete we have $H^{1,p}(X) = N^{1,p}(X)$; however, we can modify functions from $H^{1,p}(X)$ on sets of $\mu$-measure zero, whereas (as mentioned above), we cannot do so to functions from $N^{1,p}(X)$. If $X$ is not complete, then $H^{1,p}(X) = N^{1,p}(\widehat{X})$ where $\widehat{X}$ is the completion of $X$. In general, $H^{1,1}(X) \neq N^{1,1}(X)$ as $H^{1,1}(\mathbb{R}^n)$ corresponds to the class of functions of bounded variation.

As mentioned above, it is not in general true that $N^{1,p}(X)$ is reflexive. However, in the event that $1 < p < \infty$ and the measure on $X$ is doubling and supports a weak $(1, p)$-Poincaré inequality, one of the results in [8] demonstrates the reflexivity of $H^{1,p}(X)$ and hence of $N^{1,p}(X)$. To prove this, a linear derivation operator on $H^{1,p}(X)$ is constructed in [8] as follows.

**Theorem 4.1** (Cheeger)**.** *Let the measure on $X$ be doubling, $1 < p < \infty$, and assume that $X$ admits a $(1, p)$-Poincaré inequality. Then there exists a countable collection $(U_\alpha, X^\alpha)$ of measurable sets $U_\alpha$ and Lipschitz "coordinate" functions $X^\alpha = (X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha) : X \to \mathbb{R}^{k(\alpha)}$ such that $\mu\big(X \setminus \bigcup_\alpha U_\alpha\big) = 0$, $\mu(U_\alpha) > 0$, and for all $\alpha$ the following hold.*

*The functions $X_1^\alpha, \ldots, X_{k(\alpha)}^\alpha$ are linearly independent on $U_\alpha$ and $1 \leq k(\alpha) \leq N$, where $N$ is a constant depending only on the doubling constant of $\mu$ and the constant from the Poincaré inequality. If $f : X \to$*

*$\mathbb{R}$ is Lipschitz, then there exist unique bounded measurable vector-valued functions $d^\alpha f : U_\alpha \to \mathbb{R}^{k(\alpha)}$ such that for $\mu$-a.e. $x_0 \in U_\alpha$,*

$$\lim_{r\to 0^+} \sup_{x\in B(x_0,r)} \frac{|f(x) - f(x_0) - d^\alpha f(x_0)\cdot (X^\alpha(x) - X^\alpha(x_0))|}{r} = 0.$$

*Furthermore, there is a constant $C > 0$ such that for all Lipschitz functions $f$ on $X$, $\dfrac{1}{C}\, g_f \le |d^\alpha f| \le C\, g_f$ $\mu$-a.e. on $U_\alpha$ for each $\alpha$.*

Since a weak $(1,p)$-Poincaré inequality holds on $X$, Lipschitz functions form a dense subclass of $N^{1,p}(X) = H^{1,p}(X)$; see [8] and [37]. Hence the discussion by Franchi, Hajłasz, and Koskela in [9] demonstrates that the linear derivation operator $d^\alpha$ can be extended to operate also on functions in $N^{1,p}(X)$. Thus we have a natural embedding of $N^{1,p}(X)$ into $L^p(X) \times L^p(X : \mathbb{R}^N)$ (which is a uniformly convex space and hence is reflexive), resulting in $N^{1,p}(X)$ being reflexive itself. A further advantage of having this linear derivation operator for functions in $N^{1,p}(X)$ is that associated to (Cheeger) $p$-harmonic functions there is an Euler-Lagrange equation. The Euler-Lagrange equations are quite useful in the study of potential theory; see for example the discussions in [22] and [18]. It should be noted here that the map $f \mapsto \rho_f$ is rarely a linear map; also in general $\rho_{f_1 - f_2} \ne |\rho_{f_1} - \rho_{f_2}|$.

## §5. Sobolev spaces of Korevaar-Schoen, and the connection between the various Sobolev type spaces

Using the notion of energy integral proposed by Korevaar and Schoen in [29], Koskela and MacManus studied the following version of Sobolev spaces on metric measure spaces in [33] (see also [20] for a more general discussion).

**Definition 5.1.** Given $f : X \to \mathbb{R}$, we define the Korevaar-Schoen energy of $f$ to be the number $E(f)$, where

$$E(f) := \sup_B \left( \limsup_{\epsilon\to 0} \int_B \int_{B(x,\epsilon)} \frac{|f(x) - f(y)|^p}{\mu(B(x,\epsilon))\,\epsilon^p}\, d\mu(y)\, d\mu(x) \right),$$

the supremum being taken over all balls $B \subset X$. We say that $f \in KS^p(X)$ if the norm $\|f\|_{KS^{1,p}(X)} := \|f\|_{L^p(X)} + E(f)^{1/p}$ is finite.

The motivation behind such an energy construction is the theory of Dirichlet forms. The early work of Beurling and Deny in [2] and [1], applied to strongly local Dirichlet forms, yields a representation of such Dirichlet forms associated with the above energy for $p = 2$.

In the study of the relationships between the Hajłasz-Sobolev spaces, the Newtonian spaces, the collection of all pairs of functions satisfying a weak $(1,p)$-Poincaré inequality, and in the case of $p=2$ the theory of Dirichlet forms, the Sobolev spaces $KS^{1,p}(X)$ of Korevaar-Schoen play an important connective role; see [20] and [34]. The paper [34] studies the connection between $N^{1,p}(X)$ and domains of various Dirichlet forms on $X$ using the space $KS^{1,p}(X)$; a discussion of Dirichlet forms is beyond the scope of this note, but an excellent discussion can be found in the book [10] by Fukushima, Ōshima, and Takeda.

In what follows, we say that the metric measure space $X$ supports a weak $(1,p)$-Poincaré inequality if there are constants $C>0$ and $\lambda \geq 1$ such that whenever $f : X \to \mathbb{R}$ is a measurable function with $p$-weak upper gradient $\rho$ and $B(x,r)$ is a ball in $X$,

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f-f_{B(x,r)}|\,d\mu \leq C\,r\left(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)}\rho_f^p\,d\mu\right)^{1/p}.$$

In what follows, $P^{1,p}(X)$ consists of all functions $f \in L^p(X)$ for which there exists a non-negative function $g \in L^p(X)$ so that whenever $B(x,r)$ is a ball in $X$,

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|f-f_{B(x,r)}|\,d\mu \leq C\,r\left(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)}g^p\,d\mu\right)^{1/p}.$$

**Theorem 5.1.** *Fix $1<p<\infty$. If $X$ is complete and the measure on $X$ is doubling and supports a $(1,p)$-Poincaré inequality, then the natural mapping between the following spaces are isometric isomorphisms as Banach spaces:*

$$H^{1,p}(X)=N^{1,p}(X)=M^{1,p}(X)=KS^{1,p}(X)=P^{1,p}(X).$$

*If $p=1$, then $M^{1,p}(X)\subset N^{1,p}(X)\subset H^{1,p}(X)$.*

The fact that $H^{1,p}(X)=N^{1,p}(X)$ holds true even without the assumption of the doubling property of the measure nor the Poincaré inequality; see for example [37]. In [37] it is also proven that even without the assumption of a Poincaré inequality $M^{1,p}(X)\subset N^{1,p}(X)$; however, we do need the measure $\mu$ to be doubling here. It is also shown in [37] that if $X$ supports a weak $(1,q)$-Poincaré inequality in addition for some $1\leq q<p$, then $M^{1,p}(X)=N^{1,p}(X)$. The proof of this fact uses a telescoping sequence of balls concentric with points in the metric space, and when these points are Lebesgue points of the function $f\in N^{1,p}(X)$ the weak $(1,q)$-Poincaré inequality is applied to these balls

in order to control the values of $f$ at these points in terms of the Hardy-Littlewood maximal function $M\rho_f^q$ of $\rho_f^q$. If $q < p$ and $\rho_f \in L^p(X)$, then $(M\rho_f^q)^{1/q} \in L^p(X)$. We need this better Poincaré inequality $q < p$ since it is not in general true that $(M\rho_f^p)^{1/p} \in L^p(X)$. However, it is a deep result of Keith and Zhong [24] that if $X$ is complete as a metric space and the measure on $X$ is doubling and supports a weak $(1, p)$-Poincaré inequality, then there exists $1 \leq q < p$ such that $X$ supports a weak $(1, q)$-Poincaré inequality. Hence we have the validity in the above theorem of the statement that $N^{1,p}(X) = M^{1,p}(X)$ under the assumptions that $X$ is proper and supports a weak $(1, p)$-Poincaré inequality. See Theorem 4.5 of [33] for a proof of the equality $KS^{1,p}(X) = M^{1,p}(X) = P^{1,p}(X)$. Again, in [33] Koskela and MacManus require $X$ to support a weak $(1, q)$-Poincaré inequality in addition for some $1 \leq q < p$, but because of the results of Keith and Zhong in [24] we have the validity of the above theorem.

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# Quasisymmetric extension, smoothing and applications

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**Abstract.**

We discuss quasisymmetric extension of embeddings that are close to similarities, due to Tukia and Väisälä, and smooth quasiconformal approximation of such extensions. The smoothing is done by convolution with a variable kernel in conjunction with the Tukia-Väisälä extension procedure. We can apply these to the study of branch sets of smooth quasiregular maps, and quasiconformal dimension of self-similar fractals.

## §1. Branch sets of smooth quasiregular maps

A continuous mapping $f : D \to \mathbf{R}^n$ in the Sobolev space $W^{1,n}_{\mathrm{loc}}(D, \mathbf{R}^n)$ is *K-quasiregular*, $K \geq 1$, if

$$|f'(x)|^n \leq K J_f(x), \qquad \text{a.e. } x \in D.$$

Here $n \geq 2$, $D \subset \mathbf{R}^n$ is a domain, $|f'(x)|$ is the operator norm of the differential of $f$, and $J_f(x) = \det f'(x)$ is the Jacobian determinant. In the plane, 1-quasiregular maps are precisely analytic functions of a single complex variable. Quasiregular mappings were introduced by Yu. G. Reshetnyak [25] under the name "mappings of bounded distortion". A deep theorem of Reshetnyak states that nonconstant quasiregular maps are discrete and open. See [26] for historical accounts.

The branch set $B_f$ of a continuous, discrete, and open mapping $f : D \to \mathbf{R}^n$ is the closed set of points in $D$ where $f$ does not define

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a local homeomorphism. Černavskiĭ [9], [10] proved that the *topological dimensions* of the branch set and its image satisfy

$$\dim B_f = \dim f(B_f) \leq n - 2.$$

*The possible values of the topological dimension of branch sets of quasiregular maps are unknown.*

On the other hand, if $B_f$ is not empty, then $\Lambda^{n-2}(f(B_f)) > 0$ by a theorem of Martio, Rickman and Väisälä [23], and $\Lambda^{n-2}(B_f) > 0$ when $n = 3$ by a result of Martio and Rickman [22]. Here $\Lambda^r$ is the $r$-dimensional Hausdorff measure.

Branch sets of quasiregular mappings may exhibit complicated topological structure and may contain, for example, many wild Cantor sets of classical geometric topology. For recent developments and many interesting *open questions*, see [14], [15], [16], [27].

Quasiregular mappings of $\mathbf{R}^2$ can be smooth without being locally homeomorphic, for example, $f(z) = z^2$. When $n \geq 3$, sufficiently smooth nonconstant quasiregular mappings are locally homeomorphic.

**Theorem 1.1.** *Every nonconstant $C^{n/(n-2)}$-smooth quasiregular mapping must be locally homeomorphic when $n \geq 3$.*

Theorem 1.1 is due to Martio, Rickman and Väisälä [26, p. 12]; the exponent $n/(n-2)$ is derived from Morse-Sard Theorem and the theorem on $\Lambda^{n-2}(f(B_f))$ mentioned earlier. Church [11] has proved Theorem 1.1 for $C^n$ mappings.

In [34], Väisälä asked whether $C^1$-smoothness implies local homeomorphism in Theorem 1.1. Work of Bonk and Heinonen [6] showed that the exponent $n/(n-2)$ in Theorem 1.1 is sharp when $n = 3$.

**Theorem 1.2.** *For every $\epsilon > 0$, there exists a $C^{3-\epsilon}$-smooth quasiregular mapping $F : \mathbf{R}^3 \to \mathbf{R}^3$ whose branch set $B_F$ is homeomorphic to $\mathbf{R}^1$ and has Hausdorff dimension $3 - \delta(\epsilon)$ with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.*

It is proved in [19] that the exponent $n/(n-2)$ in Theorem 1.1 is sharp when $n = 4$; and the authors answered Väisälä's question in the negative for all dimensions.

**Theorem 1.3.** *For every $\epsilon > 0$, there exists a $C^{2-\epsilon}$-smooth quasiregular mapping $F : \mathbf{R}^4 \to \mathbf{R}^4$ whose branch set $B_F$ is homeomorphic to $\mathbf{R}^2$ and has Hausdorff dimension $4 - 2\epsilon$. For any $n \geq 5$, there exists $\epsilon(n) > 0$ and a $C^{1+\epsilon(n)}$-smooth quasiregular map $F : \mathbf{R}^n \to \mathbf{R}^n$ whose branch set $B_F$ is homeomorphic to $\mathbf{R}^{n-2}$.*

Bonk and Heinonen first constructed a quasiconformal mapping $g$ in $\mathbf{R}^3$ with uniformly expanding behavior on a line $L$. Then $g$ is approximated outside $L$ by a $C^\infty$-smooth quasiconformal mapping $G$ by applying a theorem of Kiikka on smoothing [21]. The map $G^{-1}$ has the correct order of smoothness on $\mathbf{R}^3$; postcomposing $G^{-1}$ with a winding map produces the desired quasiregular map $F$. As explained in [6], it is unclear how to construct a quasiconformal mapping $g$ in $\mathbf{R}^n$, $n \geq 4$, which is uniformly expanding on codimension two subspaces. Moreover, the smoothing procedure of Kiikka works in dimensions 2 and 3 only.

We discuss these issues and related topics on quasisymmetric extension, smooth approximation, existence of snowflake surfaces and quasiconformal deformation of self-similar fractals in the following sections.

## §2. Quasiconformal extensions

One of the most important results on quasiconformal mappings is the Extension Theorem.

**Theorem 2.1.** *Every quasiconformal mapping $f : \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ (quasisymmetric if $n = 2$) has a quasiconformal extension in $\mathbf{R}^n_+ = \mathbf{R}^{n-1} \times [0, \infty)$.*

This was proved by Beurling and Ahlfors [2] for $n = 2$, later by Ahlfors [1] for $n = 3$ and by Carleson [8] for $n \leq 4$. Finally, Tukia and Väisälä [30] proved the Extension Theorem for all $n \geq 2$.

Ahlfors showed that every planar quasiconformal map is a composition of mappings with dilatation arbitrarily close to 1, and that mappings in the plane with small dilatation can be extended to quasiconformal homeomorphisms of $\mathbf{R}^3$. *Whether any quasiconformal map in dimension 3 or higher can be decomposed into mappings of dilatation arbitrarily close to 1 remains unanswered.* Carleson constructed a piecewise linear approximation $g$ of $f$, extended $g$ to $\mathbf{R}^n_+$ , and performed a limiting process. The approximation of Moise used is valid for dimensions 2 and 3 only. Tukia and Väisälä extended the given map on $\mathbf{R}^{n-1}$ to a homeomorphism in $\mathbf{R}^n_+$, then applied an approximation procedure of D. Sullivan to obtain the quasiconformality.

Extending a quasisymmetric homeomorphism defined on a subset of $\mathbf{R}^n$ to a quasiconformal homeomorphism of $\mathbf{R}^n$ can be difficult and is not always possible. For example, a smooth homeomorphism from a circle onto a knotted curve in $\mathbf{R}^3$ can not be extended to a homeomorphism of $\mathbf{R}^3$ for a topological reason; certain smooth homeomorphisms between a Jordan curve with two inward spikes and a Jordan curve with one inward spike and one outward spike can not be extended to be quasiconformal on $\mathbf{R}^2$ for an analytical reason.

To study extension of quasisymmetric maps on subsets of $\mathbf{R}^n$, Tukia and Väisälä [31], [35], introduced the notion of $s$-quasisymmetric maps, a restricted class of quasisymmetric maps which are locally uniformly close to similarities, and the notion of quasisymmetric extension property .

An embedding $f : X \to Y$ of metric spaces is called *s-quasisymmetric (s-QS)*, if $f$ is quasisymmetric and satisfies

$$|f(a) - f(x)| \le (t + s)|f(b) - f(x)|$$

whenever $a, b, x \in X$ with $|a - x| \le t|b - x|$ and $t \le 1/s$ for some $s > 0$. When $X$ is a connected compact subset of $\mathbf{R}^p$ and $Y = \mathbf{R}^n$ with $1 \le p \le n$, the above definition is equivalent to the existence of a small $\varkappa > 0$ so that for every bounded $S \subset X$, there is a similarity $h : \mathbf{R}^p \to \mathbf{R}^n$ so that

$$||h - f||_S \le \varkappa L(h) \operatorname{diam} S, \tag{2.2}$$

where $L(h)$ is the similarity ratio.

A subset $A$ of $\mathbf{R}^n$ has the *quasisymmetric extension property (QSEP)* in $\mathbf{R}^n$ if every $s$-QS $f : A \to \mathbf{R}^n$ has an $s_1$-QS extension $g : \mathbf{R}^n \to \mathbf{R}^n$ whenever $0 < s \le s_0(n, A)$, where $s_1 = s_1(s, n, A) \to 0$ as $s \to 0$. See [35, p. 239]. Since the extended map is quasisymmetric, it is necessarily quasiconformal.

It is not easy to determine whether a given set possesses the extension property. Tukia and Väisälä proved the following.

**Theorem 2.3.** *Let $A$ be a subset of $\mathbf{R}^n$, $n \ge 2$, belonging to one the following classes:*

(a) $\mathbf{R}^p$ *or* $S^p$, *with* $1 \le p \le n - 1$,
(b) *$A$ a closed thick set in* $\mathbf{R}^p$,$1 \le p \le n$ *such that either $A$ or* $\mathbf{R}^p \setminus A$ *is bounded,*
(c) *a compact* $(n - 1)$*-dimensional* $C^1$*-manifold with or without boundary,*
(d) *a finite union of simplices of dimensions $n$ and $n - 1$.*

*Then $A$ has the quasisymmetric extension property in* $\mathbf{R}^n$.

A set $A \subset \mathbf{R}^p$ is *thick in* $\mathbf{R}^p$ if there are constants $r_0 > 0$ and $\beta > 0$ so that if $0 < r \le r_0$ and $y \in A$, then there is a simplex $\Delta$ in $\mathbf{R}^p$ with $\Delta^0 \subset A \cap B(y, r)$ and $\Lambda^p(\Delta) \ge \beta r^p$.

We outline the Tukia-Väisälä extension procedures in the case when $A$ is a thick set satisfying (b) in Theorem 2.3 with $p = n$. Let $f$ be an $s$-quasisymmetric map defined on $A$ and $\mathcal{W}$ be a fixed Whitney triangulation of $\mathbf{R}^n \setminus A$. At each vertex $P$ of a simplex in $\mathcal{W}$, choose $h_P$, a similarity that approximates the mapping $f$ on the ball $B(P, C \operatorname{dist}(P, A)) \cap A$, for some fixed $C > 1$, uniformly in the sense of (2.2); then define $f(P)$ to be $h_P(P)$. After $f$ has been defined at all vertices in $\mathcal{W}$, extend $f$ by the unique affine extension in each simplex in $\mathcal{W}$. Since $A$ is thick, information of $f$ on $A$ is abundant and is sufficient to show the consistency of the affine maps associated with neighboring simplices. Since $f$ is locally uniformly close to similarities and $A$ or $\mathbf{R}^n \setminus A$ is relatively compact, degree theory can then be applied to prove $f$ is injective, surjective and sense preserving when $s$ is small. Intricate estimates coupled with the thickness condition guarantee that the extension is indeed $s_1-$quasisymmetric.

The extension procedure and the estimates are sensitive to the nature of the sets; for each class of the sets in Theorem 2.3, the proof has to be somewhat altered. Examples of sets which do not have the extension property are give in [35].

*It would be interesting to know to what extent the thickness condition can be weakened. And it was asked in* [35]*, whether the manifold in (c) and the simplices in (d) can have dimension $p \leq n-2$, and whether every compact polyhedron in $\mathbf{R}^n$ has QSEP.*

Tukia-Väisälä extension procedure is especially useful in extending quasisymmetric maps on fractals, when the mappings in question are more likely to be compositions of close-to-similarities. We shall apply Theorem 2.3 to study Theorem 1.3 in section 5, and quasiconformal dimension of Sierpinski gaskets in section 6.

## §3. Smoothing

Quasiconformal mappings in $\mathbf{R}^2$ or $\mathbf{R}^3$ can be approximated by $C^\infty$-diffeomorphisms. Kiikka [21] proved the following.

**Theorem 3.1.** *Let $g : \Omega \to \Omega'$ be a $K$-quasiconformal mapping between domains in $\mathbf{R}^n$, $n = 2$ or $3$. Then for any positive continuous function $\epsilon$ on $\Omega$, there exists a $\tilde{K}$-quasiconformal $C^\infty$-diffeomorphism $\tilde{g}$ such that $|\tilde{g}(x) - g(x)| < \epsilon(x)$ for all $x \in \Omega$. The constant $\tilde{K}$ depends only on $K$.*

In the proof, Kiikka used difficult work of Moise and of Munkres on smooth approximation of piecewise differentiable homeomorphisms, when dimension is 2 or 3. *This kind of approximation for general quasiconformal maps can not exist for dimension higher than 5* [29]*, and is a long standing open question in dimension 4* [13].

Let $A$ be a set in a class described in Theorem 2.3, and $g$ be a $s$-quasisymmetric map on $A$ with a very small $s$. Tukia-Väisälä 's construction guarantees a quasiconformal extension, again called $g$, to $\mathbf{R}^n$.

Sometimes it is desirable to have a smooth extension outside $A$. To this end, we convolve $g$ with a variable kernel. Let $\delta_A$ be a regularized

$C^\infty$ distance function to $A$, see for example, [28, p. 170]. Fix a $C^\infty$ function $\varphi$ on $\mathbf{R}^n$ which is nonnegative, radial, supported in $B(0,1)$, and satisfies $\int_{\mathbf{R}^n} \varphi(x)\,dx = 1$, $\sup_{\mathbf{R}^n} \left|\frac{\partial\varphi}{\partial x_i}\right| \le C$, $\sup_{\mathbf{R}^n} \left|\frac{\partial^2\varphi}{\partial x_i \partial x_j}\right| \le C$. Then the map

$$G(x) = \begin{cases} \dfrac{1}{\delta_A^n(x)} \displaystyle\int_{\mathbf{R}^n} g(y)\varphi\Big(\frac{x-y}{\delta_A(x)}\Big)\,dy, & x \in \mathbf{R}^n \setminus A, \\ g(x), \quad x \in A \end{cases}$$

is $C^\infty$-smooth outside $A$.

Smoothing by convolution, in general, does not preserve injectivity or quasiconformality. To obtain injectivity, quasiconformality, and the correct order of smoothness, convolution must be applied in conjunction with the Tukia-Väisälä construction. Closeness to similarities uniformly at all points on $A$ and in all scales assures the essential inequalities required more or less preserved after convolution, whence the quasiconformality. See [19] for details.

Sometimes it is further necessary to know that an extension or its inverse is smooth in the entire $\mathbf{R}^n$. While this is not always possible, we discuss one particular situation when this can be done. Let $g$ be the restriction of the quasiconformal mapping in Theorem 4.1 to the hyperplane $\mathbf{R}^{n-1}$ on which the snowflake property holds, and $A$ be its image. When $\epsilon$ is very small, $g$ is $s$-QS for a very small $s$. We can re-extend $g$ to a global quasiconformal map on $\mathbf{R}^n$ following Tukia-Väisälä method; then apply convolution to this newly extended map to obtain a map $G : \mathbf{R}^n \to \mathbf{R}^n$ that agrees with $g$ on $\mathbf{R}^{n-1}$ and is $C^\infty$ outside. The snowflake property of $g$ on $\mathbf{R}^{n-1}$ ensures that $A$ is a thick set and that the gradient of $g^{-1}$ is Hölder continuous on $A$. The function $G^{-1}$ can then be shown to be $C^{1+\delta}$ in the entire $\mathbf{R}^n$ for some $\delta > 0$. Again see [19] for details.

## §4. Snowflake Embeddings

Existence of quasisymmetric embedding $f$ of $\mathbf{R}^{n-1}$ in $\mathbf{R}^n$ that has the snowflake property:

$$C^{-1}|x-y|\phi(|x-y|) \le |g(x) - g(y)| \le C|x-y|\phi(|x-y|),$$

for some $\phi(t) \to \infty$ as $t \to 0$, was raised in [17]. Existence of snowflake embeddings that can be further extended to become quasiconformal on $\mathbf{R}^n$ has been proved by Bishop [5], and David and Toro [12]. A special case of a theorem on embedding Reifenberg flat metric spaces into Euclidean spaces due to David and Toro can be stated as follows.

**Theorem 4.1.** *For each $n \geq 2$ and $0 < \epsilon < \epsilon_0(n)$, there exists a $K$-quasiconformal map $g : \mathbf{R}^n \to \mathbf{R}^n$ with*

$$C^{-1}|x-y|^{1/(1+\epsilon)} \leq |g(x) - g(y)| \leq C|x-y|^{1/(1+\epsilon)}$$

*for all $x, y \in \mathbf{R}^{n-1}$, $|x-y| \leq 1$ and some $C = C(n) > 1$ . Furthermore, $K \to 1$ as $\epsilon \to 0$.*

The exponent in Theorem 4.1 necessarily satisfies $1/(1+\epsilon) > (n-1)/n$. The method of David and Toro is incisive, however does not give estimates of the number $\epsilon_0(n)$. It is not clear whether the exponent can be made arbitrarily close to $(n-1)/n$; equivalently, whether there is a snowflake embedding of $\mathbf{R}^{n-1}$ to a surface in $\mathbf{R}^n$ having Hausdorff dimension arbitrarily close to $n$.

It is generally believed that in order to show the order of smoothness is sharp in Theorem 1.1, a snowflake embedding from $\mathbf{R}^{n-2}$ to a surface in $\mathbf{R}^n$ having Hausdorff dimension arbitrarily close to $n$ must be found. In $\mathbf{R}^4$, product of two planar snowflake curves is the image of a snowflake embedding from $\mathbf{R}^2$. The method of taking products breaks down for $n \geq 5$.

*Therefore, it is not only intrinsically interesting but also useful to know whether there is a nearly space filling snowflake embedding from $\mathbf{R}^p$ into $\mathbf{R}^n$ for every $p$, $1 \leq p < n$.* Paradoxically, this might be more easily achieved by subspaces $\mathbf{R}^p$ of a smaller dimension. Method of Bonk and Heinonen in [6] gives an affirmative answer for the case $p = 1$.

## §5. Theorem 1.3

When $n = 4$, let $\Gamma$ be a standard infinite snowflake curve of Hausdorff dimension $2 - \epsilon$ . The product set $\Gamma \times \Gamma$ is to be the branch set of a $C^{2-\epsilon}$-smooth quasiregular map $F$. Note that there is a canonical map $g$ from $\mathbf{R}^2$ to $\Gamma \times \Gamma$. This map can be written as a composition $g = g_{m-1} \circ \cdots \circ g_0$ such that each $g_j$ satisfies a snowflake property, has a product of snowflake curves as its image, and is $s$-quasisymmetric for a small $s$. Construction of Tukia and Väisälä for part (a) and (b) of Theorem 2.3 can be adapted to extend $g_j$ to be quasiconformal on $\mathbf{R}^4$. Smoothing outside products of snowflake curves via convolution with a variable kernel produces new quasiconformal maps $G_j$. The inverses $G_j^{-1}$ can be shown to be $C^{1+\epsilon_j}$ for some $\epsilon_j > 0$, in the entire $\mathbf{R}^4$, following the reasoning in Section 3. Postcompose the inverse of $G_{m-1} \circ G_{m-2} \circ \cdots \circ G_0$ with a winding map $\omega : \mathbf{R}^4 \to \mathbf{R}^4$, $\omega(x_1, x_2, r, \theta) = (x_1, x_2, r\cos 2\theta, r \sin 2\theta)$, yields the desired $C^{2-\epsilon}-$quasiregular map $F$, having $\Gamma \times \Gamma$ as its branch set.

The method of taking products does not work in $\mathbf{R}^n$, $n \geq 5$, unless there exists an appropriate embedding of the $(n-2)$-fold product $\Gamma \times \cdots \times \Gamma \to \mathbf{R}^n$. For $n \geq 5$, Theorem 4.1 of David–Toro [12] provides a snowflake-type embedding $g : \mathbf{R}^{n-2} \hookrightarrow \Sigma \subset \mathbf{R}^{n-1}$. Embed both $\mathbf{R}^{n-2}$ and the image $\Sigma$ in $\mathbf{R}^n$, and extend $g$ directly to a global quasiconformal map on $\mathbf{R}^n$ by applying part (a) of Theorem 2.3. Smooth the extension outside $\mathbf{R}^{n-2}$ by a convolution, then postcompose the inverse with a winding map. The codimension two snowflake-type surfaces $\Sigma \subset \mathbf{R}^n$ can then be realized as the branch set of a $C^{1+\epsilon(n)}$-smooth branched quasiregular map in $\mathbf{R}^n$, $n \geq 5$. This answers Väisälä's question in the negative for all dimensions.

See [19] for details.

## §6. Quasiconformal dimension of some self-similar sets

Problems on raising or lowering Hausdorff dimension of sets in $\mathbf{R}^n$ through quasiconformal homeomorphism of $\mathbf{R}^n$ have been studied for some time. Bishop [3] showed that for sets of positive dimension there is never an obstruction to raising dimension by quasiconformal maps. In fact, for any compact set $E$ in $\mathbf{R}^n$ with $\dim(E) > 0$ and any $0 < \gamma < n$ there is a quasisymmetric map $h\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\dim(h(E)) > \gamma$. On the other hand, examples of Bishop and Tyson [4] [32] showed that the corresponding statement for lowering dimension can fail.

Given a metric space $(X, d)$, the notion of conformal dimension was introduced by Pansu [24]:

$$C \dim X \equiv \inf\{\dim Y : (Y, \tilde{d}) \text{ quasisymmetrically equivalent to} (X, d)\}.$$

A variety of problems on conformal dimension has been studied; some have applications to geometric group theory. See, for example, work of Bonk-Kleiner [7] and Keith-Laakso [20] . Less studied is the quasiconformal dimension of a set $E$ in $\mathbf{R}^n$ defined as follows [33]:

$$QC \dim E \equiv \inf\{\dim f(E) : f \ \text{ quasiconformal homeomorphism of } \mathbf{R}^n\}.$$

Clearly,

$$\text{topological-}\dim E \leq C \dim E \leq QC \dim E \leq \text{Hausdorff-}\dim E.$$

Analysis on self-similar fractals has been actively pursued in recent years. Sierpinski gasket due to its simplicity, and Sierpinski carpet due to its appearance in the boundary of Gromov hyperbolic groups [18] are particularly intriguing. *One of the most challenging questions in this area is to determine the conformal dimension and the quasiconformal* *dimension of the Sierpinski carpet in* $\mathbf{R}^n$*, for any* $n \geq 2$. The analogous problem on the Sierpinski gasket $SG^n$ in $\mathbf{R}^n$ is easier [33].

**Theorem 6.1.** *For each* $n \geq 2$*,* $QC \dim SG^n = 1$.

Recall that topological dimension of $SG^n$ is 1 and Hausdorff dimension of $SG^n$ is $\frac{\log(n+1)}{\log 2}$. Theorem 6.1 says that $SG^n$ can be mapped by quasiconformal self-maps of $\mathbf{R}^n$ onto sets of Hausdorff dimension arbitrarily close to its topological dimension.

The conclusion of Theorem 6.1 remains true for the invariant sets of a large class of postcritically finite iterated function systems satisfying a so-called gasket type property [33].

We describe the role of Tukia-Väisälä extension in studying quasiconformal dimension of fractals. Depending on the nature of the invariant set $S$ in $\mathbf{R}^n$, a quasisymmetric map $f$ is selected to map $S$ onto the invariant set of an isomorphic function system having a smaller Hausdorff dimension. Selection of $f$ is largely based on intuition; this step gives an upper bound of the conformal dimension of $S$. To obtain an upper bound for the quasiconformal dimension, $f$ needs to be extended to be quasiconformal on $\mathbf{R}^n$. Imagining extending a map from the Sierpinski gasket in $\mathbf{R}^3$ to $\mathbf{R}^3$ by hand, it can be quite a task; Tukia-Väisälä extension procedure makes this process manageable. Under some extra conditions, the canonical map between invariant sets of two isomorphic systems can be decomposed into $s$-quasisymmetric maps, for a very small $s$. To do this a flow of function systems has to be produced so that the corresponding invariant sets are isotopic. Sums of orthogonal maps are not orthogonal. Therefore the flow can not be expressed algebraically as linear combinations; it has to be built geometrically and combinatorially. The construction of the flow can be quite daunting even for the Sierpinski gasket in $\mathbf{R}^3$, or a polygasket in $\mathbf{R}^2$ [33]. Finally the Tukia-Väisälä procedure is applied to each of the maps in the decomposition, then the extensions are recomposed.

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# Wiener criterion for Cheeger $p$-harmonic functions on metric spaces

**Jana Björn**

**Abstract.**

We show that for Cheeger $p$-harmonic functions on doubling metric measure spaces supporting a Poincaré inequality, the Wiener criterion is necessary and sufficient for regularity of boundary points.

## §1. Introduction

The well-known Wiener criterion in $\mathbf{R}^n$ states that a boundary point $x \in \partial\Omega$ is regular for $p$-harmonic functions (i.e. every solution of the Dirichlet problem with continuous boundary data is continuous at $x$) if and only if

$$\int_0^1 \left( \frac{\operatorname{Cap}_p(B(x,t) \setminus \Omega, B(x,2t))}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t} = \infty,$$

where $\operatorname{Cap}_p$ is the $p$-capacity on $\mathbf{R}^n$. For $p = 2$, this was proved by Wiener [30]. For $1 < p < \infty$, the sufficiency part of the Wiener criterion is due to Maz′ya [25] and has been extended to more general equations in Gariepy–Ziemer [10], Heinonen–Kilpeläinen–Martio [12] and Danielli [8]. The necessity part for $1 < p < \infty$ was proved by Kilpeläinen–Malý [19] and extended to weighted equations by Mikkonen [26]. For subelliptic operators, the Wiener criterion was proved in Trudinger–Wang [29].

In the last decade, there has been a lot of development in the theory of $p$-harmonic functions on doubling metric measure spaces supporting a Poincaré inequality. The Dirichlet problem for such $p$-harmonic

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functions has been solved for rather general boundary data (including Sobolev and continuous functions) in e.g. Cheeger [7], Shanmugalingam [27] and [28], Kinnunen–Martio [22] and Björn–Björn-Shanmugalingam [2] and [3].

In Björn–MacManus–Shanmugalingam [6], the sufficiency part of the Wiener criterion was proved in linearly locally connected spaces. The proof in [6] applies both to Cheeger $p$-harmonic functions and to $p$-harmonic functions defined using the upper gradient. In this note, we show that for Cheeger $p$-harmonic functions the assumption of linear local connectedness can be omitted. Moreover, for Cheeger $p$-harmonic functions, the Wiener condition is also necessary, i.e. we have the following result.

**Theorem 1.1.** *Let $X$ be a complete metric measure space with a doubling measure $\mu$ supporting a $p$-Poincaré inequality. Let $\Omega \subset X$ be open and bounded. Then the point $x \in \partial\Omega$ is Cheeger $p$-regular if and only if for some $\delta > 0$,*

$$\int_0^\delta \left( \frac{\mathrm{Cap}_p(B(x,t) \setminus \Omega, B(x,2t))}{t^{-p}\mu(B(x,t))} \right)^{1/(p-1)} \frac{dt}{t} = \infty. \tag{1.1}$$

Much of the theory of $p$-harmonic functions on metric spaces has been done for $p$-harmonic functions defined using the upper gradient. All those proofs go through for Cheeger $p$-harmonic functions as well (just replacing $g_u$ by $|Du|$ throughout). On the other hand, certain results and methods which apply to Cheeger $p$-harmonic functions cannot be used for $p$-harmonic functions defined using the upper gradients. The proof of Theorem 1.1 is one such example: it uses Wolff potential estimates for supersolutions, as in Kilpeläinen–Malý [19]. For other examples, see e.g. Björn–MacManus–Shanmugalingam [6] or Björn–Björn–Shanmugalingam [2].

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## §2. Preliminaries

We assume throughout the paper that $X = (X, d, \mu)$ is a complete metric space endowed with a metric $d$ and a positive complete Borel measure $\mu$ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$ (we make the convention that balls are nonempty and open). We also assume that the measure $\mu$ is *doubling*, i.e. that there exists a constant $C > 0$ such that for all balls $B = B(x,r) := \{y \in X : d(x, y) < r\}$ in $X$,

$$\mu(2B) \le C\mu(B),$$

where $\lambda B = B(x, \lambda r)$. Note that some authors assume that $X$ is proper (i.e. that closed bounded sets are compact) rather than complete, but, since $\mu$ is doubling, $X$ is complete if and only if $X$ is proper.

Throughout the paper, $1 < p < \infty$ is fixed. In [13], Heinonen and Koskela introduced upper gradients as a substitute for the modulus of the usual gradient. The advantage of this new notion is that it can easily be used in metric spaces.

**Definition 2.1.** *A nonnegative Borel function $g$ on $X$ is an* upper gradient *of an extended real-valued function $f$ on $X$ if for all nonconstant rectifiable curves $\gamma : [0, l_\gamma] \to X$, parameterized by arc length $ds$,*

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds \tag{2.1}$$

*whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If $g$ is a nonnegative measurable function on $X$ such that (2.1) holds for $p$-almost every curve, (i.e. it fails only for a curve family with zero $p$-modulus, see Definition 2.1 in Shanmugalingam* [27]*), then $g$ is a* $p$-weak upper gradient *of $f$.*

We further assume that $X$ supports a *weak $p$-Poincaré inequality*, i.e. there exist constants $C > 0$ and $\lambda \ge 1$ such that for all balls $B \subset X$, all measurable functions $f$ on $X$ and all upper gradients $g$ of $f$,

$$\fint_B |f - f_B|\, d\mu \le C(\operatorname{diam} B)\left(\fint_{\lambda B} g^p\, d\mu\right)^{1/p}, \tag{2.2}$$

where $f_B := \fint_B f\, d\mu = \mu(B)^{-1} \int_B f\, d\mu$.

By Keith–Zhong [17] it follows that $X$ supports a weak $q$-Poincaré inequality for some $q \in [1, p)$, which was earlier a standard assumption. As $X$ is complete, it suffices to require that (2.2) holds for all compactly supported Lipschitz functions, see Heinonen–Koskela [14] or Keith [15], Theorem 2. There are many spaces satisfying these assumptions, such as Riemannian manifolds with nonnegative Ricci curvature and the Heisenberg groups. For a list of examples see e.g. Björn [5], and for more detailed descriptions see Heinonen–Koskela [13] or the monograph Hajłasz–Koskela [11]. The following Sobolev type spaces were introduced in Shanmugalingam [27].

**Definition 2.2.** *For $u \in L^p(X)$, let*

$$\|u\|_{N^{1,p}(X)} = \left( \int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

*where the infimum is taken over all upper gradients of $u$. The* Newtonian space *on $X$ is the quotient space*

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

*where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.*

Every $u \in N^{1,p}(X)$ has a unique *minimal $p$-weak upper gradient* $g_u \in L^p(X)$ in the sense that for every $p$-weak upper gradient $g$ of $u$, $g_u \le g$ $\mu$-a.e., see Corollary 3.7 in Shanmugalingam [28]. Theorem 6.1 in Cheeger [7] shows that for Lipschitz $f$,

$$g_f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x,y)}.$$

Cheeger [7] uses a different definition of Sobolev spaces which leads to the same space, see Theorem 4.10 in [27]. Cheeger's definition yields the notion of partial derivatives in the following theorem, see Theorem 4.38 in [7].

**Theorem 2.3.** *Let $X$ be a metric measure space equipped with a doubling Borel regular measure $\mu$. Assume that $X$ admits a weak $p$-Poincaré inequality for some $1 < p < \infty$.*

*Then there exists $N \in \mathbf{N}$ and a countable collection $(U_\alpha, X^\alpha)$ of measurable sets $U_\alpha$ and Lipschitz "coordinate" functions $X^\alpha : X \to \mathbf{R}^{k(\alpha)}$, $1 \le k(\alpha) \le N$, such that $\mu\bigl(X \setminus \bigcup_\alpha U_\alpha\bigr) = 0$ and for every Lipschitz $f : X \to \mathbf{R}$ there exist unique bounded vector-valued functions $d^\alpha f : U_\alpha \to \mathbf{R}^{k(\alpha)}$ such that for $\mu$-a.e. $x \in U_\alpha$,*

$$\lim_{r \to 0+} \sup_{y \in B(x,r)} \frac{|f(y) - f(x) - \langle d^\alpha f(x), X^\alpha(y) - X^\alpha(x) \rangle|}{r} = 0,$$

*where $\langle \cdot\,, \cdot \rangle$ denotes the usual inner product in $\mathbf{R}^{k(\alpha)}$.*

Cheeger shows that for $\mu$-a.e. $x \in U_\alpha$, there is an inner product norm $|\cdot|_x$ on $\mathbf{R}^{k(\alpha)}$ such that for all Lipschitz $f$,

$$g_f(x)/C \le |d^\alpha f(x)|_x \le C g_f(x), \tag{2.3}$$

where $C$ is independent of $f$ and $x$, see p. 460 in [7]. We can assume that the sets $U_\alpha$ are pairwise disjoint and let $Df(x) = d^\alpha f(x)$ for $x \in U_\alpha$.

We shall in the following omit the subscript $x$ in the norms $|\cdot|_x$ and use the notation

$$|Df| = |Df(x)| := |d^\alpha f(x)|_x. \tag{2.4}$$

Thus, (2.3) can be written as

$$g_f/C \le |Df| \le Cg_f \quad \mu\text{-a.e. in } X. \tag{2.5}$$

The differential mapping $D : f \mapsto Df$ is linear and satisfies the Leibniz and chain rules. Also, $Df = 0$ $\mu$-a.e. on every set where $f$ is constant. See Cheeger [7] for these properties.

By Theorem 4.47 in [7] and Theorem 4.10 in Shanmugalingarm [27], Lipschitz functions are dense in $N^{1,p}(X)$. Using Theorem 10 in Franchi–Hajlasz–Koskela [9] or Keith [16], the "gradient" $Du$ extends uniquely to the whole $N^{1,p}(X)$ and it satisfies (2.5) for every $u \in N^{1,p}(X)$.

**Definition 2.4.** *The p-capacity of a set $E \subset X$ is the number*

$$C_p(E) := \inf_u \|u\|^p_{N^{1,p}},$$

*where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \ge 1$ on $E$.*

For various properties as well as equivalent definitions of the $p$-capacity we refer to Kilpeläinen–Kinnunen–Martio [18] and Kinnunen–Martio [20], [21]. The $p$-capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ outside a set of $p$-capacity zero. Moreover, Corollary 3.3 in Shanmugalingam [27] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ $\mu$-a.e., then $u \sim v$.

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. Let

$$N^{1,p}_0(\Omega) = \{f|_\Omega : f \in N^{1,p}(X) \text{ and } f = 0 \text{ in } X \setminus \Omega\}.$$

Throughout the paper, $\Omega \subset X$ will be a nonempty bounded open set in $X$ such that $C_p(X \setminus \Omega) > 0$. (If $X$ is unbounded then the condition $C_p(X \setminus \Omega) > 0$ is of course immediately fulfilled.)

## §3. $p$-harmonic functions and regularity

There are two ways of generalizing $p$-harmonic functions to metric spaces, one based on the scalar-valued upper gradient $g_u$ and the other using the vector-valued Cheeger gradient $Du$. In this paper, we are concerned with Cheeger $p$-harmonic functions given by the following definition.

**Definition 3.1.** *A function* $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ *is* Cheeger $p$-harmonic *in* $\Omega$ *if it is continuous and for all Lipschitz functions* $\varphi$ *with compact support in* $\Omega$*,*

$$\int_\Omega |Du|^p \, d\mu \le \int_\Omega |Du + D\varphi|^p \, d\mu, \tag{3.1}$$

*or equivalently,*

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu = 0,$$

*where* $\cdot$ *denotes the inner product giving rise to the norm* $|\cdot|$ *from (2.4) (note that it depends on* $x$*).*

As mentioned in the introduction, all properties which have been proved for $p$-harmonic functions defined using the upper gradient, also hold for Cheeger $p$-harmonic functions and will be used here without further notice. By Kinnunen–Shanmugalingam [24], every function satisfying (3.1) has a locally Hölder continuous representative which satisfies the Harnack inequality and the maximum principle. It is this representative that we call Cheeger $p$-harmonic.

The Dirichlet problem for Cheeger $p$-harmonic functions and rather general boundary data was solved using the Perron method in Björn–Björn-Shanmugalingam [3]. The construction is based on Cheeger $p$-superharmonic functions. The *upper Perron solution* for $f : \partial\Omega \to \mathbf{R}$ is

$$\overline{P} f(x) := \inf_u u(x), \quad x \in \Omega,$$

where the infimum is taken over all Cheeger $p$-superharmonic functions $u$ on $\Omega$ bounded below such that

$$\liminf_{\Omega \ni y \to x} u(y) \ge f(x) \quad \text{for all } x \in \partial\Omega.$$

The lower Perron solution is defined by $\underline{P} f = -\overline{P}(-f)$, and if both solutions coincide, we let $Pf := \overline{P} f = \underline{P} f$ and $f$ is called *resolutive.* Note that we always have $\underline{P} f \le \overline{P} f$, by Theorem 7.2 in Kinnunen–Martio [22]. The following comparison principle holds: If $f_1 \le f_2$ on $\partial\Omega$, then $Pf_1 \le Pf_2$ in $\Omega$.

The following theorem is proved in [3], Theorems 5.1 and 6.1.

**Theorem 3.2.** *Let* $f \in C(\partial\Omega)$ *or* $f \in N^{1,p}(X)$*. Then* $f$ *is resolutive. Moreover, if* $f \in N^{1,p}(X)$*, then* $Pf - f \in N^{1,p}_0(\Omega)$*.*

By Theorem 7.7 in Kinnunen–Martio [22], every Cheeger $p$-superharmonic function is a pointwise limit of an increasing sequence of $p$-supersolutions. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a *$p$-supersolution* in $\Omega$ if for all nonnegative Lipschitz functions $\varphi$ with compact support in $\Omega$,

$$\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, d\mu \geq 0.$$

We also have the following simple lemma.

**Lemma 3.3.** *Assume that $f : \partial\Omega \to \overline{\mathbf{R}}$ is resolutive. Let $\Omega' \subset \Omega$ be open and define $h : \partial\Omega' \to \overline{\mathbf{R}}$ by*

$$h(x) = \begin{cases} f(x), & \text{if } x \in \partial\Omega \cap \partial\Omega', \\ Pf(x), & \text{if } x \in \Omega \cap \partial\Omega'. \end{cases}$$

*Then $h$ is resolutive with respect to $\Omega'$ and the Perron solution for $h$ in $\Omega'$ is $P_{\Omega'} h = Pf|_{\Omega'}$.*

*Proof.* Let $u$ be a Cheeger $p$-superharmonic function admissible in the definition of $\overline{P}f = Pf$. Then it is easily verified (using the lower semicontinuity of $u$) that $\liminf_{\Omega' \ni y \to x} u(y) \geq h(x)$ for all $x \in \partial\Omega'$. Hence $u$ is admissible in the definition of the upper Perron solution $\overline{P}_{\Omega'} h$ for $h$ in $\Omega'$ and taking infimum over all such $u$ shows that $\overline{P}_{\Omega'} h \leq Pf$ in $\Omega'$. Applying the same argument to $-f$, we obtain

$$\underline{P}_{\Omega'} h = -\overline{P}_{\Omega'}(-h) \geq -P(-f) = Pf \geq \overline{P}_{\Omega'} h \geq \underline{P}_{\Omega'} h.$$

□

**Definition 3.4.** *A point $x \in \partial\Omega$ is* Cheeger $p$-regular *if*

$$\lim_{\Omega \ni y \to x} Pf(y) = f(x) \quad \text{for all } f \in C(\partial\Omega).$$

In Björn–Björn [1], regular boundary points have been characterized by means of barriers. Theorems 4.2 and 6.1 in [1] also give other equivalent characterizations of regularity. In particular, Theorem 6.1(f) in [1] shows that regularity is a local property:

**Theorem 3.5.** *Let $x \in \partial\Omega$ and $\delta > 0$. Then $x$ is Cheeger $p$-regular with respect to $\Omega$ if and only if it is Cheeger $p$-regular with respect to $\Omega \cap B(x, \delta)$.*

## §4. Proof of Theorem 1.1: sufficiency

We start by defining the relative capacity which appears in the Wiener criterion.

**Definition 4.1.** *Let $B \subset X$ be a ball and $E \subset B$. The* relative capacity *of $E$ with respect to $B$ is*

$$\operatorname{Cap}_p(E, B) = \inf_u \int_B |Du|^p \, d\mu,$$

*where the infimum is taken over all $u \in N_0^{1,p}(B)$ such that $u \geq 1$ on $E$.*

Lemma 3.3 in Björn [4] (combined with (2.5)) shows that the capacities $\operatorname{Cap}_p$ and $C_p$ are in many situations equivalent and have the same zero sets. Moreover, $\operatorname{Cap}_p(B, 2B)$ is comparable to $r^{-p}\mu(B)$.

Unless otherwise stated, the letter $C$ denotes various positive constants whose exact values are unimportant and may vary with each usage. The constant $C$ is allowed to depend on the fixed parameters associated with the geometry of the space $X$.

**Definition 4.2.** *Let $B$ be a ball and $K \subset B$ be compact. The* Cheeger $p$-potential *for $K$ with respect to $B$ is the Cheeger $p$-harmonic function in $B \setminus K$ with boundary data* 1 *on $\partial K$ and* 0 *on $\partial B$. We extend the Cheeger $p$-potential $u$ by 1 on $K$ to have $u \in N_0^{1,p}(B)$.*

Lemma 3.2 in Björn–MacManus–Shanmugalingam [6] shows that the Cheeger $p$-potential $u$ is a $p$-supersolution in $B$. Hence, by Proposition 3.5 in [6], there is a unique regular Radon measure $\nu \in N_0^{1,p}(B)^*$ such that

$$\int_B |Du|^{p-2} Du \cdot D\varphi \, d\mu = \int_B \varphi \, d\nu \quad \text{for all } \varphi \in N_0^{1,p}(B). \tag{4.1}$$

The sufficiency part of Theorem 1.1 will follow from the following lemma. It was proved in [6], Lemma 5.7, for $p$-harmonic functions defined using the upper gradient under the additional assumption that $X$ is linearly locally connected. Here we show it without this assumption, but only for Cheeger $p$-harmonic functions. Estimates of this type appeared first in Maz′ya [25], where they were used to prove the sufficiency part of the Wiener criterion for nonlinear elliptic equations.

**Lemma 4.3.** *Let $B = B(x, r)$ and $K \subset \overline{B}$ be compact. Let $u$ be the Cheeger $p$-potential for $K$ with respect to $4B$. Then for $0 < \rho \leq r$ and $y \in B(x, \rho)$,*

$$1 - u(y) \leq \exp\biggl(-C \int_\rho^r \biggl(\frac{\operatorname{Cap}_p(B(x,t) \cap K, B(x, 2t))}{t^{-p}\mu(B(x,t))}\biggr)^{1/(p-1)} \frac{dt}{t}\biggr).$$

Lemma 4.3 follows from the following lemma by iteration and the comparison principle in the same way as Lemma 5.7 in [6].

**Lemma 4.4.** *Let $B$, $K$ and $u$ be as in Lemma 4.3. Then*

$$\inf_B u \ge C \biggl( \frac{\operatorname{Cap}_p(K, 4B)}{r^{-p}\mu(B)} \biggr)^{1/(p-1)}.$$

*Proof.* Let $\nu$ be the Radon measure given by (4.1). By Lemma 3.10 in [6], we have $\operatorname{supp} \nu \subset K$ and $\nu(K) = \operatorname{Cap}_p(K, 4B)$. Lemma 4.8 in [6] then yields

$$\inf_B u \ge \inf_{2B} u + C \biggl( \frac{\nu(B)}{r^{-p}\mu(B)} \biggr)^{1/(p-1)} \ge C \biggl( \frac{\operatorname{Cap}_p(K, 4B)}{r^{-p}\mu(B)} \biggr)^{1/(p-1)}.$$

□

The following corollary is proved in a similar way as Theorem 6.18 in Heinonen–Kilpeläinen–Martio [12]. See also Maz′ya [25].

**Corollary 4.5.** *Let $f : \partial\Omega \to \mathbf{R}$ be bounded and resolutive, and $x \in \partial\Omega$. Then for all sufficiently small $0 < \rho \le r$,*

$$\begin{aligned} \sup_{\Omega \cap B(x,\rho)} (Pf - f(x)) &\le \sup_{\partial\Omega \cap B(x,4r)} (f - f(x)) \\ &+ \sup_{\partial\Omega} (f - f(x)) \exp\biggl( -C \int_\rho^r \biggl( \frac{\operatorname{Cap}_p(B(x,t) \setminus \Omega, B(x,2t))}{t^{-p}\mu(B(x,t))} \biggr)^{1/(p-1)} \frac{dt}{t} \biggr). \end{aligned}$$

*Proof.* Let $B = B(x, r)$, $m = \sup_{\partial\Omega \cap 4B} f$ and $M = \sup_{\partial\Omega} f$. Note that by the maximum principle, $Pf \le M$ in $\Omega$. We can assume that $f(x) = 0$. Let $u$ be the Cheeger $p$-potential for $K = \overline{B} \setminus \Omega$ in $4B$. Let $h$ be as in Lemma 3.3 with $\Omega' := \Omega \cap 4B$. Then it is easily verified that $h \le m + M(1 - u)$ on $\partial\Omega'$. Lemma 3.3 and the comparison principle show that

$$Pf = P_{\Omega'} h \le P_{\Omega'}(m + M(1 - u)) = m + M(1 - u) \quad \text{on } \Omega'$$

and Lemma 4.3 finishes the proof. □

To conclude the proof of the sufficiency part of Theorem 1.1, let $f \in C(\partial\Omega)$ and $\varepsilon > 0$ be arbitrary. There exists $r > 0$ such that $\sup_{\partial\Omega \cap B(x,4r)} |f - f(x)| \le \varepsilon$. Condition (1.1) and Corollary 4.5 then imply that for sufficiently small $\rho$ we have

$$\sup_{\Omega \cap B(x,\rho)} |Pf - f(x)| \le 2\varepsilon.$$

Thus, $Pf$ is continuous at $x$ and as $f \in C(\partial\Omega)$ was arbitrary, $x$ is Cheeger $p$-regular.

## §5. Proof of Theorem 1.1: necessity

To obtain the necessity part of Theorem 1.1, we first formulate an estimate for $p$-supersolutions by means of Wolff potentials. It is similar to Theorem 1.6 in Kilpeläinen–Malý [19] and Corollary 4.11 in [6].

**Lemma 5.1.** *Let $u$ be a nonnegative $p$-supersolution in $5B$, where $B = B(x,r)$. Let $\nu$ be the Radon measure given by* (4.1). *Then*

$$\lim_{\rho\to 0} \operatorname*{ess\,inf}_{B(x,\rho)} u \le C\biggl( \operatorname*{ess\,inf}_{3B} u + \int_0^r \biggl( \frac{\nu(B(x,t))}{t^{-p}\mu(B(x,t))} \biggr)^{1/(p-1)} \frac{dt}{t} \biggr).$$

*Proof.* It can be shown as in the proof of Theorem 3.13 in Mikkonen [26] that the above estimate holds with $\operatorname{ess\,inf}_{3B} u$ replaced by $\Bigl( \fint_{\frac12 B} u^\gamma \, d\mu \Bigr)^{1/\gamma}$ for all $\gamma > p-1$ (and $C$ depending on $\gamma$). Theorem 4.3 in Kinnunen–Martio [23] shows that for $\gamma$ close to $p-1$,

$$\biggl( \fint_{\frac12 B} u^\gamma \, d\mu \biggr)^{1/\gamma} \le C \operatorname*{ess\,inf}_{3B} u,$$

which concludes the proof. □

**Corollary 5.2.** *Let $u \in N_0^{1,p}(5B)$ be the Cheeger $p$-potential for a compact $K \subset \overline{B}$ in $5B$, where $B = B(x,r)$. Then*

$$\liminf_{y\to x} u(y) \le C \int_0^{2r} \biggl( \frac{\operatorname{Cap}_p(B(x,t)\cap K, B(x,2t))}{t^{-p}\mu(B(x,t))} \biggr)^{1/(p-1)} \frac{dt}{t}.$$

*Proof.* Let $\nu$ be the Radon measure given by (4.1). For $0 < t \le r$, let $\nu_t$ be the restriction of $\nu$ to $B(x,t)$ and $u_t \in N_0^{1,p}(5B)$ be the $p$-supersolution in $5B$ associated with $\nu_t$ as in (4.1), see Proposition 3.9 in Björn–MacManus–Shanmugalingam [6]. It satisfies

$$\int_{5B} |Du_t|^{p-2} Du_t \cdot D\varphi \, d\mu = \int_{5B} \varphi \, d\nu_t \quad \text{for all } \varphi \in N_0^{1,p}(5B). \tag{5.1}$$

Inserting $\varphi = (u_t - u)_+$ as a test function in both (4.1) and (5.1), a simple comparison yields $D(u_t-u)_+ = 0$ $\mu$-a.e. in $5B$ (see e.g. Lemma 2.8 in [26]). Hence $u_t \le u \le 1$ in $5B$ and Lemma 3.10 in [6] implies

$$\nu_t(B(x,t)) \le \operatorname{Cap}_p(K\cap \overline{B}(x,t), 5B) \le \operatorname{Cap}_p(K\cap B(x,2t), B(x,4t)). \tag{5.2}$$

Let $a = \inf_{3B} u$. Then $a > 0$ by the maximum principle, and Lemma 5.4 in [6] shows that

$$\operatorname{Cap}_p(3B,5B) \le \operatorname{Cap}_p(\{x : u \ge a\}, 5B) \le C a^{1-p} \operatorname{Cap}_p(K,5B).$$

It follows that

$$
\begin{aligned}
a &\le C\biggl(\frac{\operatorname{Cap}_p(K,5B)}{r^{-p}\mu(B)}\biggr)^{1/(p-1)} \\
&\le C\int_r^{2r}\biggl(\frac{\operatorname{Cap}_p(K\cap B(x,t),B(x,2t))}{t^{-p}\mu(B(x,t))}\biggr)^{1/(p-1)}\frac{dt}{t}. \qquad (5.3)
\end{aligned}
$$

Inserting (5.2) and (5.3) into Lemma 5.1 finishes the proof of the corollary. □

To conclude the proof of the necessity part of Theorem 1.1, we apply Corollary 5.2 to $K=\overline{B}(x,r)\setminus\Omega$. Let $u_r$ be the corresponding Cheeger $p$-potential with respect to $B(x,5r)$. If the integral in Theorem 1.1 converges, we can use Corollary 5.2 to find $r>0$ sufficiently small so that

$$\liminf_{y\to x} u_r(y) < 1.$$

As $u_r$ is the solution of the Dirichlet problem in $B(x,5r)\setminus K$ with the continuous boundary data 1 on $K$ and 0 on $\partial B(x,5r)$, we see that $x$ is not Cheeger $p$-regular for the open set $B(x,5r)\setminus K$. Theorem 3.5 then shows that $x$ is not Cheeger $p$-regular for $\Omega$ either.

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# On the Sharpness of Certain Approach Regions

**Fausto Di Biase**

**Abstract.**

In this survey, we describe joint work in collaboration with A. Stokolos, O. Svensson and T. Weiss. We consider the following question: How sharp is the Stolz approach region condition for the almost everywhere convergence of bounded harmonic functions? The issue was first settled in the rotation invariant case in the unit disc by Littlewood in 1927 and later examined, under less stringent conditions, by Aikawa in 1991. We show that our results are, in a precise sense, sharp.

## §1. How sharp are the Stolz approach regions?

In this survey, we describe joint work in collaboration with A. Stokolos, O. Svensson and T. Weiss. Proofs appear elsewhere [8].

### 1.1. The unit disc in the plane

Consider the space $\mathrm{H}^\infty$ of all bounded holomorphic functions in the unit disc $\mathbb{D}$ in $\mathbb{C}$. How sharp is the Stolz (nontangential) approach

$$\Gamma_\alpha(e^{i\theta}) = \left\{ z \in \mathbb{D} : \, |z - e^{i\theta}| < (1+\alpha)(1-|z|) \right\} \tag{1.1}$$

for the a. e. boundary convergence of $\mathrm{H}^\infty$ functions?

A family $\gamma = \{\gamma(\theta)\}_{\theta \in [0,2\pi)}$ of subsets of $\mathbb{D}$, called an *approach*, may have the following properties:

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**c:** each $\gamma(\theta)$ is a curve in $\mathbb{D}$ ending at $e^{i\theta}$;
**tg:** each $\gamma(\theta)$ ends tangentially at $e^{i\theta}$;
**aecv:** each $h \in \mathrm{H}^\infty$ converges a. e. along $\gamma(\theta)$ to its Stolz boundary values.

The Strong Sharpness Statement is the following claim.

(SSS) *There is no approach* $\gamma$ *satisfying* (**c**)&(**tg**)&(**aecv**).

This claim is coherent with a principle — implicit in Fatou [10] — whose first rendition is found in Littlewood [20], who showed that there is no **rotation invariant** approach $\gamma$ satisfying (**c**)&(**tg**)&(**aecv**). Another rendition of this principle (with stronger conclusions) has been given by Aikawa [1], who proved that, if (**u**) is the condition:

**u:** the curves $\{\gamma(\theta)\}_\theta$ are uniformly bi-Lipschitz equivalent;

then there is no approach $\gamma$ satisfying **(u)** and (**c**)&(**tg**)&(**aecv**).

Our first result[1] is a theorem of Littlewood type where the tangential curve is allowed to vary its shape, and we do not require uniformity in the order of tangency. Moreover, we show that, in a precise sense, Theorem 1.1 is sharp.

**Theorem 1.1** (A sharp Littlewood type theorem)**.** *Let* $\gamma : [0, 2\pi) \to 2^{\mathbb{D}}$ *be such that*

**(c$\star$):** *for each* $\theta \in [0, 2\pi)$*, the set* $\{e^{i\theta}\} \cup \gamma(\theta)$ *is connected;*
**(tg):** *for each* $\alpha > 0$ *and* $\theta \in [0, 2\pi)$ *there exists* $\delta > 0$ *such that if* $z \in \gamma(\theta) \cap \Gamma_\alpha(e^{i\theta})$ *then* $|z - e^{i\theta}| > \delta$*;*
**(reg):** *for each open subset* $O$ *of* $\mathbb{D}$ *the set*

$$\{\theta \in [0, 2\pi) : \gamma(\theta) \cap O \neq \emptyset\}$$

*is a measurable subset of* $[0, 2\pi)$*.*

*Then there exists* $h \in \mathrm{H}^\infty$ *with the property that, for almost every* $\theta \in [0, 2\pi)$*, the limit of* $h(z)$ *as* $z \to e^{i\theta}$ *and* $z \in \gamma(\theta)$ *does not exist.*

• Condition (**c$\star$**) is strictly weaker than (**c**) but it *cannot* be relaxed to the minimal condition one may ask for:

**(apprch):** $e^{i\theta}$ belongs to the closure of $\gamma(\theta)$ for all $\theta$

since Nagel and Stein [21] showed that there is a rotation invariant approach $\gamma$ satisfying (**apprch**) and (**tg**)&(**aecv**). This discovery disproved a conjecture of Rudin [24], prompted by his construction of a highly oscillating inner function in $\mathbb{D}$. Thus, (**c$\star$**) identifies the property of *curves* relevant to a theorem of Littlewood type.

[1] A preliminary version of this result was announced in Di Biase et al [7].

• It is not easy to see **(reg)** fail. The images of radii by an inner function satisfy **(reg)**: this example prompted Rudin [24] to ask about the truth value of (SSS). Observe that **(reg)** is a qualitative condition, while **(u)** is quantitative. The former is perhaps more commonly met than the latter. Furthermore, the conditions are independent of each other.

• Since our hypothesis do not impose any smoothness, neither on $\gamma(\theta)$ nor on the domain, a version of our theorem can be formulated, and proved as well, for domains with rough boundary, such as NTA domains in $\mathbb{R}^n$; see Theorem 1.3 below.

• Is it possible to prove Theorem 1.1 without assuming **(reg)**? Several theorems in Analysis do fail if we omit some regularity conditions, while others (typically those involving null sets) remain valid without 'regularity' hypothesis[2]. This question brings us back to the truth value of (SSS), and we prove the following result.

**Theorem 1.2.** *It is neither possible to prove the Strong Sharpness Statement, nor to disprove it.*

The proof uses a combination of methods of modern logic (developed after 1929) and harmonic analysis, based upon an insight about the location of the link that makes the combination possible. See Theorem 2.1, Theorem 2.2 and Theorem 2.3.

### 1.2. Nontangentially accessible domains in $\mathbb{R}^n$

Let $\mathrm{h}^\infty$ be the space of bounded harmonic functions on a bounded domain $D \subset \mathbb{R}^n$. Assume that $D$ is NTA — as defined by Jerison and Kenig [17]. How sharp is the so-called *corkscrew approach*

$$\Gamma_\alpha(w) \overset{\text{def}}{=} \{z \in D : |z - w| < (1+\alpha)\mathrm{dist}(z, \partial D)\} \tag{1.2}$$

for the boundary convergence for $\mathrm{h}^\infty$ functions, a. e. relative to harmonic measure?

Observe that $D$ may be *twisting* a. e. relative to harmonic measure. In this case, the 'corkscrew' approach (1.2) does not look like a sectorial angle at all.

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[2]A *regularity hypothesis* in a theorem is one which is not (formally) necessary to give meaning to the conclusion of the theorem. *A priori* it is not clear which theorems belong to which group. Egorov's theorem on pointwise convergence belongs to the first; see Bourbaki [2], p. 198. One example in the second group can be found in Stein [25], p. 251.

Theorem 1.1 lends itself to the task of formulating[3] the appropriate sharpness statement for NTA domains, without any further restrictions on the domain.

**Theorem 1.3.** *If $D$ is an NTA domain in $\mathbb{R}^n$ and $\gamma = \{\gamma(w)\}_{w\in\partial D}$ is a family of subsets of $D$ such that*

**(c⋆):** *for each $w \in \partial D$, $\gamma(w) \cup \{w\}$ is connected;*
**(tg):** *for each $\alpha > 0$ and $w \in \partial D$ there exists $\delta > 0$ such that if $z \in \gamma(w) \cap \Gamma_\alpha(w)$ then $|z - w| > \delta$;*
**(reg):** *for each open subset $O$ of $D$ the set*

$$\{w \in \partial D : \gamma(w) \cap O \neq \emptyset\}$$

*is a measurable subset of $\partial D$ (i. e. its characteristic function is resolutive);*

*then there exists $h \in \mathrm{h}^\infty$ such that for almost every $w \in \partial D$, with respect to harmonic measure, the limit of $h(z)$ as $z \to w$ and $z \in \gamma(w)$ does not exist.*

• A condition such as rotation invariance, in place of **(reg)**, would have no meaning, since in this context there is no group suitably acting, not even locally.

• Observe that (**c**⋆) *cannot* be relaxed to the weaker condition

$$w \text{ belongs to the closure of } \gamma(w)\,, \text{ for each } w \in \partial\mathbb{D}\,. \tag{1.3}$$

Indeed, the first-named author showed the existence, for NTA domains in $\mathbb{R}^n$, of an approach $\gamma$, satisfying (1.3) and **(tg)**, along which all $\mathrm{h}^\infty$ functions converge to their boundary values taken along (1.2), a. e. relative to harmonic measure[4].

## §2. Overview of the proofs

The core of the problem belongs to harmonic analysis, so we restrict ourselves, without loss of generality, to the space $\mathrm{h}^\infty$ of bounded harmonic functions on $\mathbb{D}$.

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[3] In formulating (and proving) our Theorem 1.1 we also had this goal in mind.

[4] In Di Biase [5], the existence is showed by reducing the problem to the discrete setting of a (not-necessarily-homogeneous) tree, rather than on the action of a group on the space. In general, in this context, there is no group suitably acting on the space.

The boundary of $\mathbb{D}$, denoted by $\partial\mathbb{D}$, is naturally identified to the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, from which it inherits the *Lebesgue measure* $m$; thus, $m(\partial\mathbb{D}) = 2\pi$.

If $h \in \mathrm{h}^\infty$, the *Fatou set of* $h$, denoted by $\mathcal{F}(h) \subset \partial\mathbb{D}$, is the set of points $w \in \partial\mathbb{D}$, such that *the limit of* $h(z)$ *as* $z \to w$ *and* $z \in \Gamma_\alpha(w)$ *exists for all* $\alpha > 0$; this limit is denoted $h_\flat(w)$. Now, $m(\mathcal{F}(h)) = 2\pi$ and $h_\flat \in \mathrm{L}^\infty(\partial\mathbb{D})$; see Fatou [10].

The *Poisson extension* $P : \mathrm{L}^\infty(\partial\mathbb{D}) \to \mathrm{h}^\infty$ recaptures $h$ from $h_\flat$, since $h = P[h_\flat]$.

If $\gamma$ is a subset of $\mathbb{D} \times \partial\mathbb{D}$ and $w \in \partial\mathbb{D}$, *the shape of* $\gamma$ *at* $w$ is the set

$$\gamma(w) \stackrel{\text{def}}{=} \{z \in \mathbb{D} : (z, w) \in \gamma\} \subset \mathbb{D} \,.$$

An *approach* is a subset $\gamma$ of $\mathbb{D} \times \partial\mathbb{D}$ such that (**apprch**) holds for all $\theta$. One may think of $\gamma$ as a family $\{\gamma(\theta)\}_{\theta\in[0,2\pi)}$ of subsets of $\mathbb{D}$. If $h \in \mathrm{h}^\infty$ and $\gamma$ is an approach, then define the following two subsets of $\partial\mathbb{D}$: $\mathrm{C}(h,\gamma)$ is the set

$$\{w \in \mathcal{F}(h);\, h(z) \text{ converges to } h_\flat(w) \text{ as } z \to w \text{ and } z \in \gamma(w)\}$$

and $\mathrm{D}(h,\gamma)$ is the subset

$$\{w \in \partial\mathbb{D};\, h(z) \text{ does not have any limit as } z \to w \text{ and } z \in \gamma(w)\}\,.$$

If $\gamma$ is an approach and $u : \mathbb{D} \to \mathbb{R}$ a function on $\mathbb{D}$, the function on $\partial\mathbb{D}$ given by

$$\gamma^\star(u)(w) \stackrel{\text{def}}{=} \sup\{|u(z)| : z \in \gamma(w)\}$$

is called the *maximal function of* $u$ *along* $\gamma$ at $w \in \partial\mathbb{D}$.

**Lemma 2.1.** *The following properties of an approach* $\gamma$ *are equivalent:*

(a) $\gamma^\star$ *maps all continuous functions (on* $\mathbb{D}$*) to measurable functions (on* $\partial\mathbb{D}$*);*

(b) *for every open* $Z \subset \mathbb{D}$*, the boundary subset*

$$\gamma^\downarrow(Z) \stackrel{\text{def}}{=} \{w \in \partial\mathbb{D} : Z \cap \gamma(w) \neq \emptyset\} \subset \partial\mathbb{D}$$

*is a measurable subset of* $\partial\mathbb{D}$*.*

The subset in (b) is called the *shadow projected by* $Z$ *along* $\gamma$. The proof of Lemma 2.1 is left to the reader[5]. The approach $\gamma$ is called: *regular* if it satisfies (a) or (b) in Lemma 2.1; *rotation invariant* if $(z, w) \in \gamma$

[5]This circle of ideas is based on the work of E. M. Stein. Cf. Fefferman and Stein [11].

implies $(e^{i\theta}z, e^{i\theta}w) \in \gamma$ for all $\theta, z, w$. A rotation invariant approach is regular.

### 2.1. The Independence Theorem

2.1.1. *Preliminary Remarks* Modern logic gives us tools that show that some statements can be neither proved nor disproved. The basic idea is familiar: if different models (or 'concrete' representations) of some axioms exhibit different properties, then these properties do not follow from those axioms. For example, the existence of a single, 'concrete' non commutative group shows that commutativity can not be derived from the group axioms. Similarly, the existence of different models of geometry shows that Euclid's Fifth Postulate does not follow from the others. Since the currently adopted system of axioms for Mathematics is ZFC[6], to prove a theorem amounts to deduce the statement from ZFC. A *model* of ZFC stands to ZFC as, say, a 'concrete' group stands to the axioms of groups. If ZFC is consistent, then it has several, different models. Gödel showed, in his *completeness theorem*, that a statement can be deduced from ZFC if and only if it holds in every model of ZFC; in particular, if it holds in some models but not in others, then it follows that it *can be neither proved nor disproved.* The tangential boundary behaviour of $\mathrm{h}^\infty$ functions is radically different in different models of ZFC[7].

2.1.2. *The Independence Result*

**Theorem 2.1.** *There is a model of ZFC in which there exists an approach $\gamma$ satisfying (**c**) and (**tg**) and such that $C(h,\gamma)$ has measure equal to $2\pi$ for every $h \in \mathrm{h}^\infty$.*

**Theorem 2.2.** *There is a model of ZFC in which for every approach satisfying (**c**$\star$) and (**tg**) there exists $h \in \mathrm{h}^\infty$ such that $D(h,\gamma)$ has outer measure equal to $2\pi$.*

Theorem 2.1 and Theorem 2.2, together with Gödel's completeness theorem, imply Theorem 1.2.

2.1.3. *A Consequence of ZFC* The following result shows that Theorem 2.2 cannot be improved[8]. Observe that while Theorem 2.1 only holds in some models of ZFC but not in others (and therefore, by Gödel's

[6]Acronym for Zermelo, Fraenkel and the Axiom of Choice. See Cohen [4], Drake [9], Jech [16], Kunen [19].

[7]Since an approach is a fairly arbitrary subset of $\mathbb{D} \times \partial\,\mathbb{D}$, *in retrospect* this result can be rationalized, but other examples in Analysis show that this rationalization is not *a priori* infallible.

[8]Theorem 2.3 in itself does not say whether (SSS) can be proved or not.

completeness theorem, the corresponding statement can not be deduced from ZFC) the following theorem can be deduced from ZFC and therefore it holds in any model of set theory (see the discussion in 2.1.1 ).

**Theorem 2.3** (A consequence of ZFC)**.** *There exists an approach $\gamma$ satisfying (**c**) and (**tg**) such that for each $h \in \mathrm{h}^\infty$, the set $C(h, \gamma)$ has outer measure equal to $2\pi$.*

*Remark* 2.1. We quote a remark made by Gödel in [12] about the Continuum Hypothesis, or Cantor's conjecture.

> Only someone who [...] denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as knows today can only mean that these axioms do not contain a complete description of this reality; and such a belief is by no means chimerical, since it is possible to points out ways in which a decision of the question, even if it is undecidable from the axioms in their present form, might nevertheless be obtained.

It seems to us that Gödel's remark applies equally well to (SSS), for those who share the Platonist viewpoint of Gödel.

## §3. How un-Stolz are the sharp approach regions in $\mathbb{C}^n$?

The theory of the boundary behaviour (from the viewpoint of the almost everywhere convergence) of holomorphic functions in the Hardy spaces, defined on a bounded pseudoconvex domain $\mathbb{D}$ with smooth boundary in $\mathbb{C}^n$, has been so far been sufficiently understood in a few cases only: the unit ball in $\mathbb{C}^n$ (Koranyi [18]; Hakim and Sibony [13]; Hirata [14]); finite type domains in $\mathbb{C}^2$ (Nagel et al [22]); convex finite type in $\mathbb{C}^n$ (Di Biase and Fischer [6]). The task is to give a precise (possibly intrinsic) description of the sharp approach, together with a proof of its sharpness, as well as a local Fatou theorem, coupled with the study of the area function, the maximal function along the sharp approach, and the $L^p$ estimates relating these operators to each other, as well as a Calderón-Stein theorem, and so forth.

In the few cases that are sufficiently understood, a family of balls in the boundary (having certain covering and doubling properties) plays an important role in the theory; see Hörmander [15], Nagel et al [23], Stein [26]. However, in general, this structure seems to be missing; see Chirka[3] (whose results appear to have a conditional nature, i.e. conditional upon the occurrence of certain covering and doubling properties of certain boundary balls, that are rather difficult to verify).

In the few cases that are sufficiently understood, two features have been observed. The first one is that the sharp approach has a shape whose section, taken along a complex tangential direction, depends on the direction itself; [18]. For example, if $\mathbb{D}$ is the unit ball in $\mathbb{C}^n$, the shape of the sharp approach at a boundary point $w$ can be described as the locus in the domain of the following inequality:

$$\frac{\operatorname{dist}(z, \partial\mathbb{D})}{\operatorname{dist}(z, w + T^c_w(\partial\mathbb{D}))} \geq C > 0$$

where $T^c_w(\partial\mathbb{D})$ is the complex tangent space at $w$. The second feature is that the shape of the approach does change near weakly pseudoconvex points and yields sharper estimates for the associated maximal operator; see Nagel et al [22], [23] for the case $n = 2$ and [6] for convex finite type domains in $\mathbb{C}^n$.

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# Continuity of weakly monotone Sobolev functions of variable exponent

**Toshihide Futamura and Yoshihiro Mizuta**

**Abstract.**

Our aim in this paper is to deal with continuity properties for weakly monotone Sobolev functions of variable exponent.

## §1. Introduction

This paper deals with continuity properties of weakly monotone Sobolev functions. We begin with the definition of weakly monotone functions. Let $D$ be an open set in the $n$-dimensional Euclidean space $\mathbf{R}^n$ ($n \geq 2$). A function $u$ in the Sobolev space $W^{1,q}_{loc}(D)$ is said to be weakly monotone in $D$ (in the sense of Manfredi [12]), if for every relatively compact subdomain $G$ of $D$ and for every pair of constants $k \leq K$ such that

$$(k-u)^+ \qquad \text{and} \qquad (u-K)^+ \in W^{1,q}_0(G),$$

we have

$$k \leq u(x) \leq K \qquad \text{for a.e. } x \in G,$$

where $v^+(x) = \max\{v(x), 0\}$. If a weakly monotone Sobolev function is continuous, then it is monotone in the sense of Lebesgue [11]. For monotone functions, see Koskela-Manfredi-Villamor [9], Manfredi-Villamor [13, 14], the second author [17], Villamor-Li [20] and Vuorinen [21, 22].

Following Kováčik and Rákosník [10], we consider a positive continuous function $p(\cdot) : D \to (1, \infty)$ and the Sobolev space $W^{1,p(\cdot)}(D)$ of

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all functions $u$ whose first (weak) derivatives belong to $L^{p(\cdot)}(D)$. In this paper we consider the function $p(\cdot)$ satisfying

$$|p(x) - p(y)| \leq \frac{a \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \frac{b}{\log(1/|x-y|)}$$

whenever $|x - y| < 1/2$, for $a \geq 0$ and $b \geq 0$.

Our first aim is to discuss the continuity for weakly monotone functions $u$ in the Sobolev space $W^{1,p(\cdot)}(D)$. For the properties of Sobolev spaces of variable exponent, we refer the reader to the papers by Diening [2], Edmunds-Rákosník [3], Kováčik-Rákosník [10] and Růžička [19].

We know that if $p(x) \geq n$ for all $x \in D$, then all weakly monotone functions in $W^{1,p(\cdot)}(D)$ are continuous in $D$ (see Manfredi [12] and Manfredi-Villamor [13]). We show that $u$ is continuous at $x_0 \in D$ when $p(\cdot)$ is of the form

$$p(x) = n - \frac{a \log(\log(1/|x-x_0|))}{\log(1/|x-x_0|)} \qquad (p(x_0) = n)$$

for $x \in B(x_0, r_0)$, where $0 < r_0 < 1/2$ and $a \leq 1$.

Our second aim is to prove the existence of boundary limits of weakly monotone Sobolev functions on the unit ball $B$, when $p(\cdot)$ satisfies the inequality

$$\left| p(x) - \left\{ n + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \leq \frac{b}{\log(e/\rho(x))}$$

for $a \geq 0$ and $b \geq 0$, where $\rho(x) = 1 - |x|$ denotes the distance of $x$ from the boundary $\partial B$. Continuity of Sobolev functions has been obtained by Harjulehto-Hästö [7] and the authors [4]. Of course, our results extend the non-variable case studied in [17].

## §2. Weakly monotone Sobolev functions

Throughout this paper, let $C$ denote various constants independent of the variables in question.

We use the notation $B(x, r)$ to denote the open ball centered at $x$ of radius $r$. If $u$ is a weakly monotone Sobolev function on $D$ and $q > n-1$, then

$$(1) \qquad |u(x) - u(x')|^q \leq C r^{q-n} \int_{A(y,2r)} |\nabla u(z)|^q dz$$

for almost every $x, x' \in B(y, r)$, whenever $B(y, 2r) \subset D$ (see [12, Theorem 1]) and $A(y, 2r) = B(y, 2r) \setminus B(y, r)$. If we define $u^*(x)$ by

$$u^*(x) = \limsup_{r \to 0} \frac{1}{|B(x, r)|} \int_{B(x,r)} u(y) dy,$$

then we see that $u^*$ satisfies (1) for all $x, x' \in B(y, r)$. Note here that $u^*$ is a quasicontinuous representative of $u$ and it is locally bounded on $D$. Hereafter, we identify $u$ with $u^*$.

EXAMPLE 2.1. Let $1 < q < \infty$ and $\mathcal{A} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be a mapping satisfying the following assumptions for some measurable function $\alpha$ and constant $\beta$ such that $0 < \alpha(x) \leq \beta < \infty$ for a.e. $x \in \mathbf{R}^n$:

(i) the mapping $x \mapsto \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbf{R}^n$,
(ii) the mapping $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbf{R}^n$,
(iii) $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha(x)|\xi|^q$ for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$,
(iv) $|\mathcal{A}(x, \xi)| \leq \beta |\xi|^{q-1}$ for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$.

Then a weak solution of the equation

$$- \operatorname{div} \mathcal{A}(x, \nabla u(x)) = 0 \tag{2}$$

in an open set $D$ is weakly monotone (see [9, Lemma 2.7]). In the special case $\alpha(x) \geq \alpha > 0$, according to the well-known book by Heinonen-Kilpeläinen-Martio [8], a weak solution of (2) is monotone in the sense of Lebesgue.

## §3. Continuity of weakly monotone functions

For an open set $G$ in $\mathbf{R}^n$, define the $L^{p(\cdot)}(G)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),G} = \inf \left\{ \lambda > 0 : \int_G \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy \leq 1 \right\}$$

and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions $f$ on $G$ with $\|f\|_{p(\cdot)} < \infty$. We denote by $W^{1,p(\cdot)}(G)$ the space of all functions $u \in L^{p(\cdot)}(G)$ whose first (weak) derivatives belong to $L^{p(\cdot)}(G)$. We define the conjugate exponent function $p'(\cdot)$ to satisfy $1/p(x) + 1/p'(x) = 1$.

Let $B(x, r)$ be the open ball centered at $x$ and radius $r > 0$, and let $B = B(0, 1)$. Consider a positive continuous function $p(\cdot)$ on $[0, 1]$ such that $\inf_{r \in [0,1]} p(r) > 1$ and

$$\left| p(r) - \left\{ n - \frac{a \log(e + \log(1/r))}{\log(e/r)} \right\} \right| \leq \frac{b}{\log(e/r)} \qquad (p(0) = n)$$

for $a \geq 0$ and $b \geq 0$.

Our aim in this section is to prove that if $a \leq 1$, then functions in $W^{1,p(\cdot)}(B)$ are continuous at the origin, in spite of the fact that $p_-(B) = \inf_{x\in B} p(x) < n$. For this purpose, we prepare the following result.

LEMMA 3.1. *Let $p(x) = p(|x|)$ for $x \in B$. Let $u$ be a weakly monotone Sobolev function in $W^{1,p(\cdot)}(B)$. If $a < 1$, then*

$$|u(x) - u(0)|^n \leq C(\log(1/r))^{a-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy,$$

*and if $a = 1$, then*

$$|u(x) - u(0)|^n \leq C(\log(\log(1/r)))^{-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy$$

*whenever $|x| < r < 1/4$, where $R = \sqrt{r}$ when $a < 1$ and $R = e^{-\sqrt{\log(1/r)}}$ when $a = 1$.*

PROOF. Let $u$ be a weakly monotone Sobolev function in $W^{1,p(\cdot)}(B)$. Set $p_1(r) = p(r)/q$, where $n - 1 < q < n$. Then, as in (1), we apply Sobolev's theorem on the sphere $S(0, r)$ to establish

$$|u(x) - u(0)|^q \leq Cr^{q-(n-1)} \int_{S(0,r)} |\nabla u(y)|^q dS(y)$$

for $|x| < r$. By Hölder's inequality we have

$$\begin{aligned} |u(x) - u(0)|^q &\leq Cr^{q-(n-1)} \left( \int_{S(0,r)} dS(y) \right)^{1/p_1'(r)} \\ &\quad \times \left( \int_{S(0,r)} |\nabla u(y)|^{qp_1(r)} dS(y) \right)^{1/p_1(r)} \\ &\leq Cr^{q-(n-1)/p_1(r)} \left( \int_{S(0,r)} |\nabla u(y)|^{p(r)} dS(y) \right)^{1/p_1(r)}, \end{aligned}$$

which yields

$$|u(x) - u(0)|^{p(r)} \leq Cr(\log(1/r))^a \int_{S(0,r)} |\nabla u(y)|^{p(y)} dS(y)$$

for $|x| < r$. Since $u$ is bounded on $B(0, 1/2)$, we see that

$$|u(x) - u(0)|^n \leq Cr(\log(1/r))^a \int_{S(0,r)} |\nabla u(y)|^{p(y)} dS(y).$$

Hence, by dividing both sides by $r(\log(1/r))^a$ and integrating them on the interval $(r, R)$, we obtain

$$|u(x) - u(0)|^n \le C(\log(1/r))^{a-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy \quad \text{when } a < 1$$

and

$$|u(x) - u(0)|^n \le C(\log(\log(1/r)))^{-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy \quad \text{when } a = 1$$

whenever $|x| < r < 1/4$. □

Lemma 3.1 yields the following result.

THEOREM 3.2. *Let $u$ be a weakly monotone Sobolev function in $W^{1,p(\cdot)}(B)$. If $a < 1$, then $u$ is continuous at the origin and it satisfies*

$$\lim_{x \to 0} (\log(1/|x|))^{(1-a)/n} |u(x) - u(0)| = 0;$$

*if $a = 1$, then*

$$\lim_{x \to 0} (\log(\log(1/|x|)))^{1/n} |u(x) - u(0)| = 0.$$

REMARK 3.3. Consider the function

$$u(x) = \frac{x_n}{|x|}$$

for $x = (x_1, ..., x_n)$. If we define $u(0) = 0$, then $u$ is a weakly monotone quasicontinuous representative in $\mathbf{R}^n$. Note that $u$ is not continuous at 0 and if $a > 1$, then

$$\int_B |\nabla u(x)|^{p(x)} dx < \infty;$$

if $a \le 1$, then

$$\int_B |\nabla u(x)|^{p(x)} dx = \infty.$$

This shows that continuity result in Theorem 3.2 is good as to the size of $a$.

REMARK 3.4. Let $\varphi$ be a nonnegative continuous function on the interval $[0, r_0]$ such that

(i) $\varphi(0) = 0$ ;
(ii) $\varphi'(t) \ge 0$ for $0 < t < r_0$ ;

(iii) $\varphi''(t) \le 0$ for $0 < t < r_0$ .

Then note that

$$\varphi(s+t) \le \varphi(s) + \varphi(t) \tag{3}$$

for $s, t \ge 0$ and $s + t \le r_0$. Consider

$$\varphi(r) = \frac{\log(\log(1/r))}{\log(1/r)}, \quad \frac{1}{\log(1/r)}$$

for $0 < r \le r_0$; set $\varphi(r) = \varphi(r_0)$ for $r > r_0$. Then we can find $r_0 > 0$ such that $\varphi$ satisfies (i) - (iii) on $[0, r_0]$, and hence (3) holds for all $s \ge 0$ and $t \ge 0$. Hence if we set

$$p(r) = n + \frac{a \log(e + \log(1/r))}{\log(e/r)} + \frac{b}{\log(e/r)},$$

then we can find $c > 0$ and $r_0 > 0$ such that

$$|p(s) - p(t)| \le \frac{|a| \log(\log(1/|s-t|))}{\log(1/|s-t|)} + \frac{c}{\log(1/|s-t|)}$$

whenever $|s - t| < r_0$.

## §4. 0-Hölder continuity of continuous Sobolev functions

Consider a positive continuous function $p(\cdot)$ on the unit ball $B$ such that $p_-(B) = \inf_{x \in B} p(x) > 1$ and

$$\left| p(x) - \left\{ p_0 + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))}$$

for all $x \in B$, where $1 < p_0 < \infty$ and $\rho(x) = 1 - |x|$ denotes the distance of $x$ from the boundary $\partial B$. Then note that

$$\begin{aligned} p'(x) - p_0' &= -\frac{p(x) - p_0}{(p(x) - 1)(p_0 - 1)} \\ &= -\frac{p(x) - p_0}{(p_0 - 1)^2} + \frac{(p(x) - p_0)^2}{(p(x) - 1)(p_0 - 1)^2}, \end{aligned}$$

where $p_0' = p_0/(p_0 - 1)$. Hence we have the following result.

LEMMA 4.1. *There exist positive constants $r_0$ and $C$ such that*

$$|p'(x) - \{p_0' - \omega(\rho(x))\}| \le C/\log(1/\rho(x))$$

*for* $x \in B$, *where* $\omega(t) = (a/(p_0 - 1)^2) \log(\log(1/t))/ \log(1/t)$ *for* $0 < r \le r_0 < 1/e$; *set* $\omega(t) = \omega(r_0)$ *for* $r > r_0$.

We see from Sobolev's theorem that all functions $u \in W^{1,p(\cdot)}(B)$ are continuous in $B$ when $p(x) > n$ in $B$. In what follows we discuss the 0-Hölder continuity of $u$. Before doing so, we need the following result.

LEMMA 4.2. *Let* $p_0 = n$ *and let* $u$ *be a continuous Sobolev function in* $W^{1,p(\cdot)}(B)$ *such that* $\| |\nabla u| \|_{p(\cdot)} \le 1$. *If* $a > n - 1$, *then*

$$\int_{B\cap B(x,r)} |x-y|^{1-n} |\nabla u(y)| \le C(\log(1/r))^{-A},$$

*where* $A = (a - n + 1)/n$.

PROOF. Let $f(y) = |\nabla u(y)|$ for $y \in B$ and $f = 0$ outside $B$. For $0 < \mu < 1$, we have

$$\begin{aligned}
&\int_{B(x,r)} |x-y|^{1-n} f(y)dy \\
\le\ & \mu \left\{ \int_{B(x,r)\cap B} (|x-y|^{1-n}/\mu)^{p'(y)} dy + \int_{B(x,r)} f(y)^{p(y)} dy \right\} \\
\le\ & \mu \left\{ \mu^{-n/(n-1)} \int_{B(x,r)\cap B} |x-y|^{(1-n)p'(y)} dy + 1 \right\}.
\end{aligned}$$

Applying polar coordinates, we have

$$\begin{aligned}
&\int_{B(x,r)\cap B} |x-y|^{(1-n)p'(y)} dy \\
\le\ & C \int_{\{t:|t-\rho(x)|<r\}} |\rho(x) - t|^{(1-n)(n'-\omega_0(t))+n-1} dt \\
=\ & C \int_{\{t:|t-\rho(x)|<r\}} |\rho(x) - t|^{(n-1)\omega_0(t)-1} dt,
\end{aligned}$$

where $\omega_0(t) = \omega(t) - C/\log(1/t)$. If $r \le \rho(x)/2$ and $|\rho(x) - t| < \rho(x)/2$, then

$$\omega_0(t) \ge \omega(r) - C/\log(1/r),$$

so that

$$\int_{\{t:|t-\rho(x)|<r\}} |\rho(x) - t|^{(n-1)\omega_0(t)-1} dt \le C(\log(1/r))^{1-a/(n-1)}.$$

If $r > \rho(x)/2$, then $|t| < 3|\rho(x) - t|$ when $|\rho(x) - t| \geq \rho(x)/2$. Hence, in this case, we obtain

$$\begin{aligned} & \int_{\{t:|t-\rho(x)|<r\}} |\rho(x) - t|^{(n-1)\omega_0(t)-1} dt \\ \leq & \int_{\{t:|t-\rho(x)|<\rho(x)/2\}} |\rho(x) - t|^{(n-1)\omega_0(t)-1} dt \\ & + C \int_{\{t:|t|<3r\}} |t|^{(n-1)\omega_0(t)-1} dt \\ \leq & \ C(\log(1/r))^{1-a/(n-1)}, \end{aligned}$$

so that

$$\int_{B(x,r)\cap B} |x-y|^{(1-n)p'(y)} dy \leq C(\log(1/r))^{1-a/(n-1)}.$$

Consequently it follows that

$$\int_{B(x,r)} |x-y|^{1-n} f(y) dy \leq \mu \left( C\mu^{-n/(n-1)} (\log(1/r))^{1-a/(n-1)} + 1 \right).$$

Now, letting $\mu^{-n/(n-1)}(\log(1/r))^{1-a/(n-1)} = 1$, we establish

$$\int_{B(x,r)} |x-y|^{1-n} f(y) dy \leq C(\log(1/r))^{(n-1-a)/n},$$

as required. $\square$

Now we are ready to show the 0-Hölder continuity of Sobolev functions in $W^{1,p(\cdot)}(B)$ .

THEOREM 4.3. *Let $p_0 = n$ and $u$ be a continuous Sobolev function in $W^{1,p(\cdot)}(B)$ such that $\||\nabla u|\|_{p(\cdot)} \leq 1$. If $a > n-1$, then*

$$|u(x) - u(y)| \leq C(\log(1/|x-y|))^{-A}$$

*whenever $x$, $y \in B$ and $|x-y| < 1/2$.*

PROOF. Let $x$, $y \in B$ and $r = |x-y| \leq \rho(x)$. Then we see from Lemma 4.2 that

$$|u(x) - u(y)| \leq C \int_{B(x,r)} |x-z|^{1-n} |\nabla u(z)| dz \leq C(\log(1/r))^{-A}.$$

If $r = |x-y| < 1/2$, $\rho(x) < r$ and $\rho(y) < r$, then we take $x_r = (1-r)x/|x|$ and $y_r = (1-r)y/|y|$ to establish

$$\begin{aligned} |u(x) - u(y)| &\le |u(x) - u(x_r)| + |u(x_r) - u(y_r)| + |u(y_r) - u(y)| \\ &\le C(\log(1/r))^{-A}, \end{aligned}$$

which proves the assertion. □

REMARK 4.4. Let $p(\cdot)$ be as above, and consider the function

$$u(x) = [\log(e + \log(1/|x-\xi|))]^{\delta},$$

where $\xi \in \partial B$ and $0 < \delta < (n-1)/n$. We see readily that $u(\xi) = \infty$ and it is monotone in $B$. Further, if $a \le n-1$, then

$$\int_B |\nabla u(x)|^{p(x)} dx < \infty,$$

so that Theorem 4.3 does not hold for $a \le n-1$.

## §5. Tangential boundary limits of weakly monotone Sobolev functions

Let $G$ be a bounded open set in $\mathbf{R}^n$. Consider a positive continuous function $p(\cdot)$ on $\mathbf{R}^n$ satisfying

(p1) $p_-(G) = \inf_G p(x) > 1$ and $p_+(G) = \sup_G p(x) < \infty$;

(p2) $|p(x) - p(y)| \le \dfrac{a \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \dfrac{b}{\log(1/|x-y|)}$
whenever $|x-y| < 1/e$, where $a \ge 0$ and $b \ge 0$.

For $E \subset G$, we define the relative $p(\cdot)$-capacity by

$$C_{p(\cdot)}(E; G) = \inf \int_G f(y)^{p(y)} dy,$$

where the infimum is taken over all nonnegative functions $f \in L^{p(\cdot)}(G)$ such that

$$\int_G |x-y|^{1-n} f(y) dy \ge 1 \qquad \text{for every } x \in E.$$

From now on we collect fundamental properties for our capacity (see Meyers [15], Adams-Hedberg [1] and the authors [6]).

LEMMA 5.1. *For $E \subset G$, $C_{p(\cdot)}(E;G) = 0$ if and only if there exists a nonnegative function $f \in L^{p(\cdot)}(G)$ such that*

$$\int_G |x-y|^{1-n} f(y)dy = \infty \qquad \text{for every } x \in E.$$

For $0 < r < 1/2$, set

$$h(r;x) = \begin{cases} r^{n-p(x)}(\log(1/r))^a & \text{if } p(x) < n, \\ (\log(1/r))^{a-(n-1)} & \text{if } p(x) = n \text{ and } a < n-1, \\ (\log(\log(1/r)))^{-a} & \text{if } p(x) = n \text{ and } a = n-1, \\ 1 & \text{if } p(x) > n \text{ or } p(x) = n,\ a > n-1 \end{cases}$$

LEMMA 5.2. *Suppose $p(x_0) \le n$ and $a \le n-1$. If $B(x_0,r) \subset G$ and $0 < r < 1/2$, then*

$$C_{p(\cdot)}(B(x_0,r);G) \le Ch(r;x_0).$$

LEMMA 5.3. *If $f$ is a nonnegative measurable function on $G$ with $\|f\|_{p(\cdot)} < \infty$, then*

$$\lim_{r\to 0+} h(r;x)^{-1} \int_{B(x,r)} f(y)^{p(y)}dy = 0$$

*holds for all $x$ except in a set $E \subset G$ with $C_{p(\cdot)}(E;G) = 0$.*

Let $p(\cdot)$ be as in Section 4; that is, we assume that $p(x) > n$ and

$$\left| p(x) - \left\{ n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))} \tag{4}$$

for $x \in B$, where $a \ge 0$ and $b > 0$. Then $p_1(x) \le p(x) \le p_2(x)$ for $x \in B$, where

$$\begin{aligned} p_1(x) &= n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} - \frac{b}{\log(e/\rho(x))}, \\ p_2(x) &= n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} + \frac{b}{\log(e/\rho(x))}. \end{aligned}$$

For simplicity, set

$$p(x) = p_1(x) = p_2(x) = n$$

outside $B$. Then we can find $b' > b$ such that for $i = 1, 2$

$$\begin{aligned} |p_i(x) - p_i(y)| &\le \frac{a \log(e + \log(1/|x-y|))}{\log(e/|x-y|)} + \frac{b}{\log(e/|x-y|)} \\ &\le \frac{a \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \frac{b'}{\log(1/|x-y|)} \end{aligned}$$

whenever $|x-y|$ is small enough, say $|x-y| < r_0 < 1/e$.

Since $G$ has finite measure, we find a constant $K > 0$ such that

$$C_{p(\cdot)}(E; G) \le K C_{p_2(\cdot)}(E; G) \tag{5}$$

whenever $E \subset G$. Hence, in view of Lemma 5.2, we obtain

$$C_{p(\cdot)}(B(x_0, r); 2B) \le C h(r; x_0) \tag{6}$$

for $x_0 \in \partial B$, where $2B = B(0, 2)$.

COROLLARY 5.4. *If $f$ is a nonnegative measurable function on $2B$ with $\|f\|_{p(\cdot)} < \infty$, then*

$$\lim_{r \to 0+} h(r; x)^{-1} \int_{B(x,r)} f(y)^{p(y)} dy = 0$$

*holds for all $x \in \partial B$ except in a set $E \subset \partial B$ with $C_{p(\cdot)}(E; 2B) = 0$.*

If $u$ is a weakly monotone function in $W^{1,p(\cdot)}(B)$, then, since $p(x) > n$ for $x \in B$ by our assumption, we see that $u$ is continuous in $B$ and hence monotone in $B$ in the sense of Lebesgue. We now show the existence of tangential boundary limits of monotone Sobolev functions $u$ in $B$ when $a \le n - 1$.

For $\xi \in \partial B$, $\gamma \ge 1$ and $c > 0$, set

$$T_\gamma(\xi, c) = \{x \in B : |x - \xi|^\gamma < c\rho(x)\}.$$

THEOREM 5.5. *Let $p(\cdot)$ be a positive continuous function on $2B$ such that $p(x) \ge n$ for $x \in 2B$ and*

$$\left| p(x) - \left\{ n + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))}$$

*for $x \in B$, where $a \ge 0$ and $b > 0$. If $u$ is a monotone function in $W^{1,p(\cdot)}(B)$ (in the sense of Lebesgue), then there exists a set $E \subset \partial B$ such that*

(i) $C_{p(\cdot)}(E; 2B) = 0$ ;

(ii) *if* $\xi \in \partial B \setminus E$, *then* $u(x)$ *has a finite limit as* $x \to \xi$ *along the sets* $T_\gamma(\xi, c)$.

If $a > n-1$, then the above function $u$ has a continuous extension on $\overline{B} = B \cup \partial B$ in view of Theorem 4.3, and hence the exceptional set $E$ can be taken as the empty set.

To prove Theorem 5.5, we may assume that

$$p(x) = n + \frac{a \log(e + \log(e/\rho(x)))}{\log(e/\rho(x))} - \frac{b}{\log(e/\rho(x))}$$

for $x \in B$.

We need the following two results. The first one follows from inequality (1) (see e.g. [9] and [5]).

Lemma 5.6. *Let* $u$ *be a monotone Sobolev function in* $W^{1,p(\cdot)}(B)$. *If* $\xi \in \partial B$, $x \in B$ *and* $n-1 < q < n$, *then*

$$|u(x) - u(\tilde{x})|^q \le C(\log(2r/\rho(x)))^{q-1} \int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy,$$

*where* $\tilde{x} = (1-r)\xi$, $r = |\xi - x|$ *and* $E(x) = \cup_{y \in \overline{x\tilde{x}}} B(y, \rho(y)/2)$ *with* $\overline{x\tilde{x}} = \{tx + (1-t)\tilde{x} : 0 < t < 1\}$.

Lemma 5.7. *Let* $u$ *be a monotone Sobolev function in* $W^{1,p(\cdot)}(B)$. *Let* $\xi \in \partial B$ *and* $a \ge 0$. *Suppose*

$$(\log(1/r))^{n-1-a} \int_{B \cap B(\xi, 2r)} |\nabla u(y)|^{p(y)} dy \le 1.$$

*If* $x \in T_\gamma(\xi, c)$, $\tilde{x} = (1-r)\xi$ *and* $r = |\xi - x|$, *then*

$$|u(x) - u(\tilde{x})|^n \le C(\log(1/r))^{n-1-a} \int_{B \cap B(\xi, 2r)} |\nabla u(y)|^{p(y)} dy.$$

Proof. First note that

$$\rho(y) \ge C(\rho(x) + |x - y|) \qquad \text{for } y \in E(x).$$

Take $q$ such that $n-1 < q < n$; when $a > 0$, assume further that $a > (n-q)/q$. Set $p_1(x) = p(x)/q$. Then we have for $\mu > 0$

$$\begin{aligned} & \int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy \\ \le \quad & \mu \left\{ \int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy + \int_{E(x)} |\nabla u(y)|^{qp_1(y)} dy \right\} \\ = \quad & \mu \left\{ \int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy + F \right\}, \end{aligned}$$

where $F = \int_{E(x)} |\nabla u(y)|^{p(y)} dy$. Note from Lemma 4.1 that

$$|p_1'(y) - \{n/(n-q) - \omega(\rho(y))\}| \le C/\log(1/\rho(y))$$

for $y \in E(x)$, where $\omega(t) = (aq^2/(n-q)^2)\log(\log(1/t))/\log(1/t)$. Hence

$$n/(n-q) - \omega_1(\rho(y)) \le p_1'(y) \le n/(n-q) - \omega_2(\rho(y)),$$

where $\omega_1(t) = \omega(t) + C/\log(1/t)$ and $\omega_2(t) = \omega(t) - C/\log(1/t)$. Suppose

$$(\log(1/r))^{-1+aq/(n-q)} F > 1.$$

Since $p_1'(y) \le n/(n-q)$, we have for $0 < \mu < 1$,

$$\begin{aligned} & \int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy \\ \le \quad & C\mu^{-n/(n-q)} \int_{E(x)} (\rho(x) + |x-y|)^{(q-n)(n/(n-q) - \omega_2(\rho(y)))} dy \\ \le \quad & C\mu^{-n/(n-q)} \int_0^{2r} (\rho(x)+t)^{-n} (\log(1/(\rho(x)+t)))^{-aq/(n-q)} t^{n-1} dt \\ \le \quad & C\mu^{-n/(n-q)} (\log(1/r))^{1-aq/(n-q)} \end{aligned}$$

whenever $x \in T_\gamma(\xi, c)$. Considering

$$\mu^{-n/(n-q)} (\log(1/r))^{1-aq/(n-q)} = F,$$

we obtain

$$\begin{aligned} & \int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy \\ \le \quad & C \left\{ (\log(1/r))^{-1+aq/(n-q)} F \right\}^{-(n-q)/n} F \\ = \quad & C \left\{ (\log(1/r))^{(n-q)/q-a} \int_{E(x)} |\nabla u(y)|^{p(y)} dy \right\}^{q/n}. \end{aligned}$$

Consequently it follows from Lemma 5.6 that

$$|u(x) - u(\tilde{x})|^n \le C(\log(1/r))^{n-1-a} \int_{B\cap B(\xi,2r)} |\nabla u(y)|^{p(y)} dy$$

whenever $x \in T_\gamma(\xi, c)$.

Next consider the case when $(\log(1/r))^{-1+aq/(n-q)} F \le 1$. Set $p^+ = \sup_{B\cap B(\xi,2r)} p(y)$ and and $p_1^+ = \sup_{B\cap B(\xi,2r)} p_1(y) = p^+/q$. For $\mu > 1$, we apply the above considerations to obtain

$$\begin{aligned} &\int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy \\ \le\ & C\mu^{-(p_1^+)'} \int_{E(x)} (\rho(x) + |x-y|)^{(q-n)(n/(n-q)-\omega_2(\rho(y)))} dy \\ \le\ & C\mu^{-(p_1^+)'} (\log(1/r))^{1-aq/(n-q)}. \end{aligned}$$

If we take $\mu$ satisfying $\mu^{-(p_1^+)'}(\log(1/r))^{1-aq/(n-q)} = F$, then we have

$$\begin{aligned} &\int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy \\ \le\ & C\Big\{(\log(1/r))^{(n-q)/q-a} \int_{E(x)} |\nabla u(y)|^{p(y)} dy\Big\}^{1/p_1^+}. \end{aligned}$$

Since $(\log(1/r))^{\omega(r)}$ is bounded above for small $r > 0$, Lemma 5.6 yields

$$|u(x) - u(\tilde{x})|^{p^+} \le C(\log(1/r))^{n-1-a} \int_{B\cap B(\xi,2r)} |\nabla u(y)|^{p(y)} dy$$

whenever $x \in T_\gamma(\xi, c)$, which proves the required assertion. $\square$

Proof of Theorem 5.5. Consider $E = E_1 \cup E_2$, where

$$E_1 = \{\xi \in \partial B : \int_B |\xi - y|^{1-n} |\nabla u(y)| dy = \infty\}$$

and

$$E_2 = \{\xi \in \partial B : \limsup_{r\to 0+} (\log(1/r))^{n-1-a} \int_{B(\xi,r)} |\nabla u(y)|^{p(y)} dy > 0\}.$$

We see from Lemma 5.1 and Corollary 5.4 that $E = E_1 \cup E_2$ is of $C_{p(\cdot)}$-capacity zero. If $\xi \notin E_1$, then we can find a line $L$ along which $u$ has a finite limit $\ell$. In view of inequality (1), we see that $u$ has a radial limit $\ell$ at $\xi$, that is, $u(r\xi)$ tends to $\ell$ as $r \to 1-0$. Now we insist from Lemma

5.7 that if $\xi \in \partial B \setminus E$, then $u(x)$ tends to $\ell$ as $x$ tends to $\xi$ along the sets $T_\gamma(\xi, c)$. $\square$

REMARK 5.8. If $a > n - 1$, then we do not need the monotonicity in Theorem 5.5, because of Theorem 4.3.

Finally we show the nontangential limit result for weakly monotone Sobolev functions. Recall that a quasicontinuous representative is locally bounded.

THEOREM 5.9. *Let $p(\cdot)$ be a positive continuous function on $B$ such that*

$$\left| p(x) - \left\{ p_0 + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))},$$

*where $-\infty < a < \infty$, $b \ge 0$ and $n - 1 < p_0 \le n$. If $u$ is a weakly monotone function in $W^{1,p(\cdot)}(B)$ (in the sense of Manfredi), then there exists a set $E \subset \partial B$ such that*

(i) *$C_{p(\cdot)}(E; 2B) = 0$ ;*
(ii) *if $\xi \in \partial B \setminus E$, then $u(x)$ has a finite limit as $x \to \xi$ along the sets $T_1(\xi, c)$.*

To prove this, we need the following lemma instead of Lemma 5.7, which can be proved by use of (1) with $q = p_- = \inf_{z \in B(x,\rho(x)/2)} p(z)$.

LEMMA 5.10. *Let $p$ and $u$ be as in Theorem 5.9. If $y \in B(x, r)$ with $r = \rho(x)/4$, then*

$$|u(x) - u(y)|^{p_-} \le C r^{p_0 - n} (\log(1/r))^{-a} \left( r^n + \int_{B(x,2r)} |\nabla u(z)|^{p(z)} dz \right).$$

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# Martin kernels of general domains

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**Abstract.**

This note consists of our recent researches on Martin kernels of general domains. In particular, minimal Martin boundary points of a John domain, the boundary behavior of quotients of Martin kernels, and comparison estimates for the Green function and the Martin kernel are studied.

## §1. Introduction

This note is a summary of our recent researches on the Martin boundary and the Martin kernel of general domains. To begin with, let us recall the notion of the Martin boundary and the Martin kernel. Let $\Omega$ be a Greenian domain in $\mathbb{R}^n$, where $n \geq 2$, possessing the Green function $G_\Omega$ for the Laplace operator. Let $x_0 \in \Omega$ be fixed, and let $\{y_j\}$ be a sequence in $\Omega$ with no limit point in $\Omega$. If $\omega$ is an open subset of $\Omega$ such that the closure $\overline{\omega}$ is compact in $\Omega$, then there exists $j_0$ such that $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}_{j=j_0}^{\infty}$ is a uniformly bounded sequence of positive harmonic functions in $\omega$. Therefore there is a subsequence of $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}_j$ converging to a positive harmonic function in $\Omega$. The collection of all such limit functions in $\Omega$ gives an ideal boundary of $\Omega$, referred to as the *Martin boundary* of $\Omega$ and denoted by $\Delta(\Omega)$. For $\zeta \in \Delta(\Omega)$, we write $K_\Omega(\cdot, \zeta)$ for the positive harmonic function in $\Omega$ corresponding to $\zeta$, and call $K_\Omega$ the *Martin kernel*. We say that a positive harmonic function $h$ is *minimal* if every positive harmonic function less than or equal to $h$ coincides with a constant multiple of $h$. The collection of all minimal elements in $\Delta(\Omega)$ is called the *minimal Martin boundary* of $\Omega$, and is denoted by $\Delta_1(\Omega)$. The importance of the Martin boundary appears in the representation theorem for positive harmonic

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functions $h$ in general domains: there exists a measure $\mu_h$ on $\Delta(\Omega)$ such that $\mu_h(\Delta(\Omega) \setminus \Delta_1(\Omega)) = 0$ and

$$h(x) = \int_{\Delta(\Omega)} K_\Omega(x,\zeta) d\mu_h(\zeta) \quad \text{for } x \in \Omega.$$

So, for general domains, it is valuable to investigate the Martin boundary and the behavior of the Martin kernel.

This note is organized as follows. In Section 2, we state the results, obtained in [2], about the number of minimal Martin boundary points of John domains. In Sections 3 and 4, we give the results, studied in [17] and [18], about the boundary behavior of the Martin kernels and comparison estimates for the Green function and the Martin kernel.

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## §2. Minimal Martin boundary points of John domains

From the viewpoint of the representation theorem, the most interesting problem is to investigate that for what kind of domains the Martin boundary and the minimal Martin boundary are homeomorphic to the Euclidean boundary. For instance, see [19] for Lipschitz domains, [20] for NTA domains, and [1] for uniform domains. However, in general, the Martin boundary need not to be homeomorphic to the Euclidean boundary. There may be even infinitely many minimal Martin boundary points at a Euclidean boundary point (cf. [22, Example 3]). Here, a Martin boundary point at $y \in \partial\Omega$ (the Euclidean boundary of $\Omega$) is a positive harmonic function in $\Omega$ which can be obtained as the limit of $\{G_\Omega(\cdot, y_j)/G_\Omega(x_0, y_j)\}_j$ for some sequence $\{y_j\}$ in $\Omega$ converging to $y$. It is also interesting to investigate that for what kind of domains the number of minimal Martin boundary points at every Euclidean boundary point is finite. For example, see [7] for Denjoy domains, [5, 6] and [13] for Lipschitz-Denjoy domains, [15] for sectorial domains, and [21] for quasi-sectorial domains. One of the main interests of these papers was to give a criterion for the number of minimal Martin boundary points at a fixed Euclidean boundary point. As a generalization of some parts of them, we study minimal Martin boundary points of John domains. A domain $\Omega$ is said to be a general John domain with John constant $c_J > 0$ and John center $K_0$, a compact subset of $\Omega$, if each point $x$ in $\Omega$ can be connected to some point in $K_0$ by a rectifiable curve $\gamma$ in $\Omega$ such that

$$\operatorname{dist}(z, \partial\Omega) \geq c_J \ell(\gamma(x, z)) \quad \text{for all } z \in \gamma,$$

where $\operatorname{dist}(z, \partial\Omega)$ stands for the distance from $z$ to $\partial\Omega$ and $\ell(\gamma(x, z))$ denotes the length of the subarc $\gamma(x, z)$ of $\gamma$ from $x$ to $z$. Note that every general John domain is bounded. We can obtain the following.

**Theorem 2.1** ([2, Theorem 1.1]). *Let $\Omega$ be a general John domain with John constant $c_J$, and let $y \in \partial\Omega$. Then the following statements hold:*

(i) *The number of minimal Martin boundary points at $y$ is bounded by a constant depending only on $c_J$ and $n$.*

(ii) *If $c_J > \sqrt{3}/2$, then the number of minimal Martin boundary points at $y$ is at most two.*

*Remark* 2.2. The bound $c_J > \sqrt{3}/2$ in Theorem 2.1(ii) is sharp. See [2, Remark 1.1].

For a class of general John domains represented as the union of open convex sets, we give a sufficient condition for the Martin boundary to be homeomorphic to the Euclidean boundary. For $0 < \theta < \pi$, we write $\Gamma_\theta(z, w) = \{x \in \mathbb{R}^n : \angle xzw < \theta\}$ for the open circular cone with vertex at $z$, axis $[z, w]$ and aperture $\theta$. Let $A_0 \geq 1$ and $\rho_0 > 0$. We consider a bounded domain $\Omega$ with the following properties:

(I) $\Omega$ is the union of a family of open convex sets $\{C_\lambda\}_{\lambda \in \Lambda}$ such that

$$B(z_\lambda, \rho_0) \subset C_\lambda \subset B(z_\lambda, A_0\rho_0);$$

(II) for each $y \in \partial\Omega$, there are positive constants $\theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 \leq \rho_0 \cos\theta_1$ such that

$$\bigcup_{\substack{w \in \Omega \\ \Gamma_{\theta_1}(y,w) \cap B(y, 2\rho_1) \subset \Omega}} \Gamma_{\theta_1}(y, w) \cap B(y, 2\rho_1) \quad \text{is connected and non-empty.}$$

Obviously, a bounded domain satisfying (I) is a general John domain with John center $\overline{\{z_\lambda\}_{\lambda \in \Lambda}}$ and John constant $A_0^{-1}$.

**Theorem 2.3** ([2, Theorem 1.2]). *Let $\Omega$ be a bounded domain satisfying (I). If $y \in \partial\Omega$ satisfies (II), then there is a unique Martin boundary point at $y$ and it is minimal. Furthermore, if every Euclidean boundary point satisfies (II), then the Martin boundary of $\Omega$ is homeomorphic to the Euclidean boundary.*

*Remark* 2.4. The bounds $\theta_1 \leq \sin^{-1}(1/A_0)$ and $\rho_1 \leq \rho_0 \cos\theta_1$ are sharp. See [2, Examples 8.1 and 8.2].

Theorem 2.3 is a generalization of Ancona's result [4]. He considered a bounded domain represented as the union of open balls with the same radius. His key lemma [4, Lemme 1] relies on the reflection with respect to a hyperplane, and is applied to a ball by the Kelvin transform. This approach is not applicable to our domains. Our approach is based on a new geometrical notion, the system of local reference points. We define the quasi-hyperbolic metric on $\Omega$ by

$$k_\Omega(x,y) = \inf_\gamma \int_\gamma \frac{ds(z)}{\operatorname{dist}(z,\partial\Omega)} \quad \text{for } x,y \in \Omega,$$

where the infimum is taken over all rectifiable curves $\gamma$ in $\Omega$ connecting $x$ to $y$ and $ds$ stands for the line element on $\gamma$. Let $N \in \mathbb{N}$ and $0 < \eta < 1$. We say that $y \in \partial\Omega$ has *a system of local reference points of order $N$ with factor $\eta$* if there exist $R_y > 0$ and $A_y > 1$ with the following property: for each positive $R < R_y$ there are $N$ points $y_1 = y_1(R), \cdots, y_N = y_N(R) \in \Omega \cap \partial B(y,R)$ such that $\operatorname{dist}(y_j,\partial\Omega) \ge A_y^{-1}R$ for $j = 1,\cdots,N$ and

$$\min_{j=1,\cdots,N}\{k_{\Omega\cap B(y,\eta^{-3}R)}(x,y_j)\} \le A_y \log\frac{R}{\operatorname{dist}(x,\partial\Omega)} + A_y$$

for $x \in \Omega \cap B(y,\eta R)$. For example, if $\Omega$ is a (sectorial) domain in $\mathbb{R}^2$ whose boundary near $y \in \partial\Omega$ lies on $m$-distributed rays emanating from $y$, then $y$ has a system of local reference points of order $N = m$. For a general John domain $\Omega$ with John constant $c_J$, we can show that

- each $y \in \partial\Omega$ has a system of local reference points of order $N$ with $N \le N(c_J,n) < \infty$. Moreover, if $c_J > \sqrt{3}/2$, then we can let $N \le 2$ by choosing a suitable factor $\eta$.
- if $\Omega$ satisfies (I) and $y \in \partial\Omega$ satisfies (II), then $y$ has a system of local reference points of order 1.

These observations played essential roles in the proofs of Theorems 2.1 and 2.3. Indeed, Theorems 2.1 and 2.3 can be reunderstood as follows.

**Proposition 2.5** ([2, Proposition 2.3]). *Let $\Omega$ be a general John domain, and suppose that $y \in \partial\Omega$ has a system of local reference points of order $N$. Then the following statements hold:*

(i) *The number of minimal Martin boundary points at $y$ is bounded by a constant depending only on $N$.*
(ii) *If $N \le 2$, then there are at most $N$ minimal Martin boundary points at $y$. Moreover, if $N = 1$, then there is a unique Martin boundary point at $y$ and it is minimal.*

In Proposition 2.5(ii), the condition $N \le 2$ may be omitted, and we expect that the number of minimal Martin boundary points at $y$ is at most $N$ even for $N \ge 3$. We raise the following question.

**Problem.** *Let $\Omega$ be a general John domain and let $N \ge 3$. Suppose that $y \in \partial\Omega$ has a system of local reference points of order $N$. Is the number of minimal Martin boundary points at $y$ at most $N$?*

## §3. Boundary behavior of quotients of Martin kernels

In [9, 10], Burdzy obtained a result on the angular derivative problem of analytic functions in a Lipschitz domain. The important step was to study the boundary behavior of the Green function. We now write $\mathbf{0}$ for the origin of $\mathbb{R}^n$ to distinguish from $0 \in \mathbb{R}$, and denote $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $e = (\mathbf{0}', 1)$. Suppose that $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function such that $\phi(\mathbf{0}') = 0$, and put $\Omega_\phi = \{(x', x_n) : x_n > \phi(x')\}$. We set

(3.1)
$$I^+ = \int_{\{|x'|<1\}} \frac{\max\{\phi(x'), 0\}}{|x'|^n} dx', \quad I^- = \int_{\{|x'|<1\}} \frac{\max\{-\phi(x'), 0\}}{|x'|^n} dx'.$$

**Theorem A.** *Let $I^+$ and $I^-$ be as in* (3.1). *Then the following statements hold:*

(i) *If $I^+ < \infty$ and $I^- = \infty$, then*

$$\lim_{t \to 0+} \frac{G_{\Omega_\phi}(te, e)}{t} = \infty.$$

(ii) *If $I^+ = \infty$ and $I^- < \infty$, then*

$$\lim_{t \to 0+} \frac{G_{\Omega_\phi}(te, e)}{t} = 0.$$

(iii) *If $I^+ < \infty$ and $I^- < \infty$, then the limit of $G_{\Omega_\phi}(te, e)/t$, as $t \to 0+$, exists and*

$$0 < \lim_{t \to 0+} \frac{G_{\Omega_\phi}(te, e)}{t} < \infty.$$

Burdzy's approach was based on probabilistic methods. Analytic proofs were given by Carroll [11, 12] and Gardiner [16]. As we see from their proofs, the convergence of the integrals $I^+$ and $I^-$ are related to the minimal thinness of the differences $\Omega_\phi \setminus \mathbb{R}^n_+$ and $\mathbb{R}^n_+ \setminus \Omega_\phi$, where

$\mathbb{R}^n_+ = \{(x', x_n) : x_n > 0\}$. A subset $E$ of $\Omega$ is said to be *minimally thin* at $\xi \in \Delta_1(\Omega)$ with respect to $\Omega$ if

$$ {}^{\Omega}\widehat{R}^{E}_{K_\Omega(\cdot,\xi)}(z) < K_\Omega(z,\xi) \quad \text{for some } z \in \Omega, $$

where ${}^{\Omega}\widehat{R}^{E}_{u}$ denotes the regularized reduced function of a positive superharmonic function $u$ relative to $E$ in $\Omega$. We say that a function $f$, defined on a minimal fine neighborhood $U$ of $\xi$, has *minimal fine limit* $l$ at $\xi$ with respect to $\Omega$ if there is a subset $E$ of $\Omega$, minimally thin at $\xi$ with respect to $\Omega$, such that $f(x) \to l$ as $x \to \xi$ along $U \setminus E$, and then we write

$$ \underset{\Omega}{\text{mf}}\text{-}\lim_{x\to\xi} f(x) = l. $$

Theorem A was shown by using Naïm's characterization [23, Théorème 11] of the minimal thinness for a difference of domains in terms of the boundary behavior of the quotient of the Green functions.

We are now interested in the boundary behavior of Martin kernels. In this case, we can not apply the Naïm's characterization. Alternatively, we can characterize the minimal thinness for a difference of domains in terms of the boundary behavior of the quotient of the Martin kernels (see [17, Lemma 3.1]), and then obtain the following general result.

**Theorem 3.1** ([17, Theorem 2.1]). *Suppose that $\Omega$ and $D$ are Greenian domains such that $\Omega \cap D$ is a non-empty domain. Let $\xi \in \Delta_1(\Omega)$, where $\xi$ is in the closure of $\Omega \cap D$ in the Martin compactification of $\Omega$. Let $\zeta \in \Delta_1(D)$, where $\zeta$ is in the closure of $\Omega \cap D$ in the Martin compactification of $D$. If $\Omega \setminus D$ is minimally thin at $\xi$ with respect to $\Omega$, then $K_D(\cdot,\zeta)/K_\Omega(\cdot,\xi)$ has a finite minimal fine limit at $\xi$ with respect to $\Omega$. Furthermore, the following statements hold:*

(i) *If $D \setminus \Omega$ is not minimally thin at $\zeta$ with respect to $D$, then*

$$ \underset{\Omega}{\text{mf}}\text{-}\lim_{x\to\xi} \frac{K_D(x,\zeta)}{K_\Omega(x,\xi)} = 0. $$

(ii) *If $D \setminus \Omega$ is minimally thin at $\zeta$ with respect to $D$, where $\zeta$ is the point such that*

$$ K_D(\cdot,\zeta) - {}^{D}\widehat{R}^{D\setminus\Omega}_{K_D(\cdot,\zeta)} = \alpha\big(K_\Omega(\cdot,\xi) - {}^{\Omega}\widehat{R}^{\Omega\setminus D}_{K_\Omega(\cdot,\xi)}\big) \quad \text{on } \Omega \cap D \tag{3.2} $$

*for some positive constant $\alpha$, then*

$$ 0 < \underset{\Omega}{\text{mf}}\text{-}\lim_{x\to\xi} \frac{K_D(x,\zeta)}{K_\Omega(x,\xi)} < \infty. $$

(iii) *If $D \setminus \Omega$ is minimally thin at $\zeta$ with respect to $D$, where $\zeta$ is a point such that* (3.2) *is not satisfied, then*

$$\operatorname*{mf-}_{\Omega}\lim_{x\to\xi}\frac{K_D(x,\zeta)}{K_\Omega(x,\xi)}=0.$$

As a consequence of Theorem 3.1, we can obtain a result corresponding to Theorem A. Note that $\Omega_\phi$ has a unique Martin boundary point at the origin $\mathbf{0}$, so we write $K_{\Omega_\phi}(\cdot,\mathbf{0})$ for the Martin kernel at $\mathbf{0}$.

**Corollary 3.2** ([17, Theorem 1.1])**.** *Let $I^+$ and $I^-$ be as in* (3.1)*. Then the following statements hold:*

(i) *If $I^+<\infty$ and $I^-=\infty$, then*

$$\lim_{t\to 0+} t^{n-1}K_{\Omega_\phi}(te,\mathbf{0})=0.$$

(ii) *If $I^+=\infty$ and $I^-<\infty$, then*

$$\lim_{t\to 0+} t^{n-1}K_{\Omega_\phi}(te,\mathbf{0})=\infty.$$

(iii) *If $I^+<\infty$ and $I^-<\infty$, then the limit of $t^{n-1}K_{\Omega_\phi}(te,\mathbf{0})$, as $t\to 0+$, exists and*

$$0<\lim_{t\to 0+} t^{n-1}K_{\Omega_\phi}(te,\mathbf{0})<\infty.$$

*Remark* 3.3. When $I^+=\infty$ and $I^-=\infty$, the limit of $t^{n-1}K_{\Omega_\phi}(te,\mathbf{0})$ may take any values 0, positive and finite, or $\infty$ (see [17, Example 1.2]).

## §4. Comparison estimates for the Green function and the Martin kernel

For two positive functions $f_1$ and $f_2$, the symbol $f_1\approx f_2$ means that there exists a constant $A>1$ such that $A^{-1}f_2\le f_1\le Af_2$. ¿From Theorem A and Corollary 3.2, we expect the following relationship between the Green function and the Martin kernel:

$$G_{\Omega_\phi}(te,e)K_{\Omega_\phi}(te,\mathbf{0})\approx t^{2-n}\quad\text{for } 0<t<2^{-1},$$

or, more generally, if $\Omega$ is a Lipschitz domain and $\xi\in\partial\Omega$, then

$$G_\Omega(x,x_0)K_\Omega(x,\xi)\approx|x-\xi|^{2-n}\quad\text{for } x\in\Gamma_\alpha(\xi)\setminus B(x_0,2^{-1}\operatorname{dist}(x_0,\partial\Omega)),\tag{4.1}$$

where $\Gamma_\alpha(\xi)=\{x\in\Omega:|x-\xi|<\alpha\operatorname{dist}(x,\partial\Omega)\}$ with $\alpha>1$ large enough. If we restrict to the case of bounded Lipschitz domains $\Omega$ in $\mathbb{R}^n$ with

$n \geq 3$, then only the upper estimate in (4.1) can be obtained from the following 3G inequality:

$$\frac{G_\Omega(x,z)G_\Omega(x,y)}{G_\Omega(z,y)} \leq A(|x-y|^{2-n} + |x-z|^{2-n}) \quad \text{for } x,y,z \in \Omega,$$

which was first proved by Cranston, Fabes and Zhao [14] in the study of conditional gauge theory for the Schrödinger operator. See also Bogdan [8]. Recently, Aikawa and Lundh [3] extended this inequality to the case of bounded uniformly John domains in $\mathbb{R}^n$ with $n \geq 3$.

Now, let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n \geq 3$ and let $\{y_j\}$ be a sequence in $\Omega$ converging to $\xi \in \partial\Omega$. Then, substituting $z = x_0$ and $y = y_j$ into the 3G inequality and letting $j \to \infty$, we obtain the upper estimate: for $x \in \Omega \setminus B(x_0, 2^{-1}\operatorname{dist}(x_0, \partial\Omega))$,

$$G_\Omega(x,x_0)K_\Omega(x,\xi) \leq A(|x-\xi|^{2-n} + |x-x_0|^{2-n}) \leq A'|x-\xi|^{2-n}.$$

The lower estimate in (4.1) does not follow from the 3G inequality, but the boundary Harnack principle would enable us to obtain (4.1). We consider (4.1) in a uniform domain. A domain $\Omega$ is said to be *uniform* if there exists a constant $A_1 > 1$ such that each pair of points $x$ and $y$ in $\Omega$ can be connected by a rectifiable curve $\gamma$ in $\Omega$ such that

$$\begin{aligned} &\ell(\gamma) \leq A_1|x-y|, \\ &\min\{\ell(\gamma(x,z)), \ell(\gamma(z,y))\} \leq A_1 \operatorname{dist}(z,\partial\Omega) \quad \text{for all } z \in \gamma. \end{aligned}$$

It is known that if $\Omega$ is a uniform domain, then there is a unique (minimal) Martin boundary point at each Euclidean boundary point (cf. [1]). As above, we write $K_\Omega(\cdot,\xi)$ for the Martin kernel at $\xi \in \partial\Omega$. Our conclusions are different between $n \geq 3$ and $n = 2$, so we state them separately. See [18] for their proofs.

**Theorem 4.1.** *Let $\Omega$ be a uniform domain in $\mathbb{R}^n$, where $n \geq 3$, and let $\xi \in \partial\Omega$. Then*

(4.2)
$$G_\Omega(x,x_0)K_\Omega(x,\xi) \approx |x-\xi|^{2-n} \quad \text{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1}\operatorname{dist}(x_0,\partial\Omega)),$$

*where the constant of comparison depends only on $\alpha$ and $\Omega$.*

When $n = 2$, the comparison estimate (4.2) does not hold in general as seen in the following example.

**Example 4.2.** Suppose that $n = 2$. Let $\Omega = B(\mathbf{0},1) \setminus \{\mathbf{0}\}$ and let $x_0 = (1/2, 0)$. Then $\Omega$ is a uniform domain, and we have for $x \in$

$B(\mathbf{0}, 1/4) \setminus \{\mathbf{0}\}$,

$$K_\Omega(x, \mathbf{0}) G_\Omega(x, x_0) = \frac{-\log|x|}{\log 2} \log\left(\frac{1}{2}\frac{|x - 4x_0|}{|x - x_0|}\right) \approx \log\frac{1}{|x|}.$$

We say that $\xi \in \partial\Omega$ satisfies the exterior condition if there exists a positive constant $\kappa$ such that for each $r > 0$ sufficiently small, there is $x_r \in B(\xi, r) \setminus \overline{\Omega}$ with $B(x_r, \kappa r) \subset \mathbb{R}^n \setminus \overline{\Omega}$.

**Theorem 4.3.** *Let $\Omega$ be a uniform domain in $\mathbb{R}^2$. Then the following statements hold:*

(i) *If $\xi \in \partial\Omega$ satisfies the exterior condition, then*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \approx 1 \quad \text{for } x \in \Gamma_\alpha(\xi) \cap B(\xi, 2^{-1} \operatorname{dist}(x_0, \partial\Omega)),$$

*where the constant of comparison depends only on $\alpha$ and $\Omega$.*

(ii) *If $\xi \in \partial\Omega$ is an isolated point and $\Omega$ is bounded, then there exists $\delta > 0$ such that*

$$G_\Omega(x, x_0) K_\Omega(x, \xi) \approx \log\frac{1}{|x - \xi|} \quad \text{for } x \in B(\xi, \delta) \setminus \{\xi\},$$

*where the constant of comparison is independent of $x$.*

Finally, we note that if $\Omega$ is a Lipschitz domain, then every $\xi \in \partial\Omega$ satisfies the exterior condition and so Theorem 4.3(i) holds.

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# Singular directions of meromorphic solutions of some non-autonomous Schröder equations

**Katsuya Ishizaki and Niro Yanagihara**

**Abstract.**

Let $s = |s|e^{2\pi\lambda i}$ be a complex constant satisfying $|s| > 1$ and $\lambda \notin \mathbb{Q}$. We show that for a transcendental meromorphic solution $f(z)$ of some non-autonomous Schröder equation $f(sz) = R(z, f(z))$, any direction is a Borel direction.

## §1. Introduction

Let $R(z, w)$ be a rational function in $z$ and $w$ of $\deg_w R(z, w)$ at least 2, and let $s \in \mathbb{C}$ be a constant of modulus bigger than 1. This note is devoted to investigate singular directions of meromorphic solutions of functional equations of the form

$$f(sz) = R(z, f(z)), \quad d = \deg_w[R(z, w)] \geq 2. \tag{1.1}$$

In this note "meromorphic" means "meromorphic in the complex plane $\mathbb{C}$", and we assume that the reader is familiar with the Nevanlinna theory, see e.g., [1], [4]. By a simple transformation, we can assume that $R(0, 0) = 0$. In order to state an existence theorem of a meromorphic solution for (1.1), we write

$$R(z, w) = \sum_{n+m \geq 1} a_{n,m} z^n w^m.$$

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Under the assumption that either

$$a_{1,0} \neq 0 \quad \text{and} \quad s^n \neq a_{0,1} \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

or

$$a_{1,0} = 0 \quad \text{and} \quad s = a_{0,1}, \tag{1.3}$$

the equation (1.1) admits the unique meromorphic solution $f(z) \not\equiv 0$ with $f(0) = 0$ for the case (1.2), and also the unique solution $f(z)$ with $f(0) = 0, f'(0) = 1$, for the case (1.3). For the proof, see [8, p.153]. When $f(z)$ is transcendental, the order of growth $\rho = \rho(f)$ is given by $\rho = \log d / \log |s|,\ d = \deg_w R(z, w)$, and there holds

$$K_1 r^\rho < T(r, f) < K_2 r^\rho, \tag{1.4}$$

with some positive constants $K_1, K_2$, where $T(r, f)$ is the Nevanlinna characteristic of $f(z)$, see [8, p.159].

Let $\mathfrak{d}_\omega = \{z = re^{i\omega},\ r > 0\}$ be a ray and $\Omega(\omega, \alpha), \alpha \in (0, \pi)$, be a sector $\Omega(\omega, \alpha) = \{z\ ;\ |\arg[z] - \omega| < \alpha\}$. When $\omega$ is fixed, we write for $\Omega(\omega, \alpha)$ simply as $\Omega_\alpha$. Further we define $\Omega_\alpha^{(r)} = \Omega_\alpha \cap \{|z| < r\}$.

Let $f(z)$ be a transcendental meromorphic function of order $\rho > 0$. Let $\mathfrak{d}_\omega$ be fixed. For any $a \in \mathbb{C} \cup \{\infty\}$, write zeros of $f(z) - a$ in $\Omega_\alpha = \Omega(\omega, \alpha)$ as $z_n^*(a, \Omega_\alpha),\ n = 0, 1, \cdots$, multiple zeros counted only once. On the other hand, zeros of $f(z) - a$, counted with multiplicity, are denoted as $z_n(a, \Omega_\alpha)$. We say $\mathfrak{d}_\omega$ to be a *Borel direction of divergence type in the sense of Tsuji* (resp. *in the sense of Valiron*), for $f(z)$ [6, p.274] (resp. [7]), if for any $a \in \mathbb{C}$, with at most two possible exception(s),

$$\sum_{n=0}^{\infty} \frac{1}{|z_n^*(a, \Omega_\alpha)|^\rho} = \infty \quad \text{for any } \alpha > 0,$$

$$\left(\text{resp.} \qquad \sum_{n=0}^{\infty} \frac{1}{|z_n(a, \Omega_\alpha)|^\rho} = \infty \quad \text{for any } \alpha > 0\right).$$

In the following, we call a *Borel direction of divergence type* simply as a *Borel direction.*

Obviously, if $c$ is a Borel exceptional value in the sense of Valiron, then $c$ is so in the sense of Tsuji, too, but the converse is not true. We write $s$ as

$$s = |s| e^{2\pi\lambda i}, \quad |s| > 1, \quad \lambda \in [0, 1). \tag{1.5}$$

In the autonomous case, i.e., $R(z, w)$ does not contain $z$, we proved [2] that for a meromorphic solution $g(z)$ of the equation $g(sz) = R(g(z))$, any direction $\mathfrak{d}_\omega$ is a Borel direction in the sense of both Valiron as well as Tsuji, supposed $\lambda \notin \mathbb{Q}$. Further, a Borel exceptional value $c$, if any, must be a Picard exceptional value, i.e. $g(z) \neq c$ for any $z \in \mathbb{C}$.

## §2. Non-autonomaus Schröder equations and a main result

In order to consider the non-autonomous case where $R(z,w)$ contains $z$, we need to make some provisions. Write $R(z,w)$ in (1.1)

$$R(z,w) = \frac{P(z,w)}{Q(z,w)},$$
$$P(z,w) = \sum_{j=0}^{p} a_j(z)w^j, \quad Q(z,w) = \sum_{k=0}^{q} b_k(z)w^k,$$

where $a_j(z), b_k(z)$ are polynomials. We have $d = \max(p,q) \geq 2$.

*Proposition* 1. By some linear transformation

$$L[w] = \frac{\alpha w + \beta}{\gamma w + \delta}, \quad \alpha,\beta,\gamma,\delta \in \mathbb{C}, \quad \alpha\delta - \beta\gamma \neq 0, \tag{2.1}$$

the equation (1.1) can be reduced to the following form

$$L[f(sz)] = R^\circ(z, L[f(z)]), \qquad R^\circ(z,w) = \frac{P^\circ(z,w)}{Q^\circ(z,w)},$$
$$P^\circ(z,w) = \sum_{j=0}^{d} a_j^\circ(z)w^j, \quad Q^\circ(z,w) = \sum_{k=0}^{d} b_k^\circ(z)w^k,$$

in which we have

$$\begin{gathered} \deg_w[P^\circ(z,w)] = \deg_w[Q^\circ(z,w)] = d, \\ \deg[a_j^\circ(z)] = \deg[b_k^\circ(z)] = D. \end{gathered} \tag{2.2}$$

We remark that the conditions (2.2) are satisfied with any quadruple $\alpha,\beta,\gamma,\delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma \neq 0$ for other than a finite number of exception.

*Proof of Proposition 1* Let

$$f(sz) = \frac{\sum_{j=0}^{p} a_j(z)f(z)^j}{\sum_{k=0}^{q} b_k(z)f(z)^k}, \quad \max(p,q) = d.$$

Put $f(z) = f_1(z) + \alpha$. Then

$$f_1(sz) = \frac{\sum_{j=0}^{d} a_j^{[1]}(z)f_1(z)^j}{\sum_{k=0}^{q} b_k^{[1]}(z)f_1(z)^k},$$

where

$$a_j^{[1]}(z) = \begin{cases} \sum_{m=j}^{p} \binom{m}{j} \alpha^{m-j} a_m(z) - \alpha \sum_{m=j}^{\min(q,j)} \binom{m}{j} \alpha^{m-j} b_m(z), \\ \qquad\qquad \text{for } j \le p, \text{ when } p \le d, \\ -\alpha \left\{ \sum_{m=j}^{d} \binom{m}{j} \alpha^{m-j} b_m(z) \right\}, \quad \text{for } j > p, \\ \qquad\qquad \text{for } j > p, \text{ when } p < q = d, \end{cases}$$

$$b_k^{[1]}(z) = \sum_{m=k}^{q} \binom{m}{k} \alpha^{m-k} b_m(z), \quad \text{for } k \le q.$$

Hence, except a finite number of values $\alpha$, we have

$$\deg[a_0^{[1]}(z)] = \max_{j,k}(\deg[a_j^{[1]}(z)], \deg[b_k^{[1]}(z)]),$$
$$\deg[b_0^{[1]}(z)] = \max_{k} \deg[b_k^{[1]}(z))].$$

Put $f_1(z) = 1/f_2(z)$. Then

$$f_2(sz) = \frac{\sum_{j=0}^{q} a_j^{[2]}(z) f_2(z)^j}{\sum_{k=0}^{d} b_k^{[2]}(z) f_2(z)^k},$$
$$\deg[b_d^{[2]}(z)] = \max_{j,k}(\deg[a_j^{[2]}(z)], \deg[b_k^{[2]}(z)]).$$

Put $f_2(z) = f_3(z) + \beta$, $f_3(z) = 1/f_4(z)$, and $f_4(z) = f_5(z) + \gamma$, then we obtain (2.2) for $a_j^{[5]}(z), b_k^{[5]}(z)$, except for a finite number of values $\beta, \gamma$. We have thus proved Proposition 1.

Write the coefficients of $w^j$ in $P(z,w)$ and those of $w^k$ in $Q(z,w)$ as

$$a_j(z) = a_D^{(j)} z^D + a_{D-1}^{(j)} z^{D-1} + \cdots + a_0^{(j)},$$
$$b_k(z) = b_D^{(k)} z^D + b_{D-1}^{(k)} z^{D-1} + \cdots + b_0^{(k)},$$

with $a_D^{(j)} \ne 0$ and $b_D^{(k)} \ne 0$ for $0 \le j, k \le d$, and put

$$\begin{aligned} P_j^*(w) &= a_D^{(j)} w^j + a_D^{(j-1)} w^{j-1} + \cdots + a_D^{(0)}, \ 0 \le j \le d, \\ Q_d^*(w) &= b_D^{(d)} w^d + b_D^{(d-1)} w^{d-1} + \cdots + b_D^{(0)}. \end{aligned} \tag{2.3}$$

The main result in this note is the following

*Theorem* 1. Suppose $\lambda \notin \mathbb{Q}$ in (1.5) and $R(z,w)$ in (1.1) satisfies (2.2). Assume that $P_d^*(w)$ and $Q_d^*(w)$ for $R(z,w)$ defined in (2.3) are relatively prime. Let (1.1) have a transcendental meromorphic solution $f(z)$. Then any $\mathfrak{d}_{\omega_0}$ ($\omega_0 \in [0, 2\pi)$) is a Borel direction in the sense of Tsuji for $f(z)$.

*Remark* 1. We can assume without losing generality that $P_j^*(w)$ and $Q_d^*(w)$ are relatively prime for each $j$, $1 \leq j \leq d-1$, which can be attained by a suitable choice of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ in (2.1).

On the contrary to autonomuous case, a Borel exceptional value need not be a Picard exceptional value (see the end of Section 1). Further, Borel exceptional value in the sense of Tsuji may be not exceptional in the sense of Valiron. We can see these in Examples 1 and 2 below.

Example 1. Consider the equation [8, p.158]

$$(2.4) \qquad f_1(sz) = \frac{1+z}{1-z} f_1(z)^2, \quad s = |s|e^{2\pi i\lambda},\ \lambda \in [0,1) \setminus \mathbb{Q},\ |s| > 2.$$

If we put $f_1(z) = 1 + h_1(z)$, then

$$h_1(sz) = \frac{2z}{1-z} + 2\frac{1+z}{1-z}h_1(z) + \frac{1+z}{1-z}h_1(z)^2 = 2z + 2h_1(z) + \cdots,$$

and we have that $a_{1,0} = 2 \neq 0$, $a_{0,1} = 2 \neq s^n$ for any $n \in \mathbb{N}$, hence there is the unique solution $h_1(z) \not\equiv 0$, $h_1(0) = 0$. Therefore, there is the unique non-trivial solution for (2.4) which is given by

$$f_1(z) = \prod_{n=1}^{\infty} \left( \frac{1 + \frac{z}{s^n}}{1 - \frac{z}{s^n}} \right)^{2^{n-1}}.$$

Hence $f_1(z)$ has two Borel exceptional values $0, \infty$ in the sense of Tsuji. They are not a Borel exceptional values in the sense of Valiron.

Example 2. Consider also the equation

$$(2.5) \qquad f_2(sz) = (1+z) f_2(z)^2, \quad s = |s|e^{2\pi i\lambda},\ \lambda \in [0,1) \setminus \mathbb{Q},\ |s| > 2.$$

As in Example 1, there is the unique solution which is given by

$$f_2(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{s^n}\right)^{2^{n-1}}.$$

$f_2(z)$ has a Borel exceptional value 0 in the sense of Tsuji, which is not exceptional in the sense of Valiron. For $f_2(z)$, $\infty$ is exceptional in the sense of Valiron (in fact, Picard exceptional).

We observe primeness in the examples above. For the equation (2.4) we have, putting $g_1(z) = 1/(f_1(z) - 1)$,

$$g_1(sz) = \frac{(-1 + \frac{1}{z})g_1(z)^2}{2g_1(z)^2 + 2(1 + \frac{1}{z})g_1(z) + (1 + \frac{1}{z})}.$$

We have $P_2^*(w) = -w^2$ and $Q_2^*(w) = 2w^2 + 2w + 1$ which are relatively prime.

For the equation (2.5), we see that $P_2^*(w)$ and $Q_2^*(w)$ are not relatively prime, and the assumption in Theorem 1 is not satisfied. But if we put $g_2(z) = 1/(f_2(z) - 1)$, then we get

$$zg_2(sz) = \frac{g_2(z)^2}{g_2(z)^2 + 2(1 + \frac{1}{z})g_2(z) + (1 + \frac{1}{z})} = R_2(z, g_2(z)).$$

For $R_2(z, w)$, we have that $P_2^*(w) = w^2$ and $Q_2^*(w) = w^2 + 2w + 1$ are relatively prime. Hence the arguments in the proof of Theorem 1 stated in Sections 4, 5 can be applied to $R_2(z, g_2(z))$. But we do not know whether for $m \in \mathbb{Z}$ with some $K > 0$,

$$T(r; \Omega_\alpha; z^m g_2(z)) \leq KT(r; \Omega_\alpha; g_2(z)) + O((\log r)^2)$$

holds or not. Therefore, our Theorem 1 can not be applied to (2.5).

## §3. Characteristic functions in a sector

Following Tsuji [6, p.272], we define the sectorial characteristic of a meromorphic function $w(z)$. Fix $\omega \in [0, 2\pi)$. With $\Omega_{\alpha_0} = \Omega(\omega, \alpha_0)$ and $\Omega_{\alpha_0}^{(r)}$ as in Section 1, we define

$$S(r; \Omega_{\alpha_0}; w) = \frac{1}{\pi} \iint_{\Omega_{\alpha_0}^{(r)}} \left( \frac{|w'(te^{i\theta})|}{1 + |w(te^{i\theta})|^2} \right)^2 t dt d\theta,$$

$$T(r; \Omega_{\alpha_0}; w) = \int_0^r \frac{S(t; \Omega_{\alpha_0}; w)}{t} dt.$$

Let $\overline{n}(r, b; \Omega_\alpha; w), \Omega_\alpha = \Omega(\omega, \alpha)$, be the number of zeros of $w(z) - b$ contained in $\Omega_\alpha^{(r)}$, multiple zeros counted only once, and put

$$\overline{N}(r, b; \Omega_\alpha; w) = \int_1^r \frac{\overline{n}(t, b; \Omega_\alpha; w)}{t} dt.$$

Then by [6, p.272, Theorem VII.3], we have with any $\alpha > \alpha_0$,

$$T(r; \Omega_{\alpha_0}; w) \leq 3 \sum_{i=1}^{3} \overline{N}(2r, b_i; \Omega_\alpha; w) + O((\log r)^2). \tag{3.1}$$

We note that (3.1) is generalized by Toda [5].

## §4. A preliminary Lemma

Let $R(z, w)$ is a rational function in $w$ whose coefficients are rational functions. Suppose that $R(z, w)$ satisfies the condition in Theorem 1.

*Lemma* 1. Write $\Omega(\omega, \alpha)$ as $\Omega_\alpha$. We have for a constant $K$

$$T(r; \Omega_\alpha; R(z, f(z))) \leq KT(r; \Omega_\alpha; f(z)) + O((\log r)^2).$$

*Proof of Lemma 1* Let $a(z)$ be a rational function satisfying $a(z) \to M \neq 0, \infty$ as $z \to \infty$. Then $|M|/2 \leq |a(z)| \leq 2|M|$ for $|z| \geq r_0$ with sufficiently large $r_0$, and we have

$$\begin{aligned}
\frac{|(af)'|}{1+|af|^2} &\leq \frac{|a'f|}{1+|af|^2} + \frac{|af'|}{1+|af|^2} \\
&\leq \frac{1}{2}\cdot\frac{|a'|}{|a|} + \frac{|af^2|}{1+|af|^2}\cdot\frac{|f'|}{1+|f|^2} + \frac{|a|}{1+|af|^2}\cdot\frac{|f'|}{1+|f|^2} \\
&\leq \frac{1}{2}\cdot\frac{|a'|}{|a|} + \frac{|af|^2}{1+|af|^2}\cdot\frac{1}{|a|}\cdot\frac{|f'|}{1+|f|^2} + 2|M|\frac{|f'|}{1+|f|^2} \\
&\leq \frac{1}{2}\cdot\frac{|a'|}{|a|} + 2(\frac{1}{|M|} + |M|)\cdot\frac{|f'|}{1+|f|^2}.
\end{aligned}$$

Hence we get

$$T(r; \Omega_\alpha; a(z)f(z)) \leq 8(|M|^{-1} + |M|)^2 T(r; \Omega_\alpha; f(z)) + O((\log r)^2). \tag{4.1}$$

Note that, if $a(z) = M$ a constant, then $O((\log r)^2)$ in (4.1) can be omitted.

We have for $c \in \mathbb{C}$

$$K_1(c)T(r; \Omega_\alpha; f) \leq T(r; \Omega_\alpha; f - c) \leq K_2(c)T(r; \Omega_\alpha; f), \tag{4.2}$$

where $K_j(c)$, $j = 1, 2$, are constants depending on $c$. In fact, (4.2) is trivial when $c = 0$. Suppose $c \neq 0$. If $|f(z)| \leq 2|c|$,

$$\frac{1}{1+9|c|^2} \leq \frac{1+|f(z)|^2}{1+|f(z)-c|^2} \leq 1 + 4|c|^2,$$

and if $|f(z)| > 2|c|$, we obtain from $|c/f(z)| < 1/2$,

$$\frac{4}{9} \leq \frac{1+|f(z)|^2}{1+|f(z)-c|^2} = \frac{1+1/|f(z)|^2}{|1-c/f(z)|^2 + 1/|f(z)|^2} \leq 4.$$

When $c(z)$ is a rational function with $c(z) \to M \in \mathbb{C}$ as $z \to \infty$, by the similar calculation as above we infer that

$$\begin{aligned} (4.3) \qquad K_1(c)T(r;\Omega_\alpha;f) &\leq T(r;\Omega_\alpha;f-c) + O((\log r)^2) \\ &\leq K_2(c)T(r;\Omega_\alpha;f) + O((\log r)^2). \end{aligned}$$

In fact, $|c(z)| \leq M_1$ with some $M_1 > |M|$, for large $|z|$. Since we have

$$\frac{|f'-c'|}{1+|f-c|^2} \leq \frac{1+|c|^2}{1+|f-c|^2}\frac{|c'|}{1+|c|^2} + \frac{1+|f|^2}{1+|f-c|^2}\frac{|f'|}{1+|f|^2}$$

and

$$\frac{1+|c|^2}{1+|f-c|^2} \leq 1+4M_1^2, \ \frac{1+|f|^2}{1+|f-c|^2} \leq \max\left(1+4M_1^2, 4+\frac{1}{M_1^2}\right),$$

we obtain the second inequality in (4.3). Thus for a meromorphic function $g(z)$, we see $T(r;\Omega_\alpha;g+c) + O((\log r)^2) \leq \tilde{K}_2(c)T(r;\Omega_\alpha;g) + O((\log r)^2)$ with a constant $\tilde{K}_2(c)$. Set $g(z) = f(z) - c(z)$ in this inequality, we get the first inequality in (4.3).

By (4.1) and (4.2), we see that, with a constant $K_L$

$$T(r;\Omega_\alpha;f) \leq K_L T(r;\Omega_\alpha;L[f]), \qquad L[w] = \frac{\alpha w+\beta}{\gamma w+\delta},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, $\alpha\delta - \beta\gamma \neq 0$.

We have for $\ell \geq 2$,

$$\frac{|(f(z)^\ell)'|}{1+|f(z)^\ell|^2} = \ell\frac{|f(z)^{\ell-1}|+|f(z)^{\ell+1}|}{1+|f(z)^\ell|^2}\frac{|f'(z)|}{1+|f(z)|^2} \leq \ell\frac{|f'(z)|}{1+|f(z)|^2},$$

since $x^{\ell-1} + x^{\ell+1} - 1 - x^{2\ell} = -(1-x^{\ell-1})(1-x^{\ell+1}) \leq 0$ for $x \geq 0$, and hence

$$T(r;\Omega_\alpha;f(z)^\ell) \leq \ell^2 T(r;\Omega_\alpha;f(z)).$$

For $R(z,w) = P(z,w)/Q(z,w)$, define $P_j^*(w)$ and $Q_d^*(w)$ as in (2.3). We assume that $P_d^*(w), Q_d^*(w)$ are relatively prime, following Theorem 1. Then, as stated in Remark 1, we can assume that $P_j^*(w), Q_d^*(w)$ are relatively prime, without losing generality. Of course we assume (2.2). Write $P(z,w) = a_d(z)P_1(z,w), Q(z,w) = b_d(z)Q_1(z,w)$ and $R_1(z,w) = P_1(z,w)/Q_1(z,w)$. Note that

$$P_1(z,w) = w^d + \sum_{j=0}^{d-1} a^{[j]}(z)w^j, \quad a^{[j]}(z) = \frac{a_j(z)}{a_d(z)} = a_0^{[j]} + \sum_{n=1}^{\infty}\frac{a_n^{[j]}}{z^n},$$

$$Q_1(z,w) = w^d + \sum_{j=0}^{d-1} b^{[j]}(z)w^j, \quad b^{[j]}(z) = \frac{b_j(z)}{b_d(z)} = b_0^{[j]} + \sum_{n=1}^{\infty}\frac{b_n^{[j]}}{z^n},$$

with $a_0^{[j]} \neq 0$ and $b_0^{[j]} \neq 0$. Since $\lim_{z\to\infty} a(z) \neq 0, \infty$, where $a(z) = a_d(z)/b_d(z)$, we have by (4.1) for a constant $K_a$

$$T(r;\Omega_\alpha;R(z,f(z))) \leq K_a T(r;\Omega_\alpha;R_1(z,f(z))) + O((\log r)^2).$$

Hence we may treat $P(z,w) = P_1(z,w)$, $Q(z,w) = Q(z,w)$. We write $R_1(z,f(z)) = w_1 + w_2$, where

$$w_1 = \frac{\sum_{j=0}^{d-1} a^{[j]}(z) f(z)^j}{Q_1(z,f(z))}, \quad w_2 = \frac{f(z)^d}{Q_1(z,f(z))}.$$

We will show that, with some constant $K_1$,

$$\text{(4.4)} \quad T(r;\Omega_\alpha;w_1+w_2) \leq K_1\{T(r;\Omega_\alpha;w_1)+T(r;\Omega_\alpha;w_2)\}+O((\log r)^2).$$

In fact, we have

$$\frac{|(w_1+w_2)'|}{1+|w_1+w_2|^2} \leq \frac{1+|w_1|^2}{1+|w_1+w_2|^2}\frac{|w_1'|}{1+|w_1|^2} + \frac{1+|w_2|^2}{1+|w_1+w_2|^2}\frac{|w_2'|}{1+|w_2|^2}.$$

If either $|w_1| \geq 2|w_2|$ or $|w_2| \geq 2|w_1|$, then

$$\frac{1+|w_1|^2}{1+|w_1+w_2|^2} \leq 4 \text{ (or } \leq 1), \quad \frac{1+|w_2|^2}{1+|w_1+w_2|^2} \leq 1 \text{ (or } \leq 4).$$

If $2|w_1| > |w_2| > (1/2)|w_1|$, then

$$|f(z)|^2 \leq 2\left(|a^{[d-1]}(z)||f(z)^{d-1}| + \cdots + |a^{[0]}(z)|\right)$$

and $a^{[j]}(z), 0 \leq j \leq d-1$, are bounded as $z \to \infty$. Hence $f(z)$ must be bounded. Since $P_d^*(w), Q_d^*(w)$ are relatively prime by the assumption, $|P_d^*(w)|^2 + |Q_d^*(w)|^2 \geq K' > 0$ with a constant $K'$. Hence $|P_1(z,f(z))|^2 + |Q_1(z,f(z))|^2 \geq K^*$ with a constant $K^* > 0$, if $|z|$ is sufficiently large. Thus we have

$$\frac{1+|w_1|^2}{1+|w_1+w_2|^2} \leq \sqrt{K_1/2}, \quad \frac{1+|w_2|^2}{1+|w_1+w_2|^2} \leq \sqrt{K_1/2}$$

with some $K_1$, for $|z| \geq r_0$ if $r_0$ is large, which shows (4.4). Next, write

$$P_2(z,w) = a^{[d-1]}(z)w^{d-1} + P_3(z,w), \quad P_3(z,w) = \sum_{j=0}^{d-2} a^{[j]}(z)w^j$$

and $w_1 = P_3(z,w)/Q_1(z,w)$ and $w_2 = a^{[d-1]}(z)w^{d-1}/Q_1(z,w)$. Since $P^*_{d-1}(w)$ and $Q^*_d(w)$ are relatively prime, we obtain (4.7) as above. Applying these arguments repeatedly, we have

$$\begin{aligned} T(r;\Omega_\alpha;R(z,f(z))) &\leq K_2 \sum_{j=0}^{d} T(r;\Omega_\alpha;\frac{f(z)^j}{Q_1(z,f(z))}) + O((\log r)^2) \\ &\leq K_2 \sum_{j=0}^{d} T(r;\Omega_\alpha;\frac{Q_1(z,f(z))}{f(z)^j}) + O((\log r)^2) \\ &\leq K_3 \sum_{0\leq j,k\leq d} T(r;\Omega_\alpha;f(z)^{|k-j|}) + O((\log r)^2) \\ &\leq KT(r;\Omega_\alpha;f(z)) + O((\log r)^2). \end{aligned}$$

with some constants $K_2, K_3, K$, making use of (4.3). We have thus proved Lemma 1.

## §5. Proof of Theorem 1

Let $T(r,f)$ be the characteristic function of $f(z)$ in the sense of Shimizu–Ahlfors. As in [6, p.274], we see from (1.4) that there is $\omega^* \in [0,2\pi)$ such that

$$\int^\infty \frac{T(r;\Omega(\omega^*,\alpha_0);f)}{r^{\rho+1}} dr = \infty,$$

for any $\alpha_0 \in (0,\pi)$. Define $R^0(z,w) = w$ and

$$R^m(z,w) = R(s^{m-1}z, R^{m-1}(z,w)) \quad \text{for} \quad m \geq 1.$$

Then we have

$$f(s^m z) = R(s^{m-1}z, f(s^{m-1}z)) = R^m(z,f(z)).$$

It is not difficult to see that $R^m(z,w)$ satisfies (2.2) from the assumption for $R(z,w)$, and also see that $P^*_{d^m}(w)$ and $Q^*_{d^m}(w)$ corresponding to $R^m(z,w)$ are relatively prime. We can assume that $P^*_j(w)$ and $Q^*_{d^m}(w)$ corresponding to $R^m(z,w)$ are relatively prime, for each $j < d^m$ by a suitable linear transformation, if necessary.

Take $\omega_0 \in [0,2\pi)$ and $\alpha \in (0,\pi)$ arbitrarily. Let $m \in \mathbb{N}$ be so large that $\alpha_0 = |\omega_0 + 2\pi m\lambda - \omega^*| < \alpha/8, \mod 2\pi$, see e.g., [3]. Then we have

$$\begin{aligned}\int^{\infty} \frac{T(r; \Omega(\omega_0, \alpha/2); R^m(z, f(z)))}{r^{\rho+1}} dr \\ = \int^{\infty} \frac{T(r; \Omega(\omega_0, \alpha/2); f(s^m z))}{r^{\rho+1}} dr \\ \geq C \int^{\infty} \frac{T(|s|^m r; \Omega(\omega^*, \alpha_0); f(z))}{r^{\rho+1}} dr = \infty.\end{aligned}$$

for some positive constant $C$. Thus by Lemma 1

$$\int^{\infty} \frac{T(r; \Omega(\omega_0, \alpha/2); f(z))}{r^{\rho+1}} dr = \infty.$$

By means of Tsuji's result (3.1), for any distinct three values $b_i \in \mathbb{C} \cup \{\infty\}, 1 \leq i \leq 3$,

$$\sum_{i=1}^{3} \int^{\infty} \frac{\overline{N}(2r, b_i; \Omega(\omega_0, \alpha); f(z))}{r^{\rho+1}} dr = \infty,$$

which implies our assertion.

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# Integral Representation for Space-Time Excessive Functions

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**Abstract.**

We study space-time excessive functions with respect to a basic submarkovian semigroup $\mathbb{P}$. It is shown that under some regularity assumptions many space-time excessive functions on a half-space have a Choquet-type integral represention by suitably choosen densities of the adjoint semigroup $\mathbb{P}^*$. If $\mathbb{P}$ is a convolution semigroup which is absolutely continuous with respect to the Haar measure, then all space-time excessive functions admit such an integral representation.

## §1. Introduction

Let $\triangle := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ denote the Laplace operator on $E := \mathbb{R}^n$. We consider the heat operator $\frac{1}{2} \triangle - \frac{\partial}{\partial t}$ on the half space $E \times ]0, \infty[$. It is well known that the positive solutions $v$ of

$$\frac{1}{2} \triangle v - \frac{\partial v}{\partial t} \leq 0$$

on $E \times ]0, \infty[$ (called supercaloric functions) admit a Choquet-type integral representation by minimal supercaloric functions (c.f. [15]). Moreover, these minimal supercaloric functions are just the densities of the Gaussian semigroup $\mathbb{P} = (P_t)_{t>0}$ which has the generator $\frac{1}{2}\triangle$.

It is a remarkable fact that all this remains true also in the degenerate case $n = 0$ (where $E = \{0\}$ is just a one-point set); in this case the above integral representation is exactly the standard correspondence between distribution functions $v$ on $]0, \infty[$ and measures $\rho$ on $[0, \infty[$ given

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by

$$v(s) = \rho([0, s[ \quad \text{for } s > 0,$$

or, written in a fancier way,

$$v(s) = \int 1_{]s_0,\infty[}(s) d\rho(s_0) \quad \text{for } s > 0,$$

where $\{1_{]s_0,\infty[} : s_0 \geq 0\}$ is the set of normalized minimal supercaloric functions.

In this paper we show that a similar result holds in great generality: We replace the above Gaussian semigroup by a general basic semigroup $\mathbb{P}$ (i.e. there exists some measure $\mu$ such that $\varepsilon_x P_t$ is absolutely continuous with respect to $\mu$ for all $x$ and $t$). Under some regularity assumptions concerning the adjoint semigroup $\mathbb{P}^*$ the appropriately choosen densities of $\mathbb{P}^*$ turn out to be minimal space-time excessive functions, which then give a Choquet-type integral representation of a large class of space-time excessive functions.

In the special setting of convolution semigroups which are absolutely continuous with respect to the Haar measure, all space-time excessive functions on a half-space are represented in this way. In particular, all excessive functions of the parabolic operator of order $\alpha$ (c.f. [7]) on the upper halpf plane admit this Choquet-type integral representation.

## §2. Notations and Preliminaries

In the following we fix the central potential theoretic notions which will be used througout. As basic references we use [5] or [3] and [6]. $(E, \mathcal{E})$ will always denote a *standard Borel measurable space*, i.e. $E$ may be identified with a Borel subset of a completely metrizable separable space equipped with it's Borel field $\mathcal{E}$.

We denote by $p\mathcal{E}$ the convex cone of positive numerical $\mathcal{E}$-measurable functions on $E$.

Remember that a *kernel* $P$ on $(E, \mathcal{E})$ is a family $(P(x, \cdot))_{x \in E}$ of measures on $(E, \mathcal{E})$ such that for $f$ in $p\mathcal{E}$ the function

$$Pf(x) = \int f(y) P(x, dy), \quad x \in E$$

is in $p\mathcal{E}$.

Then, for every measure $\mu$ on $(E, \mathcal{E})$ the measure $\mu P$ satisfies

$$\int f d(\mu P) = \int P f d\mu \quad \text{for } f \in p\mathcal{E}.$$

We assume to be given a *measurable semigroup* $\mathbb{P} = (P_t)_{t>0}$ of *substochastic* kernels on $(E, \mathcal{E})$ (i.e. we have $P_s P_t = P_{s+t}, P_t 1 \leq 1$, and $(x,t) \to P_t f(x)$ is $\mathcal{E} \otimes \mathcal{B}(]0,\infty[)$ - measurable for every $f$ in $p\mathcal{E}$).

*Examples* 2.1. i) The trivial example is given by the one-point set $E = \{0\}$ and the semigroup $P_t(0,\cdot) = \varepsilon_0$ for $t > 0$.
ii) The standard example is the *Gaussian semigroup* on $E := \mathbb{R}^n$ which is given by the Lebesgue densites $p_t(x,y) = q_t(x-y)$ with

$$q_t(x) = \frac{1}{\sqrt{2\pi t}^n} \exp(-\frac{|x|^2}{2t}), \qquad t > 0, x, y \in \mathbb{R}^n.$$

iii) More general examples are given by semigroups associated with second order linear parabolic or elliptic differential operators on a domain of $\mathbb{R}^n$ (c.f. [2]) or suitable pseudo-differential operators (c.f. [8]). In particular, absolutely continouous convolution semigroups on $\mathbb{R}^n$ fit into this setting (c.f. [1]).

In the general setting we denote by $\mathbb{V} = (V_\lambda)_{\lambda \geq 0}$ the associated *resolvent* defined by

$$V_\lambda f(x) := \int_0^\infty e^{-\lambda t} P_t f(x) dt, \qquad x \in E, f \in p\mathcal{E}, \lambda \geq 0.$$

$V := V_0$ is called the *potential kernel* of $\mathbb{P}$. The resolvent $\mathbb{V}$ determines the semigroup $\mathbb{P}$ uniquely.

Remember that a set $N \in \mathcal{E}$ is called a *set of potential zero* if $V1_N = 0$. We say that some property holds *V-a.e.* if this property holds except on a set of potential zero.

Remember that $v \in p\mathcal{E}$ is called an *excessive function* (with respect to the given semigroup $\mathbb{P}$) if $\sup_{t>0} P_t v = v$, or equivalently, $P_t v \uparrow v$ for $t \downarrow 0$. For $f$ in $p\mathcal{E}$ the *potential* $Vf$ generated by $f$ is an excessive function.

We denote $\mathcal{S} := \mathcal{S}(\mathbb{P}) := \{v : v \text{ is excessive}, v < \infty \text{ V-a.e.}\}$.
A $\sigma$-finite measure $\eta$ on $(E, \mathcal{E})$ is called an *excessive measure* if $\eta P_t \uparrow \eta$ for $t \downarrow 0$. $Exc := Exc(\mathbb{P})$ denotes the convex cone of all excessive measures.

The set of *potential measures*

$$Pot := Pot(\mathbb{P}) := \{\mu V : \mu \text{ is a measure such that } \mu V \text{ is } \sigma\text{-finite } \}$$

is a convex subcone of $Exc$.

In this paper we study space-time excessive functions. Therefore we associate with the given semigroup $\mathbb{P}$ on $E$ the *space-time semigroup* $\mathbb{Q} = (Q_t)_{t>0}$ on $E\times]0,\infty[$ defined by

$$\varepsilon_{(x,s)}Q_t = 1_{]t,\infty[}(s)(\varepsilon_x P_t \otimes \varepsilon_{s-t}), \quad x \in E, s > 0$$

for $t > 0$. $\mathbb{Q}$ is again a measurable semigroup of substochastic kernels. We denote by $\mathbb{W} = (W_\lambda)_{\lambda \geq 0}$ the resolvent associated with $\mathbb{Q}$.

Of course, there are variants of these space-time semigroups, some of them will appear later.

*Examples* 2.2. i) In the trivial example $E = \{0\}$ and $P_t = \varepsilon_0$ for $t > 0$ it is easily seen that a positive function $v$ belongs to $\mathcal{S}(\mathbb{Q})$ if and only if $v$ is finite, increasing, and left continuous (or: lower semicontinuous) on $]0,\infty[$. A measure $\eta$ belongs to Exc $(\mathbb{Q})$ if and only if $\eta$ has a finite, decreasing, and right continuous (or: lower semicontinouous) Lebesgue density $v$ with respect to Lebesgue measure $\lambda\!\!\lambda$ on $]0,\infty[$.
ii) For the Gaussian semigroup the cone of space-time excessive functions is just the cone of supercaloric functions mentioned in the introduction.
iii) In general, for $f$ in $p\mathcal{E}$ the function

$$v(x,s) := P_s f(x), \quad x \in E, s > 0$$

is excessive with respect to $\mathbb{Q}$, since

$$Q_t v(x,s) = 1_{]t,\infty[}(s) P_t P_{s-t} f(x) = 1_{]t,\infty[}(s) v(x,s) \uparrow v(x,s)$$

for $t > 0, t \downarrow 0$.

In this paper we are interested in Choquet-type integral representations for *space-time excessive* functions, i.e for functions which are excessive with respect to $\mathbb{Q}$.

*Remark* 2.1. The following general results on Choquet-type integral representations of excessive measures and functions are known:
i) Under very general assumptions every excessive measure $\eta$ has a unique representation as a mixture of minimal excessive measures, i.e.

$$\eta(A) = \int_F \nu(A) d\rho(\nu) \quad \text{ for } A \in \mathcal{E},$$

where $\rho$ is a measure on the space $F$ of suitably normalized minimal excessive measures.
Here, $\nu \epsilon$ Exc is called *minimal* if $\nu = \nu_1 + \nu_2$ for $\nu_1, \nu_2 \in$ Exc can only hold if $\nu_1$ and $\nu_2$ are proportional to $\nu$ (c.f. [17]).
ii) If the potential kernel $V$ is proper, then the corresponding integral representation of every excessive function by minimal excessive functions holds if and only if $V$ is basic, i.e. $\varepsilon_x V \ll \mu$ for all $x$ for some $\sigma$-finite measure $\mu$ (c.f. [3] and [10]).

Consequently, a Choquet-type integral representation for all space-time excessive functions exists if and only if the potential kernel $W$ is basic, i.e. for some $\sigma$-finite measure $m$ on $E \times ]0, \infty[$ all the measures $\varepsilon_{x,s} W$ are absoluteley continuous with respect to $m$. It is well known that under this assumption there exists a $\sigma$-finite measure $\mu$ on $(E, \mathcal{E})$ such that $\varepsilon_x P_t \ll \mu$ for all $x \in E, t > 0$, hence the following Assumption 3.1 is quite natural.

## §3. Choquet-type integral representation of space-time excessive functions

To obtain the wanted integral representation we need the existence of a nice dual semigroup $\mathbb{P}^*$.

**Assumption 3.1.** $\mathbb{P}$ and $\mathbb{P}^*$ are substochastic measurable semigroups on a standard Borel measurable space $(E, \mathcal{E})$ which are in duality and absolutely continuous with respect to some $\sigma$-finite measure $\mu$, i.e.

$$\varepsilon_x P_t \ll \mu, \quad \varepsilon_x P_t^* \ll \mu \qquad \text{for all } x \in E, t > 0 \text{ and}$$

$$\int P_t f \cdot g d\mu = \int f \cdot P_t^* g d\mu \qquad \text{for all } t > 0, f, g \in p\mathcal{E}.$$

From [19] we know that we can choose very nice densities for the associated space-time potential kernels $W$ and $W^*$:

**Theorem 3.1.** *There exists a unique $\mathcal{B}(]0, \infty[) \otimes \mathcal{E} \otimes \mathcal{E}$-measurable function $p : ]0, \infty[ \times E \times E \to \bar{\mathbb{R}}_+$ such that for $s, t > 0$, $x, y \in E$ and $f \in p\mathcal{E}$ the following is true:*
*i)* $P_t f(x) = \int f(z) p_t(x, z) d\mu(z)$
*ii)* $P_t^* f(x) = \int f(z) p_t(z, x) d\mu(z)$
*iii)* $p_{s+t}(x, y) = \int p_s(x, z) p_t(z, y) d\mu(z)$

**Conclusion 3.1.** *i) For $s_0 \geq 0$ and $x_0 \in E$ the function*

$$w_{x_0, s_0}(x, s) := 1_{]s_0, \infty[}(s) p_{s - s_0}(x, x_0), \quad x \in E, s > 0$$

*belongs to* $\mathcal{S}(\mathbb{Q})$*, since the Chapman-Kolmogorov equation, Theorem 3.1.iii, gives*

$$Q_t w_{x_0,s_0} = 1_{]s_0+t,\infty[} w_{x_0,s_0}.$$

*ii) By Fubini's theorem we conclude that for every* $\sigma$*-finite measure* $\rho$ *on* $E \times [0,\infty[$ *the function*

$$w^\rho(x,s) := \int 1_{]s_0,\infty[}(s) p_{s-s_0}(x,x_0) d\rho(x_0,s_0), \quad x \in E, s > 0$$

*is space-time excessive.*
*Moreover, if* $\rho$ *is concentrated on* $E \times \{0\}$*, then* $w^\rho$ *is "invariant up to the exit from* $E \times ]0,\infty[$*" for the space-time process, i.e.* $Q_t w^\rho(x,s) = w^\rho(x,s)$ *for all* $0 < t < s, x \in E$*.*

For our main result we need an additional regularity hypothesis:

**Assumption 3.2.** $\mathbb{P}^*$ is a right semigroup on $E$, i.e. there exists an associated right Markov process (c.f. [14]).

*Remark* 3.1. If $V^*$ is a proper kernel, then Assumption 3.2 is equivalent with the following potential theoretic properties of the convex cones $\mathcal{S}^*, Exc^*, Pot^*$ with respect to $\mathbb{P}^*$ (c.f. [16] for details):
i) $\mathcal{S}^*$ is inf-stable, $1 \in \mathcal{S}^*, \sigma(\mathcal{S}^*) = \mathcal{E}$,
ii) $E$ is *semisaturated, i.e. $Pot^*$ is hereditary in $Exc^*$ (i.e. for $\eta \in Exc^*$ satisfying $\eta \le \mu V^* \in Exc^*$ we have $\eta = \nu V^*$ for some measure $\nu$).
If $\mathbb{P}^*$ induces a strong harmonic space in the sense of [4], or if $\mathbb{P}^*$ induces a balayage space in the sense of [2], then Assumption 3.2 is satisfied.

In the following result we use the functions $w_{x_0,s_0}$ and $w^\rho$ introduced in Conclusion 3.1.

**Theorem 3.2.** *We assume Assumption 3.1 and Assumption 3.2. Then the following is true:*
*i) Let* $v \in \mathcal{S}(\mathbb{Q})$ *satisfy* $v \le w^{\rho_0} \in \mathcal{S}(\mathbb{Q})$ *for some measure* $\rho_0$ *on* $E \times [0,\infty[$*.*
*Then there exists a unique measure* $\rho$ *on* $E \times [0,\infty[$ *such that* $v = w^\rho$*, i.e.*

$$v(x,s) = \int w_{x_0,s_0}(x,s) d\rho(x_0,s_0), \quad x \in E, s > 0.$$

*For all* $x_0 \in E$ *and* $s_0 \ge 0$ *the function* $w_{x_0,s_0}$ *is a minimal element of* $\mathcal{S}(\mathbb{Q})$*.*
*ii) Every* $v \in \mathcal{S}(\mathbb{Q})$ *decomposes uniquely into* $v = w^\rho + v'$*, where* $\rho$ *is a unique measure on* $E \times [0,\infty[$ *and* $v' \ge w^\tau$ *holds only for the zero measure* $\tau$*.*

*Proof.* i) Let $m = \mu \otimes \lambda\!\!\lambda$. We denote by $\mathbb{Q}^*$ the semigroup on $E\times]0,\infty[$ given by

$$\varepsilon_{x,s}Q_t^* = \varepsilon_x P_t^* \otimes \varepsilon_{s+t}, \quad x \in E, s > 0, t > 0,$$

and we denote by $\mathbb{W}^*$ the associated resolvent. Then $\mathbb{W}$ is in strong duality with $\mathbb{W}^*$with respect to $m$, and

$$\Theta : v \to vm$$

is a bijection from $\mathcal{S}(Q)$ onto Exc $(\mathbb{Q}^*)$. Obviously, Exc $(\mathbb{Q}^*) =$ Exc $(\bar{\mathbb{Q}}^*)$ for the extended semigroup $\bar{\mathbb{Q}}^*$ on $E \times [0,\infty[$ given by

$$\varepsilon_{x,s}\bar{\mathbb{Q}}_t^* = \varepsilon_x P_t^* \otimes \varepsilon_{s+t}, \qquad x \in E, s \geq 0, t > 0$$

since $m(E \times \{0\}) = 0$ and since all measures in Exc$(\bar{\mathbb{Q}}^*)$ are absolutely continuous w.r. to $m$ (c.f. [3] for general details of this identification). Since $\mathbb{P}^*$ admits an associated right Markow process, this remains true for $\bar{\mathbb{Q}}^*$. Consequently, every $\bar{\mathbb{Q}}^*$-excessive measure $vm \leq w^{\rho_0}m = \rho_0 \bar{W}^*$ is of the form $vm = \rho\bar{W}^*$ for a suitable unique measure $\rho$. Inverting the mapping $\Theta$ shows that $v = w^\rho$.
Applying this to $w_{x_0,s_0} = w^\rho$ for $\rho := \varepsilon_{(x_0,s_0)}$ gives the minimality of $w_{x_0,s_0}$ for $x_0 \in E, s_0 \geq 0$.
ii) For general $v \in \mathcal{S}(\mathbb{Q})$ the measure $vm$ decomposes uniquely as $vm = \rho\bar{W}^* + v'm$, where $v'm \geq \tau\bar{W}^*$ holds only for $\tau = 0$ (i.e. $v'm$ is the harmonic part of $vm$ with respect to $\bar{\mathbb{Q}}^*$ according to [6]). Transporting this decomposition by the inverse of $\Theta$ gives the stated result. $\square$

*Remark* 3.2. Simple examples show that in general it is not true that every space-time excessive funtion admits a representation as in Theorem 3.2.i. A setting where this is true is described below in Application 3.1. Motivated by § 3 in [15] one might conjecture that it is sufficient that $E$ be thermically closed, i.e. $f \leq P_t f$ for all $t > 0$ for every $\mathbb{P}$-subharmonic $f \in p\mathcal{E}$.
In the setting of uniformly elliptic differential operators in gradient form on a domain in $\mathbb{R}^n$ Murata gave sufficient conditions for the non-existence of a non-zero positive space-time harmonic function on $E\times]0,\infty[$ with boundary values 0 on $E \times \{0\}$ (c.f. Theorem 4.2 in [12]).

Remember that $u \in \mathcal{S}$ is called *quasibounded* iff $u$ can be written as a countable sum of bounded elements of $\mathcal{S}$. It is well known that in classical potential theory associated with Laplace's equation a potential is quasibounded if and only if the associated Riesz measure does not charge polar sets. The same result is true for the potential theory associated with the heat equation according to [18].

**Corollary 3.1.** *Let $h \in \mathcal{S}(\mathbb{P})$ be invariant, i.e. $P_t h = h < \infty$ $V$-a.e. for all $t > 0$ . Then every $h$-quasibounded $v \in \mathcal{S}(\mathbb{Q})$ admits a unique integral representation*

$$v(x,s) = \int w_{x_0,s_0}(x,s) d\rho(x_0,s_0), \quad x \in E, s > 0.$$

*Here $v \in \mathcal{S}(\mathbb{Q})$ is called $h$-quasibounded if $v = \sum_{n\in\mathbb{N}} v_n$ for some sequence $(v_n) \subset \mathcal{S}(\mathbb{Q})$ such that $v_n \leq c_n h$ for suitable constants $c_n$ for all $n$ in $\mathbb{N}$.*

*Proof.* For $\rho_0 := (h\mu) \otimes \varepsilon_0$ we have obviously $h = w^{\rho_0}$, hence the stated integral representation holds for every $h$-bounded $v_n$ in $\mathcal{S}(\mathbb{Q})$. Summing these formulae for $n$ in $\mathbb{N}$ gives the wanted result. □

**Application 3.1.** Let G be a locally compact abelian group with countable base of the topology, and let $(\mu_t)_{t>0}$ be a convolution semigroup of measures on G such that all measures $\mu_t$ are absolutely continuous with respect to the Haar measure. Let $(P_t)_{t>0}$ be the associated semigroup of convolution kernels on G (c.f. [1]). The reflected measures given by

$$\int f d\mu_t^* = \int f(-x) d\mu_t(x), \quad t > 0, f \in p\mathcal{E}$$

define a dual basic convolution semigroup. $(P_t)_{t>0}$ and $(P_t^*)_{t>0}$ are strong Feller kernels. Consequently, the Assumptions 3.1 and 3.2 are satisfied (in fact, the associated Markov processes are very nice Lévy Processes). The densities of $(P_t)_{t>0}$ according to Theorem 3.1 are of the form $p_t(x,y) = q_t(x-y)$ for $t > 0, x, y \in G$ for suitable densities $q_t$ of $\mu_t$ with respect to the Haar measure on G.

In this particular case we have the following

**Result.** For every $v \in \mathcal{S}(\mathbb{Q})$ there exists a unique measure $\rho$ on $E \times [0, \infty[$ such that

$$v(s,x) = \int 1_{]s_0,\infty[}(s) q_{s-s_0}(x-x_0) d\rho(x_0,s_0), \quad x \in G, s > 0$$

*Proof.* Let $v \in \mathcal{S}(\mathbb{Q})$. We use the notations of the proof of Theorem 3.2. Then $vm$ ist in Exc $(\mathbb{Q}^*)$. Obviously, $vm$ is also an excessive measure with respect to the space-time convolution semigroup $(\mu_t^* \otimes \varepsilon_t)_{t>0}$ on the group $G \times \mathbb{R}$.
Let $\kappa^* := \int_0^\infty \mu_t^* \otimes \varepsilon_t dt$ denote the associated potential kernel measure on $G \times \mathbb{R}$. From Theorem 16.7 in [1] we know that $vm$ decomposes uniquely

as $vm = \rho * \kappa^* + \rho_1$, where $\rho_1$ is invariant with respect to $(\mu_t^* \otimes \varepsilon_t)_{t>0}$. Since $vm$ is supported by $E \times [0, \infty[$ we conclude $\rho_1 = 0$, and $\rho$ is a measure supported by $E \times [0, \infty[$. Consequently, we have $vm = \rho * \kappa^* = w^\rho m$, and the stated integral representation $v = w^\rho$ follows. □

*Examples* 3.1. As examples of Application 3.1 we obtain an explicit integral representation of all space-time excessive functions in the following cases:
i) the Brownian semigroup with densities given in Example 2.1.ii.
ii) the symmetric stable semigroup of order 1 (the Cauchy semigroup) with densities

$$q_t(x) = \frac{at}{(t^2 + |x|^2)^{\frac{n+1}{2}}}, \qquad x \in \mathbb{R}^n, t > 0$$

for $a = \Gamma(\frac{n+1}{2})/\pi^{(n+1)/2}$.
Similar results hold for more general $\alpha$-stable semigroups (c.f. [7] and [13]), except that there the densities are not elementary functions (only their Fourier transforms are explicitly given).

*Remark* 3.3. We formulated our results for a basic semigroup. The standard example for such a semigroup is determined by some second order elliptic linear partial differential operator $L$ on a domain $E$ in $\mathbb{R}^n$, where the coefficients of $L$ depend on the space variables in $E$ and have to be reasonably nice and not to degenerate. More generally, one may consider coefficients which are also time dependent. This leads to transition families $\mathbb{P} = (P_{s,t})_{s<t}$ which are no longer time homogeneous. Nevertheless, our reasoning carries over to this more general setting due to the fact that in [19] the existence of nice densities of such non-homogeneous families $\mathbb{P}$ has been proven.
Some examples of harmonic spaces associated with such time dependant differential operators appear in [9].

*Remark* 3.4. In [11] Murata proved for a large class of uniformly elliptic differential operators $L$ in gradient form on a domain $E$ in $\mathbb{R}^n$ that the Martin boundary for the associated heat operator is given by $(E \times \{0\}) \cup (\partial \times ]0, \infty[)$, where $\partial$ is the Martin boundary of $E$ with respect to $L$. It should be true in our setting that the Martin-Poisson space associated with the space-time semigroup $\mathbb{Q}$ is given by $(E \cup \partial) \times [0, \infty[)$, where $\partial$ denotes the set of suitably normalized minimal $\mathbb{P}$-harmonic functions (c.f. [3] for details).

*Remark* 3.5. It is easily verified, that the integral representation of Corollary 3.1 for $v \in \mathcal{S}(\mathbb{Q})$ holds already, if $v$ is only $w^{\rho_0}$-quasi-bounded for some general $w^{\rho_0} \in \mathcal{S}(\mathbb{Q})$. The proof is the same as that of Corollary 3.1.

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# A decomposition of the Schwartz class by a derivative space and its complementary space

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**Abstract.**

Let $\mathcal{D}(R^n)$ be the class of all $C^\infty-$functions on $R^n$ with compact support. For a multi-index $\alpha$ we denote $\mathcal{D}^\alpha(R^n) = \{D^\alpha\varphi : \varphi \in \mathcal{D}(R^n)\}$. We give a direct sum decomposition of $\mathcal{D}(R^n)$ by $\mathcal{D}^\alpha(R^n)$ and its complementary space.

## §1. Introduction

Let $R^n$ be the $n$-dimensional Euclidean space. The points of $R^n$ are ordered $n$-tuples $x = (x_1, \cdots, x_n)$, which each $x_i$ is a real number. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an $n$-tuple of nonnegative integers, then $\alpha$ is called a multi-index, and we let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. The partial derivative operators are denoted by $D_j = \partial/\partial x_j$ for $1 \leq j \leq n$, and the higher order derivatives by

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Following L.Schwartz [4] the notation $\mathcal{D}(R^n)$ (the Schwartz class) stands for the class of all infinitely differentiable functions on $R^n$ with compact support.

L.Schwartz uses the following fact about $\mathcal{D}(R^n)$ in the discussion of primitives of distributions [4: sections 4 and 5 in Chap.II].

Fact. Let $\theta_0(t) \in \mathcal{D}(R^1)$ be a function which satisfies

$$\int_{-\infty}^{\infty} \theta_0(t)dt = 1.$$

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Then $\varphi \in \mathcal{D}(R^n)$ can be decomposed uniquely as follows:

$$\varphi(x) = \chi(x) + \lambda(x_2, \cdots, x_n)\theta_0(x_1) \tag{1.1}$$

where $\chi \in \mathcal{D}^1(R^n) = \{D_1\varphi : \varphi \in \mathcal{D}(R^n)\}$ and $\lambda \in \mathcal{D}(R^{n-1})$. Namely, if we put

$$\mathcal{U}^1(R^n) = \{\lambda(x_2, \cdots, x_n)\theta_0(x_1) : \lambda \in \mathcal{D}(R^{n-1})\},$$

then

$$\mathcal{D}(R^n) = \mathcal{D}^1(R^n) \oplus \mathcal{U}^1(R^n)$$

where the symbol $\oplus$ means a direct sum.

Moreover some authors deal with orthogonal decompositions of the Lebesgue spaces and the Sobolev spaces related to certain kinds of differential operators ([1], [2], [3]).

In this article we are concerned with a direct sum decomposition of the Schwartz class $\mathcal{D}(R^n)$ related to the higher order differential operator $D^\alpha$. Namely we treat the following problem.

Problem. Let $\alpha$ be a nonzero multi-index and $\mathcal{D}^\alpha(R^n) = \{D^\alpha\varphi : \varphi \in \mathcal{D}(R^n)\}$. Give a direct sum decomposition of $\mathcal{D}(R^n)$ by means of $\mathcal{D}^\alpha(R^n)$ and its complementary space.

The next section is devoted to study of the problem.

## §2. A decomposition of the Schwartz class

As we saw in section 1, $\mathcal{D}(R^n) = \mathcal{D}^1(R^n) \oplus \mathcal{U}^1(R^n)$. In order to give a direct sum decomposition of $\mathcal{D}(R^n)$ by means of $\mathcal{D}^\alpha(R^n)$, we must construct the complementary space of $\mathcal{D}^\alpha(R^n)$. We need two preparations. For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ we set

$$M_\alpha = \{j \in \{1, \cdots, n\} : \alpha_j \neq 0\}$$

and for $j = 1, \cdots, n$ let

$$R^{n,j} = \{(x_1, \cdots, x_n) \in R^n : x_j = 0\}.$$

First we note that

**Lemma 2.1.** *(cf. [4: Section 5 in Chap. II]) Let $f \in \mathcal{D}(R^n)$ and $\alpha$ be a nonzero multi-index. Then the following two conditions are equivalent:*

(I) *There exists $u \in \mathcal{D}(R^n)$ such that $D^\alpha u = f$.*

(II)

$$\int_{-\infty}^{\infty} f(x_1, \cdots, x_j, \cdots, x_n)x_j^\ell dx_j = 0$$

*for $j \in M_\alpha, \ell = 0, \cdots, \alpha_j - 1$ and any $(x_1, \cdots, 0, \cdots, x_n) \in R^{n,j}$.*

Secondly, we need the following fact: For a nonnegative integer $m$, there exist functions $\{\theta_j\}_{j=0,1,\cdots,m} \subset \mathcal{D}(R^1)$ which satisfy the condition

$$\int_{-\infty}^{\infty} \theta_j(t)t^i dt = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \qquad (j, i = 0, 1, \cdots, m). \tag{2.1}$$

The first step is

**Lemma 2.2.** *For a nonnegative integer $m$ there exists a function $\theta(t) \in \mathcal{D}(R^1)$ such that*

$$\int_{-\infty}^{\infty} \theta(t)t^i dt = \begin{cases} 1, & i = 0 \\ 0, & i = 1, \cdots, m. \end{cases} \tag{2.2}$$

*Proof.* We take a function $\eta(t) \in \mathcal{D}(R^1)$ such that

$$\int_{-\infty}^{\infty} \eta(t)dt = 1. \tag{2.3}$$

We put

$$\theta(t) = \eta(t) + \sum_{j=1}^{m} c_j \eta^{(j)}(t)$$

where $\eta^{(j)}$ is the derivative of order $j$ of $\eta$ and $c_j$ $(j = 1, \cdots, m)$ are constants. We show that we can choose $c_j$ $(j = 1, \cdots, m)$ such that $\theta(t)$ satisfies (2.2). Since $\int_{-\infty}^{\infty} \eta^{(j)}(t)dt = 0$ $(j = 1, \cdots, m)$, by (2.3) we have

$$\int_{-\infty}^{\infty} \theta(t)dt = 1.$$

Hence we must choose the constants $c_j$ $(j = 1, \cdots, m)$ which satisfy

$$\sum_{j=1}^{m} c_j \int_{-\infty}^{\infty} \eta^{(j)}(t)t^i dt = -\int_{-\infty}^{\infty} \eta(t)t^i dt, \qquad i = 1, 2, \cdots, m. \tag{2.4}$$

We show that the linear equation (2.4) with respect to $c_j$ $(j = 1, 2, \cdots, m)$ has a solution. We consider the coefficient matrix

$$A = \left( \int_{-\infty}^{\infty} \eta^{(j)}(t)t^i dt \right)_{i,j=1,\cdots,m}.$$

We see that for $j > i$

$$\int_{-\infty}^{\infty} \eta^{(j)} t^i dt = (-1)^i i! \int_{-\infty}^{\infty} \eta^{(j-i)}(t)dt = 0$$

and

$$\int_{-\infty}^{\infty} \eta^{(j)}(t)t^j dt = (-1)^j j! \int_{-\infty}^{\infty} \eta(t)dt = (-1)^j j!.$$

Therefore the matrix $A$ is a triangular matrix and the diagonal elements are $(-1)^j j!$ $(j = 1, \cdots, m)$. Hence the determinant of $A$ is not zero. Consequently the linear equation (2.4) has a solution. Thus we obtain the lemma. □

Using Lemma 2.2 we prove

**Lemma 2.3.** *For a nonnegative integer $m$, there exist functions $\theta_j(t) \in \mathcal{D}(R^1)$ $(j = 0, 1, \cdots, m)$ which satisfy* (2.1).

*Proof.* By Lemma 2.2 there exists a function $\theta(t) \in \mathcal{D}(R^1)$ which satisfies (2.2). We put

$$\theta_j(t) = \frac{(-1)^j}{j!}\theta^{(j)}(t), \qquad j = 0, 1, \cdots, m.$$

Then for $j > i$ by integration by parts we have

$$\int_{-\infty}^{\infty} \theta_j(t)t^i dt = \frac{(-1)^j}{j!}\int_{-\infty}^{\infty} \theta^{(j)}(t)t^i dt = \frac{(-1)^{j+i}i!}{j!}\int_{-\infty}^{\infty} \theta^{(j-i)}(t)dt = 0.$$

For $j < i$ $(\leq m)$ it follows from integration by parts and (2.2) that

$$\int_{-\infty}^{\infty} \theta_j(t)t^i dt = (-1)^{2j}\begin{pmatrix} i \\ j \end{pmatrix}\int_{-\infty}^{\infty} \theta(t)t^{i-j} dt = 0$$

where $\begin{pmatrix} i \\ j \end{pmatrix} = \frac{i!}{j!(i-j)!}$. Moreover, for $j = i$ by integration by parts and (2.2) we see that

$$\int_{-\infty}^{\infty} \theta_j(t)t^j dt = (-1)^{2j}\frac{j!}{j!}\int_{-\infty}^{\infty} \theta(t)dt = 1.$$

Thus the functions $\theta_j$ $(j = 0, 1, \cdots, m)$ satisfy (2.1). The lemma was proved. □

From now on let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a nonzero multi-index with $M_\alpha = \{j_1, \cdots, j_k\}$ and $m = \max_{i=1,\cdots,n}(\alpha_i - 1)$. For the nonnegative integer $m$ we take the functions $\{\theta_j(t)\}_{j=0,1,\cdots,m} \subset \mathcal{D}(R^1)$ in Lemma 2.3.

In the decomposition (1.1) of $\varphi \in \mathcal{D}(R^n)$, $\lambda(x_2, \cdots, x_n)$ is given by

$$\lambda(x_2, \cdots, x_n) = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n)dx_1.$$

Indeed, if we put

$$\chi(x) = \varphi(x) - \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) dx_1 \theta_0(x_1),$$

then

$$\int_{-\infty}^{\infty} \chi(x_1, \cdots, x_n) dx_1$$
$$= \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) dx_1 - \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) dx_1 \int_{-\infty}^{\infty} \theta_0(x_1) dx_1 = 0$$

because of $\int_{-\infty}^{\infty} \theta_0(x_1) dx_1 = 1$. Hence by Lemma 2.1 we see that $\chi \in \mathcal{D}^1(R^n)$.

Taking the above situation into account, we introduce linear operators $T^\alpha$ and $U^\alpha$ on $\mathcal{D}(R^n)$. First, for $\varphi \in \mathcal{D}(R^n)$ and $1 \leq i \leq n, 0 \leq \ell \leq m$, we put

$$S_{i,\ell}\varphi(x) = \sum_{j=0}^{\ell} \left( \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^j dx_i \right) \theta_j(x_i).$$

We note that $S_{i,\ell}\varphi \in \mathcal{D}(R^n)$. For $1 \leq p \leq k$, let $M_{\alpha,p}$ denote the collection of subsets of $M_\alpha$ which have $p$ elements. For $\{i_1, \cdots, i_p\} \in M_{\alpha,p}$ we set

$$S^{\alpha(i_1, \cdots, i_p)} = S_{i_1, \alpha_{i_1}-1} \cdots S_{i_p, \alpha_{i_p}-1}.$$

For $\varphi \in \mathcal{D}(R^n)$, by Fubini's theorem $S^{\alpha(i_1, \cdots, i_p)}\varphi(x)$ is given by

$$S^{\alpha(i_1, \cdots, i_p)}\varphi(x) = \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_p=0}^{\alpha_{i_p}-1} \tag{2.5}$$

$$\left( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p} \right) \theta_{s_1}(x_{i_1}) \cdots \theta_{s_p}(x_{i_p}).$$

In particular, the order in the definition of $S^{\alpha(i_1, \cdots, i_p)}$ is irrelevant.

Next, for $1 \leq i \leq n$ and $0 \leq \ell \leq m$ we set

$$T_{i,\ell} = I - S_{i,\ell}.$$

Further we define

$$T^\alpha = T_{j_1, \alpha_{j_1}-1} \cdots T_{j_k, \alpha_{j_k}-1}.$$

It follows from the definition that

$$T^{\alpha} = (I - S_{j_1,\alpha_{j_1}-1}) \cdots (I - S_{j_k,\alpha_{j_k}-1}) \tag{2.6}$$
$$= I - \sum_{p=1}^{k} (-1)^{p+1} \sum_{\{i_1,\cdots,i_p\} \in M_{\alpha,p}} S^{\alpha(i_1,\cdots,i_p)}.$$

Finally we define

$$U^{\alpha} = I - T^{\alpha} = \sum_{p=1}^{k} (-1)^{p+1} \sum_{\{i_1,\cdots,i_p\} \in M_{\alpha,p}} S^{\alpha(i_1,\cdots,i_p)}. \tag{2.7}$$

For a subset $\{k_1, \cdots, k_s\} \subset \{1, \cdots, n\}$, the notation $f(\{x_{k_1}, \cdots, x_{k_s}\}^c)$ stands for a function of the remaining variables of $\{x_{k_1}, \cdots, x_{k_s}\}$. For example, $f(\{x_1\}^c) = f(x_2, \cdots, x_n)$.

Referring to (2.5) and (2.7) we define tensor product functions of order $\alpha$ (associated with $\{\theta_j\}_{j=0,1,\cdots,m}$) as follows. If a function $f \in \mathcal{D}(R^n)$ which has the following form

$$f(x) = \sum_{p=1}^{k} (-1)^{p+1} \sum_{\{i_1,\cdots,i_p\} \in M_{\alpha,p}} \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_p=0}^{\alpha_{i_p}-1} \tag{2.8}$$
$$\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c) \theta_{s_1}(x_{i_1}) \cdots \theta_{s_p}(x_{i_p})$$

satisfies the conditions

$$\text{(i)} \quad \lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p} \in \mathcal{D}(R^{n-p}), \tag{2.9}$$

$$\text{(ii)} \quad \text{for} \quad 2 \le p \le k, \{i_1, \cdots, i_p\} \in M_{\alpha,p} \text{ and} \tag{2.10}$$

$0 \le s_1 \le \alpha_{i_1} - 1, \cdots, 0 \le s_p \le \alpha_{i_p} - 1,$

$$\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \lambda_{i_\ell;s_\ell}(\{x_{i_\ell}\}^c) x_{i_1}^{s_1} \cdots \widehat{x_{i_\ell}^{s_\ell}} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots \widehat{dx_{i_\ell}} \cdots dx_{i_p}$$

$(\ell = 1, 2, \cdots, p)$,
then we call $f$ a tensor product function of order $\alpha$, where the symbol $\widehat{\ }$ indicates that the variable underneath is deleted. We denote by $\mathcal{U}^{\alpha}(R^n)$ the set of all tensor product functions of order $\alpha$.

A fundamental property of tensor product functions of order $\alpha$ is the following.

**Lemma 2.4.** *Let $f$ be a tensor product function of order $\alpha$ with the form* (2.8). *Then*

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(x_1,\cdots,x_n)x_{k_1}^{t_1}\cdots x_{k_q}^{t_q}dx_{k_1}\cdots dx_{k_q}$$
$$= \lambda_{k_1,\cdots,k_q;t_1,\cdots,t_q}(\{x_{k_1},\cdots,x_{k_q}\}^c)$$

*for* $1 \le q \le k, \{k_1,\cdots,k_q\} \in M_{\alpha,q}$ *and* $0 \le t_1 \le \alpha_{k_1}-1,\cdots,0 \le t_q \le \alpha_{k_q}-1$.

*Proof.* First we prove

$$\int_{-\infty}^{\infty} f(x_1,\cdots,x_j,\cdots,x_n)x_j^t dx_j = \lambda_{j;t}(\{x_j\}^c) \tag{2.11}$$

for $j = j_1,\cdots,j_k$ and $t = 0,1,\cdots,\alpha_j-1$. For $\{i_1,\cdots,i_p\} \in M_{\alpha,p}$ we put

$$I^{i_1,\cdots,i_p}(\{x_j\}^c) = \sum_{s_1=0}^{\alpha_{i_1}-1}\cdots\sum_{s_p=0}^{\alpha_{i_p}-1}\int_{-\infty}^{\infty}\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c)$$
$$\times\theta_{s_1}(x_{i_1})\cdots\theta_{s_p}(x_{i_p})x_j^t dx_j.$$

Then we have

$$\int_{-\infty}^{\infty} f(x_1,\cdots,x_j,\cdots,x_n)x_j^t dx_j$$
$$= \sum_{p=1}^{k}(-1)^{p+1}\sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}} I^{i_1,\cdots,i_p}(\{x_j\}^c)$$
$$= \sum_{p=1}^{k-1}(-1)^{p+1}(I_p(\{x_j\}^c)+J_p(\{x_j\}^c)) + (-1)^{k+1}I_k(\{x_j\}^c)$$

where

$$I_p(\{x_j\}^c) = \sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p},j\in\{i_1,\cdots,i_p\}} I^{i_1,\cdots,i_p}(\{x_j\}^c) \qquad (p=1,\cdots,k)$$

and

$$J_p(\{x_j\}^c) = \sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p},j\notin\{i_1,\cdots,i_p\}} I^{i_1,\cdots,i_p}(\{x_j\}^c) \qquad (p=1,\cdots,k-1).$$

We show that for $p = 1,2,\cdots,k-1$

$$J_p(\{x_j\}^c) = I_{p+1}(\{x_j\}^c). \tag{2.12}$$

First we consider $J_p(\{x_j\}^c)$. For $j \notin \{i_1, \cdots, i_p\}$, let $i_{\ell-1} < j < i_\ell$ ($\ell = 1, 2, \cdots, p+1$) with $i_0 = 0$ and $i_{p+1} = n+1$. Then by (2.10)

$$\int_{-\infty}^{\infty} \lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c)x_j^t dx_j$$
$$= \lambda_{i_1,\cdots,i_{\ell-1},j,i_\ell,\cdots,i_p;s_1,\cdots,s_{\ell-1},t,s_\ell,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_{\ell-1}},x_j,x_{i_\ell},\cdots,x_{i_p}\}^c)$$

where

$$\lambda_{i_1,\cdots,i_{\ell-1},j,i_\ell,\cdots,i_p;s_1,\cdots,s_{\ell-1},t,s_\ell,\cdots,s_p} = \begin{cases} \lambda_{j,i_1,\cdots,i_p;t,s_1,\cdots,s_p} & \text{if } \ell = 1, \\ \lambda_{i_1,\cdots,i_p,j;s_1,\cdots,s_p,t} & \text{if } \ell = p+1. \end{cases}$$

Hence

$$J_p(\{x_j\}^c) = \sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}, j\notin\{i_1,\cdots,i_p\}} \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_p=0}^{\alpha_{i_p}-1} \tag{2.13}$$
$$\lambda_{i_1,\cdots,i_{\ell-1},j,i_\ell,\cdots,i_p;s_1,\cdots,s_{\ell-1},t,s_\ell,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_{\ell-1}},x_j,x_{i_\ell},\cdots,x_{i_p}\}^c)$$
$$\times\theta_{s_1}(x_{i_1})\cdots\theta_{s_p}(x_{i_p})$$

with $i_{\ell-1} < j < i_\ell$. Next we consider $I_{p+1}(\{x_j\}^c)$. For $\{i_1, \cdots, i_{p+1}\} \in M_{\alpha,p+1}$ with $\{i_1, \cdots, i_{p+1}\} \ni j$, let $j = i_\ell$. Since

$$\int_{-\infty}^{\infty} \theta_{s_\ell}(x_{i_\ell})x_j^t dx_j = \int_{-\infty}^{\infty} \theta_{s_\ell}(x_j)x_j^t dx_j = \begin{cases} 1, & s_\ell = t, \\ 0, & s_\ell \neq t \end{cases}$$

by (2.1), we have

$$I^{i_1,\cdots,i_{p+1}}(\{x_j\}^c)$$
$$= \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_{p+1}=0}^{\alpha_{i_{p+1}}-1} \int_{-\infty}^{\infty} \lambda_{i_1,\cdots,i_{p+1};s_1,\cdots,s_{p+1}}(\{x_{i_1},\cdots,x_{i_{p+1}}\}^c)$$
$$\times\theta_{s_1}(x_{i_1})\cdots\theta_{s_{p+1}}(x_{i_{p+1}})x_j^t dx_j$$
$$= \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_{p+1}=0}^{\alpha_{i_{p+1}}-1} \lambda_{i_1,\cdots,i_{p+1};s_1,\cdots,s_{p+1}}(\{x_{i_1},\cdots,x_{i_{p+1}}\}^c)$$
$$\times\theta_{s_1}(x_{i_1})\cdots\overbrace{\theta_{s_\ell}(x_{i_\ell})}\cdots\theta_{s_{p+1}}(x_{i_{p+1}})\int_{-\infty}^{\infty} \theta_{s_\ell}(x_j)x_j^t dx_j$$
$$= \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_{\ell-1}=0}^{\alpha_{i_{\ell-1}}-1} \sum_{s_{\ell+1}=0}^{\alpha_{i_{\ell+1}}-1} \cdots \sum_{s_{p+1}=0}^{\alpha_{i_{p+1}}-1}$$
$$\lambda_{i_1,\cdots,i_\ell,\cdots,i_{p+1};s_1,\cdots,s_{\ell-1},t,s_{\ell+1},\cdots,s_{p+1}}(\{x_{i_1},\cdots,x_{i_\ell},\cdots,x_{i_{p+1}}\}^c)$$

$$\times\theta_{s_1}(x_{i_1})\cdots\theta_{s_{\ell-1}}(x_{i_{\ell-1}})\theta_{s_{\ell+1}}(x_{i_{\ell+1}})\cdots\theta_{s_{p+1}}(x_{i_{p+1}}).$$

By putting $u_1 = s_1, \cdots, u_{\ell-1} = s_{\ell-1}, u_\ell = s_{\ell+1}, \cdots, u_p = s_{p+1}$, we have

$$I^{i_1,\cdots,i_{p+1}}(\{x_j\}^c) = \sum_{u_1=0}^{\alpha_{i_1}-1}\cdots\sum_{u_{\ell-1}=0}^{\alpha_{i_{\ell-1}}-1}\sum_{u_\ell=0}^{\alpha_{i_{\ell+1}}-1}\cdots\sum_{u_p=0}^{\alpha_{i_{p+1}}-1}$$

$$\lambda_{i_1,\cdots,i_\ell,\cdots,i_{p+1};u_1,\cdots,u_{\ell-1},t,u_\ell,\cdots,u_p}(\{x_{i_1},\cdots,x_{i_\ell},\cdots,x_{i_{p+1}}\}^c)$$
$$\times\theta_{u_1}(x_{i_1})\cdots\theta_{u_{\ell-1}}(x_{i_{\ell-1}})\theta_{u_\ell}(x_{i_{\ell+1}})\cdots\theta_{u_p}(x_{i_{p+1}})$$

with $i_\ell = j$. Moreover, for $\{i_1,\cdots,i_{p+1}\}\in M_{\alpha,p+1}$ with $\{i_1,\cdots,i_{p+1}\}\ni j = i_\ell$ we put

$$m_1 = i_1,\cdots,m_{\ell-1} = i_{\ell-1}, m_\ell = i_{\ell+1},\cdots,m_p = i_{p+1}.$$

We denote $M_{\alpha,p,j} = \{\{i_1,\cdots,i_p\}\in M_{\alpha,p} : j\in\{i_1,\cdots,i_p\}\}$ and $M^c_{\alpha,p,j} = \{\{i_1,\cdots,i_p\}\in M_{\alpha,p} : j\notin\{i_1,\cdots,i_p\}\}$. We note that the above correspondence $\{i_1,\cdots,i_{p+1}\}\to\{m_1,\cdots,m_p\}$ is a one-to-one and onto mapping from $M_{\alpha,p+1,j}$ to $M^c_{\alpha,p,j}$. Hence

$$(2.14)\qquad I_{p+1}(\{x_j\}^c)$$

$$= \sum_{\{i_1,\cdots,i_{p+1}\}\in M_{\alpha,p+1}, j\in\{i_1,\cdots,i_{p+1}\}}\ \sum_{u_1=0}^{\alpha_{i_1}-1}\cdots\sum_{u_{\ell-1}=0}^{\alpha_{i_{\ell-1}}-1}\sum_{u_\ell=0}^{\alpha_{i_{\ell+1}}-1}\cdots\sum_{u_p=0}^{\alpha_{i_{p+1}}-1}$$
$$\lambda_{i_1,\cdots,i_\ell,\cdots,i_{p+1};u_1,\cdots,u_{\ell-1},t,u_\ell,\cdots,u_p}(\{x_{i_1},\cdots,x_{i_\ell},\cdots,x_{i_{p+1}}\}^c)$$
$$\times\theta_{u_1}(x_{i_1})\cdots\theta_{u_{\ell-1}}(x_{i_{\ell-1}})\theta_{u_\ell}(x_{i_{\ell+1}})\cdots\theta_{u_p}(x_{i_{p+1}})$$
$$= \sum_{\{m_1,\cdots,m_p\}\in M_{\alpha,p}, j\notin\{m_1,\cdots,m_p\}}\ \sum_{u_1=0}^{\alpha_{m_1}-1}\cdots\sum_{u_p=0}^{\alpha_{m_p}-1}$$
$$\lambda_{m_1,\cdots,m_{\ell-1},j,m_\ell,\cdots,m_p;u_1,\cdots,u_{\ell-1},t,u_\ell,\cdots,u_p}(\{x_{m_1},\cdots,x_{m_{\ell-1}},x_j,$$
$$x_{m_\ell},\cdots,x_{m_p}\}^c)\theta_{u_1}(x_{m_1})\cdots\theta_{u_p}(x_{m_p})$$

with $i_\ell = j$ and $m_{\ell-1} < j < m_\ell$. Comparing (2.13) and (2.14) we obtain (2.12). Therefore, by (2.1)

$$\int_{-\infty}^{\infty} f(x_1,\cdots,x_j,\cdots,x_n)x_j^t dt = I_1(\{x_j\}^c)$$
$$= \sum_{s=0}^{\alpha_j-1}\int_{-\infty}^{\infty}\lambda_{j;s}(\{x_j\}^c)\theta_s(x_j)x_j^t dx_j = \lambda_{j;t}(\{x_j\}^c).$$

Thus we obtain (2.11). Next, for $2 \leq q \leq k, \{k_1, \cdots, k_q\} \in M_{\alpha,q}$ and $0 \leq t_1 \leq \alpha_{k_1} - 1, \cdots, 0 \leq t_q \leq \alpha_{k_q} - 1$, by (2.10) and (2.11) we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \cdots, x_n) x_{k_1}^{t_1} \cdots x_{k_q}^{t_q} dx_{k_1} \cdots dx_{k_q} \\
&\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x_1, \cdots, x_n) x_{k_1}^{t_1} dx_{k_1} \right) x_{k_2}^{t_2} \cdots x_{k_q}^{t_q} dx_{k_2} \cdots dx_{k_q} \\
&\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \lambda_{k_1;t_1}(\{x_{k_1}\}^c) x_{k_2}^{t_2} \cdots x_{k_q}^{t_q} dx_{k_2} \cdots dx_{k_q} \\
&\quad = \lambda_{k_1,k_2,\cdots,k_q;t_1,t_2,\cdots,t_q}(\{x_{k_1}, x_{k_2}, \cdots, x_{k_q}\}^c).
\end{aligned}$$

We have completed the proof of the lemma. □

We state properties of the operators $T^\alpha$ and $U^\alpha$ which are important for a direct sum decomposition of $\mathcal{D}(R^n)$. We denote by $e_j$ the multi-index which has 1 in the $j$th spot and 0 everywhere else $(j = 1, 2, \cdots, n)$.

**Lemma 2.5.** *Let $\gamma = (\gamma_1, \cdots, \gamma_n)$ be a multi-index with $\gamma_j = 0$ and $0 \leq \ell \leq m$. Then for $\varphi \in \mathcal{D}^\gamma(R^n)$, $T_{j,\ell}\varphi \in \mathcal{D}^{\gamma+(\ell+1)e_j}(R^n)$.*

*Proof.* We note that $j \notin M_\gamma$ by the condition $\gamma_j = 0$. Let $\varphi \in \mathcal{D}^\gamma(R^n)$. In order to show that $T_{j,\ell}\varphi \in \mathcal{D}^{\gamma+(\ell+1)e_j}(R^n)$, by Lemma 2.1 it suffices to prove

$$\int_{-\infty}^{\infty} T_{j,\ell}\varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^s dx_i = 0$$

for $s = 0, 1, \cdots, \gamma_i - 1$ if $i \in M_\gamma$ and $s = 0, 1, \cdots, \ell$ if $i = j$. First, let $i \in M_\gamma$. Then by the condition $\varphi \in \mathcal{D}^\gamma(R^n)$ and Lemma 2.1 we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} T_{j,\ell}\varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^s dx_i \\
&\quad = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^s dx_i \\
&\qquad - \int_{-\infty}^{\infty} \left( \sum_{t=0}^{\ell} \left( \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) x_j^t dx_j \right) \theta_t(x_j) \right) x_i^s dx_i \\
&\quad = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_n) x_i^s dx_i \\
&\qquad - \sum_{t=0}^{\ell} \left( \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) x_i^s dx_i \right) x_j^t dx_j \right) \theta_t(x_j) \\
&\quad = 0
\end{aligned}$$

for $s = 0, 1, \cdots, \gamma_i - 1$. Next, let $i = j$. Then for $s = 0, 1, \cdots, \ell$ we see that

$$
\begin{aligned}
&\int_{-\infty}^{\infty} T_{j,\ell}\varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j \\
&\quad = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j \\
&\qquad - \int_{-\infty}^{\infty} \left( \sum_{t=0}^{\ell} \left( \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^t dx_j \right) \theta_t(x_j) \right) x_j^s dx_j \\
&\quad = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j \\
&\qquad - \sum_{t=0}^{\ell} \left( \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^t dx_j \right) \int_{-\infty}^{\infty} \theta_t(x_j) x_j^s dx_j \\
&\quad = \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j - \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_j, \cdots, x_n) x_j^s dx_j \\
&\quad = 0
\end{aligned}
$$

because the functions $\{\theta_j\}_{j=0,1,\cdots,m}$ satisfy (2.1). Thus we obtain the lemma. □

**Lemma 2.6.** (i) *If $\varphi \in \mathcal{D}(R^n)$, then $T^\alpha \varphi \in \mathcal{D}^\alpha(R^n)$.*
(ii) *If $\varphi \in \mathcal{D}(R^n)$, then $U^\alpha \varphi \in \mathcal{U}^\alpha(R^n)$.*

*Proof.* (i) By using Lemma 2.5 repeatedly we obtain (i).
(ii) Let $\varphi \in \mathcal{D}(R^n)$. By (2.5) and (2.7) $U^\alpha \varphi$ has the following form:

$$
U^\alpha \varphi(x) = \sum_{p=1}^{k} (-1)^{p+1} \sum_{\{i_1, \cdots, i_p\} \in M_{\alpha,p}} \sum_{s_1=0}^{\alpha_{i_1}-1} \cdots \sum_{s_p=0}^{\alpha_{i_p}-1}
$$

$$
\left( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p} \right) \theta_{s_1}(x_{i_1}) \cdots \theta_{s_p}(x_{i_p}).
$$

For $\{i_1, \cdots, i_p\} \in M_{\alpha,p}$ and $0 \le s_1 \le \alpha_{i_1} - 1, \cdots, 0 \le s_p \le \alpha_{i_p} - 1$ we set

$$
\begin{aligned}
&\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1}, \cdots, x_{i_p}\}^c) \\
&\quad = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(x_1, \cdots, x_n) x_{i_1}^{s_1} \cdots x_{i_p}^{s_p} dx_{i_1} \cdots dx_{i_p}.
\end{aligned}
$$

It is clear that $\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p} \in \mathcal{D}(R^{n-p})$. Moreover, for $2 \le p \le k$ and $1 \le \ell \le p$ we have

$$\begin{aligned}
&\lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c) \\
&\quad = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\varphi(x_1,\cdots,x_{i_\ell},\cdots,x_n)x_{i_\ell}^{s_\ell}dx_{i_\ell}\right) \\
&\qquad\qquad\qquad \times x_{i_1}^{s_1}\cdots\widehat{x_{i_\ell}^{s_\ell}}\cdots x_{i_p}^{s_p}dx_{i_1}\cdots\widehat{dx_{i_\ell}}\cdots dx_{i_p} \\
&\quad = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\lambda_{i_\ell;s_\ell}(\{x_{i_\ell}\}^c)x_{i_1}^{s_1}\cdots\widehat{x_{i_\ell}^{s_\ell}}\cdots x_{i_p}^{s_p}dx_{i_1}\cdots\widehat{dx_{i_\ell}}\cdots dx_{i_p}.
\end{aligned}$$

Thus $U^\alpha\varphi$ satisfies (2.9) and (2.10), and hence $U^\alpha\varphi$ is a tensor product function of order $\alpha$. The lemma was proved. □

**Lemma 2.7.** (i) *If $\varphi \in \mathcal{D}^\alpha(R^n)$, then $T^\alpha\varphi = \varphi$.*
(ii) *If $\varphi \in \mathcal{U}^\alpha(R^n)$, then $U^\alpha\varphi = \varphi$.*

*Proof.* (i) Let $\varphi \in \mathcal{D}^\alpha(R^n)$. For $p = 1, 2, \cdots, k$ and $\{i_1,\cdots,i_p\} \in M_{\alpha,p}$, since $S^{\alpha(i_1,\cdots,i_p)}\varphi$ is given by (2.5), we see that $S^{\alpha(i_1,\cdots,i_p)}\varphi = 0$ by the condition $\varphi \in \mathcal{D}^\alpha(R^n)$, Lemma 2.1 and Fubini's theorem. Hence (2.6) implies that $T^\alpha\varphi = \varphi$.

(ii) Let $\varphi \in \mathcal{U}^\alpha(R^n)$ and $\varphi$ have the form (2.8). Then by (2.5), (2.7) and Lemma 2.4 we have

$$\begin{aligned}
U^\alpha\varphi(x) &= \sum_{p=1}^{k}(-1)^{p+1}\sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}}\sum_{s_1=0}^{\alpha_{i_1}-1}\cdots\sum_{s_p=0}^{\alpha_{i_p}-1} \\
&\qquad \left(\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(x_1,\cdots,x_n)x_{i_1}^{s_1}\cdots x_{i_p}^{s_p}dx_{i_1}\cdots dx_{i_p}\right)\theta_{s_1}(x_{i_1})\cdots\theta_{s_p}(x_{i_p}) \\
&= \sum_{p=1}^{k}(-1)^{p+1}\sum_{\{i_1,\cdots,i_p\}\in M_{\alpha,p}}\sum_{s_1=0}^{\alpha_{i_1}-1}\cdots\sum_{s_p=0}^{\alpha_{i_p}-1} \\
&\qquad\qquad \lambda_{i_1,\cdots,i_p;s_1,\cdots,s_p}(\{x_{i_1},\cdots,x_{i_p}\}^c)\theta_{s_1}(x_{i_1})\cdots\theta_{s_p}(x_{i_p}) \\
&= \varphi(x).
\end{aligned}$$

Hence we obtain (ii). □

Now we establish our main result.

**Theorem 2.1.** $\mathcal{D}(R^n) = \mathcal{D}^\alpha(R^n) \oplus \mathcal{U}^\alpha(R^n)$.

*Proof.* Since $\varphi = T^\alpha\varphi + U^\alpha\varphi$ for $\varphi \in \mathcal{D}(R^n)$, Lemma 2.6 gives $\mathcal{D}(R^n) = \mathcal{D}^\alpha(R^n) + \mathcal{U}^\alpha(R^n)$. Moreover, let $\varphi \in \mathcal{D}^\alpha(R^n) \cap \mathcal{U}^\alpha(R^n)$. Then Lemma 2.7 implies

$$\varphi = T^\alpha\varphi + U^\alpha\varphi = \varphi + \varphi = 2\varphi.$$

Hence $\varphi = 0$. Therefore $\mathcal{D}^\alpha(R^n) \cap \mathcal{U}^\alpha(R^n) = \{0\}$. Thus we obtain the theorem. □

*Remark* 2.1. We note that Lemmas 2.6 (i) and 2.7 (i) (resp. Lemmas 2.6 (ii) and 2.7 (ii)) imply $T^\alpha(\mathcal{D}(R^n)) = \mathcal{D}^\alpha(R^n)$ (resp. $U^\alpha(\mathcal{D}(R^n)) = \mathcal{U}^\alpha(R^n)$). Moreover, $(T^\alpha)^{-1}(0) = \mathcal{U}^\alpha(R^n)$ and $(U^\alpha)^{-1}(0) = \mathcal{D}^\alpha(R^n)$. We give the proof of $(U^\alpha)^{-1}(0) = \mathcal{D}^\alpha(R^n)$. Let $\varphi \in \mathcal{D}^\alpha(R^n)$. Then $U^\alpha\varphi = \varphi - T^\alpha\varphi = \varphi - \varphi = 0$ by Lemma 2.7 (i). Hence $\mathcal{D}^\alpha(R^n) \subset (U^\alpha)^{-1}(0)$. Conversely let $\varphi \in (U^\alpha)^{-1}(0)$. Then $0 = U^\alpha\varphi = \varphi - T^\alpha\varphi$. Hence $\varphi = T^\alpha\varphi \in \mathcal{D}^\alpha(R^n)$ by Lemma 2.6 (i). Therefore $(U^\alpha)^{-1}(0) \subset \mathcal{D}^\alpha(R^n)$. The proof of $(T^\alpha)^{-1}(0) = \mathcal{U}^\alpha(R^n)$ is the same.

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# Kato class functions of Markov processes under ultracontractivity

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***Dedicated to Professor Shintaro Nakao on his Sixtieth Birthday***

**Abstract.**

We show that $f \in L^p(X;m)$ implies $|f|dm \in S_K^1$ for $p > D$ with $D > 0$, where $S_K^1$ is a subfamily of Kato class measures relative to a semigroup kernel $p_t(x,y)$ of a Markov process associated with a (non-symmetric) Dirichlet form on $L^2(X;m)$. We only assume that $p_t(x,y)$ satisfies the Nash type estimate of small time depending on $D$. No concrete expression of $p_t(x,y)$ is needed for the result.

## §1. Introduction

A measurable function $f$ on $\mathbb{R}^d$ is said to be in the *Kato class* $K_d$ if

$$\begin{aligned}
\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{d-2}} dy &= 0 \text{ for } d \geq 3, \\
\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|x-y|<r} (\log |x-y|^{-1}) |f(y)| dy &= 0 \text{ for } d = 2, \\
\sup_{x\in\mathbb{R}^d} \int_{|x-y|<1} |f(y)| dy &< \infty \text{ for } d = 1.
\end{aligned}$$

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Let $\mathbf{M}^{\mathrm{w}} = (\Omega, B_t, P_x)_{x\in\mathbb{R}^d}$ be a $d$-dimensional Brownian motion on $\mathbb{R}^d$. The following theorem is shown in Aizenman and Simon [1]:

**Theorem 1.1** (Theorem 1.3(ii) in [1]). *$f \in K_d$ if and only if*

$$\sup_{x\in\mathbb{R}^d} E_x\left[\int_0^t |f(B_s)|ds\right] = \sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_0^t p_s(x,y)ds\right)|f(y)|dy \xrightarrow{t\to 0} 0,$$

*where $p_t(x,y) := \frac{1}{(2\pi t)^{d/2}}\exp[-\frac{|x-y|^2}{2t}]$ is the heat kernel of $\mathbf{M}^{\mathrm{w}}$.*

Zhao [13] extends this in more general setting including a subclass of Lévy processes, but his result does not assure the low dimensional case even if the process is $\mathbf{M}^{\mathrm{w}}$. The following is also shown in [1]:

**Theorem 1.2** (cf. Theorem 1.4(iii) in [1]). *$L^p(\mathbb{R}^d) \subset K_d$ holds if $p > d/2$ with $d \geq 2$, or $p \geq 1$ with $d = 1$.*

Note that there is an $f \in L^{d/2}(\mathbb{R}^d) \setminus K_d$ for $d \geq 2$. Indeed, taking $g \in C_0([0, 2/e[ \to [0,\infty])$ with $g(r) := 1/(r^2 \log r^{-1})$ if $d \geq 3$, $:= 1/(r^2(\log r^{-1})^{1+\varepsilon})$, $\varepsilon \in ]0,1[$ if $d = 2$ for $r \in [0, 1/e]$, $f(x) := g(|x|)$ does the job through the proof of Proposition 4.10 in [1]. Here (4.10) in [1] should be changed to $\int_0^{1/e} r(\log r^{-1})|V(r)|dr < \infty$ if $d = 2$.

In the framework of strongly local regular Dirichlet forms with the notions of volume doubling and weak Poincaré inequality, Biroli and Mosco [3] gave a similar result with Theorem 1.2 (see Proposition 3.7 in [3]). Their definition of Kato class depends on the volume growth of balls. The purpose of this note is to show that Theorem 1.2 holds true in more general context replacing $K_d$ with $S^1_K$ the family of Kato class smooth measures in the strict sense in terms of semigroup kernel of Markov processes associated with (non-symmetric) Dirichlet forms (see Theorem 2.1 below).

Finally we will announce the content of [10]. In [10], we extend Theorem 1.1, that is, under some conditions, we establish $K_{d,\beta} = S^1_K$ in the framework of symmetric Markov processes which admits a semigroup kernel possessing upper and lower estimates, which includes the low dimensional case. Here $K_{d,\beta}$ is the family of Kato class measures in terms of a Green kernel depending on $d, \beta > 0$. In particular, Theorem 2.1 below can be strengthened by replacing $L^p(X;m)$ with $L^p_{\mathrm{unif}}(X;m)$.

## §2. Result

Let $X$ be a locally compact separable metric space and $m$ a positive Radon measure with full support. Let $X_\Delta := X \cup \{\Delta\}$ be a one point compactification of $X$. We consider and fix a (non-symmetric) regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(X;m)$. Then there exists a pair of Hunt processes $(\mathbf{M},\widehat{\mathbf{M}})$, $\mathbf{M}=(\Omega,X_t,\zeta,P_x)$, $\widehat{\mathbf{M}}=(\hat{\Omega},\hat{X}_t,\hat{\zeta},\hat{P}_x)$ such that for each Borel $u\in L^2(X;m)$, $T_tu(x)=E_x[u(X_t)]$ $m$-a.e. $x\in X$ and $\hat{T}_tu(x)=\hat{E}_x[u(\hat{X}_t)]$ $m$-a.e. $x\in X$ for all $t>0$, where $(T_t)_{t>0}$ (resp. $(\hat{T}_t)_{t>0}$) is the semigroup associated with $(\mathcal{E},\mathcal{F})$ (resp. $(\hat{\mathcal{E}},\mathcal{F})$), where $\hat{\mathcal{E}}(u,v):=\mathcal{E}(v,u)$ for $u,v\in\mathcal{F}$ is the dual form of $(\mathcal{E},\mathcal{F})$. Here $\zeta:=\inf\{t\ge 0\mid X_t=\Delta\}$ (resp. $\hat{\zeta}:=\inf\{t\ge 0\mid \hat{X}_t=\Delta\}$) denotes the life time of $\mathbf{M}$ (resp. $\widehat{\mathbf{M}}$). Further, we assume that there exists a kernel $p_t(x,y)$ defined for all $(t,x,y)\in]0,\infty[\times X\times X$ such that $E_x[u(X_t)]=P_tu(x):=\int_X p_t(x,y)u(y)m(dy)$ and $\hat{E}_x[u(\hat{X}_t)]=\hat{P}u(x):=\int_X \hat{p}_t(x,y)u(y)m(dy)$ for any $x\in X$, bounded Borel function $u$ and $t>0$, where $\hat{p}_t(x,y):=p_t(y,x)$. $p_t(x,y)$ is said to be a *semigroup kernel*, or sometimes called a *heat kernel* of $\mathbf{M}$ on the analogy of heat kernel of diffusions. Then $P_t$ and $\hat{P}_t$ can be extended to contractive semigroups on $L^p(X;m)$ for $p\ge 1$. The following are well-known:

(a) $p_{t+s}(x,y)=\displaystyle\int_X p_s(x,z)p_t(z,y)m(dz),\quad {}^\forall x,y\in X,\ {}^\forall t,s>0.$

(b) $p_t(x,dy)=p_t(x,y)m(dy),\quad {}^\forall x\in X,\ {}^\forall t>0.$

(c) $\displaystyle\int_X p_t(x,y)m(dy)\le 1,\quad {}^\forall x\in X,\ {}^\forall t>0.$

The same properties also hold for $\hat{p}_t(x,y)$.

**Definition 2.1** (Kato class $S_K^0$, Dynkin class $S_D^0$)**.** For a positive Borel measure $\mu$ on $X$, $\mu$ is said to be in *Kato class relative to the semigroup kernel* $p_t(x,y)$ (write $\mu\in S_K^0$) if

$$\lim_{t\to 0}\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy)=0 \tag{2.1}$$

and $\mu$ is said to be in *Dynkin class relative to the semigroup kernel* $p_t(x,y)$ (write $\mu\in S_D^0$) if

$$\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy)<\infty\ \text{ for } {}^\exists t>0. \tag{2.2}$$

Clearly, $S_K^0\subset S_D^0$. The notions $\hat{S}_K^0$ and $\hat{S}_D^0$ are similarly defined by replacing $p_t(x,y)$ with $\hat{p}_t(x,y)$.

**Definition 2.2** (Measures of finite energy integrals: $S_0$, $S_{00}$, cf. [6])**.** A Borel measure $\mu$ on $X$ is said to be *of finite energy integral* with respect to $(\mathcal{E},\mathcal{F})$ (write $\mu\in S_0$) if there exists $C>0$ such that

$$\int_X |v|d\mu\le C\sqrt{\mathcal{E}_1(v,v)},\quad {}^\forall v\in\mathcal{F}\cap C_0(X).$$

In that case, for every $\alpha > 0$, there exist $U_\alpha\mu, \hat{U}_\alpha\mu \in \mathcal{F}$ such that

$$\mathcal{E}_\alpha(U_\alpha\mu, v) = \mathcal{E}_\alpha(v, \hat{U}_\alpha\mu) = \int_X v(x)\mu(dx), \quad {}^\forall v \in \mathcal{F} \cap C_0(X).$$

Moreover we write $\mu \in S_{00}$ (resp. $\mu \in \hat{S}_{00}$) if $\mu(X) < \infty$ and $U_\alpha\mu \in \mathcal{F} \cap L^\infty(X;m)$ (resp. $\hat{U}_\alpha\mu \in \mathcal{F} \cap L^\infty(X;m)$) for some/all $\alpha > 0$.

**Definition 2.3** (Smooth measures in the strict sense: $S_1$, cf. [6])**.** A Borel measure $\mu$ on $X$ is said to be a *smooth measure in the strict sense* with respect to $(\mathcal{E}, \mathcal{F})$ (write $\mu \in S_1$) if there exists an increasing sequence $\{E_n\}$ of Borel sets such that $X = \bigcup_{n=1}^\infty E_n$, ${}^\forall n \in \mathbb{N}$, $I_{E_n}\mu \in S_{00}$ and $P_x(\lim_{n\to\infty} \sigma_{X\setminus E_n} \geq \zeta) = 1$, ${}^\forall x \in X$. Here $\zeta$ is the life time of $\mathbf{M}$. The family of *smooth measure in the strict sense* with respect to $(\hat{\mathcal{E}}, \mathcal{F})$ (write $\hat{S}_1$) can be similarly defined.

**Definition 2.4.** We define $S_K^1 := S_K^0 \cap S_1$, $S_D^1 := S_D^0 \cap S_1$, $\hat{S}_K^1 := \hat{S}_K^0 \cap \hat{S}_1$ and $\hat{S}_D^1 := \hat{S}_D^0 \cap \hat{S}_1$.

We fix $D > 0$ and assume the Nash type estimate: for each $t_0 > 0$ we have

$$(2.3) \quad {}^\exists C_{D,t_0} > 0 \text{ s.t. } \sup_{x,y\in X} p_t(x,y) \leq C_{D,t_0} t^{-D}, \quad {}^\forall t \in ]0, t_0[.$$

*Remark* 2.1. The condition (2.3) implies the following:

(a) ${}^\exists C_{D,t_0} > 0$ s.t. $||P_t||_{1\to\infty} \leq C_{D,t_0} t^{-D}$ for any $t \in ]0, t_0[$.

(b) For each $p \geq 1$, ${}^\exists C_{D,p,t_0} > 0$ s.t. $||P_t||_{p\to\infty} \leq C_{D,p,t_0} t^{-D/p}$ for any $t \in ]0, t_0[$.

If $(\mathcal{E}, \mathcal{F})$ is a symmetric Dirichlet form, (2.3) is equivalent to one (hence all) of (a),(b). If further $D > 1$, (2.3) is also equivalent to the Sobolev inequality (see [5]): there exists $C_D^* > 0$ and $\gamma > 0$ such that

(c) $||u||_{\frac{2D}{D-1}} \leq C_D^* \mathcal{E}_\gamma(u,u)$ for all $u \in \mathcal{F}$.

Next theorem extends Theorem 1.2 and the lower estimate of $p$ in this theorem is best possible as remarked after Theorem 1.2.

**Theorem 2.1.** *Suppose (2.3) and $p > D$ with $D \in [1, \infty[$ or $p \geq 1$ with $D \in ]0, 1[$. Then $f \in L^p(X;m)$ implies $|f|dm \in S_K^1 \cap \hat{S}_K^1$.*

## §3. Proof of Theorem 2.1

We set $r_\alpha(x,y) := \displaystyle\int_0^\infty e^{-\alpha t} p_t(x,y)dt$. First we show the following:

**Lemma 3.1.** *$\mu \in S_K^0$ is equivalent to*

$$\lim_{\alpha \to \infty} \sup_{x \in X} \int_X r_\alpha(x,y)\mu(dy) = 0 \tag{3.1}$$

*and $\mu \in S_D^0$ is equivalent to*

$$\sup_{x \in X} \int_X r_\alpha(x,y)\mu(dy) < \infty, \quad {}^{\exists}\alpha > 0. \tag{3.2}$$

*Proof.* We first show (2.1)⇒(3.1). Take $\alpha_0 > 0$ with $\alpha \geq \alpha_0$,

$$\begin{aligned}
&\int_X r_\alpha(x,y)\mu(dy) \\
&= \int_X \int_0^t e^{-\alpha s} p_s(x,y) ds\mu(dy) + \int_X \int_t^\infty e^{-\alpha s} p_s(x,y) ds\mu(dy) \\
&\leq \int_X \int_0^t p_s(x,y) ds\mu(dy) + e^{-(\alpha-\alpha_0)t} \int_X \int_t^\infty e^{-\alpha_0 s} p_s(x,y) ds\mu(dy).
\end{aligned}$$

Here

$$\begin{aligned}
\int_X \int_t^\infty e^{-\alpha_0 s} p_s(x,y) ds\mu(dy) &= \int_X \sum_{k=1}^\infty \int_{kt}^{(k+1)t} e^{-\alpha_0 s} p_s(x,y) ds\mu(dy) \\
&= \sum_{k=1}^\infty \int_X \int_0^t e^{-\alpha_0(u+kt)} p_{u+kt}(x,y) du\mu(dy).
\end{aligned}$$

Since $p_{u+kt}(x,y) = \displaystyle\int_X p_{kt}(x,z)p_u(z,y)m(dz)$,

$$\begin{aligned}
&\int_X \int_t^\infty e^{-\alpha_0 s} p_s(x,y) ds\mu(dy) \\
&\qquad = \sum_{k=1}^\infty e^{-\alpha_0 kt} \int_X p_{kt}(x,z) \int_X \int_0^t e^{-\alpha_0 u} p_u(z,y) du\mu(dy)m(dz) \\
&\qquad \leq \sum_{k=1}^\infty e^{-\alpha_0 kt} \int_X p_{kt}(x,z) \int_X \int_0^t p_u(z,y) du\mu(dy)m(dz).
\end{aligned}$$

From (2.1), $N_t := \displaystyle\sup_{z \in X} \int_X \int_0^t p_u(z,y) du\mu(dy) < \infty$. Then

$$\begin{aligned}
&\sup_{x \in X} \int_X r_\alpha(x,y)\mu(dy) \\
&\qquad \leq \sup_{x \in X} \int_X \int_0^t p_s(x,y) ds\mu(dy) + \frac{e^{-\alpha t}}{1 - e^{-\alpha_0 t}} N_t.
\end{aligned} \tag{3.3}$$

Therefore

$$\lim_{\alpha\to\infty}\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy)\le\sup_{x\in X}\int_X\int_0^t p_s(x,y)ds\mu(dy)\xrightarrow{t\to 0}0.$$

Next we show (3.1)⇒(2.1). We have

$$\sup_{x\in X}\int_X\int_0^t p_s(x,y)ds\mu(dy)\le e^{\alpha t}\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy). \tag{3.4}$$

Therefore

$$\lim_{t\to 0}\sup_{x\in X}\int_X\int_0^t p_s(x,y)ds\mu(dy)\le\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy)\xrightarrow{\alpha\to\infty}0.$$

The implications (3.2)⟺(2.2) are clear from (3.3) and (3.4). □

**Lemma 3.2.** *The following are equivalent to each other.*

(a) $\mu\in S_D^0$.

(b) $\displaystyle\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy)<\infty$ *for* $^\forall t>0$.

(c) $\displaystyle\sup_{x\in X}\int_X r_\alpha(x,y)\mu(dy)<\infty$ *for* $^\forall\alpha>0$.

*Proof.* We first show (a)⟹(b). Suppose that (a) holds for some $t_0>0$. For any $t>0$, we take $n\in\mathbb{N}$ with $t\le nt_0$. We have

$$\begin{aligned}
&\sup_{x\in X}\int_X\Big(\int_0^t p_s(x,y)ds\Big)\mu(dy)\\
&\qquad\le\sup_{x\in X}\sum_{k=1}^n\int_X p_{kt_0}(x,z)\Big(\int_0^{t_0}\int_X p_s(z,y)\mu(dy)ds\Big)m(dz)\\
&\qquad\le n\sup_{x\in X}\int_0^{t_0}\Big(\int_X p_s(x,y)\mu(dy)\Big)ds<\infty.
\end{aligned}$$

(b)⟹(c) is clear from (3.3) and (c)⟹(a) is clear. □

**Proposition 3.1.** *Suppose that $\mu\in S_D^0$ is a positive Radon measure on $X$. Then $\mu\in S_1$.*

*Proof.* It suffices to show that for a positive Radon measure $\mu\in S_D^0$, $I_K\mu\in S_0$ for any compact set $K$. Indeed, there exists an increasing sequence $\{G_n\}$ of relatively compact open set with $\bigcup_{n=1}^\infty G_n=X$. Then we see $I_{G_n}\mu\in S_{00}$ for each $n\in\mathbb{N}$, which implies $\mu\in S_1$ by Thoerem 5.1.7(iii) in [6]. Though the framework of Thoerem 5.1.7(iii) in [6] is symmetric, its proof only depends on the quasi-left-continuity of $\mathbf{M}$ and remains valid in the present context. We show $I_K\mu \in S_0$ for a compact set $K$. Fix $\alpha > 0$ and set $R_\alpha\mu(x) := \int_X r_\alpha(x,y)\mu(dy)$. First we show $R_\alpha(I_K\mu) \in L^2(X;m)$.

$$\begin{aligned} \|R_\alpha(I_K\mu)\|_2^2 &\le \|R_\alpha(I_K\mu)\|_\infty \|R_\alpha(I_K\mu)\|_1 \\ &= \|R_\alpha(I_K\mu)\|_\infty \langle I_K\mu, \hat{R}_\alpha 1\rangle \\ &= \frac{1}{\alpha}\|R_\alpha(I_K\mu)\|_\infty \mu(K) < \infty. \end{aligned}$$

Next we prove $R_\alpha(I_K\mu) \in \mathcal{F}$. It suffices to show

$$\sup_{\beta>0} \mathcal{E}_\alpha^{(\beta)}(R_\alpha(I_K\mu), R_\alpha(I_K\mu)) < \infty,$$

where $\mathcal{E}_\alpha^{(\beta)}(u,v) := \beta(u - \beta R_{\beta+\alpha}u, v)_m$ for $u, v \in L^2(X;m)$. Then

$$\begin{aligned} \sup_{\beta>0} \mathcal{E}_\alpha^{(\beta)}(R_\alpha(I_K\mu), R_\alpha(I_K\mu)) &= \sup_{\beta>0} \beta(R_{\beta+\alpha}(I_K\mu), R_\alpha(I_K\mu))_m \\ &= \|R_\alpha(I_K\mu)\|_\infty \sup_{\beta>0} \beta\langle I_K\mu, \hat{R}_{\beta+\alpha}1\rangle \\ &\le \|R_\alpha(I_K\mu)\|_\infty \mu(K) < \infty. \end{aligned}$$

Finally we prove $I_K\mu \in S_0$ and $R_\alpha(I_K\mu) = U_\alpha(I_K\mu)$. It suffices to show that for any $v \in \mathcal{F} \cap C_0(X)$

$$\begin{aligned} \mathcal{E}_\alpha(R_\alpha(I_K\mu), v) &= \lim_{\beta\to\infty} \mathcal{E}_\alpha^{(\beta)}(R_\alpha(I_K\mu), v) \\ &= \lim_{\beta\to\infty} \beta(R_{\beta+\alpha}(I_K\mu), v)_m \\ &= \lim_{\beta\to\infty} \beta\langle I_K\mu, \hat{R}_{\beta+\alpha}v\rangle = \langle I_K\mu, v\rangle, \end{aligned}$$

where we use the right continuity of the sample paths of $\widehat{\mathbf{M}}$. $\square$

*Proof of Theorem 2.1.* By duality, it suffices only to prove that $f \in L^p(X;m)$ implies $|f|dm \in S_K^1$. Take $p > D$ with $D \in [1,\infty[$ or $p \ge 1$ with $D \in ]0,1[$. Since $\|P_t\|_{p\to\infty} \le C_{D,p,t_0} t^{-D/p}$ for $t \in ]0, t_0[$, we have

$$\begin{aligned} &\sup_{x\in X} \int_X \Big(\int_0^t p_s(x,y)ds\Big)|f(y)|m(dy) \\ &\qquad = \sup_{x\in X} \int_0^t \Big(\int_X |f(y)|p_s(x,y)m(dy)\Big)ds \\ &\qquad \le C_{D,p,t_0}\|f\|_p \int_0^t s^{-D/p}ds \\ &\qquad = C_{D,p,t_0}\|f\|_p \frac{p}{p-D} t^{1-D/p} \xrightarrow{t\to 0} 0. \end{aligned}$$

Then $|f|dm \in S_K^0$. Since $|f|dm$ with $f \in L^p(X;m)$ is a Radon measure, we conclude $|f|dm \in S_1$ by Proposition 3.1. Therefore $|f|dm \in S_K^1$. □

## §4. Examples

**Example 4.1** (Symmetric $\alpha$-stable process). Take $\alpha \in ]0,2[$. Let $\mathbf{M}^\alpha = (\Omega, X_t, P_x)_{x\in\mathbb{R}^d}$ be the symmetric $\alpha$-stable process on $\mathbb{R}^d$, that is, Lévy process satisfying $E_0[e^{\sqrt{-1}\langle\xi,X_t\rangle}] = e^{-t|\xi|^\alpha}$. It is well-known that $\mathbf{M}^\alpha$ admits a semigroup kernel $p_t(x,y)$ satisfying the following (cf. [2],[7]): $^\exists C_i = C_i(\alpha,d) > 0$, $i=1,2$ such that for all $(t,x,y) \in ]0,\infty[\times\mathbb{R}^d\times\mathbb{R}^d$

$$\frac{C_1}{t^{d/\alpha}}\frac{1}{\left(1+\frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}} \le p_t(x,y) \le \frac{C_2}{t^{d/\alpha}}\frac{1}{\left(1+\frac{|x-y|}{t^{1/\alpha}}\right)^{d+\alpha}}.$$

Similar estimate holds for jump type process over $d$-sets (see [4]). In particular, there exists $C_2 = C_2(\alpha,d) > 0$ with $p_t(x,y) \le C_2 t^{-d/\alpha}$ for $(t,x,y) \in ]0,\infty[\times\mathbb{R}^d\times\mathbb{R}^d$. Then we have that $f \in L^p(\mathbb{R}^d)$ implies $|f(x)|dx \in S_K^1$ if $p > d/\alpha$ with $d \ge \alpha$, or $p \ge 1$ with $d < \alpha$.

**Example 4.2** (Relativistic Hamiltonian process). Let $\mathbf{M}^{\mathrm{H}}$ be the relativistic Hamiltonian process on $\mathbb{R}^d$ with mass $m > 0$, that is, $\mathbf{M}^{\mathrm{H}} = (\Omega, X_t, P_x)_{x\in\mathbb{R}^d}$ is a Lévy process satisfying

$$E_0\left[e^{\sqrt{-1}\langle\xi,X_t\rangle}\right] = e^{-t(\sqrt{|\xi|^2+m^2}-m)}.$$

It is shown in [8], the semigroup kernel $p_t(x,y)$ of $\mathbf{M}^{\mathrm{H}}$ is given by

$$p_t(x,y) = (2\pi)^{-d}\frac{t}{\sqrt{|x-y|^2+t^2}}\int_{\mathbb{R}^d} e^{mt}e^{-\sqrt{(|x-y|^2+t^2)(|z|^2+m^2)}}dz.$$

Hence we have that for each $t_0 > 0$, there exist $C_i = C_i(d) > 0$, $i = 1,2$ independent of $t_0$ such that for any $t \in ]0,t_0[$, $x,y \in \mathbb{R}^d$

$$\frac{C_1}{t^d}\frac{e^{-m|x-y|}}{\left(1+\frac{|x-y|^2}{t^2}\right)^{(d+1)/2}} \le p_t(x,y) \le \frac{C_2}{t^d}\frac{e^{mt_0}}{\left(1+\frac{|x-y|^2}{t^2}\right)^{(d+1)/2}}.$$

In particular, $\sup_{x,y\in\mathbb{R}^d} p_t(x,y) \le C_2 e^{mt_0}/t^d$ for $t \in ]0,t_0[$. Then we have that $f \in L^p(\mathbb{R}^d)$ implies $|f(x)|dx \in S_K^1$ for $p > d$.

**Example 4.3** (Brownian motion penetrating fracrals, cf. [9]). The diffusion process on $\mathbb{R}^d$ constructed in [9] admits the heat kernel $p_t(x,y)$ which has the following upper estimate: there exists $C > 0$ such that

$\sup_{x,y\in\mathbb{R}^d} p_t(x,y) \leq Ct^{-d/2}$ if $t \in ]0,1]$. Hence $f \in L^p(\mathbb{R}^d)$ implies $|f(x)|dx \in S^1_K$ for $p > d/2$ with $d \geq 2$ or $p \geq 1$ with $d = 1$.

**Example 4.4** (Diffusions with bounded drift). Let $a$ be the symmetric matrix valued measurable function such that $\lambda|\xi|^2 \leq \langle a(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$, $^\forall x, \xi \in \mathbb{R}^d$ for $0 < {}^\exists\lambda \leq {}^\exists\Lambda$. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded measurable function and assume $\operatorname{div} b \geq 0$ in the distributional sense. Consider $(\mathcal{E}^{a,b}, C_0^\infty(\mathbb{R}^d))$ defined by

$$\mathcal{E}^{a,b}(u,v) := \frac{1}{2}\int_{\mathbb{R}^d} \langle a(x)\nabla u(x), \nabla v(x)\rangle dx - \int_{\mathbb{R}^d} \langle b(x), \nabla u(x)\rangle v(x)dx$$

for $u, v \in C_0^\infty(\mathbb{R}^d)$. Then we see $\mathcal{E}^{a,b}(u,u) \geq 0$ for $u \in C_0^\infty(\mathbb{R}^d)$ and $(\mathcal{E}^{a,b}, C_0^\infty(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d)$ (see Chapter II 2(d) in [11]). We denote by $(\mathcal{E}^{a,b}, H^1(\mathbb{R}^d))$ its closure on $L^2(\mathbb{R}^d)$. $(\mathcal{E}^{a.b}, H^1(\mathbb{R}^d))$ is a non-symmetric Dirichlet form on $L^2(\mathbb{R}^d)$. Let $\{T_t^{a,b}\}_{t>0}$ be the $L^2(\mathbb{R}^d)$-semigroups associated with $(\mathcal{E}^{a,b}, H^1(\mathbb{R}^d))$. Then, by §II. 2 in [12], $T_t^{a,b}$ admits a heat kernel $p_t^{a,b}(x,y)$ on $]0,\infty[\times\mathbb{R}^d\times\mathbb{R}^d$ such that $P_t^{a,b}f(x) := \int_{\mathbb{R}^d} p_t^{a,b}(x,y)f(x)dy$ is an $m$-version of $T_t^{a,b}f$ for $f \in L^2(\mathbb{R}^d)$ and $p_t^{a,b}(x,y)$ satisfies the Aronson's estimates: (see (II. 2.4) in [12]) there exists an $M := M(\lambda, \Lambda, d) \in [1,\infty)$ such that for all $x, y \in \mathbb{R}^d, t \in ]0,1[$

$$\frac{1}{Mt^{d/2}} e^{-M(t+|x-y|^2/t)} \leq p_t^{a,b}(x,y) \leq \frac{M}{t^{d/2}} e^{Mt-|x-y|^2/Mt}. \tag{4.1}$$

In particular, $\sup_{x,y\in\mathbb{R}^d} p_t^{a,b}(x,y) \leq Me^M/t^{d/2}$ for all $t \in ]0,1[$, hence $f \in L^p(\mathbb{R}^d)$ implies $|f(x)|dx \in S^1_K \cap \hat{S}^1_K$ for $p > d/2$ with $d \geq 2$, or $p \geq 1$ with $d = 1$.

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# A subharmonic Hardy class and Bloch pullback operator norms

**Ern Gun Kwon**

**Abstract.**

We estimate the operator norm of the composition operators mapping Bloch space boundedly into Hardy spaces, $BMOA$ space, Lipschitz spaces and mean Lipschitz spaces respectively.

## §1. Introduction

This is to give a brief survey of a resent result on Bloch pullback operators, whose detailed proof will appear at [5]. Our purpose here is two-fold. One is to obtain hyperbolic version of Littlewood-Paley $g$-function equivalence, the other is to estimate the operator norm of Bloch-pullback operators. At first glance these two topics seem to be quite apart, but they are very closely related.

Let $D$ be the unit disc of the complex plane and $S = \partial D$. Let $H^p$, $0 < p < \infty$, denote the classical Hardy space defined to consist of $f$ holomorphic in $D$ for which

$$\|f\|_{H^p} = \lim_{r\to 1} \left( \int_S |f(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p} < \infty,$$

where $d\sigma$ is the rotation invariant Lebesgue probability measure (Haar measure) on $S$.

For a holomorphic function $f$ in $D$, the $g$-function of Littlewood-Paley defined as

$$g_f(\zeta) = \left( \int_0^1 (1-r)|f'(r\zeta)|^2 \, dr \right)^{1/2}, \quad \zeta \in S,$$

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satisfies the following beautiful and powerful relation

$$\|g_f\|_{L^p} \approx \|f - f(0)\|_{H^p} \tag{1.1}$$

(see [1] or [8], also see [16] for $1 < p < \infty$). Here and throughout, $L^p = L^p(S)$.

In parallel with $H^p$, there defined is $\varrho H^p$ consisting of holomorphic self map $\phi$ of $D$ for which

$$\|\phi\|_{\varrho H^p} = \lim_{r\to 1}\left(\int_S \varrho\,(\phi(r\zeta), 0)^p\, d\sigma(\zeta)\right)^{1/p} < \infty,$$

where $\varrho$ is the hyperbolic distance on $D$ :

$$\varrho(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \quad \varphi_z(w) = \frac{z-w}{1-\bar{z}w}.$$

We set $\lambda(z) = \log\frac{1}{1-|z|}$, $z \in D$. Note that if $\phi$ is a holomorphic self map of $D$, then $\lambda\circ\phi$ is subharmonic in $D$ and radial limit $\phi^*(\zeta) = \lim_{r\to 1}\phi(r\zeta)$ exists almost every $\zeta \in S$, so $\phi \in \varrho H^p$ if and only if $\lambda\circ\phi^* \in L^p(S)$. Throughout, $dA(z)$ denotes the Lebesgue area measure of $D$ normalized to be $A(D) = 1$.

Along with [6, 10] for previous results on pullback theory, we refer to [3, 16] for Hardy space theory and [2, 15] for composition operator theory.

## §2. Hyperbolic $g$-function

Our first subject is the Littlewood-Paley type $g$-function that characterizes the membership of $\varrho H^p$. See [4] and [6] for related previous works. We define, as in [4],

$$\varrho g_\phi(\zeta) = \int_0^1 (1-r)\left(\frac{|\phi'(r\zeta)|}{1-|\phi(r\zeta)|^2}\right)^2 dr, \quad \zeta \in S.$$

As our first result, we have the following hyperbolic analogue of (1.1).

**Theorem 2.1.** *Let $0 < p < \infty$. Then*

$$\|\varrho g_\phi\|_{L^p} \approx \|\lambda\circ\phi^*\|_{L^p} \tag{2.1}$$

*for all holomorphic self map $\phi$ of $D$ with $\phi(0) = 0$.*

When $p = 1$, (2.1) follows immediately from the following.

**Lemma 2.2.** *Let $\phi$ be a holomorphic self map of $D$ and $0 < p < \infty$. Then*

$$\int_D \log \frac{1}{|z|} \, \Delta(\lambda \circ \phi)^p(z) \, dA(z) \approx \|\lambda \circ \phi^*\|_{L^p}^p - (\lambda \circ \phi(0))^p .$$

For the proof of Theorem 2.1, we need several more techniques. We skip them and refer to [5].

## §3. Norm of the Bloch-pullback operators

We next pass to our second subject, the Bloch pullback. It is known that there is a Bloch function having radial limits at no points of $S$, while functions of $H^p$ should have radial limits almost everywhere on $S$. This observation give rise to the problem of characterizing holomorphic self maps $\phi$ of $D$ for which $f \circ \phi \in H^p$ for every Bloch function $f$. It is so called "Bloch - $H^p$ pullback problem" and the Bloch-pullback operator (induced by a holomorphic self map $\phi$ of $D$) means the composition operator $\mathcal{C}_\phi$ defined on the Bloch space $\mathcal{B}$ by $\mathcal{C}_\phi f = f \circ \phi$. $H^p$ is a Banach space with norm $\|f\|_{H^p}$ when $1 \le p < \infty$, while it is a Frechet space with the compatible metric $\|f\|_{H^p}^p$ when $0 < p < 1$. The following characterization of the Bloch-$H^p$ pullback operator shows a connection between Hardy space and hyperbolic Hardy class.

**Theorem A** [4, 6]. *Let $0 < p < \infty$ and $\phi$ be a holomorphic self map of $D$. Then $\mathcal{C}_\phi$ maps $\mathcal{B}$ boundedly into $H^p$ if and only if $\phi \in \varrho H^{p/2}$.*

As an application of Theorem 2.1, we moreover have the following theorem. Here, $\mathcal{B}^0$ denotes the subspace of $\mathcal{B}$ consisting of $f \in \mathcal{B}$ with $f(0) = 0$.

**Theorem 3.1.** *Let $0 < p < \infty$ and $\phi$ be a holomorphic self map of $D$ with $\phi(0) = 0$. If we set $\|\mathcal{C}_\phi\| = \sup\left\{\|\mathcal{C}_\phi f\|_{H^p} : f \in \mathcal{B}^0, \|f\|_{\mathcal{B}} \le 1\right\}$ then it satisfies*

$$\|\mathcal{C}_\phi\| \approx \|\lambda \circ \phi^*\|_{L^{p/2}}^{1/2}.$$

The assumption that $\phi(0) = 0$ is not essential restriction in the sense that if $\mathcal{C}_\phi$ is bounded (or compact) then so is $\mathcal{C}_\psi$ with $\psi = \varphi_{\phi(0)} \circ \phi$. Note also that $\mathcal{C}_\phi : \mathcal{B} \to Y$ is bounded if and only if $\mathcal{C}_\phi : \mathcal{B}^0 \to Y$ is bounded.

As a limiting space of $H^p$, a similar problem might be asked for $BMOA$. $BMOA$, the space of holomorphic functions of bounded mean oscillation, consists of holomorphic $f$ in $D$ for which

$$\|f\|_{BMOA} = \sup_{a\in D}\left\{\lim_{r\to 1}\int_S |f\circ\varphi_a(r\zeta) - f(a)|^2\, d\sigma(\zeta)\right\}^{1/2} < \infty.$$

In parallel with $BMOA$, there defined is $\varrho BMOA$ consisting of holomorphic self map $\phi$ of $D$ for which

$$\|\phi\|_{\varrho BMOA} = \sup_{a\in D}\lim_{r\to 1}\int_S \varrho\left(\phi\circ\varphi_a(r\zeta), \phi(a)\right)\, d\sigma(\zeta) < \infty.$$

The classes $\varrho BMOA$ as well as $\varrho H^p$ were defined and studied mainly as a hyperbolic counterpart of the corresponding Euclidean classes by S. Yamashita [11, 12, 13], and later studied by several authors in connection with the composition operators.

**Theorem B** [7]. *Let $\phi$ be a holomorphic self map of $D$. Then $\mathcal{C}_\phi$ maps $\mathcal{B}$ boundedly into $BMOA$ if and only if $\phi \in \varrho BMOA$.*

Noting that the Möbius invariance of $\varrho$ implies $\varrho\left(\phi\circ\varphi_a(z), \phi(a)\right) = \varrho\left(\varphi_{\phi(a)}\circ\phi\circ\varphi_a(z), 0\right)$, it follows that $\phi\in\varrho BMOA$ if and only if

$$\sup_{a\in D}\|\lambda\circ(\varphi_{\phi(a)}\circ\phi\circ\varphi_a)^*\|_{L^1} < \infty.$$

Since $\log|1-\bar{\phi}(a)\phi\circ\varphi_a|$ is harmonic in $D$,

$$\|\lambda\circ(\varphi_{\phi(a)}\circ\phi\circ\varphi_a)^*\|_{L^1} = \|\lambda\circ(\phi\circ\varphi_a)^* - \lambda\circ\phi(a)\|_{L^1}$$

[7, (3.7)], so that the next theorem gives Theorem B. Here, as the norm of $BMOA$ we take $|f(0)| + \|f\|_{BMOA}$, which makes $BMOA$ a Banach space.

**Theorem 3.2.** *Let $\phi$ be a holomorphic self map of $D$ with $\phi(0) = 0$. Then the operator norm of $\mathcal{C}_\phi$ from $\mathcal{B}^0$ boundedly into BMOA satisfies*

$$\|\mathcal{C}_\phi\| \approx \sup_{a\in D}\|\lambda\circ(\phi\circ\varphi_a)^* - \lambda\circ\phi(a)\|_{L^1}^{1/2}.$$

$VMOA$, the space of holomorphic functions of vanishing mean oscillation, consists of holomorphic $f$ in $D$ for which

$$\lim_{|a|\to 1}\lim_{r\to 1}\int_S |f\circ\varphi_a(r\zeta) - f(a)|^2\, d\sigma(\zeta) = 0.$$

In parallel to $VMOA$, $\varrho VMOA$ is defined to consist of holomorphic self map $\phi$ of $D$ for which

$$\lim_{|a|\to 1}\lim_{r\to 1}\int_S \varrho\left(\phi\circ\varphi_a(r\zeta),\phi(a)\right)\, d\sigma(\zeta) \;=\; 0.$$

We have

**Corollary C** [9]. *Let $\phi$ be a holomorphic self map of $D$. Then $\mathcal{C}_\phi$ maps $\mathcal{B}$ boundedly into $VMOA$ if and only if $\phi \in \varrho VMOA$.*

See [9] for previous study on $\varrho VMOA$.

## §4. More on Bloch-pullback operator norm

We give some more examples of Banach space $Y$ and resolve Bloch-$Y$ pullback problem by further evaluating the operator norm of $\mathcal{C}_\phi : \mathcal{B}\to Y$.

Let $\mathcal{D}$ denote the space of holomorphic functions $f$ in $D$ satisfying

$$\|f\|_{\mathcal{D}} := \left(\int_D |f'(z)|^2\; dA(z)\right)^{1/2} < \infty.$$

Then $\mathcal{D}$ is a Banach space with the norm $|f(0)| + \|f\|_{\mathcal{D}}$. Similarly, we let $\varrho\mathcal{D}$ denote the space of holomorphic self map $\phi$ of $D$ satisfying

$$\|\phi\|_{\varrho\mathcal{D}} := \left(\int_D \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^2}\; dA(z)\right)^{1/2} < \infty.$$

Then we have

**Theorem 4.1.** *Let $\phi$ be a holomorphic self map of $D$. Then $\mathcal{C}_\phi$ maps $\mathcal{B}$ boundedly into $\mathcal{D}$ if and only if $\phi\in\varrho\mathcal{D}$. Moreover, if $\phi(0)=0$ then the operator norm of $\mathcal{C}_\phi$ from $\mathcal{B}^0$ boundedly into $\mathcal{D}$ satisfies*

$$\|\mathcal{C}_\phi\| \;\approx\; \|\phi\|_{\varrho\mathcal{D}}.$$

$H^\infty$, consisting of bounded holomorphic functions, is a Banach space with the norm $\|f\|_{H^\infty} = \sup_{z\in D}|f(z)|$, while $\varrho H^\infty$ is defined to consist of holomorphic $\phi$ of $D$ for which $|\phi| < c$ for some $c < 1$.

**Theorem 4.2.** *If $\phi$ be a holomorphic self map of $D$, then $\mathcal{C}_\phi : \mathcal{B}^0 \to H^\infty$ is bounded if and only if $\phi\in\varrho H^\infty$. If $\phi(0) = 0$, then the operator norm of $\mathcal{C}_\phi$ from $\mathcal{B}^0$ boundedly into $H^\infty$ satisfies*

$$\|\mathcal{C}_\phi\| \;=\; \sup_{z\in D}\rho\circ\phi(z),$$

*where $\rho$ is defined by $\rho(w) = \rho(0,w) = \frac{1}{2}\log\frac{1+|w|}{1-|w|}$, $w\in D$.*

Beyond $H^{\infty}$, there are function spaces having smooth boundary conditions. We are going to mention about holomorphic Lipschitz spaces. For $0 < \alpha \leq 1$, we say, by definition, that $f \in Lip_{\alpha}$ if $f$ is holomorphic in $D$, $f \in C(\bar{D})$, and satisfies the Lipschitz condition:

$$\|f\|_{Lip_{\alpha}} := \sup\left\{\frac{|f(z)-f(w)|}{|z-w|^{\alpha}} : z, w \in D, z \neq w\right\} < \infty.$$

$Lip_{\alpha}$ is a Banach space equipped with the norm $|f(0)| + \|f\|_{Lip_{\alpha}}$. Several different (but essentially same) notions for $Lip_{\alpha}$ are used in the literature. We followed that of [2].

Corresponding to this, there is hyperbolic Lipschitz class of Yamashita [14]. We say, by definition, that $\phi \in \varrho Lip_{\alpha}$ if $\phi$ is a holomorphic self map of $D$, $\phi \in C(\bar{D})$, and satisfies the hyperbolic Lipschitz condition:

$$\|\phi\|_{\varrho Lip_{\alpha}} := \sup\left\{\frac{\varrho(\phi(z), \phi(w))}{|z-w|^{\alpha}} : z, w \in D, z \neq w\right\} < \infty.$$

We have

**Theorem 4.3.** *Let $0 < \alpha \leq 1$ and $\phi$ be a holomorphic self map of $D$. Then $\mathcal{C}_{\phi} : \mathcal{B} \to Lip_{\alpha}$ is bounded if and only if $\phi \in \varrho Lip_{\alpha}$. Further if $\phi(0) = 0$, then the operator norm of $\mathcal{C}_{\phi}$ from $\mathcal{B}^0$ boundedly into $Lip_{\alpha}$ satisfies*

$$\|\mathcal{C}_{\phi}\| \ = \ \|\phi\|_{\varrho Lip_{\alpha}}.$$

For $1 \leq p < \infty$ and $0 < \alpha < 1$, we say, by definition, that $f \in Lip^p_{\alpha}$ if $f \in H^p$ and satisfies the mean Lipschitz condition:

$$\|f\|_{Lip^p_{\alpha}} := \sup\left\{\frac{1}{t^{\alpha}}\left(\int_S |f(\eta\zeta) - f(\zeta)|^p d\sigma(\zeta)\right)^{\frac{1}{p}} : 0 < |1-\eta| \leq t\right\} < \infty.$$

$Lip^p_{\alpha}$ is a Banach space equipped with the norm $\|\cdot\|_{H^p} + \|\cdot\|_{Lip^p_{\alpha}}$.

Corresponding to this, there is hyperbolic mean Lipschitz class of Yamashita [14]. We say, by definition, that $\phi \in \varrho Lip^p_{\alpha}$ if $\phi$ is a holomorphic self map of $D$, $\varrho(\phi^*) \in L^p(S)$, and $\phi$ satisfies the hyperbolic mean Lipschitz condition:

$$\|\phi\|_{\varrho Lip^p_{\alpha}} := \sup\left\{\frac{1}{t^{\alpha}}\left(\int_S \varrho\left(\phi(\eta\zeta), \phi(\zeta)\right)^p d\sigma(\zeta)\right)^{\frac{1}{p}} : 0 < |1-\eta| \leq t\right\} < \infty.$$

We have

**Theorem 4.4.** *Let $1 \le p < \infty$ and $0 < \alpha < 1$. Let $\phi$ be a holomorphic self map of $D$. Then $\mathcal{C}_\phi : \mathcal{B} \to Lip^p_\alpha$ is bounded if and only if $\phi \in \varrho Lip^p_\alpha$. Furthermore, if $\phi(0) = 0$, then operator norm of $\mathcal{C}_\phi$ from $\mathcal{B}^0$ boundedly into $Lip^p_\alpha$ satisfies*

$$\|\mathcal{C}_\phi\| \approx \|\lambda \circ \phi^*\|^{1/2}_{L^{p/2}} + \|\phi\|_{\varrho Lip^p_\alpha}.$$

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# Quasiconformal mappings and minimal Martin boundary of $p$-sheeted unlimited covering surfaces of the once punctered Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type

**Hiroaki Masaoka**

***Dedicated to Professor Masakazu Shiba on his sixtieth birthday***

**Abstract.**

Let $R$ and $R'$ be $p$-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type which are quasiconformal equivalent to each other. Then the cardinal numbers of minimal Martin boundaries of $R$ and $R'$ are same.

Let $R$ be a 2-sheeted unlimited covering surface of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type and $R'$ be an open Riemann surface. If $R$ and $R'$ are quasiconformal equivalent to each other and the set of branch points of $R$ satisfies a condition, then the cardinal numbers of minimal Martin boundaries of $R$ and $R'$ are same.

## §1. Introduction.

Let $W$ be an open Riemann surface. We denote by $\Delta_1^W$ the minimal Martin boundary of $W$. In [8], it was showed that there exist open Riemann surfaces $F$ and $F'$ quasiconformally equivalent to each other such that $F'$ possesses nonconstant positive harmonic functions although $F$ does not possess nonconstant positive harmonic functions. This means that $\sharp\Delta_1^{F'} \geq 2$ although $\sharp\Delta_1^F = 1$, where $\sharp A$ stands for the cardinal

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number of a set $A$. Needless to say, the above $F$ and $F'$ are of *positive boundary*, i.e. $F$ and $F'$ admit the Green function (cf. e.g. [16]). However, in case open Riemann surfaces $F$ and $F'$ are of *null boundary* (i.e. not positive boundary), it does not seem to be known whether $\sharp\Delta_1^F = \sharp\Delta_1^{F'}$ or not if $F$ and $F'$ are quasiconformally equivalent to each other.

Consider two positive decreasing sequences $\{a_n\}$ and $\{b_n\}$ satisfying $b_{n+1} < a_n < b_n < 1$ and $\lim_{n\to\infty} a_n = 0$. Set $G = \hat{\mathbb{C}} \setminus (\{0\} \cup I)$, where $\hat{\mathbb{C}}$ is the extended complex plane, $I = \cup_{n=1}^{\infty} I_n$ and $I_n = [a_n, b_n]$. We take $p$ copies $G_1, \cdots, G_p$ of $G$ and join the upper edge of $I_n$ on $G_j$ with the lower edge of $I_n$ on $G_{j+1}$ ($j$ mod $p$) for every $n$. Then we obtain a $p$-sheeted covering surface $R$ of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ and say that $R$ is of Heins type(cf. [4]).

In this paper, we are concerned with $p$-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type. Consider $p$-sheeted unlimited covering surfaces $R$ and $R'$ of $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type which are quasiconformally equivalent to each other. Then it seems to be valid that $\sharp\Delta_1^R = \sharp\Delta_1^{R'}$ (cf. [12], [10], [18]). The first purpose of this paper is to give an answer to this conjecture. Namely,

**Theorem 1.** *Let $R$ and $R'$ be $p$-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type which are quasiconformally equivalent to each other. Then it holds that $\sharp\Delta_1^R = \sharp\Delta_1^{R'}$.*

Let $R$ be a 2-sheeted unlimited covering surface of $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type with the projection $\pi$ from $R$ onto $\hat{\mathbb{C}} \setminus \{0\}$. We have the following.

**Theorem 2.** *Suppose that $b_n - b_{n+1} \approx 2^{-n}$, that is, there exists a constant $\alpha(>1)$ with $\alpha^{-1}2^{-n} < b_n - b_{n+1} < \alpha 2^{-n}$ ($n \in \mathbb{N}$). Let $R'$ be an open Riemann surface and $f$ a quasiconformal mapping with $R' = f(R)$. Then it holds that $\sharp\Delta_1^R = \sharp\Delta_1^{R'}$.*

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## §2. Preliminaries.

In this section we consider as $R$ a general $p$-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$. Let $\Delta^R$ and $\Delta_1^R$ be as in §1, and $\pi$ the projection map from $R$ onto $\hat{\mathbb{C}} \setminus \{0\}$. Set $\mathbb{D} = \{x \in \mathbb{C} \mid |x| < 1\}$, $\mathbb{D}_0 = \mathbb{D} \setminus \{0\}$ and $R_0 = \pi^{-1}(\mathbb{D}_0)$. It is well-known that $\Delta^{R_0}$ and $\Delta_1^{R_0}$ are identified with $\Delta^R \cup \pi^{-1}(\partial\mathbb{D})$ and $\Delta_1^R \cup \pi^{-1}(\partial\mathbb{D})$, respectively, where $\partial\mathbb{D} = \{x \in \mathbb{C} \mid |x| = 1\}$. From now on we consider $\mathbb{D}_0$ (resp. $R_0$) in place of $\hat{\mathbb{C}} \setminus \{0\}$ (resp. $R$) since $\hat{\mathbb{C}} \setminus \{0\}$ (resp. $R$) does not admit the Green function. Let $g_0$ be the Green function on $\mathbb{D}$ with pole at 0.

**Definition 2.1** (cf. [2])**.** We say that a subset $E$ of $\mathbb{D}_0$ is *thin* at 0 if ${}^{\mathbb{D}}\widehat{\mathrm{R}}^E_{g_0} \neq g_0$, where ${}^{\mathbb{D}}\widehat{\mathrm{R}}^E_{g_0}$ is the balayage of $g_0$ relative to $E$ on $\mathbb{D}$ .

If $E$ is a closed subset of $\mathbb{D}$, it is well-known that $E$ is thin at 0 if and only if 0 is an irregular boundary point of $\mathbb{D} \setminus E$ in the sense of the Dirichlet problem.

The following lemma gives the quasiconformal invariance for thinness.

**Lemma 2.1** (cf. [10],[18])**.** *Let $M$ be a subdomain of $\mathbb{C}$ and $\varphi$ a quasiconformal mapping from $\mathbb{C}$ onto $\mathbb{C}$. If $\zeta$ is an irregular boundary point of $M$ in the sense of Dirichlet problem, $\varphi(\zeta)$ is an irregular boundary point of $\varphi(M)$ in the sense of Dirichlet problem.*

**Definition 2.2.** A subset $U$ in $\mathbb{D}$ which contains 0 is said to be a *fine neighborhood* of 0 if $\mathbb{D} \setminus U$ is thin at 0.

Let $k_\zeta$ be the Martin function on $R_0$ with pole at $\zeta \in \Delta^R$. If we take a sequence $\{x_n\}$ in $R_0$ such that $\lim_{n\to\infty} x_n = \zeta$, we can give a definition of $k_\zeta$ by the following.

$$k_\zeta(z) = \lim_{n\to\infty} \frac{g_{x_n}(x)}{g_{x_n}(x_0)},$$

where $x_0$ is a fixed point in $R_0$. For details we refer to [3] and [5].

**Definition 2.3.** Let $\zeta$ be a point in $\Delta^R_1$ and $E$ a subset of $R_0$. We say that $E$ is *minimally thin* at $\zeta$ if ${}^{R_0}\widehat{\mathrm{R}}^E_{k_\zeta} \neq k_\zeta$.

**Definition 2.4.** Let $\zeta$ be a point in $\Delta^R_1$ and $U$ a subset of $R_0$. We say that $U \cup \{\zeta\}$ is a *minimal fine neighborhood* of $\zeta$ if $R_0 \setminus U$ is minimally thin at $\zeta$.

The following proposition gives the characterization of $\sharp\Delta^R_1$ in terms of minimal fine topology.

**Proposition 2.1** ([11])**.** *Let $\mathcal{M}$ be the class of subdomains $M$ of $\mathbb{D}_0$ such that $M \cup \{0\}$ is a fine neighborhood of $x = 0$. Then it holds that*

$$\sharp\Delta^R_1 = \max_{M\in\mathcal{M}} n_R(M),$$

*where $n_R(M)$ is the number of connected components of $\pi^{-1}(M)$ and $\pi$ is the projection map from $R$ onto $\hat{\mathbb{C}} \setminus \{0\}$.*

## §3. Proof of Theorem 1.

In this section we first consider as $R$ a general $p$-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$. Let $\Delta^R$ and $\Delta_1^R$ be as in §1, and $\pi$ the projection map from $R$ onto $\hat{\mathbb{C}} \setminus \{0\}$. Let $\mathbb{D}$, $\mathbb{D}_0$, and $R_0$ be as in §2. The next proposition will play an important role for the proof of Theorem 1.

**Proposition 3.1.** *Let $R'$ be an open Riemann surface and $f$ a quasiconformal mapping with $R' = f(R)$. If $\sharp\Delta_1^R = p$, then $\sharp\Delta_1^R = \sharp\Delta_1^{R'}$.*

*Proof.* By Proposition 2.1 we find a subdomain $M$ of $\mathbb{D}_0$ such that $\mathbb{D}_0 \setminus M$ is thin at 0, $\partial M \setminus \{0\}$ may consist of infinitely many Jordan curves and

$$\sharp\Delta_1^R = n_R(M),$$

where $n_R(M)$ is the number of connected components of $\pi^{-1}(M)$. By the assumption of this proposition $n_R(M) = p$. Let $M_j$ $(j = 1, 2, \ldots, p)$ be components of $\pi^{-1}(M)$. Since each $M_j$ is a 1-sheeted unlimited covering surface of $M$, it is easily seen that each $M_j$ is considered as a replica of $M$. Let $g_x^{f(M_j)}$ $(j = 1, 2 \ldots, p)$ be the Green function on $f(M_j)$ with pole at $x$ and $\psi_j$ the inverse of $\pi|_M$ from $M \to M_j$. Denote by $\mu_{f\circ\psi_j}$ the complex dilatation of $f \circ \psi_j$ on $M$. Set

$$\mu_j = \begin{cases} \mu_{f\circ\psi_j} & \text{on } M \\ 0 & \text{on } \mathbb{C} \setminus M. \end{cases}$$

It is well-known that there exists a quasiconformal mapping $f_j$ from $\mathbb{C}$ onto $\mathbb{C}$ with the complex dilatation $\mu_j$ (cf. e.g. [6]). Set $V_j = f_j(M)$. By Lemma 2.1 we find that $f_j(0)$ is an irregular boundary point of $V_j$ in the sense of the usual Dirichlet problem since 0 is an irregular boundary point of $M$ in the sense of the usual Dirichlet problem. On the other hand, the function $x' \mapsto g^{f(M_j)}_{f\circ\psi_j\circ f_j^{-1}(x')} \circ f \circ \psi_j \circ f_j^{-1}(y')$ $(y' \in V_j)$ is a positive harmonic function on $V_j \setminus \{y'\}$ since $f \circ \psi_j \circ f_j^{-1}$ is conformal. Hence, by [5, Theorem 10.16], there exists a positive fine limit $\mathcal{F} - \lim_{x'\to f_j(0)} g^{f(M_j)}_{f\circ\psi_j\circ f_j^{-1}(x')} \circ f \circ \psi_j \circ f_j^{-1}$. Denote by $g_0^{V_j}$ this limit function on $V_j$ and set $g_0^{f(M_j)} = g_0^{V_j} \circ f_j \circ \psi_j^{-1} \circ f^{-1}$. We see that each $g_0^{f(M_j)}$ is a positive harmonic function on $f(M_j)$ since each $g_0^{V_j}$ is a positive harmonic function on $V_j$ and $f_j \circ \psi_j^{-1} \circ f^{-1}$ is conformal. For $j = 1, 2, \ldots, p$ set

$$S_j(g_0^{f(M_j)})(x') = \inf_s s(x'),$$

where $s$ runs over the space of positive superharmonic functions $s$ on $f(R_0)$ satisfying $s \geq g_0^{f(M_j)}$ on $f(M_j)$. By Perron-Wiener-Brelot method we find that each $S_j(g_0^{f(M_j)})$ is a positive harmonic function on $f(R_0)$. Then the following inequality

$$(*) \qquad S_j(g_0^{f(M_j)}) - {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{S_j(g_0^{f(M_j)})} \geq g_0^{f(M_j)}$$

holds on $f(M_j)$. In fact, to prove the inequality $(*)$ note that

$${}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{S_j(g_0^{f(M_j)})} = H^{f(M_j)}_{S_j(g_0^{f(M_j)})}$$

on $f(M_j)$, where $H^{f(M_j)}_{S_j(g_0^{f(M_j)})}$ is the Dirichlet solution for $S_j(g_0^{f(M_j)})$ on $f(M_j)$ (cf. e.g. [3], [5]). By definition $S_j(g_0^{f(M_j)}) \geq g_0^{f(M_j)}$ on $f(M_j)$. Hence, by the definition of the Dirichlet solution in the sense of Perron-Wiener-Brelot,

$$S_j(g_0^{f(M_j)}) - g_0^{f(M_j)} \geq H^{f(M_j)}_{S_j(g_0^{f(M_j)})}$$

on $f(M_j)$. Thus $(*)$ is proved.

We shall proceed the proof of this proposition. By [17, Theorem 3] it is known that $1 \leq \sharp\Delta_1^{R'} \leq p$. By the Martin representation theorem, there exist at most $p$ minimal functions $h_{j,1}, h_{j,2}, \ldots, h_{j,p}$ on $f(R_0)$ with $S_j(g_0^{f(M_j)}) = h_{j,1} + h_{j,2} + \ldots + h_{j,p}$ on $f(R_0)$. Hence, by the above inequality $(*)$, we have

$$\begin{aligned} & h_{j,1} + h_{j,2} + \ldots + h_{j,p} \\ = \; & S_j(g_0^{f(M_j)}) \\ \geq \; & {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_{j,1}+h_{j,2}+\ldots+h_{j,p}} + g_0^{f(M_j)} \\ > \; & {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_{j,1}} + {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_{j,2}} + \ldots + {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_{j,p}} \end{aligned}$$

on $f(M_j)$. Therefore we find that there exists a minimal function $h_j$ on $f(R_0)$ such that $h_j \neq {}^{f(R_0)}\widehat{\mathrm{R}}^{f(R_0)\setminus f(M_j)}_{h_j}$. Hence, by the definition of minimal thinness, $f(R_0) \setminus f(M_j)$ is minimally thin at the minimal boundary point corresponding to $h_j$. Since $f(M_i) \cap f(M_j) = \emptyset \;\; (i \neq j)$, we find that $\sharp\Delta_1^{R'} = p$. □

Now we give the following result which Proposition 2.1 yields.

**Theorem 3.1** (cf. [11]). *Let $R$ be a $p$-sheeted unlimited covering surfaces of the once punctured Riemann sphere $\hat{\mathbb{C}} \setminus \{0\}$ of Heins type. Then $\sharp\Delta_1^R = 1$ or $p$.*

*Proof of Theorem 1.* By Theorem 3.1 we have only to prove that $\sharp\Delta_1^{R'} = p$ if and only if $\sharp\Delta_1^R = p$. Since $f^{-1}$ is a quasiconformal mapping from $R'$ onto $R$, it is sufficient to prove that if $\sharp\Delta_1^R = p$, then $\sharp\Delta_1^{R'} = p$. Suppose that $\sharp\Delta_1^R = p$. By Proposition 3.1 $\sharp\Delta_1^{R'} = p$. We have the desired result. □

## §4. Proof of Theorem 2.

By Proposition 3.1 we find that if $\sharp\Delta_1^R = 2$, $\sharp\Delta_1^{R'} = 2$. By [17, Theorem 3] it is known that $\sharp\Delta_1^{R'} = 1$ or 2. Hence, by Theorem 3.1, it is sufficient to prove that if $\sharp\Delta_1^{R'} = 2$, $\sharp\Delta_1^R = 2$. Suppose that $\sharp\Delta_1^{R'} = 2$. Set $\Delta_1^{R'} = \{\zeta_1', \zeta_2'\}$. Let $g_{\xi'}^{f(R_0)}$ be the Green function with pole at $\xi'$ on $f(R_0)$. It is known that there exists $\lim_{y'\to\zeta_j'} g_{y'}^{f(R_0)}(x')(=: g'_{\zeta_j'}(x'))$ $(j = 1, 2)$ and $g'_{\zeta_j'}(j = 1, 2)$ is the minimal harmonic function with pole at $\zeta_j'$ $(j = 1, 2)$.

For $x \in R_0$ set

$$L = L_f = L_{x,f} = \begin{cases} \sum_{i,k=1}^2 \partial_k(J_f(x)(f'(x)^{-1}f'(x)^{-1*})_{k,i}\partial_i), \\ \qquad \text{(if there exist } f'(x) \text{ and } f'(x)^{-1}), \\ \\ \qquad\qquad \sum_{i=1}^2 \partial_i^2, \\ \qquad\qquad \text{(elsewise)}, \end{cases}$$

where $J_f(x)$ (resp. $f'(x)$) is the Jacobian (resp. Jacobi matrix) of the mapping $(u(x), v(x))$ $(f = u + iv)$, $f'(x)^{-1}$ is the inverse of $f'(x)$ and $f'(x)^{-1*}$ is the transpose of $f'(x)^{-1}$. $L$ is a elliptic second order partial differential operator of divergence type on $R$. Set $g_j^L(x) := g'_{\zeta_j'} \circ f(x)$ $(x \in R_0)$. We see that $g_j^L$ $(j = 1, 2)$ is a positive harmonic function on $R_0$ with respect to $L$. We recall the assumption that $b_n - b_{n+1} \approx 2^{-n}$, that is, there exists a constant $\alpha(> 1)$ with

$$\alpha^{-1}2^{-n} < b_n - b_{n+1} < \alpha 2^{-n} \quad (n \in \mathbb{N}).$$

For $r(> 0)$, set $C_r = \{|x| = r\}$, $B_r = \{|x| < r\}$, $\mathcal{C}_r = \pi^{-1}(C_r)$, and $\mathcal{B}_r = \pi^{-1}(B_r \setminus \{0\})$.

Suppose that there exist a constant $\alpha'(> 1)$ and a subsequence $\{n_l\}$ of $\mathbb{N} = \{n\}$ with $b_{n_l} - a_{n_l} > (\alpha')^{-1}2^{-n_l}$. Set $\mathcal{R}_l = \mathcal{B}_{(a_{n_l}+3b_{n_l})/4} \setminus Cl(\mathcal{B}_{(3a_{n_l}+b_{n_l})/4})$, where, for a set $E \subset R_0$, $Cl(E)$ stands for the closure of $E$ with respect to the usual topology on $R_0$. By the assumption that $b_{n_l} - a_{n_l} > (\alpha')^{-1}2^{-n_l}$, $Mod(\mathcal{R}_l) \approx 1$, where $Mod(\mathcal{R}_l)$ stands for the logarithmic module of $\mathcal{R}_l$ (cf. [1]), and hence, by the quasiconformal invariance of logarithmic module (cf. [6], [15]), $Mod(f(\mathcal{R}_l)) \approx 1$. Since the cardinal number of connected components of $\mathcal{R}_l$ is equal to 1, that of $f(\mathcal{R}_l)$ is so. By [17, Theorem 3], we find that $\sharp\Delta_1^{R'} = 1$. This is a contradiction. Hence we may suppose that there exists a constant $\alpha''(> 1)$ , for every integer $l$, $a_l - b_{l+1} > (\alpha'')^{-1}2^{-l}$. Set $\mathcal{A} = \cup_{l=1}^{\infty}\mathcal{A}_l$ ($\mathcal{A}_l = \mathcal{B}_{(3a_l+b_{l+1})/4} \setminus Cl(\mathcal{B}_{(a_l+3b_{l+1})/4})$), where $Cl(\mathcal{B}_{(a_l+3b_{l+1})/4})$ is the closure of $\mathcal{B}_{(a_l+3b_{l+1})/4}$ with respect to the usual topology on $R$.

**Lemma 4.1.** *On $\mathcal{A}$,*

$$g_j^L(x) + g_j^L(\iota(x)) \approx \log \frac{1}{|\pi(x)|} \ (j = 1, 2),$$

*where $\iota$ is the sheet exchange on $R$.*

*Proof.* Let $A_{l,k}$ $(k = 1, 2)$ be connected components of $\mathcal{A}_l$. Then we have

$$(\sharp) \quad {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_l)} \leq {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(A_{l,1})} + {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(A_{l,2})} \leq 2\,{}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_l)}.$$

Since ${}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_l)}$ is a Green potential on $f(R_0)$ (cf. [3]), we can find the Radon measure $\mu_{l,j}$ $(j = 1, 2)$ with

$$(\sharp\sharp) \quad {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(A_{l,j})}(x') = \int_{Cl(f(A_{l,j}))} g_{x'}^{f(R_0)} d\mu_{l,j}.$$

By the fact that ${}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_l)}(x') = 1$ for $x' \in f(\mathcal{B}_{(3a_l+b_{l+1})/4})$, letting $x'$ be to $\zeta_j'$ in $(\sharp)$, we have

$$1 \leq \int_{Cl(f(A_{l,1}))} g'_{\zeta_j'} d\mu_{l,1} + \int_{Cl(f(A_{l,2}))} g'_{\zeta_j'} d\mu_{l,2} \leq 2 \quad (j = 1, 2),$$

and hence

$$1 \leq \int_{Cl(A_{l,1})} g_j^L d(f^{-1})^*(\mu_{l,1}) + \int_{Cl(A_{l,2})} g_j^L d(f^{-1})^*(\mu_{l,2}) \leq 2 \quad (j = 1, 2),$$

where $(f^{-1})^*(\mu_{l,2})$ is the image measure of $\mu_{l,2}$ by $f^{-1}$. On the other hand, by the definition of capacitary potential, quasiconformal invariance of capacity(cf. [15, Theorem 10.10]), [9, Lemma 2.3], [1, Theorems 13C and 13D in Chap. IV] and [3, Satz 5.2 and Satz 7.2], we have

$$\begin{aligned}
(f^{-1})^*\mu_{l,j}(Cl(A_{l,j})) &= \mu_{l,j}(f(Cl(A_{l,j}))) = cap(f(Cl(A_{l,j})), f(R_0)) \\
&\approx cap(Cl(A_{l,j}), R_0) \approx cap(\pi(Cl(\mathcal{A}_l)), \mathbb{D}_0) \\
&= cap(Cl(B_{(3a_l+b_{l+1})/4}), \mathbb{D}_0) \\
&= 2\pi/\log[4/(3a_l + b_{l+1})] \approx 1/l,
\end{aligned}$$

where, for a subset $E$ of an open Riemann surface $F$ of positive boundary $cap(E,F)$ stands for the greenian capacity of $E$ on $F$. Therefore, by Harnack's inequality with respect to $L$ (cf. [13]), we have the desired result. $\square$

Set $D_I = \mathbb{D}_0 \setminus I$.

**Lemma 4.2.** *There exist components $\mathcal{D}_{I,j}$ $(j = 1,2)$ of $\pi^{-1}(D_I)$ such that*

$$g_j^L(x) \approx \log\frac{1}{|\pi(x)|}\ (x \in \mathcal{A}\cap\mathcal{D}_{I,j},\ j=1,2),$$

$$g_j^L(x) = o(\log\frac{1}{|\pi(x)|})\ (\pi(x)\to 0, x\in\mathcal{A}\cap\mathcal{D}_{I,j+(-1)^{j-1}},\ j=1,2).$$

*Proof.* Denote by $\mathcal{D}_{I,j}$ $(j = 1,2)$ components of $\pi^{-1}(D_I)$. Set $\mathcal{A}_{l,j} = \mathcal{A}_l \cap \mathcal{D}_{I,j}$. By Lemma 4.1 we may suppose that there exist subsequences $\{n_1\}$ and $\{n_2\}$ of $\mathbb{N} = \{n\}$ such that

(i) $\{n_1\}\cup\{n_2\} = \mathbb{N}$ and $\{n_1\}\cap\{n_2\} = \emptyset$;

(ii) $g_1^L(x) \approx \log\frac{1}{|\pi(x)|}\ (x \in (\cup_{n_1}\mathcal{A}_{n_1,1})\cup(\cup_{n_2}\mathcal{A}_{n_2,2})\ )$;

(iii) $g_1^L(x) = o(\log\frac{1}{|\pi(x)|})\ (\pi(x)\to 0, x\in(\cup_{n_1}\mathcal{A}_{n_1,2})\cup(\cup_{n_2}\mathcal{A}_{n_2,1})\ )$.

In fact, suppose the above does not hold. Then there exists a subsequence $\{n_3\}$ of $\mathbb{N} = \{n\}$ with

$$g_1^L(x) \approx \log\frac{1}{|\pi(x)|}\ (x \in \cup_{n_3}\mathcal{A}_{n_3}\ ).$$

On the other hand, for any $\beta(>0)$, $\{x' \in f(R_0) | g'_{\zeta'_2}(x') > \beta g'_{\zeta'_1}(x')\} \cup \{\zeta'_2\}$ is a minimal fine neighborhood of $\zeta'_2$, because, on $\{g'_{\zeta'_2} > \beta g'_{\zeta'_1}\}$, ${}^{f(R_0)}\widehat{\mathrm{R}}^{\{g'_{\zeta'_2}\le\beta g'_{\zeta'_1}\}}_{g'_{\zeta'_2}} < g'_{\zeta'_2}$, by the fact that, on $f(R_0)$,

$${}^{f(R_0)}\widehat{\mathrm{R}}^{\{g'_{\zeta'_2}\le\beta g'_{\zeta'_1}\}}_{g'_{\zeta'_2}} \le {}^{f(R_0)}\widehat{\mathrm{R}}^{\{g'_{\zeta'_2}\le\beta g'_{\zeta'_1}\}}_{\beta g'_{\zeta'_1}} \le \beta g'_{\zeta'_1}.$$

Hence, by Lemma 4.1 and by the fact that $g_j^L = g'_{\zeta'_j}\circ f$, there exists a positive $\beta_0$ with $\{x' \in f(R_0) | g'_{\zeta'_2}(x') > \beta_0 g'_{\zeta'_1}(x')\} \subset f(R_0)\setminus f(\cup_{n_3}\mathcal{A}_{n_3})$. It is well-known that we can take a connected component $G_1$ of $\{x' \in f(R_0) | g'_{\zeta'_2}(x') > \beta_0 g'_{\zeta'_1}(x')\}$ such that $G_1 \cup \{\zeta'_2\}$ is a minimal fine neighborhood of $\zeta'_2$ (cf. [14, Corollaire 2 in p.206]). This is a contradiction.

Suppose that both $\{n_1\}$ and $\{n_2\}$ are infinite sets. Let $\{m_1\}$ be a subsequence of $\{n_1\}$ with $m_1 + 1 \in \{n_2\}$. By (ii) we can find a positive constant $\kappa_1$ $(> 1)$ with

$$\kappa_1^{-1} \log \frac{1}{|\pi(x)|} \leq g_1^L(x) \leq \kappa_1 \log \frac{1}{|\pi(x)|} \ (x \in (\cup_{n_1} \mathcal{A}_{n_1,1}) \cup (\cup_{n_2} \mathcal{A}_{n_2,2})).$$

By Harnack's inequality with respect to $L$, we can find a positive constant $\kappa_2$ $(> 1)$ with

$$(\kappa_1\kappa_2)^{-1} \log \frac{1}{|\pi(x)|} \leq g_1^L(x) \leq (\kappa_1\kappa_2) \log \frac{1}{|\pi(x)|} \ (x \in \cup_{m_1} \mathcal{A}_{m_1+1,1}).$$

On the other hand, by (iii), there exists an integer $N_0$ such that,

$$g_1^L(x) < (\kappa_1\kappa_2)^{-1} \log \frac{1}{|\pi(x)|} \ (x \in \cup_{m_1 > N_0 - 1} \mathcal{A}_{m_1+1,1}).$$

This is a contradiction. Here, if necessary , by substituting $\mathcal{D}_{I,1}$ (resp. $\mathcal{D}_{I,2}$) for $\mathcal{D}_{I,2}$ (resp. $\mathcal{D}_{I,1}$), we have

$$(\flat 1) \quad g_1^L(x) \approx \log \frac{1}{|\pi(x)|} \ (x \in \cup_n \mathcal{A}_{n,1}\ )$$

$$(\flat 2) \quad g_1^L(x) = o(\log \frac{1}{|\pi(x)|}) \ (\pi(x) \to 0, x \in \cup_n \mathcal{A}_{n,2}\ ).$$

Repeating the same process for $g_1^L$ as in obtaining ($\flat$1) and ($\flat$2), we have

$$(\flat' 1) \quad g_2^L(x) \approx \log \frac{1}{|\pi(x)|} \ (x \in \cup_n \mathcal{A}_{n,2}\ )$$

$$(\flat' 2) \quad g_2^L(x) = o(\log \frac{1}{|\pi(x)|}) \ (\pi(x) \to 0, x \in \cup_n \mathcal{A}_{n,1}\ )$$

or

$$(\flat" 1) \quad g_2^L(x) \approx \log \frac{1}{|\pi(x)|} \ (x \in \cup_n \mathcal{A}_{n,1}\ )$$

$$(\flat" 2) \quad g_2^L(x) = o(\log \frac{1}{|\pi(x)|}) \ (\pi(x) \to 0, x \in \cup_n \mathcal{A}_{n,2}\ ).$$

Suppose that the estimates ($\flat$"1) and ($\flat$"2) hold. By ($\flat$1) and ($\flat$"1), we find that $f(\cup_n \mathcal{A}_{n,1})$ is minimally thin at $\zeta_1'$. In fact, there exists a positive constant $\beta_0$ such that $\beta_0 g_{\zeta_1'}^{f(R_0)} \leq g_{\zeta_2'}^{f(R_0)}$ on $f(\cup_n \mathcal{A}_{n,1})$, that is, $f(\cup_n \mathcal{A}_{n,1}) \subset \{x' \in f(R_0) | \beta_0 g_{\zeta_1'}^{f(R_0)} \leq g_{\zeta_2'}^{f(R_0)}\}$. Using the same argument as that in the former part of the proof of this lemma, we find that $\{x' \in$

$f(R_0)|\beta_0 g_{\zeta_1'}^{f(R_0)} \leq g_{\zeta_2'}^{f(R_0)}\}$ is minimally thin at $\zeta_1'$. Hence $f(\cup_n \mathcal{A}_{n,1})$ is minimally thin at $\zeta_1'$.

By ($\flat 2$) we can prove that there exists a subsequence $\{n_l\}$ of $\mathbb{N} = \{n\}$ such that $f(\cup_l \mathcal{A}_{n_l,2})$ is minimally thin at $\zeta_1'$. This fact will be proved afterwards. Hence $f(\cup_l \mathcal{A}_{n_l})$ is minimally thin at $\zeta_1'$ because $f(\cup_n \mathcal{A}_{n,1})$ is minimally thin at $\zeta_1'$. Since $[f(R_0) \setminus f(\cup_l \mathcal{A}_{n_l})] \cup \{\zeta_1'\}$ is a minimal fine neighborhood of $\zeta_1'$, we can take a connected component $G_2$ of $[f(R_0) \setminus f(\cup_l \mathcal{A}_{n_l})] \cup \{\zeta_1'\}$ such that $G_2 \cup \{\zeta_1'\}$ is a minimal fine neighborhood of $\zeta_1'$ (cf. [14, Corollaire 2 in p.206]). This is a contradiction. Hence we have the estimates ($\flat' 1$) and ($\flat' 2$).

We still remain to prove that there exists a subsequence $\{n_l\}$ of $\mathbb{N} = \{n\}$ such that $f(\cup_l \mathcal{A}_{n_l,2})$ is minimally thin at $\zeta_1'$. By ($\flat 2$) we can take a subsequence $\{n_l\}$ of $\mathbb{N} = \{n\}$ with

$$g_{\zeta_1'}^{f(R_0)}(x') \leq \frac{n_l}{l^2} \ (x' \in f(\cup_l \mathcal{A}_{n_l,2}) \ ).$$

From this estimate it follows that $f(\cup_l \mathcal{A}_{n_l,2})$ is minimally thin at $\zeta_1'$. In fact, we take a point $x_0'$ be a point of $f(R_0) \setminus Cl(f(\cup_l \mathcal{A}_{n_l,2}))$. Then, by ($\sharp\sharp$) in Lemma 4.1, the definition of capacitary potential, and the same estimate for capacity as in the latter part of the proof of Lemma 4.1, we have

$$\begin{aligned}
0 \leq {}^{f(R_0)}\widehat{\mathrm{R}}_{g_{\zeta_1'}^{f(R_0)}}^{f(\cup_{l \geq m} \mathcal{A}_{n_l,2})}(x_0') &\leq \sum_{l=m}^{\infty} {}^{f(R_0)}\widehat{\mathrm{R}}_{g_{\zeta_1'}^{f(R_0)}}^{f(\mathcal{A}_{n_l,2})}(x_0') \\
&\leq \sum_{l=m}^{\infty} \frac{n_l}{l^2} \ {}^{f(R_0)}\widehat{\mathrm{R}}_1^{f(\mathcal{A}_{n_l,2})}(x_0') \\
&\leq \sum_{l=m}^{\infty} \frac{n_l}{l^2} \int_{Cl(f(\mathcal{A}_{n_l,2}))} g_{x_0'}^{f(R_0)} d\mu_{n_l,2} \\
&\leq \alpha_0 \sum_{l=m}^{\infty} \frac{n_l}{l^2} \ \mu_{n_l,2}(Cl(f(\mathcal{A}_{n_l,2}))) \\
&\approx \sum_{l=m}^{\infty} \frac{n_l}{l^2} \ cap(Cl(f(\mathcal{A}_{n_l,2})), f(R_0)) \\
&\approx \sum_{l=m}^{\infty} \frac{n_l}{n_l l^2} \approx \sum_{l=m}^{\infty} \frac{1}{l^2} \to 0 \ (m \to +\infty),
\end{aligned}$$

where, $\alpha_0 = \sup\{g_{x_0'}^{f(R_0)}(x') | x' \in Cl(f(\cup_l \mathcal{A}_{n_l,2}))\}$. Hence we have $\lim_{m \to +\infty} {}^{f(R_0)}\widehat{\mathrm{R}}_{g_{\zeta_1'}^{f(R_0}}^{f(\cup_{l \geq m} \mathcal{A}_{n_l,2})}(x_0') = 0$. If $m$ is sufficiently large,

$f(\cup_{l\geq m}\mathcal{A}_{n_l,2})$ is minimally thin at $\zeta_1'$. Since $f(\cup_{l\leq m}\mathcal{A}_{n_l,2})$ is relatively compact, it is minimally thin at $\zeta_1'$. Hence $f(\cup_l\mathcal{A}_{n_l,2})$ is minimally thin at $\zeta_1'$.

The proof is herewith complete. $\square$

For an integer $l$, take the bounded simply connected domain $Q_l$ whose boundary in the closed polygonal line without self-intersections and which has four vertexes $((3a_l+b_{l+1})/4, (3a_l+b_{l+1})/32), ((3a_l+b_{l+1})/4, -(3a_l+b_{l+1})/32), ((a_{l-1}+3b_l)/4, -(a_{l-1}+3b_l)/32), ((a_{l-1}+3b_l)/4, (a_{l-1}+3b_l)/32)$ in positive cyclic order. Set $Q=\cup_{l=1}^{\infty}Q_l$ and $\mathcal{D}_{Q,j}=\mathcal{D}_{I,j}\backslash\pi^{-1}(Q)$ $(j=1,2)$. By Lemma 4.2 and Harnack's inequality with respect to $L$, we find that

(1) there exists a positive constant $\kappa_0$ such that
$\dfrac{1}{\kappa_0}\log\dfrac{1}{|\pi(x)|}\leq g_j^L(x)\leq\kappa_0\log\dfrac{1}{|\pi(x)|}$ $(x\in\mathcal{D}_{Q,j})$ $(j=1,2)$;

(2) $g_j^L(x)=o(\log\dfrac{1}{|\pi(x)|})$ $(\pi(x)\to 0, x\in\mathcal{D}_{Q,j+(-1)^{j-1}})$ $(j=1,2)$.

Set $E_1'=\{x'\in f(R_0)|g'_{\zeta_1'}(x')>g'_{\zeta_2'}(x')\}$, $E_2'=\{x'\in f(R_0)|g'_{\zeta_1'}(x')<g'_{\zeta_2'}(x')\}$, and $E_3'=\{x'\in f(R_0)|g'_{\zeta_1'}(x')=g'_{\zeta_2'}(x')\}$. Set $E_3=f^{-1}(E_3')=\{x\in R_0|g_1^L(x)=g_2^L(x)\}$ and $\gamma_j=\pi^{-1}(\partial Q)\cap\mathcal{D}_{I,j}$. By (1) and (2), we may suppose that there exists an integer $N_1$ such that, for any integer $n(\geq N_1)$, $E_3\cap\mathcal{B}_{(a_n+b_{n+1})/2}\subset\pi^{-1}(Q)$, $g_1^L>g_2^L$ on $\gamma_1\cap\mathcal{B}_{(a_n+b_{n+1})/2}$ and $g_1^L<g_2^L$ on $\gamma_2\cap\mathcal{B}_{(a_n+b_{n+1})/2}$. Hence, by the implicit function theorem, $E_3'\cap f(\mathcal{B}_{(a_{N_1}+b_{N_1+1})/2})$ consists of infinitely many connected components $E_{3,l}'(\subset f(\pi^{-1}(Q_l)), l\geq N_1+1)$ which are piecewise analytic closed curves because each $g'_{\zeta_j'}$ is harmonic on $f(R_0)$. Hence each $E_j'\cap f(\mathcal{B}_{(a_{N_1}+b_{N_1+1})/2})$ is a planar region, that is, each $E_j\cap\mathcal{B}_{(a_{N_1}+b_{N_1+1})/2}$ is planar region. Set $K_j=E_j\cap\mathcal{B}_{(a_{N_1}+b_{N_1+1})/2}$ and $E_{3,l}=f^{-1}(E_{3,l}')$. By Koebe's theorem and R. de Possel's theorem (cf. [20, Theorems IX.32 and IX.22], [19, Theorem 9-1]) there exist plane regions $\mathcal{E}_j$ $(j=1,2)$ of $\mathbb{C}$ and conformal mappings $\phi_j$ $(j=1,2)$ from $K_j$ onto $\mathcal{E}_j$ $(j=1,2)$ such that $\mathbb{C}\setminus\mathcal{E}_j$ $(j=1,2)$ consist of infinitely many parallel segments $\ell_{j,l}$ to the real axis with

$$\ell_{j,l}=\begin{cases}\bigcap\left\{Cl(\phi_j(M))\;\middle|\;\begin{array}{l}M\text{ is a subdomain of }E_j\text{ with}\\ Cl(M)\supset E_{3,l}\end{array}\right\}, & \\ \qquad\qquad\text{for } l>N_1, & \\ \bigcap\left\{Cl(\phi_j(M))\;\middle|\;\begin{array}{l}M\text{ is a subdomain of }E_j\text{ with}\\ Cl(M)\supset\mathcal{C}_{(a_{N_1}+b_{N_1+1})/2}\cap\mathcal{D}_{I,j}\end{array}\right\}. & \\ \qquad\qquad\text{for } l=N_1. & \end{cases}$$

Set $\ell_j = \cap_{n \geq N_1+1} Cl(\cup_{l \geq n} \ell_{j,l})$ $(j = 1, 2)$.

**Lemma 4.3.** *Each* $\ell_j$ *is a singleton.*

*Proof.* Suppose that $\sharp \ell_j \geq 2$ $(j = 1, 2)$. We remark that each $\ell_j$ is connected. In fact, suppose that $\ell_j$ is disconnected. Let $\Lambda_{j,1}$ be a component of $\ell_j$. Set $\Lambda_{j,2} = \ell_j \setminus \Lambda_{j,1}$. We can take two Jordan curves $\mathcal{C}_{j,1}$ and $\mathcal{C}_{j,2}$ in $\mathcal{E}_j$ such that, for $k = 1, 2$, each bounded region $G_{j,k,1}$ determined by $\mathcal{C}_{j,k}$ in $\mathbb{C}$ contains $\Lambda_{j,k}$, and that $Cl(G_{j,1,1}) \cap Cl(G_{j,2,1}) = \emptyset$. By the definition of $\Lambda_{j,k}$, each $G_{j,k,1}$ contains infinitely many $\ell_{j,l}$. Since $\pi \circ \phi_j^{-1}$ is continuous on $\mathcal{E}_j$ and $\mathcal{C}_{j,k}$ is a compact subset of $\mathcal{E}_j$, $\pi \circ \phi_j^{-1}(\mathcal{C}_{j,k})$ is a compact subset of $\pi(K_j)$, and hence there exists uniquely a component $M_{j,k,1}$ of $\pi(K_j) \setminus \pi \circ \phi_j^{-1}(\mathcal{C}_{j,k})$ such that $Cl(M_{j,k,1})$ is a neighborhood of the origin. Denote by $M_{j,k,2}$ the union of component of $[\pi(K_j) \setminus \pi \circ \phi_j^{-1}(\mathcal{C}_{j,k})] \cup M_{j,k,1}$. It is easily seen that $Cl(M_{j,k,1})$ (resp. $Cl(M_{j,k,2})$) contains infinitely (resp. at most finitely) many components $\pi(E_{3,l})$ of $\pi(E_3 \cap \mathcal{B}_{(a_{N_1}+b_{N_1+1})/2})$ because $\pi(E_{3,l}) \subset Q_l$ $(l \geq N_1+1)$. Let $G_{j,k,2}$ be unbounded regions determined by $\mathcal{C}_{j,k}$ in $\mathbb{C}$. We can prove that ($\natural$) $\phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \subset G_{j,k,1} \cap \mathcal{E}_j$ or ($\natural'$) $\phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \subset G_{j,k,2} \cap \mathcal{E}_j$. Suppose this fact does not hold, that is, $\phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \cap G_{j,k,1} \cap \mathcal{E}_j \neq \emptyset$ and $\phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \cap G_{j,k,2} \cap \mathcal{E}_j \neq \emptyset$. Then we can find points $\xi_{j,k,i} \in \phi_j(\pi^{-1}(M_{j,k,1}) \cap K_j) \cap G_{j,k,i} \cap \mathcal{E}_j$ $(i = 1, 2)$. Since $\pi(\phi_j^{-1}(\xi_{j,k,i})) \in M_{j,k,1}$ and $M_{j,k,1}$ is connected, we can find a curve $C$ in $M_{j,k,1}$ which joins $\pi(\phi_j^{-1}(\xi_{j,k,1}))$ to $\pi(\phi_j^{-1}(\xi_{j,k,2}))$. From the definition of component it is easily seen that the lift of $C$ in $K_j$ by $\pi$ meets $\phi_j^{-1}(C_{j,k})$ since $K_j \setminus \phi_j^{-1}(C_{j,k})$ has just two components $\phi_j^{-1}(G_{j,k,1} \cap \mathcal{E}_j)$ and $\phi_j^{-1}(G_{j,k,2} \cap \mathcal{E}_j)$. Hence $M_{j,k,1} \cap \pi \circ \phi_j^{-1}(\mathcal{C}_{j,k}) \neq \emptyset$. This is a contradiction.

We may assume that ($\natural$) holds. For, if ($\natural'$) holds, repeating the same argument as in case that ($\natural$) holds, we arrive at a contradiction. By ($\natural$) $\phi_j(\pi^{-1}(M_{j,k,2}) \cap K_j) \supset G_{j,k,2} \cap \mathcal{E}_j$. Hence $G_{j,k,1}$ (resp. $G_{j,k,2}$) contains infinitely (resp. at most finitely) many $\ell_{j,l}$ because $Cl(M_{j,k,1})$ (resp. $Cl(M_{j,k,2})$) contains infinitely (resp. at most finitely) many components $\pi(E_{3,l})$ of $\pi(E_3 \cap \mathcal{B}_{(a_{N_1}+b_{N_1+1})/2})$. Since $G_{j,k,2} \supset G_{j,k+(-1)^{k-1},1}$, $G_{j,k+(-1)^{k-1},1}$ contains at most finitely many components of $\ell_{j,l}$. This is a contradiction. Thus we conclude that each $\ell_j$ is connected.

Since each $\ell_j$ is connected, by [5, Theorem 8.26], all points of $\ell_j$ $(j = 1, 2)$ are regular boundary points of $\mathcal{E}_j$ $(j = 1, 2)$. $E_j'$ $(j = 1, 2)$ is minimally thin at $\zeta'_{j+(-1)^{j-1}}$, and hence $E_3'$ is minimally thin at $\zeta_j'$ $(j = 1, 2)$. By [14, Théorème 1 and Théorème 5], it is known that there exists a

Green potential $g_{\mu_j}(x') = \int g_{x'}^{f(R_0)} d\mu_j$ such that

$$g_{\mu_j}(\zeta'_j) < +\infty,$$

$$\lim_{x' \to \Delta^{R'}, x' \in E'_3} g_{\mu_j}(x') = +\infty,$$

because $\lim_{x' \to \zeta'_j} g_x^{f(R_0)}(x') = g_x^{f(R_0)}(\zeta'_j) < +\infty$. Since there exists an integer $N_2 (\geq N_1)$ such that $g_{\mu_j}(y') > 2 g_{\mu_j}(\zeta'_j)$ $(y' \in E'_3 \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}))$, for every $x' \in f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})$,

$$^{f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}\widehat{\mathrm{R}}_1^{E'_3 \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}(x') \leq \frac{g_{\mu_j}(x')}{2 g_{\mu_j}(\zeta'_j)}.$$

Hence

$$\begin{aligned} &\liminf_{x' (\in E'_j \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})) \to \zeta'_j} {}^{f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}\widehat{\mathrm{R}}_1^{E'_3 \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}(x') \\ \leq\ &\liminf_{x' (\in E'_j \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})) \to \zeta'_j} \frac{g_{\mu_j}(x')}{2 g_{\mu_j}(\zeta'_j)} \\ =\ &\liminf_{x' \to \zeta'_j} \frac{g_{\mu_j}(x')}{2 g_{\mu_j}(\zeta'_j)} = \frac{1}{2} < 1, \end{aligned}$$

because each $E'_j \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}))$ is not minimally thin at $\zeta'_j$. Hence there exists a sequence $\{x'_{l,j}\} (\subset E'_j \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}),\ j = 1,2)$ such that, for $j = 1,2$,

$$\lim_{l \to \infty} x'_{l,j} = \zeta'_j,$$

$$\lim_{l \to \infty} {}^{f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}\widehat{\mathrm{R}}_1^{E'_j \cap f(\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})}(x'_{l,j}) < 1.$$

Set

$$B_{N_2}^{(j)} = Int[Cl(\phi_j(E_j \cap \mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}))]\ (j = 1,2),$$

where $Int[Cl(\phi_j(E_j \cap \mathcal{B}_{(a_{N_2}+b_{N_2+1})/2}))]$ stands for the interior of the closure of $\phi_j(E_j \cap \mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})$ in $\mathbb{C}$. For $\phi_j \circ f^{-1}$ we define $L_{\phi_j \circ f^{-1}}$ as $L_f$ in the first part of this section. The above inequality implies that there exist points $z_j \in \ell_j$ $(j = 1,2)$ and sequences $\{z_{l,j}\}$ $(\subset \mathcal{E}_j \cap B_{N_2}^{(j)},\ j = 1,2)$ such that, for $j = 1,2$,

$$\lim_{l \to \infty} z_{l,j} = z_j\ (j = 1,2),$$

$$\lim_{l \to \infty} {}^{B_{N_2}^{(j)}}\widehat{\mathrm{R}}_1^{\cup_{l > N_2} \ell_{l,j}, L_{\phi_j \circ f^{-1}}}(z_{l,j}) < 1,$$

where ${}^{B_{N_2}^{(j)}}\widehat{\mathrm{R}}_1^{\cup_{l>N_2}\ell_{l,j},L_{\phi_j\circ f^{-1}}}$ stands for the balayage of 1 relative to $\cup_{l>N_2}\ell_{l,j}$ on $B_{N_2}^{(j)}$ with respect to $L_{\phi_j\circ f^{-1}}$. Hence each $z_j$ is an irregular boundary point of $\phi_j(E_j\cap\mathcal{B}_{(a_{N_2}+b_{N_2+1})/2})$ with respect to $L_{\phi_j\circ f^{-1}}$. By [7, Theorem 9.1] and [5, Theorem 10.3], each $z_j$ is an irregular boundary points of $\mathcal{E}_j$ in the usual sense. This is a contradiction. Therefore we have the desired result. $\square$

Let $N_1$ be an integer as in the definition of $\ell_j$. Let $g_\xi^{\mathcal{E}_j}$ be the Green function with pole at $\xi$ (resp. $x$) on $\mathcal{E}_j$. By Lemma 4.3, for $j=1,2$, there exists a sequence $\{\xi_{j,n}\}$ in $\mathcal{E}_j$ such that $\lim_{n\to\infty}\xi_{j,n}=z_j$ and there exists $\lim_{n\to\infty} g_{\xi_{j,n}}^{\mathcal{E}_j}$ on $\mathcal{E}_j$. For $j=1,2$, set $g_{z_j}^{\mathcal{E}_j}=\lim_{n\to\infty} g_{\xi_{j,n}}^{\mathcal{E}_j}$ and $g_j=g_{z_j}^{\mathcal{E}_j}\circ\phi_j$. Each $g_j$ is a positive harmonic function on $K_j$. For $j=1,2$, set

$$S_j(g_j)(x)=\inf_s s(x),$$

where $s$ runs over the space of positive superharmonic functions $s$ on $R_0$ satisfying $s\ge g_j$ on $K_j$. By Perron-Wiener-Brelot method each $S_j(g_j)$ is a positive harmonic function on $R_0$. Using the same argument as that in the proof of Theorem 1, we find that the following inequality

$$(**)\qquad S_j(g_j)-{}^{R_0}\widehat{\mathrm{R}}_{S_j(g_j)}^{R_0\setminus K_j}\ge g_j$$

holds on $K_j$ $(j=1,2)$. Since $\sharp\Delta_1^R=1$ or 2 by means of [17, Theorem 3], by the Martin representation theorem, we find that there exist at most two minimal functions $h_{j,k}$ $(k=1,2)$ on $R_0$ with $S_j(g_j)=h_{j,1}+h_{j,2}$ on $R_0$. Hence, by the above inequality $(**)$, we have

$$\begin{aligned} h_{j,1}+h_{j,2}=S_j(g_j) &\ge {}^{R_0}\widehat{\mathrm{R}}_{h_{j,1}+h_{j,2}}^{R_0\setminus K_j}+g_j \\ &> {}^{R_0}\widehat{\mathrm{R}}_{h_{j,1}}^{R_0\setminus K_j}+{}^{R_0}\widehat{\mathrm{R}}_{h_{j,2}}^{R_0\setminus K_j} \end{aligned}$$

on $K_j$. Therefore we find that there exists a minimal function $h_j$ $(j=1,2)$ on $R_0$ such that $h_j\ne {}^{R_0}\widehat{\mathrm{R}}_{h_j}^{R_0\setminus K_j}$. Hence, by the definition of minimal thinness, $R_0\setminus K_j$ is minimally thin at the minimal boundary point corresponding to $h_j$. Since $K_1\cap K_2=\emptyset$, we find that $\sharp\Delta_1^R=2$.

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# Hyperbolic Riemann surfaces without unbounded positive harmonic functions

**Hiroaki Masaoka and Shigeo Segawa**

**Abstract.**

Let $R$ be an open Riemann surface with Green's functions. It is proved that there exist no unbounded positive harmonic functions on $R$ if and only if the minimal Martin boundary of $R$ consists of finitely many points with positive harmonic measure.

## §1. Introduction

Denote by $O_G$ the class of open Riemann surfaces $R$ such that there exist no Green's functions on $R$. We say that an open Riemann surface $R$ is *parabolic* (resp. *hyperbolic*) if $R$ belongs (resp. does not belong) to $O_G$.

For an open Riemann surface $R$, we denote by $HP(R)$ (resp. $HB(R)$) the class of *positive* (resp. *bounded*) harmonic functions on $R$. It is well-known that if $R$ is parabolic, then $HP(R)$ and $HB(R)$ consist of constant functions (cf. [5]).

Hereafter, we consider only hyperbolic Riemann surfaces $R$. Let $\Delta = \Delta^R$ and $\Delta_1 = \Delta_1^R$ the *Martin boundary* of $R$ and the *minimal Martin boundary* of $R$, respectively. The purpose of this paper is to prove the following.

**Theorem.** *Suppose that $R$ is hyperbolic. Then the followings are equivalent*:

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(i) *there exist no unbounded positive harmonic functions on $R$, i.e. $HP(R) \subset HB(R)$,*

(ii) *the minimal Martin boundary $\Delta_1^R$ of $R$ consists of finitely many points with positive harmonic measure.*

The above theorem combined with the Martin representation theorem yields the following.

Corollary. *Suppose that $R$ is hyperbolic and there exist no unbounded positive harmonic functions on $R$. Then the linear space $HB(R)$ of bounded harmonic functions on $R$ is of finite dimension.*

Denote by $\omega_z(\cdot)$ the *harmonic measure* on $\Delta^R$ with respect to $z \in R$. We also denote by $k_\zeta(z)$ $((\zeta, z) \in (R \cup \Delta^R) \times R)$ the *Martin kernel* on $R$ with pole at $\zeta$. The following proposition, which is easily proved, plays fundamental role in the proof of the above theorem.

Proposition. *Let $\zeta$ belong to $\Delta_1^R$. Then the Martin kernel $k_\zeta(\cdot)$ with pole at $\zeta$ is bounded on $R$ if and only if the harmonic measure $\omega.(\{\zeta\})$ of the singleton $\{\zeta\}$ is positive.*

## §2. Proof of Theorem

Let $k_\zeta(\cdot)$ be the Martin kernel on $R$ with pole at $\zeta$ such that $k_\zeta(a) = 1$ for a fixed point $a \in R$. Consider the *canonical measure* $\chi$ of the harmonic function 1 in the Martin representation theorem, that is

$$1 = \int_{\Delta_1^R} k_\xi(z) d\chi(\xi). \tag{2.1}$$

As a relation between $\chi$ and harmonic measure $\omega_z$, the following is known (c.f. [1, Satz 13.4]):

$$d\omega_z(\xi) = k_\xi(z) d\chi(\xi). \tag{2.2}$$

We first give the proof of Proposition in the introduction.

*Proof of Proposition.* We assume that the Martin kernel $k_\zeta(z)$ with pole at $\zeta \in \Delta_1^R$ is bounded on $R$. Take a positive constant $M$ such that $k_\zeta(z) \leq M$ on $R$. Then, by the Martin representation theorem, we deduce that

$$\int_{\Delta_1^R} k_\xi(z) d\delta_\zeta(\xi) = k_\zeta(z) \leq M = \int_{\Delta_1^R} M k_\xi(z) d\chi(\xi),$$

where $\delta_\zeta$ is the Dirac measure on $\Delta_1^R$ supported at $\zeta$. Hence, by virtue of the fact that the mapping of $HP$ functions to their canonical measures are lattice isomorphic (cf. [1,Forgesatz 13.1]), we see that $\delta_\zeta \le M\chi$ or $(1/M)\delta_\zeta \le \chi$ on $\Delta_1^R$. From this and (2.2) it follows that

$$0 < \frac{k_\zeta(z)}{M} = k_\zeta(z)\frac{\delta_\zeta(\{\zeta\})}{M} \le k_\zeta(z)\chi(\{\zeta\}) = \omega_z(\{\zeta\}),$$

thus we have proved the '*only if part*'.

We next assume that $\omega_z(\{\zeta\}) > 0$. Then, by (2.2), we have

$$0 < \omega_z(\{\zeta\}) = k_\zeta(z)\chi(\{\zeta\}). \tag{2.3}$$

Hence $c := \chi(\{\zeta\})$ is a positive constant. On the other hand, $\omega_z(\{\zeta\}) \le 1$ on $R$. Therefore, in view of (2.3), we see that $k_\zeta(z) \le c^{-1}$ on $R$. Thus we have proved the 'if part'.

Applying Proposition proved above, we next give the proof of Theorem in the introduction.

*Proof of Theorem.* Since the implication (ii) $\Rightarrow$ (i) easily follows from Proposition and the Martin representation theorem, we only have to show the implication (i) $\Rightarrow$ (ii).

Suppose that (ii) is not the case although we are assuming that $HP(R) \subset HB(R)$. Then it easily follows from Proposition that $\Delta_1^R$ does not contain a point $\zeta$ with $\omega_\cdot(\{\zeta\}) = 0$. Therefore $\Delta_1^R$ consists of countably infinitely many points $\zeta_n$ $(n \in \mathbb{N})$ with $\omega_\cdot(\{\zeta_n\}) > 0$ and moreover each Martin kernel $k_{\zeta_n}$ is bounded on $R$. Put $M_n := \sup_{z\in R} k_{\zeta_n}(z)$. Then we deduce that

$$\int_{\Delta_1^R} k_\xi(z)d\left(\frac{1}{M_n}\right)\delta_{\zeta_n}(\xi) = \frac{k_{\zeta_n}(z)}{M_n} \le 1 = \int_{\Delta_1^R} k_\xi(z)d\chi(\xi),$$

where $\delta_{\zeta_n}$ is the Dirac measure at $\zeta_n$ and $\chi$ is the measure in (2.1). Hence, by means of lattice isomorphic determination of canonical measures, we see that $(1/M_n)\delta_{\zeta_n} \le \chi$ for every $n \in \mathbb{N}$. Since the supports $\mathrm{supp}(\delta_{\zeta_n})$ of $\{\delta_{\zeta_n}\}$ are mutually disjoint, this implies that $\sum_{n=1}^\infty (1/M_n)\delta_{\zeta_n} \le \chi$. Therefore we conclude that

$$\sum_{n=1}^\infty \frac{k_{\zeta_n}(z)}{M_n} = \int_{\Delta_1^R} k_\xi(z)d\left(\sum_{n=1}^\infty \frac{\delta_{\zeta_n}(\xi)}{M_n}\right) \le \int_{\Delta_1^R} k_\xi(z)d\chi(\xi) = 1.$$

Since $k_{\zeta_n}(a) = 1$, this yields that $\sum_{n=1}^\infty \dfrac{1}{M_n} \le 1$ and hence

$$\lim_{n\to\infty} M_n = +\infty. \tag{2.4}$$

In view of (2.4), we can choose a subsequence $\{M_{n_i}\}$ of $\{M_n\}$ such that

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{M_{n_i}}} < +\infty. \tag{2.5}$$

Put

$$h(z) := \sum_{i=1}^{\infty} \frac{1}{\sqrt{M_{n_i}}} k_{\zeta_{n_i}}(z).$$

By (2.5) and the Harnack principle, $h(z)$ is convergent and a positive harmonic function on $R$ since $k_\zeta(a) = 1$ for every $\zeta$. On the other hand, by the definition, $h(z) \geq \frac{1}{\sqrt{M_{n_i}}} k_{\zeta_{n_i}}(z)$ on $R$ and therefore

$$\sup_{z\in R} h(z) \geq \frac{1}{\sqrt{M_{n_i}}} \sup_{z\in R} k_{\zeta n_i}(z) = \frac{1}{\sqrt{M_{n_i}}} M_{n_i} = \sqrt{M_{n_i}}.$$

Hence, by means of (2.4), we see that $\sup_{z\in R} h(z) = +\infty$ or $h \notin HB(R)$. This contradicts our primary assumption $HP(R) \subset HB(R)$.

The proof is herewith complete.

## §3. Examples

In this section we will give examples of open Riemann surfaces $R$ satisfying the condition $HP(R) \subset HB(R)$ in Theorem. We can moreover require for $\Delta_1^R$ to consist of $p$ points of positive harmonic measure for an arbitrarily given integer $1 \leq p < \infty$ in advance.

Let $O_{HP}$ be the class of open Riemann surfaces on which there exists no nonconstant positive harmonic functions. Recall the class $O_G$ of open Riemann surfaces on which there exist no Green's functions. Then it holds that $O_G \subset O_{HP}$ (cf. e.g. [5]). Moreover the inclusion $O_G \subset O_{HP}$ is strict, that is, there exists an open Riemann surface $T$ belonging to $O_{HP} \setminus O_G$ (cf. [6], [5]). Since $HP(T)$ consists of only constant functions, the Martin boundary $\Delta^T$ of $T$ and hence the minimal Martin boundary $\Delta_1^T$ of $T$ also consists of a single point $\zeta_0$ and the Martin kernel $k_{\zeta_0}$ on $T$ with pole at $\zeta_0$ is equal to the constant function 1.

Consider a $p$-sheeted $(1 \leq p < \infty)$ *unlimited* (possibly branched) covering surface $\tilde{T}$ of $T$ with its projection map $\pi$. Here we say that $\tilde{T}$ is unlimited if the following condition is satisfied: for any arc $C$ in $T$ with $a$ as its initial point and any point $\tilde{a}$ over $a$, i.e. $\pi(\tilde{a}) = a$, there exists an arc $\tilde{C}$ in $\tilde{T}$ with $\tilde{a}$ as its initial point such that $\pi(\tilde{C}) = C$. By our preceeding result (cf. [2]), the minimal Martin boundary $\Delta_1^{\tilde{T}}$ of

$\tilde{T}$ consists of at most $p$ points. Moreover, there exists $\tilde{T}$ such that $\Delta_1^{\tilde{T}}$ consists of exactly $p$ points. Put $\Delta_1^{\tilde{T}} = \{\tilde{\zeta}_1, \cdots, \tilde{\zeta}_q\}$ $(1 \leq q \leq p)$ and denote by $\tilde{k}_{\tilde{\zeta}_i}$ the Martin kernel on $\tilde{T}$ with pole at $\tilde{\zeta}_i$. As a relation between $\tilde{k}_{\tilde{\zeta}_i}$ and the Martin kernel $k_{\zeta_0}$ on $T$, it holds that

$$\sum_{\tilde{z} \in \pi^{-1}(z)} \tilde{k}_{\tilde{\zeta}_i}(\tilde{z}) \leq c_i k_{\zeta_0}(z),$$

where $c_i$ is a positive constant (cf. [3]). Hence $\tilde{k}_{\tilde{\zeta}_i}$ is bounded on $\tilde{T}$ for every $i$ $(1 \leq i \leq q)$ since $k_{\zeta_0}^T = 1$. Consequently, by virtue of the Martin representation theorem, we see that $HP(\tilde{T}) \subset HB(\tilde{T})$.

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# On a covering property of rarefied sets at infnity in a cone

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**Abstract.**

This paper gives a quantitative property of rarefied sets at $\infty$ of a cone. The proof is based on the fact in which the estimations of Green potential and Poisson integral with measures are connected with a kind of densities of the measures modified from the measures.

## §1. Introduction

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n$ ($n \geq 2$) the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, y)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points $P$ and $Q$ in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin $O$ of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $S$ in $\mathbf{R}^n$ are denoted by $\partial S$ and $\bar{S}$, respectively.

We introduce a system of spherical coordinates $(r, \Theta), \Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, y)$ by $y = r \cos\theta_1$.

The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set $\Omega$, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with $\Theta$ and $\Omega$, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n;\ r \in \Lambda,\ (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half-space $\mathbf{R}_+ \times \mathbf{S}^{n-1}_+ = \{(X, y) \in \mathbf{R}^n;\ y > 0\}$ will be denoted by $\mathbf{T}_n$.

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Let $\Omega$ be a domain on $\mathbf{S}^{n-1}$ $(n \geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$\begin{aligned}(\Lambda_n + \tau)f = 0 \quad & \text{on } \Omega,\\ f = 0 \quad & \text{on } \partial\Omega,\end{aligned}$$

where $\Lambda_n$ is the spherical part of the Laplace operator $\Delta_n$

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by $\tau_\Omega$ and the normalized positive eigenfunction corresponding to $\tau_\Omega$ by $f_\Omega(\Theta)$. We denote the solutions of the equation $t^2 + (n-2)t - \tau_\Omega = 0$ by $\alpha_\Omega, -\beta_\Omega$ $(\alpha_\Omega,\ \beta_\Omega > 0)$. If $\Omega = \mathbf{S}_+^{n-1}$, then $\alpha_\Omega = 1,\ \beta_\Omega = n-1$ and $f_\Omega(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1$, where $s_n$ is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

To simplify our consideration in the following, we shall assume that if $n \geq 3$, then $\Omega$ is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ (e.g. see Gilbarg and Trudinger [7, pp.88-89] for the definition of $C^{2,\alpha}$-domain).

By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain $\Omega$ on $\mathbf{S}^{n-1}(n \geq 2)$. We call it a cone. Then $\mathbf{T}_n$ is a special cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$.

It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, and the Martin functions at $\infty$ and at $O$ with respect to a reference point chosen suitably are given by $K(P;\infty,\Omega) = r^{\alpha_\Omega} f_\Omega(\Theta)$ and $K(P;O,\Omega) = \iota r^{-\beta_\Omega} f_\Omega(\Theta)$ $(P = (r,\Theta) \in C_n(\Omega))$, respectively, where $\iota$ is a positive number.

Let $E$ be a bounded subset of $C_n(\Omega)$. Then $\hat{R}^E_{K(\cdot;\infty,\Omega)}$ is bounded on $C_n(\Omega)$ and hence the greatest harmonic minorant of $\hat{R}^E_{K(\cdot;\infty,\Omega)}$ is zero. When by $G^\Omega(P,Q)$ $(P \in C_n(\Omega), Q \in C_n(\Omega))$ and $G^\Omega\xi(P)$ $(P \in C_n(\Omega))$ we denote the Green function of $C_n(\Omega)$ and the Green potential with a positive measure $\xi$ on $C_n(\Omega)$, respectively, we see from the Riesz decomposition theorem that there exists a unique positive measure $\lambda_E$ on $C_n(\Omega)$ such that

$$\hat{R}^E_{K(\cdot;\infty,\Omega)}(P) = G^\Omega\lambda_E(P) \quad (P \in C_n(\Omega)).$$

Let $E$ be a subset of $C_n(\Omega)$ and $E_k = E \cap I_k$ $(k = 0, 1, 2, \ldots)$, where $I_k = \{P = (r,\Theta) \in \mathbf{R}^n; 2^k \leq r < 2^{k+1}\}$. A subset $E$ of $C_n(\Omega)$ is said to be *rarefied at* $\infty$ *with respect to* $C_n(\Omega)$, if

$$\sum_{k=0}^{\infty} 2^{-k\beta_\Omega}\lambda_{E_k}(C_n(\Omega)) < +\infty.$$

*Remark* 1. This definition of rarefied sets was given by Essén and Jackson [4] for sets in the half-space. This exceptional sets were originally investigated in Ahlfors and Heins [1] and Hayman [8] in connection with the regularity of value distribution of subharmonic functions in the half plane.

As in $\mathbf{T}_n$ (Essén and Jackson [4, Remark 4.4], Aikawa and Essén [2, Definition 12.4, p.74]) and in $\mathbf{T}_2$ (Hayman [9, p.474]), we proved

**Theorem A** (Miyamoto and Yoshida [10, Theorem 2]). *A subset $E$ of $C_n(\Omega)$ is rarefied at $\infty$ with respect to $C_n(\Omega)$ if and only if there exists a positive superharmonic function $v(P)$ in $C_n(\Omega)$ such that*

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K(P;\infty,\Omega)} = 0$$

*and* $E \subset \{P=(r,\Theta)\in C_n(\Omega);\ v(P)\geq r^{\alpha_\Omega}\}$.

In this paper, we shall give a quantitative property of rarefied sets at $\infty$ with respect to $C_n(\Omega)$ (Theorem 2), which extends a result obtained by Essén, Jackson and Rippon [5] with respect to $\mathbf{T}_n$ and complements Azarin's result (Corollary 1). It follows from two results. One is another characterization of rarefied sets at $\infty$ with respect to $C_n(\Omega)$ (Theorem A). The other is the fact that the value distributions of Green potential and Poisson integral with respect to any positive measure on $C_n(\Omega)$ and $\partial C_n(\Omega)$ are connected with a kind of densities of the measures modified from the measures, respectively (Theorem 1). Our proof is completely different from the way used by Essén, Jackson and Rippon [5] and is essentially based on Hayman [8], Ušakova [12] and Azarin [3].

In order to avoid complexity of our proofs, we shall assume $n\geq 3$. All our results in this paper are true, even if $n=2$.

## §2. Statements of results

In the following we denote the sets $I\times\Omega$ and $I\times\partial\Omega$ with an interval $I$ on $\mathbf{R}$ by $C_n(\Omega;I)$ and $S_n(\Omega;I)$. By $S_n(\Omega)$ we denote $S_n(\Omega;(0,+\infty))$ which is $\partial C_n(\Omega)-\{O\}$. We shall also denote a ball in $\mathbf{R}^n$ having a center $P$ and a radius $r$ by $B(P,r)$.

Let $m$ be any positive measure on $\mathbf{R^n}$. Let $q$ and $\varepsilon$ be two positive numbers. When for each $P=(r,\Theta)\in\mathbf{R}^n-\{O\}$ we set

$$M(P;m,q)=\sup_{0<\rho\leq 2^{-1}r}\frac{m(B(P,\rho))}{\rho^q},$$

the set $\{P \in \mathbf{R}^n - \{O\};\ M(P; m, q)r^q > \varepsilon\}$ is denoted by $\Psi(\varepsilon; m, q)$.

*Remark* 2. If $m(\{P\}) > 0$ ($P \neq O$), then $M(P; m, q) = +\infty$ for any positive number $q$ and hence $\{P \in \mathbf{R}^n - \{O\};\ m(\{P\}) > 0\} \subset \Psi(\varepsilon; m, q)$ for any positive number $\varepsilon$.

Let $\mu$ be any positive measure on $C_n(\Omega)$ such that $G^\Omega\mu(P) \not\equiv +\infty$ ($P \in C_n(\Omega)$). The positive measure $m_\mu^{(1)}$ on $\mathbf{R}^n$ is defined by

$$dm_\mu^{(1)}(Q) = \begin{cases} t^{-\beta_\Omega} f_\Omega(\Phi) d\mu(t, \Phi) & (Q = (t, \Phi) \in C_n(\Omega;\ (1, +\infty))) \\ 0 & (Q \in \mathbf{R}^n - C_n(\Omega;\ (1, +\infty))). \end{cases}$$

Let $\nu$ be any positive measure on $S_n(\Omega)$ such that the Poisson integral

$$\Pi^\Omega\nu(P) = \int_{S_n(\Omega)} \frac{\partial G^\Omega(P, Q)}{\partial n_Q} d\nu(Q) \not\equiv +\infty \quad (P \in C_n(\Omega)),$$

where $\frac{\partial}{\partial n_Q}$ denotes the differentiation at $Q$ along the inward normal into $C_n(\Omega)$. We define the positive measure $m_\nu^{(2)}$ on $\mathbf{R}^n$ by

$$dm_\nu^{(2)}(Q) = \begin{cases} t^{-\beta_\Omega - 1} \frac{\partial f_\Omega(\Phi)}{\partial n_\Phi} d\nu(Q) & (Q = (t, \Phi) \in S_n(\Omega;\ (1, +\infty))) \\ 0 & (Q \in \mathbf{R}^n - S_n(\Omega;\ (1, +\infty))). \end{cases}$$

*Remark* 3. We remark from Miyamoto and Yoshida [10, (i) of Lemma 1] (resp. [10, (i) of Lemma 4]) that the total mass of $m_\mu^{(1)}$ (resp. $m_\nu^{(2)}$) is finite.

The following Theorem 1 gives a way to estimate the Green potential and the Poisson integral with measures on $C_n(\Omega)$ and $S_n(\Omega)$, respectively.

**Theorem 1**. *Let $\mu$ and $\nu$ be two positive measures on $C_n(\Omega)$ and $S_n(\Omega)$ such that $G^\Omega\mu(P) \not\equiv +\infty$ and $\Pi^\Omega\nu(P) \not\equiv +\infty$ ($P \in C_n(\Omega)$), respectively. Then for a sufficiently large $L$ and a sufficiently small $\varepsilon$ we have*

$$\{P = (r, \Theta) \in C_n(\Omega; (L, +\infty));\ G^\Omega\mu(P) \geq r^{\alpha_\Omega}\} \subset \Psi(\varepsilon; m_\mu^{(1)}, n-1), \tag{2.1}$$

$$\{P \in C_n(\Omega; (L, +\infty));\ \Pi^\Omega\nu(P) \geq r^{\alpha_\Omega}\} \subset \Psi(\varepsilon; m_\nu^{(2)}, n-1). \tag{2.2}$$

As in $\mathbf{T}_n$ (Essén, Jackson and Rippon [5, p.397]) we have the following result for rarefied sets in $C_n(\Omega)$ by using Theorems A and 1.

**Theorem 2**. *If a subset $E$ of $C_n(\Omega)$ is rarefied at $\infty$ with respect to $C_n(\Omega)$, then $E$ is covered by a sequence of balls $B_k$ (k=1,2,3,...) satisfying*

$$\sum_{k=1}^{\infty}(r_k/R_k)^{n-1} < +\infty, \tag{2.3}$$

*where $r_k$ is the radius of $B_k$ and $R_k$ is the distance between the origin and the center of $B_k$.*

*Remark* 4. By giving an example we shall show that the reverse of Theorem 2 is not true. When the radius $r_k$ of a ball $B_k$ and the distance $R_k$ between the origin and the center of it are given by $r_k = 3 \cdot 2^{k-1}k^{-\frac{1}{n-2}}, R_k = 3 \cdot 2^{k-1}$ $(k = 1, 2, 3, ...)$, they satisfy

$$\sum_{k=1}^{\infty}(r_k/R_k)^{n-1} = \sum_{k=1}^{\infty} k^{-(n-1)/(n-2)} < +\infty.$$

Let $C_n(\Omega')$ be a subcone of $C_n(\Omega)$ i.e. $\overline{\Omega'} \subset \Omega$. Suppose that these balls are so located: there is an integer $k_0$ such that $B_k \subset C_n(\Omega'), r_k/R_k < 2^{-1}$ $(k \geq k_0)$. Then the set $E = \cup_{k=k_0}^{\infty} B_k$ is not rarefied. This proof will be given at the end in the last section 4.

From this Theorem 2 and Miyamoto and Yoshida [10, Theorem 3], we immediately have the following corollary.

**Corollay 1** (Azarin [3, Theorem 2]). *Let $v(P)$ be a positive superharmonic function on $C_n(\Omega)$. Then $v(P)r^{\alpha_\Omega}$ uniformly converges to $c(v)f_\Omega(\Theta)$ as $r \to +\infty$ outside a set which is covered by a sequence of balls $B_k$ satisfying (2.3), where*

$$c(v) = \inf_{P \in C_n(\Omega)} \frac{v(P)}{K(P; \infty, \Omega)}.$$

## §3. Proof of Theorem 1

All constants appearing in the expressions in the following all sections will be always written $A$, because we do not need to specify them.

Inclusion (2.1) is an analogous result to [11, Theorem 2]. Hence we shall prove only (2.2) of Theorem 1. To do it, we need two inequalities which follow from Azarin [3, Lemma 1] (also see Essén and Lewis [6, Lemma 2]) and Azarin [3, Lemma 4 and Remark]:

$$\frac{\partial}{\partial n_Q} G^{\Omega}(P,Q) \leq A r^{\alpha_\Omega - 1} t^{-\beta_\Omega} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) \tag{3.1}$$

$$(resp. \frac{\partial}{\partial n_Q} G^{\Omega}(P,Q) \leq A r^{\alpha_\Omega} t^{-\beta_\Omega - 1} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)) \tag{3.2}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in C_n(\Omega)$ satisfying $0 < t/r \leq 4/5$ (resp. $0 < r/t \leq 4/5$);

$$\frac{\partial}{\partial n_Q} G^{\Omega}(P,Q) \leq A \frac{f_\Omega(\Theta)\frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)}{t^{n-1}} + A \frac{r f_\Omega(\Theta)\frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)}{|P-Q|^n} \tag{3.3}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega;\ ((4/5)r, (5/4)r])$.

*Poof* of Theorem 1. If we can show that for a sufficiently large $L$ and a sufficiently small positive number $\varepsilon$,

$$\Pi^{\Omega}\nu(P) < r^{\alpha_\Omega} \qquad (P \in C_n(\Omega; (L,+\infty)) - \Psi(\varepsilon; m_\nu^{(2)}, n-1)), \tag{3.4}$$

then we can conclude (2.2).

For any point $P = (r,\Theta) \in C_n(\Omega)$, write $\Pi^{\Omega}\nu(P)$ as the sum

$$\Pi^{\Omega}\nu(P) = I_1(P) + I_2(P) + I_3(P), \tag{3.5}$$

where

$$I_i(P) = \int_{S_n(\Omega;J_i)} \frac{\partial}{\partial n_Q} G^{\Omega}(P,Q) d\nu(Q) \quad (i = 1,2,3),$$

where $J_1 = (0,(4/5)r]$, $J_2 = ((4/5)r, (5/4)r])$ and $J_3 = ((5/4)r, \infty)$.

From (3.1) and the boundedness of $f_\Omega(\Theta)$ $(\Theta \in \Omega)$ we first have

$$I_1(P) \leq A r^{\alpha_\Omega} (\frac{4}{5} r)^{-(\alpha_\Omega + \beta_\Omega)} \int_{S_n(\Omega;\ (0,\frac{4}{5}r])} t^{\alpha_\Omega - 1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) d\nu(Q),$$

and hence

$$I_1(P) = o(1) r^{\alpha_\Omega} \quad (r \to \infty) \tag{3.6}$$

by Miyamoto and Yoshida [10, (ii) of Lemma 4].

We similarly have

$$I_3(P) \le Ar^{\alpha_\Omega} \int_{S_n(\Omega;\ (\frac{4}{5}r,+\infty))} t^{-\beta_\Omega-1} \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) d\nu(Q),$$

from (3.2) and hence

$$I_3(P) = o(1) r^{\alpha_\Omega} \quad (r \to \infty) \tag{3.7}$$

by Remark 3.

For $I_2(P)$ we have

$$I_2(P) \le I_{2,1}(P) + I_{2,2}(P), \tag{3.8}$$

where

$$I_{2,1}(P) \le A \int_{S_n(\Omega;\ (\frac{4}{5}r,\frac{5}{4}r])} \frac{f_\Omega(\Theta) t^{\beta_\Omega+1}}{t^{n-1}} dm_\nu^{(2)}(Q),$$

$$I_{2,2}(P) = A \int_{S_n(\Omega;\ (\frac{4}{5}r,\frac{5}{4}r])} \frac{t^{\beta_\Omega+1} r f_\Omega(\Theta)}{|P-Q|^n} dm_\nu^{(2)}(Q).$$

Since $f_\Omega(\Theta)$ is bounded on $\Omega$, we first have

$$I_{2,1}(P) \le Ar^{\alpha_\Omega} \int_{S_n(\Omega;\ (\frac{4}{5}r,\frac{5}{4}r])} dm_\nu^{(2)}(Q) = o(1) r^{\alpha_\Omega} \quad (r \to \infty) \tag{3.9}$$

from Remark 3.

We shall estimate $I_{2,2}(P)$. Take a sufficiently small positive number $\kappa$ such that $S_n(\Omega;\ ((4/5)r, (5/4)r]) \subset B(P, 2^{-1}r)$ for any $P = (r, \Theta) \in \Lambda(\kappa)$, where

$$\Lambda(\kappa) = \{Q = (t, \Phi) \in C_n(\Omega);\ \inf_{Z \in \partial\Omega} |(1, \Phi) - (1, Z)| \le \kappa,\ 0 < t < +\infty\}$$

and divide $C_n(\Omega)$ into two sets $\Lambda(\kappa)$ and $C_n(\Omega) - \Lambda(\kappa)$.

If $P = (r, \Theta) \in C_n(\Omega) - \Lambda(\kappa)$, then there exists a positive constant $\kappa'$ such that $|P - Q| > \kappa' r$ for any $Q \in S_n(\Omega)$, and hence

$$I_{2,2}(P) \le Ar^{\alpha_\Omega} \int_{S_n(\Omega;\ (\frac{4}{5}r,+\infty))} dm_\nu^{(2)}(Q) = o(1) r^{\alpha_\Omega} \quad (r \to +\infty) \tag{3.10}$$

from Remark 3.

We shall consider the case where $P \in \Lambda(\kappa)$. Now put

$$W_i(P) = \{Q \in S_n(\Omega;\ ((4/5)r, (5/4)r]);\ 2^{i-1}\delta(P) \le |P - Q| < 2^i \delta(P)\},$$

where $\delta(P) = \inf_{Q \in \partial C_n(\Omega)} |P - Q|$. Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n;\ |P - Q| < \delta(P)\} = \emptyset$, we have

$$I_{2,2}(P) = \Sigma_{i=1}^{i(P)} A \int_{W_i(P)} \frac{t^{\beta_\Omega + 1} r f_\Omega(\Theta)}{|P - Q|^n} dm_\nu^{(2)}(Q),$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \leq r/2 < 2^{i(P)}\delta(P)$. Since $r f_\Omega(\Theta) \leq A\delta(P)$ $(P = (r, \Theta) \in C_n(\Omega))$, we have

$$\int_{W_i(P)} \frac{t^{\beta_\Omega + 1} r f_\Omega(\Theta)}{|P - Q|^n} dm_\nu^{(2)}(Q) \leq A r^{\alpha_\Omega} 2^{n-i} \frac{m_\nu^{(2)}(W_i(P))}{\{2^i \delta(P)\}^{n-1}}$$

for $i = 0, 1, 2, ..., i(P)$. Suppose that $P \notin \Psi(\varepsilon; m_\nu^{(2)}, n-1)$ for a positive number $\varepsilon$. Then we have

$$\frac{m_\nu^{(2)}(W_i(P))}{\{2^i \delta(P)\}^{n-1}} \leq \frac{m_\nu^{(2)}(B(P, 2^i \delta(P))}{\{2^i \delta(P)\}^{n-1}} \leq M(P; m_\nu^{(2)}, n-1) \leq \varepsilon r^{1-n}$$

for $i = 0, 1, 2, ..., i(P) - 1$ and

$$\frac{m_\nu^{(2)}(W_{i(P)}(P))}{\{2^{i(p)} \delta(P)\}^{n-1}} \leq \frac{m_\nu^{(2)}(B(P, \frac{r}{2}))}{(\frac{r}{2})^{n-1}} \leq \varepsilon r^{1-n}.$$

In this case we also have

$$I_{2,2}(P) \leq A \varepsilon r^{\alpha_\Omega}. \tag{3.11}$$

From (3.5),(3.6),(3.7),(3.8),(3.9),(3.10) and (3.11), we finally obtain that if $L$ is sufficiently large and $\varepsilon$ is sufficiently small, then $\Pi^\Omega \nu(P) < r^{\alpha_\Omega}$ for any $P \in C_n(\Omega; (L, +\infty)) - \Psi(\varepsilon; m_\nu^{(2)}, n-1)$.

## §4. Proof of Theorem 2

The following Lemma 1 is a result concerning measure theory, which was proved in Miyamoto and Yoshida [11].

**Lemma 1** . *Let $m$ be any positive measure on $\mathbf{R}^n$ having the finite total mass. Let $\varepsilon$ and $q$ be two any positive numbers. Then $\mathcal{S}(\varepsilon; m, q)$ is covered by a sequence of balls $B_j$ $(j = 1, 2, ...)$ satisfying*

$$\sum_{j=1}^{\infty} (r_j / R_j)^q < +\infty,$$

*where $r_j$ is the radius of $B_j$ and $R_j$ is the distance between the origin and the center of $B_j$.*

*Proof* of Theorem 2. Since $E$ is rarefied at $\infty$ with respect to $C_n(\Omega)$, by Theorem A there exists a positive superharmonic function $v(P)$ in $C_n(\Omega)$ such that

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K(P;\infty,\Omega)} = 0 \tag{4.1}$$

and

$$E \subset \{P=(r,\Theta)\in C_n(\Omega);\ v(P)\geq r^{\alpha_\Omega}\}. \tag{4.2}$$

By Miyamoto and Yoshida [10, Lemma 3] (also see Azarin [3, Theorem 1]) and (4.1), for this $v(P)$ there exist a unique positive measure $\mu'$ on $C_n(\Omega)$ and a unique positive measure $\nu'$ on $S_n(\Omega)$ such that

$$v(P) = c_0(v)K(P;O,\Omega) + G^{\Omega}\mu'(P) + \Pi^{\Omega}\nu'(P).$$

Let us denote the sets $\{P=(r,\Theta)\in C_n(\Omega);\ c_0(v)K(P;O,\Omega)\geq 3^{-1}r^{\alpha_\Omega}\}$, $\{P=(r,\Theta)\in C_n(\Omega);\ G^{\Omega}\mu'(P)\geq 3^{-1}r^{\alpha_\Omega}\}$ and $\{P=(r,\Theta)\in C_n(\Omega);\ \Pi^{\Omega}\nu'(P)\geq 3^{-1}r^{\alpha_\Omega}\}$ by $E^{(1)}, E^{(2)}$ and $E^{(3)}$, respectively. Then we see from (4.2) that

$$E \subset E^{(1)}\cup E^{(2)}\cup E^{(3)}. \tag{4.3}$$

For each $E^{(i)}$ $(i=1,2,3)$ we shall find a sequence of balls which covers it.

It is evident from the boundedness of $E^{(1)}$ that $E^{(1)}$ is covered by a finite ball $B_1$ satisfying

$$r_1/R_1 < +\infty, \tag{4.4}$$

where $r_1$ is the radius of $B_1$ and $R_1$ is the distance between the origin and the center of $B_1$.

When we apply Theorem 1 with the measures $\mu$ and $\nu$ defined by $\mu = 3\mu'$ and $\nu = 3\nu'$ we can find two positive constants $L$ and $\varepsilon$ such that $E^{(2)}\cap C_n(\Omega;(L,+\infty)) \subset \Psi(\varepsilon; m_\mu^{(1)}, n-1)$ and $E^{(3)}\cap C_n(\Omega;(L,+\infty)) \subset \Psi(\varepsilon; m_\nu^{(2)}, n-1)$, respectively. By Lemma 1 these $\Psi(\varepsilon; m_\mu^{(1)}, n-1)$ and $\Psi(\varepsilon; m_\nu^{(2)}, n-1)$ are covered by two sequences of balls $B_j^{(2)}$ and $B_j^{(3)} (j=1,2,...)$ satisfying

$$\sum_{j=1}^{\infty}(r_j^{(2)}/R_j^{(2)})^{n-1} < +\infty \quad \textit{and} \quad \sum_{j=1}^{\infty}(r_j^{(3)}/R_j^{(3)})^{n-1} < +\infty,$$

respectively, where $r_j^{(2)}$ (resp. $r_j^{(3)}$) is the radius of $B_j^{(2)}$ (resp. $B_j^{(3)}$) and $R_j^{(2)}$ (resp. $R_j^{(3)}$) is the distance between the origin and the center of $B_j^{(2)}$ (resp. $B_j^{(3)}$). Hence $E^{(2)}$ and $E^{(3)}$ are also covered by the sequences of balls $B_j^{(2)}$ and $B_j^{(3)}$ $(j = 0, 1, ...)$ with an additional finite ball $B_0^{(2)}$ covering $C_n(\Omega; (0, L])$ satisfying

$$(4.5) \qquad \sum_{j=0}^{\infty} (r_j^{(2)}/R_j^{(2)})^{n-1} < +\infty \quad and \quad \sum_{j=1}^{\infty} (r_j^{(3)}/R_j^{(3)})^{n-1} < +\infty,$$

respectively.

Thus by rearranging $B_1, B_j^{(2)} (j = 0, 1, ...), B_j^{(3)} (j = 1, ...)$, we have a sequence of balls $B_k$ $(k = 1, 2, ...)$ which covers $E$ from (4.3) and satisfies (2.3) from (4.4), (4.5).

*Proof* of Remark 4. Since $f_\Omega(\Theta) \geq A$ for any $\Theta \in \Omega'$ and $r_k {R_k}^{-1} < 2^{-1}$ $(k \geq k_0)$ for a positive integer $k_0$, we have that $K(P; \infty, \Omega) \geq AR_k^{\alpha\Omega}$ and hence

$$(4.6) \qquad \hat{R}^{B_k}_{K(\cdot;\infty,\Omega)}(P) \geq AR_k^{\alpha\Omega} \quad (k \geq k_0)$$

for any $P \in \overline{B_k}$ $(k \geq k_0)$.

Take a measure $\tau$ on $C_n(\Omega)$, supp $\tau \subset \overline{B_k}$, $\tau(\overline{B_k}) = 1$ such that

$$(4.7) \qquad \int_{C_n(\Omega)} |P - Q|^{2-n} d\tau(P) = \{\mathrm{Cap}(\overline{B_k})\}^{-1},$$

for any $Q \in \overline{B_k}$, where *Cap* denotes the Newtonian capacity. Since $G^\Omega(P, Q) \leq |P - Q|^{2-n}$ $(P \in C_n(\Omega),\ Q \in C_n(\Omega))$, we have

$$\int (\int G^\Omega(P, Q) d\lambda_{B_k}(Q)) d\tau(P) \leq \{\mathrm{Cap}(\overline{B_k})\}^{-1} \lambda_{B_k}(C_n(\Omega))$$

from (4.7) and

$$\int (\int G^\Omega(P, Q) d\lambda_{B_k}(Q)) d\tau(P)$$

$$= \int (\hat{R}^{B_k}_{K(\cdot;\infty,\Omega)}(P)) d\tau(P) \geq AR_k^{\alpha\Omega} \tau(\overline{B_k}) = AR_k^{\alpha\Omega}$$

from (4.6). Hence we have that $\lambda_{B_k}(C_n(\Omega)) \geq A\mathrm{Cap}(\overline{B_k}) R_k^{\alpha\Omega} \geq Ar_k^{n-2} R_k^{\alpha\Omega}$, because $\mathrm{Cap}(\overline{B_k}) = r_k^{n-2}$.

Thus if we observe $\lambda_{E_k}(C_n(\Omega)) = \lambda_{B_k}(C_n(\Omega))$, then we have

$$\sum_{k=k_0}^{\infty} 2^{-k\beta_\Omega}\lambda_{E_k}(C_n(\Omega)) \geq A\sum_{k=k_0}^{\infty}(r_k/R_k)^{n-2} = A\sum_{k=k_0}^{\infty}k^{-1} = +\infty,$$

which shows that $E$ is not rarefied.

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# The $L^p$ resolvents for elliptic systems of divergence form

**Yoichi Miyazaki**

**Abstract.**

We consider elliptic systems of divegence form in $\mathbb{R}^n$ under the limited smoothness assumptions on the coefficients. We construct $L^p$ resolvents with evaluation of their operator norms, and derive the Gaussian bounds for heat kernels and estimates for resolvent kernels. These results extend those for single operators.

## §1. Introduction

In [5] we considered a single elliptic operator of order $2m$ in divergence form, which is defined in $\mathbb{R}^n$ and has non-smooth coefficients, in the framework of $L^p$ Sobolev spaces and constructed the resolvents. In [6, 7] we extended this result to an operator defined in a general domain with the Dirichlet boundary condition. Furthermore, in [7] we showed that the heat kernels and the resolvent kernels are differentiable (we exclude the diagonal set for the resolvent kernels) up to order $m-1+\sigma$ for any $\sigma \in (0,1)$ and evaluated their derivatives. These results correspond to the results by Tanabe [8] for single operators of non-divergence form.

The purpose of this paper is to extend the above results to elliptic systems defined in $\mathbb{R}^n$.

Let $x = (x_1, \dots, x_n)$ be a generic point in $\mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index with length $|\alpha| = \alpha_1 + \dots + \alpha_n$, and

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \qquad D_j = -\sqrt{-1}\frac{\partial}{\partial x_j} \quad (j = 1, \dots, n).$$

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Let $N \geq 1$ be an integer. We consider the elliptic operator in divergence form

$$Au(x) = \sum_{|\alpha| \leq m,\, |\beta| \leq m} D^\alpha (a_{\alpha\beta}(x) D^\beta u(x)) \tag{1.1}$$

in $\mathbb{R}^n$, where $a_{\alpha\beta}(x)$ is an $N \times N$ matrix $(a^{ij}_{\alpha\beta}(x))_{1 \leq i \leq N,\, 1 \leq j \leq N}$ and $u(x) = {}^t(u_1(x), \ldots, u_N(x))$. We allow the coefficients to be complex valued, whereas many literature such as [2, 4] deals with systems with real-valued coefficients. We denote by $a(x, \xi)$ the principal symbol of $A$:

$$a(x, \xi) = \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x) \xi^{\alpha + \beta}, \quad x \in \mathbb{R}^n,\, \xi \in \mathbb{R}^n.$$

Throughout this paper we assume the following.

(H1) All the coefficients $a^{ij}_{\alpha\beta}$ are measurable and bounded in $\mathbb{R}^n$.

(H2) The coefficients $a^{ij}_{\alpha\beta}$ with $|\alpha| = |\beta| = m$ are uniformly continuous in $\mathbb{R}^n$.

(H3) The operator $A$ satisfies the Legendre-Hadamard condition, that is, there exists $\delta_A > 0$ such that

$$\operatorname{Re} {}^t\eta a(x, \xi) \eta \geq \delta_A |\xi|^{2m} |\eta|^2$$

for any $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $\eta = {}^t(\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$.

Let $1 \leq p \leq \infty$ and $\tau \in \mathbb{R}$. We denote by $L^p = L^p(\mathbb{R}^n)$ the space of $p$-integrable functions and define $H^{\tau,p}$ by

$$H^{\tau,p} = H^{\tau,p}(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \langle D \rangle^\tau f \in L^p(\mathbb{R}^n) \}$$

with norm $\|u\|_{H^{\tau,p}} = \|\langle D \rangle^\tau u\|_{L^p}$, where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

For a Banach space $X$ we define $X^N$ to be the set of all $u = {}^t(u_1, \ldots, u_N)$ such that $u_j \in X$ for $1 \leq j \leq N$ with norm $\|u\|_{X^N} = \max_{1 \leq j \leq N} \|u_j\|_X$, and $X^{N \times N}$ the set of all $N \times N$ matrices $a = (a_{ij})_{i,j}$ such that $a_{ij} \in X$ for $1 \leq i \leq N$, $1 \leq j \leq N$ with norm $\|a\|_{X^{N \times N}} = \max_{1 \leq i \leq N, 1 \leq j \leq N} \|a_{ij}\|_X$. When $X = \mathbb{R}$, we simply write $|a|$ for $\|a\|_{X^{N \times N}}$.

For an integer $k \geq 1$ it is sometimes convenient to write $f \in (H^{-k,p})^N$ as

$$f = \sum_{|\alpha| \leq k} D^\alpha f_\alpha, \quad f_\alpha \in (L^p)^N \tag{1.2}$$

and note that the norm $\inf \sum_{|\alpha| \leq k} \|f_\alpha\|_{(L^p)^N}$ is equivalent to the norm $\|f\|_{(H^{-k,p})^N}$, where the infimum is taken over all the expressions in (1.2).

We mean by $T : X \to Y$ that $T$ is a bounded linear operator from a Banach space $X$ to a Banach space $Y$.

Let $1 < p < \infty$. Since $D^\alpha : (H^{\tau,p})^N \to (H^{\tau-|\alpha|,p})^N$ for $\tau \in \mathbb{R}$ and $a_{\alpha\beta} : (L^p)^N \to (L^p)^N$, we can regard $A$ in (1.1) as a bounded operator from $(H^{m,p})^N$ to $(H^{-m,p})^N$. When we want to stress $p$, we write $A$ as $A_p$. So we have

$$A = A_p = \sum_{|\alpha|,|\beta| \le m} D^\alpha a_{\alpha\beta} D^\beta : (H^{m,p})^N \to (H^{-m,p})^N.$$

We often use the following notations:

$$\begin{aligned} M_A &= \max_{|\alpha|,\,|\beta| \le m} \|a_{\alpha\beta}\|_{(L^\infty)^{N\times N}}, \qquad \zeta_A = (n, m, N, \delta_A, M_A), \\ \omega_A(\varepsilon) &= \max_{1\le i\le N,\,1\le j\le N} \max_{|\alpha|=|\beta|=m} \\ &\qquad \sup\{|a^{ij}_{\alpha\beta}(x) - a^{ij}_{\alpha\beta}(y)| : x, y \in \mathbb{R}^n,\ |x-y| \le \varepsilon\}, \\ \Lambda(R,\theta) &= \{\lambda \in \mathbb{C} : |\lambda| \ge R,\ \theta \le \arg\lambda \le 2\pi - \theta\} \end{aligned}$$

for $\varepsilon > 0$, $R > 0$ and $0 < \theta < \pi$.

Let $\mu_j(x,\xi)$, $1 \le j \le N$ be all the eigenvalues of $a(x,\xi)$. By (H1) we have $|\mathrm{Im}\,\mu_j(x,\xi)| \le M_0|\xi|^{2m}$ with some constant $M_0$ depending only on $n$, $m$, $N$ and $M_A$. On the other hand, (H3) implies $\mathrm{Re}\,\mu_j(x,\xi) \ge \delta_A|\xi|^{2m}$. Therefore we conclude that

$$-\kappa_A \le \arg\mu_j(x,\xi) \le \kappa_A, \tag{1.3}$$

where $\kappa_A = \arctan(M_0/\delta_A) \in (0,\pi/2)$. In [5, 6] we assumed $a(x,\xi) \ge \delta_A|\xi|^{2m}$ for a single operator, which is a stronger ellipticity condition than (H3). In this case we can take $\kappa_A = 0$.

## §2. Main results

We are now ready to state the main theorems. The first theorem is concerned with the estimates of the type

$$\|(A_p - \lambda)^{-1}\|_{(H^{-i,p})^N \to (H^{j,p})^N} \le K|\lambda|^{-1+(i+j)/2m} \tag{2.1}$$

for $0 \le i \le m$ and $0 \le j \le m$ with some $K > 0$.

**Theorem 2.1.** *Let $p \in (1,\infty)$ and $\theta \in (\kappa_A, \pi/2)$. Then there exist $R = R(\theta, \zeta_A, \omega_A)$, $K_1 = K_1(p,\theta,\zeta_A)$ and $K_2 = K_2(\theta,\zeta_A)$ such that for $\lambda \in \Lambda(R,\theta)$ the resolvent $(A_p - \lambda)^{-1}$ exists and (2.1) holds for $0 \le i \le m$ and $0 \le j \le m$ with $K = K_1$, and for $0 \le i \le m-1$ and $0 \le j \le m-1$ with $K = K_2$.*

*Moreover the resolvents are consistent in the sense that*

$$(A_p - \lambda)^{-1} f = (A_q - \lambda)^{-1} f, \quad f \in (H^{-m,p})^N \cap (H^{-m,q})^N$$

*when* $\lambda \in \Lambda(R, \theta)$ *for any* $p, q \in (1, \infty)$.

For $p \in (1, \infty)$ we define the operator $A_{(p)}$ in $(L^p)^N$ by

$$\begin{aligned} D(A_{(p)}) &= \{u \in (H^{m,p})^N : A_p u \in (L^p)^N\}, \\ A_{(p)} u &= A_p u \quad \text{for } u \in D(A_{(p)}). \end{aligned}$$

It follows from Lemma 3.1 in Section 3 that $D(A_{(p)})$ is dense in $(H^{m,p})^N$, $A_{(p)}$ is a closed operator in $(L^p)^N$, and $(A_{(p)})^* = (A^*)_{(p^*)}$, where $p^* = p/(p-1)$ and $A^*$ is the dual operator of $A$.

Let $h \in \mathbb{R}^n$. We define the difference operators $\Delta_h$, $\Delta_h^{(1)}$ and $\Delta_h^{(2)}$ by $\Delta_h u(x) = u(x+h) - u(x)$, $\Delta_h^{(1)} F(x,y) = F(x+h,y) - F(x,y)$ and $\Delta_h^{(2)} F(x,y) = F(x,y+h) - F(x,y)$, respectively, for vector-valued functions $u$ of $x \in \mathbb{R}^n$ and $F$ of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. We set

$$\Delta = \{(x,x) : x \in \mathbb{R}^n\}.$$

For $t \in \mathbb{C} \setminus \{0\}$, $x \in \mathbb{R}^n$ and $C > 0$ we set

$$\Phi_m(t,x;C) = \exp\{-C(|x|^{2m}|t|^{-1})^{1/(2m-1)}\}.$$

**Theorem 2.2.** *Let* $p \in (1, \infty)$. *Then the operator* $-A_{(p)}$ *generates an analytic semigroup* $e^{-tA_{(p)}}$ *of angle* $\pi/2 - \kappa_A$ *with kernel* $U(t,x,y)$ *which is independent of* $p$ *and satisfies the following estimates. For any* $\varepsilon \in (0, \pi/2 - \kappa_A)$ *and* $\sigma \in (0,1)$ *there exist* $C_1 = C_1(\varepsilon, \zeta_A)$, $C_2 = C_2(\varepsilon, \zeta_A)$, $C_3 = C_3(\varepsilon, \zeta_A, \omega_A)$, $C_1' = C_1'(\varepsilon, \sigma, \zeta_A)$, $C_2' = C_2'(\varepsilon, \sigma, \zeta_A)$ *and* $C_3' = C_3'(\varepsilon, \sigma, \zeta_A, \omega_A)$ *such that for* $|\alpha| < m$, $|\beta| < m$ *and* $|\arg t| \leq \pi/2 - \kappa_A - \varepsilon$ *we have*

$$|\partial_x^\alpha \partial_y^\beta U(t,x,y)| \leq C_1 |t|^{-(n+|\alpha|+|\beta|)/2m} \Phi_m(t, x-y; C_2) e^{C_3|t|} \tag{2.2}$$

*for* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, *and*

$$\begin{aligned} &|\Delta_h^{(i)} \partial_x^\alpha \partial_y^\beta U(t,x,y)| \\ &\quad \leq C_1' |t|^{-(n+|\alpha|+|\beta|+\sigma)/2m} \Phi_m(t, x-y; C_2') e^{C_3'|t|} |h|^\sigma \end{aligned} \tag{2.3}$$

*for* $i \in \{1,2\}$, $h \in \mathbb{R}^n$ *and* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ *with* $2|h| \leq |x-y|$.

Theorem 2.2 extends the result for $p = 2$ obtained by Auscher and Qafsaoui [1], who used the method of Morrey-Campanato spaces.

For $x \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$, $\tau > 0$ and $C > 0$ we set

$$\Psi_m^\tau(x, \lambda; C) = \begin{cases} |\lambda|^{-1+\tau/2m} \exp(-C|\lambda|^{1/2m}|x|) & (\tau < 2m) \\ (1 + \log_+ |\lambda|^{1/2m}|x|) \exp(-C|\lambda|^{1/2m}|x|) & (\tau = 2m) \\ |x|^{2m-\tau} \exp(-C|\lambda|^{1/2m}|x|) & (\tau > 2m), \end{cases}$$

where $\log_+ s = \max\{0, \log s\}$ for $s > 0$.

**Theorem 2.3.** *Let $p \in (1, \infty)$ and $\theta \in (\kappa_A, \pi/2)$. Then there exists $R = R(\theta, \zeta_A, \omega_A)$ such that for $\lambda \in \Lambda(R, \theta)$ the resolvent $(A_{(p)} - \lambda)^{-1}$ exists and it has a kernel $G_\lambda(x, y)$ which is independent of $p$ and satisfies the following estimates. For any $\sigma \in (0, 1)$ there exist $C_1 = C_1(\theta, \zeta_A)$, $C_2 = C_2(\theta, \zeta_A)$, $C_1' = C_1'(\sigma, \theta, \zeta_A)$ and $C_2' = C_2'(\sigma, \theta, \zeta_A)$ such that for $|\alpha| < m$, $|\beta| < m$ and $\lambda \in \Lambda(R, \theta)$ we have*

$$|\partial_x^\alpha \partial_y^\beta G_\lambda(x, y)| \le C_1 \Psi_m^{n+|\alpha|+|\beta|}(x - y, \lambda; C_2) \tag{2.4}$$

*for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$, and*

$$|\Delta_h^{(i)} \partial_x^\alpha \partial_y^\beta G_\lambda(x, y)| \le C_1' \Psi_m^{n+|\alpha|+|\beta|+\sigma}(x - y, \lambda; C_2')|h|^\sigma \tag{2.5}$$

*for $i \in \{1, 2\}$, $h \in \mathbb{R}^n$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta$ with $2|h| \le |x - y|$.*

*Moreover $\partial_x^\alpha \partial_y^\beta G_\lambda(x, y)$ is continuous on $\Delta$ if $n + |\alpha| + |\beta| < 2m$.*

## §3. Partial proof of Theorem 2.1

Since $T = (T_{ij})_{i,j} : X^N \to Y^N$ and $T_{ij} : X \to Y$ for $1 \le i \le N$, $1 \le j \le N$ are equivalent, most properties of $T$ can be reduced to those of $T_{ij}$. This enables us to obtain the main results along the same line as in the case of single operators. We first derive Lemma 3.1 below, which is weaker than Theorem 2.1 for the constants $R$ and $K$ may depend on $p$. Then Lemma 3.1 leads to Thorem 2.2, from which Theorems 2.1 and 2.3 follow.

In the following we give only the outline of the proofs except Lemma 3.3 whose proof is a little complicated when $N \ge 2$. The details for the case of single operaters are found in [5, 6, 7].

**Lemma 3.1.** *Let $p \in (1, \infty)$ and $\theta \in (\kappa_A, \pi/2)$. Then there exist $R_p = R(p, \theta, \zeta_A, \omega_A)$ and $K = K(p, \theta, \zeta_A)$ such that for $\lambda \in \Lambda(R_p, \theta)$ the resolvent $(A_p - \lambda)^{-1}$ exists and (2.1) holds for $0 \le i \le m$ and $0 \le j \le m$.*

*Moreover the resolvents are consistent in the sense of Theorem 2.1.*

The proof of Lemma 3.1 is given after some preparation.

**Lemma 3.2** ([7]). *Let $\theta \in (\kappa_A, \pi/2)$. Then there exists a constant $C = C(\theta, \kappa_A) > 0$ such that*

$$|s - \lambda| \geq C(|s| + |\lambda|)$$

*for $|\arg s| \leq \kappa_A$ and $\theta \leq \arg \lambda \leq 2\pi - \theta$.*

**Lemma 3.3.** *Let $p \in (1, \infty)$, $\theta \in (\kappa_A, \pi/2)$ and fix $x_0 \in \mathbb{R}^n$. Then for $\lambda \in \Lambda(1, \theta)$ the operator $a(x_0, D) - \lambda : (H^{m,p})^N \to (H^{-m,p})^N$ has an inverse and there exists $K = K(p, \theta, \zeta_A)$ such that*

$$\|(a(x_0, D) - \lambda)^{-1}\|_{(H^{-i,p})^N \to (H^{j,p})^N} \leq K|\lambda|^{-1+(i+j)/2m} \tag{3.1}$$

*for $0 \leq i \leq m$ and $0 \leq j \leq m$.*

*Proof.* Set $b_\lambda(\xi) = (b_{\lambda ij}(\xi))_{i,j} = (a(x_0, \xi) - \lambda)^{-1}$. Then

$$b_{\lambda ij}(\xi) = (\det(a(x_0, \xi) - \lambda))^{-1} c_{\lambda ji}(\xi),$$

where $c_{\lambda ij}(\xi)$ is $(i, j)$-cofactor of the matrix $a(x_0, \xi) - \lambda$. By (H1), (1.3), Lemma 3.2 and $\operatorname{Re} \mu_j(x, \xi) \geq \delta_A |\xi|^{2m}$ we have

$$\begin{aligned} |c_{\lambda ij}(\xi)| &\leq C(|\xi|^{2m} + |\lambda|)^{N-1}, \\ |\det(a(x_0, \xi) - \lambda)| &= |\lambda - \mu_1(x_0, \xi)| \cdots |\lambda - \mu_N(x_0, \xi)| \\ &\geq C(|\xi|^{2m} + |\lambda|)^N. \end{aligned}$$

Since $\partial_\xi^\alpha b_\lambda(\xi)$ is written in the form

$$\sum_{\alpha^1 + \cdots + \alpha^k = \alpha} C_{\alpha\alpha^1 \ldots \alpha^k} b_\lambda(\xi) \cdot \partial_\xi^{\alpha^1} a(x_0, \xi) \cdots b_\lambda(\xi) \cdot \partial_\xi^{\alpha^k} a(x_0, \xi) \cdot b_\lambda(\xi)$$

with $1 \leq |\alpha^j| \leq 2m$ $(j = 1, \ldots, k)$, we have

$$\begin{aligned} |\partial_\xi^\alpha b_\lambda(\xi)| &\leq C \sum |\xi|^{2m - |\alpha^1|} \cdots |\xi|^{2m - |\alpha^k|} (|\xi|^{2m} + |\lambda|)^{-k-1} \\ &\leq C(|\xi|^{2m} + |\lambda|)^{-1 - |\alpha|/2m}. \end{aligned}$$

So we get

$$|\xi|^{|\gamma|} \left| \partial_\xi^\gamma \{\xi^{\alpha+\beta} b_\lambda(\xi)\} \right| \leq C|\lambda|^{-1 + (|\alpha| + |\beta)/2m}$$

for $|\alpha| \leq m$, $|\beta| \leq m$ and $|\gamma| \leq [n/2] + 1$. Finally, by applying Mihlin's multiplier theorem to the operator $D^\alpha b_\lambda(D) D^\beta$ we get the lemma. □

*Proof of Lemma 3.1.* For $\varepsilon \in (0,1)$ we take a family of functions $\{\eta_{s\varepsilon}(x)\}_{s\in\mathbb{Z}^n}$ in $C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned}
&\sum_{s\in\mathbb{Z}^n} \eta_{s\varepsilon}(x)^2 = 1, \qquad \operatorname{supp}\eta_{s\varepsilon} \subset \{x\in\mathbb{R}^n : |x-\varepsilon s| < \varepsilon\},\\
&|D^\alpha \eta_{s\varepsilon}(x)| \le C_{n,m}\varepsilon^{-|\alpha|} \quad \text{for } |\alpha| \le 2m,\\
&\#\{s\in\mathbb{Z}^n : \eta_{s\varepsilon}(x) \neq 0\} \le 2^n \text{ for any } x\in\mathbb{R}^n.
\end{aligned}$$

We define a parametrix for $A-\lambda$ by

$$P_\lambda = \sum_{s\in\mathbb{Z}^n} \eta_{s\varepsilon} P_{s\lambda} \eta_{s\varepsilon}, \qquad P_{s\lambda} = (a(\varepsilon s, D) - \lambda)^{-1}.$$

Using the Leibniz formula, we have

$$(A-\lambda)P_\lambda = I + R_\lambda, \qquad R_\lambda = J_1 + J_2 + J_3 + J_4,$$

where $I$ denotes the identity and

$$\begin{aligned}
J_1 &= \sum_{|\alpha|+|\beta|<2m} D^\alpha a_{\alpha\beta} D^\beta P_\lambda,\\
J_2 &= \sum_{|\alpha|=|\gamma|=m} \sum_{\beta<\gamma} C_{1\gamma\beta} D^\alpha a_{\alpha\gamma} \Big(\sum_s \eta_{s\varepsilon}^{(\gamma-\beta)} D^\beta P_{s\lambda}\eta_{s\varepsilon}\Big),\\
J_3 &= \sum_{|\alpha|=|\beta|=m} D^\alpha \Big(\sum_s (a_{\alpha\beta} - a_{\alpha\beta}(\varepsilon s))\eta_{s\varepsilon} D^\beta P_{s\lambda}\eta_{s\varepsilon}\Big),\\
J_4 &= \sum_{|\gamma|=|\beta|=m} \sum_{\alpha<\gamma} C_{0\gamma\alpha} D^\alpha \Big(\sum_s a_{\gamma\beta}(\varepsilon s)\eta_{s\varepsilon}^{(\gamma-\alpha)} D^\beta P_{s\lambda}\eta_{s\varepsilon}\Big)
\end{aligned}$$

with some constants $C_{0\gamma\alpha}$ and $C_{1\gamma\beta}$. Careful calculation yields

$$\begin{aligned}
&\|P_\lambda R_\lambda^k\|_{(H^{-i,p})^N \to (H^{j,p})^N}\\
&\quad \le K_0 K_1^k (\omega_A(\sqrt{n}\varepsilon) + \varepsilon^{-1}|\lambda|^{-1/2m})^k |\lambda|^{-1+(i+j)/2m}
\end{aligned}$$

for $0 \le i \le m$, $0 \le j \le m$ and $\lambda \in \Lambda(\varepsilon^{-2m}, \theta)$. So if we take $\varepsilon \in (0,1)$ and $R > 0$ so that

$$K_1\omega_A(\sqrt{n}\varepsilon) \le 4^{-1}, \qquad K_1\varepsilon^{-1}R^{-1/2m} \le 4^{-1}, \quad R \ge \varepsilon^{-2m},$$

then for $\lambda \in \Lambda(R,\theta)$ the series $\sum_{k=0}^\infty (-1)^k P_\lambda R_\lambda^k$ converges as an operator $(H^{-m,p})^N \to (H^{m,p})^N$ and it is a right inverse of $A-\lambda$. The duality aurgument shows that the right inverse is exactly $(A-\lambda)^{-1}$.

We also get the consistency of resolvents, since $(A-\lambda)^{-1}$ consists of three kinds of operators such as $D^\alpha$, Fourier multipliers $(a(x_0, D) - \lambda)^{-1}$, and multiplication operators by functions in $(L^\infty)^N$, which are consistent in the sense of Theorem 2.1. $\square$

## §4. Proof of Theorem 2.2

Based on Lemma 3.1, we can prove Theorem 2.2. It is seen from (2.1) that $-A_{(p)}$ generates an analytic semigroup of angle $\pi/2-\kappa_A$.

As for the heat kernel estimate, we shall first consider the case of $p=2$. By the Sobolev embedding theorem and Lemma 3.1 we have range $(A_{(p)}-\lambda)^{-1}\subset(L^p)^N\cap(L^q)^N$ for $p$, $q$ with $1<p<q$, $p^{-1}-q^{-1}<m/n$ and

$$\|(A_{(p)}-\lambda)^{-1}\|_{(L^p)^N\to(L^q)^N}\le C|\lambda|^{-1+(n/2m)(1/p-1/q)}$$

for $\lambda\in\Lambda(R_p,\theta)\cap\Lambda(R_q,\theta)$ with $\theta\in(\kappa_A,\pi/2)$.

Given $\varepsilon\in(0,2^{-1}(\pi/2-\kappa_A))$ and $\sigma\in(0,1)$, we take a sequence $\{p_j\}_{j=1}^k$ satisfying

$$2=p_k<p_{k-1}<\cdots<p_1=\max\{\tfrac{n}{1-\sigma},2\},\qquad p_j^{-1}-p_{j-1}^{-1}<m/n,$$

and assume that $\lambda\in\Lambda(\kappa_A+\varepsilon,R)$ and $|\arg t|<\pi/2-\kappa_A-2\varepsilon$ with $R=\max\{R_{p_1},\dots,R_{p_k}\}$, where each $R_{p_j}$, $j=1,\dots,k$ is the constant defined for $p=p_j$ and $\theta=\kappa_A+\varepsilon$ in Lemma 3.1. Then we have

$$(A_{(2)}-\lambda)^{-k}=(A_{(p_1)}-\lambda)^{-1}\cdots(A_{(p_k)}-\lambda)^{-1}$$

and therefore

$$\|(A_{(2)}-\lambda)^{-k}\|_{(L^2)^N\to(L^\infty)^N}\le K'|\lambda|^{-k+n/4m}.$$

This combined with the formula

$$\begin{aligned}e^{-tA_{(2)}}&=\tfrac{1}{2\pi\sqrt{-1}}\int_\Gamma e^{-t\lambda}(A_{(2)}-\lambda)^{-1}\,d\lambda\\&=\tfrac{(k-1)!}{2\pi\sqrt{-1}}t^{1-k}\int_\Gamma e^{-t\lambda}(A_{(2)}-\lambda)^{-k}\,d\lambda,\end{aligned}$$

where $\Gamma$ is a path in $\Lambda(\kappa_A+\varepsilon,R)$, gives

$$\|e^{-tA_{(2)}}\|_{(L^2)^N\to(L^\infty)^N}\le C|t|^{-n/4m}e^{(\sin(\kappa_A+\varepsilon))^{-1}R|t|}.$$

Applying the kernel theorem to $e^{-2tA_{(2)}}=e^{-tA_{(2)}}(e^{-tA^*_{(2)}})^*$, we obtain

$$|U(2t,x,y)|\le C^2|t|^{-n/2m}e^{2(\sin(\kappa_A+\varepsilon))^{-1}R|t|}. \tag{4.1}$$

The Gaussian estimate can be derived by Davies' method of exponential perturbation (cf [3]). To this end we set $A_\phi=e^{-\phi}Ae^{\phi}$, where $\phi(x)=\phi(x;\eta,R_0)$ is a $C^\infty$ function of $x$ with parameters $\eta\in\mathbb{R}^n$ and $R_0>0$

satisfying $\phi(x) = e^{x\eta}$ for $|x| \leq R_0$ and $\partial^\alpha \phi \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \leq m$. Then the heat kernel $U_\phi(t, x, y)$ for $A_\phi$ satisfies the estimate similar to (4.1). So the relation

$$U(t, x, y) = e^{(x-y)\eta} U_\phi(t, x, y)$$

for $|x| \leq R_0$ and $|y| \leq R_0$ yields the Gaussian bounds.

We can get the estimates for the derivatives of $U(t, x, y)$ and their Hölder norms by using the fact $(A_{(p_1)} - \lambda)^{-1} : (L^{p_1})^N \to (B^{m-1+\sigma})^N$ in the above argument, where $B^{m-1+\sigma}$ denotes the Hölder space of order $m - 1 + \sigma$.

Finally we shall consider the case of $p \neq 2$. The Gaussian bounds yield $\sup_y \|U(t, \cdot, y)\|_{(L^1)^{N\times N}} < \infty$ and $\sup_x \|U(t, x, \cdot)\|_{(L^1)^{N\times N}} < \infty$. So the integral operator with kernel $U(t, x, y)$ is a bounded operator in $(L^p)^N$. Hence the consistency of resolvents shows that $e^{-tA_{(p)}}$ has the same integral kernel as $e^{-tA_{(2)}}$.

## §5. Proof of Theorem 2.3

By Theorem 2.2 we have $\|e^{-tA_{(p)}}\|_{(L^p)^N \to (L^p)^N} \leq Ce^{R|t|}$ with some $C$ and $R$, and therefore

$$(A_{(p)} - \lambda)^{-1} = \int_0^\infty e^{t\lambda} e^{-tA_{(p)}}\, dt, \quad \lambda < -R.$$

Let $\theta \in (0, 2^{-1}(\pi/2 - \kappa_A))$. Deforming the integral path and using analytic continuation, we get the formulae such as

$$(A_{(p)} - \lambda)^{-1} = \int_{L_\theta} e^{t\lambda} e^{-tA_{(p)}}\, dt \tag{5.1}$$

for $\lambda$ with $|\lambda| > (\sin\theta)^{-1} R$ and $\kappa_A + 2\theta \leq \arg\lambda \leq \pi$, where $L_\theta$ is the half line which runs from 0 to $\infty e^{\sqrt{-1}(\pi/2-\kappa_A-\theta)}$. Then the estimate for the resolvent kernel $G_\lambda(x, y)$ follows from (5.1) and the Gaussian bounds.

## §6. Proof of Theorem 2.1

Let $p \in (1, \infty)$ and $\theta \in (\kappa_A, \pi/2)$. By Theorem 2.2 and (5.1) we have $\Lambda(R_0, \theta) \subset \rho(A_{(p)})$, the resolvent set of $A_{(p)}$, with some $R_0 = R_0(\theta, \zeta_A, \omega_A)$. On the other hand, by Lemma 3.1 we have $\Lambda(R_1, \theta) \subset \rho(A_p)$ with some $R_1 = R_1(p, \theta, \zeta_A, \omega_A)$. Based on these inclusions and the resolvent equation, we can take the constant $R$ independent of $p$ in Theorem 2.1. Furthermore, by using (5.1) and the Gaussian bounds we can also take the constant $K_2$ independent of $p$ in Theorem 2.1.

**Acknowledgment.** I would like to thank the referee of the paper [5] for this work is motivated by his valuable comment saying "It would also be interesting to consider operators with complex coefficients and systems."

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# Maximal functions, Riesz potentials and Sobolev's inequality in generalized Lebesgue spaces

**Yoshihiro Mizuta and Tetsu Shimomura**

**Abstract.**

Our aim in this paper is to deal with the boundedness of maximal functions in Lebesgue spaces with variable exponent. Our result extends the recent work of Diening [4], Cruz-Uribe, Fiorenza and Neugebauer [3] and the authors [8]. As an application of the boundedness of maximal functions, we show Sobolev's inequality for Riesz potentials with variable exponent.

## §1. Introduction

Sobolev functions play a significant role in many fields of analysis. In recent years, the generalized Lebesgue spaces $L^{p(\cdot)}$ and the corresponding Sobolev spaces $W^{m,p(\cdot)}$ have attracted more and more attention, in connection with the study of elasticity, fluid mechanics and differential equations with $p(\cdot)$-growth; see Ružička [16]. One of the most important results for Sobolev functions is so called Sobolev's embedding theorem, and the corresponding result has been extended to Sobolev spaces of variable exponent by many authors; see for example [2, 5, 7, 8, 12, 17]. Our main task in this study is to obtain boundedness properties for Riesz potentials. For this purpose, the boundedness of maximal functions gives a crucial tool by a trick of Hedberg [11], which is originally based on the recent work by Diening [4].

Let $\Omega$ be an open set in $\mathbf{R}^n$. We use the notation $B(x,r)$ to denote the open ball centered at $x$ of radius $r$. For a locally integrable function

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$f$ on $\Omega$, we consider the maximal function $Mf$ defined by

$$Mf(x) = \sup_B \frac{1}{|B|} \int_{\Omega \cap B} |f(y)| dy,$$

where the supremum is taken over all balls $B = B(x, r)$ and $|B|$ denotes the volume of $B$. Let $p(\cdot)$ be a positive continuous function on $\Omega$ such that $p(x) > 1$ on $\Omega$. Following Orlicz [15] and Kováčik and Rákosník [13], we define the $L^{p(\cdot)}(\Omega)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),\Omega} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(y)}{\lambda} \right|^{p(y)} dy \le 1 \right\}$$

and denote by $L^{p(\cdot)}(\Omega)$ the space of all measurable functions $f$ on $\Omega$ with $\|f\|_{p(\cdot)} < \infty$.

In this paper we are concerned with $p(\cdot)$ satisfying a condition of the form :

$$|p(x) - p(y)| \le \frac{\log(\varphi(|x-y|))}{\log(1/|x-y|)}$$

whenever $x \in \Omega$, $y \in \Omega$ and $|x - y| < 1/2$, where $\varphi$ is a positive nonincreasing function on $(0, \infty)$ of logarithmic type. Our typical example of $\varphi$ is

$$\varphi(r) = a(\log(1/r))^b (\log\log(1/r)))^c$$

for small $r > 0$, where $a > 0$, $b > 0$ and $-\infty < c < \infty$. In case $\Omega$ is not bounded, we further assume that

$$|p(x) - p_\infty| \le \frac{C}{\log(e + |x|)} \qquad \text{whenever } x \in \Omega,$$

where $1 < p_\infty < \infty$.

Our first aim in this paper is to find a function $\Phi(t, x)$ on $\mathbf{R} \times \Omega$ such that

$$\int_\Omega \Phi(Mf(x), x) dx \le C \qquad \text{whenever } \|f\|_{p(\cdot)} \le 1$$

(in Theorems 2.7 and 4.7 below). If $\varphi(r) = a(\log(e + 1/r))^b$, then our result was proved by Diening [4] (when $b = 0$ and $\Omega$ is bounded), Cruz-Uribe, Fiorenza and Neugebauer [3, Theorem 1.5] (when $b = 0$ and $\Omega$ is not bounded), and the authors [8, Theorem 2.4] (when $b > 0$ and $\Omega$ is bounded).

We consider the Riesz potential of order $\alpha$ for a locally integrable function $f$ on $\Omega$, which is defined by

$$I_\alpha f(x) = \int_\Omega |x-y|^{\alpha-n} f(y)dy.$$

Here $0 < \alpha < n$. As an application of the boundedness of maximal functions, we give Sobolev's inequality for Riesz potentials with variable exponent. We in fact find a function $\Psi(t,x)$ on $\mathbf{R} \times \Omega$ such that

$$\int_\Omega \Psi(I_\alpha f(x), x)dx \le C \qquad \text{whenever } \|f\|_{p(\cdot)} \le 1$$

(see Theorems 3.5 and 5.6 below). In case $\varphi(r) = a(\log(e + 1/r))^b$, our result was proved by Samko [17] (when $b = 0$ and $\Omega$ is bounded), Diening [5] (when $b = 0$ and $p(\cdot)$ is constant outside of a large ball), Capone, Cruz-Uribe and Fiorenza [2, Theorem 1.6] (when $b = 0$ and $\Omega$ is not bounded), and the authors [8, Theorem 3.4] (when $b > 0$ and $\Omega$ is bounded).

For related results, see also Adams-Hedberg [1], Diening [5], Edmunds-Rákosník [6], Harjulehto-Hästö-Pere[10], Kokilshvili-Samko [12], Kováčik-Rákosník [13], Nekvinda [14], Ru̇žička [16] and the authors [9].

## §2. Maximal functions

Throughout this paper, let $C$ denote various constants independent of the variables in question.

Consider a positive nonincreasing function $\varphi$ on the interval $(0,\infty)$ of logarithmic type, which has the following properties:

($\varphi$1) $\varphi(\infty) = \lim_{t\to\infty} \varphi(t) > 0$;
($\varphi$2) $(\log(1/t))^{-\varepsilon_0}\varphi(t)$ is nondecreasing on $(0, r_0)$ for some $\varepsilon_0 > 0$ and $r_0 > 0$.

*Remark* 2.1. (i) By condition ($\varphi$2), we see that

$$C^{-1}\varphi(r) \le \varphi(r^2) \le C\varphi(r) \qquad \text{whenever } r > 0,$$

which implies the doubling condition on $\varphi$.

(ii) We see from ($\varphi$2) that for each $\delta > 0$, $t^\delta\varphi(t)$ is nondecreasing on some interval $(0,T)$, $T = T(\delta) > 0$.

(iii) Our typical example of $\varphi$ is of the form

$$\varphi(t) = a(\log(1/t))^b(\log(\log(1/t)))^c$$

for small $t > 0$, where $a > 0$, $b > 0$ and $c \in \mathbf{R}$.

In this section, let $\Omega$ be an open set in $\mathbf{R}^n$. Let $p(\cdot)$ be a positive continuous function on $\Omega$ satisfying

(p1) $1 < p_-(\Omega) = \inf_\Omega p(x) \le \sup_\Omega p(x) = p_+(\Omega) < \infty$ ;
(p2) $|p(x)-p(y)| \le \log(\varphi(|x-y|))/\log(1/|x-y|)$ whenever $|x-y| < 1/2$, $x \in \Omega$ and $y \in \Omega$.

**Lemma 2.2.** *If* $0 < r_0 < 1$ *and* $\log\varphi(r_0) > \varepsilon_0$, *then* $\log\varphi(r)/\log(1/r)$ *is nondecreasing on* $(0, r_0)$.

*Proof.* Let $0 < r_1 < r_2 < r_0 < 1$. By ($\varphi$2), we have

$$\begin{aligned}
\frac{\log\varphi(r_1)}{\log(1/r_1)} &\le \varepsilon_0 \frac{\log(\log(1/r_1)) - \log(\log(1/r_2))}{\log(1/r_1)} + \frac{\log\varphi(r_2)}{\log(1/r_1)} \\
&= \frac{\log\varphi(r_2)}{\log(1/r_2)} + \frac{1}{\log(1/r_1)}\left\{\varepsilon_0 \log\left(\frac{\log(1/r_1)}{\log(1/r_2)}\right)\right. \\
&\quad \left. + \frac{\log(r_1/r_2)}{\log(1/r_2)}\log\varphi(r_2)\right\}.
\end{aligned}$$

Since $\log(1+t) < t$ for $t > 0$,

$$\log\left(\frac{\log(1/r_1)}{\log(1/r_2)}\right) \le \frac{\log(r_2/r_1)}{\log(1/r_2)},$$

so that

$$\begin{aligned}
&\frac{\log\varphi(r_1)}{\log(1/r_1)} - \frac{\log\varphi(r_2)}{\log(1/r_2)} \\
\le\ & \frac{1}{\log(1/r_1)}\left(\frac{\log(r_2/r_1)}{\log(1/r_2)}\right)(\varepsilon_0 - \log\varphi(r_2)) < 0,
\end{aligned}$$

as required. □

Let $1/p'(x) = 1 - 1/p(x)$. Then note that

$$\begin{aligned}
p'(y) - p'(x) &= \frac{p(x)-p(y)}{(p(x)-1)(p(y)-1)} \\
&= \frac{p(x)-p(y)}{(p(x)-1)^2} + \frac{(p(x)-p(y))^2}{(p(x)-1)^2(p(y)-1)}.
\end{aligned}$$

Hence, in view of ($\varphi$2), we have the following result.

**Lemma 2.3.** *There exists a positive constant $C$ such that*

$$|p'(x) - p'(y)| \le \omega(|x-y|) \qquad \text{whenever } x \in \Omega \text{ and } y \in \Omega,$$

*where*

$$\omega(r) = \omega(r; x, C) = \frac{1}{(p(x)-1)^2} \frac{\log(C\varphi(r))}{\log(1/r)}$$

*for $0 < r \le r_0$ and $\omega(r) = \omega(r_0)$ for $r \ge r_0$.*

In what follows, we may assume that $\omega(r)$ is nondecreasing as a function of $r \in (0, \infty)$. Moreover, if $f$ is a function on $\Omega$, then we set $f = 0$ outside $\Omega$.

**Lemma 2.4.** *Let $f$ be a nonnegative measurable function on $\Omega$ with $\|f\|_{p(\cdot)} \le 1$. Then*

$$\{Mf(x)\}^{p(x)} \le C\left\{Mg(x)(\varphi(Mg(x)^{-1}))^{n/p(x)} + 1\right\}$$

*for all $x \in \Omega$, where $g(y) = f(y)^{p(y)}$.*

*Proof.* For $0 < \mu \le 1$ and $r > 0$, we have by Lemma 2.3

$$\begin{aligned} f_B &\equiv \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \\ &\le \mu\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} (1/\mu)^{p'(y)}dy + \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p(y)}dy\right) \\ &\le \mu\left((1/\mu)^{p'(x)+\omega(r)} + F\right), \end{aligned}$$

where $F = |B(x,r)|^{-1}\int_{B(x,r)} f(y)^{p(y)}dy$. When $F$ is bounded, say $F \le R_0$, by considering $\mu = 1$, we have

$$f_B \le C.$$

Hence it suffices to treat the case that $F \ge R_0 > r_0^{-1}$; in this case we may assume that $0 < r < r_0$ since $\|f\|_{p(\cdot)} \le 1$. By considering $\mu = F^{-1/\{p'(x)+\omega(r)\}}$ when $F > 1$, we find

$$f_B \le 2F^{1/p(x)}F^{\omega(r)/\{p'(x)(p'(x)+\omega(r))\}} \le 2F^{1/p(x)}F^{\omega(r)/p'(x)^2}.$$

If $r \le F^{-1} < r_0$, then we see from Lemma 2.2 that

$$f_B \le CF^{1/p(x)}(\varphi(F^{-1}))^{1/p(x)^2} \le CF^{1/p(x)}(\varphi(F^{-1}))^{n/p(x)^2}.$$

If $F^{-1} < r < r_0$, then

$$
\begin{aligned}
& F^{1/p(x)+\omega(r)/p'(x)^2} \\
\leq \quad & C r^{-n/p(x)-n\omega(r)/p'(x)^2} \left( \int_{B(x,r)} f(y)^{p(y)} dy \right)^{1/p(x)+\omega(r)/p'(x)^2} .
\end{aligned}
$$

Since $r^{-n\omega(r)/p'(x)^2} \leq C\varphi(r)^{n/p(x)^2}$ and $\int_{B(x,r)} f(y)^{p(y)} dy \leq 1$ by our assumption, we obtain

$$
\begin{aligned}
& F^{1/p(x)+\omega(r)/p'(x)^2} \\
\leq \quad & C r^{-n/p(x)} \varphi(r)^{n/p(x)^2} \left( \int_{B(x,r)} f(y)^{p(y)} dy \right)^{1/p(x)+\omega(r)/p'(x)^2} \\
\leq \quad & C r^{-n/p(x)} \varphi(r)^{n/p(x)^2} \left( \int_{B(x,r)} f(y)^{p(y)} dy \right)^{1/p(x)} \\
\leq \quad & C r^{-n/p(x)} \varphi(F^{-1})^{n/p(x)^2} \left( \int_{B(x,r)} f(y)^{p(y)} dy \right)^{1/p(x)} \\
\leq \quad & C F^{1/p(x)} \varphi(F^{-1})^{n/p(x)^2}.
\end{aligned}
$$

Now it follows that

$$
f_B \leq C F^{1/p(x)} \varphi(F^{-1})^{n/p(x)^2},
$$

which completes the proof. □

**Lemma 2.5.** *For each $\delta > 0$, there exists $T_0 > e$ such that $s^\delta \varphi(s^{-1})^{-1}$ is nondecreasing on $(T_0, \infty)$.*

*Proof.* By ($\varphi 2$), it follows that $(\log s)^{\varepsilon_0} \varphi(s^{-1})^{-1}$ is nondecreasing on $(T_1, \infty)$ for some $T_1 > e$. Since

$$
s^\delta \varphi(s^{-1})^{-1} = s^\delta (\log s)^{-\varepsilon_0} \times (\log s)^{\varepsilon_0} \varphi(s^{-1})^{-1},
$$

the present lemma is obtained. □

**Lemma 2.6.** *If $\|f\|_{p(\cdot)} \leq 1$, then*

$$
\left\{ Mf(x) (\varphi(Mf(x)^{-1}))^{-n/p(x)^2} \right\}^{p(x)} \leq C\,(Mg(x) + 1)
$$

*for $x \in \Omega$.*

*Proof.* For simplicity, set $a = Mf(x)$ and $b = Mg(x)$. By Lemma 2.4, we have

$$a^p \le C\left(b\varphi(b^{-1})^{cp} + 1\right)$$

with $p = p(x)$ and $c = n/p^2$. We may assume that $a$ is large enough, that is, $a > T_0 > 1$. Using Lemma 2.5, we find

$$\left\{a\varphi(a^{-1})^{-c}\right\}^p \le Cb\varphi(b^{-1})^{cp} \times \varphi(Cb^{-1/p}\varphi(b^{-1})^{-c})^{-cp}.$$

Note from ($\varphi 2$) that

$$\varphi(Cb^{-1/p}\varphi(b^{-1})^{-c})^{-1} \le C\varphi(b^{-1})^{-1}.$$

Hence it follows that

$$\left\{a\varphi(a^{-1})^{-c}\right\}^p \le Cb$$

whenever $a > T_0$, which proves

$$\left\{a\varphi(a^{-1})^{-c}\right\}^p \le C(b+1),$$

as required. $\square$

**Theorem 2.7.** *Let $\Omega$ be an open set in $\mathbf{R}^n$ such that $|\Omega| < \infty$. If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)}\right\}^{p(x)} dx \le C$$

*whenever $f$ is a measurable function on $\Omega$ with $\|f\|_{p(\cdot)} \le 1$.*

*Proof.* Let $p_0(x) = p(x)/p_0$ for $1 < p_0 < p_-(\Omega)$. Then Lemma 2.6 yields

$$\left\{Mf(x)(\varphi(Mf(x)^{-1}))^{-n/p_0(x)^2}\right\}^{p_0(x)} \le C\left\{Mg_0(x) + 1\right\}$$

for $x \in \Omega$, where $g_0(y) = f(y)^{p_0(y)}$. Choosing $p_0 > 1$ such that $np_0^2/p(x)^2 < A(x)$, we establish

$$\left\{Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)}\right\}^{p(x)} \le C\left\{Mg_0(x) + 1\right\}^{p_0}.$$

Since $g_0 \in L^{p_0}(\Omega)$, we deduce the required inequality by the boundedness of maximal functions in $L^{p_0}$. $\square$

*Remark* 2.8. Set $\Phi(r,x) = \left\{r\varphi(r^{-1})^{-A(x)}\right\}^{p(x)}$ for $r \geq 0$ and $x \in \Omega$. Then Theorem 2.7 assures the existence of $C > 0$ such that

$$\int_\Omega \Phi(Mf(x)/C, x)dx \leq 1 \qquad \text{whenever } \|f\|_{p(\cdot)} \leq 1.$$

As in Edmunds and Rákosník [6], we define

$$\|f\|_\Phi = \|f\|_{\Phi,\Omega} = \inf\{\lambda > 0 : \int_\Omega \Phi(|f(x)|/\lambda, x)dx \leq 1\};$$

then it follows that

$$\|Mf\|_\Phi \leq C\|f\|_{p(\cdot)} \qquad \text{for } f \in L^{p(\cdot)}(\Omega).$$

If $\varphi(r) = a(\log(e+1/r))^b$, then Theorem 2.7 was proved by Diening [4] (when $b = 0$) and the authors [8, Theorem 2.4] (when $b$ is general).

*Remark* 2.9. For $0 < r < 1/2$, let

$$G = \{x = (x_1, x_2) : 0 < x_1 < 1, -1 < x_2 < 1\}$$

and

$$G(r) = \{x = (x_1, x_2) : 0 < x_1 < r, r < x_2 < 2r\}.$$

For $p(0) = p_0 > 1$, define

$$p(x_1, x_2) = \begin{cases} p_0 - \log(\varphi(x_2))/\log(1/x_2) & \text{when } 0 < x_2 \leq r_0, \\ p_0 & \text{when } x_2 \leq 0; \end{cases}$$

set $p(x_1, x_2) = p(x_1, r_0)$ when $x_2 > r_0$. Here we take $r_0 > 0$ so small that $p(x_1, r_0) > 1$. Consider

$$f_r(y) = \chi_{G(r)}(y)$$

with $\chi_E$ denoting the characteristic function of a set $E$, and set $g_r = f_r/\|f_r\|_{p(\cdot),G}$. Then we insist for $0 < r < r_0$ :

(i) $\|f_r\|_{p(\cdot),G} \leq C_1 r^{2/p_0}\varphi(r)^{-2/p_0^2}$ ;

(ii) $Mg_r(x) \geq C_2 r^{-2/p_0}\varphi(r)^{2/p_0^2}$ for $0 < x_1 < r$ and $-r < x_2 < 0$.

By integration of (ii) we see that

$$\int_G \left\{Mg_r(x)(\varphi(Mg_r(x)^{-1}))^{-2/p(x)^2}\right\}^{p(x)} dx \geq C_3,$$

which means that Theorem 2.7 does not hold for $A(x) < 2/p(x)^2$.

*Remark 2.10.* For $0 < r < 1/2$, let $G$ and $G(r)$ be as above. Define

$$p(x_1, x_2) = \begin{cases} p_0 + \log(\varphi(x_2))/\log(1/x_2) & \text{when } 0 < x_2 \le r_0, \\ p_0 & \text{when } x_2 \le 0; \end{cases}$$

and $p(x_1, x_2) = p(x_1, r_0)$ when $x_2 > r_0$. Setting

$$G'(r) = \{x = (x_1, x_2) : 0 < x_1 < r, -r < x_2 < 0\},$$

we consider

$$f'_r(y) = \chi_{G'(r)}(y)$$

and set $g'_r = f'_r/\|f'_r\|_{p(\cdot),G}$. Then we insist for $0 < r < r_0/2$ :

(i) $\|f'_r\|_{p(\cdot),G} = r^{2/p_0}$ ;

(ii) $Mg'_r(x) \ge C_1 r^{-2/p_0}$ for $0 < x_1 < r$ and $r < x_2 < 2r$ ;

(iii) $\int_G \left\{ Mg'_r(x)\varphi(Mg'_r(x)^{-1})^{-2/p(x)^2} \right\}^{p(x)} dx \ge C_2,$

as above.

## §3. Sobolev's inequality

Let $p(\cdot)$ be a continuous function on $\Omega$ satisfying (p1) and (p2). Further, suppose

$$p_+ = p_+(\Omega) < n/\alpha$$

and set

$$\frac{1}{p^\sharp(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.$$

For $0 < \alpha < n$, we consider the Riesz potential $I_\alpha f$ of measurable functions $f \in L^{p(\cdot)}(\Omega)$, which is defined by

$$I_\alpha f(x) = \int |x-y|^{\alpha-n} f(y)dy;$$

recall that we set $f = 0$ outside $\Omega$. Set

$$S_f = \{x \in \mathbf{R}^n : f(x) \neq 0\}.$$

In this section, we assume

$$|S_f| < \infty,$$

where $|E|$ denotes the $n$-dimensional measure of a measurable set $E$.

**Lemma 3.1.** *Let $f$ be a nonnegative measurable function on $\Omega$ such that $\|f\|_{p(\cdot)} \le 1$ and $|S_f| \le 1$. Then*

$$\int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\delta^{-n/p^\sharp(x)}\varphi(\delta)^{n/p(x)^2}$$

*for $x \in \Omega$ and $\delta \in (0,1)$.*

*Proof.* For $\mu > 0$, since $\|f\|_{p(\cdot)} \le 1$, we have

$$\begin{aligned}
&\int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\le\ & \mu \left( \int_{S_f \setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy + \int_{S_f \setminus B(x,\delta)} f(y)^{p(y)} dy \right) \\
\le\ & \mu \left( \int_{S_f \setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy + 1 \right).
\end{aligned}$$

Consider the set

$$E = \{y \in S_f : |x-y|^{\alpha-n} \ge \mu\} \cap \Omega.$$

Then we have

$$\int_{S_f \setminus \{E \cup B(x,\delta)\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \le |S_f| \le 1$$

by our assumption. Further, since $p'(y) \le p'(x) + \omega(|x-y|)$ by Lemma 2.3, we have

$$\begin{aligned}
&\int_{E \setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \\
\le\ & \int_{E \setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(x)+\omega(|x-y|)} dy.
\end{aligned}$$

If $\mu > 1$, then we see that

$$\begin{aligned}
&\int_{E \setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(x)+\omega(|x-y|)} dy \\
\le\ & \mu^{-p'(x)-\omega(\delta)} \int_{\mathbf{R}^n \setminus B(x,\delta)} |x-y|^{(\alpha-n)(p'(x)+\omega(|x-y|))} dy \\
\le\ & C\mu^{-p'(x)-\omega(\delta)} \delta^{(\alpha-n)(p'(x)+\omega(\delta))+n} \\
\le\ & C\mu^{-p'(x)-\omega(\delta)} \delta^{p'(x)(\alpha-n/p(x))} \varphi(\delta)^{(n-\alpha)/(p(x)-1)^2} \\
=\ & C\mu^{-p'(x)-\omega(\delta)} \delta^{-p'(x)n/p^\sharp(x)} \varphi(\delta)^{(n-\alpha)/(p(x)-1)^2}.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
&\int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\
\le\ & C\mu \left( \mu^{-p'(x)-\omega(\delta)} \delta^{-p'(x)n/p^\sharp(x)} \varphi(\delta)^{(n-\alpha)/(p(x)-1)^2} + 1 \right).
\end{aligned}$$

Considering $\mu = \delta^{-n/p^\sharp(x)} \varphi(\delta)^{n/p(x)^2}$ when $\delta$ is small, we see that

$$\int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \le C\delta^{-n/p^\sharp(x)} \varphi(\delta)^{n/p(x)^2},$$

as required. □

**Lemma 3.2.** *Let $f$ be a nonnegative measurable function on $\Omega$ such that $\|f\|_{p(\cdot)} \le 1$ and $|S_f| \le 1$. Then*

$$I_\alpha f(x) \le C\left[\{Mf(x)\}^{p(x)/p^\sharp(x)} \{\varphi(Mf(x)^{-1})\}^{\alpha/p(x)} + 1\right]$$

*for $x \in \Omega$.*

*Proof.* For $0 < \delta < 1$ we have by Lemma 3.1

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy + \int_{\Omega \setminus B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\le C\delta^\alpha Mf(x) + C\delta^{-n/p^\sharp(x)} \varphi(\delta)^{n/p(x)^2}. \end{aligned}$$

Considering $\delta = \{Mf(x)\}^{-p(x)/n} \{\varphi(Mf(x)^{-1})\}^{1/p(x)}$ when $Mf(x)$ is large enough, we see that

$$I_\alpha f(x) \le C\left[\{Mf(x)\}^{p(x)/p^\sharp(x)} \{\varphi(Mf(x)^{-1})\}^{\alpha/p(x)} + 1\right],$$

as required. □

**Lemma 3.3.** *Let $p > 1$ and $1/p^\sharp = 1/p - \alpha/n$. For $\beta > \alpha$, set $c = \beta/p$ and $d = \gamma/p^2$, where $\beta/\gamma = \alpha/n$. If $s > 0$, $t > 0$ and $s^{p^\sharp} \le C_1 \left\{t^p \varphi(t^{-1})^{cp^\sharp} + 1\right\}$, then*

$$\left\{s\varphi(s^{-1})^{-d}\right\}^{p^\sharp} \le C_2 \left\{t^p \varphi(t^{-1})^{-dp} + 1\right\},$$

*where $C_2$ is a positive constant independent of $s$ and $t$.*

*Proof.* We may assume that $t$ is large enough, that is, $t > T_0 > 1$. Using Lemma 2.5, we find

$$\left\{s\varphi(s^{-1})^{-d}\right\}^{p^\sharp} \le Ct^p \varphi(t^{-1})^{cp^\sharp} \times \varphi(t^{-p/p^\sharp} \varphi(t^{-1})^{-c})^{-dp^\sharp},$$

with $d = \gamma/p^2$. Note from ($\varphi$2) that

$$\varphi(t^{-p/p^\sharp} \varphi(t^{-1})^{-c})^{-1} \le C\varphi(t^{-1})^{-1}.$$

Hence it follows that

$$\left\{s\varphi(s^{-1})^{-d}\right\}^{p^\sharp} \le Ct^p\varphi(t^{-1})^{(c-d)p^\sharp} = Ct^p\varphi(t^{-1})^{-dp}$$

whenever $t > T_0$, which proves

$$\left\{s\varphi(s^{-1})^{-d}\right\}^{p^\sharp} \le C\left\{t^p\varphi(t^{-1})^{-dp} + 1\right\},$$

as required. □

By Lemmas 3.2 and 3.3, we have the following result.

**Corollary 3.4.** *Let $f$ be a nonnegative measurable function on $\Omega$ such that $\|f\|_{p(\cdot)} \le 1$ and $|S_f| \le 1$. If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\begin{aligned} &\left\{I_\alpha f(x)(\varphi(I_\alpha f(x)^{-1}))^{-A(x)}\right\}^{p^\sharp(x)} \\ \le\ & C\left[\left\{Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)}\right\}^{p(x)} + 1\right] \end{aligned}$$

*for $x \in \Omega$.*

Thus Theorem 2.7 and Corollary 3.4 yield the following Sobolev inequality for Riesz potentials.

**Theorem 3.5.** *Let $\Omega$ be an open set in $\mathbf{R}^n$ such that $|\Omega| < \infty$. Suppose $p_+(\Omega) < n/\alpha$. If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{I_\alpha f(x)(\varphi(I_\alpha f(x)^{-1}))^{-A(x)}\right\}^{p^\sharp(x)} dx \le C$$

*whenever $f$ is a nonnegative measurable function on $\Omega$ with $\|f\|_{p(\cdot),\Omega} \le 1$.*

*Remark 3.6.* If $\varphi(r) = a(\log(e + 1/r))^b$, then Theorem 3.5 was proved by the authors [8, Theorem 3.4]. See also Capone, Cruz-Uribe and Fiorenza [2, Theorem 1.6], Diening [4] and the authors [9, Theorem 3.3].

*Remark 3.7.* In Remark 2.9, we see that

$$I_\alpha g_r(x) \ge C_1 r^{-2/p^\sharp(x)}\varphi(r)^{2/p_0^2}$$

for $0 < x_1 < r$ and $-r < x_2 < 0$. Hence we have

$$\int_G \left\{ I_\alpha g_r(x)(\varphi(I_\alpha g_r(x)^{-1}))^{-2/p(x)^2} \right\}^{p^\sharp(x)} dx \geq C_2.$$

*Remark 3.8.* In Remark 2.10, we see that

$$I_\alpha g'_r(x) \geq C_1 r^{-2/p_0^\sharp}$$

for $0 < x_1 < r$ and $r < x_2 < 2r$. Hence we have

$$\int_G \left\{ I_\alpha g'_r(x)(\varphi(I_\alpha g'_r(x)^{-1}))^{-2/p(x)^2} \right\}^{p^\sharp(x)} dx \geq C_2.$$

In the next section, we treat the case when $\Omega$ might not be bounded, as in Cruz-Uribe, Fiorenza and Neugebauer [3].

## §4. Maximal functions on general domains

In this section we treat the boundedness of maximal functions on general domains, which gives a generalization of the result by Cruz-Uribe, Fiorenza and Neugebauer [3].

Let $\Omega$ be an open set in $\mathbf{R}^n$. Consider a positive continuous function $p(\cdot)$ on $\Omega$ such that

(p1) $1 < p_-(\Omega) = \inf_\Omega p(x) \leq \sup_\Omega p(x) = p_+(\Omega) < \infty$;
(p2) $|p(x)-p(y)| \leq \log(\varphi(|x-y|))/\log(1/|x-y|)$ whenever $|x-y| < 1/2$, $x \in \Omega$ and $y \in \Omega$;
(p3) $|p(x) - p(y)| \leq C/\log(e + |x|)$ whenever $x \in \Omega$, $y \in \Omega$ and $|y| \geq |x|$.

If (p3) holds, then $p$ has a finite limit $p_\infty$ at infinity and

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)} \qquad \text{for all } x \in \Omega. \tag{p3$'$}$$

For a nonnegative measurable function $f$ on $\Omega$, set

$$F(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)^{p(y)} dy,$$

as before. If $\|f\|_{p(\cdot)} \leq 1$ and $F(x) \geq 1$, then we have by the proof of Lemma 2.4

$$f_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy \leq CF(x)^{1/p(x)} \varphi(F(x)^{-1})^{n/p(x)^2},$$

so that

$$\left\{ f_B(\varphi(f_B^{-1}))^{-n/p(x)^2} \right\}^{p(x)} \le CF(x). \tag{4.1}$$

**Lemma 4.1.** *Let $f$ be a nonnegative measurable function on $\Omega$. If $F(x) \le 1$ and $f(y) \ge 1$ or $f(y) = 0$ for $y \in \Omega$, then*

$$(f_B)^{p(x)} \le F(x).$$

*Proof.* If $f(y) \ge 1$ or $f(y) = 0$ for $y \in \Omega$, then

$$f_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy \le \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)^{p(y)} dy = F(x).$$

Since $F(x) \le 1$, $f_B \le 1$, so that

$$(f_B)^{p(x)} \le f_B \le F(x),$$

as required. □

By (4.1) and Lemma 4.1 we have the following result.

**Corollary 4.2.** *Let $f$ be a nonnegative measurable function on $\Omega$ such that $\|f\|_{p(\cdot)} \le 1$. If $f(y) \ge 1$ or $f(y) = 0$ for $y \in \Omega$, then*

$$\left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-n/p(x)^2} \right\}^{p(x)} \le CMg(x),$$

*where $g(y) = f(y)^{p(y)}$.*

For a function $f$ on $\mathbf{R}^n$, we define the Hardy operator $H$ by

$$Hf(x) = \frac{1}{|B(0,|x|)|} \int_{B(0,|x|)} |f(y)|dy$$

for $x \in \mathbf{R}^n \setminus \{0\}$ and $Hf(0) = 0$.

**Lemma 4.3.** *Let $f$ be a nonnegative measurable function on $\Omega$. If $f \le 1$ on $\Omega$, then*

$$(f_B)^{p(x)} \le C\left[ F(x) + e(x) + \{Hf(x)\}^{p(x)} \right], \tag{4.2}$$

*where $e(x) = (e + |x|)^{-n}$.*

*Proof.* Note that $F(x) \le 1$ since $f \le 1$ on $\Omega$. If $x \in \Omega \cap B(0,1)$, then

$$f_B \le 1,$$

which proves (4.2).

For every subset $E$ of $\Omega$, we set $p_+(E) = \sup_E p(x)$ and $p_-(E) = \inf_E p(x)$. Fix $x \in \Omega \setminus B(0,1)$, and take a ball $B = B(x,r)$. We will consider two cases.

Case 1: $r \ge |x|/2$. Since $p_+ = p_+(\Omega) < \infty$, we have

$$\begin{aligned} (f_B)^{p(x)} \quad &\le \quad 2^{p_+} \left( \frac{1}{|B|} \int_{B \cap B(0,|x|)} f(y) dy \right)^{p(x)} \\ &\quad +2^{p_+} \left( \frac{1}{|B|} \int_{B \setminus B(0,|x|)} f(y) dy \right)^{p(x)}. \end{aligned}$$

Then, since $r \ge |x|/2$, we see that

$$\frac{1}{|B|} \int_{B \cap B(0,|x|)} f(y) dy \quad \le \quad CHf(x).$$

We set $E = (B \setminus B(0,|x|)) \cap \Omega$ and

$$D = \{y : f(y) \ge e(x)\}.$$

By Hölder's inequality, we have

$$\frac{1}{|B|} \int_E f(y) dy \quad \le \quad \left( \frac{1}{|B|} \int_{E \cap D} f(y)^{p_-(E)} dy \right)^{1/p_-(E)} + e(x).$$

By assumption (p3), if $y \in E$, then

$$0 \le p(y) - p_-(E) \le p_+(E) - p_-(E) \le \frac{C}{\log(e+|x|)}.$$

Therefore, if $y \in E \cap D$, then

$$\begin{aligned} f(y)^{p_-(E)} \quad &= \quad f(y)^{p(y)} f(y)^{p_-(E)-p(y)} \\ &\le \quad f(y)^{p(y)} e(x)^{-C/\log(e+|x|)} \le C f(y)^{p(y)}, \end{aligned}$$

so that

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_E f(y)dy\right)^{p(x)}\\
\le\;& C\left(\frac{1}{|B|}\int_{E\cap D} f(y)^{p(y)}dy\right)^{p(x)/p_-(E)} + Ce(x)^{p(x)}\\
\le\;& CF(x)^{p(x)/p_-(E)} + Ce(x)^{p(x)}.
\end{aligned}$$

Since $F(x) \le 1$ by our assumption and $e(x) \le 1$, we obtain

$$\left(\frac{1}{|B|}\int_E f(y)dy\right)^{p(x)} \quad\le\quad CF(x) + Ce(x),$$

which proves (4.2).

Case 2: $0 < r \le |x|/2$. In this case, we see as before that

$$0 \le p(y) - p_-(B\cap\Omega) \le p_+(B\cap\Omega) - p_-(B\cap\Omega) \le \frac{C}{\log(e+|x|)}$$

for $y \in B\cap\Omega$. Hence it follows as above that

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B f(y)dy\right)^{p(x)}\\
\le\;& C\left(\frac{1}{|B|}\int_B f(y)^{p(y)}dy\right)^{p(x)/p_-(B\cap\Omega)} + Ce(x)^{p(x)}\\
\le\;& CF(x) + Ce(x),
\end{aligned}$$

as required. □

**Lemma 4.4.** *Let $f$ be a nonnegative measurable function on $\Omega$ such that $f \le 1$ on $\Omega$. Then*

$$\{Hf(x)\}^{p(x)} \le CHg(x) + Ce(x),$$

*where $g(y) = f(y)^{p(y)}$.*

*Proof.* Let $f$ be a nonnegative measurable function on $\Omega$ such that $f \le 1$ on $\Omega$. Then, since $0 \le f \le 1$ on $\Omega$, we see that

$$H(f\chi_{B(0,r_0)})(x) \le Ce(x) \qquad \text{on } \Omega.$$

Hence we may assume that $f = 0$ on $B(0,r_0)$.

For $\mu \geq 1$ and $r = |x| > r_0$, we have

$$\frac{1}{|B(0,r)|}\int_{B(0,r)} f(y)dy \quad \leq \quad \mu\left(\frac{1}{|B(0,r)|}\int_{B(0,r)} (1/\mu)^{p'(y)}dy + G\right),$$

where $G = Hg(x)$ with $g(y) = f(y)^{p(y)}$. Then note from (p3) that

$$-p'(y) \leq -p'(x) + \omega(|y|) \qquad \text{for } y \in B(0,r),$$

where $\omega(t) = C/\log(e+t)$. If $\log \mu \leq c_1 \log r$ and $0 < m < n$, then we can find $r_1 > e$ such that

$$\mu^{-p'(y)} t^m \leq C\mu^{-p'(x)+\omega(r)} r^m$$

whenever $r_1 \leq t = |y| < r = |x|$, which yields

$$Hf(x) \quad \leq \quad \mu\left(C\mu^{-p'(x)+\omega(r)} + G\right).$$

First assume $r^{-n} < G \leq 1$. Then we set $\mu = G^{-1/\{p'(x)-\omega(r)\}}$ and, noting that $\mu \leq Cr^n$, we have

$$Hf(x) \quad \leq \quad CG^{1/p(x)}G^{-\omega(r)/\{p'(x)(p'(x)-\omega(r))\}} \leq CG^{1/p(x)}.$$

Next, if $G \leq r^{-n}$, then we set $\mu = r^{n/p'(x)}$ and obtain

$$Hf(x) \leq Ce(x)^{1/p(x)} + G^{1/p(x)} \leq Ce(x)^{1/p(x)}.$$

If $|x| \leq r_1$, then

$$Hf(x) \leq 1 \leq Ce(x),$$

which completes the proof. □

Combining Lemma 4.3 with Lemma 4.4, we obtain the following result.

**Corollary 4.5.** *Let $f$ be a nonnegative measurable function on $\Omega$. If $f \leq 1$ on $\Omega$, then*

$$\{Mf(x)\}^{p(x)} \leq C\{Mg(x) + e(x) + Hg(x)\},$$

*where $e(x) = (e+|x|)^{-n}$ and $g(y) = f(y)^{p(y)}$.*

By Hardy's inequality we can prove the following inequality (cf. Lemma 5.4).

**Lemma 4.6.** *Let $g$ be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|g\|_{p_0} \le 1$, $1 < p_0 < \infty$. Then*

$$\int \{Hg(x)\}^{p_0} dx \le C.$$

Now, as in Cruz-Uribe, Fiorenza and Neugebauer [3], we can prove the following result.

**Theorem 4.7.** *If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)} \right\}^{p(x)} dx \le C$$

*whenever $f$ is a measurable function on $\Omega$ with $\|f\|_{p(\cdot)} \le 1$.*

*Proof.* For $p_0 > 1$, set $p_0(x) = p(x)/p_0$ and $g_0(y) = f(y)^{p_0(y)}$. Then we have by Corollaries 4.2 and 4.5

$$\left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-n/p_0(x)^2} \right\}^{p_0(x)} \le C \left\{ Mg_0(x) + e(x) + Hg_0(x) \right\}.$$

If $a > np_0^2$, then

$$\begin{aligned} & \left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)} \right\}^{p(x)} \\ \le \; & C \left\{ Mg_0(x) + e(x) + Hg_0(x) \right\}^{p_0} \\ \le \; & CMg_0(x)^{p_0} + Ce(x)^{p_0} + C\{Hg_0(x)\}^{p_0}. \end{aligned}$$

Since $p_0 > 1$, $M$ is bounded on $L^{p_0}(\Omega)$ and $e(x) \in L^{p_0}(\mathbf{R}^n)$, we find

$$\int_\Omega \left\{ Mf(x)(\varphi(Mf(x)^{-1}))^{-A(x)} \right\}^{p(x)} dx \le C + C \int_{\mathbf{R}^n} \{Hg_0(x)\}^{p_0} dx.$$

Thus Lemma 4.6 yields the required inequality. $\square$

## §5. Sobolev's inequality for general domains

In this section we extend Sobolev's inequality to general domains $\Omega$. Consider a positive continuous function $p(\cdot)$ on $\Omega$ satisfying

(p1′) $1 < p_- = p_-(\Omega) \le p_+(\Omega) = p_+ < n/\alpha$;
(p2) $|p(x) - p(y)| \le \log(\varphi(|x-y|))/\log(1/|x-y|)$ whenever $x \in \Omega$, $y \in \Omega$ and $|x-y| < 1/2$;

(p3) $|p(x) - p(y)| \le C/\log(e + |x|)$ whenever $x \in \Omega$, $y \in \Omega$ and $|y| \ge |x|$.

By (p3) or (p3′) we can find $R_0 > 1$ such that

$$p(x) \le p_\infty + \frac{C}{\log(e + |x|)} < \frac{n}{\alpha} \tag{3}$$

for $x \in \Omega \setminus B(0, R_0/2)$.

**Lemma 5.1.** *If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{ I_\alpha f(x) (\varphi(I_\alpha f(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} dx \le C$$

*whenever $f$ is a nonnegative measurable function on $\Omega$ such that $f = 0$ outside $B(0, R_0)$ and $\|f\|_{p(\cdot)} \le 1$.*

*Proof.* Let $f$ be a nonnegative measurable function on $\Omega$ such that $f = 0$ on $\mathbf{R}^n \setminus B(0, R_0)$ and $\|f\|_{p(\cdot)} \le 1$. In view of Theorem 3.5, we have

$$\int_{B(0,2R_0)} \left\{ I_\alpha f(x) (\varphi(I_\alpha f(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} dx \le C.$$

If $x \in \mathbf{R}^n \setminus B(0, 2R_0)$, then

$$\begin{aligned} I_\alpha f(x) &\le (|x|/2)^{\alpha - n} \int_{B(0,R_0)} f(y) dy \\ &\le (|x|/2)^{\alpha - n} \int_{B(0,R_0)} \{1 + f(y)^{p(y)}\} dy \le C|x|^{\alpha - n}, \end{aligned}$$

so that

$$\int_{\Omega \setminus B(0,2R_0)} I_\alpha f(x)^{q_0} dx \le C$$

whenever $q_0(\alpha - n) + n < 0$. Now it follows that

$$\int_\Omega \left\{ I_\alpha f(x) (\varphi(I_\alpha f(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} dx \le C,$$

as required. □

**Lemma 5.2.** *If $f$ is a nonnegative measurable function on $\Omega$ such that $\|f\|_{p(\cdot)} \le 1$ and $f = 0$ on $B(0, R_0)$, then*

$$\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} |x - y|^{\alpha - n} f(y) dy \le C\delta^{\alpha - n/p(x)}$$

*for* $x \in \Omega \setminus B(0, R_0)$ *and* $\delta \geq 1$.

*Proof.* For $x \in \Omega \setminus B(0, R_0)$ and $\mu > 0$, since $\|f\|_{p(\cdot)} \leq 1$, we have

$$\begin{aligned} & \int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} |x-y|^{\alpha-n} f(y) dy \\ \leq \; & \mu \left( \int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \right. \\ & \left. + \int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} f(y)^{p(y)} dy \right) \\ \leq \; & \mu \left( \int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta)\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy + 1 \right). \end{aligned}$$

First consider the case $1 \leq \delta \leq 2|x|$. Let $E = \{y \in \Omega \setminus B(0, |x|/2) : |x-y|^{\alpha-n}/\mu > 1\}$. If we set

$$\beta_1 \equiv \beta_1(x) = p'(x) - \frac{C}{\log(e + |x|)},$$

then it follows from (3) that

$$p'(y) \geq \beta_1 > \frac{n}{n-\alpha} \qquad \text{for } y \in \Omega \setminus B(0, |x|/2).$$

Hence we obtain

$$\begin{aligned} & \int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta) \cup E\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \\ \leq \; & \int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta) \cup E\}} (|x-y|^{\alpha-n}/\mu)^{\beta_1} dy \\ \leq \; & \mu^{-\beta_1} \int_{\Omega \setminus B(x,\delta)} |x-y|^{(\alpha-n)\beta_1} dy \\ \leq \; & C \mu^{-\beta_1} \delta^{(\alpha-n)\beta_1 + n}. \end{aligned}$$

Considering $\mu = \delta^{\alpha - n + n/\beta_1}$, we see that

$$\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta) \cup E\}} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \leq C,$$

so that

$$\int_{\Omega \setminus \{B(0,|x|/2) \cup B(x,\delta) \cup E\}} |x-y|^{\alpha-n} f(y) dy \leq C \delta^{\alpha - n + n/\beta_1}.$$

Similarly, if we set

$$\beta_2 \equiv \beta_2(x) = p'(x) + \frac{C}{\log(e+|x|)},$$

then it follows from (3) that

$$p'(y) \le \beta_2 \qquad \text{for } y \in \Omega \setminus B(0,|x|/2).$$

Note here that

$$\begin{aligned} \int_{E\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy &\le \int_{E\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{\beta_2} dy \\ &\le \mu^{-\beta_2} \int_{\mathbf{R}^n\setminus B(x,\delta)} |x-y|^{(\alpha-n)\beta_2} dy \\ &\le C\mu^{-\beta_2}\delta^{(\alpha-n)\beta_2+n}. \end{aligned}$$

Since $\mu = \delta^{\alpha-n+n/\beta_1}$ and $\delta \ge 1$, we see that

$$\int_{E\setminus B(x,\delta)} (|x-y|^{\alpha-n}/\mu)^{p'(y)} dy \le C\delta^{n(1-\beta_2/\beta_1)} \le C,$$

so that

$$\int_{E\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \le C\delta^{\alpha-n+n/\beta_1}.$$

Therefore

$$\int_{\Omega\setminus\{B(0,|x|/2)\cup B(x,\delta)\}} |x-y|^{\alpha-n} f(y) dy \le C\delta^{\alpha-n+n/\beta_1}.$$

Since $1 \le \delta \le 2|x|$,

$$\delta^{\alpha-n+n/\beta_1} \le C\delta^{\alpha-n+n/p'(x)} = C\delta^{\alpha-n/p(x)},$$

so that

$$\int_{\Omega\setminus\{B(0,|x|/2)\cup B(x,\delta)\}} |x-y|^{\alpha-n} f(y) dy \le C\delta^{\alpha-n/p(x)}.$$

Next consider the case $\delta > 2|x| \ge 2R_0$. Then

$$\int_{\Omega\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \le C \int_{\Omega\setminus B(X_\delta,\delta/2)} |X_\delta-y|^{\alpha-n} f(y) dy,$$

where $X_\delta = (\delta/4, 0, ..., 0) \in \mathbf{R}^n$. Hence the above considerations yield

$$\int_{\Omega\setminus B(x,\delta)} |x-y|^{\alpha-n} f(y) dy \le C\delta^{\alpha-n/p(x)}.$$

Thus the proof is completed. □

For a measurable function $f$ on $\mathbf{R}^n$, we define the operator $H_\alpha$ by

$$H_\alpha f(x) = |x|^{\alpha-n} \int_{B(0,|x|)} |f(y)|dy$$

for $x \in \mathbf{R}^n \setminus \{0\}$ and $H_\alpha f(0) = 0$.

**Lemma 5.3.** *Let $f$ be a nonnegative measurable function on $\Omega$ with $\|f\|_{p(\cdot)} \le 1$. If $x \in \Omega$ and $Mf(x) \le 1$, then*

$$\{I_\alpha f(x)\}^{p^\sharp(x)} \le C\{Mf(x)\}^{p(x)} + C\{H_\alpha f(x)\}^{p^\sharp(x)}.$$

*Proof.* Let $f$ be a nonnegative measurable function on $\Omega$ with $\|f\|_{p(\cdot)} \le 1$. For $\delta \ge 1$ we have by Lemma 5.2

$$\begin{aligned} I_\alpha f(x) &= \int_{B(x,\delta)} |x-y|^{\alpha-n} f(y)dy \\ &\quad + \int_{\Omega\setminus\{B(0,|x|/2)\cup B(x,\delta)\}} |x-y|^{\alpha-n} f(y)dy \\ &\quad + \int_{B(0,|x|/2)} |x-y|^{\alpha-n} f(y)dy \\ &\le C\delta^\alpha Mf(x) + C\delta^{\alpha-n/p(x)} + CH_\alpha f(x) \end{aligned}$$

for $x \in \mathbf{R}^n$. If we set $\delta = \{Mf(x)\}^{-p(x)/n}$, then it follows that

$$I_\alpha f(x) \le C\{Mf(x)\}^{p(x)/p^\sharp(x)} + CH_\alpha f(x),$$

which yields the required inequality. □

**Lemma 5.4.** *Let $1 < p_1 < n/\beta$ and $1/q_1 = 1/p_1 - \beta/n$. Then*

$$\|H_\beta f\|_{q_1} \le C\|f\|_{p_1}.$$

This is a consequence of the usual Sobolev's inequality; see e.g. the book by Adams and Hedberg [1].

**Lemma 5.5.** *If $f$ is a nonnegative measurable function on $\Omega$ such that $\|f\|_{p(\cdot)} \le 1$ and $f = 0$ on $B(0, R_0)$, then*

$$\int_\Omega \{H_\alpha f(x)\}^{p^\sharp(x)} dx \le C.$$

*Proof.* Let $f$ be a nonnegative measurable function on $\Omega$ such that $\|f\|_{p(\cdot)} \le 1$ and $f = 0$ on $B(0, R_0)$. Write

$$f = f_1 + f_2,$$

where $f_1 = f\chi_{\{y: f(y) \ge 1\}}$ and $f_2 = f\chi_{\{y: f(y) < 1\}}$. Then we see that

$$H_\alpha f_1(x) \le |x|^{\alpha-n} \int_{B(0,|x|)} f_1(y)^{p(y)} dy \le |x|^{\alpha-n}$$

for $|x| \ge R_0$, so that

$$\int_\Omega \{H_\alpha f_1(x)\}^{p^\sharp(x)} dx \le C.$$

Thus we may assume that $f = f_2 \le 1$ on $\Omega$.

Let $1/q_\infty = 1/p_\infty - \alpha/n$ and $1/p^\sharp(x) = 1/p(x) - \alpha/n$ . For $1 < p_1 < p_-$, set $p_1(y) = p(y)/p_1$. Then for $r = |x| \ge R_0$ we have by Lemma 4.4

$$\begin{aligned} &\left( r^{\alpha-n} \int_{B(0,r)} f(y) dy \right)^{p^\sharp(x)} \\ \le \;& C \left( r^{\alpha p_\infty/p_1 - n} \int_{B(0,r)} f(y)^{p_1(y)} dy \right)^{p^\sharp(x) p_1/p(x)} + C r^{q_\infty(\alpha - np_1/p_\infty)}. \end{aligned}$$

If $\int_{B(0,r)} f(y)^{p_1(y)} dy \le 1$, then the right hand side is dominated by

$$C r^{q_\infty(\alpha - np_1/p_\infty)}.$$

Next suppose $\int_{B(0,r)} f(y)^{p_1(y)} dy > 1$. If $p^\sharp(x)p_1/p(r) \le q_\infty p_1/p_\infty$, then

$$\begin{aligned} &\left( r^{\alpha p_\infty/p_1 - n} \int_{B(0,r)} f(y)^{p_1(y)} dy \right)^{p^\sharp(x) p_1/p(r)} \\ \le \;& C \left( r^{\alpha p_\infty/p_1 - n} \int_{B(0,r)} f(y)^{p_1(y)} dy \right)^{q_\infty p_1/p_\infty}; \end{aligned}$$

if $p^\sharp(x)p_1/p(r) > q_\infty p_1/p_\infty$, then, since $r^{-n} \int_{B(0,r)} f(y)^{p_1(y)} dy \le C$, the above inequality is also true. Hence it follows that

$$\begin{aligned} &\left( r^{\alpha-n} \int_{B(0,r)} f(y) dy \right)^{p^\sharp(x)} \\ \le \;& C \left( r^{\alpha p_\infty/p_1 - n} \int_{B(0,r)} f(y)^{p_1(y)} dy \right)^{q_\infty p_1/p_\infty} + C r^{q_\infty(\alpha - np_1/p_\infty)}. \end{aligned}$$

Since $1/(q_\infty p_1/p_\infty) = 1/p_1 - (\alpha p_\infty/p_1)/n$, it follows from Lemma 5.4 that

$$\int_\Omega \{H_\alpha f(x)\}^{p^\sharp(x)} dx \le C,$$

which yields the required inequality. □

Our final goal is to establish Sobolev's inequality of Riesz potentials defined in general domains, which gives an extension of Capone, Cruz-Uribe and Fiorenza [2, Theorem 1.6].

**Theorem 5.6.** *Suppose $p_+(\Omega) < n/\alpha$. If $A(x) = a/p(x)^2$ with $a > n$, then*

$$\int_\Omega \left\{ I_\alpha f(x)(\varphi(I_\alpha f(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} dx \le C$$

*whenever $f$ is a nonnegative measurable function on $\Omega$ with $\|f\|_{p(\cdot)} \le 1$.*

*Proof.* Let $f$ be a nonnegative measurable function on $\Omega$ with $\|f\|_{p(\cdot)} \le 1$. In view of Lemma 5.1, it suffices to treat the case when $f = 0$ on $B(0, R_0)$. Set

$$f = f_1 + f_2,$$

where $f_1 = f\chi_{\{y: f(y) \ge 1\}}$ and $f_2 = f\chi_{\{y: f(y) < 1\}}$. If $Mf_1(x) \ge 1$, then Corollary 3.4 gives

$$\begin{aligned} & \left\{ I_\alpha f_1(x)(\varphi(I_\alpha f_1(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} \\ \le \quad & C \left\{ Mf_1(x)(\varphi(Mf_1(x)^{-1}))^{-A(x)} \right\}^{p(x)}, \end{aligned}$$

and if $Mf_1(x) < 1$, then Lemma 5.3 gives

$$\{I_\alpha f_1(x)\}^{p^\sharp(x)} \le C\{Mf_1(x)\}^{p(x)} + C\{H_\alpha f_1(x)\}^{p^\sharp(x)},$$

so that

$$\begin{aligned} & \left\{ I_\alpha f_1(x)(\varphi(I_\alpha f_1(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} \\ \le \quad & C \left\{ Mf_1(x)(\varphi(Mf_1(x)^{-1}))^{-A(x)} \right\}^{p(x)} + C\{H_\alpha f_1(x)\}^{p^\sharp(x)}. \end{aligned}$$

Further we have by Lemma 5.3

$$\{I_\alpha f_2(x)\}^{p^\sharp(x)} \le C\{Mf_2(x)\}^{p(x)} + C\{H_\alpha f_2(x)\}^{p^\sharp(x)},$$

which proves

$$\begin{aligned}
&\left\{ I_\alpha f_2(x)(\varphi(I_\alpha f_2(x)^{-1}))^{-A(x)} \right\}^{p^\sharp(x)} \\
\le\ & C\left\{ Mf_2(x)(\varphi(Mf_2(x)^{-1}))^{-A(x)} \right\}^{p(x)} + C\{H_\alpha f_2(x)\}^{p^\sharp(x)}.
\end{aligned}$$

Now Theorem 4.7 and Lemma 5.5 give the required inequality. □

*Remark 5.7.* As in Remark 2.10, we consider $p$ of the form:

$$p(y) = \begin{cases} p_\infty & \text{when } y_n \le 0, \\ p_\infty + \dfrac{1}{\log(e+|y|)} \dfrac{a \log(\log(1/y_n))}{\log(1/y_n)} & \text{when } 0 < y_n \le r_0, \\ p_\infty + \dfrac{1}{\log(e+|y|)} \dfrac{a \log(\log(1/r_0))}{\log(1/r_0)} & \text{when } y_n > r_0, \end{cases}$$

where $y = (y', y_n)$, $1 < p_\infty < n/\alpha$, $a > 0$ and $0 < r_0 < 1/e$. Let $B(R,r) = B(e(R), r)$ for $0 < r \le r_0$, $R > 1$ and $e(R) = (R, 0, ..., 0) \in \mathbf{R}^n$. Then Theorem 5.6 (or Theorem 3.5) implies that in case $a' > a/\log(e+R)$, we have

$$\int_{B(R,r_0)} \left\{ I_\alpha f(x)(\log(e + I_\alpha f(x)))^{-a'n/p_\infty^2} \right\}^{p^\sharp(x)} dx \le C \tag{4}$$

whenever $f$ is a nonnegative measurable function on $B(R, r_0)$ with $\|f\|_{p(\cdot)} \le 1$.

We show that this is sharp. For this purpose, consider

$$f_r = \chi_{B_-(R,r)} \qquad (B_-(R,r) = B(R,r) \setminus H),$$

where $H = \{x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R} : x_n > 0\}$. Then note that

$$\|f_r\|_{p(\cdot)} = Cr^{n/p_\infty}.$$

Setting $g_r = f_r/\|f_r\|_{p(\cdot)}$, we find

$$I_\alpha g_r(x) \ge Cr^{\alpha - n/p_\infty}$$

for $x \in B(R, r)$, so that

$$\int_{B(R,r)} \left\{ I_\alpha g_r(x)(\log(e + I_\alpha g_r(x)))^{-an/\{p_\infty^2 \log(e+R)\}} \right\}^{p^\sharp(x)} dx \ge C.$$

This implies that (4) does not hold when $a' < a/\log(e+R)$.

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# Representations of nonnegative solutions for parabolic equations

**Minoru Murata**

## §1. Introduction

This paper is an announcement of results on integral representations of nonnegative solutions to parabolic equations, and gives a representation theorem which is general and applicable to many concrete examples for establishing explicit integral representations.

We consider nonnegative solutions of a parabolic equation

$$(\partial_t + L)u = 0 \quad \text{in} \quad D \times (0, T), \tag{1.1}$$

where $T$ is a positive number, $D$ is a non-compact domain of a Riemannian manifold $M$, $\partial_t = \partial/\partial t$, and $L$ is a second order elliptic operator on $D$. We study the problem:

Determine all nonnegative solutions of the parabolic equation (1.1). This problem is closely related to the Widder type uniqueness theorem for a parabolic equation, which asserts that any nonnegative solution is determined uniquely by its initial value. (For Widder type uniqueness theorems, see [1], [5], [10], [13] and references therein.) We say that **[UP]**(i.e., uniqueness for the positive Cauchy problem) holds for (1.1) when any nonnegative solution of (1.1) with zero initial value is identically zero. When [UP] holds for (1.1) the answer to our problem is extremely simple: for any nonnegative solution of (1.1)there exists a

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unique Borel measure $\mu$ on $D$ such that

$$u(x,t) = \int_D p(x,y,t)d\mu(y), \quad x \in D,\ 0 < t < T,$$

where $p$ is the minimal fundamental solution for (1.1) (cf. [2], [1]). While [UP] does not hold, however, only few explicit integral representations of nonnegative solutions to parabolic equations are given (cf. [8], [4], [14]). On the other hand, for elliptic equations, there has been a significant progress in determining explicitly Martin boundaries in many important cases (cf. [12] and references therein). Recall that any nonnegative solution of an elliptic equation is represented by an integral of Martin kernels with respect to a Borel measure on the Martin boundary.

The aim of this paper is to give explicit integral representations of nonnegative solutions to parabolic equations for which [UP] does not hold. We give a general and sharp condition under which any nonnegative solution of (1.1) with zero initial value is represented by an integral on the product of the Martin boundary of $D$ for an elliptic operator associated with $L$ and the time interval $[0,T)$.

## §2. Main results

Let $M$ be a connected separable $n$-dimensional smooth manifold with Riemannian metric of class $C^0$. Denote by $\nu$ the Riemannian measure on $M$. $T_xM$ and $TM$ denote the tangent space to $M$ at $x \in M$ and the tangent bundle, respectively. We denote by $\mathrm{End}(T_xM)$ and $\mathrm{End}(TM)$ the set of endmorphisms in $T_xM$ and the corresponding bundle, respectively. The inner product on $TM$ is denoted by $\langle X, Y\rangle$, where $X, Y \in TM$; and $|X| = \langle X, X\rangle^{1/2}$. The divergence and gradient with respect to the metric on $M$ are denoted by div and $\nabla$, respectively. Let $D$ be a non-compact domain of $M$. Let $L$ be an elliptic differential operator on $D$ of the form

$$Lu = -m^{-1}\mathrm{div}(mA\nabla u) + Vu, \tag{2.1}$$

where $m$ is a positive measurable function on $D$ such that $m$ and $m^{-1}$ are bounded on any compact subset of $D$, $A$ is a symmetric measurable section on $D$ of $\mathrm{End}(TM)$, and $V$ is a real-valued measurable function on $D$ such that

$$V \in L^p_{\mathrm{loc}}(D, md\nu), \quad \text{for some } p > \max(\frac{n}{2}, 1).$$

Here $L^p_{\mathrm{loc}}(D, md\nu)$ is the set of real-valued functions on $D$ locally $p$-th integrable with respect to $md\nu$. We assume that $L$ is locally uniformly elliptic on $D$, i.e., for any compact set $K$ in $D$ there exists a positive constant $\lambda$ such that

$$\lambda|\xi|^2 \le \langle A\xi, \xi\rangle \le \lambda^{-1}|\xi|^2, \quad x \in K, \ (x, \xi) \in TM.$$

We assume that the quadratic form $Q$ on $C_0^\infty(D)$ defined by

$$Q[u] = \int_D (\langle A\nabla u, \nabla u\rangle + V|u|^2) m d\nu$$

is bounded from below, and put

$$\lambda_0 = \inf\{Q[u]; u \in C_0^\infty(D), \quad \int_D |u|^2 m d\nu = 1\}.$$

Denote by $L_D$ the selfadjoint operator in $L^2(D; md\nu)$ associated with the closure of $Q$. We assume that $\lambda_0$ is an eigenvalue of $L_D$. Let $\phi_0$ be the normalized positive eigenfunction for $\lambda_0$. Let $p(x, y, t)$ be the minimal fundamental solution for (1.1), which is equal to the integral kernel of the semigroup $e^{-tL_D}$ on $L^2(D, md\nu)$.

Our main assumptions are [IU] (i.e., intrinsic ultracontractivity) and [SSP] (i.e., semismall perturbation) as follows.

**[IU]** For any $t > 0$, there exists $C_t > 0$ such that

$$p(x, y, t) \le C_t \ \phi_0(x)\phi_0(y), \quad x, y \in D.$$

This condition implies that $L_D$ admits a complete orthonormal base of eigenfunctions $\{\phi_j\}_{j=0}^\infty$ with eigenvalues $\lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ repeated according to multiplicity. Furthermore,

$$p(x, y, t) = \sum_{j=0}^\infty e^{-\lambda_j t}\phi_j(x)\phi_j(y) \tag{2.2}$$

(cf. [3], [12] and references therein). Recall that if [IU] holds, then [UP] does not hold for (1.1) and the equation admits a positive solution with zero initial value (cf. [9]); and for a class of parabolic equations, [IU] is equivalent to the existence of such a solution (cf. [10]).

**[SSP]** For some $a < \lambda_0$, 1 is a semismall perturbation of $L - a$ on $D$, i.e., for any $\varepsilon > 0$ there exists a compact subset $K$ of $D$ such that for any $y \in D \setminus K$

$$\int_{D\setminus K} G(x^0, z)G(z, y)m(z)d\nu(z) \le \varepsilon G(x^0, y),$$

where $G$ is the Green function of $L - a$ on $D$, and $x^0$ is a reference point fixed in $D$.

This condition implies that for any $j = 1, 2, \cdots$ the function $\phi_j/\phi_0$ has a continuous extension $[\phi_j/\phi_0]$ up to the Martin boundary $\partial_M D$ of $D$ for $L - a$. (For semismall perturbations, see [11], [16], [12].) The union $D \cup \partial_M D$ is a compact metric space called the Martin compactification of $D$ for $L-a$. We denote by $\partial_m D$ the minimal Martin boundary of $D$ for $L-a$. This is a Borel subset of $\partial_M D$. Here, we note that $\partial_M D$ and $\partial_m D$ are independant of $a$ in the following sense: if [SSP] holds, then for any $b < \lambda_0$ there is a homeomorphism $\Phi$ from the Martin compactification of $D$ for $L-a$ onto that for $L-b$ such that $\Phi|_D = identity$ and $\Phi$ maps the Martin boundary and minimal Martin boundary of $D$ for $L-a$ onto those for $L-b$, respectively (cf. Theorem 1.4 of [11]).

Now, we are ready to state our main theorem.

**Theorem 2.1.** *Assume* [IU] *and* [SSP]. *Then, for any nonnegative solution $u$ of* (1.1) *there exists a unique pair of Borel measures $\mu$ on $D$ and $\lambda$ on $\partial_M D \times [0, T)$ such that $\lambda$ is supported by the set $\partial_m D \times [0, T)$,*

$$(2.3) \qquad \begin{aligned} u(x,t) &= \int_D p(x,y,t)\,d\mu(y) \\ &+ \int_{\partial_M D \times [0,t)} q(x,\xi,t-s)\,d\lambda(\xi,s), \end{aligned}$$

*for any $x \in D$, $0 < t < T$. Here $q(x,\xi,\tau)$ is a continuous function on $D \times \partial_M D \times (-\infty, \infty)$ defined by*

$$(2.4) \qquad \begin{aligned} q(x,\xi,\tau) &= \sum_{j=0}^{\infty} e^{-\lambda_j \tau}\phi_j(x)[\phi_j/\phi_0](\xi), \quad \tau > 0, \\ q(x,\xi,\tau) &= 0, \qquad \tau \le 0, \end{aligned}$$

*where the series in* (2.4) *converges uniformly on $K \times \partial_M D \times (\delta, \infty)$ for any compact subset $K$ of $D$ and $\delta > 0$. Furthermore,*

$$(2.5) \qquad q > 0 \quad on \quad D \times \partial_M D \times (0, \infty),$$

$$(2.6) \qquad (\partial_t + L)q(\cdot, \xi, \cdot) = 0 \quad on \quad D \times (-\infty, \infty).$$

*Conversely, for any Borel measures $\mu$ on $D$ and $\lambda$ on $\partial_M D \times [0, T)$ such that $\lambda$ is supported by $\partial_m D \times [0, T)$ and*

$$(2.7) \qquad \int_D p(x^0, y, t)\,d\mu(y) < \infty, \quad 0 < t < T,$$

$$(2.8) \qquad \int_{\partial_M D\times[0,t)} q(x^0,\xi,t-s)d\lambda(\xi,s) < \infty, \quad 0<t<T,$$

*where* $x^0$ *is a point fixed in* $D$, *the right hand side of* (2.3) *is a nonnegative solution of* (1.1).

The proof of this theorem is based upon the abstract parabolic Martin representation theorem and Choquet's theorem (cf. [7], [6], [15]), and its key step is to identify the parabolic Martin boundary.

## §3. Examples

In this section we give concrete examples as applications of Theorem 2.1.

**Example 3.1.** *Let* $\alpha \in \mathbf{R}$ *and*

$$L = -\Delta + (1+|x|^2)^{\alpha/2} \quad on \quad D = \mathbf{R}^n.$$

*Then* [UP] *holds for* (1.1) *if and only if* $\alpha \le 2$; *while* [IU] *(or* [SSP] *with* $a=-1$*) is satisfied if and only if* $\alpha > 2$ *(cf.* [10], [12]*).*

(i) *Suppose that* $\alpha \le 2$. *Then for any nonnegative solution* $u$ *of* (1.1) *there exists a unique Borel measure* $\mu$ *on* $D$ *such that*

$$(3.1) \qquad u(x,t) = \int_D p(x,y,t)d\mu(y), \quad x\in D,\ 0<t<T.$$

*Conversely, for any Borel measure* $\mu$ *on* $D$ *satisfying* (2.7), *the right hand side of* (3.1) *is a nonnegative solution of* (1.1).

(ii) *Suppose that* $\alpha > 2$. *Then the conclusions of Theorem* 2.1 *hold with*

$$(3.2) \qquad \partial_M D = \partial_m D = \infty S^{n-1},$$

*where* $\infty S^{n-1}$ *is the sphere at infinity of* $\mathbf{R}^n$, *and the Martin compactification* $D^*$ *of* $D=\mathbf{R}^n$ *with respect to* $L$ *is obtained by attaching a sphere* $S^{n-1}$ *at infinity:* $D^* = \mathbf{R}^n \sqcup \infty S^{n-1}$.

*Note that the Martin boundary* $\partial_M D$ *in the case* $-2<\alpha\le 2$ *is also equal to that for* $\alpha>2$. *Nevertheless, when* [UP] *holds, the elliptic Martin boundary disappears in the parabolic representation theorem; while it enters when* [UP] *does not hold.*

**Example 3.2.** *Let* $L=-\Delta$ *on a bounded John domain* $D\subset\mathbf{R}^n$, *i.e.* $D$ *is a bounded domain, and there exist a point* $z^0\in D$ *and a positive* *constant $c_J$ such that each $z \in D$ can be joined to $z^0$ by a rectifiable curve $\gamma(t), 0 \le t \le 1$, with $\gamma(0) = z,\ \gamma(1) = z^0,\ \gamma \subset D$, and*

$$dist(\gamma(t), \partial D) \ge c_J \ell(\gamma[0,t]), \quad 0 \le t \le 1,$$

*where $\ell(\gamma[0,t])$ is the length of the curve $\gamma(s), 0 \le s \le t$. Then the conditions* [IU] *and* [SSP] *with $a = 0$ are satisfied (cf. Example* 10.4 *of* [12]*). Thus the conclusions of Theorem* 2.1 *hold.*

*Note that the Martin boundary $\partial_M D$ of $D$ with respect to $L = -\Delta$ may be different from the topological boundary $\partial D$ in $\mathbf{R}^n$, although they are equal if $\partial D$ is not bad (for example, when $D$ is a Lipschitz domain).*

Note added in proof. It has turned out that the condition [IU] implies the condition [SSP] (see Theorem 1.1 of the paper: M. Murata and M. Tomisaki, Integral representations of nonnegative solutions for parabolic equations and elliptic Martin boundaries, Preprint, April 2006). Thus the assumption [SSP] of Theorem 2.1 in this paper is redundant.

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# Types of pasting arcs in two sheeted spheres

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**Abstract.**

Fix two disjoint nondegenerate continua $A$ and $B$ in the complex plane $\mathbb{C}$ with connected complements and choose a simple arc $\gamma$ in the complex sphere $\widehat{\mathbb{C}}$ disjoint from $A \cup B$, which we call a pasting arc for $A$ and $B$. Then form a covering Riemann surface $\widehat{\mathbb{C}}_\gamma$ over $\widehat{\mathbb{C}}$ by pasting two copies of $\widehat{\mathbb{C}} \setminus \gamma$ crosswise along the arc $\gamma$. Viewing $A$ and $B$ as embedded in the different two sheets $\widehat{\mathbb{C}} \setminus \gamma$ of $\widehat{\mathbb{C}}_\gamma$, consider the variational 2-capacity $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ of the set $A$ in $\widehat{\mathbb{C}}_\gamma$ with respect to the open subset $\widehat{\mathbb{C}}_\gamma \setminus B$ containing $A$. We are interested in the comparison of $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ with $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$. We say that the pasting arc $\gamma$ for $A$ and $B$ is subcritical, critical, or supercritical according as $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ is less than, equal to, or greater than $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$, respectively. The purpose of this paper is to show the existence of subcritical arc $\gamma$ for any arbitrarily given general pair of admissible $A$ and $B$ and then the existences of critical and also supercritical arcs $\gamma$ under the additional condition imposed upon $A$ and $B$ that each of $A$ and $B$ is symmetric about a common straight line in $\widehat{\mathbb{C}}$, which is the case e.g. if $A$ and $B$ are disjoint closed discs.

## §1. Introduction

Consider two disjoint compact subsets $A$ and $B$ in the complex plane $\mathbb{C}$ such that both of $A$ and $B$ are closures of analytic Jordan regions $A^i$ and $B^i$ in $\mathbb{C}$. Such compact subsets in $\mathbb{C}$ as $A$ and $B$ above will be referred to as being *admissible* in this paper. In actual fact, most of the results and especially those labeled as theorems are also valid for more general subsets $A$ and $B$ in $\mathbb{C}$ that are disjoint nonpolar (not necessarily

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connected) compact subsets with connected complements in $\mathbb{C}$. These results in the general setting as above can easily be deduced from the corresponding results in the present special setting proved in the sequel in this paper by the usual standard exhaustion method. However, just only for the sake of simplicity we assume that $A$ and $B$ are disjoint admissible compact subsets in the present paper. Take a simple arc $\gamma$ in $\widehat{\mathbb{C}} \setminus (A \cup B)$. We denote by $W_\gamma$ the Riemann surface obtained from $\widehat{\mathbb{C}} \setminus (A \cup \gamma)$ and $\widehat{\mathbb{C}} \setminus (B \cup \gamma)$ by pasting them crosswise along the arc $\gamma$, which is also denoted by the following impressive notation (cf. [4]):

$$W_\gamma = (\widehat{\mathbb{C}} \setminus (A \cup \gamma)) \ltimes\!\!\bigcup\nolimits_\gamma (\widehat{\mathbb{C}} \setminus (B \cup \gamma)). \tag{1}$$

The arc $\gamma$ in the above surface $W_\gamma$ is referred to as the *pasting arc* for the set $A \cup B$. The surface $W_\gamma$ may be viewed as a subsurface of the covering Riemann surface, the two sheeted sphere,

$$\widehat{\mathbb{C}}_\gamma := (\widehat{\mathbb{C}} \setminus \gamma) \ltimes\!\!\bigcup\nolimits_\gamma (\widehat{\mathbb{C}} \setminus \gamma)$$

so that

$$W_\gamma = \widehat{\mathbb{C}}_\gamma \setminus (A \cup B), \tag{2}$$

where we understand that $A$ ($B$, resp.) is situated e.g. in the upper (lower, resp.) sheet of $\widehat{\mathbb{C}}_\gamma$ although $A$ and $B$ are originally contained in the same $\mathbb{C}$. Here $\widehat{\mathbb{C}}_\gamma$ in general can be considered for any pasting arc $\gamma$ for the empty set $\emptyset$, i.e. for any simple arc in $\widehat{\mathbb{C}}$ (i.e. $\widehat{\mathbb{C}} \setminus \emptyset$). Observe that $W_\gamma$ and $\widehat{\mathbb{C}}_\gamma$ as Riemann surfaces are unchanged if the pasting arc $\gamma$ is replaced by any pasting arc homotopic to $\gamma$ in $\widehat{\mathbb{C}} \setminus (A \cup B)$ or in $\widehat{\mathbb{C}}$ : $W_\gamma = W_{\gamma'}$ and $\widehat{\mathbb{C}}_\gamma = \widehat{\mathbb{C}}_{\gamma'}$ if $\gamma$ and $\gamma'$ are homotopic in $\widehat{\mathbb{C}} \setminus (A \cup B)$ and $\widehat{\mathbb{C}}$, respectively. Here in particular $\widehat{\mathbb{C}}_\gamma$ depends only upon the initial and the terminal points of $\gamma$ and does not depends on the arc connecting these two points. In this sense we sometimes write $W_\gamma = W_{[\gamma]}$ and $\widehat{\mathbb{C}}_\gamma = \widehat{\mathbb{C}}_{[\gamma]}$, where $[\gamma]$ is the homotopy class of pasting arcs containing $\gamma$ considered in $\widehat{\mathbb{C}} \setminus (A \cup B)$ or in $\widehat{\mathbb{C}}$.

Consider next the *capacity* $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$, or more precisely the variational 2-capacity (cf. e.g. [2]), of the compact subset $A$ in $\widehat{\mathbb{C}}_\gamma$ with respect to the open subset $\widehat{\mathbb{C}}_\gamma \setminus B$ of $\widehat{\mathbb{C}}_\gamma$ containing $A$ given by

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = \inf_\varphi D_{W_\gamma}(\varphi), \tag{3}$$

where $\varphi$ in taking the infimum in (3) runs over the family of $\varphi \in C(\widehat{\mathbb{C}}_\gamma) \cap C^\infty(W_\gamma)$ with $\varphi|A = 1$ and $\varphi|B = 0$ and $D_{W_\gamma}(\varphi)$ indicates the Dirichlet integral of $\varphi$ over $W_\gamma$ defined by

$$D_{W_\gamma}(\varphi) = \int_{W_\gamma} d\varphi \wedge *d\varphi = \int_{W_\gamma} |\nabla\varphi(z)|^2 dxdy.$$

Here the second term in the above is the coordinate free expression of $D_{W_\gamma}(\varphi)$ and the third term is the expression of $D_{W_\gamma}(\varphi)$ in terms of local parameters $z = x + iy$ for $W_\gamma$ and $\nabla\varphi(z)$ is the gradient vector $(\partial\varphi(z)/\partial x, \partial\varphi(z)/\partial y)$.

The variation (3) has the unique extremal function $u_\gamma$:

$$\text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = D_{W_\gamma}(u_\gamma), \tag{4}$$

characterized by the conditions $u_\gamma \in C(\widehat{\mathbb{C}}_\gamma) \cap H(W_\gamma)$ with $u_\gamma|A = 1$ and $u_\gamma|B = 0$ (cf. e.g. [2]), where $H(X)$ denotes the class of harmonic functions defined on a Riemann surface $X$, so that the function $u_\gamma|W_\gamma$ is usually referred to as the *harmonic measure* of $\partial A$ on $W_\gamma$. In addition to two characterizations (3) and (4) of the capacity we add one more interpretation. The surface $W_\gamma$ is a doubly connected planar surface and hence $W_\gamma$ is an annulus or conformally a ring region $\{z \in \mathbb{C} : 1 \leq |z| \leq e^M\}$ $(M > 0)$ and the conformal invariant $M$ is called the *modulus* of $W_\gamma$ and denoted by mod $W_\gamma$ (cf. e.g. [6]). It is straightforward to deduce

$$\text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = 2\pi/\text{mod } W_\gamma. \tag{5}$$

The above remark concerning the dependence of the structure of $W_\gamma$ and $\widehat{\mathbb{C}}_\gamma$ on the pasting arc $\gamma$ also applies mutatis mutandis to the quantity $\text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ so that $\text{cap}(A, \widehat{\mathbb{C}}_{[\gamma]} \setminus B)$ is also meaningful.

We also consider the capacity $\text{cap}(A, \widehat{\mathbb{C}} \setminus B)$ of the set $A$ in $\widehat{\mathbb{C}}$ contained in the open subset $\widehat{\mathbb{C}} \setminus B$. It is an important task to compare $\text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ with $\text{cap}(A, \widehat{\mathbb{C}} \setminus B)$ and especially to clarify when the situation $\text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) \leq \text{cap}(A, \widehat{\mathbb{C}} \setminus B)$ occurs from the viewpoints of various applications of capacities such as those to the classical and modern type problems (cf. e.g. [5], [8], [6], [4], [3], etc.). Actually there seems to have been an expectation among people who have been concerned with this question that this is almost always the case. The general *purpose* of this paper is to investigate the above problem and to claim that the reality is not that simple and tame as to support the above expectation. Since the occurence of the situation

$$\text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = \text{cap}(A, \widehat{\mathbb{C}} \setminus B) \tag{6}$$

is very delicate in the sense that the relation is easily destroyed even if we change $\gamma$ slightly but not preserving homotopy, we say that $\gamma$ ($[\gamma]$, resp.) is a *critical* arc (homotopy class of arcs, resp.). The most desirable situation from the view point of our applications is the case in which the inequality

$$\text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) < \text{cap}(A, \widehat{\mathbb{C}} \setminus B) \tag{7}$$

holds and the pasting arc $\gamma$ (the class $[\gamma]$, resp.) which makes (7) valid will be referred to as a *subcritical* arc (homotopy class, resp.). In the rest of (6) and (7) we have

$$\text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) > \text{cap}(A, \widehat{\mathbb{C}} \setminus B) \tag{8}$$

and in this case $\gamma$ (the class $[\gamma]$) is called as a *supercritical* arc (homotopy class, resp.).

The purpose of this paper is to show that either one of the above three cases really occurs by suitably choosing a pasting arc $\gamma$ of the surface $(\widehat{\mathbb{C}} \setminus (A \cup \gamma)) \bigcup\!\!\!\times_\gamma (\widehat{\mathbb{C}} \setminus (B \cup \gamma))$ for an arbitrarily given pair of admissible compact sets $A$ and $B$ under an additional (technical) requirment that each of $A$ and $B$ is symmetric about a common straight line $l$. The simplest example of this case is the one when $A$ and $B$ are arbitrarily given disjoint closed discs. We believe that the above additional condition is only technical and not essential but at present it is merely a conjecture. More presice content of this paper is as follows. First, in entirely general case without any restriction we give two sufficient conditions for a given pasting arc to be subcritical. As a consequence of these results we see that there always exist a plenty of subcritical arcs for any general couple $(A, B)$. Then we give an example of supercritical arc for any couple $(A, B)$ satisfying the above symmetry requirment. In general, the existence of supercritical arcs always implies the existence of critical arcs as a consequence of the continuity of $\text{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$ as the function of two end points of $\gamma$, and therefore we will be able to conclude the existence of a critical arc for any couple $(A, B)$ satisfying the symmetry postulation.

## §2. Subcritical arcs

Take an arbitrarily chosen pair $(A, B)$ of two disjoint admissible compact subsets A and B in $\mathbb{C}$ as described in the introduction. In contrast with the notation $W_\gamma = \widehat{\mathbb{C}}_\gamma \setminus (A \cup B)$ we set

$$W := \widehat{\mathbb{C}} \setminus (A \cup B).$$

Recall that a pasting arc $\gamma$ for $W_\gamma$ is a simple arc in $W$ and $u_\gamma$ is the harmonic measure of $\partial A$ with respect to the region $W_\gamma$. Similarly we denote by $u$ the harmonic measure of $\partial A$ with respect to $W$. In this section we will give two sufficient conditions for a given pasting arc $\gamma$ to be subcritical.

**Theorem 1.** *For any point $a$ in $W$ and any real number $\rho > 0$ less than the distance between $a$ and $\partial W$ there exists an open disc $\Delta(a, r)$ of radius $0 < r < \rho$ centered at $a$ such that every pasting arc $\gamma$ lying in $\Delta(a, r)$ is subcritical.*

*Proof.* Since $u_\gamma = 1$ on $\partial A$ and $u_\gamma = 0$ on $\partial B$ and moreover $\partial A$ and $\partial B$ are analytic, we can find annular neighborhoods $\alpha$ and $\beta$ of $\partial A$ and $\partial B$ such that every $u_\gamma$ has a harmonic extention on $W'_\gamma := W_\gamma \cup \alpha \cup \beta$, as far as the pasting arc $\gamma$ stays in an arbitrarily chosen but then fixed open disc $\Delta(a, r_0)$ $(0 < r_0 < \rho)$. Suppose erroneously that there is a pasting arc $\gamma(r)$ in $\Delta(a, r)$ for each $0 < r < r_0$ such that

$$\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B) \leq \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma(r)} \setminus B) = D_{W_{\gamma(r)}}(u_{\gamma(r)}). \tag{9}$$

In view of $-1 \leq u_\gamma \leq 2$ on $W'_\gamma$, the family $\{u_\gamma : \gamma \subset \Delta(a, r), r_0 > r \downarrow 0\}$ forms a normal family (cf. e.g. [7], [1]) on each compact subset of

$$\left(\left((\widehat{\mathbb{C}} \setminus A) \cup \alpha\right) \setminus \{a\}\right) \bigcup \left(\left((\widehat{\mathbb{C}} \setminus B) \cup \beta\right) \setminus \{a\}\right).$$

Hence we can find a decreasing sequence $(r_n)_{n\geq 1}$ converging to zero with $r_1 < r_0$ such that $\left(u_{\gamma(r_n)}\right)_{n\geq 1}$ converges to a $v \in H(((\widehat{\mathbb{C}} \setminus A) \cup \alpha) \setminus \{a\})$ locally uniformly on $((\widehat{\mathbb{C}} \setminus A) \cup \alpha) \setminus \{a\}$ on which $-1 \leq v \leq 2$ along with each $u_{\gamma(r_n)}$. Hence, by the Riemann removability theorem $v$ has a harmonic extension on $(\widehat{\mathbb{C}} \setminus A) \cup \alpha$ and $v|\partial A = 1$ implies that $v = 1$ identically on $(\widehat{\mathbb{C}} \setminus A) \cup \alpha$ and in particular on $\alpha$. By the Green formula we see that

$$D_{W_{\gamma(r_n)}}(u_{\gamma(r_n)}) = \int_{\partial A} *du_{\gamma(r_n)} \to \int_{\partial A} *dv = 0.$$

By (9) we must conclude that $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B) = 0$, which is absurd. $\square$

We turn to the other condition for an arc $\gamma$ to be subcritical. We say that a pasting arc $\gamma \subset W$ *ranges homotopically between $\lambda$ and $\mu$* for an arbitrary given pair of real numbers $\lambda$ and $\mu$ with $0 < \lambda \leq \mu < 1$ if there is a pasting arc $\gamma'$ in the homotopy class $[\gamma]$ containing $\gamma$ in $W$ such that

$$\gamma' \subset \{z \in W : \lambda \leq u(z) \leq \mu\}. \tag{10}$$

It may be covenient to write the above fact by $[\gamma] \subset \{z \in W : \lambda \leq u(z) \leq \mu\}$. We say that a pasting arc $\gamma$ *stays homotopically at* $\lambda \in (0,1)$ if $\gamma$ ranges homotopically between $\lambda$ and itself so that there is a pasting arc $\gamma' \in [\gamma]$ such that $\gamma'$ is contained in the level line $\{z \in W : u(z) = \lambda\}$ of $u$.

**Theorem 2.** *If a pasting arc $\gamma$ ranges homotopically between $\lambda$ and $\mu$ $(0 < \lambda \leq \mu < 1)$, then*

$$(1 - (\mu - \lambda))^2 \mathrm{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) < \mathrm{cap}(A, \widehat{\mathbb{C}} \setminus B). \tag{11}$$

*In particular, if $\gamma$ stays homotopically at some $\lambda$ in $(0,1)$, then $\gamma$ is subcritical.*

*Proof.* We may assume that $\gamma \subset \{z \in W : \lambda \leq u(z) \leq \mu\}$. We denote by $X$ the connected part of $W_\gamma$ lying over $\{z \in W : \lambda \leq u(z) \leq \mu\}$. There are two connected parts $Y_1'$ and $Y_2'$ in $\widehat{\mathbb{C}}_\gamma$ lying over $\{z \in W : u(z) \geq \mu\} \cup A$ such that $A \subset Y_1'$ and $A \cap Y_2' = \emptyset$. Then we set $Y_1 := Y_1' \setminus A$ and $Y_2 := Y_2'$. Similarly there are two connected parts $Z_1'$ and $Z_2'$ in $\widehat{\mathbb{C}}_\gamma$ lying over $\{z \in W : u(z) \leq \lambda\} \cup B$ such that $B \subset Z_1'$ and $B \cap Z_2' = \emptyset$. Then we set $Z_1 := Z_1' \setminus B$ and $Z_2 := Z_2'$. Then we have the decomposition

$$W_\lambda = X \bigcup (Y_1 \cup Y_2) \bigcup (Z_1 \cup Z_2)$$

of $W_\lambda$ into 5 connected compact sets whose interiors are mutually disjoint. Consider the function $v$ given by

$$v := \begin{cases} u - (\mu - \lambda) & \text{on } \; Y_1; \\ \lambda & \text{on } \; X \cup Y_2 \cup Z_2; \\ u & \text{on } \; Z_1. \end{cases}$$

Observe that $v$ is a continuous and piecewise smooth function on $\overline{W_\gamma}$ such that $v$ is not harmonic on $W_\gamma$ and $v|\partial A = 1 - (\mu - \lambda)$ and $v|\partial B = 0$. Since the function $(1 - (\mu - \lambda))u_\gamma$ is the harmonization of $v$ on $W_\gamma$ preserving the boundary values on $\partial W_\gamma$, the Dirichlet principle assures that

$$D_{W_\gamma}((1 - (\mu - \lambda))u_\gamma) < D_{W_\gamma}(v). \tag{12}$$

We compute $D_{W_\gamma}(v)$ as follows:

$$\begin{aligned} D_{W_\gamma}(v) = D_{Y_1}(v) + D_{X \cup Y_2 \cup Z_2}(v) + D_{Z_1}(v) &= D_{Y_1}(u) + D_{X \cup Y_2 \cup Z_2}(\lambda) + D_{Z_1}(u) \\ &= D_{W \cap \{u > \mu\}}(u) + D_{W \cap \{u < \lambda\}}(u) \leq D_W(u). \end{aligned}$$

This with (12) yields $(1 - (\mu - \lambda))^2 D_{W_\gamma}(u_\gamma) < D_W(u)$, which is nothing but (11).

If $\lambda = \mu$, then (11) is reduced to the inequality defining for the arc $\gamma$ to be subcritical. $\square$

## §3. Continuity of capacity

Although the capacity $\mathrm{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) = D_{W_\gamma}(u_\gamma)$ depends not only upon the end points $z$ and $w$ of $\gamma$ but also upon the homotopy class containing $\gamma$ even if the end points $z$ and $w$ of $\gamma$ are fixed in advance. Therefore the capacity is only a multivalued function of the branch points $\tilde{z}$ and $\tilde{w}$ of $\widehat{\mathbb{C}}_\gamma$ lying over $z$ and $w$, which are the end points of $\gamma$, but it becomes a single valued function as far as we are concerned with their local behaviors. Thus fix two different points $z_1$ and $w_1$ in $W$ and two discs $U$ and $V$ given by $U := \Delta(z_1, r_1)$ ($V := \Delta(w_1, r_1)$, resp.) centered at $z_1$ ($w_1$, resp.) with radius $r_1 > 0$ such that $\overline{U} \cup \overline{V} \subset W$ and $\overline{U} \cap \overline{V} = \emptyset$. For any $z \in U$ and $w \in V$, let $\gamma_1(z, w)$ be a pasting arc joining $z$ and $w$. Then we can always find a pasting arc $\gamma(z, w) \in [\gamma_1(z, w)]$ such that $\gamma(z, w) \cap U$ ($\gamma(z, w) \cap V$, resp.) is a line segment joining $z_1 + r_1 \in \partial U$ ($w_1 + r_1 \in \partial V$, resp.) and $z \in U$ ($w \in V$, resp.) and $\gamma(z, w) \setminus U$ ($\gamma(z, w) \setminus V$, resp.) is the subarc of $\gamma(z, w)$ starting from $w$ ($w_1 + r_1$, resp.) and ending at $z_1 + r_1$ ($z$, resp.). We assume that $[\gamma(z, w) \setminus (U \cup V)]$ is a fixed homotopy class. Then

$$c(z, w) := \mathrm{cap}(A, \widehat{\mathbb{C}}_{\gamma(z,w)} \setminus B) = D_{W_{\gamma(z,w)}}(u_{\gamma(z,w)}) \tag{13}$$

is a single valued function on the polydisc $U \times V$. We maintain:

**Lemma 3.** *The function $c(z, w)$ in* (13) *is continuous on the polydisc $U \times V$ so that capacities are continuous functions of branch points.*

*Proof.* The proof is similar to that of Theorem 1 but we repeat it here since the settings or the arrangements of the stage is superficially quite different from that for Theorem 1.

Choose and then fix an arbitrary point $\sigma_0 := (z_2, w_2) \in U \times V$. We only have to show that $c(\sigma_n) \to c(\sigma_0)$ for any sequence $(\sigma_n)_{n \geq 1}$ in $U \times V$ convergent to $\sigma_0$. For the purpose we choose two annular neighborhoods $\alpha$ and $\beta$ of $\partial A$ and $\partial B$, respectively, such that $(\alpha \cup \beta) \cap (U \cup V) = \emptyset$ and every $u_{\gamma(\sigma)} \in H(W_{\gamma(\sigma)})$ can be continued to $W'_{\gamma(\sigma)} := W_{\gamma(\sigma)} \cup \alpha \cup \beta$ so as to being $u_{\gamma(\sigma)} \in H(W'_{\gamma(\sigma)})$ and $-1 \leq u_{\gamma(\sigma)} \leq 2$ on $W'_{\gamma(\sigma)}$ for every $\sigma = (z, w) \in U \times V$. The possibility of such a choice of $(\alpha, \beta)$ comes from the fact that the reflection principle is applicable as a result of $u_{\gamma(\sigma)}|\partial A = 1$ and $u_{\gamma(\sigma)}|\partial B = 0$ and the analyticity of relative boundaries of $A$ and $B$. Take an arbitrarily chosen and then fixed decreasing sequence $(r_m)_{m \geq 2}$ converging to zero with

$0 < r_m < r_1 - |z_2 - z_1|$ $(m \geq 2)$. Let $K_m := \pi^{-1}(\overline{\Delta(z_2, r_m)} \cup \overline{\Delta(w_2, r_m)})$, where the covering surface $(\widehat{\mathbb{C}}_{\gamma(\sigma_0)}, \widehat{\mathbb{C}}, \pi)$ over $\widehat{\mathbb{C}}$ with its projection $\pi$ is considered here. Then $W'_{\gamma(\sigma)} \setminus K_m = W'_{\gamma(\sigma')} \setminus K_m$ for every $(\sigma, \sigma')$ in $L_m := \pi^{-1}(\Delta(z_2, r_m)) \times \pi^{-1}(\Delta(w_2, r_m))$ for any arbitrarily fixed $m \geq 2$. We denote by $W'_m$ ($W_m$, resp.) the surface $W'_{\gamma(\sigma)} \setminus K_m$ ($W_{\gamma(\sigma)} \setminus K_m$, resp.), which does not depend on the choice of $\sigma \in L_m$. Then $\{u_{\gamma(\sigma_n)} : \sigma_n \in L_m, \sigma_n \to \sigma_0\}$ forms a normal family on $W'_m$. Hence we can find a subsequence $(\sigma_{n'})$ of any given subsequence of $(\sigma_n)$ such that $u_{\gamma(\sigma_{n'})}$ converges to a $v \in H(W'_{\gamma(\sigma_0)} \setminus \{\tilde{z_2}, \tilde{w_2}\})$ locally uniformly on $W'_{\gamma(\sigma_0)} \setminus \{\tilde{z_2}, \tilde{w_2}\}$, where $\tilde{z_2}$ and $\tilde{w_2}$ are the branch points of $W'_{\gamma(\sigma_2)}$ over $z_2$ and $w_2$. Clearly $-1 \leq v \leq 2$, $v|\partial A = 1$, and, $v|\partial B = 0$ on $W'_{\gamma(\sigma_0)} \setminus \{\tilde{z_2}, \tilde{w_2}\}$ along with each $u_{\gamma(\sigma_{n'})}$ on $W_{n'}$. Thus $v \in H(W'_{\gamma(\sigma_0)})$ so that $v = u_{\gamma(\sigma_0)}$ on $W_{\gamma(\sigma_0)}$. Hence the original sequence $u_{\gamma(\sigma_n)}$ converges to $u_{\gamma(\sigma_0)}$ locally uniformly on $W'_{\gamma(\sigma_0)} \setminus \{\tilde{z_2}, \tilde{w_2}\}$ and, in particular, not only $u_{\gamma(\sigma_n)} = 1$ converges to $u_{\gamma(\sigma_0)} = 1$ but also $*du_{\gamma(\sigma_n)}$ converges to $*du_{\gamma(\sigma_0)}$ uniformly on $\partial A$. Therefore, by the Green formula, we see that

$$c(\sigma_n) = D_{W'_{\gamma(\sigma_n)}}(u_{\gamma(\sigma_n)}) = \int_{\partial A} *du_{\gamma(\sigma_n)}$$
$$\to \int_{\partial A} *du_{\gamma(\sigma_0)} = D_{\gamma(\sigma_0)}(u_{\gamma(\sigma_0)}) = c(\sigma_0) \quad (n \to \infty),$$

which is to have been shown. □

Take an arbitrary pasting arc $\gamma$ in $W$ starting from a point $z_0$ and ending at a point $z_1$. We denote by $\gamma_z$ the subarc of $\gamma$ starting from $z_0$ and ending at some point $z \in \gamma$.

**Lemma 4.** *The range set* $\{\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) : z \in \gamma\}$ *contains the closed interval* $[0, \operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)]$ *and*

$$\lim_{z \in \gamma, z \to z_0} \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) = 0. \tag{14}$$

*Proof.* By Lemma 3, the function $z \mapsto \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B)$ is single valued and continuous on the set $\gamma$ and a fortiori the intermediate value theorem assures the validity of the first half of the aove assertion.

To prove (14) we again use the normal family argument. We can view that $\{u_{\gamma_z} : z \in \gamma, z \to z_0\}$ forms a normal family on each compact subset of $\widehat{\mathbb{C}} \setminus A^i \cup \{z_0\}$, where $A^i = A \setminus \partial A$. Then we see that $u_{\gamma_z}$ converges to a $v \in H(\widehat{\mathbb{C}} \setminus A \cup \{z_0\}) \cap C(\widehat{\mathbb{C}} \setminus A^i \cup \{z_0\})$ with $v|\partial A = 1$. Since $0 \leq v \leq 1$ on $\widehat{\mathbb{C}} \setminus A^i \cup \{z_0\}$, the Riemann removability theorem implies that $v \in H(\widehat{\mathbb{C}} \setminus A)$ so that $v|\partial A = 1$ yields that $v \equiv 1$ on $\widehat{\mathbb{C}} \setminus A^i$

and in particular $*dv = 0$ on $\partial A$. Hence $*du_{\gamma_z}$ converges to $*dv = 0$ uniformly on $\partial A$. Thus

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) = \int_{\partial A} *du_{\gamma_z} \to \int_{\partial A} *dv = 0 \quad (z \to z_0),$$

which proves (14). □

As a supplement to Theorems 1 and 2 which assure for an arc to be subcritical, we can state the following direct consequence of Lemma 4:

**Theorem 5.** *Any pasting arc $\gamma$ in $W$ contains a subarc $\gamma_z$ $(z \in \gamma)$ which is subcritical; more precisely, there exists a point $z_1 \in \gamma$ such that $\gamma_z$ is subcritical for every $z \in \gamma_{z_1}$.*

*Proof.* In view of (14) and also the first half of Lemma 4, we can find a $z_1 \in \gamma$ enough close to the initial point $z_0$ of $\gamma$ such that

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) < \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$$

for every $z \in \gamma_{z_1}$. □

**Theorem 6.** *If a pasting arc $\gamma$ starting from $z_0$ and ending at $z_1$ is supercritical, then there is a subarc $\gamma_z$ $(z \in \gamma)$ which is critical.*

*Proof.* Since $0 < \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B) < \operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B)$, Lemma 4 assures that the quantity $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$ is contained in the range set $\{\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) : z \in \gamma\}$ so that there is a point $z \in \gamma$ with $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_z} \setminus B) = \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$, which shows that $\gamma_z$ is critical. □

We define a distance $d(\gamma, \gamma')$ between two pasting arcs $\gamma$ and $\gamma'$ in $W := \widehat{\mathbb{C}} \setminus (A \cup B)$. Let $\sigma$ be an arc in $W = \widehat{\mathbb{C}} \setminus (A \cup B)$ connecting one of end points of $\gamma$ as its initial point with that of $\gamma'$ as its terminal point and $\tau$ be another arc in $W$ connecting the other end point of $\gamma$ as its initial point with that of $\gamma'$ as its terminal point such that $-\sigma + \gamma + \tau - \gamma'$ is a closed curve and $-\sigma + \gamma + \tau$ is a pasting arc homotopic to $\gamma'$. We denote by $\mathcal{F}$ the totality of pairs $(\sigma, \tau)$ of arcs $\sigma$ and $\tau$ in $W$ with the property described above and by $|\gamma''|$ the spherical length of an arc $\gamma''$ in $W$. Then the *distance* $d(\gamma, \gamma')$ of $\gamma$ and $\gamma'$ is given by

$$d(\gamma, \gamma') := \inf_{(\sigma, \tau) \in \mathcal{F}} (|\sigma| + |\tau|). \tag{15}$$

As the last consequence of Lemma 3 in this paper we state the following invariance of sub and supercriticality of pasting arcs $\gamma$ under the small perturbation of $\gamma$ in the sense of (15):

**Theorem 7.** *For any pasting arc $\gamma$ in $\widehat{\mathbb{C}} \setminus (A \cup B)$, there exists a positive number $\varepsilon$ such that any pasting arc $\gamma'$ in $\widehat{\mathbb{C}} \setminus (A \cup B)$ with $d(\gamma, \gamma') < \varepsilon$ is subcritical (supercritical, resp.) if $\gamma$ is subcritical (supercritical, resp.).*

*Proof.* Since there exist arcs $\sigma$ and $\tau$ described above in the definition of the distance $d(\gamma, \gamma')$ such that $-\sigma + \gamma + \tau$ is homotopic to $\gamma'$ and end points of $-\sigma + \gamma + \tau$ converge to the corresponding end points of $\gamma$ as $|\sigma|$ and $|\tau|$ converge to zero as a consequence of the assumption $d(\gamma, \gamma') \to 0$, Lemma 3 assures that

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma'} \setminus B) = \operatorname{cap}(A, \widehat{\mathbb{C}}_{-\sigma+\gamma+\tau} \setminus B) \to \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$$

as $|\sigma| + |\tau| \to 0$ so that

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma'} \setminus B) \to \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B) \quad (\text{as } \ d(\gamma, \gamma') \to 0).$$

Hence $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma'} \setminus B)$ is strictly greater (less, resp.) than $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$ for every $\gamma'$ with sufficiently small $d(\gamma, \gamma')$ if and only if $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$ is strictly greater (less, resp.) than $\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$. Therefore $\gamma'$ is supercritical (subcritical, resp.) for every $\gamma'$ with sufficiently small $d(\gamma, \gamma')$ if and only if $\gamma$ is supercritical (subcritical, resp.). □

## §4. Supercritical arcs

Take a pair $(A, B)$ of two disjoint admissible compact subsets $A$ and $B$ as described in Section 1 and a pasting arc $\gamma$ in $W := \widehat{\mathbb{C}} \setminus (A \cup B)$. Then, as we saw in Section 2 and also in Section 3, the capacity $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$ covers some small interval $(0, \varepsilon)$ $(\varepsilon > 0)$ by choosing $\gamma$ enough short. On the other hand we ask how large the capacity $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$ can be by a variety of choices of $\gamma$. In reality the capacity $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B)$ cannot be too large no matter how we choose $\gamma$. In general we have the following relation:

$$0 < \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma} \setminus B) < 2\operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B) \tag{16}$$

for every pasting arc $\gamma$ in $W$. The proof goes as follows. Let $(\widehat{\mathbb{C}}_{\gamma}, \widehat{\mathbb{C}}, \pi)$ be the natural two sheeted sphere (i.e. the covering surface of $\widehat{\mathbb{C}}$) with altogether two branch points over respective end points of $\gamma$ and with its projection $\pi$. Recall that $u_{\gamma}$ ($u$, resp.) is the harmonic measure of $\partial A$ on $W_{\gamma}$ ($W$, resp.) and observe that $u \circ \pi$ is a nonharmonic competing function in (3) so that the Dirichlet principle shows that

$$D_{W_{\gamma}}(u_{\gamma}) < D_{W\gamma}(u \circ \pi) = 2D_W(u),$$

which proves the above inequality (16).

The purpose of this section is to show the following central and main result of this paper: the existence of a supercritical arc $\gamma$ in $W$ characterized by the inequality $\operatorname{cap}(A, \widehat{\mathbb{C}}_\gamma \setminus B) > \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$. We believe that this result is always true for every admissible pair $(A, B)$ but at present we need to have the following additional condition on $(A, B)$ to prove the above result. We say that $A$ and $B$ are *symmetric* about a common straight line $l$ if there is a straight line $l$ in $\mathbb{C}$ such that the reflection $T$ about $l$ (i.e. the indirect conformal mapping $T$ of $\mathbb{C}$ onto itself with the property that $z$ and $T(z)$ are symmetric about $l$ for every $z \in \mathbb{C}$) maps $A$ onto itself and at the same time $B$ onto itself so that, of course, $T$ maps $\mathbb{C} \setminus (A \cup B)$ onto itself. A typical example is the case where $A$ and $B$ are disjoint closed discs; the line $l$ in this situation is the one passing through centers of $A$ and $B$.

**Theorem 8.** *Suppose that $A$ and $B$ are symmetric about a common straight line $l$. Then there exists a supercritical arc $\gamma$ in $\widehat{\mathbb{C}} \setminus (A \cup B)$ and also a critical arc in $\widehat{\mathbb{C}} \setminus (A \cup B)$ which is a subarc of $\gamma$.*

*Proof.* The last assertion on the existence of critical arc follows at once from the above theorem 6. Thus we only have to concentrate ourselves to the proof of the existence of a supercritical arc $\gamma$.

By translating and rotating $\widehat{\mathbb{C}}$ if necessary we can assume that $l$ is the real line $\{z \in \mathbb{C} : \Im z = 0\}$. Pick an arbitrary point $a \in l \setminus (A \cup B)$ and an analytic Jordan curve $\sigma$ starting and ending at $a$ and surrounding only $B$ and hence separating $B$ from $A$ such that the subarc $\langle aa' \rangle$ ($\langle a'a \rangle$, resp.) of $\sigma$ starting from $a$ ($a'$, resp.) and ending at $a'$ ($a$, resp.) is situated in the upper (lower, resp.) half plane, where $l \cap \sigma = \{a, a'\}$. We also take a line segment $-\tau$ contained in $l$ starting from $a$ and terminating at a point $b$ so that $\tau$ starts from $b$ and ending at $a$ which lies outside $\sigma$ with $\tau \subset l \setminus (A \cup B')$, where $B'$ is the region bounded by $\sigma$ so that $\overline{B} \subset B'$. For $t > 0$ let $c(t)$ be a point on $\langle a'a \rangle \subset \sigma$ and $\sigma'_t$ be the subarc of $\langle a'a \rangle \subset \sigma$ starting from $c(t)$ and ending at $a$ (i.e. $\sigma'_t = \langle c(t)a \rangle$) with $|\sigma'_t| = t$, where $|\sigma'_t|$ is the length of the arc $\sigma'_t$. Finally let

$$\gamma_t := \tau + \sigma_t \quad (t > 0),$$

where $\sigma_t := \overline{\sigma \setminus \sigma'_t}$ (i.e. $\sigma_t = (\sigma \setminus \sigma'_t) \cup \{c(t), a\}$) is the subarc of $\sigma$ starting from $a$ and ending at $c(t)$. Next consider the surface $W_t := W_{\gamma_t}$ $(t > 0)$ given by

$$W_t := (\widehat{\mathbb{C}} \setminus (A \cup \gamma_t)) \mathop{\boxtimes\!\!\!\bigcup}_{\gamma_t} \left(\widehat{\mathbb{C}} \setminus (B \cup \gamma_t)\right).$$

We denote by $\delta_t$ the segment lying over $\sigma'_t$, i.e. the union of two copies of $\sigma'_t$ with two copies of $a$ in each of the above copies being identified.

We also set

$$W_t' := W_t \setminus \delta_t \quad (t > 0),$$

which is a subsurface of any $W_s$ $(0 < s \le t)$. Consider one more surface

$$W_0 := \left(\widehat{\mathbb{C}} \setminus (A \cup B \cup \tau)\right) \bigcup_{\tau}^{\times} \left(\widehat{\mathbb{C}} \setminus \tau\right).$$

Then we see that

$$W_0 \setminus \delta_t = W_t \setminus \delta_t \ (=: W_t')$$

for every $t > 0$.

Simply we write $u_t := u_{\gamma_t}$, the harmonic measure of $\partial A$ on $W_t$. Consider the function $w_t$ on $\overline{W_t \setminus \delta_t}$ harmonic on $W_t \setminus \delta_t$ and continuous on $\overline{W_t \setminus \delta_t}$ with boundary values $w_t|\partial A = w_t|\partial B = 0$ and $w_t|\delta_t = 1$. By the standard normal family argument we see that $w_t \downarrow 0$ locally uniformly on $W_0$. Clearly

$$|u_t - u_s| < w_t$$

on $W_t'$ for every $0 < s < t$. This shows that $(u_t)_{t>0}$ converges to a continuous function $v$ on $\overline{W}_0$ harmonic on $W_0$ with $v|\partial A = 1$ and $v|\partial B = 0$ locally uniformly on $W_0 \setminus \delta_t$ for every $t > 0$, where $\overline{W}_0$ is understood here as the Carathéodory compactification of $W_0$: $\overline{W}_0 = W_0 \cup \partial A \cup \partial B$. Hence we conclude that $v$ is the harmonic measure of $\partial A$ on $W_0$. Since $*du_t$ converges uniformly on $\partial A$ to $*dv$ and $\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_t} \setminus B) = \int_{\partial A} *du_t$, we see that

$$\lim_{t\downarrow 0} \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_t} \setminus B) = \int_{\partial A} *dv. \tag{17}$$

Let $\check{v}$ be the symmetric transformation of $v$ on $\widehat{\mathbb{C}} \setminus (A \cup B \cup \tau)$ and also on $\widehat{\mathbb{C}} \setminus \tau$ about $\tau$, where values on the upper edge $\tau^+$ of $\tau$ are sent to those on the lower edge $\tau^-$ of $\tau$ and vice versa on each sheet so that $\check{v} = v \circ T$. It is not difficult to see that $\check{v}$ is also harmonic on $W_0$ along with $v$ on $W_0$. Hence now we come to the crucial conclusion in our proof: the uniqueness of the harmonic measure of $\partial A$ on $W_0$ assures that $v = \check{v}$ on $W_0$. The additional symmetry assumption is only made use of here to let this conclusion be valid. As a consequence of the above identity $v = \check{v}$ on $W_0$ , we deduce, in particular, that $v|\tau^- = v|\tau^+$, which shows that $v|(W \setminus \tau)$ can be continued to $v \in C(W)$ and $v|(\widehat{\mathbb{C}} \setminus \tau)$ to $v \in C(\widehat{\mathbb{C}})$, where $W := \widehat{\mathbb{C}} \setminus (A \cup B)$. Hence the Dirichlet principle can be applied on $W$ to deduce

$$D_W(v) \ge D_W(u) = \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B),$$

where we recall that $u$ is the harmonic measure of $\partial A$ on $W$. Then

$$\int_{\partial A} *dv = D_{W_0}(v) = D_{\widehat{\mathbb{C}}\setminus(A\cup B\cup\tau)}(v) + D_{\widehat{\mathbb{C}}\setminus\tau}(v) = D_W(v) + D_{\widehat{\mathbb{C}}}(v).$$

Thus we can conclude that

$$\int_{\partial A} *dv \geq \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B) + D_{\widehat{\mathbb{C}}}(v).$$

This with (17) implies that

$$\lim_{t\downarrow 0} \operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_t} \setminus B) > \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B)$$

since $D_{\widehat{\mathbb{C}}}(v) > 0$. This shows that, if $t > 0$ is sufficiently small, then

$$\operatorname{cap}(A, \widehat{\mathbb{C}}_{\gamma_t} \setminus B) > \operatorname{cap}(A, \widehat{\mathbb{C}} \setminus B),$$

i.e. $\gamma_t$ is supercritical. □

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# $L^p$-boundedness of Bergman projections for $\alpha$-parabolic operators

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**Abstract.**

We consider the $\alpha$-parabolic Bergman spaces on strip domains. The Bergman kernel is given by a series of derivatives of the fundamental solution. We prove the $L^p$-boundedness of the projection defined by the Bergman kernel and obtain the duality theorem for $1 < p < \infty$. At the same time, we give a new proof of the Huygens property, which enable us to verify all the results in [3] also for $n = 1$.

## §1. Introduction

For $1 \leq p \leq \infty$, we denote by $\boldsymbol{b}_\alpha^p$ the set of all $L^{(\alpha)}$-harmonic functions which are $p$-th integrable with respect to $(n+1)$-dimensional Lebesgue measure on the upper half space $H$ of the Euclidean space $\mathbf{R}^{n+1}$ and call it the $\alpha$-parabolic Bergman space. In [3], we showed that $\boldsymbol{b}_\alpha^p$ is a Banach space and discussed its dual space and the explicit formula of the Bergman kernel, where the Huygens property plays an important role.

In this note, we consider an $\alpha$-parabolic Bergman space $\boldsymbol{b}_\alpha^p(H_T)$ on the strip domain $H_T = \mathbf{R}^n \times (0, T)$ $(0 < T \leq \infty)$ where $H_\infty = H$. The main purpose of this note is to give an explicit form of the $\alpha$-parabolic Bergman kernel and to show its boundedness on $L^p(H_T)$ by using an interpolation theory. The $\alpha$-parabolic Bergman kernel has a reproducing property for $\boldsymbol{b}_\alpha^p(H_T)$. As an application, we obtain the duality $\boldsymbol{b}_\alpha^p(H_T)' \simeq \boldsymbol{b}_\alpha^q(H_T)$ for $1 < p < \infty$. Here and in the following,

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$q$ always denotes the conjugate exponent of $p$. At the same time we show the Huygens property of $\alpha$-parabolic Bergman functions for $n \geq 1$. This enables us to remove from [3] the restriction $n \geq 2$ on the space dimension.

## §2. Preliminary

We denote the $(n+1)$-dimensional Euclidean space by $\mathbf{R}^{n+1}$ $(n \geq 1)$, and its point by $(x, t)$ $(x \in \mathbf{R}^n,\ t \in \mathbf{R})$. For $0 < \alpha \leq 1$, we consider a parabolic operator $L^{(\alpha)}$ and its adjoint $\tilde{L}^{(\alpha)}$

$$L^{(\alpha)} = \frac{\partial}{\partial t} + (-\Delta)^\alpha, \qquad \tilde{L}^{(\alpha)} = -\frac{\partial}{\partial t} + (-\Delta)^\alpha$$

on $\mathbf{R}^{n+1}$. We remark that if $0 < \alpha < 1$, $(-\Delta)^\alpha$ is the convolution operator in the $x$-space $\mathbf{R}^n$ defined by $-c_{n,\alpha}\mathrm{p.f.}|x|^{-n-2\alpha}$, where $c_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$. Then for $\varphi \in C_c^\infty(\mathbf{R}^{n+1})$,

$$\begin{aligned}
&(\tilde{L}^{(\alpha)}\varphi)(x,t) = -\frac{\partial}{\partial t}\varphi(x,t) + ((-\Delta)^\alpha\varphi)(x,t) \\
&= -\frac{\partial}{\partial t}\varphi(x,t) - c_{n,\alpha}\lim_{\delta \downarrow 0}\int_{|y-x|>\delta}(\varphi(y,t) - \varphi(x,t))|x-y|^{-n-2\alpha}\,dy,
\end{aligned}$$

where we denote by $C_c^\infty(\mathbf{R}^{n+1})$ the totality of infinitely differentiable functions with compact support.

**Lemma 2.1.** *Let* $\varphi \in C_c^\infty(\mathbf{R}^{n+1})$ *with* $\mathrm{supp}(\varphi) \subset \{(x,t)|t_1 < t < t_2, |x| < r\}$. *Then* $\mathrm{supp}(\tilde{L}^{(\alpha)}\varphi) \subset \mathbf{R}^n \times (t_1, t_2)$ *and when* $0 < \alpha < 1$,

$$|(\tilde{L}^{(\alpha)}\varphi)(x,t)| \leq 2^{n+2\alpha}c_{n,\alpha}\Big(\sup_{t_1<s<t_2}\int_{\mathbf{R}^n}|\varphi(y,s)|\,dy\Big)\cdot|x|^{-n-2\alpha}$$

*for* $(x,t)$ *with* $|x| \geq 2r$.

Now we define $L^{(\alpha)}$-harmonic functions.

**Definition 2.1.** Let $D$ be an open set in $\mathbf{R}^{n+1}$. We put

$$s(D) := \{(x,t)|(y,t) \in D \text{ for some } y \in \mathbf{R}^n\}.$$

A Borel measurable function $u$ on $s(D)$ is said to be $L^{(\alpha)}$-harmonic on $D$ if it satisfies the following conditions:

(a) $u$ is continuous on $D$,

(b) $\iint_{s(D)}|u \cdot \tilde{L}^{(\alpha)}\varphi|\,dxdt < \infty$ and $\iint_{s(D)} u \cdot \tilde{L}^{(\alpha)}\varphi\,dxdt = 0$ holds for every $\varphi \in C_c^\infty(D)$.

Note that each component of $s(D)$ is a strip domain.

The fundamental solution $W^{(\alpha)}$ of $L^{(\alpha)}$ has the form :

$$W^{(\alpha)}(x,t)=\begin{cases}(2\pi)^{-n}\int_{\mathbf{R}^n}\exp(-t|\xi|^{2\alpha}+\sqrt{-1}\,x\cdot\xi)d\xi & t>0\\ 0 & t\le 0,\end{cases}$$

where $x\cdot\xi$ is the inner product of $x$ and $\xi$, and $|\xi|=(\xi\cdot\xi)^{1/2}$. Then $\tilde{W}^{(\alpha)}(x,t):=W^{(\alpha)}(x,-t)$ is the fundamental solution of $\tilde{L}^{(\alpha)}$. Note that $W^{(1)}(x,t)$ is equal to the Gauss kernel, and $W^{(1/2)}(x,t)$ is equal to the Poisson kernel.

The following estimates will be needed later.

**Lemma 2.2.** *Let $(\beta,k)$ be a multi-index, $1\le q\le\infty$ and $0<t_1<t_2<\infty$. Then there exists a constant $C$ such that*

$$\partial_x^\beta\partial_t^k W^{(\alpha)}(x,t)=t^{-\frac{n+|\beta|}{2\alpha}-k}\partial_x^\beta\partial_t^k W^{(\alpha)}(t^{-1/2\alpha}x,1), \tag{2.1}$$

$$|\partial_x^\beta\partial_t^k W^{(\alpha)}(x,t)|\le Ct^{1-k}\big(t+|x|^{2\alpha}\big)^{-\frac{n+|\beta|}{2\alpha}-1} \tag{2.2}$$

*and*

$$\|\partial_x^\beta\partial_t^k W^{(\alpha)}\|_{L^q(\mathbf{R}^n\times(t_1,t_2))}\le C(t_2-t_1)^{\frac{1}{q}}t_1^{-\frac{n(1-1/q)+|\beta|}{2\alpha}-k}. \tag{2.3}$$

*Proof.* The assertions (2.1) and (2.2) are remarked in section 3 in [3]. Then we have

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{\mathbf{R}^n}|\partial_x^\beta\partial_t^k W^{(\alpha)}(x,t)|^q dxdt\\ &\quad=\int_{t_1}^{t_2}\int_{\mathbf{R}^n}\Big(t^{-\frac{n+|\beta|}{2\alpha}-k}\Big)^q|\partial_x^\beta\partial_t^k W^{(\alpha)}(t^{-\frac{1}{2\alpha}}x,1)|^q dxdt\\ &\quad=\int_{t_1}^{t_2}\Big(t^{-\frac{n+|\beta|}{2\alpha}-k}\Big)^q\int_{\mathbf{R}^n}|\partial_x^\beta\partial_t^k W^{(\alpha)}(y,1)|^q t^{\frac{n}{2\alpha}}dydt\\ &\quad\le(t_2-t_1)\Big(t_1^{-\frac{n(1-1/q)+|\beta|}{2\alpha}-k}\Big)^q\|\partial_x^\beta\partial_t^k W^{(\alpha)}(\cdot,1)\|^q_{L^q(\mathbf{R}^n)},\end{aligned}$$

which shows (2.3) when $1\le q<\infty$. In the case of $q=\infty$, (2.3) follows from (2.1) immediately, because $\partial_x^\beta\partial_t^k W^{(\alpha)}(y,1)$ is bounded on $\mathbf{R}^n$. □

## §3. Huygens property

In our previous paper [3], we proved the Huygens property under the condition $n\ge 2$. The condition $n\ge 2$ was not able to drop because the proof of the key lemma [3, Lemma 4.3] relied on $\alpha$-harmonic function theory ([1]). In this section, we shall give another proof of the Huygens property, which is valid for all $n \geq 1$. Here we shall use the $\alpha$-parabolic dilation to estimate $L^{(\alpha)}$-harmonic measures. In [2] and [4], the notion of the $L^{(\alpha)}$-harmonic measure is introduced and discussed by using the fundamental solutions $W^{(\alpha)}$ and $\tilde{W}^{(\alpha)}$ of $L^{(\alpha)}$ and $\tilde{L}^{(\alpha)}$, respectively. We handle infinite cylinders and use the following notation.

$C_r$ $:= \{(x,t) | t \in \mathbf{R}, |x| < r\}$ : infinite cylinder.

$\varepsilon$ : the Dirac measure at the origin $(0,0)$.

$\nu_r^\alpha$ : the $L^{(\alpha)}$-harmonic measure at the origin of $C_r$.

$\omega_r^\alpha$ : the projection of $\nu_r^\alpha$ to the $x$-space $\mathbf{R}^n$.

$\tilde{\omega}_r^\alpha$ $:= \int_1^2 \omega_{\lambda r}^\alpha d\lambda$, a modified measure of $\omega_r^\alpha$.

$\tilde{W}_r^{(\alpha)} := \tilde{W}^{(\alpha)} * (\varepsilon - \nu_r^\alpha)$.

We list the properties of $\nu_r^\alpha$ in the following proposition.

**Proposition 3.1.** (1) $0 \leq \tilde{W}_r^{(\alpha)} \leq \tilde{W}^{(\alpha)}$ *and the support of* $\tilde{W}_r^{(\alpha)}$ *is in the closure of the cylinder* $C_r$.

(2) $\nu_r^\alpha$ *is rotationally invariant with respect to the space variable.*

(3) $\int d\nu_r^\alpha \leq 1$.

(4) *If* $0 < \alpha < 1$, $\nu_r^\alpha$ *is supported by* $\{(x,t) | t \leq 0, |x| \geq r\}$ *and absolutely continuous with respect to the* $(n+1)$*-dimensional Lebesgue measure on the exterior of* $C_r$. *The density of* $\nu_r^\alpha$ *is given by*

$$c_{n,\alpha} \int_{|y| \leq r} \tilde{W}_r^{(\alpha)}(y,t) |x-y|^{-n-2\alpha} dy.$$

(5) *If* $\alpha = 1$, $\operatorname{supp}(\nu_r^1) \subset \{(x,t) | t \leq 0, |x| = r\}$.

Next lemma was the key in the proof of the Huygens property ([3, Lemma 4.3]). Now we give a new proof which is valid for all $n \geq 1$.

**Lemma 3.1.** *The modified measure* $\tilde{\omega}_r^\alpha$ *is absolutely continuous with respect to the n-dimensional Lebesgue measure, whose density* $\tilde{w}_r^\alpha$ *satisfies*

$$\tilde{w}_r^\alpha(x) \leq C r^{2\alpha} |x|^{-n-2\alpha} \quad \text{and} \quad \|\tilde{w}_r^\alpha\|_{L^q(\mathbf{R}^n)} \leq C r^{-n(1-1/q)},$$

*where the constant $C$ is independent of* $r > 0$ *and* $1 \leq q \leq \infty$.

*Proof.* By Proposition 3.1, we can express $\omega_r^\alpha$ as

$$\omega_r^\alpha = w_r^\alpha(x) dx + C(r) \sigma_r, \tag{3.1}$$

where $\sigma_r$ is the surface measure of the sphere $\{|x| = r\}$, $C(r)$ is a non-negative function of $r > 0$ and

$$w_r^\alpha(x) = \begin{cases} \int_{-\infty}^0 \Big[c_{n,\alpha} \int_{|y|\le r} \tilde{W}_r^{(\alpha)}(y,t)|x-y|^{-n-2\alpha}dy\Big]dt, & 0 < \alpha < 1, \\ 0, & \alpha = 1. \end{cases}$$

Then $\tilde{\omega}_r^\alpha$ is absolutely continuous and its density is given by

$$\tilde{w}_r^\alpha(x) = \int_1^2 w_{\lambda r}^\alpha(x)d\lambda + \frac{C(|x|)}{r} 1_{\{r\le|x|\le 2r\}}(x),$$

where $1_{\{r\le|x|\le 2r\}}$ denotes the characteristic function. Considering $\alpha$-parabolic dilations $\tau_r^\alpha : (x,t) \mapsto (rx, r^{2\alpha}t)$, we have

$$W^{(\alpha)}(x,t) = r^n W^{(\alpha)}(\tau_r^\alpha(x,t)),$$

which shows that $\nu_r^\alpha$ is the image measure of $\nu_1^\alpha$ by $\tau_r^\alpha$. Thus we obtain $w_r^\alpha(x) = r^{-n} w_1^\alpha(x/r)$, $C(r)\int d\sigma_r = C(1)\int d\sigma_1$ and

$$\tilde{w}_r^\alpha(x) = r^{-n}\tilde{w}_1^\alpha(x/r).$$

In this way, we have only to estimate $\tilde{w}_1^\alpha$. First, we shall show the boundedness. For every $s \ge 1$,

$$\begin{aligned} \int \tilde{w}_1^\alpha(x)d\sigma_s(x) &\le \int\int_1^2 w_\lambda^\alpha(x)d\lambda d\sigma_s(x) + C(s)\int d\sigma_s \\ &= \int\int_1^2 \lambda^{-n} w_1^\alpha(x/\lambda)d\lambda d\sigma_s(x) + C(1)\int d\sigma_1 \\ &\le \frac{2}{s}\int_{s/2}^s \int w_1^\alpha(x)d\sigma_\lambda(x)d\lambda + C(1)\int d\sigma_1 \\ &\le 2\int d\omega_1^\alpha \le 2. \end{aligned}$$

Since $\tilde{w}_1^\alpha$ is rotationally invariant, we have the boundedness of $\tilde{w}_1^\alpha$. Next, we remark that $\tilde{w}_1^\alpha(x) \le C|x|^{-n-2\alpha}$. In fact, from (3) and (4) of Proposition 3.1, follows

$$\begin{aligned} 1 &\ge \int d\nu_1^\alpha \ge \int_{|x|>1}\int_{-\infty}^0 c_{n,\alpha}\int_{|y|\le 1} \tilde{W}_1^{(\alpha)}(y,t)|x-y|^{-n-2\alpha}dydtdx \\ &\ge c_{n,\alpha}\int_{-\infty}^0\int_{|y|\le 1}\tilde{W}_1^{(\alpha)}(y,t)\int_{|x-y|>2}|x-y|^{-n-2\alpha}dxdydt \\ &\ge c_{n,\alpha}\Big(\int_{|x|>2}|x|^{-n-2\alpha}dx\Big)\iint \tilde{W}_1^{(\alpha)}(y,t)dydt, \end{aligned}$$

which shows that $\tilde{W}_1^{(\alpha)}$ is integrable. Then taking $x$ with $|x| \geq 2$, we have $|x| \leq |x-y| + |y| \leq 2|x-y|$ and

$$\begin{aligned} w_1^\alpha(x) &= c_{n,\alpha} \int_{-\infty}^0 \int_{|y|\leq 1} \tilde{W}_1^{(\alpha)}(y,t)|x-y|^{-n-2\alpha} dydt \\ &\leq 2^{n+2\alpha} c_{n,\alpha} \|\tilde{W}_1^{(\alpha)}\|_{L^1(\mathbf{R}^{n+1})} |x|^{-n-2\alpha}. \end{aligned}$$

Thus taking $x$ with $|x| \geq 4$, we have

$$\begin{aligned} \tilde{w}_1^\alpha(x) &= \int_1^2 w_\lambda^\alpha(x) d\lambda = \int_1^2 \lambda^{-n} w_1^\alpha(x/\lambda) d\lambda \\ &\leq 2^{n+2\alpha} c_{n,\alpha} \|\tilde{W}_1^{(\alpha)}\|_{L^1(\mathbf{R}^{n+1})} \Big( \int_1^2 \lambda^{2\alpha} d\lambda \Big) |x|^{-n-2\alpha}. \end{aligned}$$

Since $\tilde{w}_1^\alpha(x)$ is bounded, we obtain

$$\tilde{w}_1^\alpha(x) \leq C|x|^{-n-2\alpha}$$

for all $x \in \mathbf{R}^n$. Therefore

$$\tilde{w}_r^\alpha(x) = r^{-n} \tilde{w}_1^\alpha(x/r) \leq C r^{2\alpha} |x|^{-n-2\alpha},$$

which also shows the norm inequality

$$\|\tilde{w}_r^\alpha\|_{L^q(\mathbf{R}^n)} \leq C r^{-n(1-1/q)},$$

because

$$\int_{|x|\geq r} \big(|x|^{-n-2\alpha}\big)^q dx = \frac{r^{-(q-1)n-2\alpha q}}{(q-1)n + 2\alpha q} \int d\sigma_1.$$

□

Using the above lemma, in the quite same manner as in the proof of Theorem 4.1 in [3], we obtain the following Huygens property. For the completeness, we give an outline of the proof.

**Theorem 3.1.** *If an $L^{(\alpha)}$-harmonic function $u$ on $H_T$ belongs to $L^p(H_T)$, then $u$ satisfies the Huygens property:*

$$u(x,t) = \int_{\mathbf{R}^n} u(y,s) W^{(\alpha)}(x-y, t-s) dy \quad \text{for} \quad 0 < s < t < T. \tag{3.2}$$

*Proof.* Let $u \in L^p(H_T)$ be an arbitrary $L^{(\alpha)}$-harmonic function with $1 \leq p \leq \infty$. Take $\delta > 0$ such that $u(\cdot, \delta) \in L^p(\mathbf{R}^n)$, and put

$$v(x,t) = u(x, t+\delta) - \int_{\mathbf{R}^n} W^{(\alpha)}(x-y, t) u(y, \delta) dy$$

and $V(x,t)=\int_0^t v(x,\tau)d\tau$. Here we remark that $\|v\|_{L^p(H_{T-\delta})}\le\|u\|_{L^p(H_T)}$ and that $V$ is $L^{(\alpha)}$-harmonic(see [3, Lemma 2.3]). For any fixed $(x,t)\in H_{T-\delta}$, taking a cylinder $\{(\xi,\tau)|0<\tau<t,|\xi-x|<r\}$ with $r>0$ and using the mean value property (cf. [4]), we have

$$\begin{aligned}|V(x,t)| &= \Big|\int_{|\xi|\ge r,-t\le\tau\le 0} V(\xi+x,\tau+t)d\nu_r^\alpha(\xi,\tau)\Big| \\ &\le \int_{|\xi|\ge r,-t\le\tau\le 0}\int_0^{\tau+t}|v(\xi+x,s)|ds d\nu_r^\alpha(\xi,\tau) \\ &= \int_0^t\int_{|\xi|\ge r,s-t\le\tau\le 0}|v(\xi+x,s)|d\nu_r^\alpha(\xi,\tau)ds \\ &\le \int_0^{T-\delta}\int|v(\xi+x,s)|d\omega_r^\alpha(\xi)ds.\end{aligned}$$

Thus we obtain

$$\begin{aligned}|V(x,t)| &\le \int_0^{T-\delta}\int|v(\xi+x,s)|\tilde{w}_r^\alpha(\xi)d\xi ds \\ &\le T^{1/q}\|v\|_{L^p(H_{T-\delta})}\|\tilde{w}_r^\alpha\|_{L^q(\mathbf{R}^n)} \\ &\le CT^{1/q}r^{-n/p}\|u\|_{L^p(H_T)},\end{aligned}$$

which shows $V(x,t)=0$ for $1\le p<\infty$, because $r>0$ is arbitrary. In this way, for $\delta<s<t<T$ and $x\in\mathbf{R}^n$, we have

$$\begin{aligned}u(x,t) &= \int_{\mathbf{R}^n} W^{(\alpha)}(x-y,t-\delta)u(y,\delta)dy \\ &= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} W^{(\alpha)}(x-z,t-s)W^{(\alpha)}(z-y,s-\delta)dz\, u(y,\delta)dy \\ &= \int_{\mathbf{R}^n} W^{(\alpha)}(x-z,t-s)u(z,s)dz.\end{aligned}$$

Since $\delta>0$ is arbitrary, we have (3.2) in the case of $1\le p<\infty$. When $p=\infty$, (3.2) follows from [4, Proposition 11] immediately. □

## §4. Some basic properties of $\alpha$-parabolic Bergman functions

In this section, for $0<T<\infty$, we define an $\alpha$-parabolic Bergman space on $H_T$.

**Definition 4.1.** Let $1\le p\le\infty$. We put

$$\boldsymbol{b}_\alpha^p(H_T) := \{u\in L^p(H_T)|L^{(\alpha)}\text{-harmonic on } H_T\},$$

which is a closed subspace of $L^p(H_T)$ (by (4.1) below) and called the $\alpha$-parabolic Bergman space on the strip domain.

*Remark* 4.1. For any $u \in \boldsymbol{b}^p_\alpha(H_T)$, the estimate

$$|u(x,t)| \le C\|u\|_{L^p(H_T)} t^{-(\frac{n}{2\alpha}+1)\frac{1}{p}} \tag{4.1}$$

holds for $(x,t) \in H_T$ in the similar way to [3, Proposition 5.2]. Therefore $u$ can be extended to an $L^{(\alpha)}$-harmonic function on the upper half space $H$ by using the Huygens property. In this paper, every $u \in \boldsymbol{b}^p_\alpha(H_T)$ is considered to be extended to the upper half space as

$$u(x, t + jT) := \int_{\mathbf{R}^n} u(y,t) W^{(\alpha)}(x-y, jT)dy \tag{4.2}$$

for $(x,t) \in H_T$ and $j \in \mathbf{N}$. We remark that the extension $u$ also satisfies the Huygens property on the whole upper half space $H$.

*Remark* 4.2. For each fixed $p$, $\boldsymbol{b}^p_\alpha(H_T)$ are the same and the $L^p(H_T)$-norm is equivalent to one another for all $0 < T < \infty$. In fact, by the Minkowski inequality, for $0 < s < t < \infty$,

$$\|u(\cdot,t)\|_{L^p(\mathbf{R}^n)} \le \|u(\cdot,s)\|_{L^p(\mathbf{R}^n)},$$

which shows the equivalence of the norms.

The Huygens property also yields the following estimate.

**Proposition 4.1.** *Let $1 \le p \le \infty$ and $(\beta,k)$ be a multi-index. Then there exists a constant $C > 0$ such that*

$$|\partial_x^\beta \partial_t^k u(x,t)| \le \begin{cases} C\|u\|_{L^p(H_T)} t^{-\left(\frac{|\beta|}{2\alpha}+k\right)-\left(\frac{n}{2\alpha}+1\right)\frac{1}{p}}, & t < T, \\ CT^{-1/p}\|u\|_{L^p(H_T)} t^{-\left(\frac{|\beta|}{2\alpha}+k\right)-\frac{n}{2\alpha}\frac{1}{p}}, & t \ge T, \end{cases}$$

*for any $u \in \boldsymbol{b}^p_\alpha(H_T)$ and $(x,t) \in H$. In particular, if $1 \le p < \infty$, $t^{\frac{|\beta|}{2\alpha}+k}\partial_x^\beta \partial_t^k u(\cdot,t)$ converges uniformly to $0$ as $t \to \infty$.*

*Proof.* If $0 < t < 2T$, we can show

$$|\partial_x^\beta \partial_t^k u(x,t)| \le C\|u\|_{L^p(H_T)} t^{-\left(\frac{|\beta|}{2\alpha}+k\right)-\left(\frac{n}{2\alpha}+1\right)\frac{1}{p}}$$

in the quite same manner as in [3, Proposition 5.4]. Next we assume $t \ge 2T$. By the Huygens property, we have

$$u(x,t) = \frac{1}{T}\iint_{H_T} u(y,s) W^{(\alpha)}(x-y, t-s)dyds$$

and hence

$$\partial_x^\beta \partial_t^k u(x,t) = \frac{1}{T} \iint_{H_T} u(y,s) \partial_x^\beta \partial_t^k W^{(\alpha)}(x-y, t-s) dyds.$$

Then by (2.3) in Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned} |\partial_x^\beta \partial_t^k u(x,t)| &= \frac{1}{T} \|u\|_{L^p(H_T)} \|\partial_x^\beta \partial_t^k W^{(\alpha)}\|_{L^q(\mathbf{R}^n \times (t-T,t))} \\ &\leq CT^{-1/p} \|u\|_{L^p(H_T)} t^{-\left(\frac{|\beta|}{2\alpha}+k\right)-\frac{n}{2\alpha}\frac{1}{p}}. \end{aligned}$$

$\square$

In the same manner as in [3, Proposition 5.5], we have the following norm inequality.

**Proposition 4.2.** *Let $1 \leq p \leq \infty$ and $(\beta, k)$ be a multi-index. Then there exists a constant $C > 0$ such that for every $u \in \boldsymbol{b}_\alpha^p(H_T)$,*

$$\|t^{\frac{|\beta|}{2\alpha}+k} \partial_x^\beta \partial_t^k u\|_{L^p(H_T)} \leq C \|u\|_{L^p(H_T)}.$$

## §5. Reproducing property of the Bergman kernel

In [3, Theorem 6.3], we have shown that the $\alpha$-parabolic Bergman kernel

$$R_\alpha(x,t;y,s) := -2\partial_t W^{(\alpha)}(x-y, t+s)$$

has a reproducing property for $\boldsymbol{b}_\alpha^p$ with $1 \leq p < \infty$.

In the case of the strip domain $H_T$ $(0 < T < \infty)$, we consider the following kernel: for $(x,t), (y,s) \in H_T$,

$$\begin{aligned} R_{\alpha,T}(x,t;y,s) : &= \sum_{j=0}^{\infty} R_\alpha(x, t+jT; y, s+jT) \\ &= -2 \sum_{j=0}^{\infty} \partial_t W^{(\alpha)}(x-y, s+t+2jT), \end{aligned}$$

which turns out to be the $\alpha$-parabolic Bergman kernel on $H_T$.

**Lemma 5.1.** *Let $(x,t) \in H_T$ be fixed. Then $R_{\alpha,T}(x,t;\cdot,\cdot) \in L^q(H_T)$ for $1 < q \leq \infty$.*

*Proof.* Let $j \geq 1$. Then by (2.3) in Lemma 2.2, we have

$$\begin{aligned} \|R_\alpha(x, t+jT; \cdot, \cdot)\|_{L^q(\mathbf{R}^n \times (jT, jT+T))} &= 2\|\partial_t W^{(\alpha)}\|_{L^q(\mathbf{R}^n \times (t+2jT, t+2jT+T))} \\ &\leq CT^{1/q}(jT)^{-\frac{n(1-1/q)}{2\alpha}-1}. \end{aligned}$$

Thus, by [3, Lemma 6.1],

$$\|R_{\alpha,T}(x,t;\cdot,\cdot)\|_{L^q(H_T)} \\ \leq \|R_\alpha(x,t;\cdot,\cdot)\|_{L^q(H)} + CT^{1/q}\sum_{j=1}^{\infty}(jT)^{-\frac{n(1-1/q)}{2\alpha}-1} < \infty.$$

□

Thus we can define the integral operator

$$R_{\alpha,T}u(x,t) := \iint_{H_T} R_{\alpha,T}(x,t;y,s)u(y,s)dyds$$

for every $u \in L^p(H_T)$ with $1 \leq p < \infty$. Next proposition shows that the kernel $R_{\alpha,T}$ has a reproducing property for $\boldsymbol{b}^p_\alpha(H_T)$.

**Proposition 5.1.** *Let $1 \leq p < \infty$. Then we have*

$$R_{\alpha,T}u(x,t) = u(x,t) \tag{5.1}$$

*for every $u \in \boldsymbol{b}^p_\alpha(H_T)$ and $(x,t) \in H_T$.*

*Proof.* Let $u \in \boldsymbol{b}^p_\alpha(H_T)$ be considered to be extended to $H$ as in (4.2). For $\delta > 0$, we put $u_\delta(x,t) := u(x,t+\delta)$. Then using the Huygens property, we have

$$\begin{aligned}
&\iint_{H_T} u_\delta(y,s)(-2)\partial_t W^{(\alpha)}(x-y,t+s+2jT)dyds \\
&\quad= \int_{\mathbf{R}^n} \Big\{ \big[u_\delta(y,s)(-2)W^{(\alpha)}(x-y,t+s+2jT)\big]_{s=0}^{T} \\
&\qquad\quad - \int_0^T \partial_t u_\delta(y,s)(-2)W^{(\alpha)}(x-y,t+s+2jT)ds\Big\}dy \\
&\quad= 2u_\delta(x,t+2jT) - 2u_\delta(x,t+2(j+1)T) \\
&\qquad\quad + \int_0^T \frac{\partial}{\partial s}\Big\{u_\delta(x,t+2s+2jT)\Big\}ds \\
&\quad= u_\delta(x,t+2jT) - u_\delta(x,t+2(j+1)T).
\end{aligned}$$

Hence, by Proposition 4.1, we obtain

$$\begin{aligned}
&\iint_{H_T} R_{\alpha,T}(x,t;y,s)u_\delta(y,s)dyds \\
&\qquad = \sum_{j=0}^{\infty}\big[u_\delta(x,t+2jT) - u_\delta(x,t+2(j+1)T)\big] = u_\delta(x,t).
\end{aligned}$$

Letting $\delta \to 0$, we have (5.1). □

Since the kernel $R_{\alpha,T}$ is symmetric and real-valued, the integral operator $R_{\alpha,T}$ is the orthogonal projection on $L^2(H_T)$ to $\boldsymbol{b}_\alpha^2(H_T)$. Therefore in particular, the operator $R_{\alpha,T}$ is bounded on $L^2(H_T)$. We call $R_{\alpha,T}$ the Bergman projection. In the next section, we discuss the boundedness for other exponents $1 < p < \infty$.

## §6. $L^p$-boundedness of the Bergman projection

In this last section, we shall prove the boundedness of the integral operator $R_{\alpha,T}$ on $L^p(H_T)$.

**Theorem 6.1.** *Let $1 < p < \infty$. Then $R_{\alpha,T}$ is a bounded operator from $L^p(H_T)$ onto $\boldsymbol{b}_\alpha^p(H_T)$.*

To prove the theorem, we introduce the following theorem from the interpolation theory. We quote the theorem from [5].

**Theorem 6.2.** [5, p.29, Theorem 1]. *Let $K \in L^2(\mathbf{R}^n)$ such that*

*(a)* $\|\hat{K}\|_{L^\infty(\mathbf{R}^n)} \le B$,
*(b)* $K \in C^1(\mathbf{R}^n \setminus \{0\})$ *and* $|\nabla K(x)| \le B|x|^{-n-1}$

*for some $B > 0$, where $\hat{K}$ denotes the Fourier transform of $K$. Then for $1 < p < \infty$, there exists a constant $A_p$, depending only on $p, B$ and $n$, such that*

$$\|K * f\|_{L^p(\mathbf{R}^n)} \le A_p \|f\|_{L^p(\mathbf{R}^n)} \tag{6.1}$$

*for every $f \in L^p(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$.*

*Remark* 6.1. In the above theorem, if in addition $K \in L^q(\mathbf{R}^n)$, the inequality (6.1) holds for every $f \in L^p(\mathbf{R}^n)$.

Now we return to the proof. For $t > 0$, we put

$$K_{T,t}(x) := -2 \sum_{j=1}^{\infty} \partial_t W^{(\alpha)}(x, t + 2jT).$$

**Lemma 6.1.** *The kernel $K_{T,t}$ satisfies the condition in Theorem 6.2 with a constant $B$ independent of $t > 0$.*

*Proof.* By the definition of $W^{(\alpha)}$, the Fourier transform of $W^{(\alpha)}$ satisfies

$$\hat{W}^{(\alpha)}(\xi, t) = (2\pi)^{-n/2} e^{-t|\xi|^{2\alpha}}, \quad \partial_t \hat{W}^{(\alpha)}(\xi, t) = -(2\pi)^{-n/2} |\xi|^{2\alpha} e^{-t|\xi|^{2\alpha}}.$$

Hence

$$\begin{aligned}\hat{K}_{T,t}(\xi) &= 2(2\pi)^{-n/2}|\xi|^{2\alpha}e^{-t|\xi|^{2\alpha}}\sum_{j=1}^{\infty}e^{-2jT|\xi|^{2\alpha}}\\ &= 2(2\pi)^{-n/2}e^{-t|\xi|^{2\alpha}}\frac{|\xi|^{2\alpha}e^{-2T|\xi|^{2\alpha}}}{1-e^{-2T|\xi|^{2\alpha}}}.\end{aligned}$$

This implies that $\hat{K}_{T,t} \in L^2(\mathbf{R}^n)$, i.e., $K_{T,t} \in L^2(\mathbf{R}^n)$, and

$$|\hat{K}_{T,t}(\xi)| \le \frac{(2\pi)^{-n/2}}{T}\sup_{s>0}\frac{se^{-s}}{1-e^{-s}} =: B < \infty.$$

Clearly, $K_{T,t}$ is of class $C^1$ and by (2.2) in Lemma 2.2,

$$\begin{aligned}|\nabla K_{T,t}(x)| &\le C\sum_{j=1}^{\infty}\big((t+2jT)+|x|^{2\alpha}\big)^{-\frac{n+1}{2\alpha}-1}\\ &\le \frac{C}{2T}\int_0^{\infty}\big((t+s)+|x|^{2\alpha}\big)^{-\frac{n+1}{2\alpha}-1}ds\\ &\le \frac{C\alpha}{T(n+1)}\big(t+|x|^{2\alpha}\big)^{-\frac{n+1}{2\alpha}} \le \frac{C\alpha}{T(n+1)}|x|^{-n-1}.\end{aligned}$$

□

*Proof of Theorem* 6.1. We decompose $R_{\alpha,T}$ as

$$R_{\alpha,T}(x,t;y,s) = R_\alpha(x,t;y,s) + K_{T,t+s}(x-y).$$

For $f \in L^p(H_T) \cap L^1(H_T)$, we put $f_s(y) := f(y,s)$ and

$$\tilde{f}(y,s) := \begin{cases} f(y,s), & 0<s<T,\\ 0, & s \ge T.\end{cases}$$

In our previous paper [3], we have shown that the integral operator $R_\alpha$ is bounded on $L^p(H)$. Then

$$\|R_\alpha\tilde{f}\|_{L^p(H_T)} \le \|R_\alpha\|\cdot\|f\|_{L^p(H_T)}.$$

Since

$$R_{\alpha,T}f(x,t) = R_\alpha\tilde{f}(x,t) + \int_0^T K_{T,t+s} * f_s(x)ds,$$

the Minkowski inequality implies

$$\|R_{\alpha,T}f(\cdot,t)\|_{L^p(\mathbf{R}^n)} \le \|R_\alpha\tilde{f}(\cdot,t)\|_{L^p(\mathbf{R}^n)} + \int_0^T \|K_{T,t+s} * f_s\|_{L^p(\mathbf{R}^n)}ds.$$

Here by Theorem 6.2, we have

$$\begin{aligned}\int_0^T \|K_{T,t+s} * f_s\|_{L^p(\mathbf{R}^n)} ds &\le A_p \int_0^T \|f_s\|_{L^p(\mathbf{R}^n)} ds \\ &\le A_p \Big( \int_0^T \|f_s\|^p_{L^p(\mathbf{R}^n)} ds \Big)^{1/p} \Big( \int_0^T ds \Big)^{1/q} \\ &\le A_p \|f\|_{L^p(H_T)} T^{1/q}.\end{aligned}$$

Taking the $L^p(0,T)$-norm, again by the Minkowski inequality, we obtain

$$\begin{aligned}\|R_{\alpha,T} f\|_{L^p(H_T)} &\le \|R_\alpha \tilde{f}\|_{L^p(H_T)} + T A_p \|f\|_{L^p(H_T)} \\ &\le (\|R_\alpha\| + T A_p) \|f\|_{L^p(H_T)}.\end{aligned}$$

This completes the proof. □

As an application, we have the following duality (cf. [3, Theorem 8.1]).

**Corollary 6.1.** *For* $1 < p < \infty$, *the following duality holds;*

$$\boldsymbol{b}^p_\alpha(H_T)' \simeq \boldsymbol{b}^q_\alpha(H_T),$$

*where the pairing is given by*

$$\langle f, g \rangle = \iint_{H_T} f(x,t) g(x,t) dx dt$$

*for* $f \in \boldsymbol{b}^p_\alpha(H_T)$ *and* $g \in \boldsymbol{b}^q_\alpha(H_T)$.

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# Vanishing theorem on the pointwise defect of a rational iteration sequence for moving targets

**Yûsuke Okuyama**

## §1. Introduction

Let $f$ be a rational map, i.e., a holomorphic endomorphism of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, of degree $d > 1$. The $k$ times iteration of $f$ is denoted by $f^k$ for $k \in \mathbb{N}$.

The Nevanlinna theory for sequences was first studied in [19], [2], [8] and [10], and recently, motivated by complex dynamics, studied in [18], [16] and [15], where the sequence of rational maps correspond to a transcendental meromorphic function. Hence the following definition is natural:

**Definition 1.1** (Picard exceptional value)**.** The point $a \in \hat{\mathbb{C}}$ is called a *Picard exceptional value* of $\{f^k\}$ if

$$\# \bigcup_{k \in \mathbb{N}} f^{-k}(a) < \infty.$$

The point $a \in \hat{\mathbb{C}}$ is a Picard exceptional value if and only if it is periodic of period at most two and $a$ and $f(a)$ are critical of order $d-1$. In particular, there exist at most two such values (cf. [9]), which is an analogue of the Picard theorem for transcendental meromorphic functions.

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**Notation 1.1.** The spherical area measure on $\hat{\mathbb{C}}$, which is normalized as $\sigma(\hat{\mathbb{C}}) = 1$, is denoted by $\sigma$, and the chordal distance between $z, w \in \hat{\mathbb{C}}$, which is normalized as $[0, \infty] = 1$, by $[z, w]$. Put $\mathbb{D}(x, r) := \{z \in \hat{\mathbb{C}}; [z, x] < r\}$ for $x \in \hat{\mathbb{C}}$ and $r > 0$.

One of the main aims of the Nevanlinna theory is to generalize the Picard theorem quantitatively by the *defects*, which are defined not only for each constant values but also for moving targets. See [14], Chapter 4 and also the recent significant result by Yamanoi [20].

Clearly, the degree $d = \int_{\hat{\mathbb{C}}} f^*(\mathrm{d}\sigma)$ of $f$ is an analogue of the order (or characteristic) function of a transcendental meromorphic function.

**Definition 1.2** (proximities and defects)**.** For a rational map $g$, the *pointwise proximity function* of $f$ is defined as

$$w(g, f) := \log \frac{1}{[g(\cdot), f(\cdot)]} : \hat{\mathbb{C}} \to [0, +\infty],$$

the *mean proximity* of $f$ as

$$m(g, f) := \int_{\hat{\mathbb{C}}} w(g, f) \mathrm{d}\sigma,$$

and the *Valiron defect* of $\{f^k\}$ as

$$\delta_V(g; \{f^k\}) := \limsup_{k \to \infty} \frac{m(g, f^k)}{d^k}.$$

**Convention 1.1.** Each point $a \in \hat{\mathbb{C}}$ is identified with the constant map $g \equiv a$.

A point $a \in \hat{\mathbb{C}}$ is called a *Valiron exceptional value* of $\{f^k\}$ if $\delta_V(a; \{f^k\}) > 0$. It is easy to see that every Picard exceptional value of $\{f^k\}$ is a Valiron one. It seems surprising that the converse is true:

**Theorem 1.1** (Valiron agrees with Picard, [12] and [13])**.** *Let $f$ be a rational map of degree $> 1$. For a point $a \in \hat{\mathbb{C}}$,*

$$\delta_V(a; \{f^k\}) = 0$$

*if and only if $a$ is not a Picard exceptional value of $\{f^k\}$.*

In [11], the following generalization of Theorem 1.1 below was shown and crucially used to obtain a new Diophantine condition for the nonlinearizability of $f$ at its irrationally indifferent cycle.

**Definition 1.3.** The *Fatou set* $F(f)$ is the set of all the points in $\hat{\mathbb{C}}$ where $\{f^k\}$ is normal, and the *Julia set* $J(f)$ is $\hat{\mathbb{C}} - F(f)$.

**Theorem 1.2** (vanishing theorem on the Valiron defects for moving targets)**.** *Let $f$ be a rational map of degree $> 1$ such that $F(f) \neq \emptyset$. Then for every non-constant rational map $g$,*

$$\delta_V(g; \{f^k\}) = 0. \tag{1}$$

In [11] we asked whether it is possible to remove the assumption $F(f) \neq \emptyset$. In the rest of this notes, we will answer affirmatively the following *pointwise* version of this problem:

**Theorem 1** (vanishing theorem on the pointwise defect)**.** *Let $f$ be a rational map of degree $d > 1$. Then for every rational map $g$,*

$$\lim_{k\to\infty} \frac{w(g, f^k)}{d^k} = 0 \tag{2}$$

*$\mu_f$-almost everywhere on $\hat{\mathbb{C}}$. Here the measure $\mu_f$ appears in Theorem* 2.1 *in* §2.

## §2. The maximal entropy measures of rational maps

In this section, we gather some useful ergodic properties of rational maps which will be used in §3.

Let $f$ be a rational map of degree $d > 1$.

**Theorem 2.1** ([6] and [5])**.** *There exists the unique maximal entropy measure $\mu_f$ for $f$, and $h_{\mu_f}(f) = \log d$, which is the topological entropy of $f$.*

Moreover, the probability measure $\mu_f$ is exponentially mixing. More quantitatively, the following holds:

**Theorem 2.2** (exponential decay of correlation [3]. See also [4])**.** *For every $\epsilon_0 > 0$, there exists $C = C(\epsilon_0) > 0$ such that for every $\psi \in L^\infty(\mu_f)$, every Lipschitz function $\phi$ on $\hat{\mathbb{C}}$, for which $\|\phi\|_{\mathrm{Lip}} := \sup_{z,w\in\hat{\mathbb{C}}, z\neq w} |\phi(z) - \phi(w)|/[z, w]$, and every $k \in \mathbb{N}$,*

$$\left|\int (\psi \circ f^k)\cdot \phi \mathrm{d}\mu_f - \int \psi \mathrm{d}\mu_f \int \phi \mathrm{d}\mu_f\right| \leq C\|\psi\|_\infty \|\phi\|_{\mathrm{Lip}} \left(\frac{1+\epsilon_0}{d}\right)^{\frac{k}{2}}. \tag{3}$$

Let us also recall several properties of $\mu_f$ proved by Mañé:

**Theorem 2.3** (Mañé [7], Theorem A)**.** *Let $\mu$ be an $f$-ergodic probability measure on $\hat{\mathbb{C}}$ with the entropy $h_\mu(f) > 0$, then*

$$\int \log |f'| \mathrm{d}\mu > 0, \tag{4}$$

*and for $\mu$-a.e. $x \in \hat{\mathbb{C}}$,*

$$\lim_{r\to 0} \frac{\log \mu(\mathbb{D}(x,r))}{\log r} = \frac{h_\mu(f)}{\int \log |f'| \mathrm{d}\mu} =: D(\mu). \tag{5}$$

Since $h_{\mu_f}(f) = \log d > 0$, Theorem 2.3 can be applied to $\mu_f$.

*Remark* 2.1. The quantity in (4) is called the *Lyapunov exponent* of $f$, which is independent of an $f$-ergodic probability measure $\mu$ on $\hat{\mathbb{C}}$. The left hand side of (5) is called the *pointwise Hausdorff dimension* of $\mu$ at $x$. By the observation of Young [21], it holds that

$$D(\mu) = \inf\{\mathrm{HD}(X); X \subset \hat{\mathbb{C}}, \mu(X) = 1\},$$

where $\mathrm{HD}(X)$ is the Hausdorff dimension of $X$.

**Theorem 2.4** (cf. Mañé [7], Lemma II.1)**.** *There exist $\rho \in (0,1]$ and $\gamma > 0$ such that for every $r \in (0,\rho)$ and every $x \in \hat{\mathbb{C}}$,*

$$\mu_f(\mathbb{D}(x,r)) \le r^\gamma. \tag{6}$$

## §3. The long fly property of a rational map

Let $f$ be a rational map of degree $d > 1$. The following is a refinement of Saussol's *long fly property* ([17]) of $(\hat{\mathbb{C}}, f, \mu_f)$ and proves Theorem 1:

**Theorem 2.** *For every rational map $g$, the following holds: for $\mu_f$-almost every $z \in \hat{\mathbb{C}}$,*

$$\log \frac{1}{[f^k(z), g(z)]} = O(\log k) \tag{7}$$

*as $k \to \infty$.*

*Proof.* We extend the argument in the proof of [17], Lemma 9.

Let $\epsilon_0 \in (0, d-1)$, $C = C(\epsilon_0), D(\mu_f), \rho, \gamma$ be the constants in Theorems 2.2, 2.3 and 2.4. Fix $\delta \in (0, \gamma/2)$, $\epsilon_1 > 0$ and $\epsilon_2 \in (0, \gamma - 2\delta)$. For each $r_0 \in (0, \rho)$, let $G(r_0)$ be the set of all such $x \in \hat{\mathbb{C}}$ that for every $r \in (0, r_0)$,

$$\frac{\log \mu_f(\mathbb{D}(x,r))}{\log r} \le D(\mu_f) + \epsilon_1, \text{ and} \tag{8}$$

$$\mu_f(\mathbb{D}(x,4r)) \le \mu_f(\mathbb{D}(x,r)) r^{-\epsilon_2}. \tag{9}$$

By Theorem 2.3 and the *weak diametrical regularity* of $\mu_f$ (cf. Barreira and Saussol [1], p452), $G(r_0)$ is increasing as $r_0 \to 0$ and

$$\mu_f(\bigcup_{r_0 \in (0,\rho)} G(r_0)) = 1,$$

by which, it is enough to show that for every sufficiently small $r_0$, (7) holds $\mu_f$-almost everywhere on $G(r_0)$

For each $m \in \mathbb{N}$, put

$$A_\delta(m; g) := \{y \in \hat{\mathbb{C}}; \inf_{k \in [e^{m\delta}, e^{(m+1)\delta}]} [f^k(y), g(y)] < e^{-m}\}.$$

Then for every $x \in \hat{\mathbb{C}}$ and every $m \in \mathbb{N}$,

$$A_\delta(m; g) \cap \mathbb{D}(x, e^{-m}) \subset \bigcup_{k \in [e^{m\delta}, e^{(m+1)\delta}]} \mathbb{D}(x, e^{-m}) \cap f^{-k}(\mathbb{D}(g(x), (K+1)e^{-m})),$$

where $K > 0$ is a constant such that $g$ is $K$-Lipschitz on $\hat{\mathbb{C}}$.

Put $\phi_{x,r}(y) := \eta_r([x, y])$, where $\eta_r : [0, \infty) \to \mathbb{R}$ is an $1/r$-Lipschitz function such that $1_{[0,r]} \le \eta_r \le 1_{[0,2r]}$. Then $\phi_{x,r}$ is $1/r$-Lipschitz on $\hat{\mathbb{C}}$ and $1_{\mathbb{D}(x,r)} \le \phi_{x,r} \le 1_{\mathbb{D}(x,2r)}$.

For every $r_0 \in (0, \rho)$ and every $r \in (0, r_0)$, from (3),

$$\begin{aligned}
&\mu_f\left(\mathbb{D}(x, r) \cap f^{-k}(\mathbb{D}(g(x), (K+1)r))\right)\\
\le &\int (1_{\mathbb{D}(g(x),(K+1)r)} \circ f^k) \cdot \phi_{x,r} \mathrm{d}\mu_f\\
\le &C \cdot 1 \cdot \frac{1}{r} \left(\frac{1+\epsilon_0}{d}\right)^{k/2} + \mu_f(\mathbb{D}(g(x), (K+1)r)) \cdot \mu_f(\mathbb{D}(x, 2r)),
\end{aligned}$$

and by (6) and (9),

$$\begin{aligned}
&\mu_f(\mathbb{D}(g(x), (K+1)r)) \cdot \mu_f(\mathbb{D}(x, 2r))\\
\le &((k+1)r)^\gamma \cdot \mu_f(\mathbb{D}(x, r/2))(r/2)^{-\epsilon_2} \le \mu_f(\mathbb{D}(x, r/2)) \cdot 2^{\epsilon_2}(K+1)^\gamma \cdot r^{\gamma-\epsilon_2}.
\end{aligned}$$

There exists so small $\rho' \in (0, \rho)$ that for every $r_0 \in (0, \rho')$, every $x \in G(r_0)$ and every $m > \log(1/r_0)$,

$$\begin{aligned}
&\mu_f\left(A_\delta(m; g) \cap \mathbb{D}(x, e^{-m})\right)\\
\le &C \cdot e^m \frac{\left(\frac{1+\epsilon_0}{d}\right)^{e^{m\delta}/2}}{1 - \left(\frac{1+\epsilon_0}{d}\right)^{1/2}} + e^{(m+1)\delta} \cdot \mu_f(\mathbb{D}(x, e^{-m}/2)) \cdot 2^{\epsilon_2}(K+1)^\gamma e^{-m(\gamma-\epsilon_2)}\\
\le &(e^{-m}/2)^{D(\mu_f)+\epsilon_1} \cdot e^{-m(\gamma-\epsilon_2-2\delta)} + \mu_f(\mathbb{D}(x, e^{-m}/2)) \cdot e^{-m(\gamma-\epsilon_2-2\delta)}\\
\le &\mu_f(\mathbb{D}(x, e^{-m}/2)) \cdot 2e^{-m(\gamma-\epsilon_2-2\delta)} \text{ (by (8))},
\end{aligned}$$

and hence for every $m > \log(1/r_0)$,

$$\begin{aligned}\mu_f\left(A_\delta(m;g)\cap G(r_0)\right) \le& \sum_{x\in S_m}\mu_f\left(A_\delta(m;g)\cap\mathbb{D}(x,e^{-m})\right)\\ \le& 2e^{-m(\gamma-\epsilon_2-2\delta)}\mu_f(\bigcup_{x\in S_m}\mathbb{D}(x,e^{-m}/2)) \le 2e^{-m(\gamma-\epsilon_2-2\delta)},\end{aligned}$$

where $S_m$ is a finite and *maximal* $e^{-m}$*-separated* set for $G(r_0)$, i.e., $G(r_0)\subset\cup_{x\in S_m}\mathbb{D}(x,e^{-m})$ and $\mathbb{D}(x,e^{-m})\cap S_m=\{x\}$ for each $x\in S_m$, and finally $\sum_{m\in\mathbb{N}}\mu_f\left(A_\delta(m;g)\cap G(r_0)\right)<\infty$.

Hence by the first Borel-Cantelli lemma, $\mu_f(\limsup_{m\to\infty}A_\delta(m;g)\cap G(r_0))=0$, that is, for $\mu_f$-almost every $z\in G(r_0)$, there exists $m(z)\in\mathbb{N}$ such that for every $m>m(z)$,

$$\inf_{k\in[e^{m\delta},e^{(m+1)\delta}]}[f^k(z),g(z)]\ge e^{-m},$$

which proves (7). □

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# Hölder continuity of solutions to quasilinear elliptic equations with measure data

**Takayori Ono**

**Abstract.**

We consider quasi-linear second order elliptic differential equations with measures date on the right hand side. In this talk, we investigate Hölder continuity of solutions of such equations.

## §1. Introduction.

Let $G$ be a bounded open set in $\mathbf{R}^N$ ($N \geq 2$) and $1 < p < N$. Suppose that $\nu$ is a signed Radon measure on $G$. We consider quasi-linear second order elliptic differential equations with measure date of the form

$$(\mathrm{E}_\nu) \qquad -\operatorname{div}\mathcal{A}(x,\, \nabla u(x)) + \mathcal{B}(x,\, u(x)) = \nu,$$

where $\mathcal{A}(x,\xi) : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ satisfies structure conditions of $p$-th order and $\mathcal{B}(x,t) : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is nondecreasing in $t$ (see section 2 below for more details).

Hölder continuity of a solution to the equation $(E_\nu)$ was investigated in [17], [8] and [6]. In these papers, they showed that the solution of $(E_\nu)$ is locally Hölder continuous with some exponent if the signed Radon measure $\nu$ satisfies the condition that there exist constants $M > 0$ and $0 < \beta < \lambda$ with

$$|\nu|(B(x_0,r)) \leq M\ r^{N-p+\beta(p-1)}$$

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whenever $B(x, 3r) \subset G$, where $\lambda$ is a number depending on $N$, $p$ and structure conditions for $\mathcal{A}$ and $\mathcal{B}$. Further, in [7], in the case $\mathcal{B} = 0$ in the equation $(E_\nu)$, namely for the equation

$$-\operatorname{div} \mathcal{A}(x, \nabla u(x)) = \nu \tag{1}$$

and $\nu$ is a nonnegative Radon measure, Kilpeläinen and Zhong showed that a solution to the equation (1) is Hölder continuous with the same exponent $\beta$. In this talk, we extend this result to the case of the equation $(\mathrm{E}_\nu)$.

Throughout this paper, we use some standard notation without explanation.

## §2. Preliminaries.

We assume that $\mathcal{A} : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}^N$ and $\mathcal{B} : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions for $1 < p < N$ :

- (A.1) $x \mapsto \mathcal{A}(x, \xi)$ is measurable on $\mathbf{R}^N$ for every $\xi \in \mathbf{R}^N$ and $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbf{R}^N$ ;
- (A.2) $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha_1 |\xi|^p$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$ with a constant $\alpha_1 > 0$;
- (A.3) $|\mathcal{A}(x, \xi)| \leq \alpha_2 |\xi|^{p-1}$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$ with a constant $\alpha_2 > 0$;
- (A.4) $\big(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)\big) \cdot \big(\xi_1 - \xi_2\big) > 0$ whenever $\xi_1,\ \xi_2 \in \mathbf{R}^N$, $\xi_1 \neq \xi_2$, for a.e. $x \in \mathbf{R}^N$;
- (B.1) $x \mapsto \mathcal{B}(x, t)$ is measurable on $\mathbf{R}^N$ for every $t \in \mathbf{R}$ and $t \mapsto \mathcal{B}(x, t)$ is continuous for a.e. $x \in \mathbf{R}^N$ ;
- (B.2) For any open set $G \Subset \mathbf{R}^N$, there is a constant $\alpha_3(G) \geq 0$ such that $|\mathcal{B}(x, t)| \leq \alpha_3(G)(|t|^{p-1} + 1)$ for all $t \in \mathbf{R}$ and a.e. $x \in G$;
- (B.3) $t \mapsto \mathcal{B}(x, t)$ is nondecreasing on $\mathbf{R}$ for a.e. $x \in \mathbf{R}^N$.

We consider elliptic quasi-linear equations of the form

$$-\operatorname{div} \mathcal{A}(x, \nabla u(x)) + \mathcal{B}(x, u(x)) = 0. \tag{E}$$

For an open subset $G$ of $\mathbf{R}^N$, we consider the Sobolev spaces $W^{1,p}(G)$, $W_0^{1,p}(G)$ and $W_{loc}^{1,p}(G)$.

Let $G$ be an open subset of $\mathbf{R}^N$. A function $u \in W_{loc}^{1,p}(G)$ is said to be a (weak) *solution* of (E) in $G$ if

$$\int_G \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx + \int_G \mathcal{B}(x, u) \varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(G)$.

A continuous solution of (E) in an open subset $G$ of $\mathbf{R}^N$ is called $(\mathcal{A},\mathcal{B})$-*harmonic* in $G$.

We can see the following proposition by the proof of [14; Theorem 4.7]. By carefully analyzing the proof of [14; Theorem 4.2 and Theorem 4.7], we can choose constants $c$ and $0<\lambda\le 1$ independent of the radius $R$ if $R\le 1$.

**Proposition 2.1.** *Let $G$ be a bounded open set. Then there are constants $c$ and $0<\lambda\le 1$ such that for $B(x_0,R)\Subset G$ and for every $(\mathcal{A},\ \mathcal{B})$-harmonic function $h$ in $G$ with $|h|\le L$ in $B(x_0,R)$,*

$$osc(h,B(x_0,r))\le c\left(\frac{r}{R}\right)^{\lambda}\big(osc(h,B(x_0,R))+R\big),$$

*whenever $0<r<R\le 1$. Here $c$ depends only on $N,p,\alpha_1,\alpha_2,\alpha_3(G)$ and $L$ and $\lambda$ depends only on $N,p,\alpha_1,\alpha_2$ and $\alpha_3(G)$.*

In the case of $\mathcal{A}(x,\xi)=|\xi|^{p-2}\xi$ and $\mathcal{B}=0$, namely for the $p$-Laplace equation, we can choose $\lambda=1$ ([4; Lemma 2.1]).

We recall the following propositions ([13; Theorem 2.2 and putting $k=0$ in Definition 2.1, and Lemma 3.1]).

**Proposition 2.2.** *Let $G$ be a bounded open set and $M_0\ge 0$. Then there is a constant $c$ such that, for every $(\mathcal{A},\mathcal{B})$-harmonic function $h$ in $G$, nonnegative $\eta\in C_0^\infty(G)$ and constant $M$ with $|M|\le M_0$,*

$$\begin{aligned}\int_{\{h>M\}}|\nabla h|^p\,\eta^p\,dx \quad &\le \quad c\int_G \max(h-M,0)^p\,(\eta^p+|\nabla\eta|^p)\,dx\\ &\qquad + c\,(M_0+1)^p\int_{\{h>M\}}\eta^p\,dx,\end{aligned}$$

*where $c$ depends only on $p$, $\alpha_1$, $\alpha_2$ and $\alpha_3(G)$.*

**Proposition 2.3.** *Let $G$ be a bounded open set, $M_0\ge 0$, $\gamma\in(0,p]$. Then there is a constant $c$ such that, for every $r\in(0,1]$ with $B(x_0,r)\Subset G$, an $(\mathcal{A},\mathcal{B})$-harmonic function $h$ in $G$ and a constant $M$ with $|M|\le M_0$,*

$$\sup_{B(x_0,r/2)}|h-M|\le c\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}|h-M|^\gamma dx\right)^{1/\gamma}+c\,r,$$

*where $c$ depends only on $p$, $\alpha_1$, $\alpha_2$, $\alpha_3(G)$, $\gamma$ and $M_0$.*

**Lemma 2.1.** *Let $G$ be a bounded open set. Then there is a constant $c$ depending only on $p,N,\alpha_1,\alpha_2$ and $\alpha_3(G)$ such that for $B(x_0,R)\subset G$*

*with $R \le 1$, $u \in W^{1,p}(B(x_0,R))$ and the $(\mathcal{A},\mathcal{B})$-harmonic function $h$ with $h-u \in W_0^{1,p}(B(x_0,R))$*

$$\left(\int_{B(x_0,R)} |\nabla h|^p\,dx\right)^{1/p} \le c\left\{\left(\int_{B(x_0,R)} |u|^p\,dx\right)^{1/p} + \left(\int_{B(x_0,R)} |\nabla u|^p\,dx\right)^{1/p} + R^{N/p}\right\}.$$

*Proof.* Fix $B = B(x_0,R) \subset G$ with $R \le 1$ and let $\|\cdot\|_{p,G}$ denote the usual $L^p(G)$-norm. It follows from (A.2), (A.3), (B.2) and (B.3) that

$$\begin{aligned}
\|\nabla h\|_{p,B}^p &\le \alpha_1^{-1}\int_B \mathcal{A}(x,\nabla h)\cdot\nabla h\,dx \\
&= \alpha_1^{-1}\left\{\int_B \mathcal{A}(x,\nabla h)\cdot\nabla u\,dx - \int_B \mathcal{B}(x,h)(h-u)\,dx\right\} \\
&\le \alpha_1^{-1}\,\alpha_2\,\|\nabla h\|_{p,B}^{p-1}\,\|\nabla u\|_{p,B} - \alpha_1^{-1}\int_B \mathcal{B}(x,u)(h-u)\,dx \\
&\le \alpha_1^{-1}\,\alpha_2\,\|\nabla h\|_{p,B}^{p-1}\,\|\nabla u\|_{p,B} \\
&\qquad + \alpha_1^{-1}\,\alpha_3(G)\,\||u|+1\|_{p,B}^{p-1}\,\|u-h\|_{p,B}.
\end{aligned}$$

Because $h-u \in W_0^{1,p}(B)$, by the Poincaré inequality we have

$$\|h-u\|_{p,B} \le c\,\|\nabla h - \nabla u\|_{p,B} \le c\,(\|\nabla h\|_{p,B} + \|\nabla u\|_{p,B}),$$

where we can take $c$ depending only on $N$ because $R \le 1$. Also,

$$\||u|+1\|_{p,B}^{p-1} \le c'\,(\|u\|_{p,B}^{p-1} + R^{N(p-1)/p}),$$

with $c' = c'(p) > 0$. Thus, by the above inequalities and Young's inequality we have

$$\begin{aligned}
\|\nabla h\|_{p,B}^p &\le c_1\,\|\nabla h\|_{p,B}^{p-1}\,\|\nabla u\|_{p,B} \\
&\qquad + c_2\,(\|u\|_{p,B}^{p-1} + R^{N(p-1)/p})\,(\|\nabla h\|_{p,B} + \|\nabla u\|_{p,B}) \\
&\le \frac{1}{2}\,\|\nabla h\|_{p,B}^p + c_3\,(\|\nabla u\|_{p,B}^p + \|u\|_{p,B}^p + R^N).
\end{aligned}$$

Hence $\|\nabla h\|_{p,B}^p \le 2c_3\,(\|\nabla u\|_{p,B}^p + \|u\|_{p,B}^p + R^N)$, which implies the desired inequality. □

**Lemma 2.2.** *Suppose that $G$ is a bounded open set and $B(x_0,R) \Subset G$. There exists a number $\lambda = \lambda(N,p,\alpha_1,\alpha_2,\alpha_3(G)) > 0$ such that for* *every* $0 < r < R \leq 1$ *and* $(\mathcal{A}, \mathcal{B})$*-harmonic function* $h$ *in* $G$ *with* $|h| \leq L$ *in* $B(x_0, R)$ *it holds that*

$$\int_{B(x_0,r)} |\nabla h|^p \; dx \leq c \; \left(\frac{r}{R}\right)^{N-p+p\lambda} \int_{B(x_0,R)} |\nabla h|^p \; dx + c \; R^N,$$

*where* $c = c(N, p, \alpha_1, \alpha_2, \alpha_3(G), L) > 0$.

*Proof.* We may assume that $0 < r < \frac{R}{4}$. From Proposition 2.2 and Proposition 2.1 we obtain

$$\begin{aligned}
&\int_{B(x_0,r)} |\nabla h|^p \, dx \leq \frac{c}{r^p} \int_{B(x_0,2r)} \{(h - \inf_{B(x_0,2r)} h)^p + (L+1)^p \, r^p\} \, dx \\
&\leq \frac{c}{r^p} \left\{ \left( \sup_{B(x_0,2r)} h - \inf_{B(x_0,2r)} h \right)^p + (L+1)^p r^p \right\} r^N \\
&\leq c \, r^{N-p} \\
&\quad \times \left[ \left\{ \left(\frac{r}{R}\right)^{\lambda} \left( \sup_{B(x_0,R/2)} h - \inf_{B(x_0,R/2)} h + R \right) \right\}^p + (L+1)^p r^p \right] \\
&\leq c \, r^{N-p} \left\{ \left(\frac{r}{R}\right)^{p\lambda} \left( \sup_{B(x_0,R/2)} h - \inf_{B(x_0,R/2)} h \right)^p + R^p \right\}.
\end{aligned}$$

On the other hand, setting

$$h_R = \frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} h \, dx,$$

by Proposition 2.3 and the Poincaré inequality, we have

$$\begin{aligned}
&\left( \sup_{B(x_0,R/2)} h - \inf_{B(x_0,R/2)} h \right)^p \\
&\quad \leq \; 2 \sup_{B(x_0,R/2)} |h - h_R|^p \\
&\quad \leq \; \frac{c}{|B(x_0,R)|} \int_{B(x_0,R)} |h - h_R|^p \, dx + c \, R^p \\
&\quad \leq \; \frac{c \, R^p}{|B(x_0,R)|} \int_{B(x_0,R)} |\nabla h|^p \, dx + c \, R^p.
\end{aligned}$$

Hence,

$$\begin{aligned}\int_{B(x_0,r)} |\nabla h|^p\,dx &\le c\,r^{N-p}\left\{\left(\frac{r}{R}\right)^{p\lambda}\left(\frac{1}{R}\right)^{N-p}\int_{B(x_0,R)}|\nabla h|^p\,dx + R^p\right\}\\ &\le c\,\left(\frac{r}{R}\right)^{N-p+p\lambda}\int_{B(x_0,R)}|\nabla h|^p\ dx + c\ R^N.\end{aligned}$$

$\square$

## §3. Hölder continuity of solutions to ($\mathbf{E}_\nu$).

In this section, we establish Hölder continuity of solutions to the equation ($\mathrm{E}_\nu$). First, we recall the following Adams' inequality ([17; Theorem 3.3]).

**Proposition 3.1.** *Suppose that $\nu$ is a nonnegative Radon measure supported in an open set $\Omega$ such that there is a constant $M$ with the property that for all $x \in \mathbf{R}^N$ and $0 < r < \infty$,*

$$\nu(B(x,r)) \le M\ r^a$$

*where $a = q(N/p-1)$, $1 < p < q < \infty$ and $p < N$. If $u \in W_0^{1,p}(\Omega)$, then*

$$\left(\int_\Omega |u|^q\,d\nu\right)^{1/q} \le c\,M^{1/q}\left(\int_\Omega |\nabla u|^p\ dx\right)^{1/p},$$

*where $c = c(p,q,N)$.*

Let $G$ be an open subset in $\mathbf{R}^N$. A function $u : G \to \mathbf{R}\cup\{\infty\}$ is said to be $(\mathcal{A},\mathcal{B})$-*superharmonic* in $G$ if it is lower semicontinuous, finite on a dense set in $G$ and, for each bounded open set $U$ and for $h \in C(\overline{U})$ which is $(\mathcal{A},\mathcal{B})$-harmonic in $U$, $u \ge h$ on $\partial U$ implies $u \ge h$ in $U$. $(\mathcal{A},\mathcal{B})$-*subharmonic* functions are similarly defined.

To show Hölder continuity of solutions to the equation ($\mathrm{E}_\nu$), we prepare the following lemma.

**Lemma 3.1.** *Suppose that $G$ is a bounded open set, $B(x_0,R) \Subset G$, $0 < \beta < 1$, $\nu$ is a signed Radon measure on $G$ such that*

$$|\nu|(B(x_0,r)) \le c_0\ r^{N-p+\beta(p-1)}$$

*for every $0 < r \le R$ and $u \in W^{1,p}_{loc}(G)$ is a solution of ($E_\nu$) in $G$ with $|u| \le L$ in $B(x_0,R)$. Then for every $0 < r \le R \le 1$ and $\varepsilon >$*

*0, there exist constants* $c_1 = c_1(N, p, \alpha_1, \alpha_2, \alpha_3(G), L) > 0$ *and* $c_2 = c_2(N, p, \alpha_1, \alpha_2, \alpha_3(G), \beta, c_0, \varepsilon, L) > 0$ *such that*

$$\begin{aligned}\int_{B(x_0,r)} |\nabla u|^p \, dx \quad &\leq \quad c_1 \left( \left(\frac{r}{R}\right)^{N-p+p\lambda} + \varepsilon \right) \int_{B(x_0,R)} |\nabla u|^p \ dx \\ &\qquad + c_2 \ R^{N-p+p\beta}.\end{aligned}$$

*where* $\lambda$ *is the constant in Lemma* 2.2.

*Proof.* We may assume that $0 < r < \frac{R}{2}$. Let $h$ be an $(\mathcal{A}, \mathcal{B})$-harmonic function with $u - h \in W_0^{1,p}(B(x, R))$. First, we will show that

$$|h| \leq L' \tag{3.1}$$

on $B(x, R)$ with $L' = L'(\mathcal{A}, \mathcal{B}, G, L)$. Let $B_0$ be a ball containing $G$. There exists an $(\mathcal{A}, \mathcal{B})$-harmonic function $h_0$ in $B_0$ belonging to $W_0^{1,p}(B_0)$ (see [10; Theorem 1.4]). Then $h_0$ is continuous on $\overline{B_0}$ and hence bounded in $G$. Let $-m_1 \leq h_0 \leq m_2$ in G with $m_1 \geq 0$ and $m_2 \geq 0$. Then, $v_1 = h_0 + m_1 + L$ is $(\mathcal{A}, \mathcal{B})$-superharmonic and $v_1 \geq L$ in $G$; and $v_2 = h_0 - m_2 - L$ is $(\mathcal{A}, \mathcal{B})$-subharmonic and $v_2 \leq -L$ in $G$. Since

$$0 \geq \min(0, v_1 - h) \geq \min(0, L - h) \geq \min(0, u - h) \in W_0^{1,p}(B(x, R)),$$

$\min(0, v_1 - h) \in W_0^{1,p}(B(x, R))$. Hence by the comparison principle (see [16; Proposition 5.1.1 and Lemma 2.2.1]), $v_1 \geq h$, so that $h \leq L + m_1 + m_2$. Similarly, we see that $v_2 \leq h$, which shows $h \geq -(L + m_1 + m_2)$. Thus, we have (3.1) with $L' = L + m_1 + m_2$.

Next, we note that $|\nu| \in (W_0^{1,p}(V))^*$ for any $V \Subset G$, that is, $|\nu|$ is in the dual space of $W_0^{1,p}(V)$. Indeed, there exists an $\mathcal{A}$-superharmonic function $U$ in $G$ satisfying

$$-\operatorname{div} \mathcal{A}(x,\, DU(x)) = |\nu|$$

with $\min(U, k) \in W_0^{1,p}(G)$ for all $k > 0$, where $DU$ is the generalized gradient of $U$ (see [5; Theorem 2.4]). Then by [6; Theorem 4.16], $U$ is locally bounded in $G$. Thus, $U \in W_{loc}^{1,p}(G)$ (see [3; Corollary 7.20]). Hence we see that $|\nu| \in (W_0^{1,p}(V))^*$ (cf. [6; p.142]). Thus, by (A.2), (A.3) and (B.3) we have

$$
\begin{aligned}
\alpha_1 \int_{B(x_0,r)} |\nabla u|^p \, dx &\le \int_{B(x_0,r)} \mathcal{A}(x,\nabla u)\cdot \nabla u \, dx \\
&= \int_{B(x_0,r)} \left(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla h)\right)\cdot(\nabla u-\nabla h)\ dx \\
&\quad + \int_{B(x_0,r)} \mathcal{A}(x,\nabla h)\cdot(\nabla u-\nabla h)\ dx \\
&\quad + \int_{B(x_0,r)} \mathcal{A}(x,\nabla u)\cdot \nabla h \, dx \\
&\le \int_{B(x_0,R)} \left(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla h)\right)\cdot(\nabla u-\nabla h)\ dx \\
&\quad + \alpha_2 \int_{B(x_0,R)} \left(|\nabla h|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla h|\right)\ dx \\
&\quad + \int_{B(x_0,R)} \left(\mathcal{B}(x,u)-\mathcal{B}(x,h)\right)(u-h)\ dx \\
&= \int_{B(x_0,R)} (u-h)\, d\nu \\
&\quad + \alpha_2 \int_{B(x_0,r)} \left(|\nabla h|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla h|\right)\ dx,
\end{aligned}
\tag{3.2}
$$

in the last inequality we have used that $u$ is a solution of $(E_\nu)$, $|\nu| \in (W_0^{1,p}(V))^*$, $h$ is $(\mathcal{A},\mathcal{B})$-harmonic and $u-h \in W_0^{1,p}(B(x,R))$. Set

$$
I_1 = \int_{B(x_0,R)} (u-h)\, d\nu
$$

and

$$
I_2 = \alpha_2 \int_{B(x_0,r)} \left(|\nabla h|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla h|\right)\ dx.
$$

Let $q = (N-p+\beta(p-1))/(\frac{N}{p}-1)$ and $1/q + 1/q' = 1$. Since $u-h \in W_0^{1,p}(B(x,R))$, by Hölder's inequality, Adams' inequality and Young's inequality we have

$$
\begin{aligned}
&\int_{B(x_0,R)} |u-h|\, d|\nu| \\
&\le \left(\int_{B(x_0,R)} |u-h|^q\, d|\nu|\right)^{1/q} \left(\int_{B(x_0,R)} d|\nu|\right)^{1/q'} \\
&\le c\left(R^{N-p+\beta(p-1)}\right)^{1/q'} \left(\int_{B(x_0,R)} |u-h|^q\, d|\nu|\right)^{1/q} \\
&\le c\, R^{\frac{p-1}{p}(N-p+\beta p)} \left(\int_{B(x_0,R)} |\nabla(u-h)|^p\, dx\right)^{1/p} \\
&\le c\, R^{\frac{p-1}{p}(N-p+\beta p)} \\
&\quad \times \left\{ \left(\int_{B(x_0,R)} |\nabla u|^p\, dx\right)^{1/p} + \left(\int_{B(x_0,R)} |\nabla h|^p\, dx\right)^{1/p} \right\} \\
&\le c\, R^{\frac{p-1}{p}(N-p+\beta p)} \\
&\quad \times \left\{ \left(\int_{B(x_0,R)} |\nabla u|^p\, dx\right)^{1/p} + \left(\int_{B(x_0,R)} |u|^p\, dx\right)^{1/p} + R^{N/p} \right\} \\
&\le c\, R^{N-p+\beta p} + \frac{\alpha_1}{2}\,\varepsilon \int_{B(x_0,R)} |\nabla u|^p\, dx + c\int_{B(x_0,R)} |u|^p\, dx + c\, R^N,
\end{aligned}
$$

where we have used Lemma 2.1. Hence we have

$$
\begin{aligned}
I_1 &\le \int_{B(x_0,R)} |u-h|\, d|\nu| \\
&\le c\, R^{N-p+\beta p} + \frac{\alpha_1}{2}\,\varepsilon \int_{B(x_0,R)} |\nabla u|^p\, dx,
\end{aligned}
\tag{3.3}
$$

where we have used that $R \le 1$ and $N-p+\beta p \le N$ imply $R^N \le R^{N-p+\beta p}$. Here $c$ depends on $N$, $p$, $\alpha_1$, $\alpha_2$, $\alpha_3(G)$, $\beta$, $c_0$, $\varepsilon$ and $L$. Also, Young's inequality, Lemma 2.2 and (3.1) yield

$$\begin{aligned}
I_2 &\le \frac{\alpha_1}{2}\int_{B(x_0,r)} |\nabla u|^p\,dx + c\int_{B(x_0,r)} |\nabla h|^p\,dx \\
&\le \frac{\alpha_1}{2}\int_{B(x_0,r)} |\nabla u|^p\,dx + c\left(\frac{r}{R}\right)^{N-p+p\lambda}\int_{B(x_0,R)} |\nabla h|^p\,dx + c\,R^N \\
&\le \frac{\alpha_1}{2}\int_{B(x_0,r)} |\nabla u|^p\,dx \\
&\quad + c\left(\frac{r}{R}\right)^{N-p+p\lambda}\left(\int_{B(x_0,R)} |\nabla u|^p\,dx + \int_{B(x_0,R)} |u|^p\,dx\right) + c\,R^N \qquad (3.4)\\
&\le \frac{\alpha_1}{2}\int_{B(x_0,r)} |\nabla u|^p\,dx \\
&\quad + c\left(\frac{r}{R}\right)^{N-p+p\lambda}\int_{B(x_0,R)} |\nabla u|^p\,dx + c\,R^{N-p+\beta p},
\end{aligned}$$

where again we have used Lemma 2.1, (3.1) and $R^N \le R^{N-p+\beta p}$. It follows from (3.2), (3.3) and (3.4) that

$$\begin{aligned}
&\int_{B(x_0,r)} |\nabla u|^p\,dx \\
&\le c_1\left(\left(\frac{r}{R}\right)^{N-p+p\lambda} + \varepsilon\right)\int_{B(x_0,R)} |\nabla u|^p\;dx + c_2\;R^{N-p+p\beta}.
\end{aligned}$$

□

To achieve the aim in this section, we need the following two propositions in [2; III Lemma 2.1 and III Theorem 1.1].

**Proposition 3.2.** *Let $A$, $\gamma_1$ and $\gamma_2$ be positive constants such that $\gamma_2 < \gamma_1$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A, \gamma_1, \gamma_2) > 0$ with the following property: if $f(t)$ is a nonnegative nondecreasing function satisfying*

$$f(r) \le A\left\{\left(\frac{r}{R}\right)^{\gamma_1} + \varepsilon\right\} f(R) + B\,R^{\gamma_2}$$

*for all $0 < r \le R \le R_0$ with $0 < \varepsilon \le \varepsilon_0$, $R_0 > 0$ and $B \ge 0$, then*

$$f(r) \le c\left\{\left(\frac{r}{R}\right)^{\gamma_2} f(R) + B\,r^{\gamma_2}\right\}$$

*for all $0 < r \le R \le R_0$ with a constant $c = c(A, \gamma_1, \gamma_2) > 0$.*

**Proposition 3.3.** *Let $u \in W^{1,p}(B(x_0, R))$, $1 \le p \le N$. Suppose that for all $x \in B(x_0, R)$, all $r$, $0 < r \le \delta(x) = R - |x - x_0|$*

$$\int_{B(x,r)} |\nabla u|^p \, dx \le L^p \ \left(\frac{r}{\delta(x)}\right)^{N-p+p\beta}$$

*holds with $0 < \beta \le 1$. Then, $u$ is Hölder continuous in $B(x_0, \rho)$ with the exponent $\beta$ for all $0 < \rho < R$.*

**Theorem 3.1.** *Let $G$ be a bounded open set and $u \in W^{1,p}_{loc}(G) \cap L^\infty_{loc}(G)$ is a solution of $(E_\nu)$ in $G$. Suppose that $\nu$ is a signed Radon measure on $G$ such that there exist constants $M > 0$ and $0 < \beta < \lambda$, where $\lambda = \lambda(N, p, \alpha_1, \alpha_2, \alpha_3(G)) > 0$ is the number in Lemma 2.2 above, with*

$$|\nu|(B(x, r)) \le M \ r^{N-p+\beta(p-1)}$$

*whenever $B(x, 3r) \subset G$. Then $u$ is locally Hölder continuous in $G$ with the exponent $\beta$.*

*Proof.* If $B(x_0, 4R) \subset G$ with $R \le 1$, then Proposition 3.2 and Lemma 3.1 yield that

$$\int_{B(x,r)} |\nabla u|^p \, dx \le c \ \left\{ \int_{B(x_0,2R)} |\nabla u|^p \, dx + 1 \right\} \left(\frac{r}{R}\right)^{N-p+p\beta},$$

whenever $x \in B(x_0, R)$ and $0 < r \le R$, where $c > 0$ depends on $N$, $p$, $\alpha_1$, $\alpha_2$, $\alpha_3(G)$, $M$, $\beta$ and $\sup_{B(x_0,2R)} |u|$. Hence, by Proposition 3.3, $u$ is Hölder continuous in $B(x_0, \rho)$ with exponent $\beta$ for $0 < \rho < R$. $\square$

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# On Davies' conjecture and strong ratio limit properties for the heat kernel

**Yehuda Pinchover**

**Abstract.**

We study strong ratio limit properties and the exact long time asymptotics of the heat kernel of a general second-order parabolic operator which is defined on a noncompact Riemannian manifold.

## §1. Introduction

Let $P$ be a linear, second-order, elliptic operator defined on a noncompact, connected, $C^3$-smooth Riemannian manifold $\mathcal{M}$ of dimension $d$ with a Riemannian measure $\mathrm{d}x$. Here $P$ is an elliptic operator with real, Hölder continuous coefficients which in any coordinate system $(U; x_1, \ldots, x_d)$ has the form

$$P(x, \partial_x) = -\sum_{i,j=1}^{d} a_{ij}(x)\partial_i\partial_j + \sum_{i=1}^{d} b_i(x)\partial_i + c(x).$$

We assume that for each $x \in \mathcal{M}$ the real quadratic form $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j$ is positive definite. The formal adjoint of $P$ is denoted by $P^*$. Denote the cone of all positive (classical) solutions of the equation $Pu=0$ in $\mathcal{M}$ by $\mathcal{C}_P(\mathcal{M})$. The *generalized principal eigenvalue* is defined by

$$\lambda_0 = \lambda_0(P, \mathcal{M}) := \sup\{\lambda \in \mathbb{R} \,:\, \mathcal{C}_{P-\lambda}(\mathcal{M}) \neq \emptyset\}.$$

Throughout this paper we always assume that $\lambda_0 \geq 0$ (actually, as it will become clear below, it is enough to assume that $\lambda_0 > -\infty$).

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We consider the parabolic operator $L$

$$Lu = u_t + Pu \qquad \text{on } \mathcal{M}\times(0,\infty). \tag{1.1}$$

We denote by $\mathcal{H}_P(\mathcal{M}\times(a,b))$ the cone of all nonnegative solutions of the equation $Lu=0$ in $\mathcal{M}\times(a,b)$. Let $k_P^{\mathcal{M}}(x,y,t)$ be the *minimal (positive) heat kernel* of the parabolic operator $L$ in $\mathcal{M}$. If for some $x\neq y$

$$\int_0^\infty k_P^{\mathcal{M}}(x,y,\tau)\,\mathrm{d}\tau<\infty \qquad \left(\text{respect.,}\ \int_0^\infty k_P^{\mathcal{M}}(x,y,\tau)\,\mathrm{d}\tau=\infty\right),$$

then $P$ is said to be a *subcritical* (respect., *critical*) operator in $\mathcal{M}$ [18].

Recall that if $\lambda<\lambda_0$, then $P$–$\lambda$ is subcritical in $\mathcal{M}$, and for $\lambda\le\lambda_0$, we have $k_{P-\lambda}^{\mathcal{M}}(x,y,t)=\mathrm{e}^{\lambda t}k_P^{\mathcal{M}}(x,y,t)$. Furthermore, $P$ is critical (respect., subcritical) in $\mathcal{M}$, if and only if $P^*$ is critical (respect., subcritical) in $\mathcal{M}$. If $P$ is critical in $\mathcal{M}$, then there exists a unique positive solution $\varphi\in\mathcal{C}_P(\mathcal{M})$ satisfying $\varphi(x_0)=1$, where $x_0\in\mathcal{M}$ is a fixed reference point. This solution is called the *ground state of the operator $P$ in $\mathcal{M}$* [15, 18]. The ground state of $P^*$ is denoted by $\varphi^*$. A critical operator $P$ is said to be *positive-critical* in $\mathcal{M}$ if $\varphi^*\varphi\in L^1(\mathcal{M})$, and *null-critical* in $\mathcal{M}$ if $\varphi^*\varphi\notin L^1(\mathcal{M})$. In [15, 17] we proved:

**Theorem 1.1.** *Let $x,y\in\mathcal{M}$. Then*

$$\lim_{t\to\infty}\mathrm{e}^{\lambda_0 t}k_P^{\mathcal{M}}(x,y,t)=\begin{cases}\dfrac{\varphi(x)\varphi^*(y)}{\int_{\mathcal{M}}\varphi(z)\varphi^*(z)\,\mathrm{d}z} & \text{if } P-\lambda_0 \text{ is positive-critical},\\ 0 & \text{otherwise}.\end{cases}$$

*Furthermore, for $\lambda<\lambda_0$, let $G_{P-\lambda}^{\mathcal{M}}(x,y):=\int_0^\infty k_{P-\lambda}^{\mathcal{M}}(x,y,\tau)\mathrm{d}\tau$ be the minimal (positive) Green function of the operator $P-\lambda$ on $\mathcal{M}$. Then*

$$\lim_{t\to\infty}\mathrm{e}^{\lambda_0 t}k_P^{\mathcal{M}}(x,y,t)=\lim_{\lambda\nearrow\lambda_0}(\lambda_0-\lambda)G_{P-\lambda}^{\mathcal{M}}(x,y). \tag{1.2}$$

Having proved that $\lim_{t\to\infty}\mathrm{e}^{\lambda_0 t}k_P^{\mathcal{M}}(x,y,t)$ always exists, we next ask how fast this limit is approached. It is natural to conjecture that the limit is approached equally fast for different points $x,y\in\mathcal{M}$. Note that in the context of Markov chains, such an *(individual) strong ratio limit property* is in general not true [5]. The following conjecture was raised by E. B. Davies [7] in the selfadjoint case.

**Conjecture 1.1.** *Let $Lu=u_t+P(x,\partial_x)u$ be a parabolic operator which is defined on a Riemannian manifold $\mathcal{M}$. Fix a reference point $x_0\in\mathcal{M}$. Then*

$$\lim_{t\to\infty}\frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_0,x_0,t)}=a(x,y) \tag{1.3}$$

*exists and is positive for all* $x, y \in \mathcal{M}$.

The aim of the present paper is to discuss Conjecture 1.1 and closely related problems, and to obtain some results under minimal assumptions.

*Remark* 1.1. Theorem 1.1 implies that Conjecture 1.1 holds true in the positive-critical case. So, we may assume in the sequel that $P$ **is not positive critical**. Also, Conjecture 1.1 does not depend on the value of $\lambda_0$, hence from now on, **we shall assume that** $\lambda_0 = 0$.

*Remark* 1.2. In the selfadjoint case, Conjecture 1.1 holds true if $\dim \mathcal{C}_P(\mathcal{M}) = 1$ [2, Corollary 2.7]. In particular, it holds true for a critical selfadjoint operator. Therefore, it would be interesting to prove Conjecture 1.1 at least under the assumption

$$\dim \mathcal{C}_P(\mathcal{M}) = \dim \mathcal{C}_{P^*}(\mathcal{M}) = 1, \tag{1.4}$$

which holds true in the critical case and in many important subcritical cases. Recently, Agmon [1] has obtained the exact asymptotics (in $(x, y, t)$) of the heat kernel for a periodic (non-selfadjoint) operator on $\mathbb{R}^d$. It follows from Agmon's results that Conjecture 1.1 holds true in this case. For a probabilistic interpretation of Conjecture 1.1, see [2].

*Remark* 1.3. Let $t_n \to \infty$. By a standard parabolic argument, we may extract a subsequence $\{t_{n_k}\}$ such that for every $x, y \in \mathcal{M}$ and $s < 0$

$$a(x, y, s) := \lim_{k\to\infty} \frac{k_P^{\mathcal{M}}(x, y, s + t_{n_k})}{k_P^{\mathcal{M}}(x_0, y_0, t_{n_k})} \tag{1.5}$$

exists. Moreover, $a(\cdot, y, \cdot) \in \mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$. Note that in the selfadjoint case, the above is valid for all $s \in \mathbb{R}$, since (2.7) holds in selfadjoint case [7, Theorem 10].

*Remark* 1.4. The example constructed in [16, Section 4] shows a case where Conjecture 1.1 holds true on $\mathcal{M}$, while the limit function $a(x, y) = 1$ is not a $\lambda_0$-invariant positive solution. Compare this with [7, Theorem 25] and the discussion therein above Lemma 26. Note also that in general, the limit function $a(x, y)$ in (1.3) need not be a *product* of solutions of the equations $Pu = 0$ and $P^*u = 0$, as is demonstrated in [6], in the hyperbolic space, and in Example 4.2.

The outline of the rest of paper is as follows. In the next section we study the existence of the strong ratio limit for the heat kernel. It turns out that if this limit exists, then it equals 1. This implies that any limit solution $u(\cdot, y, s)$ of (1.5) is time independent and is a positive solution of the equation $Pu = 0$ in $\mathcal{M}$. In Section 3 we discuss the relationship between Conjecture 1.1 and the parabolic Martin compactification of $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$, while in Section 4 we study the relation between this conjecture and the parabolic and elliptic *minimal* Martin boundaries. Finally, in Section 5 we study Conjecture 1.1 under the assumption that the uniform restricted parabolic Harnack inequality holds true.

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## §2. Strong ratio properties

In the symmetric case the function $t \mapsto k_P^{\mathcal{M}}(x,x,t)$ is log-convex, and therefore, a polarization argument implies that $\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} = 1$ for all $x, y \in \mathcal{M}$ and $s \in \mathbb{R}$ [7]. In the nonsymmetric case we have:

**Lemma 2.1.** *For every $x, y \in \mathcal{M}$ and $s \in \mathbb{R}$, we have that*

$$\liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \leq 1 \leq \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)}\,. \tag{2.1}$$

*Similarly, for any $s > 0$*

$$\liminf_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,(n\pm 1)s)}{k_P^{\mathcal{M}}(x,y,ns)} \leq 1 \leq \limsup_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,(n\pm 1)s)}{k_P^{\mathcal{M}}(x,y,ns)}\,. \tag{2.2}$$

*In particular, if* $\lim_{t\to\infty}[k_P^{\mathcal{M}}(x,y,t+s)/k_P^{\mathcal{M}}(x,y,t)]$ *exists, it equals to* $1$.

*Proof.* We may assume that $P$ is not positive-critical. Let $s < 0$. By Theorem 1.1 and the parabolic Harnack inequality we have

$$1 \leq \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \leq C(s,y). \tag{2.3}$$

Suppose that $\liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} = \ell > 1$. It follows that there exists $0 < q < 1$ and $T_s > 0$ such that

$$k_P^{\mathcal{M}}(x,y,t) < q k_P^{\mathcal{M}}(x,y,t+s) \qquad \forall\, t > T_s.$$

By induction and the Harnack inequality, we obtain that there exist $\mu < 0$ and $C > 0$ such that $k_P^{\mathcal{M}}(x,y,t) < C\mathrm{e}^{\mu t}$ for all $t > 1$, a contradiction to the assumption $\lambda_0 = 0$. Therefore, (2.1) is proved for $s < 0$, which in turn implies (2.1) also for $s > 0$. (2.2) can be proven similarly. □

*Remark* 2.1. The condition $\liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \geq 1$ for $s > 0$ is sometimes called *Lin's condition* [11].

**Corollary 2.1.** *Let $x, y \in \mathcal{M}$. Suppose that*

$$\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,(n+1)s)}{k_P^{\mathcal{M}}(x,y,ns)} \tag{2.4}$$

*exists for every $s > 0$ (i.e., the ratio limit exists for every "skeleton" sequence of the form $t_n = ns$, where $n = 1, 2, \ldots$ and $s > 0$). Then*

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+r)}{k_P^{\mathcal{M}}(x,y,t)} = 1 \qquad \forall r \in \mathbb{R}. \tag{2.5}$$

*Proof.* By Lemma 2.1, the limit in (2.4) equals 1. By induction, $\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,ns+r)}{k_P^{\mathcal{M}}(x,y,ns)} = 1$, where $r = qs$, and $q \in \mathbb{Q}$, which (by the continuity of a limiting solution) implies that it holds for $\forall r \in \mathbb{R}$. Hence, [9, Theorem 2] implies (2.5). □

*Remark* 2.2. If there exist $x_0, y_0 \in \mathcal{M}$ and $0 < s_0 < 1$ such that

$$M(x_0,y_0,s_0) := \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t)} < \infty, \tag{2.6}$$

then by the parabolic Harnack inequality, for all $x, y, z, w \in K \subset\subset \mathcal{M}$, $t > 1$, we have the following Harnack inequality of elliptic type:

$$k_P^{\mathcal{M}}(z,w,t) \le C_1 k_P^{\mathcal{M}}(x_0,y_0,t+\frac{s_0}{2}) \le C_2 k_P^{\mathcal{M}}(x_0,y_0,t-\frac{s_0}{2}) \le C_3 k_P^{\mathcal{M}}(x,y,t).$$

Similarly, (2.6) implies that for all $x, y \in \mathcal{M}$ and $r \in \mathbb{R}$:

$$0 < m(x,y,r) := \liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+r)}{k_P^{\mathcal{M}}(x_0,y_0,t)} \le$$

$$\limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+r)}{k_P^{\mathcal{M}}(x_0,y_0,t)} = M(x,y,r) < \infty. \tag{2.7}$$

**Lemma 2.2.** (a) *The following assertions are equivalent:*

(i) *For each $x, y \in \mathcal{M}$ there exists a sequence $s_j \to 0$ of negative numbers such that for all $j \ge 1$, and $s = s_j$, we have*

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} = 1. \tag{2.8}$$

(ii) *The ratio limit in* (2.8) *exists for any $x, y \in \mathcal{M}$ and $s \in \mathbb{R}$.*

(iii) *Any limit function $u(x,y,s)$ of the quotients $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)}$ with $t_n \to \infty$ does not depend on $s$ and has the form $u(x,y)$, where $u(\cdot,y) \in \mathcal{C}_P(\mathcal{M})$ for every $y \in \mathcal{M}$ and $u(x,\cdot) \in \mathcal{C}_{P^*}(\mathcal{M})$ for every $x \in \mathcal{M}$.*

(b) *If one assumes further* (1.4), *then Conjecture* 1.1 *holds true.*

*Proof.* (a) By Lemma 2.1, if the limit in (2.8) exists, then it is 1.

(i) $\Rightarrow$ (ii). Fix $x_0, y_0 \in \mathcal{M}$, and take $s_0 < 0$ for which the limit (2.8) exists. It follows that any limit solution $u(x,y,s) \in \mathcal{H}_P(\mathcal{M}\times\mathbb{R}_-)$ of a sequence $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,y_0,t_n)}$ with $t_n \to \infty$ satisfies $u(x_0,y_0,s{+}s_0){=}u(x_0,y_0,s)$ for all $s{<}0$. So, $u(x_0,y_0,\cdot)$ is $s_0$-periodic. It follow from our assumption and the continuity of $u$ that $u(x_0,y_0,\cdot)$ is the constant function. Since this holds for all $x,y \in \mathcal{M}$ and $u$, it follows that (2.8) holds for any $x,y{\in}\mathcal{M}$ and $s{\in}\mathbb{R}$.

(ii) $\Rightarrow$ (iii). Fix $y \in \mathcal{M}$. By Remark 1.3, any limit function $u$ of the sequence $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)}$ with $t_n \to \infty$ belongs to $\mathcal{H}_P(\mathcal{M}\times\mathbb{R}_-)$. Since

$$\frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x_0,x_0,t)} = \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_0,x_0,t)}\,\frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)}\,, \tag{2.9}$$

(2.8) implies that such a $u$ does not depend on $s$. Therefore, $u = u(x,y)$, where $u(\cdot,y) \in \mathcal{C}_P(\mathcal{M})$ and $u(x,\cdot) \in \mathcal{C}_{P^*}(\mathcal{M})$.

(iii) $\Rightarrow$ (i). Write

$$\frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} = \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x_0,x_0,t)}\,\frac{k_P^{\mathcal{M}}(x_0,x_0,t)}{k_P^{\mathcal{M}}(x,y,t)}\,. \tag{2.10}$$

Let $t_n \to \infty$ be a sequence such that the sequence $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)}$ converges to a solution in $\mathcal{H}_P(\mathcal{M}\times\mathbb{R}_-)$. By our assumption, we have

$$\lim_{n\to\infty}\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)} = \lim_{n\to\infty}\frac{k_P^{\mathcal{M}}(x,y,t_n)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)} = u(x,y) > 0,$$

which together with (2.10) implies (2.8) for all $s \in \mathbb{R}$.

(b) The uniqueness and (iii) imply that $\frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x_0,x_0,t)} \to \frac{u(x)u^*(y)}{u(x_0)u^*(x_0)}$, where $u{\in}\mathcal{C}_P(\mathcal{M})$ and $u^*{\in}\mathcal{C}_{P^*}(\mathcal{M})$, and Conjecture 1.1 holds. $\square$

*Remark* 2.3. Let $\mathcal{M} \subsetneqq \mathbb{R}^d$ be a smooth domain and $P$ and $P^*$ be (up to the boundary) smooth operators. Denote by $\mathcal{C}_P^0(\mathcal{M})$ the cone of all functions in $\mathcal{C}_P(\mathcal{M})$ which vanish on $\partial\mathcal{M}$. Suppose that one of the conditions (i)–(iii) of Lemma 2.2 is satisfied. Clearly, for any fixed $y$ any limit function $u(\cdot,y)$ of Lemma 2.2 belongs to the Martin boundary 'at infinity' which in this case is $\mathcal{C}_P^0(\mathcal{M})$. Therefore, Conjecture 1.1 holds true if the Martin boundaries 'at infinity' of $P$ and $P^*$ are one-dimensional. As a simple example, take $P = -\Delta$ and $\mathcal{M} = \mathbb{R}_+^d$.

**Lemma 2.3.** *Suppose that $P$ is null-critical, and for each $x, y \in \mathcal{M}$ there exists a sequence $\{s_j\}$ of negative numbers such that $s_j \to 0$, and*

$$\liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \geq 1 \tag{2.11}$$

*for $s = s_j$, $j = 1, 2, \ldots$. Then Conjecture 1.1 holds true.*

*Proof.* Let $u(x,y,s)$ be a limit function of a sequence $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)}$ with $t_n \to \infty$ and $s < 0$. By our assumption, $u(x,y,s+s_j) \geq u(x,y,s)$, and therefore, $u_s(x,y,s) \leq 0$ for all $s < 0$. Thus, $u(\cdot,y,s)$ (respect., $u(x,\cdot,s)$) is a positive supersolution of the equation $Pu = 0$ (respect., $P^*u = 0$) in $\mathcal{M}$. Since $P$ is critical, it follows that $u(\cdot,y,s) \in \mathcal{C}_P(\mathcal{M})$ (respect., $u(x,\cdot,s) \in \mathcal{C}_{P^*}(\mathcal{M})$), and hence $u_s(x,y,s) = 0$. By the uniqueness, $u$ equals to $\frac{\varphi(x)\varphi^*(y)}{\varphi(x_0)\varphi^*(x_0)}$, and Conjecture 1.1 holds true. $\square$

*Remark* 2.4. Suppose that $P$ is null-critical, and fix $x_0 \neq y_0$. Then using Theorem 1.1 and [14, Theorem 2.1] we have for $x \neq y$:

$$\text{(i)}\ \lim_{t\to\infty} k_P^{\mathcal{M}}(x,y,t) = \lim_{t\to\infty} k_P^{\mathcal{M}}(x_0,y_0,t) = 0,$$

$$\text{(ii)}\ \int_0^\infty k_P^{\mathcal{M}}(x,y,\tau)\,\mathrm{d}\tau = \int_0^\infty k_P^{\mathcal{M}}(x_0,y_0,\tau)\,\mathrm{d}\tau = \infty,$$

$$\text{(iii)}\ \lim_{\lambda\nearrow 0} \frac{\int_0^\infty \mathrm{e}^{\lambda\tau} k_P^{\mathcal{M}}(x,y,\tau)\mathrm{d}\tau}{\int_0^\infty \mathrm{e}^{\lambda\tau} k_P^{\mathcal{M}}(x_0,y_0,\tau)\mathrm{d}\tau} = \lim_{\lambda\nearrow 0} \frac{G_{P-\lambda}^{\mathcal{M}}(x,y)}{G_{P-\lambda}^{\mathcal{M}}(x_0,y_0)} = \frac{\varphi(x)\varphi^*(y)}{\varphi(x_0)\varphi^*(y_0)}.$$

Therefore, Conjecture 1.1 would follow from a strong ratio Tauberian theorem if additional Tauberian conditions are satisfied (see, [3, 19]).

## §3. The parabolic Martin boundary

The large time behavior of quotients of the heat kernel is obviously closely related to the parabolic Martin boundary (for the parabolic Martin boundary theory see [8]). Theorem 3.1 relates Conjecture 1.1 and the parabolic Martin compactification of $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$.

**Lemma 3.1.** *Fix $y \in \mathcal{M}$. The following assertions are equivalent:*

(i) *For each $x \in \mathcal{M}$ there exists a sequence $s_j \to 0$ of negative numbers such that*

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \tag{3.1}$$

*exists for $s = s_j$, $j = 1, 2, \ldots$.*

(ii) *Any parabolic Martin function in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$ corresponding to a Martin sequence $\{(y, -t_n)\}_{n=1}^\infty$, where $t_n \to \infty$, is time independent.*

*Proof.* Let $K_P^{\mathcal{M}}(x,y,s) = \lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,y,t_n)}$ be such a Martin function. The lemma follows from the identity

$$\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,y,t_n)} = \frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x,y,t_n)} \frac{k_P^{\mathcal{M}}(x,y,t_n)}{k_P^{\mathcal{M}}(x_0,y,t_n)},$$

and Lemma 2.2. □

**Theorem 3.1.** *Assume that* (2.6) *holds true for some* $x_0, y_0 \in \mathcal{M}$, *and* $s_0 > 0$. *Then the following assertions are equivalent:*

(i) *Conjecture 1.1 holds true for a fixed* $x_0 \in \mathcal{M}$.

(ii)

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_1,y_1,t)} \tag{3.2}$$

*exists, and the limit is positive for every* $x, y, x_1, y_1 \in \mathcal{M}$.

(iii)

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(y,y,t)}, \quad \textit{and} \quad \lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x,x,t)} \tag{3.3}$$

*exist, and these ratio limits are positive for every* $x, y \in \mathcal{M}$.

(iv) *For any* $y \in \mathcal{M}$ *there is a unique nonzero parabolic Martin boundary point* $\bar{y}$ *for the equation* $Lu = 0$ *in* $\mathcal{M} \times \mathbb{R}$ *which corresponds to any sequence of the form* $\{(y, -t_n)\}_{n=1}^{\infty}$ *such that* $t_n \to \infty$, *and for any* $x \in \mathcal{M}$ *there is a unique nonzero parabolic Martin boundary point* $\bar{x}$ *for the equation* $u_t + P^*u = 0$ *in* $\mathcal{M} \times \mathbb{R}$ *which corresponds to any sequence of the form* $\{(x, -t_n)\}_{n=1}^{\infty}$ *such that* $t_n \to \infty$.

*Moreover, if Conjecture 1.1 holds true, then for any fixed* $y \in \mathcal{M}$ *(respect.,* $x \in \mathcal{M}$*), the limit function* $a(\cdot, y)$ *(respect.,* $a(x, \cdot)$*) is a positive solution of the equation* $Pu = 0$ *(respect.,* $P^*u = 0$*). Furthermore, the Martin functions of part (iv) are time independent, and (2.8) holds for all* $x, y \in \mathcal{M}$ *and* $s \in \mathbb{R}$.

*Proof.* (i) ⇒ (ii) follows from the identity

$$\frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_1,y_1,t)} = \frac{k_P^{\mathcal{M}}(x,y,t)}{k_P^{\mathcal{M}}(x_0,x_0,t)} \cdot \left(\frac{k_P^{\mathcal{M}}(x_1,y_1,t)}{k_P^{\mathcal{M}}(x_0,x_0,t)}\right)^{-1}.$$

(ii) ⇒ (iii). Take $x_1 = y_1 = y$ and $x_1 = y_1 = x$, respectively.

(iii) ⇒ (iv). It is well known that the Martin compactification does not depend on the fixed reference point $x_0$. So, fix $y \in \mathcal{M}$ and take it also as a reference point. Let $\{-t_n\}$ be a sequence such that $t_n \to \infty$

and such that the Martin sequence $\frac{k_P^{\mathcal{M}}(x,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t_n)}$ converges to a Martin function $K_P^{\mathcal{M}}(x,\bar{y},t)$. By our assumption, for any $t$ we have

$$\lim_{n\to\infty}\frac{k_P^{\mathcal{M}}(x,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t+t_n)}=\lim_{\tau\to\infty}\frac{k_P^{\mathcal{M}}(x,y,\tau)}{k_P^{\mathcal{M}}(y,y,\tau)}=b(x)>0,$$

where $b$ does not depend on the sequence $\{-t_n\}$. On the other hand,

$$\lim_{n\to\infty}\frac{k_P^{\mathcal{M}}(y,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t_n)}=K_P^{\mathcal{M}}(y,\bar{y},t)=f(t).$$

Since

$$\frac{k_P^{\mathcal{M}}(x,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t_n)}=\frac{k_P^{\mathcal{M}}(x,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t+t_n)}\cdot\frac{k_P^{\mathcal{M}}(y,y,t+t_n)}{k_P^{\mathcal{M}}(y,y,t_n)},$$

we have

$$K_P^{\mathcal{M}}(x,\bar{y},t)=b(x)f(t).$$

By separation of variables, there exists a constant $\lambda$ such that

$$Pb-\lambda b=0\quad\text{on }\mathcal{M},\qquad f'+\lambda f=0\quad\text{on }\mathbb{R},\ \ f(0)=1.$$

Since $b$ does not depend on the sequence $\{-t_n\}$, it follows in particular, that $\lambda$ does not depend on this sequence. Thus, $\lim_{\tau\to\infty}\frac{k_P^{\mathcal{M}}(y,y,t+\tau)}{k_P^{\mathcal{M}}(y,y,\tau)}=f(t)=\mathrm{e}^{-\lambda t}$. Lemma 2.1 implies that $\lambda=0$. It follows that $b$ is a positive solution of the equation $Pu=0$, and

$$K_P^{\mathcal{M}}(x,\bar{y},t)=\lim_{\tau\to-\infty}\frac{k_P^{\mathcal{M}}(x,y,t-\tau)}{k_P^{\mathcal{M}}(y,y,-\tau)}=b(x). \tag{3.4}$$

The dual assertion can be proved similarly.

(iv) $\Rightarrow$ (i). Let $K_P^{\mathcal{M}}(x,\bar{y},t)$ be a Martin function, and $s_0>0$ such that $K_P^{\mathcal{M}}(x_0,\bar{y},s_0/2)>0$. Consequently, $K_P^{\mathcal{M}}(x,\bar{y},s)>0$ for $s\geq s_0$. Using the substitution $\tau=s+s_0$ we obtain

$$\lim_{\tau\to\infty}\frac{k_P^{\mathcal{M}}(x,y,\tau)}{k_P^{\mathcal{M}}(x_0,x_0,\tau)}=\lim_{s\to\infty}\left\{\frac{k_P^{\mathcal{M}}(x,y,s+s_0)}{k_P^{\mathcal{M}}(y,y,s)}\times\right.$$

$$\left.\frac{k_P^{\mathcal{M}}(y,y,s)}{k_P^{\mathcal{M}}(x_0,y,s+2s_0)}\frac{k_P^{\mathcal{M}}(x_0,y,s+2s_0)}{k_P^{\mathcal{M}}(x_0,x_0,s+s_0)}\right\}=\frac{K_P^{\mathcal{M}}(x,\bar{y},s_0)K_{P^*}^{\mathcal{M}}(\overline{x_0},y,s_0)}{K_P^{\mathcal{M}}(x_0,\bar{y},2s_0)}.$$

The last assertion of the theorem follows from (3.4) and Lemma 2.2. $\square$

## §4. Minimal positive solutions

In this section we discuss the relation between Conjecture 1.1 and the parabolic and elliptic *minimal* Martin boundaries.

*Remark* 4.1. By the parabolic Harnack inequality for $P^*$, we have for each $0 < \varepsilon < 1$

$$k_P^{\mathcal{M}}(x, y_0, t - \varepsilon) \leq C(y_0, \varepsilon) k_P^{\mathcal{M}}(x, y_0, t) \qquad \forall x \in \mathcal{M},\ t > 1. \tag{4.1}$$

Therefore, if $\{(y_0, t_n)\}$ is a nontrivial minimal Martin sequence with $t_n \to -\infty$, then one infers as in [10] that the corresponding minimal parabolic function in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$ is of the form $u(x,t) = \mathrm{e}^{-\lambda t} u_\lambda(x, y_0)$ with $\lambda \leq 0$ and $u_\lambda \in \operatorname{exr} \mathcal{C}_{P-\lambda}(\mathcal{M})$, where $\operatorname{exr} \mathcal{C}$ is the set of extreme rays of a cone $\mathcal{C}$. If further, for some $x_0 \in \mathcal{M}$ and $s < 0$ one has

$$\liminf_{t \to \infty} \frac{k_P^{\mathcal{M}}(x_0, y_0, t + s)}{k_P^{\mathcal{M}}(x_0, y_0, t)} \geq 1, \tag{4.2}$$

then $\lambda = 0$, and consequently, $u$ is also a minimal solution in $\mathcal{C}_P(\mathcal{M})$. Recall that in the selfadjoint case, the ratio *limit* in (4.2) equals 1.

**Lemma 4.1.** *Suppose that the ratio limit in* (2.8) *exists for all* $x, y \in \mathcal{M}$ *and* $s \in \mathbb{R}$. *Let* $a(x,y) := \lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x, y, t_n + s)}{k_P^{\mathcal{M}}(x_0, x_0, t_n)}$, *where* $t_n \to \infty$. *If for some* $y_0 \in \mathcal{M}$ *the function* $u(x) := a(x, y_0)$ *is minimal in* $\mathcal{C}_P(\mathcal{M})$, *then* $a(x,y) = u(x)v(y)$, *where* $v \in \mathcal{C}_{P^*}(\mathcal{M})$.

*Proof.* Fix $y \in \mathcal{M}$ and $\varepsilon > 0$. By the parabolic Harnack inequality for $P^*$ and Lemma 2.2, we have

$$\frac{k_P^{\mathcal{M}}(x, y, t - \varepsilon)}{k_P^{\mathcal{M}}(x_0, x_0, t)} \leq C(y, \varepsilon) \frac{k_P^{\mathcal{M}}(x, y_0, t)}{k_P^{\mathcal{M}}(x_0, x_0, t)} \qquad \forall x \in \mathcal{M}. \tag{4.3}$$

Therefore, $a(x,y) \leq C(y) u(x)$ which implies the claim. □

The following examples demonstrate that if Conjecture 1.1 holds true while (1.4) does not hold, then the limit function $a(\cdot, y)$ is typically a non-minimal solution in $\mathcal{C}_P(\mathcal{M})$.

*Example* 4.1. Consider a (regular) Benedicks domain $\mathcal{M} \subseteq \mathbb{R}^d$ such that the cone of positive harmonic functions which vanish on $\partial\mathcal{M}$ is of dimension two. By [6], Conjecture 1.1 holds true in this case, the limit function is not a product of two (separated) harmonic functions, and therefore, $a(\cdot, y)$ is not minimal in $\mathcal{C}_{-\Delta}(\mathcal{M})$ for any $y \in \mathcal{M}$.

*Example* 4.2. Consider a radially symmetric Schrödinger operator $H := -\Delta + V(|x|)$ on $\mathbb{R}^d$ with a bounded potential. Suppose that $\lambda_0 = 0$, and that the Martin boundary of $H$ on $\mathbb{R}^d$ is homeomorphic to $S^{d-1}$ (see [12]). Clearly, any Martin function corresponding to $\{(y_0, t_n)\}$ with $x_0 = y_0 = 0$ is radially symmetric. It follows that Davies' conjecture holds true for $x_0 = y = 0$, and the limit function is the normalized positive radial solution in $\mathcal{C}_H(\mathbb{R}^d)$. This solution is not minimal in $\mathcal{C}_H(\mathbb{R}^d)$. Thus, any limit function $u(\cdot, y)$ is not minimal in $\mathcal{C}_H(\mathbb{R}^d)$.

We conclude this section with some related problems. The following conjecture was posed by the author in [15, Conjecture 3.6].

**Conjecture 4.1.** *Suppose that $P$ is a critical operator in $\mathcal{M}$, then the ground state $\varphi$ is a minimal positive solution in the cone $\mathcal{H}_P(\mathcal{M} \times \mathbb{R})$.*

Note that if (2.11) holds true, then by Theorem 3.1, the ground state is a Martin function in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R})$.

*Example* 4.3. Consider again the example in [16, Section 4]. In that example $-\Delta$ is subcritical in $\mathcal{M}$, $\lambda_0 = 0$, and (1.4) and Conjecture 1.1 hold true. Hence, $\mathbf{1}$ is a Martin function in $\mathcal{H}_{-\Delta}(\mathcal{M} \times \mathbb{R})$. On the other hand, $\mathbf{1} \in \operatorname{exr} \mathcal{C}_{-\Delta}(\mathcal{M})$ but $\mathbf{1} \notin \operatorname{exr} \mathcal{H}_{-\Delta}(\mathcal{M} \times \mathbb{R})$. So, Conjecture 4.1 cannot be extended to the subcritical "Liouvillian" case (see also [4]).

Thus, it would be interesting to study the following problem which was raised by Burdzy and Salisbury [4] for $P = -\Delta$ and $\mathcal{M} \subset \mathbb{R}^d$.

**Question 4.1.** *Assume that $\lambda_0 = 0$. Determine which minimal positive solutions in $\mathcal{C}_P(\mathcal{M})$ are minimal in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$.*

## §5. Uniform Harnack inequality and Davies' conjecture

In this section we discuss the relationship between the parabolic Martin boundary of $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$, the elliptic Martin boundaries of $\mathcal{C}_{P-\lambda}(\mathcal{M})$, $\lambda \le \lambda_0 = 0$, and Conjecture 1.1 under a certain assumption.

**Definition 5.1.** We say that the *uniform restricted parabolic Harnack inequality* (in short, (URHI)) holds in $\mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$ if for any $\varepsilon > 0$ there exists a positive constant $C = C(\varepsilon) > 0$ such that

$$u(x, t - \varepsilon) \le C u(x, t) \quad \forall (x, t) \in \mathcal{M} \times \mathbb{R}_- \text{ and } \forall u \in \mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-). \tag{5.1}$$

It is well known that (URHI) holds true if and only if the *separation principle* (SP) holds true, that is, $u \neq 0$ is in $\operatorname{exr} \mathcal{H}_P(\mathcal{M} \times \mathbb{R}_-)$ if and only if $u$ is of the form $\mathrm{e}^{-\lambda t} v_\lambda(x)$, where $v_\lambda \in \operatorname{exr} \mathcal{C}_{P-\lambda}(\mathcal{M})$ [10, 13]. In particular, the answer to Question 4.1 is simple if (URHI) holds.

**Lemma 5.1.** (i) *Suppose that (URHI) holds true, then for any $s<0$*

$$\ell_+ := \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \leq 1 \qquad \textit{(Lin's condition).}$$

(ii) *Assume further that for some $x_0, y_0 \in \mathcal{M}$ and $s_0<0$*

$$\ell_- := \liminf_{t\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t)} \geq 1,$$

*then any limit function $u(x,y,s)$ of $\frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x_0,y_0,t_n)}$ with $t_n\to\infty$ does not depend on $s$, and has the form $u(x,y)$, where $u(\cdot,y)\in\mathcal{C}_P(\mathcal{M})$ for every $y\in\mathcal{M}$ and $u(x,\cdot)\in\mathcal{C}_{P^*}(\mathcal{M})$ for every $x\in\mathcal{M}$.*

(iii) *If one assumes further (1.4), then Conjecture 1.1 holds true.*

*Proof.* (i) By (URHI), if $u\in \operatorname{exr}\mathcal{H}_P(\mathcal{M}\times\mathbb{R}_-)$, then $u(x,t)=\mathrm{e}^{-\lambda t}u_\lambda(x)$, where $\lambda\leq 0$. Consequently, for every $u\in\mathcal{H}_P(\mathcal{M}\times\mathbb{R}_-)$

$$u(x,t+s)\leq u(x,t) \qquad \forall (x,t)\in\mathcal{M}\times\mathbb{R}_-, \text{ and } \forall s<0, \tag{5.2}$$

and equality holds for some $s<0$ and $(x,t)\in\mathcal{M}\times\mathbb{R}_-$ if and only if $u\in\mathcal{C}_P(\mathcal{M})$. Clearly, (5.2) implies that

$$\ell_+ := \limsup_{t\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t+s)}{k_P^{\mathcal{M}}(x,y,t)} \leq 1 \quad \forall x,y\in\mathcal{M} \text{ and } s<0,$$

which together with Lemma 2.1 implies $\ell_+=1$.

(ii) At the point $(x_0,y_0,s_0)$ we have $\ell_-=\ell_+=1$, therefore,

$$\lim_{t\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t)} = 1. \tag{5.3}$$

Consequently, for any sequence $t_k\to\infty$ satisfying

$$\lim_{k\to\infty} \frac{k_P^{\mathcal{M}}(x,y_0,t_k+\tau)}{k_P^{\mathcal{M}}(x_0,y_0,t_k)} = u(x,\tau) \quad \forall (x,\tau)\in\mathcal{M}\times\mathbb{R}_-,$$

we have $u(x_0,s_0)=u(x_0,2s_0)=1$, and therefore, $u\in\mathcal{C}_P(\mathcal{M})$. The other assertions of the lemma follow from Lemma 2.2. □

*Remark* 5.1. From the proof of Lemma 5.1 it follows that if (URHI) holds true, then a sequence $t_n\to\infty$ satisfies

$$\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t_n+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t_n)} = 1,$$

for some $x_0, y_0 \in \mathcal{M}$ and $s_0 \neq 0$ if and only if

$$\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t_n+s)}{k_P^{\mathcal{M}}(x,y,t_n)} = 1 \qquad \forall x,y \in \mathcal{M} \text{ and } s \in \mathbb{R}.$$

**Corollary 5.1.** *Suppose that (URHI) holds true, then there exists a sequence $t_n \to \infty$ such that $\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x,y,t_n)}{k_P^{\mathcal{M}}(x_0,x_0,t_n)} = a(x,y)$ exists and is positive for all $x,y \in \mathcal{M}$. Moreover, $a(\cdot,y) \in \mathcal{C}_P(\mathcal{M})$, and $a(\cdot,y)$ is a parabolic Martin function for all $y \in \mathcal{M}$. For each $x \in \mathcal{M}$ the function $a(x,\cdot)$ satisfies similar properties with respect to $P^*$.*

*Proof.* Take $s_0 \neq 0$ and $\{t_n\}$ such that $\lim_{n\to\infty} \frac{k_P^{\mathcal{M}}(x_0,y_0,t_n+s_0)}{k_P^{\mathcal{M}}(x_0,y_0,t_n)} = 1$, and use Remark 5.1 and a standard diagonalization argument. □

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# Some potential theoretic results on an infinite network

**Premalatha and A. K. Kalyani**

**Abstract.**

The greatest harmonic minorant of a superharmonic function is determined as the limit of a sequence of solutions for discrete Dirichlet problems on finite subnetworks. Without using the Green kernel explicitly, a positive superharmonic function is decomposed uniquely as a sum of a potential and a harmonic function. The infimum of a left directed family of harmonic functions is shown to be either $-\infty$ or harmonic. As applications, we study the reduced functions and their properties. We show the existence of the Green kernel with the aid of our reduced function.

## §1. Introduction

Let $N = \{X, Y, K, r\}$ be an infinite network which is connected and locally finite and has no self-loop. Here $X$ is a countable set of nodes, $Y$ a countable set of arcs, $K$ a node-arc incidence function and $r$ a strictly positive real function on $Y$.

We say that a network $N' = \{X', Y', K', r'\}$ is a subnetwork of $N$ if $X'$ and $Y'$ are subsets of $X$ and $Y$ respectively, $K'$ is the restriction of $K$ onto $X' \times Y'$ and $r'$ is the restriction of $r$ onto $Y'$. For simplicity, we write $N' =< X', Y' >$ in case $N' = \{X', Y', K', r'\}$ is a subnetwork of $N$. We say that $N' =< X', Y' >$ is a finite subnetwork of $N$ if $X'$ or $Y'$ is a finite set. For later use, we recall a notion of an exhaustion. We say that a sequence of finite subnetworks $\{N_n\}(N_n =< X_n, Y_n >)$ of $N$

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is an exhaustion of $N$ if

$$Y(x) := \{y \in Y; K(x,y) \neq 0\} \quad \subset \quad Y_{n+1} \text{ for all } x \in X_n,$$
$$X = \bigcup_{n=1}^{\infty} X_n \quad \text{and} \quad Y = \bigcup_{n=1}^{\infty} Y_n$$

Notice that $X_n \subset X_{n+1}$ and $Y_n \subset Y_{n+1}$. For notations and terminologies we mainly follow [2] and [3]. Let $L(X)$ be the set of all real functions on $X$ and $L^+(X)$ be the set of all non-negative functions on $X$. For $x \in X$, denote by $W_x$ the neighboring nodes of $x$, i.e.,

$$W_x = \{z \in X; K(x,y)K(z,y) = -1 \text{ for some } y \in Y(x)\}.$$

For every $u \in L(X)$, the Laplacian $\Delta u \in L(X)$ is defined by

$$\Delta u(x) = -t(x)u(x) + \sum\nolimits_{z \in W_x} t(x,z)u(z),$$

where

$$\begin{aligned} t(x) &= \sum\nolimits_{y \in Y} r(y)^{-1}|K(x,y)| \\ t(x,z) &= \sum\nolimits_{y \in Y} r(y)^{-1}|K(x,y)K(z,y)| \text{ for } z \neq x. \end{aligned}$$

Notice that $t(x,z) = t(z,x)$ and $t(x,z) = 0$ for $z \in X \setminus (W_x \cup \{x\})$

$$t(x) = \sum\nolimits_{z \in W_x} t(x,z).$$

We say that a function $u \in L(X)$ is superharmonic on a set $A \subseteq X$ if $\Delta u(x) \leq 0$ for all $x \in A$. We say that $u$ is subharmonic on $A$ if $-u$ is superharmonic on $A$. If $u$ is both superharmonic and subharmonic on $A$, we say that $u$ is harmonic on $A$. The following minimum principle and maximum principle are well-known:

**Lemma 1.1** (Minimum principle)**.** *Let $X'$ be a finite subset of $X$. If $u$ is superharmonic on $X'$ and $u(x) \geq 0$ on $X \setminus X'$, then $u(x) \geq 0$ on $X'$.*

**Lemma 1.2** (Maximum principle)**.** *Let $X'$ be a finite subset of $X$. If $u$ is subharmonic on $X'$ and $u(x) \leq 0$ on $X \setminus X'$, then $u(x) \leq 0$ on $X'$.*

**Lemma 1.3** (Harnack's principle)**.** *Let $\{X_n\}$ be a sequence of subsets of $X$ such that $X_n \subset X_{n+1}$ and $X = \cup_{n=1}^{\infty} X_n$ and let $\{u_n\}$ be a sequence of functions on $X$ such that $u_n(x) \leq u_{n+1}(x)$ on $X$. If $u_n$ is superharmonic on $X_n$ for every $n$, then the pointwise limit of $\{u_n\}$ is equal to either $\infty$ or a real valued superharmonic function.*

For a finite subnetwork $N' =< X', Y' >$ of $N$, the harmonic green function of $N'$ with pole at $a \in X'$ is the unique function $u$ determined by

$$\Delta u(x) = -\varepsilon_a(x) \text{ on } X' \text{ and } u(x) = 0 \text{ on } X \setminus X',$$

where $\varepsilon_a$ denotes the characteristic function of $\{a\}$. Denote by $g_a^{N'}$ the harmonic Green function of $N'$ with pole at $a$. Notice that $g_a^{N'}(b) = g_b^{N'}(a) > 0$ for all $a, b \in X'$(cf. [1]). For $f \in L(X)$, the Green potential $G_{N'}f$ is defined by

$$G_{N'}f(x) = \sum\nolimits_{z \in X'} g_z^{N'}(x) f(z).$$

## §2. The greatest harmonic minorant

We begin with a discrete Dirichlet problem:

**Lemma 2.1.** [1] *Let $f \in L(X)$ and $N' =< X', Y' >$ be a finite subnetwork of $N$. There exists a unique function $u'$ such that*

$$\Delta u'(x) = 0 \textit{ on } X' \textit{ and } u'(x) = f(x) \textit{ on } X \setminus X'.$$

*Proof.* The uniqueness follows from the maximum and minimum principles. We see easily that $u' = f + G_{N'}(\Delta f)$ satisfies our requirements. □

Denote by $h_f^{N'}$ the unique function $u'$ determined in Lemma 2.1.

**Corollary 2.1.** *Let $N' =< X', Y' >$ be a finite subnetwork of $N$. Then $h_{\alpha f + \beta g}^{N'} = \alpha h_f^{N'} + \beta h_g^{N'}$ for $f, g \in L(X)$ and real numbers $\alpha, \beta$.*

By Lemmas 1.1 and 1.2, we obtain

**Lemma 2.2.** *Let $N' =< X', Y' >$ be a finite subnetwork of $N$.*

(1) *If $u$ is superharmonic on $X'$, then $h_u^{N'}(x) \leq u(x)$ on $X$.*
(2) *If $u$ is subharmonic on $X'$, then $h_u^{N'}(x) \geq u(x)$ on $X$.*

**Corollary 2.2.** *If $u$ is harmonic on $X'$, then $h_u^{N'} = u$.*

**Lemma 2.3.** *Let $N' =< X', Y' >$ be a finite subnetwork of $N$ and $u_1, u_2 \in L(X)$. If $u_1(x) \leq u_2(x)$ on $X$, then $h_{u_1}^{N'}(x) \leq h_{u_2}^{N'}(x)$ on $X$.*

*Proof.* Let $v(x) = h_{u_2}^{N'}(x) - h_{u_1}^{N'}(x)$. Then $v$ is harmonic on $X'$ and $v(x) = u_2(x) - u_1(x) \geq 0$ on $X \setminus X'$. By the minimum principle,$v(x) \geq 0$ on $X'$. Hence $v(x) \geq 0$ on $X$. □

**Lemma 2.4.** *Let $N' =< X', Y' >$ be a finite subnetwork of $N$. If $u$ is a superharmonic function on $X$, then $h_u^{N'}$ is superharmonic on $X$.*

*Proof.* By Lemma 2.2, $h_u^{N'}(x) \leq u(x)$ on $X$. It suffices to show that $h_u^{N'}(x)$ is superharmonic on $X \setminus X'$. For $x \in X \setminus X'$, we have $h_u^{N'}(x) = u(x)$ and

$$\begin{aligned}\Delta h_u^{N'}(x) &= -t(x)h_u^{N'}(x) + \sum\nolimits_{z \in W_x} t(x,z)h_u^{N'}(z) \\ &\leq -t(x)u(x) + \sum\nolimits_{z \in W_x} t(x,z)u(z) = \Delta u(x) \leq 0.\end{aligned}$$

Therefore $u$ is superharmonic on $X$. □

**Lemma 2.5.** *Let $N_1 =< X_1, Y_1 >$ and $N_2 =< X_2, Y_2 >$ be finite subnetworks of $N$ such that $Y(x) \subset Y_2$ for all $x \in X_1$. If $u$ is superharmonic on $X$, then $h_u^{N_1}(x) \geq h_u^{N_2}(x)$ on $X$.*

*Proof.* Let $v(x) = h_u^{N_1}(x) - h_u^{N_2}(x)$. Then $v(x) = u(x) - h_u^{N_2}(x) \geq 0$ on $X \setminus X_1$ and $\Delta v(x) = 0$ on $X_1$. Therefore $v(x) \geq 0$ on $X$ by the minimum principle. □

**Theorem 2.1.** *Let $u$ be superharmonic on $X$ and $\{N_n\}$ be an exhaustion of $N$ and put*

$$\pi_u(x) = \lim_{n\to\infty} h_u^{N_n}(x) \text{ for each } x \in X.$$

*Then either $\pi_u = -\infty$ or $\pi_u \in L(X)$ is harmonic on $X$.*

*Proof.* Put $u_n = h_u^{N_n}$. Then $u_{n+1}(x) \leq u_n(x) \leq u(x)$ on $X$ and $u_n$ is harmonic on $X_n$. By Harnack's principle, we see that the limit $v$ of the sequence $\{-u_n\}$ is equal to either $\infty$ or a real valued superharmonic function on $X$. In case $v = \infty$, we have $\pi_u = -\infty$. Assume that $v \neq \infty$. Then we see $\pi_u = -v \in L(X)$ and $\Delta\pi_u(x) \geq 0$ on $X$. Let $x \in X$. Since $N$ is locally finite, there exists $n_0$ such that $W_x \cup \{x\} \subset X_n$ for all $n \geq n_0$. Since $u_n$ is harmonic on $X_n$ and $u_n(z) \to \pi_u(z)$ for all $z \in W_x \cup \{x\}$ as $n \to \infty$, we have

$$\begin{aligned}\Delta\pi_u(x) &= -t(x)\pi_u(x) + \sum\nolimits_{z \in W_x} t(x,z)\pi_u(z) \\ &= \lim_{n\to\infty}\{-t(x)u_n(x) + \sum\nolimits_{z \in W_x} t(x,z)u_n(z)\} \\ &= \lim_{n\to\infty} \Delta u_n(x) = 0.\end{aligned}$$

In case $\pi_u \in L(X)$, we call $\pi_u$ the harmonic part of $u$. Notice that $\pi_u$ does not depend on the choice of an exhaustion of $N$ and that $\pi_u(x) \leq u(x)$ on $X$. □

**Proposition 2.1.** *Let $u_1, u_2$ be superharmonic functions on $X$. If there exists a subharmonic minorant $v$ of* $\min(u_1, u_2)$*, then*

$$\pi_{\min(u_1,u_2)}(x) \leq \min(\pi_{u_1}(x), \pi_{u_2}(x)) \quad \textit{on } X.$$

*Proof.* Let $u = \min(u_1, u_2)$ and $\{N_n\}(N_n =< X_n, Y_n >)$ be an exhaustion of $N$. Then $u$ is superharmonic and

$$v(x) \leq h_u^{N_n}(x) \leq h_{u_k}^{N_n}(x) \ \text{ on } X \text{ for } k = 1, 2.$$

Therefore $\pi_u(x) \leq \pi_{u_k}(x)$ on $X$ for $k = 1, 2$. □

**Proposition 2.2.** *Let $u_1$ and $u_2$ be superharmonic functions on $X$. If they have subharmonic minorants, then $\pi_{u_1+u_2} = \pi_{u_1} + \pi_{u_2}$.*

*Proof.* Let $N_n$ be the same as above. We have by Corollary 2.1

$$h_{u_1+u_2}^{N_n} = h_{u_1}^{N_n} + h_{u_2}^{N_n}.$$

□

**Corollary 2.3.** *Let $u$ be a superharmonic function on $X$ with a subharmonic minorant and let $\phi$ be a harmonic function on $X$. Then $\pi_{u+\phi} = \pi_u + \phi$.*

**Theorem 2.2.** *Let $u$ be superharmonic on $X$. If $u$ has a subharmonic minorant $v$, i.e., $v$ is subharmonic on $X$ and $v(x) \leq u(x)$ on $X$, then $v(x) \leq \pi_u(x)$ on $X$. Moreover, $\pi_u$ is the greatest harmonic minorant of $u$.*

*Proof.* Let $\{N_n\}(N_n =< X_n, Y_n >)$ be an exhaustion of $N$. Since $v$ is subharmonic on $X$ and $v(x) \leq u(x)$ on $X$, we have

$$v(x) \leq h_v^{N_n}(x) \leq h_u^{N_n}(x) \text{ on } X$$

by Lemmas 2.2 and 2.3. Thus we have $v(x) \leq \pi_u(x)$ on $X$. If $s$ is a harmonic minorant of $u$, then we have $s(x) = h_s^{N_n}(x) \leq h_u^{N_n}(x)$ on $X$ by Corollary 2.2 and Lemma 2.3, so that $s(x) \leq \pi_u(x)$ on $X$. □

There are many characterizations for an infinite network $N$ to be of hyperbolic type. We say here that $N$ is of hyperbolic type (or shortly, hyperbolic) if there exists a nonconstant positive superharmonic function on $X$. It is well-known that $N$ is hyperbolic if and only if $N$ has a Green function, i.e., the limit $g_a$ of $\{g_a^{N_n}\}$ exists and satisfies the condition:$\Delta g_a(x) = -\varepsilon_a(x)$ on $X$.

Without using this Green kernel explicitly, we introduce

**Definition 2.1.** *We say that a positive superharmonic function $u$ is a potential if the greatest harmonic minorant of $u$ is zero, i.e., $\pi_u = 0$.*

Needless to say, we have $\pi_u \in L(X)$ if $u \in L^+(X)$ is superharmonic on $X$.

**Theorem 2.3.** *Let $N$ be hyperbolic.*

(1) *If $u$ is a potential, then $\lambda u$ $(\lambda > 0)$ is also a potential.*
(2) *If $u_1$ and $u_2$ are potentials, then $u_1 + u_2$ is also a potential.*
(3) *If $u_1$ is a potential and $u_2$ is a positive superharmonic function, then* $\min(u_1, u_2)$ *is a potential.*

*Proof.* (2) and (3) follow from Propositions 2.1 and 2.2. For (1), it suffices to note that $\pi_{\lambda u} = \lambda \pi_u$. □

**Theorem 2.4.** *Let $N$ be hyperbolic.*

(1) *Assume that $v$ is superharmonic on $X$ and $u$ is a potential. If $u + v \in L^+(X)$, then $v \in L^+(X)$.*
(2) *If $u$ is a potential and if $v$ is a subharmonic minorant of $u$, then $v \leq 0$.*
(3) *Assume that $u$ is a superharmonic function with a subharmonic minorant $v$. Then $u$ can be expressed uniquely as the sum of a potential and a harmonic function.*

*Proof.* Since $u \geq -v$ and $-v$ is subharmonic, we have $0 = \pi_u(x) \geq \pi_{-v}(x) \geq -v(x)$ on $X$. Thus (1) follows. The second assertion follows from the relation: $v(x) \leq \pi_v(x) \leq \pi_u(x) = 0$ on $X$. Let us prove (3). Since $u$ has a subharmonic minorant, we have $\pi_u \in L(X)$ is harmonic. We take $p = u - \pi_u$. Then $p \in L^+(X)$ and $\pi_p = 0$ by Corollary 2.3. Therefore $p$ is a potential. Assume that there exist potentials $p_1, p_2$ and harmonic functions $h_1, h_2$ satisfying the relation: $u = p_1 + h_1 = p_2 + h_2$. We have

$$p_1(x) \geq p_1(x) - p_2(x) = h_2(x) - h_1(x)$$

for all $x \in X$. We see by the above observation (2) that $h_2(x) - h_1(x) \leq 0$ on $X$. We obtain similarly $h_1(x) - h_2(x) \leq 0$ on $X$, and hence $h_1(x) = h_2(x)$. This shows the uniqueness of our decomposition. □

## §3. Sets of Superharmonic Functions

We say that a set $\Phi$ of functions on $X$ is left directed if for every $u_1, u_2 \in \Phi$, there exists $u \in \Phi$ such that $u \leq \min(u_1, u_2)$. We define $\inf \Phi$ by

$$\inf \Phi(x) = \inf\{u(x); u \in \Phi\}.$$

For simplicity, we set $X(a) = W_a \cup \{a\}$ for $a \in X$.

**Theorem 3.1.** *If $\Phi$ is a left directed family of harmonic functions on $X$, then* $\inf \Phi$ *is either equal to $-\infty$ identically or harmonic on $X$.*

*Proof.* For simplicity, put $h = \inf \Phi$. It suffices to show that $h$ is harmonic on $X$ unless $h = -\infty$. Let $a$ be any node such that $h(a) > -\infty$. Since $X(a)$ is a finite set, we can find a sequence $\{u_n\}$ in $\Phi$ such that $u_{n+1}(x) \le u_n(x)$ on $X$ and $u_n(x) \to h(x)$ as $n \to \infty$ for every $x \in X(a)$. Since $u_n(a) = \frac{\sum_{x \in W_a} t(x,a) u_n(x)}{t(a)}$, we have $h(a) = \frac{\sum_{x \in W_a} t(x,a) h(x)}{t(a)}$. Since $h(a) > -\infty$, we see that $h(x) > -\infty$ for all $x \in W(a)$ and $h$ is harmonic at $a$. Taking $b \in W(a)$ and proceeding as before we get $h(x) > -\infty$ for all $x \in W(b)$ and $h$ is harmonic at $b$. Since any point $z \in X$ is connected to $a$ by a finite number of edges we get $h(z) > -\infty$ and $h$ is harmonic at $z$. Hence we have $h$ is harmonic on $X$. □

Similarly we can prove

**Theorem 3.2.** *If $\Phi$ is a left directed family of superharmonic functions on $X$ and* $\inf \Phi \in L(X)$*, then* $\inf \Phi$ *is superharmonic on $X$.*

Let us use a discrete analogue of Poisson's integral. For $u \in L(X)$ and $a \in X$, we define the function $P_a u \in L(X)$ by

$$\begin{aligned} P_a u(x) &= u(x) \text{ if } x \neq a \\ P_a u(a) &= \sum\nolimits_{x \in X} [t(a,x)/t(a)] u(x). \end{aligned}$$

**Lemma 3.1.** *Assume that $u$ is superharmonic on $X$. Then $P_a u(x) \le u(x)$ on $X$ and $P_a u$ is superharmonic on $X$ and harmonic at $a$.*

*Proof.* Since $u$ is superharmonic at $a$, $P_a u(a) \le u(a)$, so that $P_a u(x) \le u(x)$ on $X$. For $x \notin X(a)$, it is clear that $P_a u$ is superharmonic at $x$. For $x \in W_a$, we have

$$\begin{aligned} \Delta P_a u(x) &= -t(x) P_a u(x) + \sum\nolimits_{z \in W_x} t(z,x) P_a u(z) \\ &\le -t(x) u(x) + \sum\nolimits_{z \in W_x} t(z,x) u(z) = \Delta u(x) \le 0. \end{aligned}$$

For $x = a$, we have

$$\Delta P_a u(a) = -t(a) P_a u(a) + \sum\nolimits_{z \in W_a} t(z,a) u(z) = 0.$$

□

**Theorem 3.3.** *Let $A$ be a subset of $X$ and $\Phi$ be a left directed family of superharmonic functions on $X$. If* $\inf \Phi \in L(X)$ *and $P_a u \in \Phi$ for all $a \in A$ and $u \in \Phi$, then* $\inf \Phi$ *is harmonic on $A$.*

*Proof.* Let us put $h = \inf \Phi$. Then $h$ is superharmonic on $X$ by Theorem 3.2. Let $a \in A$. Then $P_a h(x) \leq h(x)$ by Lemma 3.1. By our assumption, we have $h(a) \leq P_a u(a)$ for all $u \in \Phi$. There exists a sequence $\{u_n\}$ in $\Phi$ such that $u_n(x) \to h(x)$ as $n \to \infty$ for all $x \in X(a)$. We see easily that $P_a u_n(a) \to P_a h(a)$ as $n \to \infty$, so that $h(a) \leq P_a h(a)$. Namely, $h(a) = P_a h(a)$, i.e., $\Delta h(a) = 0$. $\square$

## §4. Reduced Functions and their properties

In this section, we always assume that $N$ is hyperbolic. Denote by $SH^+(N)$ the set of all non-negative superharmonic functions on $X$. For $f \in L^+(X)$, let us put $\mathcal{S}_f = \{u \in SH^+(N); u(x) \geq f(x) \text{ on } X\}$ and

$$R_f(x) = \inf\{u(x); u \in \mathcal{S}_f\}.$$

**Theorem 4.1.** *The function $R_f$ is superharmonic on $X$ and harmonic on the set $\{x \in X; f(x) = 0\}$.*

*Proof.* We show that $\mathcal{S}_f$ is left directed. Let $u_1, u_2 \in \mathcal{S}_f$ and $u_3(x) = \min\{u_1(x), u_2(x)\}$ for $x \in X$. Then $u_3 \in SH^+(N)$ and $u_3(x) \geq f(x)$ on $X$. Thus $u_3 \in \mathcal{S}_f$. Since $R_f(x) \geq f(x) \geq 0$ on $X$, we see by Theorem 3.2 that $R_f$ is superharmonic on $X$. Let $A = \{x \in X; f(x) = 0\}$. For any $u \in \mathcal{S}_f$, we see by Lemma 3.1 that $P_a u$ is superharmonic and $P_a u(x) = u(x) \geq f(x)$ for $x \neq a$. If $a \in A$, then $P_a u(a) \geq 0 = f(a)$. Therefore $P_a u \in \mathcal{S}_f$ for all $u \in \mathcal{S}_f$ and $a \in A$. Our assertion follows from Theorem 3.3. $\square$

Let $u \in L^+(X)$ and $A$ be a subset of $X$. The function

$$R_u^A(x) = \inf\{v(x); v \in SH^+(N), v(x) \geq u(x) \text{ on } A\}$$

is called the reduced function ( or balayage) of $u$ on $A$.

**Theorem 4.2.** *$R_u^A$ is superharmonic in $X$ and harmonic in $X \setminus A$.*

*Proof.* Consider the function $f \in L^+(X)$ defined by $f(x) = u(x)$ for $x \in A$ and $f(x) = 0$ for $x \in X \setminus A$. Then $R_u^A = R_f$ and our assertion follows from Theorem 4.1. $\square$

**Lemma 4.1.** *If $N$ is hyperbolic, there exists a potential $p$ such that $p(x) > 0$ on $X$.*

*Proof.* By our definition, there exists a non-constant positive superharmonic function $v$. Our assertion is clear if $v$ is not harmonic by Theorem 2.4. Assume that $v$ is harmonic on $X$. For $a \in X$, we consider the function $s_a \in L(X)$ defined by $s_a(x) = \min(v(x), v(a))$ for $x \in X$. Then $s_a \in SH^+(N)$, $s_a(x) \le s_a(a) = v(a)$ on $X$. If $\Delta s_a(a) = 0$, then

$$\sum\nolimits_{x \in W_a} t(x,a)[s_a(a) - s_a(x)] = 0$$

implies that $s_a(x) = s_a(a)$ on $X(a)$, i.e., $v(x) \ge v(a)$ on $X(a)$. Since $v$ is harmonic, we must have $v(x) = v(a)$ on $X(a)$. Taking $a_1 \in X(a), a \ne a_1$, we consider $s_{a_1} = \min(v, v(a_1))$. If $\Delta s_{a_1}(a_1) = 0$, we obtain $v(x) = v(a)$ on $X(a) \cup X(a_1)$. After repeating this procedure a finite number of times, we obtain $b \in X$ such that $s_b = \min(v, v(b))$ and $\Delta s_b(b) < 0$, since $v$ is non-constant. □

**Theorem 4.3.** *For any $a \in X$, there exists a unique bounded potential $G_a(x)$ such that $\Delta G_a(x) = -\varepsilon_a(x)$.*

*Proof.* We see by Theorems 4.1 and 4.2 that $u_a(x) = R_{\varepsilon_a} = R_1^{\{a\}}$ is superharmonic on $X$ and harmonic on $X \setminus \{a\}$. Since $1 \in \mathcal{S}_{\varepsilon_a}$, we have $0 \le u_a(x) \le 1$ on $X$. Since $N$ is hyperbolic, there exists a potential $p > 0$ by Lemma 4.1. Notice that $v(x) = p(x)/p(a) \in \mathcal{S}_{\varepsilon_a}$ and $v$ is also a potential. Thus $u_a(x) \le v(x)$ on $X$ and $u_a$ is a potential by Theorem 2.3. We show that $\Delta u_a(a) < 0$. Supposing the contrary, $u_a$ is harmonic on $X$. Since $u_a$ is a potential, we must have $u_a = 0$. On the other hand, we have $u_a(a) = 1$. This is a contradiction. Let us put $G_a(x) = -u_a(x)/\Delta u_a(a)$. Then $G_a$ is a bounded potential and $\Delta G_a(x) = -\varepsilon_a(x)$ on $X$.

We prove the uniqueness of $G_a$. Assume that there exists a potential $\phi$ such that $\Delta\phi(x) = -\varepsilon_a(x)$ on $X$. Let $h = \phi - G_a$. Then $\Delta h(x) = \Delta\phi(x) - \Delta G_a(x) = 0$ on $X$. Hence $h$ is harmonic on $X$ and $\phi = G_a + h$. By the uniqueness of the Riesz decomposition (Theorem 2.4(3)), we conclude that $h = 0$. Therefore $\phi = G_a$. □

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# The Littlewood-Paley Inequalities for Hardy-Orlicz Spaces of Harmonic Functions on Domains in $\mathbb{R}^n$

**Manfred Stoll**

**Abstract.**

For the unit disc $\mathbb{D}$ in $\mathbb{C}$, the harmonic Hardy spaces $\mathcal{H}^p$, $1 \le p < \infty$, are defined as the set of harmonic functions $h$ on $\mathbb{D}$ satisfying

$$\|h\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p d\theta < \infty.$$

The classical Littlewood-Paley inequalities for harmonic functions [3] in $\mathbb{D}$ are as follows: Let $h$ be harmonic on $\mathbb{D}$. Then there exist positive constants $C_1$, $C_2$, independent of $h$, such that

(a) for $1 < p \le 2$,

$$\|h\|_p^p \le C_1 \left[ |h(0)|^p + \iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \right].$$

(b) For $p \ge 2$, if $h \in \mathcal{H}^p$, then

$$\iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \le C_2 \|h\|_p^p.$$

In the paper we consider generalizations of these inequalities to Hardy-Orlicz spaces $\mathcal{H}_\psi$ of harmonic functions on domains $\Omega \subsetneq \mathbb{R}^n$, $n \ge 2$, with Green function $G$ satisfying the following: There exist constants $\alpha$ and $\beta$, $0 < \beta \le 1 \le \alpha < \infty$, such that for fixed $t_o \in \Omega$, there exist constants $C_1$ and $C_2$, depending only on $t_o$, such that $C_1 \delta(x)^\alpha \le G(t_o, x)$ for all $x \in \Omega$, and $G(t_0, x) \le C_2 \delta(x)^\beta$ for all $x \in \Omega \setminus B(t_o, \frac{1}{2}\delta(t_o))$.

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## §1. Introduction

For the unit disc $\mathbb{D}$ in $\mathbb{C}$, the harmonic Hardy spaces $\mathcal{H}^p$, $1 \le p < \infty$, are defined as the set of harmonic functions $h$ on $\mathbb{D}$ satisfying

$$\|h\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p d\theta < \infty.$$

The classical Littlewood-Paley inequalities for harmonic functions [3] in $\mathbb{D}$ are as follows: Let $h$ be harmonic on $\mathbb{D}$. Then there exist positive constants $C_1$, $C_2$, independent of $h$, such that

(a) for $1 < p \le 2$,

$$\|h\|_p^p \le C_1 \left[ |h(0)|^p + \iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \right]. \tag{1.1}$$

(b) For $p \ge 2$, if $h \in \mathcal{H}^p$, then

$$\iint_{\mathbb{D}} (1-|z|)^{p-1} |\nabla h(z)|^p dx\, dy \le C_2 \|h\|_p^p. \tag{1.2}$$

In 1956 T. M. Flett [2] proved that for analytic functions inequality (1.1) is valid for all $p$, $0 < p \le 2$. Hence if $u = \operatorname{Re} h$, $h$ analytic, then since $|\nabla u| = |h'|$ it immediately follows that inequality (1.1) also holds for harmonic functions in $\mathbb{D}$ for all $p$, $0 < p \le 2$. A short proof of the Littlewood-Paley inequalities for harmonic functions in $\mathbb{D}$ valid for all $p$, $0 < p < \infty$ has also been given recently by Pavlović in [5]. The Littlewood-Paley inequalities are also known to be valid for harmonic functions in the unit ball in $\mathbb{R}^n$. In fact Stević [7] has recently proved that for $n \ge 3$, inequality (1.1) is valid for all $p \in [\frac{n-2}{n-1}, 1]$. In [10] analogue's of the Littlewood-Paley inequalities have been proved by the author for domains $\Omega$ in $\mathbb{R}^n$ for which the Green function satisfies $G(t_o, x) \approx \delta(x)$ for all $x \in \Omega \setminus B(t_o \frac{1}{2}\delta(t_o))$, where $\delta(x)$ denotes the distance from $x$ to the boundary of $\Omega$. In the same paper it was proved that for bounded domains with $C^{1,1}$ boundary the analogue of (1.1) is also valid for all $p$, $0 < p \le 1$.

In the present paper we extend the Littlewood-Paley inequalities to harmonic functions in the Hardy–Orlicz spaces $\mathcal{H}_\psi$ on domains $\Omega \subsetneq \mathbb{R}^n$, $n \ge 2$, with Green function $G$ satisfying the following conditions: There exist constants $\alpha$ and $\beta$, $0 < \beta \le 1 \le \alpha < \infty$, such that for fixed $t_o \in \Omega$, there exist constants $C_1$ and $C_2$, depending only on $t_o$, such that

$$C_1 \delta(x)^\alpha \le G(t_o, x) \qquad \text{for all } x \in \Omega, \text{ and} \tag{1.3}$$

$$G(t_0, x) \le C_2 \delta(x)^\beta \qquad \text{for all } x \in \Omega \setminus B(t_o, \tfrac{1}{2}\delta(t_o))^1. \tag{1.4}$$

Let $\Omega$ be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$, and let $\psi$ be a non-negative increasing convex function on $[0, \infty)$ satisfying $\psi(0) = 0$ and

$$\psi(2x) \leq c\,\psi(x) \tag{1.5}$$

for some positive constant $c$. We denote by $\mathcal{H}_\psi(\Omega)$ the set of real or complex valued harmonic functions $h$ on $\Omega$ for which $\psi(|h|)$ has a harmonic majorant on $\Omega$. Since $\psi$ is convex and increasing, the function $\psi(|h|)$ is subharmonic on $\Omega$. The existence of a harmonic majorant consequently guarantees the existence of a least harmonic majorant. For $h \in \mathcal{H}_\psi$ we denote the least harmonic majorant of $\psi(|h|)$ by $H^h_\psi$, and for fixed $t_o \in \Omega$ we set

$$N_\psi(h) = H^h_\psi(t_o). \tag{1.6}$$

It is known that $N_\psi(h)$ is given by

$$N_\psi(h) = \lim_{n\to\infty} \int_{\partial\Omega_n} \psi(|h(t)|)\,d\omega_n^{t_o}(t), \tag{1.7}$$

where $\{\Omega_n\}$ is a regular exhaustion of $\Omega$ and $\omega_n^{t_o}$ is the harmonic measure on $\partial\Omega_n$ with respect to the point $t_o$. Here we assume that $t_o \in \Omega_n$ for all $n$. With $\psi(t) = t^p$, $1 \leq p < \infty$, one obtains the usual Hardy $\mathcal{H}^p$ space of harmonic functions on $\Omega$, with

$$\|h\|_p = \lim_{n\to\infty} \left( \int_{\partial\Omega_n} |h(t)|^p d\omega_n^{t_o}(t) \right)^{1/p}, \tag{1.8}$$

which is the usual norm on $\mathcal{H}^p(\Omega)$, $p \geq 1$.

In the paper we prove the following generalizations of the Littlewood-Paley inequalities.

**Theorem 1.** *Let $\Omega \subsetneq \mathbb{R}^n$ be a domain with Green function $G$ satisfying inequalities* (1.3) *and* (1.4). *Let $\psi \geq 0$ be an increasing convex $C^2$ function on $[0, \infty)$ with $\psi(0) = 0$ satisfying* (1.5). *Set $\varphi(t) = \psi(\sqrt{t})$. Then there exist positive constants $C_1$ and $C_2$ such that the following hold for all $h \in \mathcal{H}_\psi(\Omega)$.*

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[1]As in [1] [4], if $\Omega$ is a bounded $k$-Lipschitz domain, then such constants $\alpha$ and $\beta$ exist. If the boundary of $\Omega$ is $C^2$ or $C^{1,1}$, then $\alpha = \beta = 1$, and the inequalities can be established by comparing the Green function $G$ to the Green function of balls that are internally and externally tangent to the boundary of $\Omega$. By the results of Widman [11], the inequalities are also valid with $\alpha = \beta = 1$ for domains with $C^{1,\alpha}$ or Liapunov-Dini boundaries.

(a) *If $\varphi$ is concave on $[0,\infty)$, then*

$$\mathcal{N}_\psi(h) \le C_1 \left[\psi(|h(t_o)|) + \int_\Omega \delta(x)^{\beta-2}\psi(\delta(x)|\nabla h(x)|)\, dx\right].$$

(b) *If $\varphi$ is convex on $[0,\infty)$, then*

$$\psi(|h(t_o)|) + \int_\Omega \delta(x)^{\alpha-2}\psi(\delta(x)|\nabla h(x)|)\, dx \le C_2\mathcal{N}_\psi(h).$$

An immediate consequence of the previous theorem with $\psi(t) = t^p$, $1 \le p < \infty$, is the following:

**Theorem 2.** *Let $\Omega \subsetneq \mathbb{R}^n$ be a domain with Green function $G$ satisfying inequalities* (1.3) *and* (1.4), *and let $1 \le p < \infty$. Then there exist positive constants $C_1$ and $C_2$ such that the following hold for all $h \in \mathcal{H}^p(\Omega)$.*

(a) *For $1 \le p \le 2$,*

$$\|h\|_p^p \le C_1 \left[|h(t_o)|^p + \int_\Omega \delta(x)^{\beta+p-2}|\nabla h(x)|^p\, dx\right].$$

(b) *For $2 \le p < \infty$,*

$$|h(t_o)|^p + \int_\Omega \delta(x)^{\alpha+p-2}|\nabla h(x)|^p\, dx \le C_2\|h\|_p^p.$$

## §2. Preliminaries

Our setting throughout the paper is $\mathbb{R}^n$, $n \ge 2$, the points of which are denoted by $x = (x_1, ..., x_n)$ with euclidean norm $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. For $r > 0$ and $x \in \mathbb{R}^n$, set $B_r(x) = B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $S_r(x) = S(x,r) = \{y \in \mathbb{R}^n : |x-y| = r\}$. For convenience we denote the ball $B(0,\rho)$ by $B_\rho$, and the unit sphere $S_1(0)$ by $S$. Lebesgue measure in $\mathbb{R}^n$ will be denoted by $d\lambda$ or simply $dx$, and the normalized surface measure on $S$ by $d\sigma$. The volume of the unit ball $B_1$ in $\mathbb{R}^n$ will be denoted by $\omega_n$. For an integrable function $f$ on $\mathbb{R}^n$ we have

$$\int_{\mathbb{R}^n} f(x)dx = n\omega_n \int_0^\infty r^{n-1}\int_S f(r\zeta)\, d\sigma(\zeta)\, dr.$$

Finally, for a real (or complex) valued $C^1$ function $f$, the gradient of $f$ is denoted by $\nabla f$, and if $f$ is $C^2$, the Laplacian $\Delta f$ of $f$ is given by

$$\Delta f = \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}.$$

Let $\Omega$ be an open subset of $\mathbb{R}^n$, $n \geq 2$, with $\Omega \subsetneq \mathbb{R}^n$. For $x \in \Omega$, let $\delta(x)$ denote the distance from $x$ to the boundary of $\Omega$, and set

$$B(x) = B(x, \tfrac{1}{2}\delta(x)) = \left\{ y \in \Omega : |y - x| < \tfrac{1}{2}\delta(x) \right\}. \tag{2.1}$$

Then for all $y \in B(x)$ we have

$$\tfrac{1}{2}\delta(x) \leq \delta(y) \leq \tfrac{3}{2}\delta(x). \tag{2.2}$$

For the proof of Theorem 1 we require several preliminary lemmas.

**Lemma 1.** *For $f \in L^1(\Omega)$ and $\gamma \in \mathbb{R}$,*

$$\int_\Omega \delta(x)^\gamma |f(x)|\, dx \approx \int_\Omega \delta(w)^{\gamma - n} \left[ \int_{B(w)} |f(x)|\, dx \right] dw.$$

**Note.** The notation $A \approx B$ means that there exist constants $c_1$ and $c_2$ such that $c_1 A \leq B \leq c_2 A$.

*Proof.* The proof is a straightforward application of Tonelli's theorem, and consequently is omitted. Details may be found in [10]. □

**Lemma 2.** *For $u \in C^2(\overline{B_\rho})$, $\rho > 0$,*

$$\int_S u(\rho\zeta)\, d\sigma(\zeta) = u(0) + \int_{B_\rho} \Delta u(x) G_\rho(x)\, dx,$$

*where*

$$G_\rho(x) = \begin{cases} \dfrac{1}{n(n-2)\omega_n} \left[ \dfrac{1}{|x|^{n-2}} - \dfrac{1}{\rho^{n-2}} \right], & 0 < |x| \leq \rho, \quad n \geq 3, \\ \dfrac{1}{2\pi} \log \dfrac{\rho}{|x|}, & 0 < |x| \leq \rho, \quad n = 2, \end{cases} \tag{2.3}$$

*is the Green function of $B_\rho$ with singularity at* $0$.

*Proof.* The proof is an immediate consequence of Green's formula and hence is omitted. □

**Lemma 3.** *Let $\varphi$ be an increasing absolutely continuous function on $[0, \infty)$ with $\varphi(0) = 0$.*

(a) *If $\varphi$ is convex, then $\varphi(x) + \varphi(y) \leq \varphi(x + y)$ for all $x$, $y \in [0, \infty)$.*

(b) *If $\varphi$ is concave, then $\varphi(x) + \varphi(y) \geq \varphi(x + y)$ for all $x$, $y \in [0, \infty)$.*

*Proof.* (a) Suppose $\varphi$ is convex. Since $\varphi$ is absolutely continuous and increasing, $\varphi(x) = \int_0^x \varphi'$ where $\varphi' \geq 0$. Hence

$$\varphi(x+y) = \int_0^{x+y} \varphi' = \varphi(x) + \int_x^{x+y} \varphi'.$$

But

$$\int_x^{x+y} \varphi'(t)dt = \int_0^y \varphi'(x+t)dt.$$

Since $\varphi$ is convex, $\varphi'$ is increasing. Thus

$$\int_0^y \varphi'(x+t)dt \geq \int_0^y \varphi'(t)dt = \varphi(y),$$

from which the result follows. The proof of (b) is similar. □

**Lemma 4.** *Suppose $\varphi$ is an increasing $C^2$ function on $(0,\infty)$ with $\varphi(0) = 0$ and*

$$2t\varphi''(t) + \varphi'(t) \geq 0, \quad t > 0. \tag{2.4}$$

*Let $h$ be a harmonic function on $\overline{B}_\rho$, $\rho > 0$.*

*(a) If $\varphi$ is concave, then*

$$\int_{B_{\rho/4}} \rho^2 \Delta\varphi(|h|^2)dx \leq C \int_{B_\rho} \varphi(\rho^2|\nabla h|^2)dx.$$

*(b) If $\varphi$ is convex and satisfies inequality (1.5), then*

$$\int_{B_\rho} \rho^2 \Delta\varphi(|h|^2)dx \geq C \int_{B_{\rho/2}} \varphi(\rho^2|\nabla h|^2)dx.$$

**Remark.** If $u$ is a positive real-valued $C^2$ function, then

$$\Delta\varphi(u^2) = 2|\nabla u|^2 \left[2\varphi''(u^2)u^2 + \varphi'(u^2)\right] + 2\varphi'(u^2)u\Delta u.$$

Thus the hypothesis $2t\varphi''(t) + \varphi'(t) \geq 0$ guarantees that $\varphi(u^2)$ is subharmonic whenever $u$ is subharmonic. For $\psi_p(t) = t^p$, the function $\varphi_p(t) = \psi_p(\sqrt{t}) = t^{p/2}$ satisfies inequality (2.4) if and only if $p \geq 1$.

*Proof.* We only prove the Lemma for $n \geq 3$, the special case $n = 2$ is similar. (a) Suppose $\varphi$ is concave. Set $\epsilon = \rho/4$, $\delta = \rho/2$, and let $G_\delta$ be the Green function of $B_\delta$ with singularity at 0. For $|x| \leq \epsilon$,

$$\begin{aligned} G_\delta(x) &= \frac{1}{n(n-2)\omega_n}\left[\frac{1}{|x|^{n-2}} - \frac{1}{\delta^{n-2}}\right] \\ &\geq \frac{1}{n(n-2)\omega_n}\left[\frac{4^{n-2}}{\rho^{n-2}} - \frac{2^{n-2}}{\rho^{n-2}}\right] = c_n\rho^{2-n}. \end{aligned}$$

Hence

$$I_1 = \int_{B_\epsilon} \Delta\varphi(|h|^2)dx \le C\rho^{n-2} \int_{B_\delta} \Delta\varphi(|h(x)|^2)G_\delta(x)dx,$$

which by Lemma 2

$$= C\rho^{n-2}\left[\int_S \varphi(|h(\delta\zeta)|^2)d\sigma(\zeta) - \varphi(|h(0)|^2)\right].$$

Since $\varphi$ is concave, $\int_S \varphi(|h|^2)d\sigma \le \varphi\left(\int_S |h|^2 d\sigma\right)$. Thus

$$I_1 \le C\rho^{n-2}\left[\varphi\left(\int_S |h(\delta\zeta)|^2 d\sigma(\zeta)\right) - \varphi(|h(0)|^2)\right].$$

Since $\varphi$ is concave and increasing with $\varphi(0) = 0$, by Lemma 3

$$\varphi(b) - \varphi(a) \le \varphi(b-a), \quad 0 < a \le b.$$

Therefore

$$I_1 \le C\rho^{n-2}\varphi\left(\int_S |h(\delta\zeta)|^2 d\sigma(\zeta) - |h(0)|^2\right),$$

which by Green's identity (Lemma 2)

$$= C\rho^{n-2}\varphi\left(2\int_{B_\delta} |\nabla h(x)|^2 G_\delta(x)dx\right).$$

Hence

$$I_1 \le C\rho^{n-2}\varphi\left(2\sup_{x\in B_\delta} |\nabla h(x)|^2 \int_{B_\delta} G_\delta(x)dx\right).$$

But

$$\int_{B_\delta} G_\delta(x)dx = \frac{1}{2n}\delta^2.$$

Therefore since $\delta = \frac{1}{2}\rho$,

$$I_1 \le C\rho^{n-2}\varphi\left(\frac{\rho^2}{4n}\sup_{x\in B_\delta} |\nabla h(x)|^2\right),$$

which since $\varphi$ is increasing

$$\le C\rho^{n-2}\sup_{x\in B_\delta} \varphi(\rho^2|\nabla h(x)|^2).$$

But since $x \to \varphi(\rho^2|\nabla h(x)|^2)$ is subharmonic,

$$\varphi(\rho^2|\nabla h(x)|^2) \leq \frac{C}{\rho^n}\int_{B_\rho}\varphi(\rho^2|\nabla h(y)|^2)dy.$$

for all $x \in B_\delta$. Therefore, combining the above we have

$$\int_{B_{\rho/4}} \rho^2\Delta\varphi(|h|^2)\,d\lambda \leq C\int_{B_\rho}\varphi(\rho^2|\nabla h|^2)\,d\lambda.$$

(b) Suppose $\varphi$ is convex and satisfies inequality (1.5). By Lemma 2

$$\int_{B_\delta}\Delta\varphi(|h(x)|^2)G_\delta(x)\,dx = \int_S\varphi(|h(\delta\zeta)|^2)\,d\sigma(\zeta) - \varphi(|h(0)|^2),$$

which since $\varphi$ is convex

$$\geq \varphi\left(\int_S|h(\delta\zeta)|^2d\sigma(\zeta)\right) - \varphi(|h(0)|^2) = I_2.$$

But by Lemma 3,

$$I_2 \geq \varphi\left(\int_S|h(\delta\zeta)|^2d\sigma(\zeta) - |h(0)|^2\right).$$

Thus by Lemma 2,

$$\int_{B_\delta}\Delta\varphi(|h(x)|^2)G_\delta(x)\,dx \geq \varphi\left(2\int_{B_\delta}|\nabla h(x)|^2G_\delta(x)\,dx\right).$$

For $|x| \leq \epsilon$ and $n \geq 3$, $G_\delta(x) \geq c_n\rho^{2-n}$, where $c_n = 2^{2n-5}/n(n-2)\omega_n$. Therefore

$$2\int_{B_\delta}|\nabla h(x)|^2G_\delta(x)\,dx \geq \frac{2^{2n-4}\rho^{2-n}}{n(n-2)\omega_n}\int_{B_\epsilon}|\nabla h(x)|^2dx,$$

which since $|\nabla h(x)|^2$ is subharmonic and $\epsilon = \rho/4$

$$\geq \frac{1}{2^4n(n-2)}\rho^2|\nabla h(0)|^2 \geq \frac{1}{2^{n+3}}\rho^2|\nabla h(0)|^2.$$

By inequality (1.5)

$$\varphi\left(\frac{1}{2^{n+3}}\rho^2|\nabla h(0)|^2\right) \geq \frac{1}{c^{n+3}}\varphi(\rho^2|\nabla h(0)|^2),$$

where $c$ is the constant in inequality (1.5). Combining the above gives

$$\varphi(\rho^2|\nabla h(0)|^2) \le c^{n+3} \int_{B_\delta} \Delta\varphi(|h(x)|^2) G_\delta(x)\, dx.$$

Since $G_\delta(x) \le C_n |x|^{2-n}$ we have

$$\varphi(\rho^2|\nabla h(0)|^2) \le C_n \int_{B_\delta} \Delta\varphi(|h(x)|^2)|x|^{2-n}\, dx,$$

where $C_n$ is a constant depending only on $n$.

For $w \in B_\delta$, set $h_w(x) = h(w+x)$. Thus

$$\varphi(\rho^2|\nabla h(w)|^2) \le C_n \int_{B_\delta} \Delta_x\varphi(|h_w(x)|^2)|x|^{2-n}\, dx,$$

which by the change of variable $y = w + x$

$$= C_n \int_{B_\delta(w)} \Delta\varphi(|h(y)|^2)|y-w|^{2-n} dy.$$

Therefore,

$$\int_{B_\delta} \varphi(\rho^2|\nabla h(w)|^2)\, dw \le C_n \int_{B_\delta} \int_{B_\delta(w)} \Delta\varphi(|h(y)|^2)|y-w|^{2-n}\, dy\, dw,$$

which by Fubini's theorem

$$\le C_n \int_{B_{2\delta}} \Delta\varphi(|h(y)|^2) \left( \int_{B_\delta(y)} |y-w|^{2-n} dw \right) dy.$$

But

$$\int_{B_\delta(y)} |y-w|^{2-n} dw = \int_{B_\delta} |x|^{2-n} dx = n\omega_n \frac{\rho^2}{4}.$$

Therefore,

$$\int_{B_\delta} \varphi(\rho^2|\nabla h|^2)\, d\lambda \le C_n \rho^2 \int_{B_{2\delta}} \Delta\varphi(|h|^2)\, d\lambda,$$

which completes the proof. □

**Lemma 5.** *Let $\psi$ and $\varphi$ be as in Theorem 1, and let $h$ be harmonic on $\Omega$. Assume that $\psi(|h|) \in C^2(\Omega)$. Then for $\gamma \in \mathbb{R}$, the following hold:*

*(a) If $\varphi$ is concave, then*

$$\int_\Omega \delta(x)^\gamma \Delta\psi(|h(x)|)dx \le C \int_\Omega \delta(x)^{\gamma-2}\psi(\delta(x)|\nabla h(x)|)dx.$$

*(b) If $\varphi$ is convex and satisfies inequality (1.5), then*

$$\int_\Omega \delta(x)^\gamma \Delta\psi(|h(x)|)dx \ge C \int_\Omega \delta(x)^{\gamma-2}\psi(\delta(x)|\nabla h(x)|)dx.$$

*Proof.* (a) By Lemma 1

$$\begin{aligned}&\int_\Omega \delta(x)^\gamma \Delta\psi(|h(x)|)dx \\ &\quad\le C \int_\Omega \delta(w)^{\gamma-n}\left[\int_{B(w,\frac{1}{8}\delta(w))} \Delta\psi(|h(y)|)dy\right] dw.\end{aligned}$$

Set $\rho = \frac{1}{2}\delta(w)$ and $u(x) = h(w+x)$. Then

$$\int_{B(w,\frac{1}{8}\delta(w))} \Delta\psi(|h(y)|)dy = \int_{B_{\rho/4}} \Delta\psi(|u(x)|)dx,$$

which by Lemma 4

$$\begin{aligned}&\le C\rho^{-2}\int_{B_\rho} \psi(\rho|\nabla u(x)|)dx \\ &= C\delta(w)^{-2}\int_{B_\rho(w)} \psi(\tfrac{1}{2}\delta(w)|\nabla h(y)|)dy.\end{aligned}$$

But $\frac{1}{2}\delta(w) \le \delta(y)$ for all $y \in B_\rho(w)$. Hence since $\psi$ is increasing, $\psi(\frac{1}{2}\delta(w)|\nabla h(y)|) \le \psi(\delta(y)|\nabla h(y)|)$, and thus

$$\int_{B(w,\frac{1}{8}\delta(w))} \Delta\psi(|h(y)|)dy \le C\delta(w)^{-2}\int_{B(w)} \psi(\delta(y)|\nabla h(y)|)dy.$$

Finally, by Lemma 1,

$$\begin{aligned}&\int_\Omega \delta(w)^{\gamma-n-2}\left[\int_{B(w)} \psi(\delta(y)|\nabla h(y)|)dy\right] dw \\ &\quad\le C\int_\Omega \delta(x)^{\gamma-2}\psi(\delta(x)|\nabla h(x)|)dx,\end{aligned}$$

which proves (a). The proof of part (b) proceeds in the same manner, except that this case also requires inequality (1.5). □

## §3. Proof of Theorem 1

Before proving Theorem 1 we require two preliminary results about subharmonic functions. Let $\mathcal{S}^+(\Omega)$ denote the set of non-negative subharmonic functions on $\Omega$ that have a harmonic majorant on $\Omega$. As in the Introduction, for $f \in \mathcal{S}^+(\Omega)$ we let $H_f$ denote the least harmonic majorant of $f$ on $\Omega$. For convenience we will assume that $f \in C^2(\Omega)$. As in [8],[9] we have the following.

**Lemma 6.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$, with Green function $G$, and let $f \in C^2(\Omega)$. Then $f \in \mathcal{S}^+(\Omega)$ if and only if there exists $t_o \in \Omega$ such that*

$$\int_\Omega G(t_o, x)\Delta f(x)\, dx < \infty. \tag{3.1}$$

*If this is the case, then by the Riesz decomposition theorem*

$$H_f(x) = f(x) + \int_\Omega G(x, y)\Delta f(y)\, dy \tag{3.2}$$

If the subharmonic function $f$ is not $C^2$, then the quantity $\Delta f(x)\, dx$ may be replaced by $d\mu_f$, where $\mu_f$ is the Riesz measure of the subharmonic function $f$.

**Lemma 7.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$, with Green function $G$ satisfying* (1.3) *and* (1.4). *Let $t_o \in \Omega$ be fixed, and let $\alpha$ and $\beta$ be as in inequalities* (1.3) *and* (1.4) *respectively. Then there exists constants $C_1$ and $C_2$, depending only on $t_o$ and $\Omega$, such that for all $f \in \mathcal{S}^+(\Omega) \cap C^2(\Omega)$,*

$$C_1 \left[ f(t_o) + \int\limits_\Omega \delta(x)^\alpha \Delta f(x) dx \right] \leq H_f(t_o)$$
$$\leq C_2 \left[ \int\limits_{B(t_o)} f(x) dx + \int\limits_\Omega \delta(x)^\beta \Delta f(x) dx \right].$$

*Proof.* The left side of the previous inequality is an immediate consequence of identity (3.2) and inequality (1.3). For the right side, integrating equation (3.2) over $B(t_o)$ gives

$$H_f(t_o) = \frac{1}{\omega_n \rho_o^n} \int_{B(t_o)} f(x)\, dx + \frac{1}{\omega_n \rho_o^n} \int_{B(t_o)} \int_\Omega G(x, y)\Delta f(y)\, dy\, dx,$$

where $\rho_o = \frac{1}{2}\delta(t_o)$. By Fubini's theorem,

$$\frac{1}{\omega_n \rho_o^n} \int_{B(t_o)} \int_\Omega G(x,y)\Delta f(y)\,dy\,dx = \frac{1}{\omega_n \rho_o^n} \int_\Omega \Delta f(y) \int_{B(t_o)} G(x,y)\,dx\,dy.$$

Set

$$I(y) = \frac{1}{\omega_n \rho_o^n} \int_{B(t_o)} G(x,y)\,dx.$$

To complete the proof it remains to be shown that $I(y) \le C\,\delta(y)^\beta$.

If $y \notin B(t_o)$, then since $x \to G(x,y)$ is harmonic on $B(t_o)$ and $G$ satisfies inequality (1.4),

$$I(y) = G(t_o, y) \le C_2 \delta(y)^\beta.$$

Suppose $y \in B(t_o)$ and $n \ge 3$. Then since $G(x,y) \le c_n |x-y|^{2-n}$,

$$I(y) \le \frac{c_n}{\omega_n \rho_o^n} \int_{B(t_o)} |x-y|^{2-n} dx \le \frac{c_n}{\omega_n \rho_o^n} \int_{B(y,2\rho_o)} |x-y|^{2-n} dx = 2nc_n \rho_o^{2-n}.$$

But for $y \in B(t_o)$, $\rho_o \le 2\delta(y)$. Thus

$$I(y) \le 2nc_n 2^\beta \delta(y)^\beta \rho_o^{2-n-\beta} = C\delta(y)^\beta,$$

where $C$ is a constant depending only on $t_o$ and $\Omega$. □

**Proof of Theorem 1.** (a) Let $\psi$ be as in the statement of the theorem, and let $h$ be a real-valued harmonic function on $\Omega$. Set $h_\epsilon(x) = h(x) + i\epsilon$. Then $h_\epsilon$ is harmonic on $\Omega$ and $\psi(|h_\epsilon|) \in C^2(\Omega)$. Hence by Lemma 7,

$$N_\psi(h_\epsilon) \le C_2 \left[ \int_{B(t_o)} \psi(|h_\epsilon(x)|)\,dx + \int_\Omega \delta(x)^\beta \Delta\psi(|h_\epsilon(x)|)\,dx \right],$$

which by Lemma 5(a)

$$\le C_2 \left[ \int_{B(t_o)} \psi(|h_\epsilon(x)|)\,dx + \int_\Omega \delta(x)^{\beta-2} \psi(\delta(x)|\nabla h(x)|)\,dx \right].$$

Letting $\epsilon \to 0^+$ gives

$$N_\psi(h) \le C_2 \left[ \max_{x \in B(t_o)} \psi(|h(x)|) + \int_\Omega \delta(x)^{\beta-2} \psi(\delta(x)|\nabla h(x)|)\,dx \right].$$

It remains to be shown that

(3.3)
$$\max_{x\in B(t_o)} \psi(|h(x)|) \le C\left[\psi(|h(t_o)|) + \int_\Omega \delta(x)^{\beta-2}\psi(\delta(x)|\nabla h(x)|)\,dx\right].$$

Without loss of generality we take $t_o = 0$. As a consequence of the Fundamental Theorem of Calculus, for all $x \in B(t_o)$,

$$|h(x)| \le |h(0)| + \rho_o \max_{y\in B(t_o)} |\nabla h(y)|.$$

Since $\psi$ is increasing, convex, and continuous, and satisfies property (1.5)

$$\psi(|h(x)|) \le \frac{c}{2}\left[\psi(|h(0)|) + \max_{y\in B(t_o)} \psi(\rho_o|\nabla h(y)|)\right].$$

Also, since $y \to \psi(\rho_o|\nabla h(y)|)$ is subharmonic,

$$\psi(\rho_o|\nabla h(y)|) \le \frac{2^n}{\omega_n\rho_o^n}\int_{B(y,\frac{1}{2}\rho_o)} \psi(\rho_o|\nabla h(x)|)\,dx$$

But $\rho_o \le \delta(y) \le 3\rho_o$ for all $y \in B(t_o)$, and $\frac{1}{2}\delta(y) \le \delta(x) \le \frac{3}{2}\delta(y)$ for all $x \in B(y, \frac{1}{2}\rho_o)$. Thus

$$\psi(\rho_o|\nabla h(y)|) \le C(\rho_o)\int_\Omega \delta(x)^{\beta-2}\psi(\delta(x)|\nabla h(x)|)\,dx,$$

from which inequality (3.3) now follows. This completes the proof of (a). The proof of (b) is an immediate consequence of Lemma 7 and Lemma 5(b).

## §4. Remarks

The techniques employed in this paper may also be used to prove analogue's of Theorems 1 and 2 for Hardy-Orlicz spaces of holomorphic functions on a domain $\Omega \subsetneq \mathbb{C}^n$, $n \ge 1$.

In this setting the spaces $\mathcal{H}_\psi$ are traditionally defined as in [6, page 83]. For a non-negative, non-decreasing convex function $\psi$ on $(-\infty, \infty)$ with $\lim_{t\to-\infty}\psi(t) = 0$, the Hardy-Orlicz space $\mathcal{H}_\psi(\Omega)$ is defined as the set of holomorphic functions $f$ on $\Omega$ for which $\psi(\log|f|)$ has a harmonic majorant on $\Omega$. As in (1.5) we set $N_\psi(f) = H_\psi^f(t_o)$, where $H_\psi^f$ denotes the least harmonic majorant of $\psi(\log|f|)$. With $\psi(t) = e^{pt}$, $0 < p < \infty$, one obtains the usual Hardy $\mathcal{H}^p$ space of holomorphic functions on $\Omega$.

To obtain the analogue of Theorem 1 one considers the function $\varphi(t) = \psi(\frac{1}{2}\log t)$. In this setting, hypothesis (2.4) can be replaced by

$$x\varphi''(x) + \varphi'(x) \geq 0 \tag{4.1}$$

for all $x \in (0, \infty)$. If the above holds, then it is easily shown that for $f$ holomorphic on $\Omega$, $\varphi(|f|^2)$ is plurisubharmonic on $\Omega$, hence also subharmonic. Clearly $\varphi(x) = \psi(\frac{1}{2}\log x)$ satisfies (4.1) whenever $\psi$ is convex. The details of the statements and proofs of the appropriate theorems are left to the reader.

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# Estimates of maximal functions by Hausdorff contents in a metric space

**Hisako Watanabe**

**Abstract.**

Let $M$ be the Hardy-Littlewood maximal operator in a quasi-metric space $X$. We give the estimates of $Mf$ with weak type and strong type with respect to the $\alpha$-Hausdorff content. To do these, we use the dyadic balls introduced by E. Sawyer and R. L. Wheeden.

## §1. Introduction

In analysis many operators are dominated by constant multiples of the Hardy-Littlewood maximal operators. In $\mathbf{R}^n$ the maximal function $Mf$ of $f$ is defined by

$$Mf(x) = \sup \frac{1}{|B|} \int_B |f| dx,$$

where the supremum is taken over all balls $B$ containing $x$ and $|B|$ stands for the $n$-dimensional volume of $B$.

In 1988 D. R. Adams considered the estimates of the maximal functions with respect to the $\alpha$-Hausdorff content $H_\infty^\alpha$ and proved the following strong type inequality (cf. [1]).

**Theorem A.** *Let $0 < \alpha < n$. Then there is a constant $c$ such that*

$$\int Mf dH_\infty^\alpha \le c \int |f| dH_\infty^\alpha.$$

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In this theorem, the integral of a nonnegative function $g$ with respect to $H_\infty^\alpha$ is in the sense of Choquet and is defined by

$$\int g dH_\infty^\alpha := \int_0^\infty H_\infty^\alpha(\{x \in \mathbf{R}^n : g(x) > t\})dt.$$

In 1998 J. Orobitg and J. Verdera generalized Theorem A as follows (cf. [5]).

**Theorem B.** *Let $0 < \alpha < n$. Then, for some constant $c$ depending only on $\alpha$ and $n$,*

(i) $\displaystyle\int (Mf)^p dH_\infty^\alpha \le c \int |f|^p dH_\infty^\alpha, \quad \alpha/n < p,$

(ii) $\displaystyle H_\infty^\alpha(\{x; Mf(x) > t\}) \le ct^{-\alpha/n} \int |f|^{\alpha/n} dH_\infty^\alpha.$

To prove Theorem A and Theorem B, the authors considered the maximal function and the $\alpha$-Hausdorff content restricted to dyadic cubes. More precisely, let us define $\tilde{M}f$ and $\tilde{H}_\infty^\alpha$ in $\mathbf{R}^n$.

For each $x$

$$\tilde{M}f(x) := \sup \frac{1}{|Q|} \int_Q |f| dy,$$

where the supremum is taken over all dyadic cubes containing $x$ and for a subset $E$ of $\mathbf{R}^n$

$$\tilde{H}_\infty^\alpha(E) := \inf \sum_{j=1}^\infty l(Q_j)^\alpha,$$

where the infimum is taken over all coverings of $E$ by countable families of dyadic cubes and $l(Q_j)$ stands for the side length of $Q_j$.

We see that $Mf$ and $H_\infty^\alpha(E)$ are comparable to $\tilde{M}f$ and $\tilde{H}_\infty^\alpha(E)$, respectively. So they used $\tilde{M}$ and $\tilde{H}_\infty^\alpha$ instead of $M$ and $H_\infty^\alpha$.

In [2] D. R. Adams defined a Choquet-Lorentz space $L^{q,p}(H_\infty^\delta)$ of the Lorentz type with respect to the Hausdorff capacity $H_\infty^\delta$ in $\mathbf{R}^n$ and gave the estimates of the fractional maximal functions of order $\alpha$ in term of $L^{q,p}(H_\infty^\delta)$ (cf. Theorem 7 in [2]).

In this paper we estimate the Hardy-Littlewood maximal functions by Hausdorff contents in a quasi-metric space.

Recall that $(X, \rho)$ is called a quasi-metric space if the mapping $\rho$ from $X \times X$ to $[0, \infty)$ has the following three properties;

(i) $\rho(x, y) = 0$ if and only if $x = y$,
(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$,
(iii) There is a constant $K \ge 1$ such that

$$\rho(x, y) \le K(\rho(x, z) + \rho(z, y)) \quad \text{for all } x, y, z \in X. \tag{1.1}$$

In addition, we assume that the diameter of $X$ is finite and set

$$\operatorname{diam} X = R.$$

Let $M$ be the Hardy-Littlewood maximal operator and let $H^\alpha_\infty$ be the $\alpha$-Hausdorff content. Furthermore we suppose that there are a non-negative Borel measure $\mu$ on $X$ and a positive number $d$ such that

$$b_1 r^d \le \mu(B(x,r)) \le b_2 r^d \tag{1.2}$$

for all positive $r \le R$, where

$$B(x,r) := \{y \in X : \rho(x,y) < r\}.$$

In a quasi-metric space there is no dyadic cube. Instead of dyadic cubes E. Sawyer and R. L. Wheeden [6] constructed a family of balls as follows:

**Theorem C.** *Put $\lambda = K + 2K^2$. Then, for each integer $k$, there exists a sequence $\{B^k_j\}_j$ ($B^k_j = B(x_{jk}, \lambda^k)$) of balls of radius $\lambda^k$ having the following properties:*

(i) *Every ball of radius $\lambda^{k-1}$ is contained in at least one of the balls $B^k_j$,*

(ii) $\sum_j \chi_{B^k_j} \le M$ *for all $k$ in $\mathbf{Z}$,*

(iii) $\hat{B}^k_i \cap \hat{B}^k_j = \emptyset$ *for $i \ne j$, $k \in \mathbf{Z}$, where $\hat{B}^k_j = B(x_{jk}, \lambda^{k-1})$.*

They call these balls $B^k_j$ dyadic balls. Denote by $\mathcal{B}_d$ the family of all dyadic balls. Using dyadic balls, we give the estimates of the maximal operator $M$ in a quasi-metric space $X$ by the integral with respect to $H^\alpha_\infty$, corresponding to the results of Orobitg–Verdera.

**Theorem 1.** *Let $(X,\rho)$ be a quasi-metric space with diam $X < \infty$. Suppose that there are a positive number $d$ and a Borel measure $\mu$ on $X$ satisfying* (1.2) *for every ball $B(x,r) \subset X$. Furthermore, let $0 < \alpha < d$. Then*

$$H^\alpha_\infty(\{x : Mf(x) > t\}) \le c t^{-\alpha/d} \int |f|^{\alpha/d} dH^\alpha_\infty$$

*for every $f$ and $t > 0$.*

**Theorem 2.** *Assume that $X$ and $\mu$ satisfy the same conditions as Theorem 1. Let $\alpha/d < p$. Then*

$$\int (Mf)^p dH^\alpha_\infty \le c \int |f|^p dH^\alpha_\infty \quad \text{for every } f.$$

We note that, for a nonnegative function $g$ and a subset $G$ of $X$,

$$\int_G g dH^\alpha_\infty := \int_0^\infty H^\alpha_\infty(\{x \in G : g(x) > t\})dt$$

and

$$\int_G g d\mu := \int_0^\infty \mu(\{x \in G : g(x) > t\})dt.$$

If $g \in L^1(\mu)$ and $G$ is $\mu$-measurable, then the integral with respect to the measure $\mu$ coincides with the usual one.

## §2. Dyadic balls in a quasi-metric space

Throughout this paper let $(X, \rho)$ be a quasi-metric space. The function $\rho$ is called a quasi-metric. We assume that the diameter of $X$ is finite and diam $X = R$. Furthermore we assume that there exists a positive Radon measure $\mu$ on $X$ with $\mu(X) < \infty$ and satisfying (1.2) for some $d$. We note that, if (1.2) holds for all positive $r \leq R$, then (1.2) holds for all positive $r \leq 2(K + 2K^2)^2 R$ by changing the constants. So we may assume that (1.2) holds for all positive $r \leq 2(K + 2K^2)^2 R$. Consequently $\mu$ satisfies the doubling condition, i.e., there is a constant $c > 0$ such that

$$\mu(B(x, 2r)) \leq c\mu(B(x, r))$$

for $x \in X$ and $r \leq 2(K+2K^2)^2 R$. So $X$ is a space of homogeneous type (See [3] on more precise properties on a space of homogeneous type).

For any quasi-metric $\rho$ there exists an equivalent quasi-metric $\rho'$ such that all balls with respect to $\rho'$ are open (cf. [4]). Consequently we may assume that all balls $B(x, r)$ in $X$ are open.

Let $B = B(x, r)$ be a ball and $b$ be a positive real number. The notation $bB$ stands for the ball of radius $br$ centered at $x$ and $r(B)$ stands for the radius of $B$. We often use the following value $\lambda$ defined by

$$\lambda = 2K^2 + K,$$

where $K$ is the constant in (1.1).

We begin with the following lemma.

**Lemma 2.1.** *Let $B$ be a ball and $\{B_j\}$ be a sequence of disjoint balls. Put*

$$E = \{j : B \cap \lambda^{-1} B_j \neq \emptyset,\ r(B) \leq r(B_j)\}.$$

*Then $\#E \leq N$, where $N$ is a constant independent of $B$ and $\{B_j\}$.*

*Proof.* Case 1. We first consider the case where there exists $B_i \in \{B_j\}_j$ satisfying $B \cap \lambda^{-1} B_i \neq \emptyset$ and $r(B) \leq \lambda^{-1} r(B_i)$.

Let $w \in B$ and $x_i$ be the center of $B_i$. Then, for $z \in B \cap \lambda^{-1} B_i$,

$$\begin{aligned} \rho(w, x_i) &\leq K(\rho(w, z) + \rho(z, x_i)) \\ &< 2K^2 r(B) + K\lambda^{-1} r(B_i) \leq r(B_i). \end{aligned}$$

Hence $B \subset B_i$. Noting that $\{B_j\}$ are disjoint, we conclude that $\#E = 1$.

Case 2. We next consider the case where $r(B) > \lambda^{-1} r(B_j)$ for all $j \in E$. Let $x$ be the center of $B$. Since $B_j \subset B(x, 2\lambda K r(B))$, we have

$$\cup_{j \in E} B_j \subset B(x, 2\lambda K r(B)).$$

Note that $\{B_j\}$ are disjoint and $r(B) \leq r(B_j)$ for all $j \in E$.

Let $\#E = n$. From (1.2), we deduce

$$\begin{aligned} & n\mu(B(x, 2K\lambda r(B)) \leq n b_2 (2K\lambda r(B))^d \leq \frac{b_2}{b_1} (2K\lambda)^d \sum_{j \in E} \mu(B_j) \\ = \; & \frac{b_2}{b_1} (2K\lambda)^d \mu(\cup_{j \in E} B_j) \leq \frac{b_2}{b_1} (2K\lambda)^d \mu(B(x, 2K\lambda r(B))). \end{aligned}$$

Thus $n \leq \frac{b_2}{b_1} (2K\lambda)^d$. This leads to the conclusion. □

We have the following lemma for dyadic balls.

**Lemma 2.2.** *Let* $\{B_j^k\} \subset \mathcal{B}_d$ *and* $B_j^k = B(x_{jk}, \lambda^k)$. *Then there is a constant* $N_1$, *independent of* $j$ *and* $k$, *such that*

$$\sum_j \chi_{\lambda B_j^k} \leq N_1.$$

*Proof.* Assume that $x \in \cap_{j=1}^n \lambda B_j^k$. Then $\hat{B}_j^k \subset B(x, 2K\lambda^{k+1})$. Similarly $B(x, \lambda^k) \subset B(x_{jk}, K\lambda^k(1 + \lambda))$. Hence, by (1.2),

$$\begin{aligned} \mu(B(x, 2K\lambda^{k+1})) &\leq c_1 \mu(B(x, \lambda^k)) \leq c_1 \mu(B(x_{jk}, K\lambda^k(1 + \lambda))) \\ &\leq c_2 \mu(B(x_{jk}, \lambda^{k-1})) = c_2 \mu(\hat{B}_j^k) \end{aligned}$$

for $j$. Noting that $\{\hat{B}_j^k\}$ are disjoint, we have

$$\frac{n}{c_2} \mu(B(x, 2K\lambda^{k+1})) \leq \sum_{j=1}^n \mu(\hat{B}_j^k) = \mu(\cup_{j=1}^n \hat{B}_j^k) \leq \mu(B(x, 2K\lambda^{k+1})),$$

whence $n \leq c_2$. Thus we have the conclusion. □

A sequence $\{B_j\}$ of balls is called maximal by inclusion if each $B_j$ includes no $B_i$ for $i \neq j$.

**Lemma 2.3.** *Let $\{B_j\} \subset \mathcal{B}_d$. If $\{\lambda^2 B_j\}$ is a maximal sequence by inclusion, then there is a constant $N_1$ such that*

$$\sum_j \chi_{\lambda B_j} \leq N_1.$$

*Proof.* Let $\{B_{jl}^k\}_l$ be the subfamily of $\{B_j\}$ having radius $\lambda^k$. Lemma 2.2 yields that

$$\sum_l \chi_{\lambda B_{jl}^k} \leq N_1.$$

We next consider two balls $B_j = B_j^k$ and $B_i = B_i^l$, $l < k$, in $\{B_j\}$. If $\lambda B_j^k \cap \lambda B_i^l \neq \emptyset$, then we pick $z \in \lambda B_j^k \cap \lambda B_i^l$. Let $w \in \lambda^2 B_i^l$. Writing $B_j^k = B(x_{jk}, \lambda^k)$ and $B_i^l = B(x_{il}, \lambda^l)$, we have

$$\begin{aligned} \rho(x_{jk}, w) &\leq K(\rho(x_{jk}, z) + K(\rho(z, x_{il}) + \rho(x_{il}, w))) \\ &< K\lambda^{k+1} + 2K^2\lambda^{l+2} \leq \lambda^{k+2}, \end{aligned}$$

whence $\lambda^2 B_i^l \subset \lambda^2 B_j^k$. This contradicts that $\{\lambda^2 B_j\}$ is maximal. Therefore we conclude that $\lambda B_j^k \cap \lambda B_i^l = \emptyset$. □

Using this lemma, we have

**Lemma 2.4.** *Let $\{B_j\} \subset \mathcal{B}_d$ such that $\{\lambda^2 B_j\}$ is a maximal sequence by inclusion. Furthermore let $B \in \mathcal{B}_d$. Put*

$$F = \{j : B \cap B_j \neq \emptyset,\ r(B) \leq r(B_j)\}.$$

*Then $\#F \leq N_1$.*

*Proof.* If $j \in F$, then $B \subset \lambda B_j$. Lemma 2.3 yields

$$\sum_j \chi_{\lambda B_j} \leq N_1.$$

Hence $\#F \leq N_1$.

□

Let $\{B_j\}$ be a (finite or infinite) sequence of subsets of $X$. Using it, we can construct a maximal sequence by inclusion. Indeed, we consider $\{B_1, B_2\}$ and, if $B_1 \subset B_2$ or $B_2 \subset B_1$, then we remove the less one from $\{B_1, B_2\}$ and denote by $B_1'$ the big one. Otherwise, put

$$B_1' = B_1 \text{ and } B_2' = B_2.$$

We next assume that $\{B'_1, \cdots, B'_m\}$ has been constructed by using $\{B_1, \cdots, B_n\}$. Then we consider $\{B'_1, \cdots, B'_m, B_{n+1}\}$, remove all sets which are included by the other sets and make a new family $\{B'_1, \cdots, B'_l\}$ of all balls which remain. Thus we inductively construct a subsequence $\{B_1, B_2, \cdots\}$ of $\{B_j\}$, which is a maximal sequence by inclusion, and call it the maximal sequence of $\{B_j\}$.

We are ready to prove our main lemma.

**Lemma 2.5.** *Let $\{B_j\} \subset \mathcal{B}_d$ and $\alpha > 0$. Then there exists a (finite or infinite) subsequence $\{B_{j_k}\}$ of $\{B_j\}$ having the following properties:*

(i)

$$\sum_{j_k \in S_B} r(B_{j_k})^\alpha \le 2r(B)^\alpha \quad \textit{for each } B \in \mathcal{B}_d,$$

*where $S_B = \{j_k : B_{j_k} \cap B \ne \emptyset,\ r(B_{j_k}) \le r(B)\}$.*

(ii) *For a positive number $b$ there is a constant $c$ such that*

$$H^\alpha_\infty(\cup_j bB_j) \le c \sum_k r(B_{j_k})^\alpha,$$

*where $c$ is independent of $\{B_j\}$.*

*Proof.* We construct a subsequence $\{B_{j_k}\}$ of $\{B_j\}$ by induction. First, put $j_1 = 1$. The set $\{B_{j_1}\}$ has the property (i). Next, assume that $\{j_1, \cdots, j_m\}$ $(j_1 < \cdots < j_m)$ have been chosen so that (i) holds for $\{B_{j_1}, \cdots, B_{j_m}\}$. We set $j_{m+1}$ the first number $j$ such that $j_m < j$ and $\{B_{j_1}, \cdots, B_{j_m}, B_j\}$ satisfies (i). We note that, if $S_B = \emptyset$, then the left-hand side of the inequality in (i) is regarded as 0. Thus we construct $j_1, \cdots, j_n, \cdots$.

We next show that $\{B_{j_k}\}$ also satisfies (ii). Let $j'$ be a number satisfying $j_m < j' < j_{m+1}$. Then there is a ball $C_{j'} \in \mathcal{B}_d$ such that $B_{j'} \cap C_{j'} \ne \emptyset$, $r(B_{j'}) \le r(C_{j'})$ and

$$\sum_{j_k \in S_{C_{j'}}} r(B_{j_k})^\alpha + r(B_{j'})^\alpha > 2r(C_{j'})^\alpha.$$

From this it follows that

$$\sum_{j_k \in S_{C_{j'}}} r(B_{j_k})^\alpha > r(C_{j'})^\alpha. \tag{2.1}$$

To prove (ii), we may suppose that $\sum_k r(B_{j_k})^\alpha < \infty$. We denote by $\{D_i\}$ the maximal sequence of $\{\lambda^2 C_{j'}\}$. Since $B_{j'} \cap C_{j'} \ne \emptyset$ and

$r(B_{j'}) \leq r(C_{j'})$, we have $B_{j'} \subset \lambda C_{j'}$. Hence $B_{j'} \subset D_i$ for some $i$. Noting that

$$\cup_j bB_j \subset \cup_k bB_{j_k} \cup (\cup_{j'} bB_{j'}) \subset \cup_k bB_{j_k} \cup (\cup_i bD_i),$$

we have

$$H^\alpha_\infty(\cup_j bB_j) \leq b^\alpha \sum_k r(B_{j_k})^\alpha + b^\alpha \lambda^{2\alpha} \sum_i r(\lambda^{-2} D_i)^\alpha.$$

The inequality (2.1) implies

$$\begin{aligned}\sum_i r(\lambda^{-2}D_i)^\alpha &\leq \sum_i \sum_{j_k \in S_{\lambda^{-2}D_i}} r(B_{j_k})^\alpha \\ &= \sum_k \sum_{B_{j_k} \cap \lambda^{-2} D_i \neq \emptyset, r(B_{j_k}) \leq \lambda^{-2} r(D_i)} r(B_{j_k})^\alpha.\end{aligned}$$

Fix a natural number $k$. We see by Lemma 2.4 that the number of $\lambda^{-2}D_i$ satisfying $B_{j_k} \cap \lambda^{-2} D_i \neq \emptyset$ and $r(B_{j_k}) \leq \lambda^{-2} r(D_i)$ is at most $N_1$. Hence

$$\begin{aligned}H^\alpha_\infty(\cup_j bB_j) &\leq b^\alpha \sum_k r(B_{j_k})^\alpha + b^\alpha \lambda^{2\alpha} N_1 \sum_k r(B_{j_k})^\alpha \\ &= b^\alpha (1 + N_1 \lambda^{2\alpha}) \sum_k r(B_{j_k})^\alpha.\end{aligned}$$

We may put $c = b^\alpha(1 + N_1\lambda^{2\alpha})$. Thus we have the assertion (ii). □

## §3. Maximal functions and Hausdorff contents with respect to dyadic balls

In this section we introduce maximal functions and Hausdorff contents with respect to dyadic balls. We begin with maximal functions. For a function $f$ we define

$$\tilde{M} f(x) = \sup \frac{1}{\mu(B)} \int_B |f| d\mu,$$

where the supremum is taken over all dyadic balls containing $x$. Here we note that, for a nonnegative function $g$,

$$\int g d\mu := \int_0^\infty \mu(\{x : g(x) > t\}) dt.$$

Using the properties in Theorem C, we can show the following lemma.

**Lemma 3.1.** *Let $f$ be a function on $X$. Then there is a constant $c$ independent of $f$ such that*

$$\tilde{M}f(x) \leq Mf(x) \leq c\tilde{M}f(x)$$

*for all $x \in X$.*

Fix $\alpha$ satisfying $0 < \alpha < d$. Similarly we define, for $E \subset X$,

$$\tilde{H}_\infty^\alpha(E) = \inf \sum_j r(B_j)^\alpha,$$

where the infimum is taken over all coverings $\{B_j\}$ of $E$ by dyadic balls $B_j$. Similarly we can show the following lemma.

**Lemma 3.2.** *Let $0 < \alpha < d$. Then there is a positive constant $c$ such that*

$$c\tilde{H}_\infty^\alpha(E) \leq H_\infty^\alpha(E) \leq \tilde{H}_\infty^\alpha(E).$$

## §4. Proofs of Theorem 1 and Theorem 2

In this section we will prove Theorem 1 and Theorem 2. To do these, we estimate the integral of a nonnegative function $f$ with respect to the measure $\mu$ by the integral of $f$ with respect to $H_\infty^\alpha$.

**Lemma 4.1.** *Let $0 < \alpha \leq d$ and $f$ be a nonnegative function on $X$. Then*

$$\int f d\mu \leq c \left( \int f^{\alpha/d} dH_\infty^\alpha \right)^{d/\alpha},$$

*where $c$ is a positive constant independent of $f$.*

*Proof.* Noting that $\mu$ satisfies (1.2), we can prove this lemma by the same method as in the proof of Lemma 3 in [5]. □

We note that $H_\infty^\alpha(\{x : f(x) > t\})$ is abbreviated to $H_\infty^\alpha(\{f > t\})$ in the proofs of Theorem 1 and Theorem 2.

*Proof of Theorem 1.* We may assume that $f \geq 0$. Put

$$E_t = \{x : \tilde{M}f(x) > t\}$$

for $t > 0$. For each $x \in E_t$ there is a ball $B_x \in \mathcal{B}_d$ such that

$$\frac{1}{\mu(B_x)} \int_{B_x} f d\mu > t. \tag{4.1}$$

Then $E_t \subset \cup_{x \in E_t} B_x \subset \cup_{x \in E_t} \lambda B_x$.

By Théorèm (1.2) on p.69 in [3] we can choose a countable family $\{\lambda B_j\} \subset \{\lambda B_x\}_{x\in E_t}$ such that $\{\lambda B_j\}$ ($B_j = B(x_j, r_j)$) are disjoint and $E_t \subset \cup_j B(x_j, h\lambda r_j)$ for some $h \geq 1$. Then, by Lemma 4.1 and (4.1),

$$r(B_j)^\alpha \leq \left(\frac{1}{b_1 t}\int_{B_j} f d\mu\right)^{\alpha/d} \leq c_1 t^{-\alpha/d}\int_{B_j} f^{\alpha/d} dH^\alpha_\infty. \tag{4.2}$$

Applying Lemma 2.5 to the sequence $\{B_j\}$, we choose a subsequence $\{B_{j_k}\}$ satisfying (i) and (ii) in Lemma 2.5 for $b = \lambda h$. Writing $B_{j_k} = B(x_k, r_k)$, we have, by (4.2),

$$H^\alpha_\infty(E_t) \leq H^\alpha_\infty(\cup_j B(x_j, \lambda h r_j)) \leq c_2 \sum_k r_k^\alpha \leq c_3 \sum_k t^{-\alpha/d}\int_{B_{j_k}} f^{\alpha/d} dH^\alpha_\infty.$$

We claim that

$$\sum_k \int_{B_{j_k}} f^{\alpha/d} dH^\alpha_\infty \leq c_4 \int f^{\alpha/d} dH^\alpha_\infty. \tag{4.3}$$

Indeed, if $\int f^{\alpha/d} dH^\alpha_\infty = +\infty$, then it is clear that (4.3) holds. Assume that $\int f^{\alpha/d} dH^\alpha_\infty < +\infty$. Since

$$\int_0^\infty H^\alpha_\infty(\{f^{\alpha/d} > \tau\}) d\tau < \infty,$$

we have

$$H^\alpha_\infty(\{f^{\alpha/d} > \tau\}) < \infty \quad \text{for a.e. } \tau$$

and hence, by Lemma 3.2,

$$\tilde{H}^\alpha_\infty(\{f^{\alpha/d} > \tau\}) < \infty \quad \text{for a.e. } \tau.$$

Fix $\tau$ satisfying $\tilde{H}^\alpha_\infty(\{f^{\alpha/d} > \tau\}) < \infty$. For $\epsilon > 0$ we take balls $Q_i \in \mathcal{B}_d$ such that

$$\{x : f(x)^{\alpha/d} > \tau\} \subset \cup_i Q_i$$

and

$$\sum_i r(Q_i)^\alpha < \tilde{H}^\alpha_\infty(\{f^{\alpha/d} > \tau\}) + \epsilon. \tag{4.4}$$

Since $\{\lambda B_{j_k}\}$ are disjoint, we see, by Lemma 2.1, that for each $Q_i$ the number of $B_{j_k}$ satisfying $Q_i \cap B_{j_k} \neq \emptyset$ and $r(Q_i) \leq \lambda r(B_{j_k})$ is at most

$N$. Hence

$$
\begin{aligned}
3\sum_i r(Q_i)^\alpha &= 2\sum_i r(Q_i)^\alpha + \sum_i r(Q_i)^\alpha \\
\geq\ & \sum_i \left( \sum_{\substack{Q_i\cap B_{j_k}\neq\emptyset \\ r(Q_i)>r(B_{j_k})}} r(B_{j_k})^\alpha + \frac{1}{N} N r(Q_i)^\alpha \right) \\
\geq\ & \sum_k \left( \sum_{\substack{Q_i\cap B_{j_k}\neq\emptyset \\ r(Q_i)>r(B_{j_k})}} r(B_{j_k})^\alpha + \frac{1}{N} \sum_{\substack{Q_i\cap B_{j_k}\neq\emptyset \\ r(Q_i)\leq r(B_{j_k})}} r(Q_i)^\alpha \right) \\
\geq\ & \frac{1}{N}\sum_k \tilde{H}^\alpha_\infty(B_{j_k}\cap(\cup_i Q_i)) \\
\geq\ & \frac{1}{N}\sum_k \tilde{H}^\alpha_\infty(B_{j_k}\cap\{f^{\alpha/d}>\tau\}).
\end{aligned}
$$

Hence, by (4.4),

$$
\tilde{H}^\alpha_\infty(\{f^{\alpha/d}>\tau\}) + \epsilon \geq \frac{1}{3N}\sum_k \tilde{H}^\alpha_\infty(B_{j_k}\cap\{f^{\alpha/d}>\tau\}).
$$

Thus, by Lemma 3.2, we have the claim (4.3). Therefore

$$
H^\alpha_\infty(E_t) \leq c_3\sum_k t^{-\alpha/d}\int_{B_{j_k}} f^{\alpha/d}\,dH^\alpha_\infty \leq c_5 t^{-\alpha/d}\int f^{\alpha/d}\,dH^\alpha_\infty.
$$

This is the desired inequality. □

We next prove Theorem 2.

*Proof of Theorem* 2. Define

$$
f_1(x) = \begin{cases} f(x) & |f(x)| > \frac{t}{2}, \\ 0 & \text{otherwise.} \end{cases}
$$

Then

$$
|f(x)| \leq |f_1(x)| + t/2 \quad \text{and } Mf(x) \leq Mf_1(x) + t/2.
$$

Hence, by Theorem 1,

$$\begin{aligned} H_\infty^\alpha(\{x : Mf(x) > t\}) &\leq H_\infty^\alpha(\{x : Mf_1(x) > t/2\}) \\ &\leq c_1 t^{-\alpha/d} \int_{|f|>t/2} |f|^{\alpha/d} dH_\infty^\alpha. \end{aligned}$$

Therefore we write

$$\begin{aligned} \int (Mf)^p dH_\infty^\alpha &= \int_0^\infty H_\infty^\alpha(\{(Mf)^p > t\}) dt \\ &= p \int_0^\infty H_\infty^\alpha(\{Mf > t\}) t^{p-1} dt \\ &\leq c_1 p \int_0^\infty t^{p-1} t^{-\alpha/d} dt \int_{|f|>t/2} |f|^{\alpha/d} dH_\infty^\alpha \\ &\leq I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= c_1 p \int_0^\infty t^{p-1} t^{-\alpha/d} dt \int_0^\infty H_\infty^\alpha(\{|f| > s^{d/\alpha}\}) \chi_{\{s^{d/\alpha} \geq t/2\}} ds, \\ I_2 &= c_1 p \int_0^\infty t^{p-1} t^{-\alpha/d} dt \int_0^\infty H_\infty^\alpha(\{|f| > t/2\}) \chi_{\{s^{d/\alpha} < t/2\}} ds. \end{aligned}$$

Using Fubini's theorem, we have

$$\begin{aligned} I_1 &\leq c_1 p \int_0^\infty H_\infty^\alpha(\{|f| > s^{d/\alpha}\}) ds \int_0^{2s^{d/\alpha}} t^{p-1-\alpha/d} dt \\ &= c_2 \int_0^\infty (s^{d/\alpha})^{p-\alpha/d} H_\infty^\alpha(\{|f| > s^{d/\alpha}\}) ds. \end{aligned}$$

Putting $t' = s^{dp/\alpha}$, we have

$$I_1 \leq c_3 \int_0^\infty H_\infty^\alpha(\{|f|^p > t'\}) dt' = c_3 \int |f|^p dH_\infty^\alpha.$$

We next estimate $I_2$. Note

$$\begin{aligned} I_2 &\leq c_1 p \int_0^\infty t^{p-1-\alpha/d} H_\infty^\alpha(\{|f| > t/2\}) dt \int_0^{(t/2)^{\alpha/d}} ds \\ &= c_1 p \int_0^\infty t^{p-1-\alpha/d} (t/2)^{\alpha/d} H_\infty^\alpha(\{|f| > t/2\}) dt. \end{aligned}$$

Put $t' = t/2$. Then

$$I_2 \leq c_4 \int_0^\infty (t')^{p-1} H_\infty^\alpha(\{|f| > t'\}) dt' = c_5 \int |f|^p dH_\infty^\alpha.$$

Thus we have the conclusion. □

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# Harmonic conjugates of parabolic Bergman functions

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**Abstract.**

The parabolic Bergman space is the Banach space of solutions of some parabolic equations on the upper half space which have finite $L^p$ norms. We introduce and study $L^{(\alpha)}$-harmonic conjugates of parabolic Bergman functions, and give a sufficient condition for a parabolic Bergman space to have unique $L^{(\alpha)}$-harmonic conjugates.

## 1. Introduction

Recently, Nishio, Shimomura, and Suzuki [4] have introduced parabolic Bergman spaces on the upper half-space and proved many interesting properties of these spaces. Parabolic Bergman spaces contain harmonic Bergman spaces studied by Ramey and Yi [6]. In this paper, we introduce and study $L^{(\alpha)}$-harmonic conjugates of parabolic Bergman functions, which are a generalized notion of usual harmonic conjugates of harmonic Bergman functions.

We describe the definition of parabolic Bergman spaces. Let $H$ be the upper half-space of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$, that is, $H = \{(x,t) \in \mathbb{R}^{n+1} \ ; \ x \in \mathbb{R}^n, \ t > 0\}$. For $1 \le p < \infty$, the Lebesgue space $L^p(H, dV)$ is defined to be the Banach space of Lebesgue measurable functions on $H$ with

$$\| u \|_p = \left( \int_H |u(x,t)|^p dV(x,t) \right)^{1/p} < \infty,$$

where $dV$ is the Lebesgue volume measure on $H$. For $0 < \alpha \le 1$, We define $L^{(\alpha)}$-harmonic functions on $H$. For $0 < \alpha < 1$, $(-\Delta)^\alpha$ is the

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convolution operator defined by

$$((-\Delta)^\alpha \varphi)(x,t) = -C_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y-x|>\delta} (\varphi(y,t) - \varphi(x,t))|y-x|^{-n-2\alpha} dy$$

for all $\varphi \in C_0^\infty(H)$, where $C_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$, and $\Delta$ is the Laplace operator with respect to $x$. For $0 < \alpha \leq 1$, a parabolic operator $L^{(\alpha)}$ is defined by $L^{(\alpha)} = \frac{\partial}{\partial t} + (-\Delta)^\alpha$. (We note that when $\alpha = 1$, $L^{(1)}$ is the heat operator.) A continuous function $u$ on $H$ is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions, that is, $u \cdot \tilde{L}^{(\alpha)}\varphi \in L^1(H, dV)$ and $\int u \cdot \tilde{L}^{(\alpha)}\varphi dV = 0$ for all $\varphi \in C_0^\infty(H)$, where $\tilde{L}^{(\alpha)} = -\frac{\partial}{\partial t} + (-\Delta)^\alpha$ is the adjoint operator of $L^{(\alpha)}$. For $1 \leq p < \infty$ and $0 < \alpha \leq 1$, the parabolic Bergman space $b^p_\alpha$ is the set of all $L^{(\alpha)}$-harmonic functions on $H$ which belong to $L^p(H, dV)$, and it is a Banach space with the $L^p$ norm. It is known that $b^p_\alpha \subset C^\infty(H)$ (see Theorem 5.4 of [4]), and when $\alpha = 1/2$, $b^p_{1/2}$ coincides with harmonic Bergman spaces of Ramey and Yi (see Corollary 4.4 of [4]).

We introduce the definition of $L^{(\alpha)}$-harmonic conjugates of parabolic Bergman functions. For a function $u$ on $H$ such that $\partial u/\partial x_j$ and $\partial u/\partial t$ exist at every $(x,t) = (x_1, \ldots, x_n, t) \in H$, we write $\partial_{x_j} u = \partial u/\partial x_j$ and $\partial_t u = \partial u/\partial t$, respectively.

Definition 1.1. *For a function $u \in b^p_\alpha$, the functions $v_1, \ldots, v_n$ are called $L^{(\alpha)}$-harmonic conjugates of $u$ if $v_1, \ldots, v_n$ satisfy the following conditions:*

*(1) $v_1, \ldots, v_n$ are $L^{(\alpha)}$-harmonic on $H$,*

*(2) $\partial_{x_j} v_k = \partial_{x_k} v_j$ and $\partial_{x_j} u = \partial_t v_j$ $(1 \leq j, k \leq n)$.*

Usually, given a harmonic function $u$ on $H$, the functions $v_1, \ldots, v_n$ on $H$ are called harmonic conjugates of $u$ if $(v_1, \ldots, v_n, u) = \nabla f$ for some harmonic function $f$ on $H$. As mentioned above, $b^p_{1/2}$ coincide with harmonic Bergman spaces, and it is easy to see that when $\alpha = 1/2$ the conditions (1) and (2) of Definition 1.1 are equivalent to the definition of usual harmonic conjugates of harmonic Bergman functions. Hence, $L^{(\alpha)}$-harmonic conjugates are generalization of harmonic conjugates.

Many authors have studied and proved interesting and important results concerning properties of harmonic conjugates, (for instance, see Chapter III of [2]). One of the fundamental problems of harmonic conjugates is the boundedness of the conjugation operator. It is known that when $\alpha = 1/2$ there are unique harmonic conjugates $v_1, \ldots, v_n$ of a function $u \in b^p_{1/2}$ such that $v_j \in b^p_{1/2}$ ( see Theorem 6.1 of [6] ), and thus the conjugation operator is bounded on the harmonic Bergman spaces for all $1 \le p < \infty$. In this paper, we prove the following result (see Theorem 4.1): Let $0 < \alpha \le 1$ and $1 \le p < \infty$. If $\lambda = p(\frac{1}{2\alpha} - 1) > -1$ and $u \in b^p_\alpha$, then there exist unique $L^{(\alpha)}$-harmonic conjugates $v_1, \dots, v_n$ of $u$ such that $v_j \in b^p_\alpha(\lambda)$, where $b^p_\alpha(\lambda)$ is the weighted parabolic Bergman spaces (see section 3 for the definition). Hence, we obtain the conjugation operator from $b^p_\alpha$ into $b^p_\alpha(\lambda)$ is bounded whenever $\lambda = p(\frac{1}{2\alpha} - 1) > -1$.

Throughout this paper, $C$ will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

## 2. Existence of $L^{(\alpha)}$-harmonic conjugates

When $\alpha = 1/2$, there are unique harmonic conjugates $v_1, \dots, v_n$ of a function $u \in b^p_{1/2}$ such that $v_j \in b^p_{1/2}$ ( see Theorem 6.1 of [6] ). In this section, we show that there exist $L^{(\alpha)}$-harmonic conjugates $v_1, \dots, v_n$ of a function $u \in b^p_\alpha$ such that $t^{\frac{1}{2\alpha}-1} v_j \in L^p(H, dV)$ whenever $p(\frac{1}{2\alpha} - 1) > -1$.

A fundamental solution of the parabolic operator $L^{(\alpha)}$ plays an important role for studying parabolic Bergman spaces. We define the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, let

$$W^{(\alpha)}(x,t) = \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i\, x \cdot \xi)\, d\xi & t > 0 \\ 0 & t \le 0, \end{cases} \tag{2.1}$$

where $x \cdot \xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi \cdot \xi)^{1/2}$. The function $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ and $L^{(\alpha)}$-harmonic on $H$. We describe some properties of $W^{(\alpha)}$. We note that $W^{(\alpha)}(x,t) \ge 0$ and

$$\int_{\mathbb{R}^n} W^{(\alpha)}(x-y, s)dy = 1 \tag{2.2}$$

for all $x \in \mathbb{R}^n$ and $s > 0$. If $u \in b^p_\alpha$, then $u$ satisfies the Huygens property, that is,

$$u(x,t) = \int_{\mathbb{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s)dy \tag{2.3}$$

holds for all $x \in \mathbb{R}^n$ and $0 < s < t < \infty$ ( see Theorem 4.1 of [4] ). By (2.1), the fundamental solution $W^{(\alpha)}$ is in $C^\infty(H)$. Let $k \in \mathbb{N}_0$ and $\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}_0^n$ be a multi-index, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then, we define $\partial_x^\beta \partial_t^k = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n} \partial_t^k = \partial^{|\beta|+k} / \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n} \partial t^k$. Clearly, we have

$$\partial_x^\beta \partial_t^k W^{(\alpha)}(x-y, t+s) = (-1)^{|\beta|} \partial_y^\beta \partial_s^k W^{(\alpha)}(x-y, t+s) \tag{2.4}$$

for all $(x,t),(y,s)\in H$. The following estimate is (1) of Proposition 1 of [5] : there exists a constant $C>0$ such that

$$|\partial_x^\beta \partial_t^k W^{(\alpha)}(x,t)| \le \frac{Ct^{-k+1}}{(t+|x|^{2\alpha})^{\frac{n+|\beta|}{2\alpha}+1}} \ . \tag{2.5}$$

The following lemma is an immediate consequence of Theorem 1 of [5].

LEMMA 2.1. *Let $0<\alpha\le 1$, $1\le q<\infty$, $\theta\in\mathbb{R}$, $\beta\in\mathbb{N}_0^n$ be a multi-index, and $k\in\mathbb{N}$. If $\big(\frac{n+|\beta|}{2\alpha}+k\big)q-\big(\frac{n}{2\alpha}+1\big)>\theta>-1$, then there exists a constant $C>0$ such that*

$$\begin{aligned}&\int_H t^\theta |\partial_x^\beta \partial_t^k W^{(\alpha)}(x-y,t+s)|^q dV(x,t)\\ \le\ & Cs^{\frac{n}{2\alpha}+1-(\frac{n+|\beta|}{2\alpha}+k)q+\theta}\end{aligned}$$

*for all$(y,s)\in H$.*

Let $c_k=\frac{(-2)^k}{k!}$. The following lemma is Theorem 6.7 of [4].

LEMMA 2.2. *Let $0<\alpha\le 1$ and $1\le p<\infty$. If $u\in b_\alpha^p$ and $(y,s)\in H$, then*

$$u(y,s)=-2c_{m+j}\int_H \partial_t^m u(x,t)\ t^{m+j}\partial_t^{j+1}W^{(\alpha)}(x-y,t+s)dV(x,t)$$

*for all $m,\ j\in\mathbb{N}_0$.*

PROPOSITION 2.3. *Let $0<\alpha\le 1$ and $1\le p<\infty$. If $\lambda=p(\frac{1}{2\alpha}-1)>-1-\frac{n}{2\alpha}$ and $u\in b_\alpha^p$, then there exist $L^{(\alpha)}$-harmonic conjugates $v_1,\dots,v_n$ of $u$.*

PROOF. For each $1\le j\le n$, let $v_j$ be a function on $H$ defined by

$$v_j(y,s)=2c_1\int_H u(x,t)\ t\ \partial_{x_j}\partial_t W^{(\alpha)}(x-y,t+s)dV(x,t). \tag{2.6}$$

Since $p(\frac{1}{2\alpha}-1)>-1-\frac{n}{2\alpha}$, Lemma 2.1 implies that

$$t\partial_{x_j}\partial_t W^{(\alpha)}(\,\cdot-y,\ \cdot+s)\in L^q(H,dV),$$

where $q$ is the exponent conjugate to $p$. Hence, the function $v_j$ is well defined for all $(y,s)\in H$ when $p(\frac{1}{2\alpha}-1)>-1-\frac{n}{2\alpha}$. We show that

$v_1, \dots, v_n$ are the $L^{(\alpha)}$-harmonic conjugates of $u$. Since $W^{(\alpha)}$ is $L^{(\alpha)}$-harmonic, so is $v_j$. Moreover, since (2.4) and Lemma 2.1 imply that

$$t\partial_{y_k}\partial_{x_j}\partial_t W^{(\alpha)}(\,\cdot - y,\ \cdot + s) \in L^q(H, dV)$$

for all $1 < q \leq \infty$, we can differentiate through the integral (2.6) with respect to $y_k$. Therefore we obtain $\partial_{y_k} v_j = \partial_{y_j} v_k$. Similarly, Lemmas 2.1 and 2.2 imply that $\partial_s v_j = \partial_{y_j} u$. □

REMARK 2.4. We note that when $0 < \alpha \leq \frac{1}{2}$, the assumption $\lambda = p(\frac{1}{2\alpha} - 1) > -1 - \frac{n}{2\alpha}$ of Proposition 2.3 always holds for all $1 \leq p < \infty$.

We consider an integrability condition of the function $v_j$ which is defined in (2.6).

THEOREM 2.5. *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. If $\lambda = p(\frac{1}{2\alpha} - 1) > -1$, then there exists a constant $C > 0$ such that*

$$\| t^{\frac{1}{2\alpha}-1} v_j \|_p \leq C \| u \|_p$$

*for all $u \in b^p_\alpha$ and $1 \leq j \leq n$, where $v_j$ is defined in* (2.6).

PROOF. Let $c = \frac{1}{2\alpha} - 1$. We suppose that $p = 1$ (we note that when $p = 1$, $\lambda > -1$ for all $0 < \alpha \leq 1$). Then, (2.6) and the Fubini theorem imply that there exists a constant $C > 0$ such that

$$\begin{aligned}
&\int_H |s^c v_j(y,s)| dV(y,s) \\
\leq\ & C \int_H |u(x,t)|\ t \int_H s^c |\partial_{x_j}\partial_t W^{(\alpha)}(x-y, t+s)| dV(y,s) dV(x,t).
\end{aligned}$$

Therefore, Lemma 2.1 implies that $\| t^{\frac{1}{2\alpha}-1} v_j \|_1 \leq C \| u \|_1$.

Suppose that $p > 1$, and let $q$ be the exponent conjugate to $p$. Then, the Hölder inequality shows that there exists a constant $C > 0$ such that

$$\begin{aligned}
&|v_j(y,s)| \\
\leq\ & C \int_H |u(x,t)|\ t^{\frac{1}{pq}+\frac{1}{p}}\ t^{-\frac{1}{pq}+\frac{1}{q}} \\
& \times |\partial_{x_j}\partial_t W^{(\alpha)}(x-y,t+s)|^{\frac{1}{p}+\frac{1}{q}} dV(x,t) \\
\leq\ & C \left( \int_H |u(x,t)|^p t^{\frac{1}{q}+1} |\partial_{x_j}\partial_t W^{(\alpha)}(x-y,t+s)| dV(x,t) \right)^{\frac{1}{p}} \\
& \times \left( \int_H t^{-\frac{1}{p}+1} |\partial_{x_j}\partial_t W^{(\alpha)}(x-y,t+s)| dV(x,t) \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\lambda = p(\frac{1}{2\alpha} - 1) > -1$, Lemma 2.1 implies that

$$\int_H t^{-\frac{1}{p}+1} |\partial_{x_j} \partial_t W^{(\alpha)}(x-y, t+s)| dV(x,t) \leq C s^{-(\frac{1}{2\alpha}+\frac{1}{p}-1)}.$$

Thus, by the Fubini theorem we have

$$\begin{aligned} &\int_H |s^c v_j(y,s)|^p dV(y,s) \\ \leq\ & C \int_H |u(x,t)|^p t^{\frac{1}{q}+1} \\ & \times \int_H s^{cp-(\frac{1}{2\alpha}+\frac{1}{p}-1)\frac{p}{q}} |\partial_{x_j} \partial_t W^{(\alpha)}(x-y, t+s)| dV(y,s) dV(x,t). \end{aligned}$$

Lemma 2.1 also implies that

$$\int_H s^{cp-(\frac{1}{2\alpha}+\frac{1}{p}-1)\frac{p}{q}} |\partial_{x_j} \partial_t W^{(\alpha)}(x-y, t+s)| dV(y,s) \leq C t^{-(\frac{1}{q}+1)}.$$

Therefore, we obtain $\| t^{\frac{1}{2\alpha}-1} v_j \|_p \leq C \| u \|_p$. □

## 3. Weighted parabolic Bergman spaces

In Proposition 2.3 and Theorem 2.5, we prove that the function $v_j$ which is defined in (2.6) is $L^{(\alpha)}$-harmonic and in $L^p(H, t^\lambda dV)$, where $\lambda = p(\frac{1}{2\alpha} - 1)$. In order to study the $L^{(\alpha)}$-harmonic conjugates, we define weighted parabolic Bergman spaces. For any $\lambda > -1$, the weighted parabolic Bergman space $b^p_\alpha(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions on $H$ which belong to $L^p(H, t^\lambda dV)$. We note that any function $u \in L^p(H, t^\lambda dV)$ satisfies $u \cdot \tilde{L}^{(\alpha)} \varphi \in L^1(H, dV)$ for all $\varphi \in C_0^\infty(H)$. In fact, it is known that $u \cdot \tilde{L}^{(\alpha)} \varphi \in L^1(H, dV)$ for all $\varphi \in C_0^\infty(H)$ if and only if

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u(x,t)| (1+|x|)^{-n-2\alpha} dV(x,t) < \infty$$

for all $t_2 > t_1 > 0$ ( see Remark 2.2 of [4] ). If $u \in L^p(H, t^\lambda dV)$ for some $1 \leq p < \infty$ and $\lambda > -1$, then elementary calculations show that $\int_{t_1}^{t_2} \int_{\mathbb{R}^n} |u(x,t)|(1+|x|)^{-n-2\alpha} dV(x,t) < \infty$ for all $t_2 > t_1 > 0$. Hence, $u \in L^p(H, t^\lambda dV)$ satisfies the integrability condition in the definition of $L^{(\alpha)}$-harmonic functions.

We give some properties of the weighted parabolic Bergman spaces. When $\lambda = 0$, the following lemma is Theorem 4.1 of [4]. We claim that $u \in b^p_\alpha(\lambda)$ also satisfies the Huygens property.

Lemma 3.1. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$ and $\lambda > -1$. If $u \in b^p_\alpha(\lambda)$, then*

$$u(x,t) = \int_{\mathbb{R}^n} u(x-y, t-s) W^{(\alpha)}(y,s) dy$$

*for all $x \in \mathbb{R}^n$ and $0 < s < t < \infty$.*

Proof. In the proof of Theorem 4.1 of [4], the Huygens property for $u \in b^p_\alpha$ derives from an $L^{(\alpha)}$-harmonicity of $u$ and a local integrability of a function $U(t) = \int_{\mathbb{R}^n} |u(x,t)|^p dx$ on $(0,\infty)$. If $u \in b^p_\alpha(\lambda)$, then it is easy to check that the function $U(t)$ is also locally integrable on $(0,\infty)$. Therefore, $u$ satisfies the Huygens property. □

Remark 3.2. It was known that for $u \in b^p_\alpha$ the function $U(t) = \int_{\mathbb{R}^n} |u(x,t)|^p dx$ is decreasing on $(0,\infty)$ ( see Lemma 5.6 of [4] ). By Lemma 3.1 and the Minkowski inequality, for any $\lambda > -1$ the same result holds for $u \in b^p_\alpha(\lambda)$.

When $\lambda = 0$, the following lemma is Proposition 5.2 of [4].

Lemma 3.3. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$ and $\lambda > -1$. Then there exists a constant $C > 0$ such that*

$$|u(x,t)| \leq C t^{-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}} \left( \int_H |u(y,s)|^p s^\lambda dV(y,s) \right)^{\frac{1}{p}}$$

*for all $(x,t) \in H$ and $u \in b^p_\alpha(\lambda)$.*

Proof. Since the proof of Lemma 3.3 is analogous to that of Proposition 5.2 of [4], we describe the outline of the proof. For fixed $0 < a_1 < a_2 < 1$, Lemma 3.1 implies that

$$u(x,t) = \frac{1}{(a_2 - a_1)t} \int_{a_1 t}^{a_2 t} \int_{\mathbb{R}^n} u(y, t-s) W^{(\alpha)}(x-y, s) dy ds.$$

Then, using the Jensen inequality and (2.5), we have

$$\begin{aligned}
&|u(x,t)| \\
\leq\ & Ct^{-(\frac{n}{2\alpha}+1)\frac{1}{p}} \left( \int_{a_1 t}^{a_2 t} \int_{\mathbb{R}^n} |u(y,t-s)|^p dyds \right)^{\frac{1}{p}} \\
=\ & Ct^{-(\frac{n}{2\alpha}+1)\frac{1}{p}} \left( \int_{a_1 t}^{a_2 t} (t-s)^{-\lambda}(t-s)^{\lambda} \int_{\mathbb{R}^n} |u(y,t-s)|^p dyds \right)^{\frac{1}{p}} \\
\leq\ & Ct^{-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}} \left( \int_{a_1 t}^{a_2 t} (t-s)^{\lambda} \int_{\mathbb{R}^n} |u(y,t-s)|^p dyds \right)^{\frac{1}{p}},
\end{aligned}$$

because $(1-a_2)t < t-s < (1-a_1)t$ whenever $a_1 t < s < a_2 t$. Hence, we obtain

$$|u(x,t)| \leq Ct^{-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}} \left( \int_0^{\infty} s^{\lambda} \int_{\mathbb{R}^n} |u(y,s)|^p dyds \right)^{\frac{1}{p}}.$$

□

By Lemma 3.1, $u \in b^p_\alpha(\lambda)$ is in $C^\infty(H)$. Thus, as in the proof of Lemma 3.3, we have the following lemma, which is Theorem 5.4 of [4] when $\lambda = 0$.

Lemma 3.4. *Let $0 < \alpha \leq 1$, $1 \leq p < \infty$ and $\lambda > -1$. If $\beta \in \mathbb{N}_0^n$ is a multi-index and $k \in \mathbb{N}_0$, then there exists a constant $C > 0$ such that*

$$|\partial_x^\beta \partial_t^k u(x,t)| \leq Ct^{-(\frac{|\beta|}{2\alpha}+k)-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}} \left( \int_H |u(y,s)|^p s^\lambda dV(y,s) \right)^{\frac{1}{p}}$$

*for all $(x,t) \in H$ and $u \in b^p_\alpha(\lambda)$.*

For $\delta > 0$ and a function $u$ on $H$, we write $u_\delta(x,t) = u(x,t+\delta)$. We note that if $u \in b^p_\alpha(\lambda)$ then $u_\delta \in b^p_\alpha(\lambda)$ for all $\delta > 0$. In fact, if $u \in b^p_\alpha(\lambda)$, then

$$\begin{aligned}
& \int_1^\infty t^\lambda \int_{\mathbb{R}^n} |u(x,t+\delta)|^p dxdt \\
\leq\ & C \int_1^\infty (t+\delta)^\lambda \int_{\mathbb{R}^n} |u(x,t+\delta)|^p dxdt \\
\leq\ & C \int_H |u(x,t)|^p t^\lambda dV.
\end{aligned}$$

Moreover, Remark 3.2 implies that

$$\int_0^1 t^\lambda \int_{\mathbb{R}^n} |u(x,t+\delta)|^p dxdt \le U(\delta) \int_0^1 t^\lambda dt < \infty.$$

Hence, we have $u_\delta \in b^p_\alpha(\lambda)$.

When $\lambda = 0$, the following lemma is Lemma 6.6 of [4].

LEMMA 3.5. *Let $0 < \alpha \le 1$, $1 \le p < \infty$ and $\lambda > -1$. If $u \in b^p_\alpha(\lambda)$ and $(y,s) \in H$, then*

(3.1)

$$u_\delta(y,s) = -2c_{m+j} \int_H \partial_t^m u_\delta(x,t)\ t^{m+j} \partial_t^{j+1} W^{(\alpha)}(x-y,t+s)dV(x,t)$$

*for all $m,\ j \in \mathbb{N}_0$ and $\delta > 0$.*

PROOF. The proof of Lemma 3.5 is analogous to that of Lemma 6.6 of [4]. We only show that the integral (3.1) is well defined. By Lemma 3.4, there exist constants $C > 0$ and $0 < \varepsilon < 1$ such that

$$|\partial_t^m u_\delta(x,t)| \le C(t+\delta)^{-m-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}} \le Ct^{-m-\varepsilon}\ \delta^{\varepsilon-(\frac{n}{2\alpha}+\lambda+1)\frac{1}{p}}.$$

Therefore, we have

$$|\partial_t^m u_\delta(x,t) t^{m+j} \partial_t^{j+1} W^{(\alpha)}(x-y,t+s)| \le Ct^{j-\varepsilon} |\partial_t^{j+1} W^{(\alpha)}(x-y,t+s)|.$$

Hence, Lemma 2.1 implies that $\partial_t^m u_\delta(x,t) t^{m+j} \partial_t^{j+1} W^{(\alpha)}(x-y,t+s) \in L^1(H,dV)$. □

THEOREM 3.6. *Let $0 < \alpha \le 1$, $1 \le p < \infty$, and $\lambda > -1$. If $\gamma > -1$ and non-negative integers $\ell, m$ satisfy*

$$\gamma + (\ell - m)p > -1, \tag{3.2}$$

*then there exists a constant $C > 0$ such that*

$$\int_H t^{\gamma+(\ell-m)p} |\partial_t^\ell u_\delta|^p dV \le C \int_H t^\gamma |\partial_t^m u_\delta|^p dV \tag{3.3}$$

*for all $u \in b^p_\alpha(\lambda)$ and $\delta > 0$.*

PROOF. Suppose that $p > 1$, and let $q$ be the exponent conjugate to $p$. By (3.2), we can choose a constant $\eta > 0$ such that

$$\gamma + (\ell - m)p - \frac{p}{q}\eta > -1 \tag{3.4}$$

Moreover, let $j$ be a non-negative integer such that

$$-\eta + \ell + j > -1 \tag{3.5}$$

and

$$\ell + j > \gamma + (\ell - m)p - \frac{p}{q}\eta. \tag{3.6}$$

Since, as in the proof of Lemma 3.5, there exist constants $C > 0$ and $0 < \varepsilon < 1$ such that

$$\begin{aligned} & |\partial_t^m u_\delta(x,t) t^{m+j} \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| \\ \leq \ & C t^{j-\varepsilon} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)|, \end{aligned}$$

Lemma 2.1 implies that

$$\partial_t^m u_\delta(x,t) t^{m+j} \partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s) \in L^1(H, dV).$$

Therefore, by Lemma 3.5 we have

(3.7)

$$\partial_s^\ell u_\delta(y,s) = -2c_{m+j} \int_H \partial_t^m u_\delta(x,t)\, t^{m+j} \partial_t^{\ell+j+1} W^{(\alpha)}(x-y, t+s) dV(x,t).$$

As in the proof of Theorem 2.5, the Hölder inequality implies that there exists a constant $C > 0$ such that

$$\begin{aligned} & |\partial_s^\ell u_\delta(y,s)|^p \\ \leq \ & C \left( \int_H t^{-\eta+\ell+j} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(x,t) \right)^{\frac{p}{q}} \\ \times & \int_H |\partial_t^m u_\delta(x,t)|^p t^{\frac{p(\eta+m-\ell)}{q}+m+j} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(x,t). \end{aligned}$$

By (3.5), Lemma 2.1 and the Fubini theorem imply that

$$\begin{aligned} & \int_H s^{\gamma+(\ell-m)p} |\partial_s^\ell u_\delta(y,s)|^p dV(y,s) \\ \leq \ & C \int_H s^{\gamma+(\ell-m)p-\frac{p}{q}\eta} \int_H |\partial_t^m u_\delta(x,t)|^p t^{\frac{p(\eta+m-\ell)}{q}+m+j} \\ & \times |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(x,t) dV(y,s) \\ = \ & C \int_H |\partial_t^m u_\delta(x,t)|^p t^{\frac{p(\eta+m-\ell)}{q}+m+j} \\ & \times \int_H s^{\gamma+(\ell-m)p-\frac{p}{q}\eta} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(y,s) dV(x,t). \end{aligned}$$

By (3.4) and (3.6), Lemma 2.1 also implies that

$$\begin{aligned} & \int_H s^{\gamma+(\ell-m)p-\frac{p}{q}\eta} |\partial_t^{\ell+j+1} W^{(\alpha)}(x-y,t+s)| dV(y,s) \\ \leq \ & C t^{\gamma+(\ell-m)p-\frac{p}{q}\eta-(\ell+j)}. \end{aligned}$$

Hence, we obtain

$$\int_H s^{\gamma+(\ell-m)p}|\partial_s^\ell u_\delta(y,s)|^p dV(y,s) \le C\int_H t^\gamma |\partial_t^m u_\delta(x,t)|^p dV(x,t).$$

We suppose that $p=1$. Then, using (3.7) and the Fubini theorem, we have

$$\begin{aligned}&\int_H s^{\gamma+\ell-m}|\partial_s^\ell u_\delta(y,s)|dV(y,s)\\ \le\ & C\int_H |\partial_t^m u_\delta(x,t)|t^{m+j}\\ &\times\int_H s^{\gamma+\ell-m}|\partial_t^{\ell+j+1}W^{(\alpha)}(x-y,t+s)|dV(y,s)dV(x,t).\end{aligned}$$

Since we can choose a non-negative integer $j$ such that $\gamma-m-j<0$, Lemma 2.1 implies that

$$\int_H s^{\gamma+\ell-m}|\partial_t^{\ell+j+1}W^{(\alpha)}(x-y,t+s)|dV(y,s)\le Ct^{\gamma-m-j}.$$

Hence, we have the theorem. □

For a function $u\in L^p(H,t^\lambda dV)$, define $\| u \|_{p,\lambda}=(\int_H |u|^p t^\lambda dV)^{1/p}$. We have the following inequalities.

COROLLARY 3.7. *Let $0<\alpha\le 1$, $1\le p<\infty$, and $\lambda>-1$. Then, there exists a constant $C>0$ such that*

$$C^{-1}\| u_\delta \|_{p,\lambda}\le\| t^\ell\partial_t^\ell u_\delta \|_{p,\lambda}\le C\| u_\delta \|_{p,\lambda} \tag{3.8}$$

*for all $u\in b_\alpha^p(\lambda)$, $\delta>0$, and $\ell\in\mathbb{N}_0$.*

## 4. Uniqueness of $L^{(\alpha)}$-harmonic conjugates

In this section, we show that $L^{(\alpha)}$-harmonic conjugates of $u\in b_\alpha^p$ are unique whenever $\lambda=p(\frac{1}{2\alpha}-1)>-1$.

THEOREM 4.1. *Let $0<\alpha\le 1$ and $1\le p<\infty$. If $\lambda=p(\frac{1}{2\alpha}-1)>-1$ and $u\in b_\alpha^p$, then there exist unique $L^{(\alpha)}$-harmonic conjugates $v_1,\dots,v_n$ of $u$ on $H$ such that $v_j\in b_\alpha^p(\lambda)$.*

PROOF. By Proposition 2.3 and Theorem 2.5, it suffices to prove the uniqueness of $L^{(\alpha)}$-harmonic conjugates of $u\in b_\alpha^p$ that belong to $b_\alpha^p(\lambda)$.

Suppose that $u_1, \ldots, u_n$ are also $L^{(\alpha)}$-harmonic conjugates of $u$ such that $u_j \in b^p_\alpha(\lambda)$. Take arbitrary $\delta > 0$. Then by Corollary 3.7, there exists a constant $C > 0$ such that

$$\| t^{\frac{1}{2\alpha}-1}(v_j - u_j)_\delta \|_p \leq C \| t^{\frac{1}{2\alpha}} \partial_t (v_j - u_j)_\delta \|_p . \tag{4.1}$$

By the hypothesis and the definition of $L^{(\alpha)}$-harmonic conjugates, we have

$$\partial_t (v_j - u_j)_\delta = \partial_{x_j} u_\delta - \partial_{x_j} u_\delta \equiv 0.$$

Therefore, (4.1) and the continuity of $v_j - u_j$ imply that $v_j(x, t+\delta) = u_j(x, t+\delta)$ for all $(x,t) \in H$. Since $\delta > 0$ is arbitrary, we obtain $v_j = u_j$ as desired. $\square$

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# On the behavior at infinity for non-negative superharmonic functions in a cone

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**Abstract.**

This paper shows that a positive superharmonic function on a cone behaves regularly outside an $a$-minimally thin set in a cone. This fact is known for a half space which is a special cone.

## §1. Introduction

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n$ ($n \geq 2$) the $n$-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, y)$, $X = (x_1, x_2, \dots, x_{n-1})$. The Euclidean distance of two points $P$ and $Q$ in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin $O$ of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $S$ in $\mathbf{R}^n$ are denoted by $\partial S$ and $\bar{S}$, respectively.

We introduce spherical coordinates $(r, \Theta)$, $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \dots, x_{n-1}, y)$ by

$$x_1 = r(\Pi_{j=1}^{n-1} \sin\theta_j) \quad (n \geq 2), \qquad y = r\cos\theta_1,$$

and if $n \geq 3$, then

$$x_{n+1-k} = r(\Pi_{j=1}^{k-1} \sin\theta_j)\cos\theta_k \quad (2 \leq k \leq n-1),$$

where $0 \leq r < +\infty$, $-\frac{1}{2}\pi \leq \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi$ $(1 \leq j \leq n-2)$.

The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the

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set $\{\Theta; (1,\Theta) \in \Omega\}$ for a set $\Omega$, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with $\Theta$ and $\Omega$, respectively. For two sets $\Lambda \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r,\Theta) \in \mathbf{R}^n;\ r \in \Lambda,\ (1,\Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Lambda \times \Omega$. In particular, the half-space $\mathbf{R}_+ \times \mathbf{S}_+^{n-1} = \{(X,y) \in \mathbf{R}^n;\ y > 0\}$ will be denoted by $\mathbf{T}_n$. By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain $\Omega$ on $\mathbf{S}^{n-1}(n \geq 2)$ having smooth boundary. We call it a cone. Then $\mathbf{T}_n$ is a special cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$.

Let $\Omega$ be a domain on $\mathbf{S}^{n-1}(n \geq 2)$ with smooth boundary. Consider the Dirichlet problem

$$\begin{aligned} (\Lambda_n + \tau)f &= 0 \quad \text{on } \Omega \\ f &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Lambda_n$ is the spherical part of the Laplace operator $\Delta_n$

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by $\tau_\Omega$ and the normalized positive eigenfunction corresponding to $\tau_\Omega$ by $f_\Omega(\Theta)$; $\int_\Omega f_\Omega^2(\Theta)d\sigma_\Theta = 1$, where $d\sigma_\Theta$ is the surface element on $\mathbf{S}^{n-1}$. We denote the solutions of the equation $t^2 + (n-2)t - \tau_\Omega = 0$ by $\alpha_\Omega, -\beta_\Omega$ $(\alpha_\Omega, \beta_\Omega > 0)$. If $\Omega = \mathbf{S}_+^{n-1}$, then $\alpha_\Omega = 1$, $\beta_\Omega = n-1$ and $f_\Omega(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1$, where $s_n$ is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

In the following, we shall assume that if $n \geq 3$, then $\Omega$ is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ (e.g. see Gilbarg and Trudinger [4] for the definition of $C^{2,\alpha}$-domain).

It is known that the Martin boundary of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$, each of which is a minimal Martin boundary point. When we denote the Martin kernel by $\tilde{K}(P,Q)$ $(P \in C_n(\Omega), Q \in \partial C_n(\Omega) \cup \{\infty\})$ with respect to a reference point chosen suitably, we know

$$\tilde{K}(P,\infty) = r^{\alpha_\Omega} f_\Omega(\Theta), \quad \tilde{K}(P,O) = \kappa r^{-\beta_\Omega} f_\Omega(\Theta) \quad (P \in C_n(\Omega)),$$

where $\kappa$ is a positive constant (Yoshida [8, p.292]).

Let $u(P)$ be a non-negative superharmonic function on $\mathbf{T}_n$, and let $c(u) = \inf_{P=(X,y)\in\mathbf{T}_n} u(P)/y$. Aikawa [1] introduced the notion of $a$-minimal thinness $(0 \leq a \leq 1)$, which is identical to minimal thinness when $a = 1$ and which is identical to rarefiedness when $a = 0$, and showed that

$$\lim_{|P|\to\infty,\ P\in\mathbf{T}_n\setminus E} \frac{u(P) - c(u)y}{y^a|P|^{1-a}} = 0, \tag{1.1}$$

with a set $E$ in $\mathbf{T}_n$ which is $a$-minimally thin at $\infty$. Aikawa also showed that if $E \subset \mathbf{T}_n$ is unbounded and a-minimally thin at $\infty$ in $\mathbf{T}_n$, then there exists a non-negative superharmonic function $u$ on $\mathbf{T}_n$ such that

$$\lim_{|P|\to\infty,\ P\in E} \frac{u(P)-c(u)y}{y^a|P|^{1-a}} = +\infty, \tag{1.2}$$

and showed that (1.1) is the best possible as to the size of the exceptional set. The cases of $a = 1$ in (1.1) and (1.2) give the result of Lelong-Ferrand [6, pp. 134-143], and the cases of $a = 0$ in (1.1) and (1.2) give the result of Essén and Jackson [3, Theorem 4.6].

For a non-negative superharmonic function in a cone, the results corresponding to $a = 1$ of (1.1) and (1.2) are showed by the Fatou boundary limit theorem for Martin space (Miyamoto and Yoshida [7, Remark 2]). In detail, for a non-negative superharmonic function $u$ on $C_n(\Omega)$, there exists a set $E \subset C_n(\Omega)$ which is minimally thin at $\infty$ such that

$$\lim_{|P|\to+\infty,\ P\in C_n(\Omega)\setminus E} \frac{u(P)-c_\infty(u)\tilde{K}(P,\infty)}{\tilde{K}(P,\infty)} = 0, \tag{1.3}$$

where we put $c_\infty(u) = \inf_{P\in C_n(\Omega)} \frac{u(P)}{\tilde{K}(P,\infty)}$. On the other hand, Miyamoto and Yoshida [7, Theorem 3] introduced the notion of rarefiedness at $\infty$ with respect to $C_n(\Omega)$, and showed that for a non-negative superharmonic function $u$ on $C_n(\Omega)$, there exists a set $E \subset C_n(\Omega)$ which is rarefied at $\infty$ such that

$$\lim_{|P|\to+\infty,\ P\in C_n(\Omega)\setminus E} \frac{u(P)-c_\infty(u)\tilde{K}(P,\infty)}{|P|^{\alpha_\Omega}} = 0. \tag{1.4}$$

(1.4) gives the extension of the case $a = 0$ in (1.1).

From these results, in this paper we shall introduce the notion of $a$-mimal thinness ($0 \le a \le 1$) at $\infty$ with respect to a cone and extend the above results for a cone ((1.3) and (1.4)). We shall also extend the results (1.1) and (1.2) bacause our main result contains (1.1) and (1.2) as the case $\Omega = \mathbf{S}_+^{n-1}$. The results of this paper are proved by modifying the methods of Aikawa [1] and Essén and Jackson [3].

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## §2. Preliminaries

We denote by $G(P,Q)$ $(P \in C_n(\Omega), Q \in C_n(\Omega))$ the Green function of $C_n(\Omega)$, and let $G\mu(P) = \int_{C_n(\Omega)} G(P,Q)d\mu(Q)$ be the Green potential at $P \in C_n(\Omega)$ of a positive Radon measure $\mu$.

Let $S_n(\Omega)$ be the set $\partial C_n(\Omega) \setminus \{O\}$. Now we shall define the Martin type kernel $K(P,Q)$ $(P = (r,\Theta) \in C_n(\Omega), Q = (t,\Phi) \in \overline{C_n(\Omega)} \cup \{\infty\})$ as follows:

$$K(P,Q) = \begin{cases} \dfrac{G(P,Q)}{t^{\alpha_\Omega} f_\Omega(\Phi)} & \text{on } C_n(\Omega) \times C_n(\Omega) \\ \dfrac{\partial G(P,Q)}{\partial n_Q} \left\{ t^{\alpha_\Omega - 1} \dfrac{\partial}{\partial n_\Phi} f_\Omega(\Phi) \right\}^{-1} & \text{on } C_n(\Omega) \times S_n(\Omega) \\ r^{\alpha_\Omega} f_\Omega(\Theta) & \text{on } C_n(\Omega) \times \{\infty\} \\ \kappa r^{-\beta_\Omega} f_\Omega(\Theta) & \text{on } C_n(\Omega) \times \{O\}, \end{cases}$$

where $\partial/\partial n_Q$ denotes the differentiation at $Q$ along the inward normal into $C_n(\Omega)$. We note that $K_P(Q) = K(P,Q)$ is continuous in the extended sence on $C_n(\Omega) \cup S_n(\Omega)$. Following Brelot [2, p.31], we let $K^*(P,Q) = K(Q,P)$ be the associated kernel of $K$ on $(\overline{C_n(\Omega)} \cup \{\infty\}) \times C_n(\Omega)$.

If $\mu$ is a measure on $\overline{C_n(\Omega)} \cup \{\infty\}$, we abbreviate $\int_{\overline{C_n(\Omega)} \cup \{\infty\}} K(P,Q)d\mu(Q)$ to $K\mu(P)$ and also $\int_{C_n(\Omega)} K^*(P,Q)d\nu(Q)$ to $K^*\nu(P)$ for a measure $\nu$ on $C_n(\Omega)$.

Let $u$ be a non-negative superharmonic function on $C_n(\Omega)$ and put $c_O(u) = \inf_{P \in C_n(\Omega)} \frac{u(P)}{K(P,O)}$. Then from Miyamoto and Yoshida [7, Lemma 3], we see that there exists a unique measure $\mu_u$ on $\overline{C_n(\Omega)} \cup \{\infty\}$ such that $u = K\mu_u$. When we denote by $\mu'_u$ the restriction of the measure $\mu_u$ on $C_n(\Omega)$, we have $u(P) = c_\infty(u)K(P,\infty) + c_O(u)K(P,O) + K\mu'_u(P)$.

For a number $a$, $0 \le a \le 1$, we define the positive superharmonic function $g_a$ by $g_a(P) = (K(P,\infty))^a \quad (P \in C_n(\Omega))$.

For a non-negative function $v$ on $C_n(\Omega)$ and $E \subset C_n(\Omega)$, let $\hat{R}_v^E$ be the regularized reduced function of $v$ relative to $E$ (Helms [5, p.116]).

Let $E$ be a bounded subset of $C_n(\Omega)$. We define the a-mass of $E$ by $\lambda_E^a(\overline{C_n(\Omega)})$ for $0 \le a \le 1$, where $\lambda_E^a$ is the measure on $\overline{C_n(\Omega)}$ such that $K\lambda_E^a = \hat{R}_{g_a}^E$.

Let $E \subset C_n(\Omega)$ be bounded. Then there exists a unique measure $\lambda_E$ on $C_n(\Omega)$ such that $\hat{R}_{\tilde{K}(\cdot,\infty)}^E = G\lambda_E$ on $C_n(\Omega)$. If $0 < a \le 1$, then following Yoshida [8, Corollary 5.3] we see the greatest harmonic minorant of $\hat{R}_{g_a}^E$ is zero, so that $\lambda_E^a(\partial C_n(\Omega)) = 0$. Then according to the proof of Aikawa [1, Lemma 2.1] we can similarly have

$$\lambda_E^a(\overline{C_n(\Omega)}) = \int_{C_n(\Omega)} g_a d\lambda_E. \tag{2.1}$$

In particular $\lambda_E^1(\overline{C_n(\Omega)}) = \int G\lambda_E d\lambda_E$ and $\lambda_E^0(\overline{C_n(\Omega)}) = \lambda_E(C_n(\Omega))$.

Let $E$ be a subset of $C_n(\Omega)$ and $E_k = E \cap I_k$, where

$$I_k = \{P \in C_n(\Omega);\ 2^k \le |P| < 2^{k+1}\} \quad (k = 0, 1, 2, \ldots).$$

We say that $E \subset C_n(\Omega)$ is $a$-minimally thin at $\infty$ in $C_n(\Omega)$ if

$$\sum_{k=0}^{\infty} \lambda_{E_k}^a(\overline{C_n(\Omega)}) 2^{-k(a\alpha_\Omega + \beta_\Omega)} < +\infty.$$

*Remark* 2.1. From Theorems 1 and 2 of Miyamoto and Yoshida [7] and (2.1), we see that the notion of $a$-minimal thinness contains the notions of minimal thinness and rarefiedness.

In the following we set

$$C_n(\Omega;\ a,\ b) = \{P = (r, \Theta) \in C_n(\Omega);\ a < r < b\} \qquad (0 < a < b \le +\infty),$$
$$S_n(\Omega;\ a,\ b) = \{P = (r, \Theta) \in S_n(\Omega);\ a < r < b\} \qquad (0 < a < b \le +\infty).$$

As far as we are concerned with $a$-minimal thinness in the following, we shall restrict a subset $E$ of $C_n(\Omega)$ to the set located in $C_n(\Omega; 1, +\infty)$, because the part of $E$ separated from $\infty$ is unessential to $a$-minimal thinness.

## §3. Statements of results

Let $\eta$ be a real number satisfying $(2-n)\frac{1}{\alpha_\Omega} - 1 < \eta \le 1$. We define the positive superharmonic function $h_\eta$ on $C_n(\Omega)$ by $h_\eta(P) = K(P, \infty)|P|^{\{(2-n)\frac{1}{\alpha_\Omega} - 1 - \eta\}\alpha_\Omega}$. Since $K(P, \infty)$ is a minimal harmonic function on $C_n(\Omega)$, we see that there exists a measure $\nu_\eta$ on $C_n(\Omega)$ such that $G\nu_\eta(P) = \min(K(P, \infty), h_\eta(P))$.

Let $\mathfrak{F}_\eta$ be the class of all non-negative superharmonic functions $u$ on $C_n(\Omega)$ such that $c_\infty(u) = 0$ and

$$\int_{C_n(\Omega;1,+\infty) \cup S_n(\Omega;1,+\infty)} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega} - 1 - \eta\}\alpha_\Omega} d\mu_u(Q) < +\infty. \tag{3.1}$$

*Remark* 3.1. If $P \in C_n(\Omega)$, then $K^*\nu_\eta(P) = G\nu_\eta(P)/K(P,\infty)$. If $P \in S_n(\Omega)$, then $K^*\nu_\eta(P) = \liminf_{Q \to P, Q \in C_n(\Omega)} K^*\nu_\eta(Q)$ (cf. Essén and Jackson [3, p.240]). Hence for a point $P \in C_n(\Omega) \cup S_n(\Omega)$, we have

$$K^*\nu_\eta(P) = \begin{cases} 1 & \text{for } 0 < |P| < 1, \\ |P|^{\{(2-n)\frac{1}{\alpha_\Omega} - 1 - \eta\}\alpha_\Omega} & \text{for } |P| \geq 1. \end{cases} \tag{3.2}$$

Let $u \in \mathfrak{F}_\eta$. From (3.2) we see that (3.1) is equivalent to the following condition;

$$\int_{C_n(\Omega)} \{u(P) - c_O(u)K(P,O)\} d\nu_\eta(P) < +\infty.$$

If $u_1$, $u_2 \in \mathfrak{F}_\eta$ and $c$ is a positive constant, then $u_1 + u_2$, $cu_1 \in \mathfrak{F}_\eta$.

Let $v \in \mathfrak{F}_\eta$ such that $c_O(v) = 0$, and let $u$ be a non-negative superharmonic function such that $c_O(u) = 0$. Then $0 \leq u \leq v$ on $C_n(\Omega)$ implies $u \in \mathfrak{F}_\eta$ (cf. Aikawa [1, Lemma 3.1]).

We define the function $h_{\eta,a}(P) = K(P,\infty)^a |P|^{(\eta-a)\alpha_\Omega}$ $(P \in C_n(\Omega))$.

**Theorem 3.1.** *If $u(P) \in \mathfrak{F}_\eta$, then there exists a set $E \subset C_n(\Omega)$ which is a-minimally thin at $\infty$ with respect to $C_n(\Omega)$ such that*

$$\lim_{|P| \to +\infty,\ P \in C_n(\Omega) \setminus E} \frac{u(P)}{h_{\eta,a}(P)} = 0.$$

*Conversely, if $E$ is unbounded and a-minimally thin at $\infty$ with respect to $C_n(\Omega)$, then there exists $u(P) \in \mathfrak{F}_\eta$ such that*

$$\lim_{|P| \to +\infty,\ P \in E} \frac{u(P)}{h_{\eta,a}(P)} = +\infty.$$

When $\Omega = \mathbf{S}_+^{n-1}$, we obtain the result of Aikawa [1, Theorem 3.2].

Let $u(P)$ be a non-negative superharmonic function on $C_n(\Omega)$. Since $u_1(P) = u(P) - c_\infty(u)K(P,\infty)$ belongs to $\mathfrak{F}_1$, we obtain the following Corollary 3.1 by applying Theorem 3.1 of the case $\eta = 1$ to $u_1$.

**Corollary 3.1.** *Let $u(P)$ be a non-negative superharmonic function on $C_n(\Omega)$. Then there exists a set $E \subset C_n(\Omega)$ which is a-minimally thin at $\infty$ with respect to $C_n(\Omega)$ such that*

$$\lim_{|P| \to +\infty,\ P \in C_n(\Omega) \setminus E} \frac{u(P) - c_\infty(u)K(P,\infty)}{K(P,\infty)^a |P|^{(1-a)\alpha_\Omega}} = 0.$$

*Conversely, if $E$ is unbounded and a-minimally thin at $\infty$ with respect to $C_n(\Omega)$, then there exists a non-negative superharmonic function $u(P)$*

*such that*

$$\lim_{|P|\to+\infty,\ P\in E} \frac{u(P) - c_\infty(u)K(P,\infty)}{K(P,\infty)^a |P|^{(1-a)\alpha_\Omega}} = +\infty.$$

The case $a = 0$ in Corollary 3.1 gives the result of Miyamoto and Yoshida [7, Theorem 3].

## §4. Proof of Theorem 3.1

We remark that

$$G(P,Q) \leq M_1 r^{\alpha_\Omega} t^{-\beta_\Omega} f_\Omega(\Theta) f_\Omega(\Phi) \tag{4.1}$$

$$(\text{resp. } G(P,Q) \leq M_2 t^{\alpha_\Omega} r^{-\beta_\Omega} f_\Omega(\Theta) f_\Omega(\Phi)) \tag{4.2}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in C_n(\Omega)$ satisfying $0 < \frac{r}{t} \leq \frac{1}{2}$ (resp. $0 < \frac{t}{r} \leq \frac{1}{2}$), where $M_1$ (resp. $M_2$) is a positive constant. From (4.1) and (4.2) we have the following inequalities:

$$\frac{\partial G(P,Q)}{\partial n_Q} \leq M_3 r^{\alpha_\Omega} t^{-\beta_\Omega - 1} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi) \tag{4.3}$$

$$(\text{resp. } \frac{\partial G(P,Q)}{\partial n_Q} \leq M_4 t^{\alpha_\Omega - 1} r^{-\beta_\Omega} f_\Omega(\Theta) \frac{\partial}{\partial n_\Phi} f_\Omega(\Phi)) \tag{4.4}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ and any $Q = (t,\Phi) \in S_n(\Omega)$ satisfying $0 < \frac{r}{t} \leq \frac{1}{2}$ (resp. $0 < \frac{t}{r} \leq \frac{1}{2}$), where $M_3$ (resp. $M_4$) is a positive constant and $\partial/\partial n_\Phi$ denotes the differntiation at $\Phi \in \partial\Omega$ along the inward normal into $\Omega$ (Miyamoto and Yoshida [7]).

For two positive functions $u$ and $v$, we shall write $u \approx v$ if and only if there exist constants $A$, $B$, $0 < A \leq B$, such that $Av \leq u \leq Bv$ everywhere on $C_n(\Omega)$.

**Lemma 4.1.** *$E \subset C_n(\Omega; 1, +\infty)$ is a-minimally thin at $\infty$ if and only if $\sum_{k=0}^{\infty} \hat{R}^{E_k}_{h_{\eta,a}} \in \mathfrak{F}_\eta$.*

*Proof.* We note that for every $k = 0, 1, 2, \ldots$,

$$\hat{R}^{E_k}_{g_a} \approx 2^{-k(\eta-a)\alpha_\Omega} \hat{R}^{E_k}_{h_{\eta,a}},$$

$$\lambda^a_{E_k}(\overline{C_n(\Omega)}) \approx 2^{-k\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} \int_{C_n(\Omega)\cup S_n(\Omega)} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\lambda^a_{E_k}(Q),$$

where the constants of comparison are independent of $k$. Since

$$\int_{C_n(\Omega)} \hat{R}_{g_a}^{E_k}(P)d\nu_\eta(P) = \int_{C_n(\Omega)} K\lambda_{E_k}^a(P)d\nu_\eta(P)$$
$$= \int_{C_n(\Omega)\cup S_n(\Omega)} K^*\nu_\eta(Q)d\lambda_{E_k}^a(Q) = \int_{C_n(\Omega)\cup S_n(\Omega)} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\lambda_{E_k}^a(Q),$$

we have $2^{k(-a\alpha_\Omega-\beta_\Omega)}\lambda_{E_k}^a(\overline{C_n(\Omega)}) \approx \int_{C_n(\Omega)} \hat{R}_{h_{\eta,a}}^{E_k}(P)d\nu_\eta(P)$ where the constants of comparison are independent of $k$, which gives the conclusion. □

**Lemma 4.2.** *Let $E$ be a set in $C_n(\Omega;1,+\infty)$. If $\hat{R}_{h_{\eta,a}}^E \in \mathfrak{F}_\eta$, then $E$ is a-minimally thin at $\infty$.*

*Proof.* Since $h_{\eta,a}(P)$ satisfies

$$\liminf_{|P|\to\infty} \frac{h_{\eta,a}(P)}{K(P,\infty)|P|^{(\eta-1)\alpha_\Omega}} > 0,$$

we find a positive constant $C'$ and a natural number $N_1$ such that $h_{\eta,a}(P) \geq C'K(P,\infty)|P|^{(\eta-1)\alpha_\Omega}$ for $|P| > 2^{N_1}$. Let $C_1 = M_1/C'$, $C_2 = M_2/C'$, $C_3 = M_3/C'$ and $C_4 = M_4/C'$. And put $C = \max_{1\leq i\leq 4}\{C_i\}$.

Let $\hat{R}_{h_{\eta,a}}^E = K\mu$, where $\mu$ satisfies (3.1). Noting (3.1), we put $A = \int_{C_n(\Omega;1,+\infty)\cup S_n(\Omega;1,+\infty)} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q) < +\infty$. We take a natural number $N_2$ such that $4AC < 2^{-N_2\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega}$. Then there exists a natural number $k_0$ such that

$$C\int_{\{Q\in C_n(\Omega)\cup S_n(\Omega);\ |Q|\geq 2^{k+N_2+1}\}} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q) < \frac{1}{4}$$

for $k \geq k_0$. Let $N = \max\{N_1, N_2, k_0\}$. Hence it is sufficient to prove $\sum_{k>N} \hat{R}_{h_{\eta,a}}^{E_k} \in \mathfrak{F}_\eta$ beacause $\sum_{k=0}^N \hat{R}_{h_{\eta,a}}^{E_k} \leq (N+1)\hat{R}_{h_{\eta,a}}^E \in \mathfrak{F}_\eta$. We set $J_k = I_{k-N_2} \cup \cdots \cup I_k \cup \cdots \cup I_{k+N_2}$. Let $k > N$ and let $P = (r,\Theta) \in E_k$. If $Q \in C_n(\Omega)$ and $|Q| \leq 2^{k-N_2}$, then from (4.2) we have

$$K(P,Q) = \frac{G(P,Q)}{t^{\alpha_\Omega} f_\Omega(\Phi)} \leq M_2 r^{-\beta_\Omega} f_\Omega(\Theta).$$

Hence

$$\int_{\{Q\in C_n(\Omega);\ |Q|\leq 2^{k-N_2}\}} K(P,Q)d\mu(Q) \leq C_2 h_{\eta,a}(P) r^{-(\eta\alpha_\Omega+\beta_\Omega)} \int_{1\leq|Q|\leq 2^{k-N_2}} d\mu(Q)$$
$$\leq C_2 h_{\eta,a}(P) \int_{1\leq|Q|\leq 2^{k-N_2}} |P|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q).$$

On the other hand, if $Q \in C_n(\Omega)$ and $|Q| \geq 2^{k+N_2+1}$, then from (4.1) we have

$$\int\limits_{\{Q \in C_n(\Omega);\ |Q| \geq 2^{k+N_2+1}\}} K(P,Q)d\mu(Q) \leq C_1 h_{\eta,a}(P) r^{-(\eta-1)\alpha_\Omega} \int\limits_{|Q| \geq 2^{k+N_2+1}} |Q|^{-(\alpha_\Omega+\beta_\Omega)} d\mu(Q)$$
$$\leq C_1 h_{\eta,a}(P) \int\limits_{|Q| \geq 2^{k+N_2+1}} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q).$$

If $Q \in S_n(\Omega)$ and $|Q| \leq 2^{k-N_2}$ or $Q \in S_n(\Omega)$ and $|Q| \geq 2^{k+N_2+1}$, then from (4.4) or (4.3) we have similar inequalities. From these inequalities, we have

$$C^{-1} \int\limits_{\overline{C_n(\Omega)} \setminus \bar{J}_k} K(P,Q)d\mu(Q) \leq h_{\eta,a}(P) \int\limits_{|Q| \leq 2^{k-N_2}} |P|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q)$$
$$+ h_{\eta,a}(P) \int\limits_{|Q| \geq 2^{k+N_2+1}} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q).$$

Since $4AC < 2^{-N_2\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega}$, we see that

$$C \int\limits_{|Q| \leq 2^{k-N_2}} |P|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q) \leq \frac{1}{4A} \int\limits_{|Q| \leq 2^{k-N_2}} \left(\frac{|P|}{2^{N_2}}\right)^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q)$$
$$\leq \frac{1}{4A} \int\limits_{|Q| \leq 2^{k-N_2}} |Q|^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} d\mu(Q) \leq \frac{1}{4}.$$

So we have $\int_{\overline{C_n(\Omega)} \setminus \bar{J}_k} K(P,Q)d\mu(Q) \leq \frac{1}{2} h_{\eta,a}(P)$ on $E_k$, which implies that

$$h_{\eta,a}(P) \leq \hat{R}^E_{h_{\eta,a}}(P) \leq \int_{\bar{J}_k} K(P,Q)d\mu(Q) + \frac{1}{2} h_{\eta,a}(P)$$

q.e. on $E_k$. Hence $h_{\eta,a}(P) \leq 2\int_{\bar{J}_k} K(P,Q)d\mu(Q)$ q.e. on $E_k$. Therefore $\hat{R}^{E_k}_{h_{\eta,a}}(P) \leq 2\int_{J_k} K(P,Q)d\mu(Q)$ on $C_n(\Omega)$, by the definition of $\hat{R}^{E_k}_{h_{\eta,a}}$. If we sum up $\hat{R}^{E_k}_{h_{\eta,a}}$ over $k > N$, we obtain $\sum_{k>N}^{\infty} \hat{R}^{E_k}_{h_{\eta,a}} \leq 2(2N_2+1)\hat{R}^E_{h_{\eta,a}}$. By Remark 3.1 we see $\sum_{k>N} \hat{R}^{E_k}_{h_{\eta,a}} \in \mathfrak{F}_\eta$. Thus the lemma follows from Lemma 4.1. □

*Proof of Theorem* 3.1. Let $u_1(P) = u(P) - c_O(u)K(P,O)$ $(P \in C_n(\Omega))$, then we see $u_1 \in \mathfrak{F}_\eta$. For each non-negative integer $j$, we set $A_j = \{P \in C_n(\Omega; 1, +\infty);\ u_1(P)/h_{\eta,a}(P) \geq (j+1)^{-1}\}$. Since $\hat{R}^{A_j}_{h_{\eta,a}} \leq$

$(j+1)u_1 \in \mathfrak{F}_\eta$, we see from Remark 3.1 that $\hat{R}^{A_j}_{h_{\eta,a}} \in \mathfrak{F}_\eta$, and then $A_j$ is a-minimally thin by Lemma 4.2. Following Aikawa [1, Lemma 3.4], we can similarly find an increasing sequence $\{m(j)\}$ of natural numbers such that $\sum_j \hat{R}^{\cup_{k\geq m(j)}(A_j\cap I_k)}_{h_{\eta,a}} \in \mathfrak{F}_\eta$. Set $\cup_{j=0}^{\infty} \cup_{k\geq m(j)} (A_j \cap I_k) = E$. Since $\hat{R}^{E}_{h_{\eta,a}} \leq \sum_j \hat{R}^{\cup_{k\geq m(j)}(A_j\cap I_k)}_{h_{\eta,a}}$, $E$ is a-minimally thin by Lemma 4.2. If $P \notin E$, then $P \notin \cup_{k\geq m(j)}(A_j \cap I_k)$ for every $j$. It follows that if $|P| \geq 2^{m(j)}$, then $P \notin A_j$. This implies that $u_1(P)/h_{\eta,a}(P) < (j+1)^{-1}$. Hence we have $u_1(P)/h_{\eta,a}(P) \to 0$ as $|P| \to \infty$, $P \in C_n(\Omega) \setminus E$. On the other hand, we see $K(P,O)/h_{\eta,a}(P) = \kappa r^{\{(2-n)\frac{1}{\alpha_\Omega}-1-\eta\}\alpha_\Omega} f_\Omega(\Theta)^{1-a} \to 0$ as $|P| \to \infty$. Thus we have

$$\frac{u(P)}{h_{\eta,a}(P)} = \frac{u_1(P) + c_O(u)K(P,O)}{h_{\eta,a}(P)} \to 0 \quad (|P| \to \infty, P \in C_n(\Omega) \setminus E).$$

For the converse we take an unbounded and a-minimally thin set $E$. As in the proof of Aikawa [1, Lemma 2.4 (iv)], we see that if $U$ is bounded, then $\lambda^a_U(\overline{C_n(\Omega)}) = \inf\{\lambda^a_O(\overline{C_n(\Omega)}); U \subset O, O \text{ is open}\}$. By applying the above property to $E_k$ ($k = 0, 1, 2, \dots,$), we obtain an open set $O \supset E$ such that O is a-minimally thin. By Lemma 4.1 we have $\sum_{k=0}^{\infty} \hat{R}^{O_k}_{h_{\eta,a}}(P) \in \mathfrak{F}_\eta$, where $O_k = O \cap I_k$, which implies $\sum_k \int \hat{R}^{O_k}_{h_{\eta,a}}(P) d\nu_\eta(P) < +\infty$. We find an increasing sequence $\{c_k\}$ of positive numbers such that $c_k \nearrow \infty$ and $\sum_k c_k \int \hat{R}^{O_k}_{h_{\eta,a}}(P) d\nu_\eta(P) < +\infty$. Set $u(P) = \sum_{k=0}^{\infty} c_k \hat{R}^{O_k}_{h_{\eta,a}}(P)$. By Lebesgue's monotone convergence theorem, we see that $u \in \mathfrak{F}_\eta$. Since $O_k$ is included in the interior of $O_{k-1} \cup O_k$,

$$\hat{R}^{O_{k-1}}_{h_{\eta,a}}(P) + \hat{R}^{O_k}_{h_{\eta,a}}(P) \geq \hat{R}^{O_{k-1}\cup O_k}_{h_{\eta,a}}(P) \geq h_{\eta,a}(P)$$

for $P \in O_k$. Hence, if $P \in E_k \subset O_k$, then

$$u(P) \geq c_{k-1}\hat{R}^{O_{k-1}}_{h_{\eta,a}}(P) + c_k \hat{R}^{O_k}_{h_{\eta,a}}(P) \geq c_{k-1} h_{\eta,a}(P).$$

Therefore

$$\lim_{|P|\to+\infty,\ P\in E} \frac{u(P)}{h_{\eta,a}(P)} = +\infty.$$

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