# Moduli spaces of twisted sheaves on a projective variety 

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## Appendix by Daniel Huybrechts and Paolo Stellari

## §0. Introduction

Let $X$ be a smooth projective variety over $\mathbb{C}$. Let $\alpha:=\left\{\alpha_{i j k} \in\right.$ $\left.H^{0}\left(U_{i} \cap U_{j} \cap U_{k}, \mathcal{O}_{X}^{\times}\right)\right\}$be a 2 -cocycle representing a torsion class $[\alpha] \in$ $H^{2}\left(X, \mathcal{O}_{X}^{\times}\right)$. An $\alpha$-twisted sheaf $E:=\left\{\left(E_{i}, \varphi_{i j}\right)\right\}$ is a collection of sheaves $E_{i}$ on $U_{i}$ and isomorphisms $\varphi_{i j}: E_{i \mid U_{i} \cap U_{j}} \rightarrow E_{j \mid U_{i} \cap U_{j}}$ such that $\varphi_{i i}=\operatorname{id}_{E_{i}}, \varphi_{j i}=\varphi_{i j}^{-1}$ and $\varphi_{k i} \circ \varphi_{j k} \circ \varphi_{i j}=\alpha_{i j k} \operatorname{id}_{E_{i}}$. We assume that there is a locally free $\alpha$-twisted sheaf, that is, $\alpha$ gives an element of the Brauer group $\operatorname{Br}(X)$. A twisted sheaf naturally appears if we consider a non-fine moduli space $M$ of the usual stable sheaves on $X$. Indeed the transition functions of the local universal families satisfy the patching condition up to the multiplication by constants and gives a twisted sheaf. If the patching condition is satisfied, i.e., the moduli space $M$ is fine, than the universal family defines an integral functor on the bounded derived categories of coherent sheaves $\mathbf{D}(M) \rightarrow \mathbf{D}(X)$. Assume that $X$ is a $K 3$ surface and $\operatorname{dim} M=\operatorname{dim} X$. Then Mukai, Orlov and Bridgeland showed that the integral functor is the Fourier-Mukai functor, i.e., it is an equivalence of the categories. In his thesis [C2], Căldăraru studied the category of twisted sheaves and its bounded derived category. In particular, he generalized Mukai, Orlov and Bridgeland's results on the Fourier-Mukai transforms to non-fine moduli spaces on a $K 3$ surface. For the usual derived category, Orlov [Or] showed that the equivalence class is described in terms of the Hodge structure of the Mukai lattice. Căldăraru conjectured that a similar result also holds for the derived

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category of twisted sheaves. Recently this conjecture was modified and proved by Huybrechts and Stellari, if $\rho(X) \geq 12$ in [H-St]. As is wellknown, twisted sheaves also appear if we consider a projective bundle over $X$.

In this paper, we define a notion of the stability for a twisted sheaf and construct the moduli space of stable twisted sheaves on $X$. We also construct a projective compactification of the moduli space by adding the $S$-equivalence classes of semi-stable twisted sheaves. In particular if $H^{1}\left(X, \mathcal{O}_{X}\right)=0$ (e.g. $X$ is a $K 3$ surface), then the moduli space of locally free twisted sheaves is the moduli space of projective bundles over $X$. Thus we compactify the moduli space of projective bundles by using twisted sheaves. The idea of the construction is as follows. We consider a twisted sheaf as a usual sheaf on the Brauer-Severi variety. Instead of using the Hilbert polynomial associated to an ample line bundle on the Brauer-Severi variety, we use the Hilbert polynomial associated to a line bundle coming from $X$ in order to define the stability. Then the construction is a modification of Simpson's construction of the moduli space of usual sheaves (cf. [Y3]). M. Lieblich informed us that our stability condition coincides with Simpson's stability for modules over the associated Azumaya algebra via Morita equivalence. Hence the construction also follows from Simpson's moduli space [S, Thm. 4.7] and the valuative criterion for properness.

In section 3, we consider the moduli space of twisted sheaves on a $K 3$ surface. We generalize known results on the moduli space of usual stable sheaves to the moduli spaces of twisted stable sheaves (cf. [Mu2], [Y1]). In particular, we consider the non-emptyness, the deformation type and the weight 2 Hodge structure. Then we can consider twisted version of the Fourier-Mukai transform by using 2 dimensional moduli spaces, which is done in section 4. As an application of our results, Huybrechts and Stellari prove Căldăraru's conjecture generally (see Appendix).

Since our main example of twisted sheaves are those on $K 3$ surfaces or abelian surfaces, we consider twisted sheaves over $\mathbb{C}$. But some of the results (e.g., subsection 2.2 ) also hold over any field.
E. Markman and D. Huybrechts communicated to the author that M. Lieblich independently constructed the moduli of twisted sheaves. In his paper [Li], Lieblich developed a general theory of twisted sheaves in terms of algebraic stack and constructed the moduli space intrinsic way. He also studied the moduli spaces of twisted sheaves on surfaces. There are also some overlap with a paper by N. Hoffmann and U. Stuhler [Ho-St]. They also constructed the moduli space of rank 1 twisted sheaves and studied the symplectic structure of the moduli space.

## §1. Twisted sheaves

Notation: For a locally free sheaf $E$ on a variety $X, \mathbb{P}(E) \rightarrow X$ denotes the projective bundle in the sense of Grothendieck, that is, $\mathbb{P}(E)=\operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} S^{n}(E)\right)$.

Let $X$ be a smooth projective variety over $\mathbb{C}$. Let $\alpha:=\left\{\alpha_{i j k} \in\right.$ $\left.H^{0}\left(U_{i} \cap U_{j} \cap U_{k}, \mathcal{O}_{X}^{\times}\right)\right\}$be a 2-cocycle representing a torsion class $[\alpha] \in$ $H^{2}\left(X, \mathcal{O}_{X}^{\times}\right)$. An $\alpha$-twisted sheaf $E:=\left\{\left(E_{i}, \varphi_{i j}\right)\right\}$ is a collection of sheaves $E_{i}$ on $U_{i}$ and isomorphisms $\varphi_{i j}: E_{i \mid U_{i} \cap U_{j}} \rightarrow E_{j \mid U_{i} \cap U_{j}}$ such that $\varphi_{i i}=\operatorname{id}_{E_{i}}, \varphi_{j i}=\varphi_{i j}^{-1}$ and $\varphi_{k i} \circ \varphi_{j k} \circ \varphi_{i j}=\alpha_{i j k} \mathrm{id}_{E_{i}}$. If all $E_{i}$ are coherent, then we say that $E$ is coherent. Let $\operatorname{Coh}(X, \alpha)$ be the category of coherent $\alpha$-twisted sheaves on $X$.

If $E_{i}$ are locally free for all $i$, then we can glue $\mathbb{P}\left(E_{i}^{\vee}\right)$ together and get a projective bundle $p: Y \rightarrow X$ with $\delta([Y])=[\alpha]$, where $[Y] \in H^{1}(X, P G L(r))$ is the corresponding cohomology class of $Y$ and $\delta: H^{1}(X, P G L(r)) \rightarrow H^{2}\left(X, \mathcal{O}_{X}^{\times}\right)$is the connecting homomorphism induced by the exact sequence

$$
1 \rightarrow \mathcal{O}_{X}^{\times} \rightarrow G L(r) \rightarrow P G L(r) \rightarrow 1
$$

Thus [ $\alpha$ ] belongs to the Brauer group $\operatorname{Br}(X)$. If $X$ is a smooth projective surface, then $\operatorname{Br}(X)$ coincides with the torsion part of $H^{2}\left(X, \mathcal{O}_{X}^{\times}\right)$. Let $\mathcal{O}_{\mathbb{P}\left(E_{i}^{\vee}\right)}\left(\lambda_{i}\right)$ be the tautological line bundle on $\mathbb{P}\left(E_{i}^{\vee}\right)$. Then, $\varphi_{i j}$ induces an isomorphism $\widetilde{\varphi}_{i j}: \mathcal{O}_{\mathbb{P}\left(E_{i}^{\vee}\right)}\left(\lambda_{i}\right)_{\mid p^{-1}\left(U_{i} \cap U_{j}\right)} \rightarrow \mathcal{O}_{\mathbb{P}\left(E_{j}^{\vee}\right)}\left(\lambda_{j}\right)_{\mid p^{-1}\left(U_{i} \cap U_{j}\right)}$. $\mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right):=\left\{\left(\mathcal{O}_{\mathbb{P}\left(E_{i}^{\vee}\right)}\left(\lambda_{i}\right), \widetilde{\varphi}_{i j}\right)\right\}$ is an $p^{*}\left(\alpha^{-1}\right)$-twisted line bundle on $Y$.

### 1.1. Sheaves on a projective bundle

In this subsection, we shall interpret twisted sheaves as usual sheaves on a Brauer-Severi variety. Let $p: Y \rightarrow X$ be a projective bundle. Let $X=\cup_{i} U_{i}$ be an analytic open covering of $X$ such that $p^{-1}\left(U_{i}\right) \cong$ $U_{i} \times \mathbb{P}^{r-1}$. We set $Y_{i}:=p^{-1}\left(U_{i}\right)$. We fix a collection of tautological line bundles $\mathcal{O}_{Y_{i}}\left(\lambda_{i}\right)$ on $Y_{i}$ and isomorphisms $\phi_{j i}: \mathcal{O}_{Y_{i} \cap Y_{j}}\left(\lambda_{j}\right) \rightarrow \mathcal{O}_{Y_{i} \cap Y_{j}}\left(\lambda_{i}\right)$. We set $G_{i}:=p_{*}\left(\mathcal{O}_{Y_{i}}\left(\lambda_{i}\right)\right)^{\vee}$. Then $G_{i}$ are vector bundles on $U_{i}$ and $p^{*}\left(G_{i}\right)\left(\lambda_{i}\right)$ defines a vector bundle $G$ of rank $r$ on $Y$. We have the Euler sequence

$$
0 \rightarrow \mathcal{O}_{Y} \rightarrow G \rightarrow T_{Y / X} \rightarrow 0
$$

Thus $G$ is a non-trivial extension of $T_{Y / X}$ by $\mathcal{O}_{Y}$.
Lemma 1.1. $\operatorname{Ext}^{1}\left(T_{Y / X}, \mathcal{O}_{Y}\right)=\mathbb{C}$. Thus $G$ is characterized as a non-trivial extension of $T_{Y / X}$ by $\mathcal{O}_{Y}$. In particular, $G$ does not depend on the choice of the local trivialization of $p$.

Proof. Since $\mathbf{R} p_{*}\left(G^{\vee}\right)=0$, the Euler sequence inplies that

$$
\operatorname{Ext}^{1}\left(T_{Y / X}, \mathcal{O}_{Y}\right) \cong H^{0}\left(Y, \mathcal{O}_{Y}\right) \cong \mathbb{C}
$$

Q.E.D.

Definition 1.1. For a projective bundle $p: Y \rightarrow X$, let $\epsilon(Y)(:=G)$ be a vector bundle on $Y$ which is a non-trivial extension

$$
0 \rightarrow \mathcal{O}_{Y} \rightarrow \epsilon(Y) \rightarrow T_{Y / X} \rightarrow 0
$$

By the exact sequence $0 \rightarrow \mu_{r} \rightarrow S L(r) \rightarrow P G L(r) \rightarrow 1$, we have a connecting homomorphism $\delta^{\prime}: H^{1}(X, P G L(r)) \rightarrow H^{2}\left(X, \mu_{r}\right)$. Let $o: H^{2}\left(X, \mu_{r}\right) \rightarrow H^{2}\left(X, \mathcal{O}_{X}^{\times}\right)$be the homomorphism induced by the inclusion $\mu_{r} \hookrightarrow \mathcal{O}_{X}^{\times}$. Then we have $\delta=o \circ \delta^{\prime}$.

Definition 1.2. For a $\mathbb{P}^{r-1}$-bundle $p: Y \rightarrow X$ corresponding to $[Y] \in H^{1}(X, P G L(r))$, we set $w(Y):=\delta^{\prime}([Y]) \in H^{2}\left(X, \mu_{r}\right)$.

Lemma 1.2 ([C1],[H-Sc]). If $p: Y \rightarrow X$ is a $\mathbb{P}^{r-1}$-bundle associated to a vector bundle $E$ on $X$, i.e., $Y=\mathbb{P}\left(E^{\vee}\right)$, then

$$
w(Y)=\left[c_{1}(E) \quad \bmod r\right]
$$

Lemma 1.3. $\left[c_{1}(G) \bmod r\right]=p^{*}(w(Y)) \in H^{2}\left(Y, \mu_{r}\right)$.
Proof. There is a line bundle $L$ on $Y \times_{X} Y$ such that $L_{\mid Y_{i} \times_{U_{i}} Y_{i}} \cong$ $p_{1}^{*}\left(\mathcal{O}_{Y_{i}}\left(-\lambda_{i}\right)\right) \otimes p_{2}^{*}\left(\mathcal{O}_{Y_{i}}\left(\lambda_{i}\right)\right)$, where $p_{i}: Y \times_{X} Y \rightarrow Y, i=1,2$ are $i$-th projections. By the definition of $G, p_{1 *}(L) \cong G^{\vee}$. Hence $p_{1}: Y \times_{X} Y \rightarrow$ $Y$ is the projective bundle $\mathbb{P}\left(G^{\vee}\right) \rightarrow Y$. Then we get

$$
-\left[c_{1}\left(G^{\vee}\right) \bmod r\right]=w\left(Y \times_{X} Y\right)=p^{*}(w(Y))
$$

Q.E.D.

Lemma 1.4. Let $p: Y \rightarrow X$ be a $\mathbb{P}^{r-1}$-bundle. Then the following conditions are equivalent.
(1) $\quad Y=\mathbb{P}\left(E^{\vee}\right)$ for a vector bundle on $X$.
(2) $\quad w(Y) \in \mathrm{NS}(X) \otimes \mu_{r}$.
(3) There is a line bundle $L$ on $Y$ such that $L_{\mid p^{-1}(x)} \cong \mathcal{O}_{p^{-1}(x)}(1)$.

Proof. $\quad(2) \Rightarrow(3)$ : If $w(Y)=[D \bmod r], D \in \operatorname{NS}(X)$, then $c_{1}(\epsilon(Y))-p^{*}(D) \equiv 0 \bmod r$. We take a line bundle $L$ on $Y$ with $c_{1}(\epsilon(Y))-p^{*}(D)=r c_{1}(L)$. (3) $\Rightarrow(1)$ : We set $E^{\vee}:=p_{*}(L)$. Then $Y=\mathbb{P}\left(E^{\vee}\right)$.
Q.E.D.

Definition 1.3. $\operatorname{Coh}(X, Y)$ is a subcategory of $\operatorname{Coh}(Y)$ such that $E \in \operatorname{Coh}(X, Y)$ if and only if

$$
E_{\mid Y_{i}} \cong p^{*}\left(E_{i}\right) \otimes \mathcal{O}_{Y_{i}}\left(\lambda_{i}\right)
$$

for $E_{i} \in \operatorname{Coh}\left(U_{i}\right)$. For simplicity, we call $E \in \operatorname{Coh}(X, Y)$ a $Y$-sheaf.
By this definition, $\left\{\left(U_{i}, E_{i}\right)\right\}$ gives a twisted sheaf on $X$. Thus we have an equivalence

$$
\begin{array}{rll}
\Lambda^{\mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right)}: \quad \operatorname{Coh}(X, Y) & \cong \quad \operatorname{Coh}(X, \alpha)  \tag{1.1}\\
E & \mapsto \quad p_{*}\left(E \otimes L^{\vee}\right)
\end{array}
$$

where $\mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right):=\left\{\left(\mathcal{O}_{Y_{i}}\left(\lambda_{i}\right), \phi_{i j}\right)\right\}$ is a twisted line bundle on $Y$ and $\alpha_{i j k}^{-1} \operatorname{id}_{\mathcal{O}_{Y_{i}}\left(\lambda_{i}\right)}=\phi_{k i} \circ \phi_{j k} \circ \phi_{i j}$.

We have the following relations:

$$
\begin{gathered}
p_{*}\left(G^{\vee} \otimes E\right)_{\mid U_{i}}=p_{*}\left(p^{*}\left(G_{i}^{\vee}\right) \otimes \mathcal{O}_{Y_{i}}\left(-\lambda_{i}\right) \otimes p^{*}\left(E_{i}\right) \otimes \mathcal{O}_{Y_{i}}\left(\lambda_{i}\right)\right) \\
=p_{*} p^{*}\left(G_{i}^{\vee} \otimes E_{i}\right)=G_{i}^{\vee} \otimes E_{i} \\
p_{*}(E)_{\mid U_{i}}= \\
=p_{*}\left(p^{*}\left(E_{i}\right) \otimes \mathcal{O}_{Y_{i}}\left(\lambda_{i}\right)\right) \\
\quad=E_{i} \otimes p_{*}\left(\mathcal{O}_{Y_{i}}\left(\lambda_{i}\right)\right)=G_{i}^{\vee} \otimes E_{i}
\end{gathered}
$$

Lemma 1.5. A coherent sheaf $E$ on $Y$ belongs to $\operatorname{Coh}(X, Y)$ if and only if $\phi: p^{*} p_{*}\left(G^{\vee} \otimes E\right) \rightarrow G^{\vee} \otimes E$ is an isomorphism. In particular $E \in \operatorname{Coh}(X, Y)$ is an open condition.

Proof. $\quad \phi_{\mid Y_{i}}$ is the homomorphism

$$
p^{*} G_{i}^{\vee} \otimes p^{*} p_{*}\left(E\left(-\lambda_{i}\right)\right) \rightarrow p^{*} G_{i}^{\vee} \otimes E\left(-\lambda_{i}\right)
$$

Hence $\phi_{\mid Y_{i}}$ is an isomorphism if and only if $p^{*} p_{*}\left(E\left(-\lambda_{i}\right)\right) \rightarrow E\left(-\lambda_{i}\right)$ is an isomorphism, which is equivalent to $E \in \operatorname{Coh}(X, Y)$.
Q.E.D.

Lemma 1.6. Assume that $H^{3}(X, \mathbb{Z})_{\text {tor }}=0$. Then $H^{*}(Y, \mathbb{Z}) \cong$ $H^{*}(X, \mathbb{Z})[x] /(f(x))$, where $f(x) \in H^{*}(X, \mathbb{Z})[x]$ is a monic polynomial of degree $r$. In particular, $H^{2}(X, \mathbb{Z}) \otimes \mu_{r^{\prime}} \rightarrow H^{2}(Y, \mathbb{Z}) \otimes \mu_{r^{\prime}}$ is injective for all $r^{\prime}$.

Proof. $\quad R^{2} p_{*} \mathbb{Z}$ is a local system of rank 1. Since $c_{1}\left(K_{Y / X}\right)$ is a section of this local system, $R^{2} p_{*} \mathbb{Z} \cong \mathbb{Z}$. Let $h$ be the generator. Then $R^{2 i} p_{*} \mathbb{Z} \cong \mathbb{Z} h^{i}$. Since $H^{3}(X, \mathbb{Z})_{\text {tor }}=0$, by the Leray spectral sequence, we get a surjective homomorphism $H^{2}(Y, \mathbb{Z}) \rightarrow H^{0}\left(X, R^{2} p_{*} \mathbb{Z}\right)$. Let $x \in H^{2}(Y, \mathbb{Z})$ be a lifting of $h$. Then $x^{i}$ is a lifting of $h^{i} \in H^{0}\left(X, R^{2 i} p_{*} \mathbb{Z}\right)$. Therefore the Leray-Hirsch theorem implies that

$$
H^{*}(Y, \mathbb{Z}) \cong H^{*}(X, \mathbb{Z})[x] /(f(x))
$$

Q.E.D.

Lemma 1.7. Assume that $o(w(Y))=o\left(w\left(Y^{\prime}\right)\right)$.
(i) Then there is a line bundle $L$ on $Y^{\prime} \times_{X} Y$ such that

$$
L_{\mid p^{\prime-1}(x) \times p^{-1}(x)} \cong \mathcal{O}_{p^{\prime-1}(x)}(1) \boxtimes \mathcal{O}_{p^{-1}(x)}(-1)
$$

for all $x \in X$. If $L^{\prime} \in \operatorname{Pic}\left(Y^{\prime} \times_{X} Y\right)$ also satisfies the property, then $L^{\prime}=L \otimes q^{*}(P), P \in \operatorname{Pic}(X)$, where $q: Y^{\prime} \times_{X} Y \rightarrow X$ is the projection.
(ii) We have an equivalence

$$
\begin{array}{ccc}
\Xi_{Y \rightarrow Y^{\prime}}^{L}: \quad \operatorname{Coh}(X, Y) & \rightarrow & \operatorname{Coh}\left(X, Y^{\prime}\right) \\
E & \mapsto & p_{Y^{\prime} *}\left(p_{Y}^{\prime *}(E) \otimes L\right),
\end{array}
$$

where $p_{Y^{\prime}}: Y^{\prime} \times_{X} Y \rightarrow Y^{\prime}$ and $p_{Y}^{\prime}: Y^{\prime} \times_{X} Y \rightarrow Y$ are projections.
Remark 1.1. We also see that $E$ is a $Y$-sheaf if and only if $p_{Y}^{\prime *}(E) \otimes$ $L \cong p_{Y^{\prime}}^{*}\left(E^{\prime}\right)$ for a sheaf $E^{\prime}$ on $Y^{\prime}$.

Definition 1.4. Assume that $H^{3}(X, \mathbb{Z})_{t o r}=0$. For a $Y$-sheaf $E$ of rank $r^{\prime},\left[c_{1}(E) \bmod r^{\prime}\right] \in H^{2}\left(Y, \mu_{r^{\prime}}\right)$ belongs to $p^{*}\left(H^{2}\left(X, \mu_{r^{\prime}}\right)\right)$. We set

$$
w(E):=\left(p^{*}\right)^{-1}\left(\left[c_{1}(E) \quad \bmod r^{\prime}\right]\right) \in H^{2}\left(X, \mu_{r^{\prime}}\right)
$$

By Lemmas 1.3 and 1.7, we see that
Lemma 1.8. (i) By the functor $\Xi_{Y \rightarrow Y^{\prime}}^{L}$ in Lemma 1.7,

$$
w\left(\Xi_{Y \rightarrow Y^{\prime}}^{L}(E)\right)=w(E), \quad \text { for } E \in \operatorname{Coh}(X, Y)
$$

(ii) $\quad w(\epsilon(Y))=w(Y)$.

## §2. Moduli of twisted sheaves

### 2.1. Definition of the stability

Let $\left(X, \mathcal{O}_{X}(1)\right)$ be a pair of a projective scheme $X$ and an ample line bundle $\mathcal{O}_{X}(1)$ on $X$. Let $p: Y \rightarrow X$ be a projective bundle over $X$.

Definition 2.1. A $Y$-sheaf $E$ is of dimension $d$, if $p_{*}(E)$ is of dimension $d$.

For a coherent sheaf $F$ of dimension $d$ on $X$, we define $a_{i}(F) \in \mathbb{Z}$ by the coefficient of the Hilbert polynomial of $F$ :

$$
\chi(F(m))=\sum_{i=0}^{d} a_{i}(F)\binom{m+i}{i}
$$

Let $G$ be a locally free $Y$-sheaf. For a $Y$-sheaf $E$ of dimension $d$, we set $a_{i}^{G}(E):=a_{i}\left(p_{*}\left(G^{\vee} \otimes E\right)\right)$. Thus we have

$$
\chi\left(G, E \otimes p^{*} \mathcal{O}_{X}(m)\right)=\chi\left(p_{*}\left(G^{\vee} \otimes E\right)(m)\right)=\sum_{i=0}^{d} a_{i}^{G}(E)\binom{m+i}{i}
$$

Definition 2.2. Let $E$ be $Y$-sheaf of dimension $d$. Then $E$ is ( $G$ twisted) semi-stable (with respect to $\mathcal{O}_{X}(1)$ ), if
(i) $E$ is of pure dimension $d$,
(ii)

$$
\begin{equation*}
\frac{\chi\left(p_{*}\left(G^{\vee} \otimes F\right)(m)\right)}{a_{d}^{G}(F)} \leq \frac{\chi\left(p_{*}\left(G^{\vee} \otimes E\right)(m)\right)}{a_{d}^{G}(E)}, m \gg 0 \tag{2.1}
\end{equation*}
$$

for all subsheaf $F \neq 0$ of $E$.
If the inequality in (2.1) is strict for all proper subsheaf $F \neq 0$ of $E$, then $E$ is ( $G$-twisted) stable with respect to $\mathcal{O}_{X}(1)$.

Theorem 2.1. Let $p: Y \rightarrow X$ be a projective bundle. There is a coarse moduli scheme $\bar{M}_{X / \mathbb{C}}^{h}$ parametrizing $S$-equivalence classes of $G$ twisted semi-stable $Y$-sheaves $E$ with the $G$-twisted Hilbert polynomial h. $\bar{M}_{X / \mathbb{C}}^{h}$ is a projective scheme.

Remark 2.1. The construction also works for a projective bundle $Y \rightarrow X$ over any field and also for a family of projective bundles, by the fundamental work of Langer [L].

Lemma 2.2. Let $p^{\prime}: Y^{\prime} \rightarrow X$ be a projective bundle with $o\left(w\left(Y^{\prime}\right)\right)=$ $o(w(Y))$ and $\Xi_{Y \rightarrow Y^{\prime}}^{L}$ the correspondence in Lemma 1.7. Then a $Y$-sheaf $E$ is $G$-twisted semi-stable if and only if $\Xi_{Y \rightarrow Y^{\prime}}^{L}(E) \in \operatorname{Coh}\left(X, Y^{\prime}\right)$ is $\Xi_{Y \rightarrow Y^{\prime}}^{L}(G)$-twisted semi-stable. In particular, we have an isomorphism of the corresponding moduli spaces.

Indeed, since $\Xi_{Y \times S \rightarrow Y^{\prime} \times S}^{L \boxtimes \mathcal{O}_{S}}(*)_{s}=\Xi_{Y \rightarrow Y^{\prime}}^{L}(* \otimes k(s))$, if we have a flat family of $Y$-sheaves $\left\{\mathcal{E}_{s}\right\}_{s \in S}, \mathcal{E} \in \operatorname{Coh}(Y \times S)$, then $\left\{\mathcal{E}_{s}^{\prime}\right\}_{s \in S}$ is also a flat family of $Y^{\prime}$-sheaves, where $\mathcal{E}^{\prime}:=\Xi_{Y \times S \rightarrow Y^{\prime} \times S}^{L \boxtimes \mathcal{O}_{S}}(\mathcal{E})$.

Remark 2.2. For a locally free $Y$-sheaf $G$, we have a projective bundle $Y^{\prime} \rightarrow X$ with $\epsilon\left(Y^{\prime}\right)=\Xi_{Y \rightarrow Y^{\prime}}^{L}(G)$. Hence it is sufficient to study the $\epsilon(Y)$-twisted semi-stability.

Remark 2.3. This definition is the same as in [C1]. If $Y=\mathbb{P}\left(G^{\vee}\right)$ for a vector bundle $G$ on $X$, then $\operatorname{Coh}(X, Y)$ is equivalent to $\operatorname{Coh}(X)$ and $G$-twisted stability is nothing but the twisted semi-stability in [Y3].

Definition 2.3. Let $\lambda$ be a rational number. Let $E$ be a $Y$-sheaf of dimension $d$. Then $E$ is of type $\lambda$ with respect to the $G$-twisted semi-stability, if
(i) $E$ is of pure dimension $d$,
(ii)

$$
\frac{a_{d-1}^{G}(F)}{a_{d}^{G}(F)} \leq \frac{a_{d-1}^{G}(E)}{a_{d}^{G}(E)}+\lambda
$$

for all subsheaf $F$ of $E$.
If $\lambda=0$, then $E$ is $\mu$-semi-stable.

### 2.2. Construction of the moduli space

From now on, we assume that $G=\epsilon(Y)$ (cf. Remark 2.2). Let $P(x)$ be a numerical polynomial. We shall construct the moduli space of $G$-twisted semi-stable $Y$-sheaves $E$ with $\chi\left(p_{*}\left(G^{\vee} \otimes E\right)(n)\right)=P(n)$.
2.2.1. Boundedness Let $E$ be a $Y$-sheaf. Then

$$
p^{*} p_{*}\left(G^{\vee} \otimes E\right) \otimes G \rightarrow E
$$

is surjective. Indeed $p^{*} p_{*}\left(G^{\vee} \otimes E\right) \rightarrow G^{\vee} \otimes E$ is an isomorphism and $G \otimes G^{\vee} \rightarrow \mathcal{O}_{Y}$ is surjective.

We take a surjective homomorphism $\mathcal{O}_{X}\left(-n_{G}\right)^{\oplus N} \rightarrow p_{*}\left(G^{\vee} \otimes G\right)$, $n_{G} \gg 0$. Then we have a surjective homomorphism $p^{*}\left(\mathcal{O}_{X}\left(-n_{G}\right)\right)^{\oplus N} \rightarrow$ $G^{\vee} \otimes G$.

Lemma 2.3. Let $E$ be a $Y$-sheaf of pure dimension $d$. If

$$
\begin{equation*}
a_{d-1}^{G}(F) \geq a_{d}^{G}(F)\left(\frac{a_{d-1}^{G}(E)}{a_{d}^{G}(E)}-\nu\right) \tag{2.2}
\end{equation*}
$$

for all quotient $E \rightarrow F$, then $a_{d-1}\left(F^{\prime}\right) \geq a_{d}\left(F^{\prime}\right)\left(\frac{a_{d-1}^{G}(E)}{a_{d}^{G(E)}}-\nu-n_{G}\right)$ for all quotient $p_{*}\left(G^{\vee} \otimes E\right) \rightarrow F^{\prime}$. In particular

$$
S_{\nu}:=\left\{\begin{array}{l|l}
E \in \operatorname{Coh}(X, Y) & \begin{array}{c}
E \text { satisfies }(2.2) \text { and } \\
\chi\left(p_{*}\left(G^{\vee} \otimes E\right)(n H)\right)=P(n)
\end{array}
\end{array}\right\}
$$

is bounded.
Proof. Since $p^{*} p_{*}\left(G^{\vee} \otimes E\right) \cong G^{\vee} \otimes E$, we have a surjective homomorphism

$$
p^{*}\left(\mathcal{O}_{X}\left(-n_{G} H\right)\right)^{\oplus N} \otimes E \rightarrow G \otimes p^{*} p_{*}\left(G^{\vee} \otimes E\right) \rightarrow G \otimes p^{*}\left(F^{\prime}\right)
$$

By our assumption, we get

$$
\begin{aligned}
& a_{d-1}\left(p_{*}\left(G^{\vee} \otimes G\right) \otimes F^{\prime}\right) \\
\geq & a_{d}\left(p_{*}\left(G^{\vee} \otimes G\right) \otimes F^{\prime}\right)\left(\frac{a_{d-1}\left(p_{*}\left(G^{\vee} \otimes E\right)\right)}{a_{d}\left(p_{*}\left(G^{\vee} \otimes E\right)\right)}-n_{G}-\nu\right) .
\end{aligned}
$$

Since $a_{d-1}\left(p_{*}\left(G^{\vee} \otimes G\right) \otimes F^{\prime}\right)=\operatorname{rk}(G)^{2} a_{d-1}\left(F^{\prime}\right)$ and $a_{d}\left(p_{*}\left(G^{\vee} \otimes G\right) \otimes\right.$ $\left.F^{\prime}\right)=\operatorname{rk}(G)^{2} a_{d}\left(F^{\prime}\right)$, we get our claim. The boundedness of $S_{\nu}$ follows from the boundedness of $\left\{p_{*}\left(G^{\vee} \otimes E\right) \mid E \in S_{\nu}\right\}$ and Lemma 2.4 below. Q.E.D.

Lemma 2.4. Let $S$ be a bounded subset of $\operatorname{Coh}(X)$. Then $T:=$ $\left\{E \in \operatorname{Coh}(X, Y) \mid p_{*}\left(G^{\vee} \otimes E\right) \in S\right\}$ is also bounded.

Proof. For $E \in T$, we set $I(E):=\operatorname{ker}\left(p^{*} p_{*}\left(G^{\vee} \otimes E\right) \otimes G \rightarrow E\right)$. We shall show that $T^{\prime}:=\{I(E) \mid E \in T\}$ is bounded. We note that $I(E) \in \operatorname{Coh}(X, Y)$ and we have an exact sequence

$$
0 \rightarrow p_{*}\left(G^{\vee} \otimes I(E)\right) \rightarrow p_{*}\left(G^{\vee} \otimes E\right) \otimes p_{*}\left(G \otimes G^{\vee}\right) \rightarrow p_{*}\left(G^{\vee} \otimes E\right) \rightarrow 0
$$

Since $p_{*}\left(G^{\vee} \otimes E\right) \in S,\left\{p_{*}\left(G^{\vee} \otimes I(E)\right) \mid E \in T\right\}$ is also bounded. Since $p^{*} p_{*}\left(G^{\vee} \otimes I(E)\right) \otimes G \rightarrow I(E)$ is surjective and $I(E)$ is a subsheaf of $p^{*} p_{*}\left(G^{\vee} \otimes E\right) \otimes G, T^{\prime}$ is bounded.
Q.E.D.

Corollary 2.5. Under the same assumption (2.2), there is a rational number $\nu^{\prime}$ which depends on $\nu$ such that

$$
a_{d-1}\left(F^{\prime}\right) \leq a_{d}\left(F^{\prime}\right)\left(\frac{a_{d-1}^{G}(E)}{a_{d}^{G}(E)}+\nu^{\prime}\right)
$$

for a subsheaf $F^{\prime} \subset p_{*}\left(G^{\vee} \otimes E\right)$.
Combining this with Langer's important result [L, Cor. 3.4], we have the following

Lemma 2.6. Under the same assumption (2.2),

$$
\frac{h^{0}(G, E)}{a_{d}^{G}(E)} \leq\left[\frac{1}{d!}\left(\frac{a_{d-1}^{G}(E)}{a_{d}^{G}(E)}+\nu^{\prime}+c\right)^{d}\right]_{+}
$$

where $c$ depends only on $\left(X, \mathcal{O}_{X}(1)\right), G, d$ and $a_{d}^{G}(E)$.
2.2.2. A quot-scheme Since $p_{*}\left(G^{\vee} \otimes E\right)(n), n \gg 0$ is generated by global sections,

$$
H^{0}\left(G^{\vee} \otimes E \otimes p^{*} \mathcal{O}_{X}(n)\right) \otimes G \rightarrow E \otimes p^{*} \mathcal{O}_{X}(n)
$$

is surjective. Since $R^{i} p_{*}\left(G^{\vee} \otimes E\right)=0$ for $i>0$, we also see that $H^{i}\left(G^{\vee} \otimes E \otimes p^{*} \mathcal{O}_{X}(n)\right)=0, i>0$ and $n \gg 0$.

We fix a sufficiently large integer $n_{0}$. We set $N:=\chi\left(p_{*}\left(G^{\vee} \otimes\right.\right.$ $\left.E)\left(n_{0}\right)\right)=P\left(n_{0}\right)$. We set $V:=\mathbb{C}^{N}$. We consider the quot-scheme $\mathfrak{Q}$ parametrizing all quotients

$$
\phi: V \otimes G \rightarrow E
$$

such that $E \in \operatorname{Coh}(X, Y)$ and $\chi\left(p_{*}\left(G^{\vee} \otimes E\right)(n)\right)=P\left(n_{0}+n\right)$. By Lemma 2.4, $\mathfrak{Q}$ is bounded, in particular, it is a quasi-projective scheme.

Lemma 2.7. $\mathfrak{Q}$ is complete.
Proof. We prove our claim by using the valuative criterion. Let $R$ be a discrete valuation ring and $K$ the quotient field of $R$. Let $\phi: V_{R} \otimes$ $G \rightarrow \mathcal{E}$ be a $R$-flat family of quotients such that $\mathcal{E} \otimes_{R} K \in \operatorname{Coh}(X, Y)$, where $V_{R}:=V \otimes_{\mathbb{C}} R$. We set $\mathcal{I}:=\operatorname{ker} \phi$. We have an exact and commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & p^{*} p_{*}\left(\mathcal{I} \otimes G^{\vee}\right) & \rightarrow & V_{R} \otimes G \otimes G^{\vee} & \rightarrow & p^{*} p_{*}\left(\mathcal{E} \otimes G^{\vee}\right) & \rightarrow & 0 \\
& & \downarrow & & \| & & \downarrow \psi & & \\
0 & \rightarrow & \mathcal{I} \otimes G^{\vee} & \rightarrow & V_{R} \otimes G \otimes G^{\vee} & \rightarrow & \mathcal{E} \otimes G^{\vee} & \rightarrow & 0
\end{array}
$$

We shall show that $\psi$ is an isomorphism. Obviously $\psi$ is surjective. Since $\mathcal{E}$ is $R$-flat, $\mathcal{E}$ has no $R$-torsion, which implies that $p^{*} p_{*}\left(\mathcal{E} \otimes G^{\vee}\right)$ is a torsion free $R$-module. Hence $\operatorname{ker} \psi$ is also torsion free. On the other hand, our choice of $\mathcal{E}$ implies that $\psi \otimes K$ is an isomorphism. Therefore $\operatorname{ker} \psi=0$.
Q.E.D.

Since ker $\phi \in \operatorname{Coh}(X, Y)$, we have a surjective homomorphism

$$
V \otimes \operatorname{Hom}\left(G, G \otimes p^{*} \mathcal{O}_{X}(n)\right) \rightarrow \operatorname{Hom}\left(G, E \otimes p^{*} \mathcal{O}_{X}(n)\right)
$$

for $n \gg 0$. Thus we can embed $\mathfrak{Q}$ as a subscheme of an Grassmann variety $\operatorname{Gr}\left(V \otimes W, P\left(n_{0}+n\right)\right)$, where $W=\operatorname{Hom}\left(G, G \otimes p^{*} \mathcal{O}_{X}(n)\right)$. Since all semi-stable $Y$-sheaf are pure, we may replace $\mathfrak{Q}$ by the closure of the open subset parametrizing pure quotient $Y$-sheaves. The same arguments in [Y3] imply that $\mathfrak{Q} / / G L(V)$ is the moduli space of $G$-twisted semi-stable sheaves. The details are left to the reader. For the proof, we also use the following.

Let $(R, \mathfrak{m})$ be a discrete valuation ring $R$ and the maximal ideal $\mathfrak{m}$. Let $K$ be the fractional field and $k$ the residue field. Let $\mathcal{E}$ be a $R$-flat family of $Y \otimes R$-sheaves such that $\mathcal{E} \otimes_{R} K$ is pure.

Lemma 2.8. There is a $R$-flat family of coherent $Y \otimes R$-sheaves $\mathcal{F}$ and a homomorphism $\psi: \mathcal{E} \rightarrow \mathcal{F}$ such that $\mathcal{F} \otimes_{R} k$ is pure, $\psi_{K}$ is an isomorphism and $\psi_{k}$ is an isomorphic at generic points of $\operatorname{Supp}\left(\mathcal{F} \otimes_{R} k\right)$.

By using [S, Lem. 1.17] or [H-L, Prop. 4.4.2], we first construct $\mathcal{F}$ as a usual family of sheaves. Then the very construction of it, $\mathcal{F}$ becomes a $Y \otimes R$-sheaf.

### 2.3. A family of $Y$-sheaves on a projective bundle over $M_{X / \mathbb{C}}^{h}$

Assume that $\mathfrak{Q}^{s s}$ consists of stable points. Then $\mathfrak{Q}^{s s} \rightarrow \bar{M}_{X / \mathbb{C}}^{h}$ is a principal $P G L(N)$-bundle. For a scheme $S, f_{S}: Y \times S \rightarrow S$ denotes the projection. Let $\mathcal{Q}$ be the universal quotient sheaf on $Y \times \mathfrak{Q}^{s s}$. $V:=\operatorname{Hom}_{f_{\mathfrak{Q}^{s s}}}\left(G \boxtimes \mathcal{O}_{\mathfrak{Q}^{s s}}, \mathcal{Q}\right)$ is a locally free sheaf on $\mathfrak{Q}^{s s}$. We consider the projective bundle $\mathfrak{q}: \mathbb{P}(V) \rightarrow \mathfrak{Q}^{s s}$. Since $\mathcal{Q}$ is $G L(N)$-linearized, $V$ is also $G L(N)$-linearized. Then we have a quotient $\psi: \mathbb{P}(V) \rightarrow$ $\mathbb{P}(V) / P G L(N)$ with the commutative diagram:


Since $\left(1_{Y} \times \mathfrak{q}\right)^{*}(\mathcal{Q}) \otimes f_{\mathbb{P}(V)}^{*}\left(\mathcal{O}_{\mathbb{P}(V)}(-1)\right)$ is $P G L(N)$-linearlized, we have a family of $G$-twisted stable $Y$-sheaves $\mathcal{E}$ on $Y \times \widetilde{\bar{M}_{X / \mathbb{C}}^{h}}$ with

$$
\left(1_{Y} \times \psi\right)^{*}(\mathcal{E})=\left(1_{Y} \times \mathfrak{q}\right)^{*}(\mathcal{Q}) \otimes f_{\mathbb{P}(V)}^{*}\left(\mathcal{O}_{\mathbb{P}(V)}(-1)\right)
$$

Hence $\mathcal{E}^{\vee} \in \operatorname{Coh}\left(Y \times \bar{M}_{X / \mathbb{C}}^{h}, Y \times{\widetilde{\bar{M}_{X / \mathbb{C}}}}^{h}\right.$ ) (if $\mathcal{E}$ is locally free). Let $W$ be a locally free sheaf on $\bar{M}_{X / \mathbb{C}}^{h}$ such that $\psi^{*}(W)=\mathfrak{q}^{*}(V)(-1)$. Then we also have $W^{\vee}=\epsilon\left(\widetilde{\bar{M}_{X / \mathbb{C}}^{h}}\right) \in \operatorname{Coh}\left(\bar{M}_{X / \mathbb{C}}^{h}, \widetilde{\bar{M}_{X / \mathbb{C}}^{h}}\right)$ and $\mathcal{E} \otimes f_{\widetilde{\bar{M}_{X / \mathbb{C}}^{h}}}^{*}\left(W^{\vee}\right)$ descends to a sheaf on $Y \times \bar{M}_{X / \mathbb{C}}^{h}$.

Remark 2.4. There is also a family of $G$-twisted stable $Y$-sheaves $\mathcal{E}^{\prime}$ on $Y \times \mathbb{P}\left(V^{\vee}\right) / P G L(N)$ such that

$$
\mathcal{E}^{\prime} \in \operatorname{Coh}\left(Y \times \bar{M}_{X / \mathbb{C}}^{h}, Y \times \mathbb{P}\left(V^{\vee}\right) / P G L(N)\right)
$$

## §3. Twisted sheaves on a projective $K 3$ surface

### 3.1. Basic properties

Let $X$ be a projective $K 3$ surface and $p: Y \rightarrow X$ a projective bundle.

Lemma 3.1. For a locally free $Y$-sheaf $E$ of rank r,

$$
c_{2}\left(\mathbf{R} p_{*}\left(E^{\vee} \otimes E\right)\right) \equiv-(r-1)\left(w(E)^{2}\right) \quad \bmod 2 r
$$

Proof. First we note that $(r-1)\left(D^{2}\right) \bmod 2 r$ is well-defined for $D \in H^{2}\left(Z, \mu_{r}\right), Z=X, Y$. We take a representative $\alpha \in H^{2}(X, \mathbb{Z})$ of $w(E)$. Then $c_{1}(E) \equiv p^{*}(\alpha) \bmod r$. Hence $c_{2}\left(p^{*}\left(\mathbf{R} p_{*}\left(E^{\vee} \otimes E\right)\right)\right)=$ $2 r c_{2}(E)-(r-1)\left(c_{1}(E)^{2}\right) \equiv-(r-1)\left(p^{*}\left(\alpha^{2}\right)\right) \bmod 2 r$. Since $H^{4}(X, \mathbb{Z})$ is a direct summand of $H^{4}(Y, \mathbb{Z})$,

$$
c_{2}\left(\mathbf{R} p_{*}\left(E^{\vee} \otimes E\right)\right) \equiv-(r-1)\left(\alpha^{2}\right) \quad \bmod 2 r
$$

Q.E.D.

Let $K(X, Y)$ be the Grothendieck group of $Y$-sheaves.
Lemma 3.2. (1) There is a locally free $Y$-sheaf $E_{0}$ such that

$$
\operatorname{rk} E_{0}=\min \{\operatorname{rk} E>0 \mid E \in \operatorname{Coh}(X, Y)\}
$$

(2) $K(X, Y)=\mathbb{Z} E_{0} \oplus K(X, Y)_{\leq 1}$, where $K(X, Y)_{\leq 1}$ is the submodule of $K(X, Y)$ generated by $E \in \operatorname{Coh}(X, Y)$ of $\operatorname{dim} E \leq 1$.

Proof. (1) Let $F$ be a $Y$-sheaf such that rk $F=\min \{\operatorname{rk} E>0 \mid E \in$ $\operatorname{Coh}(X, Y)\}$. Then $E_{0}:=F^{\vee \vee}$ satisfies the required properties. (2) We shall show that the image of $E \in \operatorname{Coh}(X, Y)$ in $K(X, Y)$ belongs to $\mathbb{Z} E_{0} \oplus K(X, Y) \leq 1$ by the induction of $\mathrm{rk} E$. We may assume that $\operatorname{rk} E>0$. Let $T$ be the torsion submodule of $E$. Then $E=T+E / T$ in $K(X, Y)$. Since $\operatorname{Hom}\left(E_{0}(-n H), E / T\right) \neq 0$ for $n \gg 0$, we have a nonzero homomorphism $\varphi: E_{0}(-n H) \rightarrow E / T$. By our choice of $E_{0}, \varphi$ is injective. Since $E_{0}(-n H)=E_{0}-E_{0 \mid n H}$ in $K(X, Y), E=\left((E / T) / E_{0}+\right.$ $\left.E_{0}\right)+\left(T-E_{0 \mid n H}\right)$. Since $\operatorname{rk}(E / T) / E_{0}<\operatorname{rk} E$, we get $(E / T) / E_{0} \in \mathbb{Z} E_{0} \oplus$ $K(X, Y)_{\leq 1}$, and hence $E$ also belongs to $\mathbb{Z} E_{0} \oplus K(X, Y)_{\leq 1}$. Q.E.D.

Remark 3.1. rk $E_{0}$ is the order of the Brauer class of $Y$.
Let $\langle$,$\rangle be the Mukai pairing on H^{*}(X, \mathbb{Z})$ :

$$
\langle x, y\rangle=-\int_{X} x^{\vee} y, \quad x, y \in H^{*}(X, \mathbb{Z})
$$

Definition 3.1. Let $G$ be a locally free $Y$-sheaf. For a $Y$-sheaf $E$, we define a Mukai vector of $E$ as

$$
\begin{align*}
v_{G}(E): & =\frac{\operatorname{ch}\left(\mathbf{R} p_{*}\left(E \otimes G^{\vee}\right)\right)}{\sqrt{\operatorname{ch}\left(\mathbf{R} p_{*}\left(G \otimes G^{\vee}\right)\right)}} \sqrt{\operatorname{td}_{X}}  \tag{3.1}\\
& =(\operatorname{rk}(E), \zeta, b) \in H^{*}(X, \mathbb{Q}),
\end{align*}
$$

where $p^{*}(\zeta)=c_{1}(E)-\operatorname{rk}(E) \frac{c_{1}(G)}{\operatorname{rk} G}$ and $b \in \mathbb{Q}$. More generally, for $G \in \operatorname{Coh}(X, Y)$ with $\mathrm{rk} G>0$, we define $v_{G}(E)$ by (3.1).

Since

$$
\begin{aligned}
\mathbf{R} p_{*}\left(E_{1} \otimes G^{\vee}\right) \otimes & \mathbf{R} p_{*}\left(E_{2} \otimes G^{\vee}\right)^{\vee}=\mathbf{R} p_{*}\left(E_{1} \otimes E_{2}^{\vee}\right) \otimes \mathbf{R} p_{*}\left(G \otimes G^{\vee}\right) \\
\left\langle v_{G}\left(E_{1}\right), v_{G}\left(E_{2}\right)\right\rangle & =-\int_{X} \frac{\operatorname{ch}\left(\mathbf{R} p_{*}\left(E_{1} \otimes G^{\vee}\right)\right) \operatorname{ch}\left(\mathbf{R} p_{*}\left(E_{2} \otimes G^{\vee}\right)\right)^{\vee}}{\operatorname{ch}\left(\mathbf{R} p_{*}\left(G \otimes G^{\vee}\right)\right)} \operatorname{td}{ }_{X} \\
& =-\int_{X} \operatorname{ch}\left(\mathbf{R} p_{*}\left(E_{1} \otimes E_{2}^{\vee}\right)\right) \operatorname{td} X_{X} \\
& =-\chi\left(E_{2}, E_{1}\right)
\end{aligned}
$$

We define an integral structure on $H^{*}(X, \mathbb{Q})$ such that $v_{G}(E)$ is integral. This is due to Huybrechts and Stellari [H-St]. For a positive integer $r$ and $\xi \in H^{2}(X, \mathbb{Z})$, we consider an injective homomorphism

$$
\begin{array}{cccc}
T_{-\xi / r}: & H^{*}(X, \mathbb{Z}) & \rightarrow \quad H^{*}(X, \mathbb{Q}) \\
x & \mapsto & e^{-\xi / r} x
\end{array}
$$

$T_{-\xi / r}$ preserves the bilinear form $\langle$,$\rangle .$
Lemma 3.3. We take a representative $\xi \in H^{2}(X, \mathbb{Z})$ of $w(G) \in$ $H^{2}\left(X, \mu_{r}\right)$, where $\operatorname{rk}(G)=r$. We set $(\operatorname{rk}(E), D, a):=e^{\xi / r} v_{G}(E)$. Then $(\operatorname{rk}(E), D, a)$ belongs to $H^{*}(X, \mathbb{Z})$ and $[D \bmod \operatorname{rk}(E)]=w(E)$.

Proof. We set $\sigma:=\left(c_{1}(G)-p^{*}(\xi)\right) / r \in H^{2}(Y, \mathbb{Z})$. Since $p^{*}(D)=$ $p^{*}(\zeta)+\operatorname{rk}(E) p^{*}(\xi) / \operatorname{rk}(G)=c_{1}(E)-\operatorname{rk}(E) \sigma \in H^{2}(Y, \mathbb{Z})$, we get $D \in$ $H^{2}(X, \mathbb{Z})$. By Lemma 3.1, we see that

$$
\begin{aligned}
\left\langle e^{\xi / r} v_{G}(E), e^{\xi / r} v_{G}(E)\right\rangle & =\left\langle v_{G}(E), v_{G}(E)\right\rangle \\
& =c_{2}\left(\mathbf{R} p_{*}\left(E \otimes E^{\vee}\right)\right)-2 \operatorname{rk}(E)^{2} \\
& \equiv\left(D^{2}\right) \quad \bmod 2 \operatorname{rk}(E) .
\end{aligned}
$$

Hence $a \in \mathbb{Z}$. The last claim is obvious.
Q.E.D.

Remark 3.2. $e^{\xi / r} v_{G}(E)$ is the same as the Mukai vector defined by the rational $B$-field $\xi / r$ in [H-St]. More precisely, there is a topological line bundle $L$ on $Y$ with $c_{1}(L)=\sigma$ and $E \otimes L^{-1}$ is the pull-back of a topological sheaf $E_{\xi / r}$ on $X$. Then we see that $e^{\xi / r} v_{G}(E)=\operatorname{ch}\left(E_{\xi / r}\right) \sqrt{\operatorname{td}_{X}}$ (we use $H^{i}(X, \mathbb{Q})=0$ for $i>4$, or we deform $X$ so that $L$ becomes holomorphic).

Definition 3.2. [H-St] We define a weight 2 Hodge structure on the lattice $\left(H^{*}(X, \mathbb{Z}),\langle\rangle,\right)$ as

$$
\begin{aligned}
& H^{2,0}\left(H^{*}(X, \mathbb{Z}) \otimes \mathbb{C}\right):=T_{-\xi / r}^{-1}\left(H^{2,0}(X)\right) \\
& H^{1,1}\left(H^{*}(X, \mathbb{Z}) \otimes \mathbb{C}\right):=T_{-\xi / r}^{-1}\left(\bigoplus_{p=0}^{2} H^{p, p}(X)\right) \\
& H^{0,2}\left(H^{*}(X, \mathbb{Z}) \otimes \mathbb{C}\right):=T_{-\xi / r}^{-1}\left(H^{0,2}(X)\right)
\end{aligned}
$$

We denote this polarized Hodge structure by $\left(H^{*}(X, \mathbb{Z}),\langle\rangle,,-\frac{\xi}{r}\right)$.
Lemma 3.4. The Hodge structure $\left(H^{*}(X, \mathbb{Z}),\langle\rangle,,-\frac{\xi}{r}\right)$ depends only on the Brauer class $\delta^{\prime}\left(\left[\begin{array}{l}\bmod r]) \text {. }\end{array}\right.\right.$

Proof. If $\delta^{\prime}([\xi \bmod r])=\delta^{\prime}\left(\left[\xi^{\prime} \bmod r^{\prime}\right]\right) \in H^{2}\left(X, \mathcal{O}_{X}^{\times}\right)$, then we have $r^{\prime} \xi-r \xi^{\prime}=L+r r^{\prime} N$, where $L \in \operatorname{NS}(X)$ and $N \in H^{2}(X, \mathbb{Z})$. Then we have the following commutative diagram:

$$
\begin{array}{ll}
H^{*}(X, \mathbb{Z}) \xrightarrow{e^{-\frac{\xi}{r}}} & H^{*}(X, \mathbb{Q}) \\
e^{-N} \downarrow & \\
H^{*}(X, \mathbb{Z}) \xrightarrow[e^{-\frac{\xi^{\prime}}{r^{\prime}}}]{ } & H^{*}(X, \mathbb{Q})
\end{array}
$$

Thus we have an isometry of Hodge structures

$$
\left(H^{*}(X, \mathbb{Z}),\langle,\rangle,-\frac{\xi}{r}\right) \cong\left(H^{*}(X, \mathbb{Z}),\langle,\rangle,-\frac{\xi^{\prime}}{r^{\prime}}\right)
$$

Q.E.D.

Definition 3.3. Let $Y \rightarrow X$ be a projective bundle and $G$ a locally free $Y$-sheaf. Let $\xi \in H^{2}(X, \mathbb{Z})$ be a lifting of $w(G) \in H^{2}\left(X, \mu_{r}\right)$, where $r=\operatorname{rk}(G)$.
(i) We define an integral Hodge structure of $H^{*}(X, \mathbb{Q})$ as

$$
T_{-\xi / r}\left(\left(H^{*}(X, \mathbb{Z}),\langle,\rangle,-\frac{\xi}{r}\right)\right)
$$

(ii) $\quad v:=(r, \zeta, b)$ is a Mukai vector, if $v \in T_{-\xi / r}\left(H^{*}(X, \mathbb{Z})\right)$ and $\zeta \in \operatorname{Pic}(X) \otimes \mathbb{Q}$. Moreover if $v$ is primitive in $T_{-\xi / r}\left(H^{*}(X, \mathbb{Z})\right)$, then $v$ is primitive.

Definition 3.4. Let $v:=(r, \zeta, b) \in H^{*}(X, \mathbb{Q})$ be a Mukai vector.
(i) $\bar{M}_{H}^{Y, G}(r, \zeta, b)$ (resp. $\left.M_{H}^{Y, G}(r, \zeta, b)\right)$ denotes the coarse moduli space of $S$-equivalence classes of $G$-twisted semi-stable (resp. stable) $Y$-sheaves $E$ with $v_{G}(E)=v$.
(ii) $\mathcal{M}_{H}^{Y, G}(r, \zeta, b)^{s s}$ (resp. $\left.\mathcal{M}_{H}^{Y, G}(r, \zeta, b)^{s}\right)$ denotes the moduli stack of $G$-twisted semi-stable (resp. stable ) $Y$-sheaves $E$ with $v_{G}(E)=v$.

Lemma 3.5. Assume that $o(w(Y))=o\left(w\left(Y^{\prime}\right)\right)$. Then $\Xi_{Y \rightarrow Y^{\prime}}^{L}$ induces an isomorphism

$$
\mathcal{M}_{H}^{Y, G}(v)^{s s} \cong \mathcal{M}_{H}^{Y^{\prime}, G^{\prime}}(v)^{s s},
$$

where $G^{\prime}:=\Xi_{Y \rightarrow Y^{\prime}}^{L}(G)$. Moreover if $\operatorname{dim} Y=\operatorname{dim} Y^{\prime}$ and $w(Y)=$ $w\left(Y^{\prime}\right)$, then $\mathcal{M}_{H}^{Y, \epsilon(Y)}(v)^{s s} \cong \mathcal{M}_{H}^{Y^{\prime}, \epsilon\left(Y^{\prime}\right)}(v)^{s s}$.

Proof. We use the notation in Lemma 1.7. For a $Y$-sheaf $E$, we set $E^{\prime}:=\Xi_{Y \rightarrow Y^{\prime}}^{L}(E)$. Then $p_{Y}^{\prime}{ }^{*}\left(E \otimes G^{\vee}\right) \cong p_{Y^{\prime}}^{*}\left(E^{\prime} \otimes G^{\prime \vee}\right)$. Hence $v_{G}(E)=v_{G^{\prime}}\left(E^{\prime}\right)$. If $\operatorname{dim} Y=\operatorname{dim} Y^{\prime}$ and $w(Y)=w\left(Y^{\prime}\right)$, then since $w(\epsilon(Y))=w\left(\epsilon\left(Y^{\prime}\right)\right)$, replacing $L$ by $L \otimes q^{*}(P), P \in \operatorname{Pic}(X)$, we may assume that $c_{1}\left(\Xi_{Y \rightarrow Y^{\prime}}^{L}(\epsilon(Y))\right)=c_{1}(\epsilon(Y))$. Thus $\Xi_{Y \rightarrow Y^{\prime}}^{L}(\epsilon(Y))=\epsilon(Y)+$ $T$ in $K\left(X, Y^{\prime}\right)$, where $T$ is a $Y$-sheaf with $\operatorname{dim} T=0$. From this fact, we get $\mathcal{M}_{H}^{Y^{\prime}, \Xi_{Y \rightarrow Y^{\prime}}^{L}(\epsilon(Y))}(v)^{s s}=\mathcal{M}_{H}^{Y^{\prime}, \epsilon\left(Y^{\prime}\right)}(v)^{s s}$.
Q.E.D.

Let $E$ be a $Y$-sheaf. Then the Zariski tangent space of the Kuranishi space is $\operatorname{Ext}^{1}(E, E)$ and the obstruction space is the kernel $\operatorname{Ext}^{2}(E, E)_{0}$ of the trace map

$$
\operatorname{tr}: \operatorname{Ext}^{2}(E, E) \rightarrow H^{2}\left(Y, \mathcal{O}_{Y}\right) \cong H^{2}\left(X, \mathcal{O}_{X}\right)
$$

Hence as in the usual sheaves on a $K 3$ surfaces [Mu1], we get the following.

Proposition 3.6. Let $E$ be a simple $Y$-sheaf. Then the Kuranishi space is smooth of dimension $\left\langle v_{G}(E)^{2}\right\rangle+2$ with a holomorphic symplectic form. In particular, $\left\langle v_{G}(E)^{2}\right\rangle \geq-2$.

Corollary 3.7. Let $E$ be a $\mu$-semi-stable $Y$-sheaf such that $E=$ $l E_{0}+F \in K(X, Y), F \in K(X, Y)_{\leq 1}$. Then $\left\langle v_{G}(E)^{2}\right\rangle \geq-2 l^{2}$.
3.1.1. Wall and Chamber In this subsection, we generalize the notion of the wall and the chamber for the usual stable sheaves to the twisted case.

Lemma 3.8. Assume that there is an exact sequence of twisted sheaves

$$
\begin{equation*}
0 \rightarrow E_{1} \rightarrow E \rightarrow E_{2} \rightarrow 0 \tag{3.2}
\end{equation*}
$$

such that $E_{i}, i=1,2$ are $\mu$-semi-stable $Y$-sheaves. We set $E_{i}=l_{i} E_{0}+$ $F_{i} \in K(X, Y)$ with $F_{i} \in K(X, Y)_{\leq 1}$. Then we have

$$
\frac{\left\langle v_{G}(E)^{2}\right\rangle}{l}+2 l \geq-\frac{\left(l_{2} v_{G}\left(F_{1}\right)-l_{1} v_{G}\left(F_{2}\right)\right)^{2}}{l l_{1} l_{2}}
$$

This lemma easily follows from Corollary 3.7 and the following lemma.
Lemma 3.9. Let $E_{0}$ be a locally free $Y$-sheaf such that $\mathrm{rk} E_{0}=$ $\min \{\mathrm{rk} E>0 \mid E \in \operatorname{Coh}(X, Y)\}$. For an exact sequence of twisted sheaves

$$
\begin{equation*}
0 \rightarrow E_{1} \rightarrow E \rightarrow E_{2} \rightarrow 0 \tag{3.3}
\end{equation*}
$$

we have

$$
\frac{\left\langle v_{G}\left(E_{1}\right)^{2}\right\rangle}{l_{1}}+\frac{\left\langle v_{G}\left(E_{2}\right)^{2}\right\rangle}{l_{2}}-\frac{\left\langle v_{G}(E)^{2}\right\rangle}{l}=\frac{\left(l_{2} v_{G}\left(F_{1}\right)-l_{1} v_{G}\left(F_{2}\right)\right)^{2}}{l l_{1} l_{2}}
$$

where $E_{i}=l_{i} E_{0}+F_{i}$ and $E=l E_{0}+F$ in $K(X, Y)$ with $F_{i}, F \in$ $K(X, Y)_{\leq 1}$.

Proof.

$$
\begin{aligned}
& \frac{\left\langle v_{G}\left(E_{1}\right)^{2}\right\rangle}{l_{1}}+\frac{\left\langle v_{G}\left(E_{2}\right)^{2}\right\rangle}{l_{2}}-\frac{\left\langle v_{G}(E)^{2}\right\rangle}{l} \\
= & \left(l_{1}\left\langle v_{G}\left(E_{0}\right)^{2}\right\rangle+2\left\langle v_{G}\left(E_{0}\right), v_{G}\left(F_{1}\right)\right\rangle+\frac{\left\langle v_{G}\left(F_{1}\right), v_{G}\left(F_{1}\right)\right\rangle}{l_{1}}\right) \\
& +\left(l_{2}\left\langle v_{G}\left(E_{0}\right)^{2}\right\rangle+2\left\langle v_{G}\left(E_{0}\right), v_{G}\left(F_{2}\right)\right\rangle+\frac{\left\langle v_{G}\left(F_{2}\right), v_{G}\left(F_{2}\right)\right\rangle}{l_{2}}\right) \\
& \quad-\left(l\left\langle v_{G}\left(E_{0}\right)^{2}\right\rangle+2\left\langle v_{G}\left(E_{0}\right), v_{G}(F)\right\rangle+\frac{\left\langle v_{G}(F), v_{G}(F)\right\rangle}{l}\right) \\
= & \frac{\left\langle v_{G}\left(F_{1}\right), v_{G}\left(F_{1}\right)\right\rangle}{l_{1}}+\frac{\left\langle v_{G}\left(F_{2}\right), v_{G}\left(F_{2}\right)\right\rangle}{l_{2}}-\frac{\left\langle v_{G}(F), v_{G}(F)\right\rangle}{l} \\
= & \frac{\left(l_{2} v_{G}\left(F_{1}\right)-l_{1} v_{G}\left(F_{2}\right)\right)^{2}}{l l_{1} l_{2}} .
\end{aligned}
$$

Q.E.D.

Definition 3.5. We set $v=v_{G}\left(l E_{0}+F\right)$, where $F$ is of dimension 1 or 0 .
(i) For a $\xi \in \operatorname{NS}(X)$ with $0<-\left(\xi^{2}\right) \leq \frac{l^{2}}{4}\left(2 l^{2}+\left\langle v^{2}\right\rangle\right)$, we define a wall $W_{\xi}$ as

$$
W_{\xi}:=\{L \in \operatorname{Amp}(X) \otimes \mathbb{R} \mid(\xi, L)=0\}
$$

(ii) A chamber with respect to $v$ is a connected component of $\operatorname{Amp}(X) \otimes \mathbb{R} \backslash \bigcup_{\xi} W_{\xi}$.
(iii) A polarization $H$ is general with respect to $v$, if $H$ does not lie on any wall.
Remark 3.3. The concept of chambers and walls are determined by $\operatorname{rk}\left(l E_{0}+F\right)$ and $\left\langle v^{2}\right\rangle$. Thus they do not depend on the choice of $Y$ and $G$.

Proposition 3.10. Keep notation as above.
(i) If $H$ and $H^{\prime}$ belong to the same chamber, then $\mathcal{M}_{H}^{Y, G}(v)^{s s} \cong$ $\mathcal{M}_{H^{\prime}}^{Y, G}(v)^{s s}$.
(ii) If $H$ is general, then $\mathcal{M}_{H}^{Y, G}\left(v_{G}(F)\right)^{s s} \cong \mathcal{M}_{H}^{Y, G^{\prime}}\left(v_{G^{\prime}}(F)\right)^{s s}$ for $F \in K(X, Y)$ with $\mathrm{rk} F>0$. Thus $\mathcal{M}_{H}^{Y, G}\left(v_{G}(F)\right)^{\text {ss }}$ does not depend on the choice of a $Y$-sheaf $G$.
(iii) If

$$
\min \left\{-\left(D^{2}\right)>0 \mid D \in \mathrm{NS}(X),(D, H)=0\right\}>\frac{l^{2}}{4}\left(2 l^{2}+\left\langle v^{2}\right\rangle\right)
$$

then $H$ is general with respect to $v$.
The proof is standard (cf. [H-L]) and is left to the reader. By Proposition 3.10 and Proposition 3.6, we have

Theorem 3.11. Assume that $v$ is a primitive Mukai vector and $H$ is general with respect to $v$. Then all $G$-twisted semi-stable $Y$-sheaves $E$ with $v_{G}(E)=v$ are $G$-twisted stable. In particular $M_{H}^{Y, G}(v)$ is a projective manifold, if it is not empty.

In the next subsection, we show the non-emptyness of the moduli space. We also show that $M_{H}^{Y, G}(v)$ is a $K 3$ surface, if $\left\langle v^{2}\right\rangle=0$.

Proposition 3.12. (cf. [Mu3, Prop. 3.14]) Assume that $\operatorname{Pic}(X)=$ $\mathbb{Z} H$. Let $E$ be a simple twisted sheaf with $\left\langle v_{G}(E)^{2}\right\rangle \leq 0$. Then $E$ is stable.

For the proof, we use Lemma 3.9 and the following:
Lemma 3.13. [Mu3, Cor. 2.8] If $\operatorname{Hom}\left(E_{1}, E_{2}\right)=0$, then

$$
\operatorname{dim} \operatorname{Ext}^{1}\left(E_{1}, E_{1}\right)+\operatorname{dim} \operatorname{Ext}^{1}\left(E_{2}, E_{2}\right) \leq \operatorname{dim} \operatorname{Ext}^{1}(E, E)
$$

### 3.2. Existence of stable sheaves

In this subsection, we shall show that the moduli space of twisted sheaves is deformation equivalent to the usual one. In particular we show the non-emptyness of the moduli space.

Theorem 3.14. [H-Sc] $H^{1}(X, P G L(r)) \rightarrow H^{2}\left(X, \mu_{r}\right)$ is surjective.
Proposition 3.15. For a $w \in H^{2}\left(X, \mu_{r}\right)$, there is a $\mathbb{P}^{r-1}$-bundle $p: Z \rightarrow X$ such that $w(Z)=w$ and $\epsilon(Z)$ is $\mu$-stable.
D. Huybrechts informed us that the claim follows from the proof of Theorem 3.14. Here we give another proof which works for other surfaces.

Proof. Let $p: Y \rightarrow X$ be a $\mathbb{P}^{r-1}$-bundle with $w(Y)=w$. We set $E_{0}:=\epsilon(Y)$. In order to prove our claim, it is sufficient to find a $\mu$ stable locally free $Y$-sheaf $E$ of rank $r$ with $c_{1}(E)=c_{1}\left(E_{0}\right)$. For points $x_{1}, x_{2}, \ldots, x_{n} \in X$, let $F$ be a $Y$-sheaf which is the kernel of a surjection $E_{0} \rightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{p^{-1}\left(x_{i}\right)}(1)$. We take a smooth divisor $D \in|m H|, m \gg 0$. We set $\widetilde{D}:=p^{-1}(D)$. Let $\operatorname{Ext}^{i}(F, F(-\widetilde{D}))_{0}$ be the kernel of the trace map

$$
\operatorname{Ext}^{i}(F, F(-\widetilde{D})) \rightarrow H^{i}\left(Y, \mathcal{O}_{Y}(-\widetilde{D})\right) \cong H^{i}\left(X, \mathcal{O}_{X}(-D)\right)
$$

If $n \gg 0$, then by the Serre duality,

$$
\operatorname{Ext}^{2}(F, F(-\widetilde{D}))_{0} \cong \operatorname{Hom}(F, F(\widetilde{D}))_{0}=0
$$

Hence $\operatorname{Ext}^{1}(F, F)_{0} \rightarrow \operatorname{Ext}^{1}\left(F_{\mid \widetilde{D}}, F_{\mid \widetilde{D}}\right)_{0}$ is surjective. Since $F_{\mid \widetilde{D}}$ deforms to a $\mu$-stable vector bundle on $\widetilde{D}, F$ deforms to a $Y$-sheaf $F^{\prime}$ such that $F_{\mid \widetilde{D}}^{\prime}$ is $\mu$-stable. Then $F^{\prime}$ is also $\mu$-stable. Then $E:=\left(F^{\prime}\right)^{\vee \vee}$ satisfies required properties.
Q.E.D.

Theorem 3.16. Let $Y \rightarrow X$ be a projective bundle and $G$ a locally free $Y$-sheaf. Let $v_{G}:=(r, \zeta, b)$ be a primitive Mukai vector with $r>0$. Then $M_{H}^{Y, G}\left(v_{G}\right)$ is an irreducible symplectic manifold which is deformation equivalent to $\operatorname{Hilb}_{X}^{\left\langle v_{G}^{2}\right\rangle / 2+1}$ for a general polarization $H$. In particular
(1) $M_{H}^{Y, G}\left(v_{G}\right) \neq \emptyset$ if and only if $\left\langle v_{G}^{2}\right\rangle \geq-2$.
(2) If $\left\langle v_{G}^{2}\right\rangle=0$, then $M_{H}^{Y, G}\left(v_{G}\right)$ is a K3 surface.

We divide the proof into several steps.
Step 1 (Reduction to $\left.M_{H}^{Y, \epsilon(Y)}(r, 0,-a)\right)$ : Let $\xi$ be a lifting of $w(G)$. Then $e^{\xi / \mathrm{rk}(G)} v_{G}=\left(r, D, b^{\prime}\right) \in H^{*}(X, \mathbb{Z})$. By Theorem 3.14, there is
a projective bundle $Y^{\prime} \rightarrow X$ such that $w\left(Y^{\prime}\right)=[D \bmod r]$. Since $D / r-\xi / \operatorname{rk}(G)=\zeta / r \in \operatorname{Pic}(X) \otimes \mathbb{Q}, o\left(w\left(Y^{\prime}\right)\right)=o(w(Y))$. Let $G^{\prime}$ be a locally free $Y$-sheaf such that $\Xi_{Y \rightarrow Y^{\prime}}^{L}\left(G^{\prime}\right)=\epsilon\left(Y^{\prime}\right)$, where we use the notation in Lemma 1.7. By Lemma 1.8, $w\left(G^{\prime}\right)=w\left(\epsilon\left(Y^{\prime}\right)\right)=[D$ $\bmod r]$. Then replacing $L$ by $L \otimes q^{*}(P), P \in \operatorname{Pic}(X)$, we may assume that $e^{\xi / \mathrm{rk} G} v_{G}\left(G^{\prime}\right)=(r, D, c), c \in \mathbb{Z}$. Hence $v_{G^{\prime}}(E)=(r, 0,-a)$ for a $Y$-sheaf $E$ with $v_{G}(E)=(r, \zeta, b)$. Since $H$ is general with respect to $(r, \zeta, b)$, Proposition 3.10 implies that $M_{H}^{Y, G}(r, \zeta, b) \cong M_{H}^{Y, G^{\prime}}(r, 0,-a)$. By Lemma 3.5, $M_{H}^{Y, G^{\prime}}(r, 0,-a) \cong M_{H}^{Y^{\prime}, \epsilon\left(Y^{\prime}\right)}(r, 0,-a)$. Therefore replac$\operatorname{ing}(Y, G)$ by $\left(Y^{\prime}, \epsilon\left(Y^{\prime}\right)\right)$, we shall prove the assertion for $M_{H}^{Y, G}(r, 0,-a)$ with $G=\epsilon(Y)$.

Step 2: First we assume that $w(Y) \in \mathrm{NS}(X) \otimes \mu_{r} \subset H^{2}\left(X, \mu_{r}\right)$. Then the Brauer class of $Y$ is trivial, that is, $Y=\mathbb{P}(F)$ for a locally free sheaf $F$ on $X$. Since $H$ is general with respect to $(r, 0,-a)$, Proposition 3.10 (ii) and Lemma 3.5 imply that $M_{H}^{Y, G}(r, 0,-a) \cong M_{H}^{X, \mathcal{O}_{X}}(r, D, c)$ with $2 r a=\left(D^{2}\right)-2 r c$. By [Y1, Thm. 8.1], $M_{H}^{X, \mathcal{O}_{X}}(r, D, c)$ is deformation equivalent to $\operatorname{Hilb}_{X}^{r a+1}$.

We next treat the general cases. We shall deform the projective bundle $Y \rightarrow X$ to a projective bundle in Step 2.

Step 3: We first construct a local family of projective bundles.

Proposition 3.17. Let $f:(\mathcal{X}, \mathcal{H}) \rightarrow T$ be a family of polarized $K 3$ surfaces. Let $p: Y \rightarrow \mathcal{X}_{t_{0}}$ be a projective bundle associated to a stable $Y$-sheaf $E$. Then there is a smooth morphism $U \rightarrow T$ whose image contains $t_{0}$ and a projective bundle $p: \mathcal{Y} \rightarrow \mathcal{X} \times_{T} U$ such that $\mathcal{Y}_{t_{0}} \cong Y$.

Proof. We note that $p_{*}\left(K_{Y / \mathcal{X}_{t_{0}}}^{\vee}\right)$ is a vector bundle on $\mathcal{X}_{t_{0}}$ and we have an embedding $Y \hookrightarrow \mathbb{P}\left(p_{*}\left(K_{Y / \mathcal{X}_{t_{0}}}^{\vee}\right)\right)$. We take an embedding $\mathbb{P}\left(p_{*}\left(K_{Y / \mathcal{X}_{t_{0}}}^{\vee}\right)\right) \hookrightarrow \mathbb{P}^{N-1} \times \mathcal{X}_{t_{0}}$ by a suitable quotient $\mathcal{O}_{\mathcal{X}_{t_{0}}}\left(-n \mathcal{H}_{t_{0}}\right)^{\oplus N} \rightarrow$ $p_{*}\left(K_{Y / \mathcal{X}_{t_{0}}}^{\vee}\right)$. More generally, let $\mathcal{Y}_{S} \rightarrow \mathcal{X} \times_{T} S$ be a projective bundle and a surjective homomorphism $\mathcal{O}_{\mathcal{X}_{\times_{T}} S}(-n \mathcal{H})^{\oplus N} \rightarrow p_{*}\left(K_{\mathcal{Y}_{S} / \mathcal{X}_{\times_{T}} S}^{\vee}\right)$. Then we have an embedding $\mathcal{Y}_{S} \hookrightarrow \mathbb{P}^{N-1} \times \mathcal{X} \times_{T} S$.

Let $\mathfrak{Y}$ be a connected component of the Hilbert scheme $\operatorname{Hilb}_{\mathbb{P}^{N-1} \times \mathcal{X} / T}$ containing $Y$. Let $\mathcal{Y} \subset \mathbb{P}^{N-1} \times \mathcal{X} \times_{T} \mathfrak{Y}$ be the universal subscheme. Let $\varphi: \mathcal{Y} \rightarrow \mathcal{X} \times_{T} \mathfrak{Y}$ be the projection. Let $\mathfrak{Y}^{0}$ be an open subscheme of $\mathfrak{Y}$ such that $\varphi_{\mid \mathcal{X} \times_{T}\{t\}}$ is smooth and $H^{1}\left(T_{\varphi^{-1}(x, t)}\right)=0$ for $(x, t) \in \mathcal{X} \times_{T} \mathfrak{Y}^{0}$. Since $Y \in \mathfrak{Y}^{0}$, it is non-empty. Then $\varphi$ is locally trivial on $\mathcal{X} \times{ }_{T} \mathfrak{Y}^{0}$. Thus $\mathcal{Y} \rightarrow \mathcal{X} \times_{T} \mathfrak{Y}^{0}$ is a projective bundle.

If $Y$ is a projective bundle associated to a twisted vector bundle $E$, then the obstruction for the infinitesimal liftings belongs to

$$
H^{2}\left(\mathcal{E} n d(E) / \mathcal{O}_{X}\right) \cong H^{0}\left(\mathcal{E} n d(E)_{0}\right)^{\vee}
$$

where $\mathcal{E} n d(E)_{0}$ is the trace free part of $\mathcal{E} n d(E)$. Hence if $E$ is simple (and $\mathrm{rk} E$ is not divisible by the characteristic), then there is no obstruction for the infinitesimal liftings. In particular $\mathfrak{Y}^{0} \rightarrow T$ is smooth at $Y$.
Q.E.D.

Step 4 (A relative moduli space of twisted sheaves): Let $f:(\mathcal{X}, \mathcal{H}) \rightarrow$ $T$ be a family of polarized $K 3$ surfaces and $p: \mathcal{Y} \rightarrow \mathcal{X}$ a projective bundle on $\mathcal{X}$. We set $g:=f \circ p$. We note that $H^{i}\left(\mathcal{Y}_{t}, \Omega_{\mathcal{Y}_{t} / \mathcal{X}_{t}}\right)=0$, $i \neq 1$ and $H^{1}\left(\mathcal{Y}_{t}, \Omega_{\mathcal{Y}_{t} / \mathcal{X}_{t}}\right)=\mathbb{C}$ for $t \in T$. Hence $L:=\operatorname{Ext}_{g}^{1}\left(T_{\mathcal{Y} / \mathcal{X}}, \mathcal{O}_{\mathcal{Y}}\right) \cong$ $R^{1} g_{*}\left(\Omega_{\mathcal{Y} / \mathcal{X}}\right)$ is a line bundle on $T$. By the local-global spectral sequence, we have an isomorphism

$$
\operatorname{Ext}^{1}\left(T_{\mathcal{Y} / \mathcal{X}}, g^{*}\left(L^{\vee}\right)\right) \cong H^{0}\left(T, \operatorname{Ext}_{g}^{1}\left(T_{\mathcal{Y} / \mathcal{X}}, g^{*}\left(L^{\vee}\right)\right)\right) \cong H^{0}\left(T, \mathcal{O}_{T}\right)
$$

We take the extension corresponding to $1 \in H^{0}\left(T, \mathcal{O}_{T}\right)$ :

$$
0 \rightarrow g^{*}\left(L^{\vee}\right) \rightarrow \mathcal{G} \rightarrow T_{\mathcal{Y} / \mathcal{X}} \rightarrow 0
$$

such that $\mathcal{G}_{t}=\epsilon\left(\mathcal{Y}_{t}\right)$. Let $v:=(r, \zeta, b) \in R^{*} f_{*} \mathbb{Q}$ be a family of Mukai vectors with $\zeta \in \operatorname{NS}(\mathcal{X} / T) \otimes \mathbb{Q}$. Then as in the absolute case, we have a family of the moduli spaces of semi-stable twisted sheaves $\bar{M}_{(\mathcal{X}, \mathcal{H}) / T}^{\mathcal{Y}, \mathcal{G}}(v) \rightarrow T$ parametrizing $\mathcal{G}_{t}$-twisted semi-stable $\mathcal{Y}_{t}$-sheaves $E$ on $\mathcal{X}_{t}, t \in T$ with $v_{\mathcal{G}_{t}}(E)=v_{t} . \bar{M}_{(\mathcal{X}, \mathcal{H}) / T}^{\mathcal{Y}, \mathcal{G}}(v) \rightarrow T$ is a projective morphism. Let $E$ be a $\mathcal{G}_{t}$-twisted stable $\mathcal{Y}_{t}$-sheaf. By our choice of $\zeta$, $\operatorname{det}(E)$ is unobstructed under deformations over $T$, and hence $E$ itself is unobstructed. Therefore $M_{(\mathcal{X}, \mathcal{H}) / T}^{\mathcal{Y}, \mathcal{G}}(v)$ is smooth over $T$.

Step 5 (A family of $K 3$ surfaces): Let $\mathcal{M}_{d}$ be the moduli space of the polarized $K 3$ surfaces $(X, H)$ with $\left(H^{2}\right)=2 d . \mathcal{M}_{d}$ is constructed as a quotient of an open subscheme $T$ of a suitable Hilbert scheme $\operatorname{Hilb}_{\mathbb{P}^{N} / \mathbb{C}}$. Let $(\mathcal{X}, \mathcal{H}) \rightarrow T$ be the universal family. Let $\Gamma$ be the abstruct $K 3$ lattice and $h$ a primitive vector with $\left(h^{2}\right)=2 d$. Let $\mathcal{D}$ be the period domain for polarized $K 3$ surfaces $(X, H)$. Let $\tau: \widetilde{T} \rightarrow T$ be the universal covering and $\phi_{\tilde{t}}: H^{2}\left(\mathcal{X}_{\tau(\tilde{t})}, \mathbb{Z}\right) \rightarrow \Gamma, \tilde{t} \in \widetilde{T}$ a trivialization on $\widetilde{T}$. We may assume that $\phi_{\tilde{t}}\left(\mathcal{H}_{\tau(\tilde{t})}\right)=h$. Then we have a period map $\mathfrak{p}: \widetilde{T} \rightarrow \mathcal{D}$. By the surjectivity of the period map, we can show that $\mathfrak{p}$ is surjective: Let $U$ be a suitable analytic neighborhood of a point $x \in \mathcal{D}$. Then we have a family of polarized $K 3$ surfaces $\left(\mathcal{X}_{U}, \mathcal{H}_{U}\right) \rightarrow U$ and an embedding of $\mathcal{X}$ as a subscheme of $\mathbb{P}^{N} \times U$. Thus we have a morphism $h: U \rightarrow T$. The
embedding is unique up to the action of $\operatorname{PGL}(N+1)$. Moreover if there is a point $\tilde{t}_{0} \in \widetilde{T}$ such that $\mathfrak{p}\left(\tilde{t}_{0}\right) \in U$, then we have a lifting $\widetilde{h}: U \rightarrow \widetilde{T}$ of $h: U \rightarrow T$ such that $\tilde{t}_{0}=\widetilde{h}\left(\mathfrak{p}\left(\tilde{t}_{0}\right)\right)$. Then $U \rightarrow \widetilde{T} \rightarrow \mathcal{D}$ is the identity. Hence we can construct a lifting of any path on $\mathcal{D}$ intersecting $\mathfrak{p}(\widetilde{T})$. Since $\mathcal{D}$ is connected, we get the assertion.

Step 6 (Reduction to step 2): We take a point $\widetilde{t} \in \widetilde{T}$. We set $(X, H):=\left(\mathcal{X}_{\tau(\widetilde{t})}, \mathcal{H}_{\tau(\tilde{t})}\right)$. Let $p: Y \rightarrow X$ be a $\mathbb{P}^{r-1}$-bundle. Assume that $H$ is general with respect to $v:=(r, 0,-a)$. We take a $D \in \Gamma$ with $[D \bmod r]=\bar{\phi}_{\tilde{t}}(w(Y))$. Let $e_{1}, e_{2}, \ldots, e_{22}$ be a $\mathbb{Z}$-basis of $\Gamma$ such that $e_{1}=\phi_{\overparen{t}}\left(\mathcal{H}_{\tau(\tilde{t})}\right)$ and $D=a e_{1}+b e_{2}$. For an $\eta \in \oplus_{i=3}^{22} \mathbb{Z} e_{i}$ with $\left(e_{1}^{2}\right)\left(\eta^{2}\right)-\left(e_{1}, \eta\right)^{2}<0$, we set $\widetilde{\eta}:=e_{2}+r k \eta \in \Gamma, k \gg 0$. Since $\operatorname{det}\left(\begin{array}{cc}\left(e_{1}^{2}\right) & \left(e_{1}, e_{2}+r k \eta\right) \\ \left(e_{1}, e_{2}+r k \eta\right) & \left(\left(e_{2}+r k \eta\right)^{2}\right)\end{array}\right) \ll 0$ for $k \gg 0$, the signature of the primitive sublattice $L:=\mathbb{Z} e_{1} \oplus \mathbb{Z} \widetilde{\eta}$ of $\Gamma$ is of type $(1,1)$. Moreover $e_{1}^{\perp} \cap L$ does not contain a (-2)-vector. We take a general $\omega \in L^{\perp} \cap \Gamma \otimes \mathbb{C}$ with $(\omega, \omega)=0$ and $(\omega, \bar{\omega})>0$. Then $\omega^{\perp} \cap \Gamma=L$. Replacing $\omega$ by its complex conjugate if necessary, we may assume that $\omega \in \mathcal{D}$. Since $\mathfrak{p}$ is surjective, there is a point $\tilde{t}_{1} \in \widetilde{\mathfrak{H}}$ such that $\mathfrak{p}\left(\tilde{t}_{1}\right)=\omega$. Then $\mathcal{X}_{\tau\left(\widetilde{t}_{1}\right)}$ is a $K 3$ surface with $\operatorname{Pic}\left(\mathcal{X}_{\tau\left(\widetilde{t}_{1}\right)}\right)=\mathbb{Z} \mathcal{H}_{\tau\left(\widetilde{t}_{1}\right)} \oplus \mathbb{Z} \phi_{\widetilde{t}_{1}}^{-1}\left(e_{2}+r k \eta\right)$. Hence $\left[\phi_{\tilde{t}_{1}}^{-1}(D) \bmod r\right]=\left[\phi_{\tilde{t}_{1}}^{-1}\left(a e_{1}+b \widetilde{\eta}\right) \bmod r\right] \in \operatorname{Pic}\left(\mathcal{X}_{\tau\left(\tilde{t}_{1}\right)}\right) \otimes \mu_{r}$. Since

$$
\min \left\{-\left(L^{2}\right) \mid 0 \neq L \in \operatorname{Pic}\left(\mathcal{X}_{\tau\left(\tilde{t}_{1}\right)}\right),\left(L, \mathcal{H}_{\tau\left(\tilde{t}_{1}\right)}\right)=0\right\} \gg \frac{r^{2}}{4}\left(2 r^{2}+\left\langle v^{2}\right\rangle\right)
$$

Proposition 3.10 (iii) implies that $\mathcal{H}_{\tau\left(\tilde{t}_{1}\right)}$ is a general polarization with respect to $v$. Then by the following lemma, we can reduce the proof to Step 2. Therefore we complete the proof of Theorem 3.16.

Lemma 3.18. For $\widetilde{t}_{1}, \tilde{t}_{2} \in \widetilde{T}$, let $Y^{i} \rightarrow \mathcal{X}_{\tau\left(\tilde{t}_{i}\right)}, i=1,2$ be $\mathbb{P}^{r-1}$ _ bundles with $w\left(Y^{i}\right)=\left[\phi_{\tilde{t}_{i}}^{-1}(D) \bmod r\right]$ and $G_{i}:=\epsilon\left(Y^{i}\right)$. Let $v=$ $(r, 0,-a)$ be a primitive Mukai vector. Assume that $\mathcal{H}_{\tau\left(\tilde{t}_{i}\right)}, i=1,2$ are general polarization. Then $M_{\mathcal{H}_{\tau\left(\tilde{t}_{1}\right)}}^{Y^{1}, G_{1}}(r, 0,-a)$ is deformation equivalent to $M_{\mathcal{H}_{\tau\left(\overline{t_{2}}\right)}}^{Y^{2}, G_{2}}(r, 0,-a)$.

Proof. In order to simplify the notation, we denote $M_{\mathcal{H}_{t}}^{Y, \epsilon(Y)}(r, 0,-a)$ by $M(Y)$ for a projective bundle $Y$ over $\left(\mathcal{X}_{t}, \mathcal{H}_{t}\right)$. By Proposition 3.15 and Lemma 3.5, we may assume that $\epsilon\left(Y^{i}\right)(i=1,2)$ is $\mu$-stable. Let $\widetilde{\gamma}:[0,1] \rightarrow \widetilde{T}$ be a path from $\tilde{t}_{1}=\widetilde{\gamma}(0)$ to $\tilde{t}_{2}=\widetilde{\gamma}(1)$ and $\gamma:=\tau \circ \widetilde{\gamma}$. Then we have a trivialization $\bar{\phi}_{s}: H^{2}\left(\mathcal{X}_{\gamma(s)}, \mu_{r}\right) \rightarrow \Gamma \otimes_{\mathbb{Z}} \mu_{r}$. By Proposition 3.15 , there is a projective bundle $Y_{s} \rightarrow \mathcal{X}_{\gamma(s)}$ such that $\bar{\phi}_{s}\left(w\left(Y_{s}\right)\right)=[D$
$\bmod r]$ and $\epsilon\left(Y_{s}\right)$ is $\mu$-stable for each $s \in[0,1]$. By Proposition 3.17, we have a family of projective bundles $\mathcal{Y}^{s} \rightarrow \mathcal{X} \times_{T} \mathfrak{Y}^{s}$ over a $T$-scheme $\psi^{s}: \mathfrak{Y}^{s} \rightarrow T$ such that there is a point $y^{s} \in\left(\psi^{s}\right)^{-1}(\gamma(s)) \subset \mathfrak{Y}^{s}$ with $Y_{s}=\mathcal{Y}_{y^{s}}^{s}$ and $\psi^{s}$ is smooth at $y^{s}$. Then we have a family of moduli spaces $\bar{M}_{\left(\mathcal{X} \times{ }_{T} \mathfrak{Y}^{s}, \widetilde{\mathcal{H}}\right) / \mathfrak{Y}^{s}}^{\mathcal{Y}^{s} \mathcal{G}^{s}}(r, 0,-a) \rightarrow \mathfrak{Y}^{s}$, where $\widetilde{\mathcal{H}}$ is the pull-back of $\mathcal{H}$ to $\mathcal{X} \times{ }_{T} \mathfrak{Y}^{s}$ (Step 4). Since $\psi^{s}$ is smooth, $\psi^{s}\left(\mathfrak{Y}^{s}\right)$ is an open subscheme of $T$ containing $\gamma(s)$. We take an analytic open neighborhood $U_{s}$ of $\gamma(s)$ such that $U_{s}$ is contractible and has a section $\sigma_{s}: U_{s} \rightarrow \mathfrak{Y}^{s}$ with $\sigma_{s}(\gamma(s))=$ $y^{s}$. Let $V_{s}$ be a connected neighborhood of $s$ which is contained in $\gamma^{-1}\left(U_{s}\right)$. Since $[0,1]$ is compact, we can take a finite open covering of $[0,1]:[0,1]=\cup_{j=1}^{n} V_{s_{j}}, s_{1}<s_{2}<\cdots<s_{n}$. Since $\left\{t \in T \mid \operatorname{rkPic}\left(\mathcal{X}_{t}\right)=1\right\}$ is a dense subset of $T$, there is a point $t_{j} \in U_{s_{j}} \cap U_{s_{j+1}}$ such that $t_{j}$ is sufficiently close to a point $\gamma\left(s_{j, j+1}\right), s_{j, j+1} \in V_{s_{j}} \cap V_{s_{j+1}}$ and $\operatorname{Pic}\left(\mathcal{X}_{t_{j}}\right)=\mathbb{Z} \mathcal{H}_{t_{j}}$. Under the identification $H^{2}\left(\mathcal{X}_{t}, \mu_{r}\right) \cong H^{2}\left(\mathcal{X}_{\gamma(s)}, \mu_{r}\right)$ for $t \in U_{s}$, we have $w\left(\mathcal{Y}_{\sigma_{j}\left(t_{j}\right)}^{s_{j}}\right)=w\left(\mathcal{Y}_{y^{j}}^{s_{j}}\right)$ and $w\left(\mathcal{Y}_{\sigma_{j+1}\left(t_{j}\right)}^{s_{j+1}}\right)=w\left(\mathcal{Y}_{y^{j+1}}^{s_{j+1}}\right)$, where we set $\sigma_{j}:=\sigma_{s_{j}}$ and $y^{j}:=y^{s_{j}}$. Since $t_{j}$ is sufficiently close to the point $\gamma\left(s_{j, j+1}\right)$, we have $w\left(\mathcal{Y}_{\sigma_{j}\left(t_{j}\right)}^{s_{j}}\right)=w\left(\mathcal{Y}_{\sigma_{j+1}\left(t_{j}\right)}^{s_{j+1}}\right)$. Hence by Lemma $3.5, M\left(\mathcal{Y}_{\sigma_{j}\left(t_{j}\right)}^{s_{j}}\right)$ is isomorphic to $M\left(\mathcal{Y}_{\sigma_{j+1}\left(t_{j}\right)}^{s_{j+1}}\right)$. By Step 4, $M\left(\mathcal{Y}_{\sigma_{j}\left(t_{j-1}\right)}^{s_{j}}\right)$ is deformation equivalent to $M\left(\mathcal{Y}_{\sigma_{j}\left(t_{j}\right)}^{s_{j}}\right)$. Therefore $M\left(\mathcal{Y}_{\sigma_{1}\left(t_{1}\right)}^{s_{1}}\right)$ is deformation equivalent to $M\left(\mathcal{Y}_{\sigma_{n}\left(t_{n-1}\right)}^{s_{n}}\right)$. By using Step 4 again, we also see that $M\left(Y^{1}\right)=M\left(\mathcal{Y}_{y^{0}}^{0}\right)$ is deformation equivalent to $M\left(\mathcal{Y}_{\sigma_{1}\left(t_{1}\right)}^{s_{1}}\right)$ and $M\left(Y^{2}\right)=M\left(\mathcal{Y}_{y^{1}}^{1}\right)$ is deformation equivalent to $M\left(\mathcal{Y}_{\sigma_{n}\left(t_{n-1}\right)}^{s_{n}}\right)$. Therefore our claim holds.
Q.E.D.

Remark 3.4. Let $v_{G}:=(r, \zeta, b)$ be a Mukai vector with $r,\left\langle v_{G}^{2}\right\rangle>0$ which is not necessary primitive. By the same proof, we can also show that $\bar{M}_{H}^{Y, G}\left(v_{G}\right)$ is an irreducible normal variety for a general $H$ (cf. [Y2]).

### 3.3. The second cohomology groups of moduli spaces

By Theorem 3.16, $M_{H}^{Y, G}\left(v_{G}\right)$ is an irreducible symplectic manifold, if $v_{G}$ is primitive and $H$ is general. Then $H^{2}\left(M_{H}^{Y, G}\left(v_{G}\right), \mathbb{Z}\right)$ is equipped with a bilinear form called the Beauville form. In this subsection, we shall describe the Beauville form in terms of the Mukai lattice.

Let $p: Y \rightarrow X$ be a projective bundle with $w(Y)=[\xi \bmod r]$ and set $G:=\epsilon(Y)$. We consider a Mukai lattice with a Hodge structure $\left(H^{*}(X, \mathbb{Z}),\langle\quad, \quad\rangle,-\frac{\xi}{r}\right)$ in this subsection. We set $w:=r\left(1,0, \frac{a}{r}-\right.$ $\left.\frac{1}{2} \frac{\left(\xi^{2}\right)}{r^{2}}\right), a \in \mathbb{Z}$. In this subsection, we assume that $w$ is primitive, that is, $\operatorname{gcd}(r, \xi, a)=1$. We set $v:=w e^{\xi / r}=(r, \xi, a) \in H^{*}(X, \mathbb{Z})$. Then $v$ is algebraic.

Let $q: \widetilde{M_{H}^{Y, G}(w)} \rightarrow M_{H}^{Y, G}(w)$ be a projective bundle in subsection 2.3 and $\mathcal{E}$ the family of twisted sheaves on $Y \times \widetilde{M_{H}^{Y, G}(w)}$. We set $W^{\vee}:=$ $\epsilon\left(\widetilde{M_{H}^{Y, G}(w)}\right)$. Let $\widetilde{\pi}_{M_{H}^{Y, G}(w)}: Y \times \widetilde{M_{H}^{Y, G}(w)} \rightarrow \widetilde{M_{H}^{Y, G}(w)}$ and $\widetilde{\pi}_{Y}: Y \times$ $\widetilde{M_{H}^{Y, G}(w)} \rightarrow Y$ be projections. Then $\left(1_{Y} \times q\right)_{*}\left(\mathcal{E} \otimes \widetilde{\pi}_{M_{H}^{*} \widetilde{Y, G}(w)}\left(W^{\vee}\right)\right)$ is a quasi-universal family on $Y \times M_{H}^{Y, G}(w)$.

Let $\pi_{X}: X \times M_{H}^{Y, G}(w) \rightarrow X$ be the projection. We define a homo$\operatorname{morphism} \theta_{v}^{G}: v^{\perp} \rightarrow H^{*}\left(M_{H}^{Y, G}(w), \mathbb{Q}\right)$ by

$$
\theta_{v}^{G}(u):=\int_{X}\left[\mathcal{Q}^{\vee} \pi_{X}^{*}\left(e^{-\xi / r} u\right)\right]_{3}
$$

where $[\ldots]_{3}$ means the degree 6 part and

$$
\left.\left.\begin{array}{l}
\mathcal{Q}:=\frac{\sqrt{\operatorname{td}_{X}}}{\sqrt{\operatorname{ch}\left(\mathbf{R} p_{*}\left(G^{\vee} \otimes G\right)\right)}} \frac{\sqrt{\operatorname{td}_{M_{H}^{Y, G}(w)}}}{\sqrt{\operatorname{ch}\left(\mathbf{R} q_{*}\left(W^{\vee} \otimes W\right)\right)}} \\
\cdot \operatorname{ch}\left(\mathbf { R } ( p \times q ) _ { * } \left(\widetilde{\pi}_{Y}^{*}\left(G^{\vee}\right) \otimes \mathcal{E} \otimes \widetilde{\pi}_{M_{H}^{*}}^{Y, G}(w)\right.\right.
\end{array}\left(W^{\vee}\right)\right)\right),
$$

Remark 3.5. If $\xi$ is algebraic, then $Y$ is isomorphic to the projective bundle $\mathbb{P}\left(F^{\vee}\right)$ and $G=F^{\vee} \otimes \mathcal{O}_{Y}(1)$, where $F$ is a vector bundle of rank $r$ on $X$ with $c_{1}(F)=-\xi$. In this case, $M_{H}^{Y, G}(w)$ is the usual moduli space of stable sheaves $F$ with the Mukai vector $v$ and $\mathbf{R}(p \times$ $q)_{*}\left(\widetilde{\pi}_{Y}^{*}\left(\mathcal{O}_{Y}(-1)\right) \otimes \mathcal{E} \otimes \widetilde{\pi}_{M_{H}^{*, G}(w)}\left(W^{\vee}\right)\right)$ is a quasi-universal family. Since $\operatorname{ch} F / \sqrt{\operatorname{ch}\left(F \otimes F^{\vee}\right)}=e^{-\xi / r}$, we have

$$
\begin{aligned}
\mathcal{Q}=e^{-\frac{\xi}{r}} & \sqrt{\operatorname{td}_{X}} \frac{\sqrt{\operatorname{td}_{M_{H}^{Y, G}(w)}}}{\sqrt{\operatorname{ch}\left(\mathbf{R} q_{*}\left(W^{\vee} \otimes W\right)\right)}} \\
& \cdot \operatorname{ch}\left(\mathbf{R}(p \times q)_{*}\left(\widetilde{\pi}_{Y}^{*}\left(\mathcal{O}_{Y}(-1)\right) \otimes \mathcal{E} \otimes \widetilde{\pi}_{M_{H}^{*} \widetilde{Y, G}(w)}\left(W^{\vee}\right)\right)\right) .
\end{aligned}
$$

Hence $\theta_{v}^{G}$ is the usual Mukai homomorphism, which is defined over $\mathbb{Z}$.
Let $p^{\prime}: Y^{\prime} \rightarrow X$ be another $\mathbb{P}^{r-1}$-bundle with $w\left(Y^{\prime}\right)=w(Y)$. Then by the proof of Lemma 3.5, we see that the following diagram is
commutative:

where $G^{\prime}:=\Xi_{Y \rightarrow Y^{\prime}}^{L}(G)=\epsilon\left(Y^{\prime}\right)$. Since $\mathcal{Q}$ is algebraic, $\theta_{v}^{G}$ preserves the Hodge structure. By the deformation argument, Remark 3.5 implies that $\theta_{v}^{G}$ is defined over $\mathbb{Z}$. Moreover it preserves the bilinear forms.

Theorem 3.19. For $\xi \in H^{2}(X, \mathbb{Z})$ with $[\xi \bmod r]=w(Y)$, we set $v=w e^{\xi / r}$.
(i) If $\left\langle v^{2}\right\rangle>0$, then $\theta_{v}^{G}: v^{\perp} \rightarrow H^{2}\left(M_{H}^{Y, G}(w), \mathbb{Z}\right)$ is an isometry of the Hodge structures.
(ii) If $\left\langle v^{2}\right\rangle=0$, then $\theta_{v}^{G}$ induces an isometry of the Hodge structures $v^{\perp} / \mathbb{Z} v \rightarrow H^{2}\left(M_{H}^{Y, G}(w), \mathbb{Z}\right)$.
The second claim is due to Mukai [Mu4].

## §4. Fourier-Mukai transform

### 4.1. Integral functor

Let $p: Y \rightarrow X$ be a projective bundle such that $\delta([Y])=[\alpha] \in$ $\operatorname{Br}(X)$ and $p^{\prime}: Y^{\prime} \rightarrow X^{\prime}$ a projective bundle such that $\delta\left(\left[Y^{\prime}\right]\right)=\left[\alpha^{\prime}\right] \in$ $\operatorname{Br}\left(X^{\prime}\right)$. Let $\pi_{X}: X^{\prime} \times X \rightarrow X$ and $\pi_{X^{\prime}}: X^{\prime} \times X \rightarrow X^{\prime}$ be projections. We also let $\widetilde{\pi}_{Y}: Y^{\prime} \times Y \rightarrow Y$ and $\widetilde{\pi}_{Y^{\prime}}: Y^{\prime} \times Y \rightarrow Y^{\prime}$ be projections. We set $G:=\epsilon(Y)$ and $G^{\prime}:=\epsilon\left(Y^{\prime}\right)$.

Definition 4.1. Let $\operatorname{Coh}\left(X^{\prime} \times X, Y^{\prime}, Y\right)$ be the subcategory of $\operatorname{Coh}\left(Y^{\prime} \times Y\right)$ such that $Q \in \operatorname{Coh}\left(Y^{\prime} \times Y\right)$ belongs to $\operatorname{Coh}\left(X^{\prime} \times X, Y^{\prime}, Y\right)$ if and only if $\left(p^{\prime} \times p\right)^{*}\left(p^{\prime} \times p\right)_{*}\left(G^{\prime} \otimes Q \otimes G^{\vee}\right) \cong G^{\prime} \otimes Q \otimes G^{\vee}$. In terms of local trivialization of $p, p^{\prime}$, this is equivalent to

$$
Q_{\mid Y_{i}^{\prime} \times Y_{j}} \cong \mathcal{O}_{Y_{i}^{\prime}}\left(-\lambda_{i}^{\prime}\right) \boxtimes \mathcal{O}_{Y_{j}}\left(\lambda_{j}\right) \otimes\left(p^{\prime} \times p\right)^{*}\left(Q_{i j}\right)
$$

$Q_{i j} \in \operatorname{Coh}\left(U_{i}^{\prime} \times U_{j}\right) . \operatorname{Coh}\left(X^{\prime} \times X, Y^{\prime}, Y\right)$ is equivalent to $\operatorname{Coh}\left(X^{\prime} \times\right.$ $\left.X, \alpha^{\prime-1} \times \alpha\right)$.

Remark 4.1. We take twisted line bundles $\mathcal{L}\left(p^{\prime *}\left(\alpha^{\prime-1}\right)\right)$ on $Y^{\prime}$ and $\mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right)$ on $Y$ respectively which give equivalences $\Lambda^{\mathcal{L}\left(p^{\prime *}\left(\alpha^{\prime-1}\right)\right)}$ : $\operatorname{Coh}\left(X^{\prime}, Y^{\prime}\right) \cong \operatorname{Coh}\left(X^{\prime}, \alpha^{\prime}\right)$ and $\Lambda^{\mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right)}: \operatorname{Coh}(X, Y) \cong \operatorname{Coh}(X, \alpha)$ in (1.1). Then we have an equivalence $\Lambda^{\mathcal{L}\left(p^{\prime *}\left(\alpha^{\prime-1}\right)\right)^{\vee}} \times \Lambda^{\mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right)}$ :

$$
\begin{array}{ccc}
\operatorname{Coh}\left(X^{\prime} \times X, Y^{\prime}, Y\right) & \rightarrow & \operatorname{Coh}\left(X^{\prime} \times X, \alpha^{\prime-1} \times \alpha\right) \\
Q & \mapsto & \left(p^{\prime} \times p\right)_{*}\left(\mathcal{L}\left(p^{\prime *}\left(\alpha^{\prime-1}\right)\right) \otimes Q \otimes \mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right)^{\vee}\right) .
\end{array}
$$

Let $\mathbf{D}\left(X^{\prime} \times X, Y^{\prime}, Y\right) \cong \mathbf{D}\left(X^{\prime} \times X, \alpha^{\prime-1} \times \alpha\right)$ be the bounded derived category of $\operatorname{Coh}\left(X^{\prime} \times X, Y^{\prime}, Y\right)$. For $\mathcal{Q} \in \mathbf{D}\left(X^{\prime} \times X, Y^{\prime}, Y\right)$, we define an integral functor

$$
\begin{array}{cccc}
\Phi_{X^{\prime} \rightarrow X}^{\widetilde{\mathcal{Q}}}: & \mathbf{D}\left(X^{\prime}, Y^{\prime}\right) & \rightarrow & \mathbf{D}(X, Y) \\
x & \mapsto & \mathbf{R} \widetilde{\pi}_{Y *}\left(\mathcal{Q} \otimes \widetilde{\pi}_{Y^{\prime}}^{*}(x)\right)
\end{array}
$$

For $\mathcal{Q} \in \mathbf{D}\left(X^{\prime} \times X, Y^{\prime}, Y\right)$ and $\mathcal{R} \in \mathbf{D}\left(X^{\prime \prime} \times X^{\prime}, Y^{\prime \prime}, Y^{\prime}\right)$, we have

$$
\Phi_{X^{\prime} \rightarrow X}^{\mathcal{Q}} \circ \Phi_{X^{\prime \prime} \rightarrow X^{\prime}}^{\mathcal{R}}=\Phi_{X^{\prime \prime} \rightarrow X}^{\mathcal{S}}
$$

where $\mathcal{S}=\mathbf{R} \widetilde{\pi}_{Y^{\prime \prime} \times Y *}\left(\widetilde{\pi}_{Y^{\prime \prime} \times Y^{\prime}}^{*}(\mathcal{R}) \otimes \widetilde{\pi}_{Y^{\prime} \times Y}^{*}(\mathcal{Q})\right)$ and $\left.\widetilde{\pi}_{( }^{*}\right): Y^{\prime \prime} \times Y^{\prime} \times Y \rightarrow$ ( ) is the projection.
4.1.1. Cohomological correspondence For simplicity, we denote the pull-backs of $G$ and $G^{\prime}$ to $Y^{\prime} \times Y$ by the same letters. For example $G^{\prime} \otimes \mathcal{Q} \otimes G^{\vee}$ implies $\pi_{Y^{\prime}}^{*}\left(G^{\prime}\right) \otimes \mathcal{Q} \otimes \pi_{Y}\left(G^{\vee}\right)$. We note that

$$
\mathbf{R}\left(p^{\prime} \times p\right)_{*}\left(G^{\prime} \otimes \mathcal{Q} \otimes G^{\vee}\right) \in \mathbf{D}\left(X^{\prime} \times X\right)
$$

satisfies

$$
\left(p^{\prime} \times p\right)^{*}\left(\mathbf{R}\left(p^{\prime} \times p\right)_{*}\left(G^{\prime} \otimes \mathcal{Q} \otimes G^{\vee}\right)\right)=G^{\prime} \otimes \mathcal{Q} \otimes G^{\vee}
$$

We define a homomorphism

$$
\Psi_{X^{\prime} \rightarrow X}^{\mathcal{Q}}: H^{*}\left(X^{\prime}, \mathbb{Q}\right) \rightarrow H^{*}(X, \mathbb{Q})
$$

by

$$
\begin{aligned}
& \Psi_{X^{\prime} \rightarrow X}^{\mathcal{Q}}(y) \\
&:= \pi_{X *} \circ\left(p^{\prime} \times p\right)_{*}\left(\left(p^{\prime} \times p\right)^{*} \circ \pi_{X^{\prime}}^{*}(y) \operatorname{ch}\left(G^{\prime}\right) \operatorname{ch}(\mathcal{Q}) \operatorname{ch}\left(G^{\vee}\right)\right. \\
&\left.\cdot \frac{\sqrt{\operatorname{td}_{X^{\prime}}} \operatorname{td}_{Y^{\prime} / X^{\prime}}}{\sqrt{\operatorname{ch}\left(G^{\prime \vee} \otimes G^{\prime}\right)}} \frac{\sqrt{\operatorname{td}_{X}} \operatorname{td}_{Y / X}}{\sqrt{\operatorname{ch}\left(G^{\vee} \otimes G\right)}}\right) \\
&=\pi_{X *}\left(\pi_{X^{\prime}}^{*}(y) \frac{\sqrt{\operatorname{td}_{X^{\prime}}}}{\sqrt{\operatorname{ch}\left(\mathbf{R} p_{*}^{\prime}\left(G^{\prime \vee} \otimes G^{\prime}\right)\right)}} \frac{\sqrt{\operatorname{td} X_{X}}}{\sqrt{\operatorname{ch}\left(\mathbf{R} p_{*}\left(G^{\vee} \otimes G\right)\right)}}\right. \\
&\left.\cdot \operatorname{ch}\left(\mathbf{R}\left(p^{\prime} \times p\right)_{*}\left(G^{\prime} \otimes \mathcal{Q} \otimes G^{\vee}\right)\right)\right),
\end{aligned}
$$

where $\operatorname{td}_{X}, \operatorname{td}_{X^{\prime}}, \ldots$ are identified with their pull-backs.
Lemma 4.1. $\Psi_{X^{\prime \prime} \rightarrow X}^{\mathcal{S}}=\Psi_{X^{\prime} \rightarrow X}^{\mathcal{Q}} \circ \Psi_{X^{\prime \prime} \rightarrow X^{\prime}}^{\mathcal{R}}$.

Proof. $\left.\pi_{( }\right): X^{\prime \prime} \times X^{\prime} \times X \rightarrow()$ denotes the projection to ( ). We note that

$$
\begin{aligned}
& \pi_{X^{\prime \prime} \times X}^{*}\left(\mathbf{R}\left(p^{\prime \prime} \times p^{\prime}\right)_{*}\left(G^{\prime \prime} \otimes \mathcal{R} \otimes G^{\vee}\right)\right) \\
& \otimes \pi_{X^{\prime} \times X}^{*}\left(\mathbf{R}\left(p^{\prime} \times p\right)_{*}\left(G^{\prime} \otimes \mathcal{Q} \otimes G^{\vee}\right)\right) \\
= & \mathbf{R}\left(p^{\prime \prime} \times p^{\prime} \times p\right)_{*}\left(G^{\prime \prime} \otimes \mathcal{R} \otimes \mathcal{Q} \otimes G^{\vee}\right) \otimes \pi_{X^{\prime}}^{*}\left(\mathbf{R} p_{*}^{\prime}\left(G^{\prime \vee} \otimes G^{\prime}\right)\right) .
\end{aligned}
$$

Then

$$
\begin{aligned}
& \pi_{X^{\prime \prime} \times X}^{*}\left(\operatorname{ch}\left(\mathbf{R}\left(p^{\prime \prime} \times p^{\prime}\right)_{*}\left(G^{\prime \prime} \otimes \mathcal{R} \otimes G^{\prime \vee}\right)\right)\right) \\
& \pi_{X^{\prime} \times X}^{*}\left(\operatorname{ch}\left(\mathbf{R}\left(p^{\prime} \times p\right)_{*}\left(G^{\prime} \otimes \mathcal{Q} \otimes G^{\vee}\right)\right)\right) \pi_{X^{\prime}}^{*}\left(\frac{\operatorname{td}_{X^{\prime}}}{\operatorname{ch}\left(\mathbf{R} p_{*}^{\prime}\left(G^{\prime \vee} \otimes G^{\prime}\right)\right)}\right) \\
= & \operatorname{ch}\left(\mathbf{R}\left(p^{\prime \prime} \times p^{\prime} \times p\right)_{*}\left(G^{\prime \prime} \otimes \mathcal{R} \otimes \mathcal{Q} \otimes G^{\vee}\right)\right) \pi_{X^{\prime}}^{*}\left(\operatorname{td}_{X^{\prime}}\right) .
\end{aligned}
$$

Since

$$
\begin{aligned}
& \pi_{X^{\prime \prime} \times X *}\left(\operatorname{ch}\left(\mathbf{R}\left(p^{\prime \prime} \times p^{\prime} \times p\right)_{*}\left(G^{\prime \prime} \otimes \mathcal{R} \otimes \mathcal{Q} \otimes G^{\vee}\right)\right) \pi_{X^{\prime}}^{*}\left(\operatorname{td}_{X^{\prime}}\right)\right) \\
= & \operatorname{ch}\left(\mathbf{R} \pi_{X^{\prime \prime} \times X *}\left(\mathbf{R}\left(p^{\prime \prime} \times p^{\prime} \times p\right)_{*}\left(G^{\prime \prime} \otimes \mathcal{R} \otimes \mathcal{Q} \otimes G^{\vee}\right)\right)\right) \\
= & \operatorname{ch}\left(\mathbf{R}\left(p^{\prime \prime} \times p\right)_{*} \circ \mathbf{R} \tilde{\pi}_{Y^{\prime \prime} \times Y *}\left(G^{\prime \prime} \otimes \mathcal{R} \otimes \mathcal{Q} \otimes G^{\vee}\right)\right) \\
= & \operatorname{ch}\left(\mathbf{R}\left(p^{\prime \prime} \times p\right)_{*}\left(G^{\prime \prime} \otimes \mathcal{S} \otimes G^{\vee}\right)\right),
\end{aligned}
$$

we get

$$
\begin{aligned}
& \Psi_{X^{\prime \prime} \rightarrow X}^{\mathcal{S}}(z)= \pi_{X *}\left(\pi_{X^{\prime \prime}}^{*}(z) \operatorname{ch}\left(\mathbf{R}\left(p^{\prime \prime} \times p\right)_{*}\left(G^{\prime \prime} \otimes \mathcal{S} \otimes G^{\vee}\right)\right)\right. \\
& \cdot \frac{\left.\sqrt{{\operatorname{td} X^{\prime \prime}}} \frac{\sqrt{\operatorname{td} X}}{\sqrt{\operatorname{ch}\left(\mathbf{R} p_{*}^{\prime \prime}\left(G^{\prime \prime} \otimes G^{\prime \prime}\right)\right)}} \frac{\sqrt{\operatorname{ch}\left(\mathbf{R} p_{*}\left(G^{\vee} \otimes G\right)\right)}}{}\right)}{} \\
&=\Psi_{X^{\prime} \rightarrow X}^{\mathcal{Q}} \circ \Psi_{X^{\prime \prime} \rightarrow X^{\prime}}^{\mathcal{R}}(z)
\end{aligned}
$$

Q.E.D.

Lemma 4.2. Assume that the canonical bundles $K_{X}, K_{X^{\prime}}$ are trivial. Then

$$
\left\langle x, \Psi_{X^{\prime} \rightarrow X}^{\mathcal{Q}}(y)\right\rangle=\left\langle\Psi_{\tilde{X} \rightarrow X^{\prime}}^{\mathcal{Q}^{\vee}}(x), y\right\rangle, \quad x \in H^{*}(X, \mathbb{Q}), y \in H^{*}\left(X^{\prime}, \mathbb{Q}\right),
$$

where $\langle$,$\rangle is the Mukai pairing.$

Proof.

$$
\begin{aligned}
& \left\langle x, \Psi_{X^{\prime} \rightarrow X}^{\mathcal{Q}}(y)\right\rangle \\
= & -\int_{X} x \Psi_{X^{\prime} \rightarrow X}^{\mathcal{Q}}(y)^{\vee} \\
= & -\int_{X^{\prime} \times X} \pi_{X}^{*}(x)\left(\pi_{X^{\prime}}^{*}(y) \frac{\sqrt{\operatorname{td} X^{\prime}}}{\sqrt{\operatorname{ch}\left({\mathbf{R} p_{*}^{\prime}}_{\prime}\left(G^{\prime \vee} \otimes G^{\prime}\right)\right)}} \frac{\sqrt{\operatorname{td} X_{X}}}{\sqrt{\operatorname{ch}\left(\mathbf{R} p_{*}\left(G^{\vee} \otimes G\right)\right)}}\right. \\
= & \left.\cdot \operatorname{ch}\left(\mathbf{R}\left(p^{\prime} \times p\right)_{*}\left(G^{\prime} \otimes \mathcal{Q} \otimes G^{\vee}\right)\right)\right)^{\vee} \\
= & \int_{X^{\prime} \times X}\left(\frac{\sqrt{\operatorname{td}_{X^{\prime}}}}{\sqrt{\operatorname{ch}\left(\mathbf{R} p_{*}^{\prime}\left(G^{\prime \vee} \otimes G^{\prime}\right)\right)}} \frac{\sqrt{\operatorname{td} X_{X}}}{\sqrt{\operatorname{ch}\left(\mathbf{R} p_{*}\left(G^{\vee} \otimes G\right)\right)}}\right. \\
= & \left.\cdot \operatorname{ch}\left(\mathbf{R}\left(p^{\prime} \times p\right)_{*}\left(G^{\prime \vee} \otimes \mathcal{Q}^{\vee} \otimes G\right)\right) \pi_{X}^{*}(x)\right) \pi_{X^{\prime}}^{*}\left(y^{\vee}\right) \\
= & \int_{X^{\prime}} \Psi_{X \rightarrow X^{\prime}}^{\mathcal{Q}^{\vee}}(x) y^{\vee} \\
= & \left\langle\Psi_{\mathcal{X}^{\vee} \rightarrow X^{\prime}}(x), y\right\rangle .
\end{aligned}
$$

Q.E.D.

### 4.2. Fourier-Mukai transform induced by stable twisted sheaves

Let $p: Y \rightarrow X$ be a projective bundle over an abelian surface or a $K 3$ surface. Let $G$ be a locally free $Y$-sheaf. Assume that $X^{\prime}:=\bar{M}_{H}^{Y, G}(v)$ is a surface and consists of stable sheaves. We set $Y^{\prime}:=\widehat{\bar{M}_{H}^{Y, G}}(v)$. Let $\mathcal{E}$ be the family on $Y^{\prime} \times Y$.

We consider integral functors

$$
\begin{array}{cccc}
\Phi_{X^{\prime} \rightarrow X}^{\mathcal{E}}: & \mathbf{D}\left(X^{\prime}, Y^{\prime}\right) & \rightarrow & \mathbf{D}(X, Y) \\
& x & \mapsto & \mathbf{R} \widetilde{\pi}_{Y *}\left(\mathcal{E} \otimes \widetilde{\pi}_{Y^{\prime}}^{*}(x)\right), \\
\Phi_{X \rightarrow X^{\prime}}^{\mathcal{E}^{\vee}}[2]: & \mathbf{D}(X, X) & \rightarrow & \mathbf{D}\left(X^{\prime}, Y^{\prime}\right) \\
y & \mapsto & \mathbf{R} \widetilde{\pi}_{Y^{\prime} *}\left(\mathcal{E}^{\vee} \otimes \widetilde{\pi}_{Y}^{*}(y)[2]\right) .
\end{array}
$$

Remark 4.2. Let $\mathcal{L}\left(p^{\prime *}\left(\alpha^{-1}\right)\right)$ and $\mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right)$ be twisted line bundles on $Y^{\prime}$ and $Y$ respectively in (1.1). Then $\Lambda^{\mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right)} \circ \Phi_{X^{\prime} \rightarrow X}^{\mathcal{E}} \circ$ $\left(\Lambda^{\mathcal{L}\left(p^{\prime *}\left(\alpha^{\prime-1}\right)\right)}\right)^{-1}: \mathbf{D}\left(X^{\prime}, \alpha^{\prime}\right) \rightarrow \mathbf{D}(X, \alpha)$ is an integral functor with the kernel $\mathbf{R}\left(p^{\prime} \times p\right)_{*}\left(\mathcal{L}\left(p^{\prime *}\left(\alpha^{\prime-1}\right)\right) \otimes \mathcal{E} \otimes \mathcal{L}\left(p^{*}\left(\alpha^{-1}\right)\right)^{\vee}\right) \in \mathbf{D}\left(X^{\prime} \times X, \alpha^{\prime-1} \times \alpha\right)$.

Căldăraru [C2] developed a theory of derived category of twisted sheaves. In particular, Grothendieck-Serre duality holds. Then we see that $\Phi_{X \rightarrow X^{\prime}}^{\mathcal{E}}[2]$ is the adjoint of $\Phi_{X^{\prime} \rightarrow X}^{\mathcal{E}}$. As in the usual Fourier-Mukai functor, we see that the following theorem holds (see [Br], [C1]).

Theorem 4.3. $\Phi_{X \rightarrow X^{\prime}}^{\mathcal{E}}[2] \circ \Phi_{X^{\prime} \rightarrow X}^{\mathcal{E}} \cong 1$ and $\Phi_{X^{\prime} \rightarrow X^{\mathcal{E}}}^{\mathcal{E}} \circ \Phi_{X \rightarrow X^{\prime}}^{\mathcal{E}}[2] \cong 1$. Thus $\Phi_{X^{\prime} \rightarrow X}^{\mathcal{E}}$ is an equivalence.

Then we have the following which also follows from a more general statement [H-St, Thm. 0.4].

Corollary 4.4. $\Psi_{X^{\prime} \rightarrow X}^{\mathcal{E}}$ induces an isometry of the Hodge structures:

$$
\left(H^{*}\left(X^{\prime}, \mathbb{Z}\right),\langle,\rangle,-\frac{\xi^{\prime}}{r}\right) \cong\left(H^{*}(X, \mathbb{Z}),\langle,\rangle,-\frac{\xi}{r}\right)
$$

Proof. Obviously $\Psi_{X^{\prime} \rightarrow X}^{\mathcal{E}}$ induces an isometry of the Hodge structures over $\mathbb{Q}$. If $X$ is a $K 3$ surface such that $w(Y) \in \mathrm{NS}(X) \otimes \mu_{r}$ and $X^{\prime}$ is a fine moduli space, then $\Psi_{X^{\prime} \rightarrow X}^{\mathcal{E}}$ is defined over $\mathbb{Z}$. For a general case, we use the deformation arguments.
Q.E.D.

We also have the following which is used in [Y4].
Corollary 4.5. Assume that $X^{\prime}$ consists of locally free $Y$-sheaves. Then $\mathcal{E}_{\mid Y^{\prime} \times\{y\}}^{\vee}, y \in Y$ is a simple $Y^{\prime}$-sheaf. If $\operatorname{NS}(X) \cong \mathbb{Z} H$, then $\mathcal{E}_{\mid Y^{\prime} \times\{y\}}^{\vee}, y \in Y$ is a stable $Y^{\prime}$-sheaf.

Proof. Since $\Phi_{X \rightarrow X^{\prime}}^{\mathcal{E}}[2]$ is an equivalence, $\Phi_{X \rightarrow X^{\prime}}^{\mathcal{E}}\left(\mathcal{O}_{p^{-1}(p(y))}(1)\right)=$ $\mathcal{E}_{\mid Y^{\prime} \times\{y\}}^{\vee}$ is a simple $Y^{\prime}$-sheaf. If $\operatorname{NS}(X) \cong \mathbb{Z}$, then Proposition 3.12 implies the stability of $\mathcal{E}_{\mid Y^{\prime} \times\{y\}}^{\vee}$.
Q.E.D.

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# Appendix : Proof of Căldăraru's conjecture 

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In this short note we show how to combine Yoshioka's recent results on moduli spaces of twisted sheaves on K3 surfaces with more or less standard methods to prove Căldăraru's conjecture on the equivalence of twisted derived categories of projective K3 surfaces. More precisely, we shall show

Theorem 0.1. Let $X$ and $X^{\prime}$ be two projective $K 3$ surfaces endowed with $B$-fields $B \in H^{2}(X, \mathbb{Q})$ respectively $B^{\prime} \in H^{2}\left(X^{\prime}, \mathbb{Q}\right)$. Suppose there exists a Hodge isometry

$$
g: \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}\left(X^{\prime}, B^{\prime}, \mathbb{Z}\right)
$$

that preserves the natural orientation of the four positive directions. Then there exists a Fourier-Mukai equivalence

$$
\Phi: \mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha^{\prime}\right)
$$

such that the induced action $\Phi_{*}^{B, B^{\prime}}$ on cohomology equals $g$.
Here, $\alpha:=\alpha_{B}$ and $\alpha^{\prime}:=\alpha_{B^{\prime}}$ are the Brauer classes induced by $B$ respectively $B^{\prime}$.

The twisted Hodge structures and the cohomological Fourier-Mukai transform (based on the notion of twisted Chern character), indispensable for the formulation of the conjecture, were introduced in [4]. For a complete discussion of the natural orientation of the positive directions and the cohomological Fourier-Mukai transform $\Phi_{*}^{B, B^{\prime}}$ we also refer to [4]. Note that Căldăraru's conjecture was originally formulated purely in terms of the transcendental lattice. But, as has been explained in [4], in the twisted case passing from the transcendental part to the full cohomology is not always possible, so that the original formulation had to be changed slightly to the above one.

Also note that any Fourier-Mukai equivalence

$$
\Phi: \mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha^{\prime}\right)
$$

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induces a Hodge isometry as above, but for the time being we cannot prove that this Hodge isometry also preserves the natural orientation. In the untwisted case this is harmless, for a given orientation reversing Hodge isometry can always be turned into an orientation preserving one by composing with $-\mathrm{id}_{H^{2}}$. In the twisted setting this cannot always be guaranteed, so that we cannot yet exclude the case of Fourier-Mukai equivalent twisted K 3 surfaces $\left(X, \alpha_{B}\right)$ and $\left(X^{\prime}, \alpha_{B^{\prime}}\right)$ which only admit an orientation reversing Hodge isometry $\widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}\left(X^{\prime}, B^{\prime}, \mathbb{Z}\right)$. Of course, this is related to the question whether any Fourier-Mukai equivalence is orientation preserving which seems to be a difficult question even in the untwisted case (see $[3,11]$ ).

From Yoshioka's paper [12] we shall use the following
Theorem 0.2. (Yoshioka) Let $X$ be a $K 3$ surface with a rational $B$-field $B \in H^{2}(X, \mathbb{Q})$ and $v \in \widetilde{H}^{1,1}(X, B, \mathbb{Z})$ a primitive vector with $\langle v, v\rangle=0$. Then there exists a moduli space $M(v)$ of stable (with respect to a generic polarizations) $\alpha_{B}$-twisted sheaves $E$ with $\operatorname{ch}^{B}(E) \sqrt{t d(X)}=$ $v$ such that:
i) Either $M(v)$ is empty or a K3 surface. The latter holds true if the degree zero part of $v$ is positive.
ii) On $X^{\prime}:=M(v)$ one finds a B-field $B^{\prime} \in H^{2}\left(X^{\prime}, \mathbb{Q}\right)$ such that there exists a universal family $\mathcal{E}$ on $X \times X^{\prime}$ which is an $\alpha_{B}^{-1} \boxtimes \alpha_{B^{\prime}}$ twisted sheaf.
iii) The twisted sheaf $\mathcal{E}$ induces a Fourier-Mukai equivalence

$$
\mathrm{D}^{\mathrm{b}}\left(X, \alpha_{B}\right) \cong \mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha_{B^{\prime}}\right)
$$

The existence of the moduli space of semistable twisted sheaves has been proved by Yoshioka for arbitrary projective varieties. Instead of considering twisted sheaves, he works with coherent sheaves on a BrauerSeveri variety. Using the equivalence between twisted sheaves and modules over Azumaya algebras, one can in fact view these moduli spaces also as a special case of Simpson's general construction [10]. (The two stability conditions are indeed equivalent.) In his thesis [5] M. Lieblich considers similar moduli spaces. (See also [2] for the rank one case.)

The crucial part for the application to Căldăraru's conjecture is i), in particular the non-emptiness. Yoshioka follows Mukai's approach, which also yields ii). Part iii) is a rather formal consequence of the usual criteria for the equivalence of Fourier-Mukai transforms already applied to the twisted case in [1].

In the last section we provide a dictionary between the different versions of twisted Chern characters and the various notions of twisted sheaves. Only parts of it is actually used in the proof of the conjecture.

The rest is meant to complement [4] and to facilitate the comparison of [4], [5], and [12].

Acknowledgements: It should be clear that the lion's share of the proof of Căldăraru's conjecture in the above form is in fact contained in K. Yoshioka's paper. We are grateful to him for informing us about his work and comments on the various versions of this note. Thanks also to M. Lieblich who elucidated the relation between the different ways of constructing moduli spaces of twisted sheaves. Our proof follows the arguments in the untwisted case, due to S. Mukai [6] and D. Orlov [8] (see also $[3,7]$ ), although some modifications were necessary. During the preparation of this paper the second named author was partially supported by the MIUR of the Italian government in the framework of the National Research Project "Algebraic Varieties" (Cofin 2002).

## §1. Examples

Let $X$ and $X^{\prime}$ be projective K3 surfaces (always over $\mathbb{C}$ ) with B-fields $B \in H^{2}(X, \mathbb{Q})$ respectively $B^{\prime} \in H^{2}\left(X^{\prime}, \mathbb{Q}\right)$. We denote the induced Brauer classes by $\alpha:=\alpha_{B}:=\exp \left(B^{0,2}\right) \in H^{2}\left(X, \mathcal{O}_{X}^{*}\right)$ respectively $\alpha^{\prime}:=\alpha_{B^{\prime}} \in H^{2}\left(X^{\prime}, \mathcal{O}_{X^{\prime}}^{*}\right)$. We start out with introducing a few examples of equivalences between the bounded derived categories $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ respectively $\mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha^{\prime}\right)$ of the abelian categories of $\alpha$-twisted (resp. $\alpha^{\prime}$ twisted) sheaves.
i) Let $f: X \cong X^{\prime}$ be an automorphism with $f^{*} \alpha^{\prime}=\alpha$. Then $\Phi:=$ $f_{*}: \mathrm{D}^{\mathrm{b}}(X, \alpha) \rightarrow \mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha^{\prime}\right), E \mapsto R f_{*} E$ is a Fourier-Mukai equivalence with kernel $\mathcal{O}_{\Gamma_{f}}$ viewed as an $\alpha^{-1} \boxtimes \alpha^{\prime}$-twisted sheaf on $X \times X^{\prime}$.

If in addition $f_{*}(B)=B^{\prime}$ then $\Phi_{*}^{B, B^{\prime}}=f_{*}$.
ii) Let $L \in \operatorname{Pic}(X)$ be a(n untwisted) line bundle on $X$. Then $E \mapsto L \otimes E$ defines a Fourier-Mukai equivalence $L \otimes(): \mathrm{D}^{\mathrm{b}}(X, \alpha) \cong$ $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ with kernel $i_{*} L$ considered as an $\alpha^{-1} \boxtimes \alpha$-twisted sheaf on $X \times$ $X$. Here, $i: X \hookrightarrow X \times X$ denotes the diagonal embedding. The induced cohomological Fourier-Mukai transform $(L \otimes())_{*}^{B, B}: \widetilde{H}(X, B, \mathbb{Z}) \cong$ $\widetilde{H}(X, B, \mathbb{Z})$ is given by multiplication with $\exp \left(\mathrm{c}_{1}(L)\right)$.
iii) Let $b \in H^{2}(X, \mathbb{Z})$. Then $\alpha_{B}=\alpha_{B+b}$. The identity

$$
\text { id }: \mathrm{D}^{\mathrm{b}}\left(X, \alpha_{B}\right)=\mathrm{D}^{\mathrm{b}}\left(X, \alpha_{B+b}\right)
$$

descends to $\mathrm{id}_{*}^{B, B+b}: \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}(X, B+b, \mathbb{Z})$ which is given by the multiplication with $\exp (b)$. This follows from the formula $\operatorname{ch}^{B+b}(E)=$ $\operatorname{ch}^{B}(E) \cdot \operatorname{ch}^{b}(\mathcal{O})=\operatorname{ch}^{B}(E) \cdot \exp (b)($ see [4, Prop. 1.2] $)$.
iv) Changing the given B-field $B$ by a class $b \in H^{1,1}(X, \mathbb{Q})$ does not affect $\widetilde{H}(X, B, \mathbb{Z})$. Thus, the identity can be considered as an orientation preserving Hodge isometry $\widetilde{H}(X, B, \mathbb{Z})=\widetilde{H}(X, B+b, \mathbb{Z})$.

As shall be explained in the last section, this can be lifted to a Fourier-Mukai equivalence. More precisely, there is an exact functor $\Phi: \mathbf{C o h}\left(X, \alpha_{B}\right) \cong \mathbf{C o h}\left(X, \alpha_{B+b}\right)$, whose derived functor, again denoted by $\Phi: \mathrm{D}^{\mathrm{b}}\left(X, \alpha_{B}\right) \cong \mathrm{D}^{\mathrm{b}}\left(X, \alpha_{B+b}\right)$, is of Fourier-Mukai type and such that $\Phi_{*}^{B, B+b}=\mathrm{id}$.
v) Let $E \in \mathrm{D}^{\mathrm{b}}(X, \alpha)$ be a spherical object, i.e. $\operatorname{Ext}^{i}(E, E)=0$ for all $i$ except for $i=0,2$ when it is of dimension one. Then the twist functor $T_{E}$ that sends $F \in \mathrm{D}^{\mathrm{b}}(X, \alpha)$ to the cone of $\operatorname{Hom}(E, F) \otimes E \rightarrow F$ defines a Fourier-Mukai autoequivalence $T_{E}: \mathrm{D}^{\mathrm{b}}(X, \alpha) \cong \mathrm{D}^{\mathrm{b}}(X, \alpha)$. The kernel of $T_{E}$ is given by the cone of the natural map

$$
E^{*} \boxtimes E \longrightarrow \mathcal{O}_{\Delta}
$$

where $\mathcal{O}_{\Delta}$ is considered as an $\alpha^{-1} \boxtimes \alpha$-twisted sheaf on $X \times X$. The result in the untwisted case goes back to Seidel and Thomas [9]. The following short proof of this, which carries over to the twisted case, has been communicated to us by D. Ploog [7]. Consider the class $\Omega \subset \mathrm{D}^{\mathrm{b}}(X, \alpha)$ of objects $F$ that are either isomorphic to $E$ or contained in its orthogonal complement $E^{\perp}$, i.e. $\operatorname{Ext}^{i}(E, F)=0$ for all $i$. It is straightforward to check that this class is spanning. Since $T_{E}(E) \cong E[-1]$ and $T_{E}(F) \cong F$ for $F \in E^{\perp}$, one easily verifies that $\operatorname{Ext}^{i}\left(F_{1}, F_{2}\right)=\operatorname{Ext}^{i}\left(T_{E}\left(F_{1}\right), T_{E}\left(F_{2}\right)\right)$ for all $F_{1}, F_{2} \in \Omega$.

In other words, $T_{E}$ is fully faithful on the spanning class $\Omega$ and hence fully faithful. By the usual argument, the Fourier-Mukai functor $T_{E}$ is then an equivalence.

As in the untwisted case, one proves that the induced action on cohomology is the reflection $\alpha \mapsto \alpha+\left\langle\alpha, v^{B}(E)\right\rangle \cdot v^{B}(E)$. Here, $v^{B}(E)$ is the Mukai vector $v^{B}(E):=\operatorname{ch}^{B}(E) \sqrt{\operatorname{td}(X)}$.

Special cases of this construction are:

- Let $\mathbb{P}^{1} \cong C \subset X$ be a smooth rational curve. As $H^{2}\left(C, \mathcal{O}_{C}^{*}\right)$ is trivial, its structure sheaf $\mathcal{O}_{C}$ and any twist $\mathcal{O}_{C}(k)$ can naturally be considered as $\alpha$-twisted sheaves. The Mukai vector for $k=-1$ is given by $v\left(\mathcal{O}_{C}(-1)\right)=(0,[C], 0)$.
- In the untwisted case, the trivial line bundle $\mathcal{O}$ (and in fact any line bundle) provides an example of a spherical object. Its Mukai vector is $(1,0,1)$ and has, in particular, a non-trivial degree zero component. It is the latter property that is of importance for the proof in the untwisted case. So the original argument goes through if at least one
spherical object of non-trivial rank can be found. Unfortunately, spherical object (in particular those of positive rank) might not exist at all in the twisted case. In fact, any spherical object $E$ has a Mukai vector $v^{B}(E) \in \operatorname{Pic}(X, B)$ of square $\left\langle v^{B}(E), v^{B}(E)\right\rangle=-2$ and it is not difficult to find examples of rational B-fields $B \neq 0$ such that such a vector does not exist.
vi) Let $\ell \in \operatorname{Pic}(X)$ be a nef class with $\langle\ell, \ell\rangle=0$. If $w=(0, \ell, s)$ is a primitive vector, then the moduli space $M(w)$ of $\alpha_{B}$-twisted sheaves which are stable with respect to a generic polarization is non-empty. Indeed, in this case $\ell$ is a multiple $n \cdot f$ of a fibre class $f$ of an elliptic fibration $\pi: X \rightarrow \mathbb{P}^{1}$. As $\operatorname{gcd}(n, s)=1$, there exists a stable rank $n$ vector bundle of degree $s$ on a smooth fibre of $\pi$ which yields a point in $M(w)$.

If $\ell$ is the fibre class of an elliptic fibration $X \rightarrow \mathbb{P}^{1}$, we can think of $M(w)$ as the relative Jacobian $\mathcal{J}^{s}\left(X / \mathbb{P}^{1}\right) \rightarrow \mathbb{P}^{1}$.

In any case, $M(w)$ is a K3 surface and the universal twisted sheaf provides an equivalence $\Phi: \mathrm{D}^{\mathrm{b}}\left(M(w), \alpha_{B^{\prime}}\right) \cong \mathrm{D}^{\mathrm{b}}\left(X, \alpha_{B}\right)$ (for some Bfield $B^{\prime}$ on $\left.M(w)\right)$ inducing a Hodge isometry $\Phi_{*}^{B^{\prime}, B}: \widetilde{H}\left(M(w), B^{\prime}, \mathbb{Z}\right) \cong$ $\widetilde{H}(X, B, \mathbb{Z})$ that sends $(0,0,1)$ to $w$.

## §2. The proof

Let $g: \widetilde{H}(X, B, \mathbb{Z}) \cong \widetilde{H}\left(X^{\prime}, B^{\prime}, \mathbb{Z}\right)$ be an orientation preserving Hodge isometry. The Mukai vector of $k(x)$ with $x \in X$ is $v^{B}(k(x))=$ $v(k(x))=(0,0,1)$. We shall denote its image under $g$ by $w:=g(0,0,1)=$ $(r, \ell, s)$.

1st step. In the first step we assume that $r=0$ and $\ell=0$, i.e. $g(0,0,1)= \pm(0,0,1)$, and that furthermore $g(1,0,0)= \pm(1,0,0)$. By composing with -id we may actually assume $g(0,0,1)=(0,0,1)$ and $g(1,0,0)=(1,0,0)$.

In particular, $g$ preserves the grading of $\widetilde{H}$ and induces a Hodge isometry $H^{2}(X, \mathbb{Z}) \cong H^{2}\left(X^{\prime}, \mathbb{Z}\right)$. Denote $b:=g(B)-B^{\prime} \in H^{2}(X, \mathbb{Q})$. As $g$ respects the Hodge structure, it maps $\sigma+B \wedge \sigma$ to $\sigma^{\prime}+B^{\prime} \wedge \sigma^{\prime}$ and, therefore $\langle\sigma, B\rangle=\left\langle\sigma^{\prime}, B^{\prime}\right\rangle$. On the other hand, as $g$ is an isometry, one has $\langle\sigma, B\rangle=\left\langle\sigma^{\prime}, g(B)\right\rangle$. Altogether this yields $\left\langle\sigma^{\prime}, b\right\rangle=0$, i.e. $b \in$ $H^{1,1}(X, \mathbb{Q})$.

Now compose $g$ with the orientation preserving Hodge isometry given by the identity $\widetilde{H}\left(X^{\prime}, B^{\prime}, \mathbb{Z}\right)=\widetilde{H}\left(X^{\prime}, g(B)=B^{\prime}+b, \mathbb{Z}\right)$. As the latter can be lifted to a Fourier-Mukai equivalence $\mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha_{B^{\prime}}\right) \cong$ $\mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha_{g(B)}\right)$ (see example iv)), it suffices to show that $g$ viewed as a

Hodge isometry $\widetilde{H}(X, B, \mathbb{Z})=\widetilde{H}(X, g(B), \mathbb{Z})$ can be lifted. So we may from now on assume that $B^{\prime}=g(B)$.

As $g$ is orientation preserving, its degree two component defines a Hodge isometry that maps the positive cone $\mathcal{C}_{X} \subset H^{1,1}(X)$ onto the positive cone $\mathcal{C}_{X^{\prime}} \subset H^{1,1}\left(X^{\prime}\right)$.

If $g$ maps an ample class to an ample class, then by the Global Torelli Theorem $g$ can be lifted to an isomorphism $f: X \cong X^{\prime}$ which in turn yields a Fourier-Mukai equivalence $\Phi:=f_{*}: \mathrm{D}^{\mathrm{b}}\left(X, \alpha_{B}\right) \cong \mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha_{B^{\prime}}\right)$. Obviously, with this definition $\Phi_{*}^{B, B^{\prime}}=g$ (use $f_{*}(B)=g(B)=B^{\prime}$ ).

If $g$ does not preserve ampleness, then the argument has to be modified as follows: After a finite number of reflections $s_{C_{i}}$ in hyperplanes orthogonal to $(-2)$-classes $\left[C_{i}\right]$ we may assume that $s_{C_{1}}\left(\ldots s_{C_{n}}(h(a)) \ldots\right)$ is an ample class. As the reflections $s_{C_{i}}$ are induced by the twist functors $T_{\mathcal{O}_{C_{i}}(-1)}: \mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha_{B^{\prime}}\right) \cong \mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha_{B^{\prime}}\right)$ (see the explanations in the last section), the Hodge isometry $g$ is induced by a Fourier-Mukai equivalence if and only if the composition $s_{C_{1}} \circ \ldots s_{C_{n}} \circ g$ is. Thus, we have reduced the problem to the case already treated above.

In the following steps we shall explain how the general case can be reduced to the case just considered.

2nd step. Suppose $g(0,0,1)= \pm(0,0,1)$ but $g(1,0,0) \neq \pm(1,0,0)$. Again, by composing with -id we may reduce to $g(0,0,1)=(0,0,1)$ and $g(1,0,0) \neq(1,0,0)$. Then $g(1,0,0)$ is necessarily of the form $\exp (b)$ for some $b \in H^{2}\left(X^{\prime}, \mathbb{Z}\right)$. Hence, we may compose $g$ with the Hodge isometry $\exp (-b): \widetilde{H}\left(X^{\prime}, B^{\prime}, \mathbb{Z}\right) \cong \widetilde{H}\left(X^{\prime}, B^{\prime}-b, \mathbb{Z}\right)$ (that preserves the orientation) which can be lifted to a Fourier-Mukai equivalence according to example iii). This reduces the problem to the situation studied in the previous step.

3rd step. Suppose that $r>0$. Using Theorem 0.2 one finds a K3 surface $X_{0}$ with a B-field $B_{0} \in H^{2}\left(X_{0}, \mathbb{Q}\right)$ such that over $X_{0} \times X^{\prime}$ there exists a universal $\alpha_{B_{0}}^{-1} \boxtimes \alpha_{B^{\prime}}$-twisted sheaf parametrizing stable $\alpha^{\prime}$-twisted sheaves on $X^{\prime}$ with Mukai vector $v^{B^{\prime}}=w$. In particular, $\mathcal{E}$ induces an equivalence $\Phi_{\mathcal{E}}: \mathrm{D}^{\mathrm{b}}\left(X_{0}, \alpha_{B_{0}}\right) \cong \mathrm{D}^{\mathrm{b}}\left(X^{\prime}, \alpha_{B^{\prime}}\right)$ and $\Phi_{\mathcal{E} *}^{B_{0}, B^{\prime}}(0,0,1)=w$.

Thus, the composition $g_{0}:=\left(\Phi_{\mathcal{E} *}^{B_{0}, B^{\prime}}\right)^{-1} \circ g$ yields an orientation preserving (!) Hodge isometry $\widetilde{H}(X, B, \mathbb{Z}) \cong H\left(X_{0}, B_{0}, \mathbb{Z}\right)$. (The proof that the universal family of stable sheaves induces an orientation preserving Hodge isometry is analogous to the untwisted case. This seems to be widely known [3, 11]. For an explicit proof see [4].) Clearly, $g$ can be lifted to a Fourier-Mukai equivalence if and only if $g_{0}$ can. The latter follows from step one.

4th step. Suppose $g$ is given with $r<0$. Then compose with the orientation preserving Hodge isometry -id of $\widetilde{H}\left(X^{\prime}, B^{\prime}, \mathbb{Z}\right)$ which is lifted to the shift functor $E \mapsto E[1]$. Thus, it is enough to lift the composition -id $\circ g$ which can be achieved according to step three.

5th step The remaining case is $r=0$ and $\ell \neq 0$. One applies the construction of example vi) in Section 1 and proceeds as in step 3. The class $\ell$ can be made nef by applying -id if necessary to make it effective (i.e. contained in the closure of the positive cone) and then composing it with reflections $s_{C}$ as in step one.

## §3. The various twisted categories and their Chern characters

Let $\alpha \in H^{2}\left(X, \mathcal{O}_{X}^{*}\right)$ be a Brauer class represented by a Čech cocycle $\left\{\alpha_{i j k}\right\}$.

1. The abelian category $\operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right)$ of $\left\{\alpha_{i j k}\right\}$-twisted coherent sheaves only depends on the class $\alpha \in H^{2}\left(X, \mathcal{O}_{X}^{*}\right)$. More precisely, for any other choice of a Čech-cocycle $\left\{\alpha_{i j k}^{\prime}\right\}$ representing $\alpha$, there exists an equivalence

$$
\operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right) \xrightarrow{\Psi_{\left\{\lambda_{i j}\right\}}} \operatorname{Coh}\left(X,\left\{\alpha_{i j k}^{\prime}\right\}\right), \quad\left\{E_{i}, \varphi_{i j}\right\} \mapsto\left\{E_{i}, \varphi_{i j} \cdot \lambda_{i j}\right\},
$$

where $\left\{\lambda_{i j} \in \mathcal{O}^{*}\left(U_{i j}\right)\right\}$ satisfies $\alpha_{i j k}^{\prime} \alpha_{i j k}^{-1}=\lambda_{i j} \cdot \lambda_{j k} \cdot \lambda_{k i}$. Clearly, $\left\{\lambda_{i j}\right\}$ exists, as $\left\{\alpha_{i j k}\right\}$ and $\left\{\alpha_{i j k}^{\prime}\right\}$ define the same Brauer class, but it is far from being unique. In other words, the above equivalence $\Psi_{\left\{\lambda_{i j}\right\}}$ is not canonical. In order to make this more precise, choose a second $\left\{\lambda_{i j}^{\prime}\right\}$. Then $\gamma_{i j}:=\lambda_{i j}^{\prime} \cdot \lambda_{i j}^{-1}$ can be viewed as the transition function of a holomorphic line bundle $\mathcal{L}_{\lambda \lambda^{\prime}}$. With this notation one finds

$$
\Psi_{\left\{\lambda_{i j}^{\prime}\right\}}=\left(\mathcal{L}_{\lambda \lambda^{\prime}} \otimes()\right) \circ \Psi_{\left\{\lambda_{i j}\right\}}
$$

A very special case of this is the equivalence

$$
\mathcal{L} \otimes(): \operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right) \longrightarrow \mathbf{C o h}\left(X,\left\{\alpha_{i j k}\right\}\right)
$$

that is induced by the tensor product with a holomorphic line bundle $\mathcal{L}$ given by a cocycle $\left\{\gamma_{i j}\right\}$.

Despite this ambiguity in identifying these categories for different choices of the Čech-representative, $\operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right)$ is often simply denoted $\operatorname{Coh}(X, \alpha)$.
2. Now fix a B-field $B \in H^{2}(X, \mathbb{Q})$ together with a Čech-representative $\left\{B_{i j k}\right\}$. The induced Brauer class $\alpha:=\exp \left(B^{0,2}\right) \in H^{2}\left(X, \mathcal{O}_{X}^{*}\right)$ is represented by the Čech-cocycle $\left\{\alpha_{i j k}:=\exp \left(B_{i j k}\right)\right\}$.

In [4] we introduced

$$
\operatorname{ch}^{B}: \operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right) \longrightarrow H^{*, *}(X, \mathbb{Q})
$$

The construction makes use of a further choice of $\mathcal{C}^{\infty}$-functions $a_{i j}$ with $-B_{i j k}=a_{i j}+a_{j k}+a_{k i}$, but the result does not depend on it. Indeed, by definition, $\operatorname{ch}^{B}\left(\left\{E_{i}, \varphi_{i j}\right\}\right)=\operatorname{ch}\left(\left\{E_{i}, \varphi_{i j} \cdot \exp \left(a_{i j}\right)\right\}\right)$. Thus, if we pass from $a_{i j}$ to $a_{i j}+c_{i j}$ with $c_{i j}+c_{j k}+c_{k i}=0$, then $\operatorname{ch}^{B}\left(\left\{E_{i j}, \varphi_{i j}\right\}\right)$ changes by $\exp \left(c_{1}(\mathcal{L})\right)$, where $\mathcal{L}$ is given by the transition functions $\left\{\exp \left(c_{i j}\right)\right\}$. But by the very definition of the first Chern class, one has $\mathrm{c}_{1}(\mathcal{L})=\left[\left\{c_{i j}+c_{j k}+c_{k i}\right\}\right]=0$.

More generally, we may change the class $B$ by a class $b \in H^{2}(X, \mathbb{Q})$ represented by $\left\{b_{i j k}\right\}$. Suppose $\alpha_{B+b}=\alpha_{B} \in H^{2}\left(X, \mathcal{O}_{X}^{*}\right)$. We denote the Cech-representative $\exp \left(B_{i j k}+b_{i j k}\right)$ by $\alpha_{i j k}^{\prime}$. As before, we write $-B_{i j k}=a_{i j}+a_{j k}+a_{k i}$ and $-b_{i j k}=c_{i j}+c_{j k}+c_{k i}$.

The Chern characters $\operatorname{ch}^{B}$ and $\operatorname{ch}^{B+b}$ fit into the following commutative diagram


Unfortunately, we cannot replace $\mathbf{S h}$ by $\mathbf{C o h}$, for $\exp \left(c_{i j}\right)$ are only differentiable functions. Nevertheless, there exist $\beta_{i j} \in \mathcal{O}^{*}\left(U_{i j}\right)$, nonunique, with $\alpha_{i j k}^{\prime}=\alpha_{i j k} \cdot\left(\beta_{i j} \cdot \beta_{j k} \cdot \beta_{k i}\right)$. Using these one finds a commutative diagram


Here, $\mathcal{L}$ is the line bundle given by the transition functions $\beta_{i j} \cdot \exp \left(c_{i j}\right)$.
It is not difficult to see that $\mathrm{c}_{1}(\mathcal{L}) \in H^{1,1}(X)$ whenever one has $b \in H^{1,1}(X, \mathbb{Q})$. Indeed, $c_{1}(\mathcal{L})=\left\{d \log \left(\beta_{i j}\right)\right\}+b$, which is of type $(1,1)$, as $b$ is $(1,1)$ by assumption and the functions $\beta_{i j}$ are holomorphic.

Thus, in this case there exists a holomorphic line bundle $\widetilde{\mathcal{L}}$ with $\mathrm{c}_{1}(\widetilde{\mathcal{L}})=\mathrm{c}_{1}(\mathcal{L})$. Now consider the composition $\Phi:=\left(\widetilde{\mathcal{L}^{*}} \otimes()\right) \circ \Psi_{\left\{\beta_{i j}\right\}}:$ $\operatorname{Coh}\left(X, \alpha_{B}:=\left\{\alpha_{i j k}\right\}\right) \cong \mathbf{C o h}\left(X, \alpha_{B+b}:=\left\{\alpha_{i j k}^{\prime}\right\}\right)$, which is an exact
equivalence, and denote the derived one again by $\Phi: \mathrm{D}^{\mathrm{b}}\left(X, \alpha_{B}\right) \cong$ $\mathrm{D}^{\mathrm{b}}\left(X, \alpha_{B+b}\right)$. Then the above calculation of the twisted Chern character implies that $\Phi_{*}^{B, B+b}=\mathrm{id}$.
3. Consider again the abelian category $\operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right)$. For any locally free $G=\left\{G_{i}, \varphi_{i j}\right\} \in \operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right)$ one defines an Azumaya algebra $\mathcal{A}_{G}:=\mathcal{E} n d\left(G^{\vee}\right)$. The abelian category of left $\mathcal{A}_{G}$-modules will be denoted $\operatorname{Coh}\left(\mathcal{A}_{G}\right)$. An equivalence of abelian categories is given by

$$
\operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right) \longrightarrow \operatorname{Coh}\left(\mathcal{A}_{G}\right), E \longmapsto G^{\vee} \otimes E
$$

In [12] Yoshioka considers yet another abelian category $\operatorname{Coh}(X, Y)$ of certain coherent sheaves on a projective bundle $Y \rightarrow X$ realizing the Brauer class $\alpha$. As is explained in detail in [12], one again has an equivalence of abelian categories $\operatorname{Coh}(X, Y) \cong \operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right)$. In order to define an appropriate notion of stability, Yoshioka defines a Hilbert polynomial for objects $E \in \mathbf{C o h}(X, Y)$. It is straightforward to see that under the composition

$$
\operatorname{Coh}(X, Y) \xrightarrow{\sim} \operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right) \xrightarrow{\sim} \operatorname{Coh}\left(\mathcal{A}_{G}\right)
$$

his Hilbert polynomial corresponds to the usual Hilbert polynomial for sheaves $F \in \operatorname{Coh}\left(\mathcal{A}_{G}\right)$ viewed as $\mathcal{O}_{X}$-modules. The additional choice of the locally free object $G$ in $\operatorname{Coh}(X, Y)$ or equivalently in $\operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right)$ needed to define the Hilbert polynomial in [12] enters this comparison via the equivalence $\operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right) \cong \operatorname{Coh}\left(\mathcal{A}_{G}\right)$. From here it is easy to see that the stability conditions considered in [10, 12] are actually equivalent.

We would like to define a twisted Chern character for objects in $\operatorname{Coh}\left(\mathcal{A}_{G}\right)$. Of course, as any $F \in \operatorname{Coh}\left(\mathcal{A}_{G}\right)$ is in particular an ordinary sheaf, $\operatorname{ch}(F)$ is well defined. In order to define something that takes into account the $\mathcal{A}_{G}$-module structure, one fixes $B=\left\{B_{i j k}\right\}$ and assumes $\alpha_{i j k}=\exp \left(B_{i j k}\right)$. Then we introduce

$$
\operatorname{ch}_{G}^{B}: \operatorname{Coh}\left(\mathcal{A}_{G}\right) \longrightarrow H^{*}(X, \mathbb{Q}), F \longmapsto \frac{\operatorname{ch}(F)}{\operatorname{ch}^{-B}\left(G^{V}\right)}
$$

Note that a priori the definition depends on $B$ and $G$, but the dependence on the latter is well-behaved as will be explained shortly.

Here are the main compatibilities for this new Chern character:
i) The following diagram is commutative:


Indeed, $\operatorname{ch}\left(G^{\vee} \otimes E\right)=\operatorname{ch}^{-B}\left(G^{\vee}\right) \cdot \operatorname{ch}^{B}(E)$.
ii) Let $H$ be a locally free coherent sheaf and $G^{\prime}:=G \otimes H \in$ $\operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right)$. Then the natural equivalence $\operatorname{Coh}\left(\mathcal{A}_{G}\right) \rightarrow \operatorname{Coh}\left(\mathcal{A}_{G^{\prime}}\right)$, $F \mapsto H^{\vee} \otimes F$ fits in the commutative diagram


This roughly says that the new Chern character is independent of $G$.
iii) If $E_{1}, E_{2} \in \operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right)$ and $F_{i}:=G^{\vee} \otimes E_{i} \in \operatorname{Coh}\left(\mathcal{A}_{G}\right)$ then $\chi\left(E_{1}, E_{2}\right):=\sum(-1)^{i} \operatorname{dim} \operatorname{Ext}^{i}\left(E_{1}, E_{2}\right)$ is well-defined and equals $\chi\left(F_{1}, F_{2}\right):=\sum(-1)^{i} \operatorname{dim} \operatorname{Ext}_{\mathcal{A}_{G}}^{i}\left(F_{1}, F_{2}\right)$. Both expressions can be computed in terms of the twisted Chern characters introduced above and the Mukai pairing. Concretely,

$$
\chi\left(F_{1}, F_{2}\right)=-\left\langle\operatorname{ch}_{G}^{B}\left(F_{1}\right) \cdot \sqrt{\operatorname{td}(X)}, \operatorname{ch}_{G}^{B}\left(F_{2}\right) \cdot \sqrt{\operatorname{td}(X)}\right\rangle .
$$

Here $\langle$,$\rangle denotes the generalized Mukai pairing and$

$$
\chi\left(F_{1}, F_{2}\right):=-\left\langle\operatorname{ch}^{B}\left(E_{1}\right) \cdot \sqrt{\operatorname{td}(X)}, \operatorname{ch}^{B}\left(E_{2}\right) \cdot \sqrt{\operatorname{td}(X)}\right\rangle .
$$

(Be aware of the different sign conventions for K3 surfaces and the general case.)
4. There is yet another way to define a twisted Chern character which is implicitly used in [12]. We use the above notations and define $\operatorname{ch}_{G}: \operatorname{Coh}\left(\mathcal{A}_{G}\right) \rightarrow H^{*}(X, \mathbb{Q})$ by $\operatorname{ch}_{G}(F):=\frac{\operatorname{ch}(F)}{\sqrt{\operatorname{ch}\left(\mathcal{A}_{G}\right)}}$, where $F$ and $\mathcal{A}_{G}$ are considered as ordinary $\mathcal{O}_{X}$-modules. Using the natural identifications explained earlier, namely $\boldsymbol{\operatorname { C o h }}(X, Y) \cong \operatorname{Coh}\left(X,\left\{\alpha_{i j k}\right\}\right) \cong$ $\operatorname{Coh}\left(\mathcal{A}_{G}\right)$, this Chern character can also be viewed as a Chern character on the other abelian categories.

Although the definition $\mathrm{ch}_{G}$ seems very natural, it does not behave nicely under change of $G$. More precisely, in general $\operatorname{ch}_{G \otimes H}\left(H^{\vee} \otimes F\right) \neq$ $\operatorname{ch}_{G}(F)$.

Fortunately, the situation is less critical for K3 surfaces. Here, the relation between $\operatorname{ch}_{G}$ and $\operatorname{ch}_{G}^{B}$ can be described explicitly and using the results in 3. one deduces from this a formula for the change of $\mathrm{ch}_{G}$ under $G \mapsto G \otimes H$. In fact, it is straightforward to see that the following diagram commutes:


Here $B_{G}:=\frac{\mathrm{c}_{1}^{B}(G)}{\operatorname{rk}(G)}$, where $\mathrm{c}_{1}^{B}(G)$ is the degree two part of $\operatorname{ch}^{B}(G)$. Note that $B$ and $B_{G}$ define the same Brauer class. In particular, the Hodge structures $\widetilde{H}^{*}(X, B, \mathbb{Z})$ and $\widetilde{H}^{*}\left(X, B_{G}, \mathbb{Z}\right)$ are isomorphic.

This relation between $\operatorname{ch}_{G}^{B}$ and $\operatorname{ch}_{G}$ can be used to compare the two versions of the cohomogical Fourier-Mukai transform in [4] and [12]. With $v^{B}:=\operatorname{ch}^{B} \cdot \sqrt{\operatorname{td}(X)}$ and $v_{G}:=\operatorname{ch}_{G} \cdot \sqrt{\operatorname{td}(X)}$ and the implicit identification $\operatorname{Coh}(X, \alpha)=\operatorname{Coh}\left(\mathcal{A}_{G}\right)$ the following diagram is commutative:


Here, the central isomorphism $H^{*}(X, \mathbb{Q}) \cong H^{*}\left(X^{\prime}, \mathbb{Q}\right)$ is the correspondence defined by $v_{G^{\vee} \boxtimes G^{\prime}}(\mathcal{E})$ with $\mathcal{E} \in \mathrm{D}^{\mathrm{b}}\left(X \times X^{\prime}, \alpha_{B}^{-1} \boxtimes \alpha_{B^{\prime}}\right)$ the kernel defining $\Phi$.

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# On integral Hodge classes on uniruled or Calabi-Yau threefolds 

Claire Voisin

To Masaki Maruyama, on his 60th birthday

## §0. Introduction

Let $X$ be a smooth complex projective variety of dimension $n$. The Hodge conjecture is then true for rational Hodge classes of degree $2 n-2$, that is, the degree $2 n-2$ rational cohomology classes of $X$ which are of Hodge type $(n-1, n-1)$ are algebraic, which means that they are the cohomology classes of algebraic cycles with $\mathbb{Q}$-coefficients. Indeed, this follows from the hard Lefschetz theorem, which provides an isomorphism:

$$
\cup c_{1}(L)^{n-2}: H^{2}(X, \mathbb{Q}) \cong H^{2 n-2}(X, \mathbb{Q})
$$

from the fact that the isomorphism above sends the space of rational Hodge classes of degree 2 onto the space of rational Hodge classes of degree $2 n-2$, and from the Lefschetz theorem on ( 1,1 )-classes.

For integral Hodge classes, Kollár [11], (see also [14]) gave examples of smooth complex projective manifolds which do not satisfy the Hodge conjecture for integral degree $2 n-2$ Hodge classes, for any $n \geq 3$. The examples are smooth general hypersurfaces $X$ of certain degrees in $\mathbb{P}^{n+1}$. By the Lefschetz restriction theorem, such a variety satisfies

$$
H^{2}(X, \mathbb{Z})=\mathbb{Z} H, H=c_{1}\left(\mathcal{O}_{X}(1)\right)
$$

and

$$
H^{2 n-2}(X, \mathbb{Z})=\mathbb{Z} \alpha,<\alpha, H>=1
$$

Plane sections $C$ of $X$ have cohomology class $[C]=d \alpha, d=\operatorname{deg} X$, because

$$
\operatorname{deg} C=d=<[C], H>
$$

Kollár [11] proves the following :

Theorem 1. Consider hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree $d$, where $n \geq 4$. Assume $d$ satisfies the property that $p^{n}$ divides d, for some integer $p$ coprime to $n$ !. Then for a general $X$, any curve $C$ in $X$ has degree divisible by $p$, hence its cohomology class is a multiple of $p \alpha$. Thus the class $\alpha$ is not algebraic, that is, it is not the cohomology class of an algebraic cycle with integral coefficients.

The condition on the degree makes the canonical bundle of $X$ very ample, since the smallest possible degree available by this construction is $\geq 2^{n}$. It is thus natural to try to understand whether this is an artificial consequence of the method of construction, or whether the positivity of the canonical bundle is essential.

Another reason to ask whether one could find examples above with Kodaira dimension equal to $-\infty$ is the remark made in [14] :

Lemma 1. Let $X$ be a smooth rational complex projective manifold. Then the Hodge conjecture is true for integral Hodge classes of degree $2 n-2$.
(Note that the whole degree $2 n-2$ cohomology of such an $X$ is of type $(n-1, n-1)$, so the statement is that classes of curves generate $H^{2 n-2}(X, \mathbb{Z})$ for a rational variety $X$.)

One can thus ask whether this criterion could be used to produce new examples of unirational or rationally connected, but non rational varieties (we refer to [5], [1], [9] for other criteria). Namely, it would suffice to produce a smooth projective rationally connected variety which does not satisfy the Hodge conjecture for degree $2 n-2$ integral cohomology classes. The main result of this paper implies that in dimension 3 , this cannot be done:

Theorem 2. Let $X$ be a smooth complex projective threefold which either is uniruled, or satisfies

$$
K_{X} \cong \mathcal{O}_{X}, H^{2}\left(X, \mathcal{O}_{X}\right)=0 .
$$

Then the Hodge conjecture is true for integral degree 4 Hodge classes on $X$.

Remark 1. Recall [12] that a complex projective threefold is uniruled, that is swept out by rational curves, if and only if it has Kodaira dimension equal to $-\infty$. Thus our condition is that either $\kappa(X)=-\infty$ or $K_{X}=\mathcal{O}_{X}$ and $H^{2}\left(X, \mathcal{O}_{X}\right)=0$.

Note that as an obvious corollary, we get the following:

Corollary 1. Let $X$ be a smooth complex projective n-fold. Assume $X$ contains a subvariety $Y$ which is a smooth 3 -dimensional complete intersection of ample divisors, and satisfies one of the conditions in Theorem 2. Then the Hodge conjecture is true for integral degree $2 n-2$ Hodge classes on $X$.

Indeed, let $j$ be the inclusion of $Y$ into $X$. By Lefschetz restriction theorem, the map

$$
j_{*}: H^{4}(Y, \mathbb{Z})=H_{2}(Y, \mathbb{Z}) \rightarrow H_{2}(X, \mathbb{Z})=H^{2 n-2}(X, \mathbb{Z})
$$

is an isomorphism. Thus the Hodge conjecture for integral Hodge classes of degree 4 on $Y$ implies the Hodge conjecture for integral Hodge classes of degree $2 n-2$ on $X$.

Note that in higher dimensions, there are two possible generalizations of the problem studied above. Namely, one can study the Hodge conjecture for integral Hodge classes in degree 4 or $2 n-2$. Both problems are birationally invariant, in the sense that the two groups

$$
H d g^{4}(X, \mathbb{Z}) /<[Z]>, H d g^{2 n-2}(X, \mathbb{Z}) /<[Z]>
$$

where $<[Z]>$ denotes the subgroups generated by classes of algebraic cycles with integral coefficients, are birational invariants of a smooth complex projective manifold $X$ of dimension $n$ (see [14]). For both problems, it is clear that the assumption "uniruled" will not be sufficient in higher dimension to guarantee that the groups above vanish. Indeed, starting from one of Kollár's 3-dimensional example of a pair $X, \alpha \in$ $H^{4}(X, \mathbb{Z})$, where $\alpha$ is a non-algebraic integral Hodge class (Theorem 1), we can consider the product

$$
Y=X \times \mathbb{P}^{1}
$$

and both classes

$$
p r_{1}^{*} \alpha, p r_{1}^{*} \alpha \cup p r_{2}^{*}([p t])
$$

in degree 4 and $6=2 n-2$ respectively will give examples of non-algebraic integral Hodge classes.

However, one may wonder if the analogue of Theorem 2 holds for $X$ rationally connected, and for integral Hodge classes of degree 4 or $2 n-2$ on $X, n=\operatorname{dim} X$.

The proof of Theorem 2 uses the Noether-Lefschetz locus for surfaces $S$ in an adequately chosen ample linear system on $X$. This leads to simple criteria which guarantee that integral degree 2 cohomology classes on a given $S$ are generated over $\mathbb{Z}$ by those which become algebraic
on some small deformation $S_{t}$ of $S$. The Lefschetz hyperplane section Theorem allows then to conclude.

In section 1, we state this criterion, which is an algebraic criterion concerning the infinitesimal variation of Hodge structure on $H^{2}(S)$, for varieties $X$ with $H^{2}\left(X, \mathcal{O}_{X}\right)=0$. In section 2, we prove that this criterion is satisfied for uniruled or $K$-trivial varieties with trivial $H^{2}\left(X, \mathcal{O}_{X}\right)$. In the case of $K$-trivial varieties, the criterion had been also checked in [15], but the proof given here is substantially simpler. In section 3, a refinement of this criterion for uniruled threefolds with $H^{2}\left(X, \mathcal{O}_{X}\right) \neq 0$ is given and proven to hold for an adequate choice of linear system.

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## §1. An infinitesimal criterion

Let $X$ be a smooth complex projective $n$-fold. Let $j: S \hookrightarrow X$ be a surface which is a smooth complete intersection of ample divisors. Thus by Lefschetz theorem, the Gysin map:

$$
j_{*}: H^{2}(S, \mathbb{Z}) \rightarrow H^{2 n-2}(X, \mathbb{Z})
$$

is surjective.
We assume that the Hilbert scheme $\mathcal{H}$ of deformations of $S$ in $X$ is smooth near $S$. This is the case if $S$ is a smooth complete intersection of sufficiently ample divisors. The space $H^{0}\left(S, N_{S / X}\right)$ is the tangent space to $\mathcal{H}$ at $S$. Let $\rho: H^{0}\left(S, N_{S / X}\right) \rightarrow H^{1}\left(S, T_{S}\right)$ be the Kodaira-Spencer map, which is the classifying map for the first order deformations of the complex structure on $S$ induced by the universal family $\pi: \mathcal{S} \rightarrow \mathcal{H}$ of surfaces parameterized by $\mathcal{H}$.

For $u \in H^{1}\left(S, T_{S}\right)$ we have the interior product with $u$ :

$$
u\lrcorner: H^{1}\left(S, \Omega_{S}\right) \rightarrow H^{2}\left(S, \mathcal{O}_{S}\right)
$$

The criterion we shall use is the following:
Proposition 1. Assume there exists $a \lambda \in H^{1}\left(S, \Omega_{S}\right)$ such that the map

$$
\begin{array}{r}
\mu_{\lambda}: H^{0}\left(S, N_{S / X}\right) \rightarrow H^{2}\left(S, \mathcal{O}_{S}\right)  \tag{1.1}\\
\left.\mu_{\lambda}(n)=\rho(n)\right\lrcorner \lambda,
\end{array}
$$

is surjective. Then any class $\alpha \in H^{2 n-2}(X, \mathbb{Z})$ is algebraic.
Remark 2. Our assumptions imply immediately that the cohomology $H^{2 n-2}(X, \mathbb{C})$ is of type $(n-1, n-1)$. Indeed, this last fact is equivalent to the vanishing of the space $H^{n}\left(X, \Omega_{X}^{n-2}\right)$. On the other hand, interpreting the map $\mu_{\lambda}$ above in terms of infinitesimal variations of Hodge structures on the degree 2 cohomology of the surfaces $S_{t}$ parameterized by $\mathcal{H}$, one sees that $\operatorname{Im} \mu_{\lambda}$ is contained in

$$
\operatorname{Ker}\left(j_{*}: H^{2}\left(S, \mathcal{O}_{S}\right) \rightarrow H^{n}\left(X, \Omega_{X}^{n-2}\right)\right)
$$

Thus the assumptions imply that this last map $j_{*}$ is 0 , and as it is surjective by Lefschetz theorem, it follows that $H^{n}\left(X, \Omega_{X}^{n-2}\right)=0$.

Remark 3. The assumption of Proposition 1 is exactly the assumption of Green's infinitesimal criterion for the density of the NoetherLefschetz locus (see [19], 5.3.4), which allows to conclude that real degree 2 cohomology classes on $S$ can be approximated by rational algebraic cohomology classes on nearby fibers $S_{t}$. It had been already used in [16], [17] to construct interesting algebraic cycles on Calabi-Yau threefolds.

Proof of Proposition. We refer to [19], chapter 5, for more details on infinitesimal variations of Hodge structures. On a simply connected neighborhood $U$ in $\mathcal{H}$ of the point $0 \in \mathcal{H}$ parameterizing $S \subset X$, the restricted family

$$
\pi: \mathcal{S}_{U} \rightarrow U
$$

is differentiably trivial, and in particular the local system

$$
H_{\mathbb{Z}}^{2}:=R^{2} \pi_{*} \mathbb{Z} / \text { torsion }
$$

is trivial. Thus the locally free sheaf

$$
\mathcal{H}^{2}:=H_{\mathbb{Z}}^{2} \otimes \mathcal{O}_{U}
$$

is canonically trivial, and denoting by $H^{2}$ the corresponding vector bundle on $U$, we get a canonical isomorphism

$$
H^{2} \cong U \times H^{2}(S, \mathbb{C})
$$

since the fiber of $H^{2}$ at 0 is canonically isomorphic to $H^{2}(S, \mathbb{C})$. Composing with the second projection gives us a holomorphic map

$$
\tau: H^{2} \rightarrow H^{2}(S, \mathbb{C})
$$

which on each fiber $H_{t}^{2}=H^{2}\left(S_{t}, \mathbb{C}\right)$ is the natural identification

$$
H^{2}\left(S_{t}, \mathbb{C}\right) \cong H^{2}(S, \mathbb{C})
$$

Next the vector bundle $H^{2}$ contains a holomorphic subbundle $F^{1} H^{2}$, which at the point $t \in U$ has for fibre the subspace

$$
F^{1} H^{2}\left(S_{t}\right):=H^{2,0}\left(S_{t}\right) \oplus H^{1,1}\left(S_{t}\right) \subset H^{2}\left(S_{t}, \mathbb{C}\right)
$$

We shall denote by

$$
\tau_{1}: F^{1} H^{2} \rightarrow H^{2}(S, \mathbb{C})
$$

the restriction of $\tau$ to $F^{1} H^{2}$.
The key point is the following fact, for which we refer to [19], 5.3.4:
Lemma 2. For $\lambda \in H^{1}\left(S_{t}, \Omega_{S_{t}}\right)$, choose any lifting $\tilde{\lambda} \in F^{1} H_{t}^{2}$ of $\lambda$. Then the surjectivity of the map

$$
\mu_{\lambda}: H^{0}\left(S_{t}, N_{S_{t} / X}\right) \rightarrow H^{2}\left(S_{t}, \mathcal{O}_{S_{t}}\right)
$$

is equivalent to the fact that the map $\tau_{1}$ is a submersion at $\tilde{\lambda}$.
Having this, we conclude as follows: First of all, we observe that the assumption of Proposition 1 is a Zariski open condition on $\lambda \in$ $H^{1}\left(S, \Omega_{S}\right)$. Now, the space $H^{1}\left(S, \Omega_{S}\right)=H^{1,1}(S)$ has a real structure, namely

$$
H^{1,1}(S)=H^{1,1}(S)_{\mathbb{R}} \otimes \mathbb{C}
$$

where $H^{1,1}(S)_{\mathbb{R}}=H^{1,1}(S) \cap H^{2}(S, \mathbb{R})$. Thus if the assumption is satisfied for one $\lambda \in H^{1,1}(S)$, it is satisfied for one real $\lambda \in H^{1,1}(S)_{\mathbb{R}}$.

In the lemma above, choose for lifting $\tilde{\lambda}$ the class $\lambda$ itself. Thus $\tilde{\lambda}$ is real, and so is $\tau_{1}(\tilde{\lambda})$. As the assumption on $\lambda$ and the lemma imply that $\tau_{1}$ is a submersion at $\tilde{\lambda}$, so is the restriction

$$
\tau_{1, \mathbb{R}}: H_{\mathbb{R}}^{1,1} \rightarrow H^{2}(S, \mathbb{R})
$$

of $\tau_{1}$ to $\tau_{1}^{-1}\left(H^{2}(S, \mathbb{R})\right)$. Here we identified $\tau_{1}^{-1}\left(H^{2}(S, \mathbb{R})\right)$ to

$$
\cup_{t \in U} F^{1} H^{2}\left(S_{t}\right) \cap H^{2}\left(S_{t}, \mathbb{R}\right)=\cup_{t \in U} H^{1,1}\left(S_{t}\right)_{\mathbb{R}}=: H_{\mathbb{R}}^{1,1}
$$

As $\tau_{1, \mathbb{R}}$ is a submersion at $\tilde{\lambda}$, and $H^{1,1}(S)_{\mathbb{R}}$ is a smooth real manifold, because it is a real vector bundle on $U$ and $U$ is smooth, $\operatorname{Im} \tau_{1, \mathbb{R}}$ contains a non-empty open set of $H^{2}(S, \mathbb{R})$. On the other hand $\operatorname{Im} \tau_{1, \mathbb{R}}$ is a cone. We use now the following elementary lemma:

Lemma 3. Let $V_{\mathbb{Z}}$ be a lattice, and let $C$ be a non-empty open cone in $V_{\mathbb{R}}:=V_{\mathbb{Z}} \otimes \mathbb{R}$. Then $V_{\mathbb{Z}}$ is generated over $\mathbb{Z}$ by the points in $V_{\mathbb{Z}} \cap C$.

Proof. $V_{\mathbb{Z}} \cap C$ is non-empty because $V_{\mathbb{Q}}$ is dense, and $C$ is a nonempty open cone in $V_{\mathbb{R}}$. Let $u \in V_{\mathbb{Z}}$, and let $u^{\prime} \in V_{\mathbb{Z}} \cap C$. For $q$ a large integer, we have $\frac{1}{q} u+u^{\prime} \in C$, because $C$ is open. Then $u+q u^{\prime}:=v^{\prime} \in$ $V_{\mathbb{Z}} \cap C$. Thus $u=v^{\prime}-q u^{\prime}$ is in the sublattice generated over $\mathbb{Z}$ by the points in $V_{\mathbb{Z}} \cap C$.

We apply Lemma 3 to $V_{\mathbb{Z}}=H^{2}(S, \mathbb{Z}) /$ torsion and to $C$ an open cone contained in $\operatorname{Im} \tau_{1, \mathbb{R} \text {. Thus we conclude that } H^{2}(S, \mathbb{Z}) / \text { torsion is }}$ generated over $\mathbb{Z}$ by classes $\alpha \in \operatorname{Im} \tau_{1, \mathbb{R}}$. But by definition of $\tau_{1}$, if an integral cohomology class $\alpha \in H^{2}(S, \mathbb{Z}) /$ torsion is equal to $\tau_{1, \mathbb{R}}\left(\lambda_{t}\right)$, for some

$$
\lambda_{t} \in H^{1,1}\left(S_{t}\right)_{\mathbb{R}} \subset H^{2}\left(S_{t}, \mathbb{R}\right)
$$

the corresponding class

$$
\alpha_{t} \in H^{2}\left(S_{t}, \mathbb{Z}\right) / \text { torsion } \subset H^{2}\left(S_{t}, \mathbb{R}\right)
$$

is equal to $\lambda_{t}$ in $H^{2}\left(S_{t}, \mathbb{R}\right)$. Thus the class

$$
\alpha_{t}=\lambda_{t} \in H^{1,1}\left(S_{t}\right) \cap H^{2}\left(S_{t}, \mathbb{Z}\right) / \text { torsion }
$$

is algebraic on $S_{t}$ by Lefschetz theorem on $(1,1)$-classes.
The conclusion is that, under the assumptions of Proposition 1, the lattice $H^{2}(S, \mathbb{Z}) /$ torsion is generated over $\mathbb{Z}$ by integral classes which become algebraic (i.e. are the class of a divisor) on some nearby fiber $S_{t}$. As the torsion of $H^{2}(S, \mathbb{Z})$ is algebraic, the same conclusion holds for $H^{2}(S, \mathbb{Z})$.

Finally, as the map $j_{*}: H^{2}(S, \mathbb{Z}) \rightarrow H^{2 n-2}(X, \mathbb{Z})$ is surjective, we conclude that $H^{2 n-2}(X, \mathbb{Z})$ is generated over $\mathbb{Z}$ by classes of 1 -cycles in $X$.

## §2. Proof of Theorem 2 when $H^{2}\left(X, \mathcal{O}_{X}\right)=0$

In this section, we assume that $H^{2}\left(X, \mathcal{O}_{X}\right)=0$ and $X$ either has trivial canonical bundle or is uniruled.

In case where $X$ is uniruled, we have the following result:
Lemma 4. Let $X$ be a uniruled threefold. Then a smooth birational model $X^{\prime}$ of $X$ carries an ample line bundle $H$ such that

$$
H^{2} K_{X^{\prime}}<0
$$

Proof. As $X$ is uniruled, $X$ is birationally equivalent to a $\mathbb{Q}$ Gorenstein threefold $Y$ which is either a singular Fano threefold, or a Del Pezzo fibration over a smooth curve, or a conic bundle over a $\mathbb{Q}$-Gorenstein surface. Let us first prove the existence of an ample line bundle $H_{Y}$ on $Y$ such that $K_{Y} H_{Y}^{2}<0$ :
a) If $Y$ is Fano, $-K_{Y}$ is ample, so we can take for $H_{Y}$ an integral multiple of $-K_{Y}$.
b) Otherwise there is a morphism

$$
\pi: Y \rightarrow B
$$

where $B$ is $\mathbb{Q}$-Gorenstein of dimension 1 or 2 , and the relative canonical bundle $K_{\pi}$ has the property that $-K_{\pi}$ is a relatively ample $\mathbb{Q}$-divisor. Let $H_{B}$ be an ample line bundle on $B$, and choose for $H_{Y}$ the $\mathbb{Q}$-divisor

$$
H_{Y}=\pi^{*} H_{B}-\epsilon K_{\pi}
$$

where $\epsilon$ is a small rational number. As $-K_{\pi}$ is relatively ample, $H_{Y}$ is ample for small enough $\epsilon$. We compute now:

$$
\begin{gathered}
H_{Y}^{2} K_{Y}=\left(\pi^{*} H_{B}-\epsilon K_{\pi}\right)^{2}\left(\pi^{*} K_{B}+K_{\pi}\right) \\
=\pi^{*} H_{B}^{2} K_{\pi}-2 \epsilon K_{\pi} \pi^{*} H_{B}\left(\pi^{*} K_{B}+K_{\pi}\right)+O\left(\epsilon^{2}\right) .
\end{gathered}
$$

If $\operatorname{dim} B=2$, the term $\pi^{*} H_{B}^{2} K_{\pi}$ is negative, so that for small $\epsilon$,

$$
H_{Y}^{2} K_{Y}<0
$$

If $\operatorname{dim} B=1$, the first term vanishes but the second term is equal to $-2 \epsilon K_{\pi}^{2} \pi^{*} H_{B}$ and this is negative because $-K_{\pi}$ is relatively ample.

Let now $Y, H_{Y}$ be as above, and let $\tau: X^{\prime} \rightarrow Y$ be a desingularization of $Y$. Thus $X^{\prime}$ is a smooth birational model of $X$. Then there is a relatively ample divisor $E$ on $X^{\prime}$ which is supported on the exceptional divisor of $\tau$. Consider the $\mathbb{Q}$-divisor

$$
H=\tau^{*} H_{Y}+\epsilon E
$$

for $\epsilon$ a sufficiently small rational number. Then we have $K_{X^{\prime}}=\tau^{*} K_{Y}+F$ where $F$ is supported on the exceptional divisor of $\tau$. This gives

$$
H^{2} K_{X^{\prime}}=\left(\tau^{*} H_{Y}+\epsilon E\right)^{2}\left(\tau^{*} K_{Y}+F\right)
$$

As $\tau^{*} H_{Y}^{2} F=0$, the dominating term is equal to

$$
\tau^{*} H_{Y}^{2} \tau^{*} K_{Y}=H_{Y}^{2} K_{Y}<0
$$

Thus for small $\epsilon$ we have $H^{2} K_{X^{\prime}}<0$.

From now on, we will, in the uniruled case, consider $X^{\prime}$ instead of $X$, which can be done since the statement of Theorem 2 is invariant under birational equivalence, and we will assume that $H$ satisfies the conclusion of Lemma 4.

Our aim in this section is to prove the following Proposition, which by Proposition 1 implies Theorem 2 for uniruled and Calabi-Yau threefolds $X$ with $H^{2}\left(X, \mathcal{O}_{X}\right)=0$.

Proposition 2. Let $X$ be a smooth projective uniruled or CalabiYau threefold such that $H^{2}\left(X, \mathcal{O}_{X}\right)=0$. Let $H$ be an ample line bundle on $X$. In the uniruled case, assume that $H$ satisfies $H^{2} K_{X}<0$. Then for $n$ large enough, and for $S$ a generic surface in $|n H|$, there is a $\lambda \in H^{1}\left(S, \Omega_{S}\right)$ which satisfies the property that

$$
\mu_{\lambda}: H^{0}\left(S, \mathcal{O}_{S}(n H)\right) \rightarrow H^{2}\left(S, \mathcal{O}_{S}\right)
$$

is surjective.
To see that this is a reasonable statement, note that in the $K$ trivial case, the spaces $H^{0}\left(S, \mathcal{O}_{S}(n H)\right)$ and $H^{2}\left(S, \mathcal{O}_{S}\right)$ have the same dimension, since, if $S \in|n H|$, we have by adjunction

$$
H^{0}\left(S, K_{S}\right)=H^{0}\left(S, \mathcal{O}_{S}(S)\right)=H^{0}\left(S, N_{S / X}\right)
$$

with $H^{0}\left(S, K_{S}\right)=H^{2}\left(S, \mathcal{O}_{S}\right)^{*}$. Thus the two spaces involved in Proposition 2 have the same dimension. In the uniruled case, we have:

Lemma 5. Assume $X, H$ satisfies $H^{2} K_{X}<0$, then for $S \in|n H|$, we have

$$
h^{0}\left(\mathcal{O}_{S}(S)\right)=h^{0}\left(K_{S}\right)+\phi(n)
$$

where $\phi(n)=\alpha n^{2}+o\left(n^{2}\right), \alpha>0$.
Proof. We have $K_{S}=K_{X}(S)_{\mid S}$. Thus

$$
\begin{gathered}
\chi\left(\mathcal{O}_{S}(S)\right)=\chi\left(K_{S}\left(-K_{X}\right)\right) \\
=\chi\left(K_{X \mid S}\right)=\chi\left(\mathcal{O}_{S}\right)+\frac{1}{2}\left(K_{X \mid S}^{2}-K_{X \mid S}\left(K_{X \mid S}+n H_{\mid S}\right)\right) \\
=\chi\left(K_{S}\right)+\frac{1}{2}\left(n H K_{X}^{2}-n H K_{X}\left(n H+K_{X}\right)\right)
\end{gathered}
$$

It follows that

$$
\chi\left(\mathcal{O}_{S}(S)\right)-\chi\left(K_{S}\right)=-\frac{1}{2} n^{2} H^{2} K_{X}+\text { affine linear term in } n
$$

On the other hand, for large $n$, the ranks

$$
\begin{gathered}
h^{1}\left(\mathcal{O}_{S}(S)\right)=h^{2}\left(\mathcal{O}_{X}\right), h^{2}\left(\mathcal{O}_{S}(S)\right)=h^{3}\left(\mathcal{O}_{X}\right) \\
h^{1}\left(K_{S}\right)=h^{1}\left(\mathcal{O}_{S}\right)=h^{1}\left(\mathcal{O}_{X}\right), h^{2}\left(K_{S}\right)=\mathbb{C}
\end{gathered}
$$

do not depend on $n$. It follows that we also have

$$
h^{0}\left(\mathcal{O}_{S}(S)\right)-h^{0}\left(K_{S}\right)=-\frac{1}{2} n^{2} H^{2} K_{X}+\text { affine linear term in } n
$$

which proves the result with $\alpha=-\frac{1}{2} H^{2} K_{X}>0$.

By this Lemma, we conclude that in the $K$-trivial case and in the uniruled case, we can assume that we have for $n$ large enough, and $S \in|n H|$,

$$
h^{0}\left(N_{S / X}\right)=h^{0}\left(S, \mathcal{O}_{S}(S)\right) \geq h^{0}\left(K_{S}\right)=h^{2}\left(\mathcal{O}_{S}\right)
$$

This makes possible the surjectivity of the map

$$
\mu_{\lambda}: H^{0}\left(S, N_{S / X}\right) \rightarrow H^{2}\left(S, \mathcal{O}_{S}\right)
$$

of (1.1), and also says that $\mu_{\lambda}$ is surjective if and only if it has maximal rank.

Another way to see this is to introduce

$$
V:=H^{0}\left(S, K_{S}\right), V^{\prime}:=H^{0}\left(S, N_{S / X}\right)
$$

The bilinear map

$$
\begin{array}{r}
\mu: V \times V^{\prime} \rightarrow H^{1}\left(S, \Omega_{S}\right)  \tag{2.2}\\
\left.\mu\left(v, v^{\prime}\right)=\rho\left(v^{\prime}\right)\right\lrcorner v
\end{array}
$$

and Serre's duality $H^{1}\left(S, \Omega_{S}\right) \cong H^{1}\left(S, \Omega_{S}\right)^{*}$ give a dual map

$$
q=\mu^{*}: H^{1}\left(S, \Omega_{S}\right) \rightarrow\left(V \otimes V^{\prime}\right)^{*}=H^{0}\left(\mathbb{P}(V) \times \mathbb{P}\left(V^{\prime}\right), \mathcal{O}(1,1)\right)
$$

given by

$$
q(\lambda)\left(v \otimes v^{\prime}\right)=<\lambda, \mu\left(v \times v^{\prime}\right)>
$$

As we have

$$
\left.\left.<\lambda, \rho\left(v^{\prime}\right)\right\lrcorner v>=-<\rho\left(v^{\prime}\right)\right\lrcorner \lambda, v>
$$

where the $<,>$ stand for Serre's duality on $H^{1}\left(S, \Omega_{S}\right)$ on the left and between $H^{0}\left(S, K_{S}\right)$ and $H^{2}\left(S, \mathcal{O}_{S}\right)$ on the right, we see that $q(\lambda)$ identifies to $\mu_{\lambda} \in \operatorname{Hom}\left(V, V^{*}\right)$.

Thus the condition that $\mu_{\lambda}$ has maximal rank for generic $\lambda$ is equivalent to the condition that the hypersurface of $\mathbb{P}(V) \times \mathbb{P}\left(V^{\prime}\right)$ defined by $q(\lambda)$ is non singular.

We shall use the following criterion:
Lemma 6. Given $\mu$ as in (2.2), the generic hypersurface defined by $q(\lambda)$ is non singular if the following set

$$
\begin{equation*}
Z=\left\{\left(v, v^{\prime}\right) \in \mathbb{P}(V) \times \mathbb{P}\left(V^{\prime}\right), \mu\left(v \times v^{\prime}\right)=0 \in H^{1}\left(S, \Omega_{S}\right)\right\} \tag{2.3}
\end{equation*}
$$

satisfies

$$
\operatorname{dim} Z<\operatorname{dim} \mathbb{P}\left(V^{\prime}\right)
$$

Proof. Assume to the contrary that the generic $q(\lambda)$ is singular. Let

$$
\begin{gathered}
Z^{\prime} \subset \mathbb{P}\left(H^{1}\left(S, \Omega_{S}\right)\right) \times \mathbb{P}(V) \\
Z^{\prime}=\left\{(\lambda, v), q(\lambda) \text { is singular at }\left(v, v^{\prime}\right) \text { for some } v^{\prime} \in \mathbb{P}\left(V^{\prime}\right)\right\}
\end{gathered}
$$

By assumption $Z^{\prime}$ dominates $\mathbb{P}\left(H^{1}\left(S, \Omega_{S}\right)\right)$. Clearly there is only one irreducible component $Z_{g}^{\prime}$ of $Z^{\prime}$ which dominates $\mathbb{P}\left(H^{1}\left(S, \Omega_{S}\right)\right)$. Let $Z_{1}^{\prime}$ be the second projection of $Z_{g}^{\prime}$ in $\mathbb{P}(V)$.

As $Z_{g}^{\prime}$ dominates $\mathbb{P}\left(H^{1}\left(S, \Omega_{S}\right)\right)$ we have

$$
\operatorname{dim} Z_{g}^{\prime} \geq r k H^{1}\left(S, \Omega_{S}\right)-1
$$

On the other hand, the fiber of $Z_{g}^{\prime}$ over the generic point $v_{g}$ of $Z_{1}^{\prime}$ is equal to

$$
\mu\left(v_{g} \times V^{\prime}\right)^{\perp}
$$

Thus we have

$$
\operatorname{dim} Z_{g}^{\prime}=\operatorname{dim} Z_{1}^{\prime}+r k H^{1}\left(S, \Omega_{S}\right)-1-r k \mu_{v_{g}}
$$

where $\mu_{v_{g}}: V^{\prime} \rightarrow H^{1}\left(S, \Omega_{S}\right)$ is the map $v^{\prime} \mapsto \mu\left(v_{g} \times v^{\prime}\right)$.
The condition $\operatorname{dim} Z_{g}^{\prime} \geq r k H^{1}\left(S, \Omega_{S}\right)-1$ is thus equivalent to

$$
\begin{equation*}
\operatorname{dim} Z_{1}^{\prime} \geq r k \mu_{v_{g}} \tag{2.4}
\end{equation*}
$$

But on the other hand, the unique irreducible component $Z_{0}$ of $Z$ which dominates $Z_{1}^{\prime}$ has dimension equal to $\operatorname{dim} Z_{1}^{\prime}+\operatorname{dim} \mathbb{P}\left(V^{\prime}\right)-r k \mu_{v_{g}}$ and inequality (2.4) implies that this is $\geq \operatorname{dim} \mathbb{P}\left(V^{\prime}\right)$.

Our first task will be thus to study the set $Z$ introduced in (2.3). To this effect, we degenerate the surface $S \in|n H|$ to a surface with many nodes. The reason for doing that is the following fact (cf [15]):

Lemma 7. Let $\mathcal{S} \rightarrow \Delta$ be a Lefschetz degeneration of surfaces $S_{t}$ in $|n H|$, where the central fiber has ordinary double points $x_{1}, \ldots, x_{N}$ as singularities. Then the limiting space

$$
\lim _{t \rightarrow 0} \operatorname{Im}\left(q_{t}: H^{1}\left(S_{t}, \Omega_{S_{t}}\right) \rightarrow\left(H^{0}\left(S_{t}, K_{S_{t}}\right) \otimes H^{0}\left(S_{t}, \mathcal{O}_{S_{t}}(n H)\right)\right)^{*}\right)
$$

which is a subspace of $\left(H^{0}\left(S_{0}, K_{S_{0}}\right) \otimes H^{0}\left(S_{0}, \mathcal{O}_{S_{0}}(n H)\right)\right)^{*}$, contains for each $i=1, \ldots, N$ the multiplication-evaluation map which is the composite:

$$
\begin{gathered}
H^{0}\left(S_{0}, K_{S_{0}}\right) \otimes H^{0}\left(S_{0}, \mathcal{O}_{S_{0}}(n H)\right) \xrightarrow{m u l t} H^{0}\left(S_{0}, K_{S_{0}}(n H)\right) \\
\xrightarrow{e v_{x_{i}}} H^{0}\left(K_{S_{0}}(n H)_{\mid x_{i}}\right) .
\end{gathered}
$$

As we want to use this lemma to bound the dimension of the space $Z$ of (2.3) for a generic surface, it is natural to degenerate the generic surface to a surface with many nodes. To get surfaces with many nodes, we use discriminant surfaces as in [2]. We assume here that $H$ is very ample on $X$, and we consider a generic symmetric $n$ by $n$ matrix $A$ whose entries $A_{i j}$ are in $H^{0}\left(X, \mathcal{O}_{X}(H)\right)$. Let $\sigma_{A}:=\operatorname{discr} A \in H^{0}\left(X, \mathcal{O}_{X}(n H)\right)$ and $S_{A}$ be the surface defined by $\sigma_{A}$.

Theorem 3. (Barth [2]) The surface $S_{A}$ has $N$ ordinary double points as singularities, with

$$
N=\binom{n+1}{3} H^{3}
$$

Note that for large $n$, this grows like $\frac{n^{3}}{6} H^{3}$ while both dimensions $h^{0}\left(K_{S}\right), h^{0}\left(\mathcal{O}_{S}(n H)\right)$ of the spaces $V, V^{\prime}$ grow like $h^{0}\left(\mathcal{O}_{X}(n H)\right)$, that is like $\frac{n^{3}}{6} H^{3}$ by Riemann-Roch.

Next we have the following lemma, which might well be known already, but for which we could not find a reference:

Lemma 8. Let $X$ be a smooth projective threefold, and $H$ a very ample line bundle on $X$ which satisfies the property that

$$
H^{i}\left(X, \mathcal{O}_{X}(l H)\right)=0, \text { for } i>0, l>0
$$

Let $S_{A} \in|n H|$ be a generic discriminant surface as above, and let $W \subset$ $X$ be its singular set. Then the cohomology group $H^{1}\left(X, \mathcal{I}_{W}((n+2) H)\right)$ vanishes.

Proof. Let $G=\operatorname{Grass}(2, n)$ be the Grassmannian of 2-dimensional vector subspaces of $K:=\mathbb{C}^{n}$. The matrix $A$ as above can be seen as a family of quadrics $A_{x}$ on $\mathbb{P}(K)$ parameterized by $x \in X$, the surface $S_{A}$ corresponds to singular quadrics and the singular set $W$ parameterizes quadrics of rank $n-2$. Thus $W$ is via the second projection in one-to-one correspondence with the following algebraic set:

$$
\widetilde{W}:=\left\{(l, x) \in G \times X, A_{x} \text { is singular along } l\right\}
$$

Let $\mathcal{E}$ be the tautological rank 2 quotient bundle on $G$, whose fiber at $l$ is $H^{0}\left(\mathcal{O}_{\Delta_{l}}(1)\right) . \mathcal{E}$ is a quotient of $K^{*} \otimes \mathcal{O}_{G}$, and there is the natural map

$$
e: S^{2} K^{*} \otimes \mathcal{O}_{G} \rightarrow K^{*} \otimes \mathcal{E}
$$

Let

$$
\begin{equation*}
\mathcal{F}:=\operatorname{Im} e . \tag{2.5}
\end{equation*}
$$

Clearly, a quadric $A \in S^{2} K^{*}$ on $\mathbb{P}(K)$ is singular along $\Delta_{l}$ if and only if it vanishes under the map $e$ at the point $l$. Thus the set $\widetilde{W}$ is the zero locus of a section of the vector bundle

$$
\mathcal{F} \boxtimes \mathcal{O}_{X}(H)
$$

which is of rank $2 n-1$ on $G \times X$. Note that the cokernel of $e$ identifies to $\bigwedge^{2} \mathcal{E}=: \mathcal{L}$ where $\mathcal{L}$ is the Plücker line bundle on $G$. Thus we have an exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{F} \rightarrow K^{*} \otimes \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0 \tag{2.6}
\end{equation*}
$$

on $G$.
As $\widetilde{W}$ is the zero set of a transverse section of a rank $2 n-1$ vector bundle on $G \times X$, its ideal sheaf admits the Koszul resolution:

$$
0 \rightarrow \bigwedge^{2 n-1} \mathcal{F}^{*} \boxtimes \mathcal{O}_{X}((-2 n+1) H) \rightarrow \ldots \rightarrow \mathcal{F}^{*} \boxtimes \mathcal{O}_{X}(-H) \rightarrow \mathcal{I}_{\widetilde{W}} \rightarrow 0
$$

Thus the space $H^{1}\left(X, \mathcal{I}_{W}((n+2) H)\right)=H^{1}\left(G \times X, \mathcal{I}_{\widetilde{W}} \otimes p r_{2}^{*}((n+2) H)\right)$ is the abutment of a spectral sequence whose $E_{1}$-term is equal to

$$
H^{i}\left(G \times X, \bigwedge^{i} \mathcal{F}^{*} \boxtimes \mathcal{O}_{X}((n+2-i) H)\right), i \geq 1
$$

By Künneth decomposition and the vanishing assumptions, these spaces split as:

$$
\begin{gathered}
H^{i}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right) \otimes H^{0}(X,(n+2-i) H), n+2-i>0 \\
H^{i}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right) \otimes H^{0}\left(X, \mathcal{O}_{X}\right) \oplus H^{i-1}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right) \otimes H^{1}\left(X, \mathcal{O}_{X}\right) \\
\oplus H^{i-2}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right) \otimes H^{2}\left(X, \mathcal{O}_{X}\right) \oplus H^{i-3}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right) \otimes H^{3}\left(X, \mathcal{O}_{X}\right), i=n+2, \\
H^{i-3}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right) \otimes H^{3}(X,(n+2-i) H), n+2-i<0
\end{gathered}
$$

The proof of Lemma 8 is thus concluded by the following lemma, which implies that the $E_{1}$-terms of the spectral sequence above all vanish.

Lemma 9. On the Grassmannian $G=\operatorname{Grass}(2, n)$, the bundle $\mathcal{F}$ being defined as in (2.5), we have the vanishings:
(1) $H^{i}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right)=0, n+2-i \geq 0, i \geq 1$
(2) $H^{i-1}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right)=0, n+2-i=0$.
(3) $H^{i-2}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right)=0, n+2-i=0$.
(4) $H^{i-3}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right)=0, n+2-i \leq 0, i \geq 1$.

The proof of this last lemma is postponed to an Appendix.
As an immediate corollary, we get the following:
Corollary 2. Under the same assumptions as in Lemma 8, the numbers

$$
r k H^{1}\left(X, K_{X} \otimes \mathcal{I}_{W}(n H)\right), r k H^{1}\left(X, \mathcal{I}_{W}(n H)\right)
$$

are bounded by $C n^{2}$ for some constant $C$.
Combining Corollary 2 with Riemann-Roch and Barth's Theorem 3 , we get the following corollary:

Corollary 3. The spaces $H^{0}\left(X, K_{X}(n H) \otimes \mathcal{I}_{W}\right)$ and $H^{0}\left(X, \mathcal{I}_{W}(n H)\right)$ have dimension bounded by $\mathrm{cn}^{2}$ for some constant $c$.

We shall use the following consequence of the uniform position principle of Harris:

Lemma 10. Let $A$ be generic and let $W^{\prime} \subset W$ be a subset of $W=$ Sing $S_{A}$. Then if

$$
H^{0}\left(X, K_{X}(n H) \otimes \mathcal{I}_{W^{\prime}}\right) \neq H^{0}\left(X, K_{X}(n H) \otimes \mathcal{I}_{W}\right)
$$

$W^{\prime}$ imposes card $W^{\prime}$ independent conditions to $H^{0}\left(X, K_{X}(n H)\right)$. Similarly, if

$$
H^{0}\left(X, \mathcal{O}_{X}(n H) \otimes \mathcal{I}_{W^{\prime}}\right) \neq H^{0}\left(X, \mathcal{O}_{X}(n H) \otimes \mathcal{I}_{W}\right)
$$

$W^{\prime}$ imposes card $W^{\prime}$ independent conditions to $H^{0}\left(X, \mathcal{O}_{X}(n H)\right)$.
Proof. Indeed, we represented in the previous proof the set $W$ as the projection in $X$ of a 0-dimensional subscheme $\widetilde{W}$ of $G \times X$, defined as the zero set of a generic transverse section of the vector bundle $\mathcal{F} \boxtimes$ $\mathcal{O}_{X}(H)$ on $G \times X$. One verifies that the uniform position principle [8] applies to $\widetilde{W}$, and this allows to conclude that all subsets of $W$ of given cardinality impose the same number of independent conditions to $H^{0}\left(X, K_{X}(n H)\right)$ or $H^{0}\left(X, \mathcal{O}_{X}(n H)\right)$. This number is then obviously equal to

$$
\operatorname{Min}\left(\operatorname{card} W^{\prime}, a\right)
$$

where $a=r k\left(\right.$ rest : $\left.H^{0}\left(X, K_{X}(n H)\right) \rightarrow H^{0}\left(W, K_{X}(n H)_{\mid W}\right)\right)$ ), resp.

$$
a=r k\left(r e s t: H^{0}\left(X, \mathcal{O}_{X}(n H)\right) \rightarrow H^{0}\left(W, \mathcal{O}_{W}(n H)\right)\right)
$$

in the second case.

From now on, we will treat separately the uniruled and the $K$-trivial cases.

The uniruled case. We may assume $(X, H)$ satisfies the inequality $H^{2} K_{X}<0$ of Lemma 4. We want to study the set $Z$ of (2.3) for a generic surface $S \in|n H|$, and more precisely the irreducible components $Z^{\prime}$ of $Z$ which are of dimension $\geq \operatorname{dim} \mathbb{P}\left(V^{\prime}\right)$.

Degenerating $S$ to $S_{A}$ and applying Lemma 7, we find that the specialization $Z_{s}^{\prime}$ of $Z^{\prime}$ is contained in

$$
Z_{0}:=\left\{\left(v, v^{\prime}\right) \in \mathbb{P}\left(V_{A}\right) \times \mathbb{P}\left(V_{A}^{\prime}\right), v v_{\mid W}^{\prime}=0\right\}
$$

where

$$
V_{A}=H^{0}\left(S_{A}, K_{S_{A}}\right), V_{A}^{\prime}=H^{0}\left(S_{A}, \mathcal{O}_{S_{A}}(n H)\right)
$$

Lemma 11. $Z_{s}^{\prime}$ is contained in the union

$$
\begin{array}{r}
\mathbb{P} H^{0}\left(S_{A}, K_{S_{A}} \otimes \mathcal{I}_{W}\right) \times \mathbb{P}\left(V_{A}^{\prime}\right)  \tag{2.7}\\
\cup \mathbb{P}\left(V_{A}\right) \times \mathbb{P} H^{0}\left(S_{A}, \mathcal{O}_{S_{A}}(n H) \otimes \mathcal{I}_{W}\right)
\end{array}
$$

Proof. We observe that $Z_{0}$ is a union of irreducible components indexed by subsets $W^{\prime} \subset W$, with complementary set $W^{\prime \prime}:=W \backslash W^{\prime}$ :

$$
\begin{gathered}
Z_{0}=\cup_{W^{\prime} \subset W} Z_{W^{\prime}} \\
Z_{W^{\prime}}:=\mathbb{P} H^{0}\left(S_{A}, K_{S_{A}} \otimes \mathcal{I}_{W^{\prime}}\right) \times \mathbb{P} H^{0}\left(S_{A}, \mathcal{O}_{S_{A}}(n H) \otimes \mathcal{I}_{W^{\prime \prime}}\right)
\end{gathered}
$$

We use now Lemma 10: it says that if both conditions

$$
\begin{aligned}
& H^{0}\left(X, K_{X}(n H) \otimes \mathcal{I}_{W^{\prime}}\right) \neq H^{0}\left(X, K_{X}(n H) \otimes \mathcal{I}_{W}\right) \\
& H^{0}\left(X, \mathcal{O}_{X}(n H) \otimes \mathcal{I}_{W^{\prime \prime}}\right) \neq H^{0}\left(X, \mathcal{O}_{X}(n H) \otimes \mathcal{I}_{W}\right)
\end{aligned}
$$

hold, then $W^{\prime}$ imposes card $W^{\prime}$ independent conditions to the linear system $H^{0}\left(X, K_{X}(n H)\right)$ and $W^{\prime \prime}$ imposes card $W^{\prime \prime}$ independent conditions to $H^{0}\left(X, \mathcal{O}_{X}(n H)\right)$. Thus the codimension of $Z_{W^{\prime}}$ in $\mathbb{P}\left(V_{A}\right) \times \mathbb{P}\left(V_{A}^{\prime}\right)$ is equal to $\operatorname{card} W^{\prime}+\operatorname{card} W^{\prime \prime}=\operatorname{card} W$. But $\operatorname{card} W$ is equal to $\frac{n\left(n^{2}-1\right)}{6} H^{3}$ by Theorem 3 , while the dimension of $V_{A}=H^{0}\left(S_{A}, K_{S_{A}}\right) \cong$ $H^{0}\left(X, K_{X}(n H)\right)$ is equal to

$$
\frac{1}{6} n^{3} H^{3}+\frac{1}{4} n^{2} K_{X} H^{2}+\text { affine linear term in } n
$$

by Riemann-Roch.

As $K_{X} H^{2}<0$, we conclude that for $n$ large enough, if $W^{\prime}$ is as above, we have

$$
\operatorname{codim} Z^{\prime}<\operatorname{dim} \mathbb{P}\left(V_{A}\right),
$$

and thus

$$
\operatorname{dim} Z_{W^{\prime}}<\operatorname{dim} \mathbb{P}\left(V_{A}^{\prime}\right)
$$

Thus, for large $n$, the only components of $Z_{0}$ which may have dimension $\geq \operatorname{dim} \mathbb{P}\left(V_{A}^{\prime}\right)$ are the two components $\mathbb{P} H^{0}\left(S_{A}, K_{S_{A}} \otimes \mathcal{I}_{W}\right) \times \mathbb{P}\left(V_{A}^{\prime}\right)$ and $\mathbb{P}\left(V_{A}\right) \times \mathbb{P} H^{0}\left(S_{A}, \mathcal{O}_{S_{A}}(n H) \otimes \mathcal{I}_{W}\right)$.

Corollary 4. Assume $S$ is generic and $Z^{\prime} \subset \mathbb{P}(V) \times \mathbb{P}\left(V^{\prime}\right)$ is an irreducible component of $Z$ which has dimension $\geq \operatorname{dim} \mathbb{P}\left(V^{\prime}\right)$. Then either
i) $\operatorname{dimpr} r_{1}\left(Z^{\prime}\right) \leq c n^{2}$ or
ii) $\operatorname{dimpr} r_{2}\left(Z^{\prime}\right) \leq c n^{2}$,
where $c$ is the constant of Corollary 3.
Proof. By Lemma 11, the specialization $Z_{s}^{\prime}$ of $Z^{\prime}$ is contained in the union (2.7). As we have by Corollary 3

$$
\operatorname{dim} \mathbb{P} H^{0}\left(X, K_{X}(n H) \otimes \mathcal{I}_{W}\right)<c n^{2}, \operatorname{dim} \mathbb{P} H^{0}\left(X, \mathcal{I}_{W}(n H)\right)<c n^{2}
$$

this implies that the cycle $Z_{s}^{\prime}$ satisfies:

$$
h_{1}^{n c^{2}} h_{2}^{n c^{2}}\left[Z_{s}^{\prime}\right]=0 \text { in } H^{*}\left(\mathbb{P}\left(V_{A}\right) \times \mathbb{P}\left(V_{A}^{\prime}\right), \mathbb{Z}\right)
$$

where

$$
h_{1}:=p r_{1}^{*} c_{1}\left(\mathcal{O}_{\mathbb{P}\left(V_{A}\right)}(1)\right), h_{2}:=p r_{2}^{*} c_{1}\left(\mathcal{O}_{\mathbb{P}\left(V_{A}^{\prime}\right)}(1)\right),
$$

and $\left[Z_{s}^{\prime}\right]$ is the cohomology class of the cycle $Z_{s}^{\prime}$.
It follows that we also have

$$
\begin{equation*}
h_{1}^{n c^{2}} h_{2}^{n c^{2}}\left[Z^{\prime}\right]=0 \text { in } H^{*}\left(\mathbb{P}(V) \times \mathbb{P}\left(V^{\prime}\right), \mathbb{Z}\right) \tag{2.8}
\end{equation*}
$$

We claim that this implies that i) or ii) holds. Indeed, as $Z^{\prime}$ is irreducible, there are well defined generic ranks $k_{1}, k_{2}$ of the projection $p r_{1 \mid Z^{\prime}}, p r_{2 \mid Z^{\prime}}$ respectively, which are also the generic ranks of the pull-back of the $(1,1)$-forms $p r_{1}^{*} \omega_{1}, p r_{2}^{*} \omega_{2}$ to $Z^{\prime}$, where $\omega_{i}$ are the Fubini-Study $(1,1)$ forms on $\mathbb{P}(V), \mathbb{P}\left(V^{\prime}\right)$. As the form

$$
p r_{1}^{*} \omega_{1}^{n c^{2}} \wedge p r_{1}^{*} \omega_{2}^{n c^{2}}
$$

is semi-positive on $Z^{\prime}$, the condition (2.8) implies that everywhere on $Z$, we have

$$
p r_{1}^{*} \omega_{1}^{n c^{2}} \wedge p r_{2}^{*} \omega_{2}^{n c^{2}}=0
$$

As $\operatorname{dim} Z^{\prime} \geq 2 c n^{2}$ and $\left(p r_{1}, p r_{2}\right)$ is an immersion on the smooth locus of $Z^{\prime}$, this implies easily that either $k_{1}=r k p r_{1}$ or $k_{2}=r k p r_{2}$ has to be $<c n^{2}$, that is i) or ii).

Corollary 5. With the same assumptions as in the previous corollary, if $\left(v, v^{\prime}\right) \in Z^{\prime}$, one has either
i) $r k \mu_{v}: V^{\prime} \rightarrow H^{1}\left(S, \Omega_{S}\right)<c n^{2}$, or
ii) $r k \mu_{v^{\prime}}: V \rightarrow H^{1}\left(S, \Omega_{S}\right)<c n^{2}$,
where i) and ii) refer to the two cases of Corollary 4 and

$$
\mu_{v}(\cdot)=\mu(v \otimes \cdot), \mu_{v^{\prime}}(\cdot)=\mu\left(\cdot \otimes v^{\prime}\right)
$$

Proof. Indeed, assume case i) of Corollary 4 holds. As $\operatorname{dim} Z^{\prime} \geq$ $\operatorname{dim} \mathbb{P}\left(V^{\prime}\right)$, the generic fibre of $p r_{1}: Z^{\prime} \rightarrow \mathbb{P}\left(V^{\prime}\right)$ has dimension $>$ $\operatorname{dim} \mathbb{P}\left(V^{\prime}\right)-c n^{2}$. But the generic fibre is, by definition of $Z$, equal to $\mathbb{P}\left(\operatorname{Ker} \mu_{v}\right)$. Thus rank $\mu_{v}<c n^{2}$.

In case ii), we can do the same reasoning, as we have

$$
\operatorname{dim} Z^{\prime} \geq \operatorname{dim} \mathbb{P}\left(V^{\prime}\right) \geq \operatorname{dim} \mathbb{P}(V)
$$

The proof that such a $Z^{\prime}$ does not exist, and thus, the proof of Proposition 2 in the uniruled case, concludes now with the following two Lemmas :

Lemma 12. Let $S \in|n H|$ be generic, with $n$ large enough. Let $c$ be any positive constant. Then there exists a constant $A$ such that the sets

$$
\begin{array}{r}
\Gamma=\left\{v \in \mathbb{P}(V), r k \mu_{v}<c n^{2}\right\} \\
\Gamma^{\prime}=\left\{v^{\prime} \in \mathbb{P}\left(V^{\prime}\right), r k \mu_{v^{\prime}}<c n^{2}\right\} \tag{2.10}
\end{array}
$$

both have dimension bounded by $A$.
Lemma 13. Let $A$ be any positive constant. Let $S \in|n H|$ be generic, with $n$ large enough (depending on $A$ ). Then the set

$$
B=\left\{v \in V, r k \mu_{v}<A\right\}
$$

reduces to 0 .
Indeed, we know by Corollary 5 that our set $Z^{\prime}$ should satisfy either $p r_{1}\left(Z^{\prime}\right) \subset \Gamma$ (case i) or $p r_{2}\left(Z^{\prime}\right) \subset \Gamma^{\prime}$ (case ii). Thus by Lemma 12, one concludes that in case i), $\operatorname{dimpr} r_{1}\left(Z^{\prime}\right) \leq A$ and in case ii), $\operatorname{dimpr} r_{2}\left(Z^{\prime}\right) \leq$ $A$, where $A$ does not depend on $n$.

In case ii), it follows that $\operatorname{dim} Z \leq \operatorname{dim} \mathbb{P}(V)+A$ and as we have $\operatorname{dim} \mathbb{P}(V)+A<\operatorname{dim} \mathbb{P}\left(V^{\prime}\right)$ by Lemma 5 , this gives a contradiction.

In case i), it follows, arguing as in the proof of Corollary 5, that for $\left(v, v^{\prime}\right) \in Z^{\prime}$, one has $r k \mu_{v}<A$. This is impossible unless $Z^{\prime}$ is empty by Lemma 13. Thus, assuming Lemmas 12 and 13, Proposition 2 is proved for uniruled threefolds with $H^{2}\left(X, \mathcal{O}_{X}\right)=0$.

Proof of Lemma 12. Our first step is to reduce the statement to the case where $S$ is a surface in $\mathbb{P}^{3}$. This is done as follows: we choose once for all a morphism

$$
f: X \rightarrow \mathbb{P}^{3}
$$

given by 4 sections of $H$, so that $f^{*} \mathcal{O}_{\mathbb{P}^{3}}(1)=H$. We shall prove the result for surfaces of the form $S=f^{-1}(\Sigma)$, where $\Sigma$ is a generic smooth surface of degree $n$ in $\mathbb{P}^{3}$. Let $f_{S}: S \rightarrow \Sigma$ be the restriction of $f$ to $S$. We have trace maps

$$
\begin{aligned}
& f_{S *}: H^{1}\left(S, \Omega_{S}(s H)\right) \rightarrow H^{1}\left(\Sigma, \Omega_{\Sigma}(s)\right), \\
& f_{S *}: H^{0}\left(S, K_{S}(s H)\right) \rightarrow H^{0}\left(\Sigma, K_{\Sigma}(s)\right)
\end{aligned}
$$

for all integers $s$. We note now that the map $\mu$ admits obvious twists that we shall also denote by $\mu$ :

$$
\mu: H^{0}\left(S, K_{S}(l H)\right) \otimes H^{0}\left(S, \mathcal{O}_{S}(n H)\right) \rightarrow H^{1}\left(S, \Omega_{S}(l H)\right)
$$

Furthermore, we have similarly defined bilinear maps $\mu^{\Sigma}$ :

$$
\mu^{\Sigma}: H^{0}\left(\Sigma, K_{\Sigma}(l)\right) \otimes H^{0}\left(\Sigma, \mathcal{O}_{\Sigma}(n)\right) \rightarrow H^{1}\left(\Sigma, \Omega_{\Sigma}(l)\right)
$$

All the maps $\mu$ can be defined using the maps

$$
\delta: H^{0}\left(S, K_{S}(l H)\right) \hookrightarrow H^{1}\left(S, \Omega_{S}((-n+l) H)\right)
$$

induced by the exact sequence (which is itself a twist of the normal exact sequence)

$$
0 \rightarrow \Omega_{S}(-n H) \rightarrow \Omega_{X \mid S}^{2} \rightarrow K_{S} \rightarrow 0
$$

twisted by $l H$, and then the product map

$$
H^{1}\left(S, \Omega_{S}((-n+l) H)\right) \otimes H^{0}\left(S, \mathcal{O}_{S}(n H)\right) \rightarrow H^{1}\left(S, \Omega_{S}(l H)\right)
$$

The same is true for the maps $\mu_{\Sigma}$.

As there is a commutative diagram of normal exact sequences

$$
\begin{array}{ccccccccc}
0 & \rightarrow & T_{S} & \rightarrow & T_{X \mid S} & \rightarrow & \mathcal{O}_{S}(n H) & \rightarrow & 0 \\
& & f_{S *} \downarrow & & f_{*} \downarrow \\
& \rightarrow & & \| & & \\
f^{*} T_{\Sigma} & \rightarrow & f^{*} T_{\mathbb{P}^{3} \mid S} & \rightarrow & \mathcal{O}_{S}(n H) & \rightarrow & 0
\end{array},
$$

where the bottom line is the normal bundle sequence of $\Sigma$ pulled-back to $S$, it follows that for $v \in H^{0}\left(S, K_{S}(l)\right)$ and $\eta \in H^{0}\left(\Sigma, \mathcal{O}_{\Sigma}(n)\right)$, we have:

$$
\begin{equation*}
f_{S *}\left(\mu_{v}\left(f_{S}^{*} \eta\right)\right)=\mu_{f_{S *}(v)}^{\Sigma}(\eta) \tag{2.11}
\end{equation*}
$$

Equation (2.11) implies that

$$
\begin{aligned}
& r k\left(\mu_{f_{S *}(v)}^{\Sigma}: H^{0}\left(\Sigma, \mathcal{O}_{\Sigma}(n)\right) \rightarrow H^{1}\left(\Sigma, \Omega_{\Sigma}(l)\right)\right) \\
& \leq r k\left(\mu_{v}: H^{0}\left(S, \mathcal{O}_{S}(n)\right) \rightarrow H^{1}\left(S, \Omega_{S}(l)\right)\right) .
\end{aligned}
$$

Let us now prove the first case of Lemma 12, namely for the set $\Gamma$. The second proof is done similarly.

Starting from a sufficiently ample $H$, one finds that $H^{0}\left(X, \mathcal{O}_{X}(4 H)\right)$ restricts surjectively onto $H^{0}\left(X_{u}, \mathcal{O}_{X_{u}}(4 H)\right)$, for any $u \in \mathbb{P}^{3}$, where $X_{u}:=f^{-1}(u)$.

We have the following Lemma:
Lemma 14. The image $\Gamma_{\Sigma}$ of the composed map

$$
\Gamma \times H^{0}\left(X, \mathcal{O}_{X}(4 H)\right) \xrightarrow{\nu} H^{0}\left(S, K_{S}(4 H)\right) \xrightarrow{f_{S *}} H^{0}\left(\Sigma, K_{\Sigma}(4)\right),
$$

where $\nu$ is the product, has dimension at least equal to $\frac{1}{N} \operatorname{dim} \Gamma$, where

$$
N:=r k H^{0}\left(X, \mathcal{O}_{X}(4 H)\right)
$$

Proof. Indeed, as the restriction map

$$
H^{0}\left(X, \mathcal{O}_{X}(4 H)\right) \rightarrow H^{0}\left(X_{u}, \mathcal{O}_{X_{u}}(4 H)\right)
$$

is surjective, if $e_{i}$ is a basis of $H^{0}\left(X, \mathcal{O}_{X}(4 H)\right)$, the map

$$
\Gamma \rightarrow \Gamma_{\Sigma}^{D}, \quad \gamma \mapsto f_{S *}\left(\gamma e_{i}\right),
$$

is injective. Thus $\operatorname{dim} \Gamma \leq N \operatorname{dim} \Gamma_{\Sigma}$.

On the other hand, if $v \in \Gamma, \alpha \in H^{0}\left(X, \mathcal{O}_{X}(4 H)\right)$, we have

$$
r k \mu_{\alpha v} \leq r k \mu_{v}
$$

because $\mu_{\alpha v}=\alpha \mu_{v}$. Thus we conclude that the following hold:

$$
\begin{gathered}
\operatorname{dim} \Gamma_{\Sigma} \geq \frac{1}{N} \operatorname{dim} \Gamma \\
r k \mu_{w}^{\Sigma} \leq r k \mu_{v} \leq c n^{2}
\end{gathered}
$$

for all $w \in \Gamma_{\Sigma}$.
As $N$ does not depend on $n$, it suffices to show the result for generic $\Sigma$ in $\mathbb{P}^{3}$ and for the product

$$
\mu^{\Sigma}: H^{0}\left(\Sigma, K_{\Sigma}(4)\right) \times H^{0}\left(\Sigma, \mathcal{O}_{\Sigma}(n)\right) \rightarrow H^{1}\left(\Sigma, \Omega_{\Sigma}(4)\right)
$$

This last product is well known (cf [19],6.1.3) to identify to the multiplication in the Jacobian ring of $\Sigma$ :

$$
\mu_{\Sigma}: H^{0}\left(\Sigma, \mathcal{O}_{\Sigma}(n)\right) \times H^{0}\left(\Sigma, \mathcal{O}_{\Sigma}(n)\right) \rightarrow R_{\Sigma}^{2 n}
$$

Thus we have to show that for generic $\Sigma$, the set

$$
\Gamma_{\Sigma}:=\left\{v \in H^{0}\left(\Sigma, \mathcal{O}_{\Sigma}(n)\right), r k \mu_{\Sigma, v} \leq c n^{2}\right\}
$$

has dimension bounded by a constant which is independent of $n$.
For this, we specialize to the case where $\Sigma$ is the Fermat surface, that is, its defining equation is $\sigma=\sum_{0}^{3} X_{i}^{n}$. The Jacobian ideal of $\Sigma$ is then generated by the $X_{i}^{n-1}$, and there is thus a natural action of the torus $\left(\mathbb{C}^{*}\right)^{4}$ on the Jacobian ring $R_{\Sigma}$, by multiplication of the coordinates by a scalar. The subspace

$$
\Gamma_{\Sigma} \subset R_{\Sigma}^{n-1}
$$

is thus invariant under $\left(\mathbb{C}^{*}\right)^{4}$. Note that the fixed points of the induced action on $\mathbb{P}\left(R_{\Sigma}^{n-1}\right)$ are the monomials, and are thus isolated. It follows that we have the inequality

$$
\operatorname{dim} \bar{\Gamma}_{\Sigma} \leq \text { number of fixed points on } \bar{\Gamma}_{\Sigma}
$$

Thus we have to bound the number of monomials

$$
X_{I}=X_{0}^{i_{0}} X_{1}^{i_{1}} X_{2}^{i_{2}} X_{3}^{i_{3}}, i_{0}+i_{2}+i_{3}+i_{4}=n
$$

such that

$$
r k X_{I}: R_{\Sigma}^{n} \rightarrow R_{\Sigma}^{2 n-1} \leq c n^{2}
$$

But the kernel of the multiplication by $X_{I}$ above is equal to the ideal

$$
X_{0}^{n-i_{0}} S^{i_{0}}+\ldots X_{3}^{n-i_{3}} S^{i_{3}}
$$

where $S^{l}:=H^{0}\left(\mathbb{P}^{3}, \mathcal{O}_{\mathbb{P}^{3}}(l)\right)$, and thus has dimension $\leq \sum_{k} r k S^{i_{k}}$. Hence, if $r k X_{I} \leq c n^{2}$, we must have

$$
\begin{equation*}
\sum_{k} r k S^{i_{k}} \geq r k S^{n}-c n^{2} \tag{2.12}
\end{equation*}
$$

with $\sum_{k} i_{k}=n$. It is not hard to see that there exists an integer $l>0$ such that, if $n$ is large enough and (2.12) holds for $I$, $n$, one of the $i_{k}^{\prime} s$ has to be $\geq n-l$. Thus the other $i_{j}$ 's have to be non greater than $l$. This shows immediately that the number of such monomials is bounded by a constant independent of $n$ and concludes the proof of Lemma 12 .

Proof of Lemma 13. The key point is the following fact from [6].
Proposition 3. Let $X$ be any projective manifold and $H$ be a very ample line bundle on $X$. Let $A$ be a given constant, and for $n>A$, let $M \subset H^{0}\left(X, \mathcal{O}_{X}(n H)\right)$ be a subspace of codimension $\leq A$. Then

$$
H^{0}\left(X, \mathcal{O}_{X}(H)\right) \cdot M \subset H^{0}\left(X, \mathcal{O}_{X}((n+1) H)\right)
$$

has codimension $\leq A$, with strict inequality if $M$ has no base-point.
Assume $v \in V$ satisfies the condition that $r k \mu_{v}<A$. Let $M:=$ Ker $\mu_{v} \subset H^{0}\left(S, \mathcal{O}_{S}(n H)\right)$. By Proposition 3, we conclude that if $n>A$, we have

$$
H^{0}\left(S, \mathcal{O}_{S}(H)\right) \cdot M \subset H^{0}\left(S, \mathcal{O}_{S}((n+1) H)\right)
$$

has codimension $<A$. Next, we consider for each $l$ the map

$$
\mu_{v}^{l}: H^{0}\left(S, \mathcal{O}_{S}((n+l) H)\right) \rightarrow H^{1}\left(S, \Omega_{S}(l)\right)
$$

obtained as the composite of the twisted Kodaira-Spencer map

$$
H^{0}\left(S, \mathcal{O}_{S}((n+l) H)\right) \rightarrow H^{1}\left(S, T_{S}(l)\right)
$$

and the contraction with $v$, using the contraction map

$$
H^{0}\left(S, K_{S}\right) \otimes H^{1}\left(S, T_{S}(l)\right) \rightarrow H^{1}\left(S, \Omega_{S}(l)\right)
$$

We note that the kernel $M_{l}$ of the map $\mu_{v}^{l}$ contains

$$
M_{1} \cdot H^{0}\left(S, \mathcal{O}_{S}((l-1) H)\right)
$$

On the other hand, $M_{1}$ also contains the image of the map

$$
H^{0}\left(S, T_{X}(H)_{\mid S}\right) \rightarrow H^{0}\left(S, \mathcal{O}_{S}((n+1) H)\right)
$$

induced by the normal bundle sequence twisted by $H$. We may assume that $H$ is ample enough so that $H^{0}\left(X, T_{X}(1)\right)$ is generated by global sections, and then $M_{1}$ has no base-point. Proposition 3 thus implies that if $n>A$, the numbers corank $M_{l}$ are strictly decreasing, starting from $l \geq 1$. Hence we conclude that

$$
M_{A}=H^{0}\left(S, \mathcal{O}_{S}((n+A) H)\right)
$$

As $n$ is large and $A$ is fixed, we may assume that
$H^{0}\left(X, K_{X}((2 n-A) H)\right) \otimes H^{0}\left(X, \mathcal{O}_{X}((n+A) H)\right) \rightarrow H^{0}\left(X, K_{X}(3 n H)\right)$
is surjective, and that the same is true after restriction to $S$. Thus we conclude that

$$
M_{A} \cdot H^{0}\left(S, K_{X}((2 n-A) H)_{\mid S}\right)=H^{0}\left(S, K_{X}(3 n H)_{\mid S}\right)
$$

We use now the definition of $M_{A}$, and the compatibility of the twisted Kodaira-Spencer maps and the maps $\lrcorner v$ with multiplication. This implies that for any $P \in H^{0}\left(S, K_{X}((3 n) H)_{\mid S}\right)$, sending to

$$
\bar{P} \in H^{1}\left(S, T_{S}\left(K_{S}(2 n H)\right)\right)
$$

via the map induced by the twisted normal bundle sequence

$$
0 \rightarrow T_{S}\left(K_{S}(2 n H)\right) \rightarrow T_{X \mid S}\left(K_{S}(2 n H)\right) \rightarrow K_{X}(3 n H)_{\mid S} \rightarrow 0
$$

we have

$$
\begin{equation*}
\bar{P}\lrcorner v=0 \text { in } H^{1}\left(S, \Omega_{S}\left(K_{S}(n H)\right)\right) . \tag{2.13}
\end{equation*}
$$

We have now a map

$$
\delta: H^{1}\left(S, \Omega_{S}\left(K_{S}(n H)\right)\right) \rightarrow H^{2}\left(S, K_{S}\right)
$$

induced by the exact sequence

$$
0 \rightarrow K_{S} \rightarrow \Omega_{X}\left(K_{X}(2 n H)\right)_{\mid S} \rightarrow \Omega_{S}\left(K_{S}(n H)\right) \rightarrow 0
$$

and one knows (cf [4]) that up to a multiplicative coefficient, one has

$$
\begin{equation*}
\delta(\bar{P}\lrcorner v)=<v, \operatorname{res}_{S}(P)> \tag{2.14}
\end{equation*}
$$

where on the right, $<,>$ is Serre duality between $H^{0}\left(S, K_{S}\right)$ and $H^{2}\left(S, \mathcal{O}_{S}\right)$, and the Griffiths residue map

$$
\begin{equation*}
H^{0}\left(X, K_{X}(3 n H)\right) \xrightarrow{\text { ress }_{S}} H^{2}\left(S, \mathcal{O}_{S}\right) \tag{2.15}
\end{equation*}
$$

is described in [19], 6.1.2. The key point for us is that, because in our case $H^{3}\left(X, \Omega_{X}\right)=0$ and because $n$ is large enough, the residue map (2.15) is surjective, and thus (2.13) together with (2.14) imply that, for all $\eta \in H^{2}\left(S, \mathcal{O}_{S}\right)$, one has

$$
<\eta, v>=0
$$

which implies that $v=0$.
The Calabi-Yau case. Here $X$ has trivial canonical bundle and satisfies $H^{2}\left(X, \mathcal{O}_{X}\right)=0$. We use in this case a variant of Lemma 6. As $K_{X}$ is trivial, the spaces $V$ and $V^{\prime}$ are equal, and the pairing $\mu$ : $V \times V^{\prime} \rightarrow H^{1}\left(S, \Omega_{S}\right)$ is symmetric. Thus, using Bertini, Lemma 6 can be refined as follows (cf [15]):

Lemma 15. Let $\mu: V \otimes V^{\prime} \rightarrow H^{1}\left(S, \Omega_{S}\right)$ be symmetric and $q:$ $H^{1}\left(S, \Omega_{S}\right) \rightarrow S^{2} V^{*}$ be its dual. Then the generic quadric in Im $q$ is nonsingular if the following condition holds. There is no subset $Z \subset \mathbb{P}(V)$ contained in the base-locus of $\operatorname{Im} q$ and satisfying:

$$
r k \mu_{v} \leq \operatorname{dim} Z, \forall v \in Z
$$

We have to verify that such a $Z$ does not exist for generic $S \in|n H|$, $n$ large enough. Degenerating $S$ to $S_{A}$ as before, the base-locus of $\operatorname{Im} q$ specializes to a subspace of the base-locus of $\operatorname{Im} q_{A}$. We now use Lemma 7 , together with Corollary 3, to conclude that the base-locus of $\operatorname{Im} q_{A}$ has dimension $\leq c n^{2}$, for some $c$ independent of $n$.

Thus the base-locus of $\operatorname{Im} q$ also has dimension bounded by $c n^{2}$, for generic $S$.

By definition of $Z$, it follows that for $v \in Z$ one has

$$
r k\left(\mu_{v}: V \rightarrow H^{1}\left(S, \Omega_{S}\right)\right) \leq c n^{2}
$$

Using Lemma 12, it follows that $\operatorname{dim} Z \leq A$ for some constant $A$ independent of $N$. But then, for $v \in Z$, one has

$$
r k\left(\mu_{v}: V \rightarrow H^{1}\left(S, \Omega_{S}\right)\right) \leq A
$$

which implies that $Z$ is empty by Lemma 13 . This concludes the proof of Proposition 2 when $X$ is a Calabi-Yau threefold.

## $\S$ 3. The case where $H^{2}\left(X, \mathcal{O}_{X}\right) \neq 0$

In this section, we show how to adapt the previous proof to the case where $X$ is uniruled with $H^{2}\left(X, \mathcal{O}_{X}\right) \neq 0$.

In this case, a smooth birational model of $X$ admits a map $\phi$ : $X^{\prime} \longrightarrow \Sigma$, with generic fibre isomorphic to $\mathbb{P}^{1}$, where $\Sigma$ is a smooth surface. Note that $\phi_{*}$ sends $H^{3}\left(X, \Omega_{X}\right)$ isomorphically to $H^{2}\left(\Sigma, \mathcal{O}_{\Sigma}\right)$.

We may assume that $X^{\prime}$ carries a line bundle $H$ such that

$$
H^{2} K_{X^{\prime}}<0
$$

because there is a smooth birational model of $X$ on which such an $H$ exists, and by blowing-up this $X^{\prime}$ to an $\widetilde{X}$ with exceptional relatively anti-ample divisor $E$, we may assume that $\phi$ becomes defined, while an $\widetilde{H}$ of the form $\tau^{*} H-\epsilon E$ with small $\epsilon$ will still satisfy the property $\widetilde{H}^{2} . K_{\tilde{X}}<0$.

In the sequel $X, H, \phi$ will satisfy the properties above. For $S$ a smooth surface in $|n H|$, we have the Gysin maps:

$$
\begin{aligned}
\phi_{*}: H^{1}\left(S, \Omega_{S}\right) \rightarrow & H^{1}\left(\Sigma, \Omega_{\Sigma}\right), \phi_{*}: H^{2}\left(S, \mathcal{O}_{S}\right) \rightarrow H^{2}\left(\Sigma, \mathcal{O}_{\Sigma}\right) \\
& \phi_{*}: H^{2}(S, \mathbb{Z}) \rightarrow H^{2}(\Sigma, \mathbb{Z})
\end{aligned}
$$

We will denote by

$$
H^{1}\left(S, \Omega_{S}\right)_{\Sigma}, H^{2}\left(S, \mathcal{O}_{S}\right)_{\Sigma}, H^{2}(S, \mathbb{Z})_{\Sigma}
$$

the respective kernels of these maps. The proof will use the following variant of Proposition 1:

Proposition 4. Assume there is a $S \in|n H|$, and $a \lambda \in H^{1}\left(S, \Omega_{S}\right)_{\Sigma}$ such that the natural map

$$
\mu_{\lambda}: H^{0}\left(S, \mathcal{O}_{S}(n H)\right) \rightarrow H^{2}\left(S, \mathcal{O}_{S}\right)_{\Sigma}
$$

defined as in (1.1) is surjective. Then the Hodge conjecture is true for integral Hodge classes on $X$.

Proof. We consider a simply connected open set in $|n H|$ parameterizing smooth surfaces and containing the point $0 \in|n H|$ which is the parameter for $S$. We study the infinitesimal variation of Hodge structure on $H^{2}\left(S_{t}, \mathbb{Z}\right)_{\Sigma}$ for $t \in B$.

By the same reasoning as in the proof of Proposition 1, the existence of $\lambda$ satisfying the property above implies that at some point $\lambda \in H^{1,1}(S)_{\mathbb{R}, \Sigma}$, the natural map

$$
\psi: H_{\mathbb{R}, \Sigma}^{1,1} \rightarrow H^{2}(S, \mathbb{R})_{\Sigma}
$$

is a submersion. Here on the left hand side, we have the real vector bundle with fibre $H^{1,1}\left(S_{t}\right)_{\mathbb{R}, \Sigma}$ at the point $t$, and on each fibre $H^{1,1}\left(S_{t}\right)_{\mathbb{R}, \Sigma}, \psi$ is the inclusion $H^{1,1}\left(S_{t}\right)_{\mathbb{R}, \Sigma} \subset H^{2}\left(S_{t}, \mathbb{R}\right)_{\Sigma}$, followed by the topological isomorphism $H^{2}\left(S_{t}, \mathbb{R}\right)_{\Sigma} \cong H^{2}(S, \mathbb{R})_{\Sigma}$.

This implies that the image of $\psi$ contains an open cone and we deduce from this as in the proof of Proposition 1 that $H^{2}(S, \mathbb{Z})_{\Sigma}$ is generated over $\mathbb{Z}$ by classes $\alpha$ which are algebraic on some nearby fiber $S_{t}$.

Consider now the inclusion $j: S \rightarrow X$. It induces a surjective Gysin map $j_{*}: H^{2}(S, \mathbb{Z}) \rightarrow H^{4}(X, \mathbb{Z})$ by Lefschetz hyperplane theorem. On the other hand, we have a commutative diagram of Gysin maps:

$$
\begin{array}{ccc}
H^{2}(S, \mathbb{Z}) & \xrightarrow{j_{*}} & H^{4}(X, \mathbb{Z}) \\
\phi_{*} \downarrow & & \phi_{*} \downarrow \\
H^{2}(\Sigma, \mathbb{Z}) & = & H^{2}(\Sigma, \mathbb{Z})
\end{array} .
$$

From this and the previous conclusion, we deduce that the group

$$
\operatorname{Ker}\left(\phi_{*}: H^{4}(X, \mathbb{Z}) \rightarrow H^{2}(\Sigma, \mathbb{Z})\right)=j_{*} H^{2}(S, \mathbb{Z})_{\Sigma}
$$

is generated by classes of algebraic cycles on $X$.
Proposition 4 is then a consequence of the following:
Lemma 16. Let $\alpha$ be an integral Hodge class of degree 2 on $\Sigma$. Then there is an algebraic 1-cycle $Z$ on $X$ such that $\alpha=\phi_{*}([Z])$.

Indeed, assuming this lemma, if $\alpha$ is an integral Hodge class on $X$ of degree $4, \phi_{*} \alpha$ is an integral Hodge class of degree 2 on $\Sigma$, hence is equal to $\phi_{*}([Z])$ for some $Z$. Hence $\alpha-[Z]$ belongs to $\operatorname{Ker} \phi_{*}$ and thus it is algebraic as we already proved. This proves the Proposition.

Proof of Lemma 16. We may assume by Lefschetz $(1,1)$ theorem and because $\Sigma$ is algebraic, that $\alpha$ is the class of a curve $C \subset S$ which is in general position. Thus

$$
\phi_{C}: X_{C}:=\phi^{-1}(C) \rightarrow C
$$

is a geometrically ruled surface, which admits a section $C^{\prime} \subset X_{C}$ (see [3], or [7] for a more general statement).

But then the curve $C^{\prime} \subset X$ satisfies $\phi_{*}\left[C^{\prime}\right]=[C]$.
By Proposition 4, the proof of Theorem 2 in case where $X$ is uniruled and satisfies $H^{2}\left(X, \mathcal{O}_{X}\right) \neq 0$ will now be a consequence of the following proposition.

Proposition 5. Let the pair $(X, H)$ satisfy the inequality

$$
H^{2} K_{X}<0
$$

Then for $n$ large enough, for $S$ a generic surface in $|n H|$, there is a $\lambda \in H^{1}\left(S, \Omega_{S}\right)_{\Sigma}$ which satisfies the property that

$$
\mu_{\lambda}: H^{0}\left(S, \mathcal{O}_{S}(n H)\right) \rightarrow H^{2}\left(S, \mathcal{O}_{S}\right)_{\Sigma}
$$

is surjective.
The proof works exactly as the proof of Proposition 2 in the uniruled case. The only thing to note is the fact that the analogue of Proposition 13 still holds in this case, with $V=H^{0}\left(S, K_{S}\right)_{\Sigma}, V^{\prime}=$ $H^{0}\left(S, \mathcal{O}_{S}(n H)\right)$. This is indeed the only place where we used the assumption $H^{2}\left(X, \mathcal{O}_{X}\right)=0$.

In this case, we have an isomorphism

$$
\phi_{*}: H^{3}\left(\Omega_{X}\right) \cong H^{2}\left(\Sigma, \mathcal{O}_{\Sigma}\right),
$$

so that for $S \subset X$ a smooth surface

$$
H^{2}\left(S, \mathcal{O}_{S}\right)_{\Sigma}=\operatorname{Ker}\left(j_{*}: H^{2}\left(S, \mathcal{O}_{S}\right) \rightarrow H^{3}\left(X, \Omega_{X}\right)\right)
$$

where $j$ is the inclusion of $S$ into $X$.
But the theory of Griffiths residues shows that the last kernel is precisely generated by residues $\operatorname{res}_{S} \omega, \omega \in H^{0}\left(X, K_{X}(3 n H)\right)$. Thus, the arguments of Lemma 13 will show in this case that if $v \in H^{0}\left(S, K_{S}\right)$ satisfies rank $\mu_{v} \leq A$, where $A$ is a given constant, and $S \in|n H|$ with $n$ large enough, then

$$
v \in\left(\operatorname{Ker} j_{*}\right)^{\perp}
$$

where $\perp$ refers to Serre duality between $H^{0}\left(S, K_{S}\right)$ and $H^{2}\left(S, \mathcal{O}_{S}\right)$. But as $\operatorname{Ker} \phi_{*}=\operatorname{Ker} j_{*}$, we have

$$
\left(\text { Ker } j_{*}\right)^{\perp}=\phi^{*} H^{0}\left(\Sigma, K_{\Sigma}\right)
$$

Thus if furthermore $v \in H^{0}\left(S, K_{S}\right)_{\Sigma}$, we must have $v=0$ because

$$
H^{0}\left(S, K_{S}\right)_{\Sigma} \cap \phi^{*} H^{0}\left(\Sigma, K_{\Sigma}\right)=0
$$

## §4. Appendix

We give for the convenience of the reader the proof of the vanishing Lemma 9. Recall that we want to prove the vanishing of the spaces:
(1) $H^{i}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right), n+2-i \geq 0, i \geq 1$
(2) $H^{i-1}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right), n+2-i=0$.
(3) $H^{i-2}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right), n+2-i=0$.
(4) $H^{i-3}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right), n+2-i \leq 0$.

We use first the dual of the exact sequence (2.6) to get a resolution of $\bigwedge^{i} \mathcal{F}^{*}$ :

$$
\ldots \rightarrow \bigwedge^{i-1}\left(K \otimes \mathcal{E}^{*}\right) \otimes \mathcal{L}^{-1} \rightarrow \bigwedge^{i}(K \otimes \mathcal{E})^{*} \rightarrow \bigwedge^{i} \mathcal{F}^{*} \rightarrow 0
$$

This induces a spectral sequence converging to

$$
H^{i}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right), H^{i-1}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right), H^{i-2}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right), H^{i-3}\left(G, \bigwedge^{i} \mathcal{F}^{*}\right)
$$

whose $E_{1}$ terms are
Case $1 \quad H^{i+s}\left(G, \bigwedge^{i-s}\left(K \otimes \mathcal{E}^{*}\right) \otimes \mathcal{L}^{-s}\right), n+2 \geq i \geq 1, i \geq s \geq 0$,

Case $2 \quad H^{i+s-1}\left(G, \bigwedge^{i-s}\left(K \otimes \mathcal{E}^{*}\right) \otimes \mathcal{L}^{-s}\right), i=n+2, i \geq s \geq 0$,
Case $3 \quad H^{i+s-2}\left(G, \bigwedge^{i-s}\left(K \otimes \mathcal{E}^{*}\right) \otimes \mathcal{L}^{-s}\right), i=n+2 i \geq s \geq 0$
Case $4 \quad H^{i+s-3}\left(G, \bigwedge^{i-s}\left(K \otimes \mathcal{E}^{*}\right) \otimes \mathcal{L}^{-s}\right), n+2 \leq i, i \geq s \geq 0$ respectively.

Let $P \subset \mathbb{P}(K) \times G$ be the incidence scheme, so $P$ is a $\mathbb{P}^{1}$-bundle over $G$. Let $p r_{i}, i=1,2$ denote the projections from $P$ to $\mathbb{P}(K)$ and $G$ respectively. Let $H:=p r_{1}^{*} \mathcal{O}(1)$ and denote also by $\mathcal{L}$ the pull-back of $\mathcal{L}$ to $P$. Then $p r_{2}^{*} \mathcal{E}^{*}$ fits into an exact sequence:

$$
0 \rightarrow H^{-1} \rightarrow p r_{2}^{*} \mathcal{E}^{*} \rightarrow H \otimes \mathcal{L}^{-1} \rightarrow 0
$$

Thus the bundle

$$
p r_{2}^{*}\left(\bigwedge^{i-s}\left(K \otimes \mathcal{E}^{*}\right) \otimes \mathcal{L}^{-s}\right)
$$

admits a filtration whose successive quotients are line bundles of the form

$$
H^{-\alpha} \otimes\left(H \otimes \mathcal{L}^{-1}\right)^{\beta} \otimes \mathcal{L}^{-s}=H^{-\alpha+\beta} \otimes \mathcal{L}^{-\beta-s}
$$

where $\alpha+\beta=i-s, \alpha \geq 0, \beta \geq 0$. As we are interested in

$$
H^{*}\left(G, \bigwedge_{\bigwedge}^{i-s}\left(K \otimes \mathcal{E}^{*}\right) \otimes \mathcal{L}^{-s}\right)=H^{*}\left(G, R^{0} p r_{2 *}\left(p r_{2}^{*}\left(\bigwedge \bigwedge\left(K \otimes \mathcal{E}^{*}\right) \otimes \mathcal{L}^{-s}\right)\right)\right)
$$

it suffices to study the cohomology groups

$$
H^{*}\left(P, H^{-\alpha+\beta} \otimes \mathcal{L}^{-\beta-s}\right)
$$

with $-\alpha+\beta \geq 0$. These groups are equal to the groups

$$
H^{*}\left(G, S^{-\alpha+\beta} \mathcal{E} \otimes \mathcal{L}^{-\beta-s}\right)
$$

which are partially computed in [18]. The conclusion is the following:
Lemma 17. a) These groups vanish for $* \neq n-2,2(n-2)$ and for $\beta+s \leq n-2$.
b) For $*=n-2$, these groups vanish if $-s-\alpha+1<0$.
c) For $*=2(n-2)$, these groups vanish if $-s-\alpha \geq-n+1$.

Case 1. Here $*=i+s$, and the following inequalities hold:

$$
\begin{equation*}
\beta \geq \alpha \geq 0, \beta+s \geq n-1 \tag{4.16}
\end{equation*}
$$

and furthermore

$$
1 \leq i \leq n+2, \alpha+\beta=i-s
$$

According to Lemma 17, in order to get a non trivial cohomology group, we have only two possibilities:
a) $i+s=n-2,-s-\alpha+1 \geq 0$.
b) $i+s=2(n-2),-s-\alpha<-n+1$.

In case a), we have $\beta+s \geq n-1$ and $\alpha+\beta+2 s=i+s=n-2$, which is clearly a contradiction as $\alpha+s \geq 0$.

In case b ), we have $\beta+s \geq n-1, \alpha+s \geq n$ and thus

$$
2 n-1 \leq \alpha+\beta+2 s=i+s=2(n-2)
$$

which is clearly a contradiction.
Case 2. Now $*=i+s-1$ and $i=n+2$. We have again the inequalities (4.16) and furthermore

$$
i=n+2, \alpha+\beta=i-s
$$

By Lemma 17, in order to get a non trivial cohomology group, we have only two possibilities:
a) $i+s-1=n-2,-s-\alpha+1 \geq 0$.
b) $i+s-1=2(n-2), s+\alpha \geq n$.

In case a), we have $i=n+2$ and $s \geq 0$, hence $i+s-1=n-2$ is impossible.

In case b), we have $i+s=2 n-3$, while $s+\alpha \geq n$ and $\beta+s \geq n-1$ give $\alpha+\beta+2 s=i+s \geq 2 n-1$, contradiction.

Case 3. Now $*=i+s-2$ and $i=n+2$. We have again the inequalities (4.16) and furthermore $i=n+2, \alpha+\beta=i-s$. As before, in order to get a non trivial cohomology group, we have only two possibilities:
a) $i+s-2=n-2,-s-\alpha+1 \geq 0$.
b) $i+s-2=2(n-2), s+\alpha \geq n$.

In case a), we have $i=n+2$ and $s \geq 0$, hence $i+s-2=n-2$ is impossible.

In case b), we have $i+s=2 n-2$, while $s+\alpha \geq n$ and $\beta+s \geq n-1$ give $\alpha+\beta+2 s=i+s \geq 2 n-1$, contradiction.

Case 4. Now $*=i+s-3$ and $i \geq n+2$. We have again the inequalities (4.16) and furthermore $i \geq n+2, \alpha+\beta=i-s$. As before, in order to get a non trivial cohomology group, we have only two possibilities:
a) $i+s-3=n-2,-s-\alpha+1 \geq 0$.
b) $i+s-3=2(n-2), s+\alpha \geq n$.

In case a), we have $i \geq n+2$ and $s \geq 0$ thus $i+s-3=n-2$ is impossible.

In case b), we have $i+s=2 n-1$, while $s+\alpha \geq n$ and $\beta+s \geq n-1$ give $\alpha+\beta+2 s=i+s \geq 2 n-1$. Thus we must have the two equalities

$$
s+\alpha=n, \beta+s=n-1
$$

This contradicts the fact that $\beta \geq \alpha$.

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# Birational geometry of symplectic resolutions of nilpotent orbits 

Yoshinori Namikawa<br>Dedicated to Professor Masaki Maruyama on his sixtieth birthday


#### Abstract

. We shall give a complete description of the relatively ample cones and the relatively movable cones of symplectic resolutions of the closures of the nilpotent orbits in complex simple Lie algebras. Moreover, we shall prove that all symplectic resolutions of such nilpotent orbit closures are connected by finite numbers of Mukai flops of type $A, D$ and $E_{6}$.


## §1. Introduction

Let $G$ be a complex simple Lie group and let $\mathfrak{g}$ be its Lie algebra. Then $G$ has the adjoint action on $\mathfrak{g}$. The orbit $\mathcal{O}_{x}$ of a nilpotent element $x \in \mathfrak{g}$ is called a nilpotent orbit. A nilpotent orbit $\mathcal{O}_{x}$ admits a nondegenerate closed 2-form $\omega$ called the Kostant-Kirillov symplectic form. The closure $\overline{\mathcal{O}}_{x}$ of $\mathcal{O}_{x}$ then becomes a symplectic singularity. In other words, the 2 -form $\omega$ extends to a holomorphic 2 -form on a resolution of $\overline{\mathcal{O}}_{x}$. A resolution of $\overline{\mathcal{O}}_{x}$ is called a symplectic resolution if this extended form is everywhere non-degenerate on the resolution. For a parabolic subgroup $P$ of $G$, one can find a unique nilpotent orbit $\mathcal{O}$ such that $\mathcal{O} \cap n(\mathfrak{p})$ is an open dense subset of $n(\mathfrak{p})$. Here $n(\mathfrak{p})$ is the nil-radical of $\mathfrak{p}:=\operatorname{Lie}(P)$. This orbit is called the Richardson orbit for $P$. Conversely, $P$ is called a polarization of $\mathcal{O}$. We then have a generically finite proper surjective map

$$
\mu: T^{*}(G / P) \rightarrow \overline{\mathcal{O}}
$$

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Here $T^{*}(G / P)$ is the cotangent bundle of the homogenous space $G / P$. When $\operatorname{deg}(\mu)=1, \mu$ becomes a symplectic resolution of $\overline{\mathcal{O}}$. We call it a Springer resolution. Recently, Fu [Fu 1] (see also some corrections in its e-print version) has shown that, if a nilpotent orbit $\overline{\mathcal{O}}$ has a (projective) symplectic resolution $f$, then $\mathcal{O}$ has a polarization $P$ such that $f$ coincides with the Springer resolution for $P$. However, there is a nilpotent orbit with no polarizations. Moreover, even if $\mathcal{O}$ has a polarization, it is not unique and we may possibly have $\operatorname{deg}(\mu)>1$. Spaltenstein [S2] and Hesselink [He] obtained a necessary and sufficient condition for $\overline{\mathcal{O}}$ to have a Springer resolution when $\mathfrak{g}$ is a classical simple Lie algebra. Moreover, [He] gave an explicit number of such parabolics $P$ up to conjugacy class that give Springer resolutions of $\overline{\mathcal{O}}_{x}$ (cf. §4). In this paper we shall deal with an arbitrary simple Lie algebra. First we introduce an equivalence relation in the set of parabolic subgroups of $G$ in terms of marked Dynkin diagrams (Definition 1, §5). The following is one of main results of this paper.

Theorem(cf. Theorem 6.1): Let $\mathcal{O}$ be a nilpotent orbit of a complex simple Lie algebra. Assume that $\overline{\mathcal{O}}$ has a Springer resolution $Y_{P_{0}}:=$ $T^{*}\left(G / P_{0}\right)$. Then, for any parabolic subgroup $P$ equivalent to $P_{0}, Y_{P}:=$ $T^{*}(G / P)$ is a Springer resolution of $\overline{\mathcal{O}}$. Moreover, any projective symplectic resolution of $\overline{\mathcal{O}}$ has this form. All $Y_{P}\left(P \sim P_{0}\right)$ are connected by Mukai flops of type $A, D$, and $E_{6}$.

A Mukai flop of type $A$ is a kind of Springer resolutions; let $x \in \mathfrak{s l}(n)$ be a nilpotent element of Jordan type $\left[2^{k}, 1^{n-2 k}\right]$ with $2 k<n$. Then a Mukai flop of type A is the diagram of two Springer resolutions of $\overline{\mathcal{O}}_{x}$ :

$$
T^{*} G(k, n) \rightarrow \overline{\mathcal{O}}_{x} \leftarrow T^{*} G(n-k, n)
$$

where $G(k, n)$ (resp. $G(n-k, n))$ is the Grassmannian which parametrizes $k$-dimensional (resp. $n-k$-dimensional) subspaces of $\mathbf{C}^{n}$. This flop naturally appears in the wall-crossing of the moduli spaces of various objects (eg. stable sheaves on K3 surfaces, quiver varieties and so on). On the other hand, a Mukai flop of type $D$ comes from an orbit of a simple Lie algebra of type D . Let $x \in \mathfrak{s o}(2 k)$ be a nilpotent element of type $\left[2^{k-1}, 1^{2}\right]$, where $k$ is an odd integer with $k \geq 3$. Then $\overline{\mathcal{O}}_{x}$ admits two Springer resolutions

$$
T^{*} G_{i s o}^{+}(k, 2 k) \rightarrow \overline{\mathcal{O}}_{x} \leftarrow T^{*} G_{i s o}^{-}(k, 2 k)
$$

where $G_{i s o}^{+}(k, 2 k)$ and $G_{i s o}^{-}(k, 2 k)$ are two connected components of the orthogonal Grassmannian $G_{i s o}(k, 2 k)$. Finally, there are two Mukai flops of type $E_{6}$. We call them of type $E_{6, I}$ and of type $E_{6, I I}$. The Mukai flop of type $E_{6, I}$ (resp. $E_{6, I I}$ ) consists of two resolutions of the nilpotent
orbit closure $\overline{\mathcal{O}}_{2 A_{1}}$ (resp. $\overline{\mathcal{O}}_{A_{2}+2 A_{1}}$ ) in $E_{6}$. For details on these flops, see §5. Let us consider a family of Mukai flops parametrized by a variety $T: Y \rightarrow W \leftarrow Y^{\prime}$. By definition, there is a bundle map $W \rightarrow T$ with a typical fiber $\overline{\mathcal{O}}_{x}$ such that, for each $t \in T, Y_{t} \rightarrow \overline{\mathcal{O}}_{x} \leftarrow Y_{t}^{\prime}$ is a Mukai flop. A flop

$$
Z \rightarrow X \leftarrow Z^{\prime}
$$

is called a locally trivial family of Mukai flop if there is a smooth surjective map $X \rightarrow W$ and it is the pull-back by this map of the family of Mukai flops above. The last statement of Theorem claims that, for any two $Y_{P}$ and $Y_{P^{\prime}}$, the birational map $Y_{P}-\rightarrow Y_{P^{\prime}}$ is decomposed into diagrams $Y_{i} \rightarrow X_{i} \leftarrow Y_{i+1}(i=1, \ldots, m-1)$ with $Y_{1}=Y_{P}$ and $Y_{m}=Y_{P^{\prime}}$ so that each diagram is a locally trivial family of Mukai flops.

In the course of the proof of Theorem, we describe the ample cones and movable cones of symplectic resolutions of $\overline{\mathcal{O}}$. Even when $\mathfrak{g}$ is classical, it would clarify the geometric meaning of the results of Spaltenstein and Hesselink. To illustrate these, three examples will be given (see Examples 6.7, 6.8, 6.9).

Another purpose of this paper is to give an affirmative answer to the following conjecture in the case of (the normalization of) a nilpotent orbit closure in a simple Lie algebra (Theorem 7.9).

Conjecture([F-N]): Let $W$ be a normal symplectic singularity. Then for any two symplectic resolutions $f_{i}: X_{i} \rightarrow W, i=1,2$, there are deformations $\mathcal{X}_{i} \xrightarrow{F_{i}} \mathcal{W}$ of $f_{i}$ over a parameter space $S$ such that, for $s \in S-\{0\}, F_{i, s}: \mathcal{X}_{i, s} \rightarrow \mathcal{W}_{s}$ are isomorphisms. In particular, $X_{1}$ and $X_{2}$ are deformation equivalent.

This conjecture is already proved in $[\mathrm{F}-\mathrm{N}]$ when $W$ is a nilpotent orbit closure in $\mathfrak{s l}(n)$. On the other hand, a weaker version of this conjecture is proved in [Fu 2] when $W$ is the normalization of a nilpotent orbit closure in a classical simple Lie algebra. According to the idea of Borho and Kraft [B-K], we shall define a deformation of $\overline{\mathcal{O}}_{x}$ by using a Dixmier sheet. Corresonding to each parabolic subgroup $P$, this deformation has a simultaneous resolution. These simultaneous resolutions would give the desired deformations of the conjecture. Details on the construction of them can be found in $\S 7$.

The content of this paper is as follows. Main body of the paper are $\S \S .5,6,7$. The first three sections $\S \S .2,3,4$ are preliminaries for the later sections. In the proof of Proposition 5.1, we shall use Springer's correspondence (cf. Theorem 3.1, Proposition 4.3) to calculate the dimension of fibers of Springer maps. The proofs of Theorem 6.1 are written in an abstract way so that they are valid for exceptional Lie algebras. One
can, however, find a more explicit treatment in Example 6.5 when $\mathfrak{g}$ is classical.

Finally, the author would like to thank S. Mukai for an important comment on an earlier version of the present paper and he would like to thank D. Alvis for sending him the paper [Al].

Notation. (1) A partition $\mathbf{d}$ of $n$ is a set of positive integers $\left[d_{1}, \ldots, d_{k}\right]$ such that $\Sigma d_{i}=n$ and $d_{1} \geq d_{2} \geq \ldots \geq d_{k}$. We mean by $\left[d_{1}^{j_{1}}, \ldots, d_{k}^{j_{k}}\right]$ the partition where $d_{i}$ appear in $j_{i}$ multiplicity. If ( $p_{1}, \ldots, p_{s}$ ) is a sequence of positive integers, then we define the partition $\mathbf{d}=$ $\operatorname{ord}\left(p_{1}, \ldots, p_{s}\right)$ by $d_{i}:=\sharp\left\{j ; p_{j} \geq i\right\}$. In particular, for a partition $\mathbf{d},{ }^{t} \mathbf{d}:=\operatorname{ord}\left(d_{1}, \ldots, d_{k}\right)$ is called the dual partition of $\mathbf{d}$. We define $d^{i}:=\left({ }^{t} d\right)_{i}$.
(2) For a proper birational map $f$ of algebraic varieties, we say that $f$ is divisorial if $\operatorname{Exc}(f)$ contains a divisor, and otherwise, we say that $f$ is small. Note that the terminology of "small" is, for example, different from that in $[\mathrm{B}-\mathrm{M}]$.

## §2. Classification of nilpotent orbits

Let $G$ be a complex simple Lie group and let $\mathfrak{g}$ be its Lie algebra. $G$ has the adjoint action on $\mathfrak{g}$. The orbit $\mathcal{O}_{x}$ of a nilpotent element $x \in \mathfrak{g}$ for this action is called a nilpotent orbit. This orbit carries a natural closed non-degenerate 2-form (Kostant-Kirillov form) $\omega$ (cf. [C-G], Prop. 1.1.5, $[\mathrm{C}-\mathrm{M}], 1.3$ ), and its closure $\overline{\mathcal{O}}_{x}$ becomes a symplectic singularity, that is, the symplectic 2 -form $\omega$ extends to a holomorphic 2 -form on a resolution $Y$ of $\overline{\mathcal{O}}_{x}$. When $\mathfrak{g}$ is classical, $\mathfrak{g}$ is naturally a Lie subalgebra of $\operatorname{End}(V)$ for a complex vector space $V$. Then we can attach a partition $\mathbf{d}$ of $n:=\operatorname{dim} V$ to each orbit as the Jordan type of an element contained in the orbit. Here a partition $\mathbf{d}:=\left[d_{1}, d_{2}, \ldots, d_{k}\right]$ of $n$ is a set of positive integers with $\Sigma d_{i}=n$ and $d_{1} \geq d_{2} \geq \ldots \geq d_{k}$. When a number $e$ appears in the partition $\mathbf{d}$, we say that $e$ is a part of $\mathbf{d}$. We call $\mathbf{d}$ very even when d consists with only even parts, each having even multiplicity. The following result can be found, for example, in [C-M, $\S 5]$.

Proposition 2.1. Let $\mathcal{N} o(\mathfrak{g})$ be the set of nilpotent orbits of $\mathfrak{g}$.
(1) $\left(A_{n-1}\right)$ : When $\mathfrak{g}=\mathfrak{s l}(n)$, there is a bijection between $\mathcal{N} o(\mathfrak{g})$ and the set of partitions $\mathbf{d}$ of $n$.
(2) $\left(B_{n}\right):$ When $\mathfrak{g}=\mathfrak{s o}(2 n+1)$, there is a bijection between $\mathcal{N} o(\mathfrak{g})$ and the set of partitions $\mathbf{d}$ of $2 n+1$ such that even parts occur with even multiplicity.
(3) $\left(C_{n}\right)$ : When $\mathfrak{g}=\mathfrak{s p}(2 n)$, there is a bijection between $\mathcal{N} o(\mathfrak{g})$ and the set of partitions $\mathbf{d}$ of $2 n$ such that odd parts occur with even multiplicity
$(4)\left(D_{n}\right):$ When $\mathfrak{g}=\mathfrak{s o}(2 n)$, there is a surjection $f$ from $\mathcal{N} o(\mathfrak{g})$ to the set of partitions $\mathbf{d}$ of $2 n$ such that even parts occur with even multiplicity. For a partition $\mathbf{d}$ which is not very even, $f^{-1}(\mathbf{d})$ consists of exactly one orbit, but, for very even $\mathbf{d}, f^{-1}(\mathbf{d})$ consists of exactly two different orbits.

When $\mathfrak{g}$ is of exceptional type, we need different methods to classify nilpotent orbits. Dynkin [D] associates a weighted Dynkin diagram with each nilpotent orbit. The weighted Dynkin diagram uniquely determines a nilpotent orbit. However, all weighted Dynkin diagrams do not come from nilpotent orbits. Bala and Carter [B-L] has classified which weighted Dynkin diagram is realized, and they give a label (Bala-Carter label) to each nilpotent orbit. We shall use these labels to indicate nilpotent orbits in an exceptional Lie algebra $\mathfrak{g}$ (cf. [B-C],[C-M]).

## §3. Springer's correspondence

Let $G$ be a complex simple Lie group and let $B$ be a Borel subgroup of $G$. Let $\mathfrak{g}$ (resp. b) be the Lie algebra of $G$ (resp. B). The set of nilpotent elements $\mathcal{N}$ of $\mathfrak{g}$ is called the nilpotent variety. It coincides with the closure of the regular nilpotent orbit in $\mathfrak{g}$. The (original) Springer resolution

$$
\pi: T^{*}(G / B) \rightarrow \mathcal{N}
$$

is constructed as follows. Let $n(\mathfrak{b})$ be the nil-radical of $\mathfrak{b}$. Then the cotangent bundle $T^{*}(G / B)$ of $G / B$ is identified with $G \times{ }^{B} n(\mathfrak{b})$, which is, by definition, the quotient space of $G \times n(\mathfrak{b})$ by the equivalence relation $\sim$. Here $(g, x) \sim\left(g^{\prime}, x^{\prime}\right)$ if $g^{\prime}=g b$ and $x^{\prime}=A d_{b^{-1}}(x)$ for some $b \in B$. Then we define $\pi([g, x]):=A d_{g}(x)$. According to BorhoMacPherson [B-M], we shall briefly review Springer's correspondence $[\mathrm{Sp}]$. The nilpotent variety $\mathcal{N}$ is decomposed into the disjoint union of nilpotent orbits $\mathcal{O}_{x}$, where $x$ is a distinguished base point of the orbit $\mathcal{O}_{x}$. We put $d_{x}:=\operatorname{dim} \pi^{-1}(x)$. Now $\pi_{1}\left(\mathcal{O}_{x}\right)$ acts on $H^{2 d_{x}}\left(\pi^{-1}(x), \mathbf{Q}\right)$ by monodromy. Decompose $H^{2 d_{x}}\left(\pi^{-1}(x), \mathbf{Q}\right)$ into irreducible representations of $\pi_{1}\left(\mathcal{O}_{x}\right)$ :

$$
H^{2 d_{x}}\left(\pi^{-1}(x), \mathbf{Q}\right)=\oplus_{\phi}\left(V_{\phi} \otimes V_{(x, \phi)}\right),
$$

where $\phi: \pi_{1}\left(\mathcal{O}_{x}\right) \rightarrow \operatorname{End}\left(V_{\phi}\right)$ are irreducible representations and $V_{(x, \phi)}=$ $\operatorname{Hom}_{\pi_{1}\left(\mathcal{O}_{x}\right)}\left(V_{\phi}, H^{2 d_{x}}\left(\pi^{-1}(x), \mathbf{Q}\right)\right)$. By definition, $\operatorname{dim} V_{(x, \phi)}$ coincides with
the multiplicity of $\phi$ in $H^{2 d_{x}}\left(\pi^{-1}(x), \mathbf{Q}\right)$. We call $(x, \phi)$ is $\pi$-relevant if $V_{(x, \phi)} \neq 0$. Fix a maximal torus $T$ in $B$, and let $W$ be the Weyl group relative to $T$. Then there is a natural action of $W$ on $H^{2 d_{x}}\left(\pi^{-1}(x), \mathbf{Q}\right)$ commuting with the action of $\pi_{1}\left(\mathcal{O}_{x}\right)$. Each factor $V_{\phi} \otimes V_{(x, \phi)}$ becomes a $W$-module, where $W$ acts trivially on $V_{\phi}$ and $V_{(x, \phi)}$ is an irreducible representation of $W$. These representations were originally constructed by Springer. In $[B-M]$, they are given in terms of the decomposition theorem of intersection cohomology by Beilinson, Bernstein, Deligne and Gabber. The following theorem is called Springer's correspondence:

Theorem 3.1. Any irreducible representaion of $W$ is isomorphic to $V_{(x, \phi)}$ for a unique $\pi$-relevant pair $(x, \phi)$.

One can find the tables on Springer's correspondence in [C, 13.3] for each simple Lie group (see also [A-L], [B-L]).

## §4. Parabolic subgroups and Springer maps

Let $G$ be a complex reductive Lie group and let $\mathfrak{g}$ be its Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let

$$
\mathfrak{g}=\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}
$$

be the root space decomposition. Let $\Delta \subset \Phi$ be a base of $\Phi$ and denote by $\Phi^{+}$(resp. $\Phi^{-}$) the set of positive roots (resp. negative root). We define a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ as

$$
\mathfrak{b}:=\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^{+}} \mathfrak{g}_{\alpha} .
$$

For a subset $\Theta \subset \Delta$, let $\langle\Theta>$ be the sub-root system generated by $\Theta$. We put $<\Theta>^{+}:=<\Theta>\cap \Phi^{+}$and $<\Theta>^{-}:=<\Theta>\cap \Phi^{-}$. We define

$$
\mathfrak{p}_{\Theta}:=\mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^{+}} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\alpha \in\langle\Theta\rangle^{-}} \mathfrak{g}_{\alpha} .
$$

By definition, $\mathfrak{p}_{\Theta}$ is a parabolic subalgebra containing $\mathfrak{b}$. Moreover, any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $G$-conjugate to $\mathfrak{p}_{\Theta}$ for some $\Theta \subset \Delta$. $\mathfrak{p}_{\Theta}$ and $\mathfrak{p}_{\Theta^{\prime}}$ are $G$-conjugate if and only if $\Theta=\Theta^{\prime}$. Therefore, there is a one-to-one correspondence between subsets of $\Delta$ and the conjugacy classes of parabolic subalgebras of $\mathfrak{g}$. An element of $\Delta$ is called a simple root, which corresponds to a vertex of the Dynkin diagram attached to $\mathfrak{g}$. A Dynkin diagram with some vertices being marked is called a marked Dynkin diagram. If $\Theta \subset \Delta$ is given, we have a marked Dynkin
diagram by marking the vertices which correspond to $\Delta \backslash \Theta$. A marked Dynkin diagram with only one marked vertex is called a single marked Dynkin diagram. A conjugacy class of parabolic subgroups $P \subset G$ with $b_{2}(G / P)=1$ corresponds to a single marked Dynkin diagram.

Example 4.1. When $G=S L(n)$, the parabolic subgroup of flag type $(k, n-k)$ corresponds to the marked Dynkin diagram


Example 4.2. Let $\epsilon$ denote the number 0 or 1. Assume that $V$ is a $\mathbf{C}$-vector space equipped with a non-degenerate bilinear form $<,>$ such that

$$
<v, w>=(-1)^{\epsilon}<w, v>,(v, w \in V)
$$

When $\epsilon=0$ (resp. $\epsilon=1$ ), this means that the bilinear form is symmetric (resp. skew-symmetric). We shall describe parabolic subgroups of $S O(V)$ and $\operatorname{Sp}(V)$. We put

$$
H:=\{x \in G L(V) ;<x v, x w>=<v, w>,(v, w \in V)\}
$$

and

$$
G:=\{x \in H ; \operatorname{det}(x)=1\} .
$$

Note that

$$
H=\left\{\begin{aligned}
O(V) & (\epsilon=0) \\
S p(V) & (\epsilon=1)
\end{aligned}\right.
$$

and

$$
G=\left\{\begin{aligned}
S O(V) & (\epsilon=0) \\
S p(V) & (\epsilon=1)
\end{aligned}\right.
$$

A flag $F:=\left\{F_{i}\right\}_{1 \leq i \leq s}$ of $V$ is called isotropic if $F_{i}^{\perp}=F_{s-i}$ for $1 \leq$ $i \leq s$. An isotropic flag $F$ is admissible if the stabilizer group $P$ of $F$ has no finner stabilized flag than $F$. In other words, let $P_{F}:=\left\{g \in G ; g F_{i} \subset\right.$ $\left.F_{i} \forall i\right\}$. Then, for any $i$, there is no $P_{F}$-invariant subspace $F_{i}^{\prime}$ such that $F_{i} \subset F_{i}^{\prime} \subset F_{i+1}$ with $F_{i}^{\prime} \neq F_{i}, F_{i+1}$. When the length $s$ of an isotropic flag $F$ is even, one can write the type of $F$ as $\left(p_{1}, \ldots, p_{k}, p_{k}, \ldots, p_{1}\right)$ with $k=s / 2$. On the other hand, when $s$ is odd, one can write the type of $F$ as $\left(p_{1}, \ldots, p_{k}, q, p_{k}, \ldots, p_{1}\right)$ with $k=(s-1) / 2$. For the consistency, we shall write the flag type of $F$ as $\left(p_{1}, \ldots, p_{k}, 0, p_{k}, \ldots, p_{1}\right)$ when $s$ is even. An isotropic flag $F$ is not admissible when $\epsilon=0$ and $q=2$. In fact, one can always find a $P_{F}$-invariant subspace $F_{k}^{\prime}$ such that $F_{k} \subset F_{k}^{\prime} \subset$ $F_{k+1}$ and $\operatorname{dim}\left(F_{k}^{\prime} / F_{k}\right)=1$. This is the only case where an isotropic
flag is not admissible. The stabilizer group of an admissible isotropic flag becomes a parabolic subgroup of $G$. If a parabolic subgroup of $G$ has a stabilized (admissible) flag $F$ of type ( $p_{1}, \ldots, p_{k}, q, p_{k}, \ldots, p_{1}$ ), then $\pi:=\operatorname{ord}\left(p_{1}, \ldots, p_{k}, q, p_{k}, \ldots, p_{1}\right)$ is called the Levi type of $P$.

When $G=S O(2 n+1)$, the parabolic subgroup of flag type $(k, 2 n-$ $2 k+1, k)$ corresponds to the marked Dynkin diagram


When $G=\operatorname{Sp}(2 n)$, the parabolic subgroup of flag type $(k, 2 n-2 k, k)$ corresponds to the marked Dynkin diagram


Finally, assume that $G=S O(2 n)$. Then the parabolic subgroup corresponding to the marked Dynkin diagram ( $k \geq 3$ )

has flag type $(n-k+1,2 k-2, n-k+1)$. On the other hand, two marked Dynkin diagrams

both give parabolic subgroups of flag type $(n, 0, n)$ which are not $G$ conjugate.

For a parabolic subgroup $P$ of $G$, let $\mathfrak{p}$ be its Lie algebra and let $n(\mathfrak{p})$ be the nil-radical of $\mathfrak{p}$. There is a unique nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ such that $\mathcal{O} \cap n(\mathfrak{p})$ is an open dense subset of $n(\mathfrak{p})$. This nilpotent orbit is called the Richardson orbit for $P$. Conversely, such parabolic subgroup $P$ is called a polarization of $\mathcal{O}$. When $x \in n(\mathfrak{p})$ and $P$ is a polarization of $\mathcal{O}_{x}$, we call $P$ a polarization of $x$. A parabolic subgroup $P$ is a polarization of $x$ if and only if $x \in n(\mathfrak{p})$ and $\operatorname{dim} \mathcal{O}_{x}=2 \operatorname{dim}(G / P)$ (cf.
[He]). The cotangent bundle $T^{*}(G / P)$ of the homogenous space $G / P$ is naturally isomorphic to $G \times{ }^{P} n(\mathfrak{p})$, which is the quotient space of $G \times n(\mathfrak{p})$ by the equivalence relation $\sim$. Here $(g, x) \sim\left(g^{\prime}, x^{\prime}\right)$ if $g^{\prime}=g p$ and $x^{\prime}=A d_{p^{-1}}(x)$ for some $p \in P$. The Springer map

$$
\mu: T^{*}(G / P) \rightarrow \overline{\mathcal{O}}
$$

is defined as $\mu([g, x])=A d_{g}(x)$. The Springer map $\mu$ is a generically finite surjective proper map. When $\operatorname{deg} \mu=1$, it is called a Springer resolution. For a nilpotent orbit $\mathcal{O}_{x} \subset \overline{\mathcal{O}}$, we call $\mathcal{O}_{x}$ is $\mu$-relevant if

$$
\operatorname{dim} \mu^{-1}(x)=\operatorname{codim}\left(\mathcal{O}_{x} \subset \overline{\mathcal{O}}\right) / 2
$$

From now on, we assume that $\mathfrak{g}$ is a simple Lie algebra. For the Springer resolution $\pi$ for a Borel subgroup $B$, every nilpotent orbit is $\pi$-relevant. However, this is not the case for a general parabolic subgroup $P$. The $\mu$-relevancy is closely related to Springer's correspondence. In order to state the result, we shall prepare some terminology. Let $L$ be a Levi subgroup of $P$. Fix a maximal torus $T$ of $L$. Then $T$ is also a maximal torus of $G$. Let $W(L)$ be the Weyl group for $L$ relative to $T$ and let $W$ be the Weyl group for $G$ relative to $T$. Now we have a natural inclusion $W(L) \subset W$. Let $\epsilon_{W(L)}$ be the sign representation of $W(L)$. Denote by $\epsilon_{W(L)}^{W}$ the induced representation of $\epsilon_{W(L)}$ to $W$. By Theorem 3.1, every irreducible representation of $W$ has the form $V_{(x, \phi)}$ for a $\pi$-relevant pair $(x, \phi)$. Recall that $\phi$ is an irreducible representation of $\pi_{1}\left(\mathcal{O}_{x}\right)$. Denote by 1 the trivial representation. Then $(x, 1)$ is a $\pi$-relevant pair (cf. [B-M, Lemma 1.2]).

Proposition 4.3. A nilpotent orbit $\mathcal{O}_{x} \subset \overline{\mathcal{O}}$ is $\mu$-relevant if and only if $V_{(x, 1)}$ occurs in $\epsilon_{W(L)}^{W}$.

Proof. See [B-M, Collorary 3.5, (b)].
In the remainder of this section we shall review some results on Richardson orbits and polarizations when $\mathfrak{g}$ is a complex classical Lie algebra. Let $x \in \mathfrak{g}$ be a nilpotent element and denote by $\operatorname{Pol}(x)$ the set of polarizations of $x$.

Theorem 4.4. Let $x \in \mathfrak{s l}(n)$ be a nilpotent element. Then $\operatorname{Pol}(x) \neq$ Ø. Assume that $x$ is of type $\mathbf{d}=\left[d_{1}, \ldots, d_{k}\right]$. Then $P \in \operatorname{Pol}(x)$ has the flag type $\left(p_{1}, \ldots, p_{s}\right)$ such that $\operatorname{ord}\left(p_{1}, \ldots, p_{s}\right)=\mathbf{d}$. Conversely, for any sequence $\left(p_{1}, \ldots, p_{s}\right)$ with $\operatorname{ord}\left(p_{1}, \ldots, p_{s}\right)=\mathbf{d}$, there is a unique polarization $P \in \operatorname{Pol}(x)$ which has the flag type $\left(p_{1}, \ldots, p_{s}\right)$.

Proof. We shall construct a flag $F$ of type $\left(p_{1}, \ldots, p_{s}\right)$ such that $x F_{i} \subset F_{i-1}$ for all $i$. We identify the partition $\mathbf{d}$ with a Young table
consisting of $n$ boxes, where the $i$-th row consists of $d_{i}$ boxes for each $i$. We denote by $(i, j)$ the box of $\mathbf{d}$ lying on the $i$-th row and on the $j$-th column. Let $e(i, j),(i, j) \in \mathbf{d}$ be a Jordan basis of $V:=\mathbf{C}^{n}$ such that $x e(i, j)=e(i-1, j)$. We consruct a flag by the induction on $n$. Define first $F_{1}:=\Sigma_{1 \leq j \leq p_{1}} \mathbf{C e}(1, j)$. Then $x$ induces a nilpotent endomorphism $\bar{x}$ of $V / F_{1}$. The Jordan type of $\bar{x}$ is $\left[d_{1}-1, \ldots, d_{p_{1}}-1, d_{p_{1}+1}, \ldots, d_{k}\right]$. Note that this coincides with $\operatorname{ord}\left(p_{2}, \ldots, p_{k}\right)$. By the induction hypothesis, we already have a flag of type $\left(p_{2}, \ldots, p_{k}\right)$ on $V / F_{1}$ stabilized by $\bar{x}$; hence we have a desired flag $F$. Let $P$ be the stabilizer group of $F$. Then it is clear that $x \in \mathfrak{n}(P)$. By an explicit calculation $\operatorname{dim} \mathcal{O}_{x}=2 \operatorname{dim} G / P$. Q.E.D.

Next consider simple Lie algebras of type $B, C$ or $D$. Let $V$ be an $n$ dimensional $\mathbf{C}$-vector space with a non-degenerate symmetric (skewsymmetric) form. As in Example $4.2, \epsilon=0$ when this form is symmetric and $\epsilon=1$ when this form is skew-symmetric. Let $P_{\epsilon}(n)$ be the set of partitions d of $n$ such that $\sharp\left\{i ; d_{i}=m\right\}$ is even for every integer $m$ with $m \equiv \epsilon(\bmod 2)$. Note that these partitions are nothing but those which appear as the Jordan types of nilpotent elements of $\mathfrak{s o}(n)$ or of $\mathfrak{s p}(n)$. Next, let $q$ be a non-negative integer and assume moreover that $q \neq 2$ when $\epsilon=0$. We define $\operatorname{Pai}(n, q)$ to be the set of partitions $\pi$ of $n$ such that $\pi_{i} \equiv 1(\bmod 2)$ if $i \leq q$ and $\pi_{i} \equiv 0(\bmod 2)$ if $i>q$. Note that, if $\left(p_{1}, \ldots, p_{k}, q, p_{k}, \ldots, p_{1}\right)$ is the type of an admissible flag of $V$, then $\operatorname{ord}\left(p_{1}, \ldots, p_{k}, q, p_{k}, \ldots, p_{1}\right) \in \operatorname{Pai}(n, q)$. Now we shall define the Spaltenstein map $S$ from $\operatorname{Pai}(n, q)$ to $P_{\epsilon}(n)$. For $\pi \in \operatorname{Pai}(n, q)$, let

$$
I(\pi):=\left\{j \in \mathbf{N} \mid j \not \equiv n(\bmod 2), \pi_{j} \equiv \epsilon(\bmod 2), \pi_{j} \geq \pi_{j+1}+2\right\}
$$

Then the Spaltenstein map (cf. [He])

$$
S: \operatorname{Pai}(n, q) \rightarrow P_{\epsilon}(n)
$$

is defined as

$$
S(\pi)_{j}:=\left\{\begin{aligned}
\pi_{j}-1 & (j \in I(\pi)) \\
\pi_{j}+1 & (j-1 \in I(\pi)) \\
\pi_{j} & (\text { otherwise })
\end{aligned}\right.
$$

Theorem 4.5. Let $G$ be $S O(V)$ or $S p(V)$ according as $\epsilon=0$ or $\epsilon=$ 1. Let $x \in \mathfrak{g}$ be a nilpotent element of type $\mathbf{d} \in P_{\epsilon}(n)$. For $\pi \in \operatorname{Pai}(n, q)$ , define $\operatorname{Pol}(x, \pi)$ to be the set of polarizations of $x$ with Levi type $\pi$ (cf. Example 4.2). Then $\operatorname{Pol}(x, \pi) \neq \emptyset$ if and only if $S(\pi)=\mathbf{d}$.

Proof. The proof of this theorem can be found in [He], Theorem 7.1, (a). But we prove here that $\operatorname{Pol}(x, \pi) \neq \emptyset$ if $S(\pi)=\mathbf{d}$ because we will
later use this argument. There is a basis $\{e(i, j)\}$ of $V$ indexed by the Young diagram d with the following properties (cf. [S-S], p.259, see also [C-M], 5.1.)
(i) $\{e(i, j)\}$ is a Jordan basis of $x$, that is, $x e(i, j)=e(i-1, j)$ for $(i, j) \in \mathbf{d}$.
(ii) $<e(i, j), e(p, q)>\neq 0$ if and only if $p=d_{j}-i+1$ and $q=$ $\beta(j)$, where $\beta$ is a permutation of $\left\{1,2, \ldots, d^{1}\right\}$ which satiesfies: $\beta^{2}=i d$, $d_{\beta(j)}=d_{j}$, and $\beta(j) \not \equiv j(\bmod 2)$ if $d_{j} \not \equiv \epsilon(\bmod 2)$. One can choose an arbitrary $\beta$ within these restrictions.

For a sequence $\left(p_{1}, \ldots, p_{s}\right)$ with $\pi=\operatorname{ord}\left(p_{1}, \ldots, p_{s}\right)$ and $p_{i}=p_{s+1-i}$, $(1 \leq i \leq s)$, we shall construct an admissible flag $F$ of type $\left(p_{1}, \ldots, p_{s}\right)$ such that $x F_{i} \subset F_{i-1}$ for all $i$. We proceed by the induction on $s$. When $s=1, \pi=\left[1^{n}\right]$ and $\pi=\mathbf{d}$. In this case, $x=0$ and $F$ is a trivial flag $F_{1}=$ $V$. When $s>1$, we shall construct an isotropic flag $0 \subset F_{1} \subset F_{s-1} \subset V$. Put $p:=p_{1}\left(=p_{s}\right)$ and let $\rho:=\operatorname{ord}\left(p_{2}, \ldots, p_{s-1}\right) \in \operatorname{Pai}(n-2 p, q)$. Then we have

$$
\rho_{j}:=\left\{\begin{aligned}
\pi_{j}-2 & (j \leq p) \\
\pi_{j} & (j>p)
\end{aligned}\right.
$$

Let

$$
S^{\prime}: \operatorname{Pai}(n-2 p, q) \rightarrow P_{\epsilon}(n-2 p)
$$

be the Spatenstein map and we put $\mu:=S^{\prime}(\rho)$. There are two cases (A) and (B). The first case (A) is when $i(\pi)=\{p\} \cup I(\rho)$ and $p \notin I(\rho)$. In this case, $p \not \equiv n(\bmod 2), \pi_{p} \equiv \epsilon(\bmod 2)$ and $\pi_{p}=\pi_{p+1}+2$. Now we have

$$
\begin{gathered}
\mu_{j}=d_{j}-2,(j<p), \\
\mu_{p}=d_{p}-1 \\
\mu_{p+1}=d_{p+1}-1 \\
\mu_{j}=d_{j},(j>p+1),
\end{gathered}
$$

where $d_{p}=d_{p+1}$. The second case is exactly when (A) does not occur. In this case, $I(\pi)=I(\rho)$ and

$$
\begin{gathered}
\mu_{j}=d_{j}-2,(j \leq p) \\
\mu_{j}=d_{j},(j>p)
\end{gathered}
$$

Let us assume that the case (A) occurs. We choose the basis $e(i, j)$ of $V$ in such a way that the permutaion $\beta$ satisfies $\beta(p)=p+1$. There are two choices for $F_{1}$. The first one is to put

$$
F_{1}=\Sigma_{1 \leq j \leq p} \mathbf{C} e(1, j)
$$

The second one is to put

$$
F_{1}=\Sigma_{1 \leq j \leq p+1, j \neq p} \mathbf{C e}(1, j) .
$$

In any case, we put $F_{s-1}=F_{1}^{\perp}$. Then $x$ induces a nilpotent endomorphism of $F_{s-1} / F_{1}$ of type $\mu$. Next assume that the case (B) occurs. In this case, we put

$$
F_{1}=\Sigma_{1 \leq j \leq p} \mathbf{C} e(1, j)
$$

and $F_{s-1}=F_{1}^{\perp}$. Then $x$ induces a nilpotent endomorphism of $F_{s-1} / F_{1}$ of type $\mu$. By the induction on $s$, we have an admissible filtration $0 \subset$ $F_{1} \subset \ldots \subset F_{s-1} \subset V$ with desired properties. Let $P$ be the stabilizer group of $F$. Then it is clear that $x \in \mathfrak{n}(P)$. By an explicit calculation $\operatorname{dim} \mathcal{O}_{x}=2 \operatorname{dim} G / P$.

Theorem 4.6. Let $G$ and $\mathfrak{g}$ be the same as Theorem 4.5. Let $x \in \mathfrak{g}$ be a nilpotent element of type $\mathbf{d}$ and denote by $\mathcal{O}$ the orbit containing $x$. Assume that $P$ is a polarization of $x$ with Levi type $\pi$. Let

$$
\mu: T^{*}(G / P) \rightarrow \overline{\mathcal{O}}
$$

be the Springer map. Then

$$
\operatorname{deg}(\mu):=\left\{\begin{aligned}
2^{\sharp I(\pi)-1} & \left(q=\epsilon=0, \pi^{i} \not \equiv 0(\bmod 2) \exists i\right) \\
2^{\sharp I(\pi)} & \left(q+\epsilon \geq 1 \text { or } q=\epsilon=0, \pi^{i} \equiv 0(\bmod 2) \forall i\right)
\end{aligned}\right.
$$

Moreover, if $\operatorname{deg}(\mu)=1$, then the Levi type of $P$ is unique. In other words, if two polarizations of $x$ respectively give Springer resolutions of $\overline{\mathcal{O}}$, then they have the same Levi type.

Proof. The first part is [He], Theorem 7.1, (d) (cf. [He], §1). The proof of the second part is rather technical, but for the completeness, we include it here. Let

$$
B(\mathbf{d})=\left\{j \in \mathbf{N} ; d_{j}>d_{j+1}, d_{j} \not \equiv \epsilon(\bmod 2)\right\} .
$$

Note that $S(\pi)=\mathbf{d}$, where $S$ is the Spaltenstein map. When $\epsilon=0$, $B(\mathbf{d})=\emptyset$ if and only if $q=0$ and $d^{i} \equiv 0(\bmod 2)$ for all $i$. Assume that $B(\mathbf{d})=\emptyset$. Since $\operatorname{deg}(\mu)=1$, by the first part of our theorem, $\sharp I(\pi)=0$. Then $\pi=\mathbf{d}$. Assume that $B(\mathbf{d}) \neq \emptyset$. If $q \neq 0$ for our $\pi$ or $\epsilon=1$, then $\sharp I(\pi)=0$; hence $\pi=\mathbf{d}$. If $\epsilon=0$ and $q=0$ for $\pi$, then $\sharp I(\pi)=1$. Since $\sharp I(\pi)=1 / 2 \sharp\left\{j ; d_{j} \equiv 1(\bmod 2)\right\}$ by [He], Lemma 6.3, (b). This implies that $\sharp\left\{j ; d_{j} \equiv 1(\bmod 2)\right\}=2$. Note that $\pi$ with $q=0$ is uniquely determined by d because the Spaltenstein map is injective ([He], Prop. 6.5, (a)).

Now let us prove the second part of our theorem. When $\epsilon=1$, we should have $\pi=\mathbf{d}$ by the argument above. Next consider the case where $\epsilon=0$. Assume that there exist two polarizations $P_{1}$ and $P_{2}$ giving Springer resolutions. Let $\pi_{1}$ and $\pi_{2}$ be their Levi types. Assume that $\pi_{1} \in \operatorname{Pai}(n, 0)$ and $\pi_{2} \in \operatorname{Pai}\left(n, q_{2}\right)$ with $q_{2}>0$. By the argument above, we see that $\sharp\left\{j ; d_{j} \equiv 1(\bmod 2)\right\}=2$. On the other hand, since $q_{2}>0$, $\pi_{2}=\mathbf{d}$. This shows that $q_{2}=2$; but, when $\epsilon=0, q_{2} \neq 2$ by Example 4.2, which is a contradiction. Hence, in this case, $\pi$ is also uniquely determined by $\mathbf{d}$.

## §5. Equivalence relation in the set of parabolic subgroups

Proposition 5.1. Let $G$ be a complex simple Lie group. Assume that $b_{2}(G / P)=1$. Then the following are equivalent.
(i) $\operatorname{deg} \mu=1$ and $\operatorname{Codim}(\operatorname{Exc}(\mu)) \geq 2$,
(ii) The single marked Dynkin diagram associated with $P$ is one of the following:
$A_{n-1}(k<n / 2)$

$D_{n}(n:$ odd $\geq 4)$


$E_{6, I}:$


$E_{6, I I}$ :


Remark 5.2. In (ii) there are exactly two different markings for each Dynkin diagram $A_{n-1}$ with $k<n / 2, D_{n}, E_{6, I}$ or $E_{6, I I}$. They are called dual marked Dynkin diagrams. Let $P$ and $P^{\prime}$ be the corresponding (conjugacy classes of) parabolic subgroups of $G$. Then $\mathfrak{p}$ and $\mathfrak{p}^{\prime}$ have conjugate Levi factors by Proposition 6.3 of $[B-C]$. This implies that $P$ and $P^{\prime}$ have the same Richardson orbit.

Proof of Proposition 5.1. Assume that the single marked Dynkin diagram is one of first two series in (ii). For this case we shall prove in Lemmas 5.4 and 5.6, that the Springer map $\mu: T^{*}(G / P) \rightarrow \overline{\mathcal{O}}$ becomes a small resolution (cf. Notation (2)). If the single marked diagram is of type $E_{6, I}$, then the Richardson orbit $\mathcal{O}$ of $P$ coincides with orbit $\mathcal{O}_{2 A_{1}}$ in the list of $[\mathrm{C}-\mathrm{M}], \mathrm{p} .129$, which has dimension 32. The maximal orbit contained in $\overline{\mathcal{O}}_{2 A_{1}}-\mathcal{O}_{2 A_{1}}$ is $\mathcal{O}_{A_{1}}$, which has dimension 22 . This shows that $\operatorname{Sing}(\overline{\mathcal{O}})$ has codimension $\geq 10$ in $\overline{\mathcal{O}}$. On the other hand, since $\pi_{1}\left(\mathcal{O}_{2 A_{1}}\right)=1$ (cf. [C-M], p.129), $\operatorname{deg}(\mu)=1$. If $\mu$ is a divisorial birational contraction, then $\operatorname{Codim}(\operatorname{Sing}(\overline{\mathcal{O}}) \subset \overline{\mathcal{O}})=2$ (cf. [Na 1, Cor. 1.5]), which is absurd. Hence $\mu$ should be a small resolution. If the single marked diagram is of type $E_{6, I I}$, then the Richardson orbit $\mathcal{O}$ of $P$ coincides with the orbit $\mathcal{O}_{A_{2}+2 A_{1}}$ in the list of [C-M], p.129, which has dimension 50. Moreover, $\pi_{1}\left(\mathcal{O}_{A_{2}+2 A_{1}}\right)=1$. By looking at the closure ordering of $E_{6}$ orbits (cf. [C], p.441), we see that the maximal orbit contained in $\overline{\mathcal{O}}_{A_{2}+2 A_{1}}-\mathcal{O}_{A_{2}+2 A_{1}}$ is the orbit $\mathcal{O}_{A_{2}+A_{1}}$, which has dimension 46. By the same argument as above, $\mu$ becomes a small resolution.

To prove the implication (i) $\Rightarrow$ (ii), let us assume that the single marked Dynkin diagram is not contained in the list of (ii). Let $\mathcal{O}$ be the corresponding Richardson orbit. We shall first prove that $\overline{\mathcal{O}}$ contains a nilpotent orbit $\mathcal{O}^{\prime}$ of codimension 2 (STEP 1). Next we shall prove that $\mathcal{O}^{\prime}$ is $\mu$-relevant(STEP 2). These imply that $\mu$ is a divisorial birational contraction map if $\operatorname{deg}(\mu)=1$.

STEP 1: Assume that $\mathfrak{g}$ is classical. If $\mathfrak{g}$ is of type $A_{n-1}$, then we must look at the single marked Dynkin diagram with $k=n / 2$. In this case, we will see in Remark 5.5 that $\mu$ is a divisorial birational contraction map.

When $\mathfrak{g}$ is of type $B_{n}, C_{n}$ or $D_{n}$, the parabolic subgroup $P$ is a stabilizer group of an admissible isotropic flag. Its flag type is written as $(k, q, k)$. When $\mathfrak{g}$ is of type $B_{n}$, we have $k>0, q>0$ and $2 k+q=2 n+1$. When $\mathfrak{g}$ is of type $C_{n}$ or of type $D_{n}$, we have $k>0, q \geq 0$ and $2 k+q=2 n$. Denote by $\pi$ the dual partition of $\operatorname{ord}(k, q, k)$ and call $\pi$ the Levi type of $P$.

Assume that $\mathfrak{g}$ is of type $B_{n}$. The Levi type of $P$ is given by

$$
\pi:=\left\{\begin{aligned}
{\left[3^{2 n+1-2 k}, 2^{3 k-2 n-1}\right] } & (k>(2 n+1) / 3) \\
{\left[3^{k}, 1^{2 n-3 k+1}\right] } & (k \leq(2 n+1) / 3)
\end{aligned}\right.
$$

When $k>(2 n+1) / 3, k$ must be an odd number. In fact, if $k$ is even, then $I(\pi) \neq \emptyset$ and $\operatorname{deg}(\mu)>1$ (cf. Theorem 4.6). Recall that the Richardson orbit $\mathcal{O}$ of $P$ has the Jordan type $S(\pi)$, where $S$ is the Spaltenstein map (cf. Theorem 4.5). Since now $I(\pi)=\emptyset$, $S(\pi)=\pi$. Let us consider the nilpotent orbit $\mathcal{O}^{\prime}$ of the Jordan type $\left[3^{2 n+1-2 k}, 2^{3 k-2 n-3}, 1^{4}\right]$ (resp. $\left[3^{k-1}, 2^{2}, 1^{2 n-3 k}\right],\left[3^{k-1}, 1^{3}\right]$ ) when $k>$ $(2 n+1) / 3$ (resp. $k<(2 n+1) / 3, k=(2 n+1) / 3)$. In any case, we have $\mathcal{O}^{\prime} \subset \overline{\mathcal{O}}$. By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that $\operatorname{dim} \mathcal{O}^{\prime}=\operatorname{dim} \mathcal{O}-2$.

Assume that $\mathfrak{g}$ is of type $C_{n}$. The Levi type of $P$ is given by

$$
\pi:=\left\{\begin{aligned}
{\left[3^{2 n-2 k}, 2^{3 k-2 n}\right] } & (k>2 n / 3) \\
{\left[3^{k}, 1^{2 n-3 k}\right] } & (k \leq 2 n / 3)
\end{aligned}\right.
$$

When $k \leq 2 n / 3, k$ must be an even number. In fact, if $k$ is odd, then $I(\pi) \neq \emptyset$ and $\operatorname{deg}(\mu)>1$ (cf. Theorem 4.6). The Richardson orbit $\mathcal{O}$ has the Jordan type $\pi$. Let us consider the nilpotent orbit $\mathcal{O}^{\prime}$ of the Jordan type $\left[3^{2 n-2 k}, 2^{3 k-2 n-1}, 1^{2}\right]$ (resp. $\left[3^{k-2}, 2^{4}, 1^{2 n-3 k-2}\right],\left[3^{k-2}, 2^{3}\right]$ ) when $k>2 n / 3$ (resp. $k<2 n / 3, k=2 n / 3$ ). In any case, we have $\mathcal{O}^{\prime} \subset \overline{\mathcal{O}}$. By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that $\operatorname{dim} \mathcal{O}^{\prime}=\operatorname{dim} \mathcal{O}-2$.

Assume that $\mathfrak{g}$ is of type $D_{n}$. First assume that the Levi type of $P$ is $\left[2^{k}\right]$. The single marked Dynkin diagram is not contained in the list of (ii) exactly when $k$ is even. In this case, we will see in Remark 5.7 that $\mu$ is a divisorial birational contraction map. We next assume $k<n$. In this case, the Levi type of $P$ is given by

$$
\pi:=\left\{\begin{aligned}
{\left[3^{2 n-2 k}, 2^{3 k-2 n}\right] } & (n>k>2 n / 3) \\
{\left[3^{k}, 1^{2 n-3 k}\right] } & (k \leq 2 n / 3)
\end{aligned}\right.
$$

When $k>2 n / 3, k$ must be an even number. In fact, if $k$ is odd, then $I(\pi) \neq \emptyset$ and $\operatorname{deg}(\mu)>1$ (cf. Theorem 4.6). Recall that the Richardson orbit $\mathcal{O}$ of $P$ has the Jordan type $S(\pi)$, where $S$ is the Spaltenstein map
(cf. Theorem 4.5). Since now $I(\pi)=\emptyset, S(\pi)=\pi$. Let us consider the nilpotent orbit $\mathcal{O}^{\prime}$ of the Jordan type $\left[3^{2 n-2 k}, 2^{3 k-2 n-2}, 1^{4}\right]$ (resp. $\left[3^{k-1}, 2^{2}, 1^{2 n-3 k-1}\right],\left[3^{k-1}, 1^{3}\right]$ ) when $k>2 n / 3$ (resp. $k<2 n / 3, k=$ $2 n / 3)$. In any case, we have $\mathcal{O}^{\prime} \subset \overline{\mathcal{O}}$. By the dimension formula of nilpotent orbits ([C-M, Corollary 6.1.4]), we see that $\operatorname{dim} \mathcal{O}^{\prime}=\operatorname{dim} \mathcal{O}-2$.

When $\mathfrak{g}$ is of type $G_{2}$, there are exactly two single marked Dynkin diagrams. In the table of $G_{2}$ nilpotent orbits in [C-M, p.128], $\mathcal{O}_{G_{2}\left(a_{1}\right)}$ is the Richardson orbit of the parabolic subgroups corresonding to these diagrams. The orbit $\mathcal{O}_{\tilde{A}_{1}}$ is contained in $\overline{\mathcal{O}}_{G_{2}\left(a_{1}\right)}$. Note that $\operatorname{dim} \mathcal{O}_{G_{2}\left(a_{1}\right)}=$ 10 and $\operatorname{dim} \mathcal{O}_{\tilde{A}_{1}}=8$.

When $\mathfrak{g}$ is of type $F_{4}$, there are exactly four single marked Dynkin diagrams. Richardson orbits of the parabolic subgroups corresponding to them are $\mathcal{O}_{A_{2}}, \mathcal{O}_{\tilde{A}_{2}}, \mathcal{O}_{F_{4}\left(a_{3}\right)}$ in the table of [C-M, p.128]. Note that two non-conjugate parabolic subgroups have the same Richardson orbit $\mathcal{O}_{F_{4}\left(a_{3}\right)}$. By looking at the closure ordering of $F_{4}$ orbits [C, p.440], we see that the closure of each orbit contain a codimension 2 orbit.

When $\mathfrak{g}$ is of type $E_{6}$, there are exactly 6 single marked Dynkin diagrams. Four of them are already contained in the list of (ii). The Richardson orbits corresponding to other diagrams are $\mathcal{O}_{A_{2}}$ and $\mathcal{O}_{D_{4}\left(a_{1}\right)}$ in the list of $E_{6}$ nilpotent orbits in [C-M, p.129]. $\overline{\mathcal{O}}_{A_{2}}$ contains a codimension 2 orbit $\mathcal{O}_{3 A_{1}}$. $\overline{\mathcal{O}}_{D_{4}\left(a_{1}\right)}$ contains a codimension 2 orbit $\mathcal{O}_{A_{3}+A_{1}}$.

When $\mathfrak{g}$ is of type $E_{7}$, there are exactly 7 single marked Dynkin diagrams. Richardson orbits of the parabolic subgroups corresponding to them are $\mathcal{O}_{\left(3 A_{1}\right)^{\prime \prime}}, \mathcal{O}_{A_{2}}, \mathcal{O}_{2 A_{2}}, \mathcal{O}_{A_{2}+3 A_{1}}, \mathcal{O}_{D_{4}\left(a_{1}\right)}, \mathcal{O}_{A_{3}+A_{2}+A_{1}}$ and $\mathcal{O}_{A_{4}+A_{2}}$ in the table of [C-M, p.130-p.131]. By looking at the closure ordering of $E_{7}$ orbits [C, p.442], we see that the closure of each orbit contains a codimension 2 orbit.

When $\mathfrak{g}$ is of type $E_{8}$, there are exactly 8 single marked Dynkin diagrams. In the table of [C-M, p.132-p.134], Richardson orbits of the parabolic subgroups corresponding to them are $\mathcal{O}_{A_{2}}, \mathcal{O}_{2 A_{2}}, \mathcal{O}_{D_{4}\left(a_{1}\right)}$, $\mathcal{O}_{D_{4}\left(a_{1}\right)+A_{2}}, \mathcal{O}_{A_{4}+A_{2}}, \mathcal{O}_{A_{4}+A_{2}+A_{1}}, \mathcal{O}_{E_{8}\left(a_{7}\right)}$ and $\mathcal{O}_{A_{6}+A_{1}}$. By looking at the closure ordering of $E_{8}$ orbits, we see that the closure of each orbit contains a codimension 2 orbit.

STEP 2: Assume that $\mathfrak{g}$ is classical. Let $f: \tilde{\mathcal{O}} \rightarrow \overline{\mathcal{O}}$ be the normalization map. By STEP 1 we may assume that $\overline{\mathcal{O}}$ contains a codimension 2 orbit $\mathcal{O}^{\prime}$. In the classical case, by [K-P, 14], we see that $\tilde{\mathcal{O}}$ has actually singularities along $f^{-1}\left(\mathcal{O}^{\prime}\right)$. The Springer map $\mu$ is factorized as

$$
T^{*}(G / P) \xrightarrow{\mu^{\prime}} \tilde{\mathcal{O}} \xrightarrow{f} \overline{\mathcal{O}} .
$$

If $\operatorname{deg}(\mu)=1$, then $\mu^{\prime}$ is a birational maps of normal varieties. Then, by Zariski's main theorem, $\mu^{\prime}$ must have a positive dimensional fiber over a point of $f^{-1}\left(\mathcal{O}^{\prime}\right)$. This implies that $\mu$ is a divisorial birational map.

Assume that $\mathfrak{g}$ is of exceptional type. As explained above, the codimension 2 orbit $\mathcal{O}^{\prime}$ of $\overline{\mathcal{O}}$ can be specified. It is enough to show that $\mathcal{O}^{\prime}$ is $\mu$-relevant. By the previous proposition, we have to check that $V_{(x, 1)}$ occurs in $\epsilon_{W(L)}^{W}$ for $x \in \mathcal{O}^{\prime}$. In [Al], Alvis describes an irreducible decomposition of the induced representation $\operatorname{Ind}_{W(L)}^{W}(\rho)$ for any irreducible representation $\rho$ of $W(L)$. Hence, this can be done by using the tables of [Al] (see also the tables in [A-L], [B-L] and [C, 13.3]). Note that Spaltenstein [S1] (cf. the footnote of p.68, [B-M]) has already checked that a special orbit is $\mu$-relevant by using these tables. Hence it is enough to check for non-special orbits $\mathcal{O}^{\prime}$. One can find which orbits are nonspecial in the tables of [C-M, 8.4]. Q.E.D.

Example 5.3. (Mukai flops): Let $P$ and $P^{\prime}$ be two parabolic subgroups of $G$ which correspond to dual marked Dynkin diagrams in the proposition above. Let $\mathcal{O}$ be the Richardson orbit of them. Then we have a diagram

$$
T^{*}(G / P) \xrightarrow{\mu} \overline{\mathcal{O}} \stackrel{\mu^{\prime}}{\leftarrow} T^{*}\left(G / P^{\prime}\right)
$$

The birational maps $\mu$ and $\mu^{\prime}$ are both small by the proposition, Lemmas 5.4 and 5.6. Moreover, $T^{*}(G / P)--\rightarrow T^{*}\left(G / P^{\prime}\right)$ is not an isomorphism. In fact, $T^{*}(G / P), T^{*}\left(G / P^{\prime}\right)$ and $\overline{\mathcal{O}}$ all have $G$ actions, and $\mu$ and $\mu^{\prime}$ are $G$-equivariant. If the birational map is an isomorphism, this would become a $G$-equivariant isomorphism. This implies that $G / P$ and $G / P^{\prime}$ are isomorphic as $G$-varieties. In particular, $P$ and $P^{\prime}$ are $G$-conjugate, which is absurd. Since the relative Picard numbers $\rho\left(T^{*}(G / P) / \overline{\mathcal{O}}\right)$ and $\rho\left(T^{*}\left(G / P^{\prime}\right) / \overline{\mathcal{O}}\right)$ equal 1 , we see that the diagram above is a flop. The diagram is called a Mukai flop of type $A_{n-1, k}$ (resp. $D_{n}, E_{6, I}, E_{6, I I}$ ) according to the type of the corresponding marked Dynkin diagram.

We shall describe Mukai flops of type A and D in terms of flags.
Mukai flop of type A. Let $x \in \mathfrak{s l}(n)$ be a nilpotent element of type $\left[2^{k}, 1^{n-2 k}\right]$ and let $\mathcal{O}$ be the nilpotent orbit containing $x$. By Theorem 4.4 , there are two polarizations $P$ and $P^{\prime}$ of $x$, where $P$ has the flag type $(k, n-k)$ and $P^{\prime}$ has the flag type $(n-k, k)$. The closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ admits two Springer resolutions

$$
T^{*}(S L(n) / P) \xrightarrow{\pi} \overline{\mathcal{O}} \stackrel{\pi^{\prime}}{\leftarrow} T^{*}\left(S L(n) / P^{\prime}\right)
$$

Note that $S L(n) / P$ is isomorphic to the Grassmannian $G(k, n)$ and $S L(n) / P^{\prime}$ is isomorphic to $G(n-k, n)$.

Lemma 5.4. When $k<n / 2, \pi$ and $\pi^{\prime}$ are both small birational maps and the diagram becomes a flop.

Proof. The closure $\overline{\mathcal{O}}$ consists of finite number of orbits $\left\{\mathcal{O}_{\left[2^{i}, 1^{n-2 i}\right]}\right\}_{0 \leq i \leq k}$. The main orbit $\mathcal{O}_{\left[2^{k}, 1^{n-2 k}\right]}$ is an open set of $\overline{\mathcal{O}}$. A fiber of $\pi$ (resp. $\pi^{\prime}$ ) over a point of $\mathcal{O}_{\left[2^{i}, 1^{n-2 i}\right]}$ is isomorphic to the Grassmannian $G(k-i, n-2 i)$ (resp. $G(n-i-k, n-2 i)$ ). By a simple dimension count, if $k<n / 2$, then $\pi$ and $\pi^{\prime}$ are both small birational maps. Next let us prove that the diagram is a flop. This is already proved in Example 5.3. But, we shall give here a more explicit proof. Let $\tau \subset \mathcal{O}_{G(k, n)}^{\oplus n}$ (resp. $\left.\tau^{\prime} \subset \mathcal{O}_{G(n-k, n)}^{\oplus n}\right)$ be the universal subbundle. Denote by $T$ (resp. $T^{\prime}$ ) the pull-back of $\tau$ (resp. $\tau^{\prime}$ ) by the projection $T^{*} G(k, n) \rightarrow G(k, n)$ (resp. $\left.T^{*} G(n-k, n) \rightarrow G(n-k, n)\right)$. We shall describe the strict transform of $\wedge^{k} T$ by the birational map $T^{*} G(k, n)-$ $-\rightarrow T^{*} G(n-k, n)$. Take a point $y \in \mathcal{O}_{\left[2^{k}, 1^{n-2 k}\right]}$. Note that $T^{*} G(k, n)$ is naturally embedded in $G(k, n) \times \overline{\mathcal{O}}$. Then the fiber $\pi^{-1}(y)$ consists of one point $([\operatorname{Im}(y)], y) \in G(k, n) \times \overline{\mathcal{O}}$. The fiber $T_{\pi^{-1}(y)}$ of the vector bundle $T$ over $\pi^{-1}(y)$ coincides with the vector space $\operatorname{Im}(y)$. Hence $\left(\wedge^{k} T\right)_{\pi^{-1}(y)}$ is isomorphic to $\wedge^{k} \operatorname{Im}(y)$. Now let $L$ be the strict transform of $\wedge^{k} T$ by $T^{*} G(k, n)--\rightarrow T^{*} G(n-k, n)$. First note that $\left(\pi^{\prime}\right)^{-1}(y)$ also consists of one point $([\operatorname{Ker}(y)], y) \in G(n-k, n) \times \overline{\mathcal{O}}$. Then, by definition, $L_{\left(\pi^{\prime}\right)^{-1}(y)}=\wedge^{k} \operatorname{Im}(y)$. Since $\wedge^{k} \operatorname{Im}(y) \cong\left(\wedge^{n-k} \operatorname{Ker}(y)\right)^{*}$, we see that $L \cong\left(\wedge^{n-k} T^{\prime}\right)^{-1}$. Now $\wedge^{k} T$ is a negative line bundle. On the other hand, its strict transform $L$ becomes an ample line bundle. This implies that our diagram is a flop. Q.E.D.

Remark 5.5. When $k=n / 2, \pi$ and $\pi^{\prime}$ are both divisorial birational contraction maps. Moreover, two resolutions are isomorphic.

Mukai flop of type $\mathbf{D}$. Assume that $k$ is an odd integer with $k \geq 3$. Let $V$ be a $\mathbf{C}$-vector space of $\operatorname{dim} 2 k$ with a non-degenerate symmetric form $<,>$. Let $x \in \mathfrak{s o}(V)$ be a nilpotent element of type $\left[2^{k-1}, 1^{2}\right]$ and let $\mathcal{O}$ be the nilpotent orbit containing $x$. Let $S: \operatorname{Pai}(2 k, 0) \rightarrow P_{\epsilon}(2 k)$ be the Spaltenstein map, where $\epsilon=0$ in our case. Then, for $\pi:=$ $\left(2^{k}\right) \in \operatorname{Pai}(2 k, 0), S(\pi)=\left[2^{k-1}, 1^{2}\right]$. Let us recall the construction of the stabilized flags by the polarizations of $x$ in the proof of Theorem 4.5. Since $I(\pi)=\{k\}$, the case (A) occurs (cf. the proof of Theorem 4.5); hence there are two choices of the flags. We denote by $P^{+}$the stabilizer subgroup of $S O(V)$ of one flag, and denote by $P^{-}$the stabilizer subgroup of another one. Let $G_{i s o}(k, V)$ be the orthogonal Grassmannian which parametrizes $k$ dimensional isotropic subspaces of $V . G_{i s o}(k, V)$
has two connected components $G^{+}{ }_{i s o}(k, V)$ and $G^{-}{ }_{i s o}(k, V)$. Note that $S O(V) / P^{+} \cong G^{+}{ }_{i s o}(k, V)$ and $S O(V) / P^{-} \cong G^{-}{ }_{i s o}(k, V)$. The closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ admits two Springer resolutions

$$
T^{*}\left(S O(V) / P^{+}\right) \xrightarrow{\pi^{+}} \overline{\mathcal{O}} \stackrel{\pi^{-}}{\leftarrow} T^{*}\left(S O(V) / P^{-}\right)
$$

Lemma 5.6. $\pi^{+}$and $\pi^{-}$are both small birational maps and the diagram becomes a flop.

Proof. The closure $\overline{\mathcal{O}}$ consists of the orbits $\left\{\mathcal{O}_{\left[2^{k-2 i-1}, 1^{4 i+2}\right]}\right\}_{1 \leq i \leq 1 / 2(k-1)}$. The main orbit is an open set of $\overline{\mathcal{O}}$. A fiber of $\pi^{+}$(resp. $\pi^{-}$) over a point of $\mathcal{O}_{\left[2^{k-2 i-1}, 1^{4 i+2}\right]}$ is isomorphic to $G^{+}{ }_{i s o}(2 i+1,4 i+2)$ (resp. $G^{-}{ }_{i s o}(2 i+1,4 i+2)$ ). By dimension counts of each orbit and of each fiber, we see that $\pi^{+}$and $\pi^{-}$are both small birational maps. Next let us prove that the diagram is a flop. This is already proved in Example 5.3. But, we shall give here a more explicit proof. Let $\tau^{+} \subset \mathcal{O}_{G^{+}{ }_{i s o}(k, V)}^{\oplus 2 k}$ (resp. $\left.\tau^{-} \subset \mathcal{O}_{G^{-}{ }_{i s o}(k, V)}^{\oplus 2 k}\right)$ be the universal subbundle. Denote by $T^{+}$(resp. $T^{-}$) the pull-back of $\tau^{+}$ (resp. $\tau^{-}$) by the projection $T^{*}\left(G^{+}{ }_{i s o}(k, V)\right) \rightarrow G^{+}{ }_{i s o}(k, V)$ (resp. $\left.T^{*}\left(G^{-}{ }_{i s o}(k, V)\right) \rightarrow G^{-}{ }_{i s o}(k, V)\right)$. We shall describe the strict transform of $\wedge^{k} T^{-}$by the birational map $T^{*}\left(G^{-}{ }_{i s o}(k, V)\right)--\rightarrow T^{*}\left(G^{+}{ }_{i s o}(k, V)\right)$. Take a point $y \in \mathcal{O}_{\left[2^{k-1}, 1^{2}\right]}$. Let $g \in S O(V)$ be an element such that $g x g^{-1}=y$. Note that $T^{*}\left(G^{+}{ }_{i s o}(k, V)\right)\left(\right.$ resp. $\left.T^{*}\left(G^{-}{ }_{i s o}(k, V)\right)\right)$ is naturally embedded in $G^{+}{ }_{i s o}(k, V) \times \overline{\mathcal{O}}$ (resp.
$\left.G^{-}{ }_{i s o}(k, V) \times \overline{\mathcal{O}}\right)$. Then the fiber $\left(\pi^{+}\right)^{-1}(y)\left(\right.$ resp. $\left.\left(\pi^{-}\right)^{-1}(y)\right)$ consists of one point $\left(\left[F_{y}^{+}\right], y\right) \in G^{+}{ }_{i s o}(k, V) \times \overline{\mathcal{O}}\left(\right.$ resp. $\left.\left(\left[F_{y}^{-}\right], y\right) \in G^{-}{ }_{i s o}(k, V) \times \overline{\mathcal{O}}\right)$ where $F_{y}^{+} \subset V\left(\right.$ resp. $\left.F_{y}^{-} \subset V\right)$ is the flag stabilized by $g P^{+} g^{-1}$ (resp. $g P^{-} g^{-1}$ ). Note that $g P^{+} g^{-1}$ and $g P^{-} g^{-1}$ are both polarizations of $y$. Let us recall the construction of flags in the proof of Theorem 4.5. For $y$ we choose a Jordan basis $\{e(i, j)\}$ of $V$ as in the proof of Theorem 4.5. Since $\mathbf{d}=\left[2^{k-1}, 1^{2}\right], \beta$ is a permutation of $\{1,2, \ldots, k, k+1\}$. But it preserves the subsets $\{1,2, \ldots, k-1\}$ and $\{k, k+1\}$ respectively. We assume that $\beta(k)=k+1$ and $\beta(k+1)=k$. In our situation, the case (A) occurs. There are two choices of the flags:

$$
\Sigma_{1 \leq j \leq k-1} \mathbf{C} e(1, j)+\mathbf{C} e(1, k)
$$

and

$$
\Sigma_{1 \leq j \leq k-1} \mathbf{C} e(1, j)+\mathbf{C} e(1, k+1)
$$

Note that one of these is stabilized by $g P^{+} g^{-1}$ and another one is stabilized by $g P^{-} g^{-1}$. We may assume that

$$
F_{y}^{+}=\Sigma_{1 \leq j \leq k-1} \mathbf{C} e(1, j)+\mathbf{C} e(1, k)
$$

and

$$
F_{y}^{-}=\Sigma_{1 \leq j \leq k-1} \mathbf{C} e(1, j)+\mathbf{C} e(1, k+1) .
$$

Since $\operatorname{Ker}(y)=\Sigma_{1 \leq j \leq k+1} e(1, j)$ and $\operatorname{Im}(y)=\Sigma_{1 \leq j \leq k-1} e(1, j)$, we have two exact sequences

$$
0 \rightarrow \operatorname{Ker}(y) / F_{y}^{+} \rightarrow V / F_{y}^{+} \rightarrow \operatorname{Im}(y) \rightarrow 0
$$

and

$$
0 \rightarrow \operatorname{Im}(y) \rightarrow F_{y}^{-} \rightarrow F_{y}^{-} / \operatorname{Im}(y) \rightarrow 0
$$

Since $F_{y}^{-} / \operatorname{Im}(y) \cong \operatorname{Ker}(y) / F_{y}^{+}$, we conclude that

$$
\wedge^{k} F_{y}^{-} \cong \wedge^{k}\left(V / F_{y}^{+}\right)
$$

Let $L$ be the strict transform of $\wedge^{k} T^{-}$by the birational map $T^{*}\left(G^{-}{ }_{i s o}(k, V)\right)--\rightarrow T^{*}\left(G^{+}{ }_{i s o}(k, V)\right)$. The fiber $T_{\left(\pi^{-}\right)^{-1}(y)}^{-}$of the vector bundle $T^{-}$is isomorphic to the vector space $\wedge^{k} F_{y}^{-}$. Hence, by the definition of $L, L_{\left(\pi^{+}\right)^{-1}(y)}=\wedge^{k} F_{y}^{-}$. By the observation above, we see that $L_{\left(\pi^{+}\right)^{-1}(y)}=\wedge^{k}\left(V / F_{y}^{+}\right)$. This shows that $L \cong\left(\wedge^{k} T^{+}\right)^{-1}$. Now $\wedge^{k} T^{-}$is a negative line bundle. On the other hand, its strict transform $L$ is an ample line bundle. This implies that our diagram is a flop. Q.E.D.

Remark 5.7. When $k$ is an even integer with $k \geq 2$, there are two nilpotent orbits $\mathcal{O}^{+}$and $\mathcal{O}^{-}$with Jordan type $\left[2^{k}\right]$. They have Springer resolutions

$$
T^{*}\left(G^{+}{ }_{i s o}(k, 2 k)\right) \rightarrow \overline{\mathcal{O}}^{+}
$$

and

$$
T^{*}\left(G^{-}{ }_{i s o}(k, 2 k)\right) \rightarrow \overline{\mathcal{O}}^{-}
$$

These resolutions are both divisorial birational contraction maps. When $k=1$, three varieties $T^{*}\left(G^{+}{ }_{\text {iso }}(1,2)\right), T^{*}\left(G^{-}{ }_{\text {iso }}(1,2)\right)$ and $\overline{\mathcal{O}}$ are all isomorphic.

Let us return to the general situation. The following notion will play important roles in the later section.

Definition 1. (i) Let $\mathcal{D}$ be a marked Dynkin diagram with exactly $l$ marked vertices. Choose $l-1$ marked vertices from them. Making the remained one vertex unmarked, we have a new marked Dynkin diagram $\overline{\mathcal{D}}$. This procedure is called a contraction of a marked Dynkin diagram. Next remove from $\mathcal{D}$ these $l-1$ vertices and edges touching these vertices. We then have a (non-connected) diagram; one of its connected component is a single marked Dynkin diagram. Assume that this
single marked Dynkin diagram is one of those listed in Proposition 5.1. Replace this single marked Dynkin diagram by its dual and leave other components untouched. Connecting again removed edges and vertices as before, we obtain a new marked Dynkin diagram $\mathcal{D}^{\prime}$. Note that $\mathcal{D}^{\prime}$ (resp. $\overline{\mathcal{D}})$ has exactly $l$ (resp. $l-1$ ) marked vertices. Now we say that $\mathcal{D}^{\prime}$ is adjacent to $\mathcal{D}$ by means of $\overline{\mathcal{D}}$.
(ii) Two marked Dynkin diagrams $\mathcal{D}$ and $\mathcal{D}^{\prime}$ are called equivalent and are written as $\mathcal{D} \sim \mathcal{D}^{\prime}$ if there is a finite chain of adjacent diagrams connecting $\mathcal{D}$ and $\mathcal{D}^{\prime}$.
(iii) Let $P$ be a parabolic subgroup of $G$ and let $\mathcal{D}_{P}$ be the corresponding marked Dynkin diagram. Two parabolic subgroups $P$ and $P^{\prime}$ of $G$ are called equivalent and are written as $P \sim P^{\prime}$ if $\mathcal{D}_{P} \sim \mathcal{D}_{P^{\prime}}$.

Example 5.8. Let us consider the marked Dynkin diagram
D :

where vertices 2 and 3 are marked. We choose the vertex 3. Making the remained one vertex (= the vertex 2) unmarked, we have a marked Dynkin diagram
$\overline{\mathcal{D}}:$


Now the following marked Dynkin diagram $\mathcal{D}^{\prime}$ is adjacent to $\mathcal{D}$ by $\overline{\mathcal{D}}$.
$\mathcal{D}^{\prime}:$


## §6. Main Theorem

The following is our main theorem. For the notion of a relative ample cone and a relative movable cone, see [Ka 1], where some elementary roles of these cones in birational geometry are discussed.

Theorem 6.1. Let $\mathcal{O} \subset \mathfrak{g}$ be a nilpotent orbit of a complex simple Lie algebra $\mathfrak{g}$. Assume that its closure $\overline{\mathcal{O}}$ has a Springer resolution $\mu_{P_{0}}$ : $T^{*}\left(G / P_{0}\right) \rightarrow \overline{\mathcal{O}}$. Then the following hold.
(i) For a parabolic subgroup $P$ of $G$ such that $P \sim P_{0}, Y_{P}:=$ $T^{*}(G / P)$ gives a symplectic resolution of $\overline{\mathcal{O}}$. Conversely, any symplectic resolution is a Springer resolution of this form.
(ii) The closure $\overline{\operatorname{Amp}}\left(Y_{P} / \overline{\mathcal{O}}\right)$ of the relative ample cone is a simplicial polyhedral cone.
(iii) $\overline{\operatorname{Mov}}\left(Y_{P_{0}} / \overline{\mathcal{O}}\right)=\cup_{P \sim P_{0}} \overline{\operatorname{Amp}}\left(Y_{P} / \overline{\mathcal{O}}\right)$, where $\overline{\operatorname{Mov}}\left(Y_{P_{0}} / \overline{\mathcal{O}}\right)$ is the closure of the relative movable cone of $Y_{P_{0}}$ over $\overline{\mathcal{O}}$.
(iv) A codimension 1 face of $\overline{\operatorname{Amp}}\left(Y_{P} / \overline{\mathcal{O}}\right)$ corresponds to a small birational contraction map when it is a face of another ample cone, and corresponds to a divisorial contraction map when it is not a face of any other ample cone.
(v) $\left\{Y_{P}\right\}_{P \sim P_{0}}$ are connected by Mukai flops of type $A, D, E_{6, I}$ and $E_{6, I I}$.

Remark 6.2. For a classical complex Lie algebra, it is already known which nilpotent orbit closure has a Springer resolution (cf. Theorems 4.5 and 4.6). When $\mathfrak{g}$ is $G_{2}$, there are exactly 2 nilpotent orbits $\mathcal{O}_{G_{2}}$ and $\mathcal{O}_{G_{2}\left(a_{1}\right)}$ whose closures admit Springer resolutions. When $\mathfrak{g}$ is $F_{4}$, such orbits are $\mathcal{O}_{A_{2}}, \mathcal{O}_{\tilde{A}_{2}}, \mathcal{O}_{F_{4}\left(a_{3}\right)}, \mathcal{O}_{B_{3}}, \mathcal{O}_{C_{3}}, \mathcal{O}_{F_{4}\left(a_{2}\right)}, \mathcal{O}_{F_{4}\left(a_{1}\right)}$ and $\mathcal{O}_{F_{4}}$. When $\mathfrak{g}$ is $E_{6}$, such orbits are $\mathcal{O}_{2 A_{1}}, \mathcal{O}_{A_{2}}, \mathcal{O}_{2 A_{2}}, \mathcal{O}_{A_{2}+2 A_{1}}, \mathcal{O}_{A_{3}}$, $\mathcal{O}_{D_{4}\left(a_{1}\right)}, \mathcal{O}_{A_{4}}, \mathcal{O}_{D_{4}}, \mathcal{O}_{A_{4}+A_{1}}, \mathcal{O}_{D_{5}\left(a_{1}\right)}, \mathcal{O}_{E_{6}\left(a_{3}\right)}, \mathcal{O}_{D_{5}}, \mathcal{O}_{E_{6}\left(a_{1}\right)}$, and $\mathcal{O}_{E_{6}}$.

The statement (ii) of Theorem 6.1 follows from the next Lemma.
Lemma 6.3. Let $G$ be a complex simple Lie group and let $P$ be a parabolic subgroup. Let $\hat{\mathcal{O}}$ be the Stein factorization of a Springer map $\mu: Y_{P}:=T^{*}(G / P) \rightarrow \overline{\mathcal{O}}$. Then $\overline{\operatorname{Amp}}\left(Y_{P} / \hat{\mathcal{O}}\right)$ is a simplicial polyhedral cone.

Proof. Let $\mathcal{D}$ be the marked Dynkin diagram corresponding to $P$. Assume that $\mathcal{D}$ has $k$ marked vertices, say, $v_{1}, \ldots, v_{k}$. Then $b_{2}(G / P)=k$. Choose $l$ vertices $v_{i_{1}}, \ldots, v_{i_{l}}, 1 \leq i_{1}<\ldots<i_{l} \leq k$ and let $\mathcal{D}_{i_{1}, \ldots, i_{l}}$ be the marked Dynkin diagram such that exactly these $l$ vertices are marked and its underlying diagram is the same as $\mathcal{D}$. We denote by $X_{i_{1}, \ldots, i_{l}}$ the image of $Y_{P} \subset G / P \times \overline{\mathcal{O}}$ by the projection

$$
G / P \times \overline{\mathcal{O}} \rightarrow G / P_{1_{1}, \ldots, i_{l}} \times \overline{\mathcal{O}}
$$

Let

$$
\nu_{i_{1}, \ldots, i_{l}}: Y_{P} \rightarrow X_{i_{1}, \ldots, i_{l}}
$$

be the induced map. Then the Stein factorization of $\nu_{i_{1}, \ldots, i_{l}}$ is a birational contraction map, which corresponds to a codimension $k-l$ face of $\overline{\operatorname{Amp}}\left(Y_{P} / \hat{\mathcal{O}}\right)$. We shall denote by $F_{i_{1}, \ldots, i_{l}}$ this face. Then $\overline{\operatorname{Amp}}\left(Y_{P} / \hat{\mathcal{O}}\right)$
is a simplicial polyhedral cone generated by $F_{1}, F_{2}, \ldots$, and $F_{k}$. In fact, any $l$ dimensional face generated by $F_{i_{1}}, \ldots, F_{i_{l}}$ corresponds to the Stein factorization of $\nu_{i_{1}, \ldots, i_{l}}$, which is not an isomorphism. Q.E.D.

Next assume that two marked Dynkin diagrams $\mathcal{D}$ and $\mathcal{D}^{\prime}$ are adjacent by means of $\overline{\mathcal{D}}$. We have three parabolic subgroups $P, P^{\prime}$ and $\bar{P}$ of $G$ corresponding to $\mathcal{D}, \mathcal{D}^{\prime}$ and $\overline{\mathcal{D}}$ respectively. One can assume that these subgroups contain the same Borel subgroup $B$ of $G$ and $\bar{P}$ contains both $P$ and $P^{\prime}$. Let $\mu: T^{*}(G / P) \rightarrow \mathfrak{g}$ and $\mu^{\prime}: T^{*}\left(G / P^{\prime}\right) \rightarrow \mathfrak{g}$ be the Springer maps.

Proposition 6.4. (i) The Richardson orbits $\mathcal{O}$ of $P$ is the Richardson orbit of $P^{\prime}$
(ii) Let $\nu$ be the composed map

$$
T^{*}(G / P) \rightarrow G / P \times \overline{\mathcal{O}} \rightarrow G / \bar{P} \times \overline{\mathcal{O}}
$$

and let $\nu^{\prime}$ be the composed map

$$
T^{*}\left(G / P^{\prime}\right) \rightarrow G / P^{\prime} \times \overline{\mathcal{O}} \rightarrow G / \bar{P} \times \overline{\mathcal{O}}
$$

Then $\operatorname{Im}(\nu)=\operatorname{Im}\left(\nu^{\prime}\right)$.
(iii) If we put $X:=\operatorname{Im}(\nu)$, then

$$
T^{*}(G / P) \rightarrow X \leftarrow T^{*}\left(G / P^{\prime}\right)
$$

is a locally trivial family of Mukai flops of type $A, D, E_{6, I}$ or $E_{6, I I}$. In particular, $\nu$ and $\nu^{\prime}$ are both small birational maps. If $\operatorname{deg}(\mu)=1$, then $\operatorname{deg}\left(\mu^{\prime}\right)=1$.

Proof. (i): Take a Levi decomposition

$$
\overline{\mathfrak{p}}=l(\overline{\mathfrak{p}}) \oplus n(\overline{\mathfrak{p}}) .
$$

In the reductive Lie algebra $l(\overline{\mathfrak{p}}), \mathfrak{p} \cap l(\overline{\mathfrak{p}})$ and $\mathfrak{p}^{\prime} \cap l(\overline{\mathfrak{p}})$ are parabolic subalgebras corresponding to dual marked Dynkin diagrams in Proposition 5.1. Hence they have conjugate Levi factors by Remark 5.2. On the other hand, we have

$$
l(\mathfrak{p})=l(\mathfrak{p} \cap l(\overline{\mathfrak{p}})),
$$

and

$$
l\left(\mathfrak{p}^{\prime}\right)=l\left(\mathfrak{p}^{\prime} \cap l(\overline{\mathfrak{p}})\right)
$$

Therefore, $l(\mathfrak{p})$ and $l\left(\mathfrak{p}^{\prime}\right)$ are conjugate. Since $\mathfrak{p}$ and $\mathfrak{p}^{\prime}$ have conjugate Levi factors, their Richardson orbits coincide (cf. [C-M, Theorem 7.1.3]).
(ii): Let $\mathcal{O}$ be the Richardson orbit of $\mathfrak{p}$ and $\mathfrak{p}^{\prime}$. Springer maps $\mu: T^{*}(G / P) \rightarrow \overline{\mathcal{O}}$ and $\mu^{\prime}: T^{*}\left(G / P^{\prime}\right) \rightarrow \overline{\mathcal{O}}$ are both $G$-equivariant with respect to natural $G$-actions. Then $U:=\mu^{-1}(\mathcal{O})$ and $U^{\prime}:=\left(\mu^{\prime}\right)^{-1}(\mathcal{O})$ are open dense orbits of $T^{*}(G / P)$ and $T^{*}\left(G / P^{\prime}\right)$ respectively. Since $\nu$ and $\nu^{\prime}$ are proper maps, $\operatorname{Im}(\nu)=\overline{\nu(U)}$ and $\operatorname{Im}\left(\nu^{\prime}\right)=\overline{\nu^{\prime}\left(U^{\prime}\right)}$. In the following we shall prove that $\nu(U)=\nu^{\prime}\left(U^{\prime}\right)$.
(ii-1): We regard $T^{*}(G / P)$ (resp. $T^{*}\left(G / P^{\prime}\right)$ ) as a closed subvariety of $G / P \times \overline{\mathcal{O}}$ (resp. $G / P^{\prime} \times \overline{\mathcal{O}}$ ). By replacing $P^{\prime}$ by a suitable conjugate in $\bar{P}$, we may assume that there exists an element $x \in \mathcal{O}$ such that $([P], x) \in U$ and $\left(\left[P^{\prime}\right], x\right) \in U^{\prime}$. In fact, for a Levi decomposition

$$
\overline{\mathfrak{p}}=l(\overline{\mathfrak{p}}) \oplus n(\overline{\mathfrak{p}}),
$$

we have a direct sum decomposition

$$
n(\mathfrak{p})=n(\mathfrak{p} \cap l(\overline{\mathfrak{p}})) \oplus n(\overline{\mathfrak{p}}) .
$$

Let $p_{1}: n(\mathfrak{p}) \rightarrow n(\mathfrak{p} \cap l(\overline{\mathfrak{p}}))$ be the 1-st projection. Let $\mathcal{O}^{\prime} \subset l(\overline{\mathfrak{p}})$ be the Richardson orbit of the parabolic subalgebra $\mathfrak{p} \cap l(\overline{\mathfrak{p}})$ of $l(\overline{\mathfrak{p}})$. Since $p_{1}^{-1}\left(n(\mathfrak{p}) \cap \mathcal{O}^{\prime}\right)$ and $n(\mathfrak{p}) \cap \mathcal{O}$ are both Zariski open subsets of $n(\mathfrak{p})$, we can take an element

$$
x \in p_{1}^{-1}\left(n(\mathfrak{p}) \cap \mathcal{O}^{\prime}\right) \cap(n(\mathfrak{p}) \cap \mathcal{O})
$$

Since $x \in n(\mathfrak{p}) \cap \mathcal{O}$, we have $([P], x) \in U$. Decompose $x=x_{1}+x_{2}$ according to the direct sum decomposition. Then $x_{1} \in \mathcal{O}^{\prime}$. The orbit $\mathcal{O}^{\prime}$ is also the Richardson orbit of $\mathfrak{p}^{\prime} \cap l(\overline{\mathfrak{p}})$. Therefore, for some $g \in L(\bar{P})$ (the Levi factor of $\bar{P}$ corresponding to $l(\bar{P})$ ),

$$
x_{1} \in n\left(A d_{g}\left(\mathfrak{p}^{\prime} \cap l(\overline{\mathfrak{p}})\right)\right) .
$$

The Levi decomposition of $\overline{\mathfrak{p}}$ induces a direct sum decomposition

$$
n\left(A d_{g}\left(\mathfrak{p}^{\prime}\right)\right)=n\left(A d_{g}\left(\mathfrak{p}^{\prime}\right) \cap l(\overline{\mathfrak{p}})\right) \oplus n(\overline{\mathfrak{p}})
$$

Note that $A d_{g}\left(\mathfrak{p}^{\prime}\right) \cap l(\overline{\mathfrak{p}})=A d_{g}\left(\mathfrak{p}^{\prime} \cap l(\overline{\mathfrak{p}})\right)$. Hence we see that $x_{1}+x_{2} \in$ $n\left(A d_{g}\left(\mathfrak{p}^{\prime}\right)\right)$. Now, for $A d_{g}\left(P^{\prime}\right) \subset \bar{P}$, we have $\left(\left[\operatorname{Ad}_{g}\left(P^{\prime}\right)\right], x\right) \in U^{\prime}$.
(ii-2): Any element of $U$ can be written as $\left([g P], A d_{g}(x)\right)$ for some $g \in G$. Then

$$
\nu\left([g P], A d_{g}(x)\right)=\left([g \bar{P}], A d_{g}(x)\right) .
$$

For the same $g \in G$, we have $\left(\left[g P^{\prime}\right], A d_{g}(x)\right) \in U^{\prime}$ and

$$
\nu^{\prime}\left(\left[g P^{\prime}\right], A d_{g}(x)\right)=\left([g \bar{P}], A d_{g}(x)\right)
$$

Therefore, $\nu(U) \subset \nu^{\prime}\left(U^{\prime}\right)$. By the same argument, we also have $\nu^{\prime}\left(U^{\prime}\right) \subset$ $\nu(U)$.
(iii): For $g \in G, A d_{g}(n(\overline{\mathfrak{p}}))$ is the nil-radical of $A d_{g}(\overline{\mathfrak{p}})$. Since $A d_{g}(\overline{\mathfrak{p}})$ depends only on the class $[g] \in G / \bar{P}, \operatorname{Ad}_{g}(n(\overline{\mathfrak{p}}))$ also depends on the class $[g] \in G / \bar{P}$. We denote by $A d_{g}(l(\overline{\mathfrak{p}}))$ the quotient of $A d_{g}(\overline{\mathfrak{p}})$ by its nil-radical $A d_{g}(n(\overline{\mathfrak{p}}))$. Let us consider the vector bundle over $G / \bar{P}$

$$
\cup_{[g] \in G / \bar{P}} A d_{g}(\overline{\mathfrak{p}}) \rightarrow G / \bar{P} .
$$

Let $\mathcal{L}$ be its quotient bundle whose fiber over $[g] \in G / \bar{P}$ is $A d_{g}(l(\overline{\mathfrak{p}}))$. We call $\mathcal{L}$ the Levi bundle. Let $\mathcal{O}^{\prime}$ be the Richardson orbit of the parabolic subalgebra $\mathfrak{p} \cap l(\overline{\mathfrak{p}})$ of $l(\overline{\mathfrak{p}})$. Note that $\mathcal{O}^{\prime}$ is also the Richardson orbit of $\mathfrak{p}^{\prime} \cap l(\overline{\mathfrak{p}})$. In $\mathcal{L}$, we consider the fiber bundle

$$
W:=\cup_{[g] \in G / \bar{P}} A d_{g}\left(\overline{\mathcal{O}}^{\prime}\right)
$$

whose fiber over $[g] \in G / \bar{P}$ is $\operatorname{Ad}_{g}\left(\overline{\mathcal{O}}^{\prime}\right)$. Put $X:=\operatorname{Im}(\nu)$. Define a map

$$
f: X \rightarrow W
$$

as $f([g], x):=\left([g], x_{1}\right)$, where $x_{1}$ is the first factor of $x$ under the direct sum decomposition

$$
A d_{g}(\overline{\mathfrak{p}})=A d_{g}(l(\overline{\mathfrak{p}})) \oplus n\left(A d_{g}(\overline{\mathfrak{p}})\right)
$$

Note that $x_{1} \in A d_{g}\left(\overline{\mathcal{O}}^{\prime}\right)$. In fact, in the direct sum decomposition, we have

$$
n\left(A d_{g}(\mathfrak{p})\right)=n\left(A d_{g}(\mathfrak{p}) \cap A d_{g}(l(\overline{\mathfrak{p}}))\right) \oplus n\left(A d_{g}(\overline{\mathfrak{p}})\right)
$$

Therefore

$$
x_{1} \in n\left(A d_{g}(\mathfrak{p}) \cap A d_{g}(l(\overline{\mathfrak{p}}))\right) \subset A d_{g}\left(\overline{\mathcal{O}}^{\prime}\right)
$$

Since $W \rightarrow G / \bar{P}$ is an $\overline{\mathcal{O}^{\prime}}$ bundle, we have a family of Mukai flops parametrized by $G / \bar{P}$ :

$$
Y \rightarrow W \leftarrow Y^{\prime}
$$

By pulling back this diagram by $f: X \rightarrow W$, we have the diagram

$$
T^{*}(G / P) \rightarrow X \leftarrow T^{*}\left(G / P^{\prime}\right)
$$

Q.E.D.

Example 6.5. Let $\mathfrak{g}$ be a simple Lie algebra of type $B, C$ or $D$. This long example will explain what actually goes on in the proof of Proposition 6.4. The example consists of two claims.

Claim 6.5.1. Let $V$ be a $\mathbf{C}$-vector space of $\operatorname{dim} n$ with a nondegenerate bilinear form such that $\left\langle v, w>=(-1)^{\epsilon}<w, v>\right.$ for all $v, w \in V$. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{s o}(V)$ or $\mathfrak{s p}(V)$ according as $\epsilon=0$ or $\epsilon=1$. Let $x \in \mathfrak{g}$ be a nilpotent element of type d. Suppose that for $\pi \in \operatorname{Pai}(n, q), \mathbf{d}=S(\pi)$ where $S$ is the Spaltenstein map. Let $\left(p_{1}, \ldots, p_{k}, q, p_{k}, \ldots, p_{1}\right)$ be a sequence of integers such that $\pi=\operatorname{ord}\left(p_{1}, \ldots, p_{k}, q, p_{k}, \ldots, p_{1}\right)$. Fix an admissible flag $F$ of type $\left(p_{1}, \ldots, p_{k}, q, p_{k}, \ldots, p_{1}\right)$ such that $x F_{i} \subset F_{i-1}$ for all $i$.
(i) Assume that $p_{j-1} \neq p_{j}$ for an index $1 \leq j \leq k$. Then we obtain a new flag $F^{\prime}$ of type $\left(p_{1}, \ldots, p_{j}, p_{j-1}, \ldots, p_{k}, q, p_{k}, \ldots, p_{j-1}, p_{j}, \ldots, p_{1}\right)$ from $F$ such that $x F_{i}^{\prime} \subset F_{i-1}^{\prime}$ for all $i$ by the following operation.
(The case where $p_{j-1}<p_{j}$ ): $x$ induces an endomorphism $\bar{x} \in$ $\operatorname{End}\left(F_{j} / F_{j-2}\right)$. For the projection $\phi: F_{j} \rightarrow F_{j} / F_{j-2}$, we put $F_{j-1}^{\prime}:=$ $\phi^{-1}(\operatorname{Ker}(\bar{x}))$. We then put

$$
F_{i}^{\prime}:=\left\{\begin{aligned}
F_{i} & (i \neq j-1,2 k+2-j) \\
F_{j-1}^{\prime} & (i=j-1) \\
\left(F_{j-1}^{\prime}\right)^{\perp} & (i=2 k+2-j)
\end{aligned}\right.
$$

(The case where $p_{j-1}>p_{j}$ ): $x$ induces an endomorphism $\bar{x} \in$ $\operatorname{End}\left(F_{j} / F_{j-2}\right)$. For the projection $\phi: F_{j} \rightarrow F_{j} / F_{j-2}$, we put $F_{j-1}^{\prime}:=$ $\phi^{-1}(\operatorname{Im}(\bar{x}))$. We then put

$$
F_{i}^{\prime}:=\left\{\begin{aligned}
F_{i} & (i \neq j-1,2 k+2-j) \\
F_{j-1}^{\prime} & (i=j-1) \\
\left(F_{j-1}^{\prime}\right)^{\perp} & (i=2 k+2-j)
\end{aligned}\right.
$$

(ii) Assume that $q=0$ and $p_{k}$ is odd. Then there is an admissible flag $F^{\prime}$ of $V$ of type $\left(p_{1}, \ldots, p_{k}, p_{k}, \ldots, p_{1}\right)$ such that
$x F_{i}^{\prime} \subset F_{i-1}^{\prime}$ for all $i$,
$F_{i}^{\prime}=F_{i}$ for $i \neq k$ and
$F_{k}^{\prime} \neq F_{k}$.
Proof. (i): When $p_{j-1}<p_{j}, \operatorname{rank}(\bar{x})=p_{j-1}$ for $\bar{x} \in \operatorname{End}\left(F_{j} / F_{j-2}\right)$. In fact, since $x F_{j} \subset F_{j-1}, \operatorname{rank}(\bar{x}) \leq p_{j-1}$. Assume that $\operatorname{rank}(\bar{x})<$ $p_{j-1}$. Then we can construct a new flag from $F$ by replacing $F_{j-1}$ with a subspace $F_{j-1}^{\prime}$ containing $F_{j-2}$ such that

$$
\operatorname{Im}(\bar{x}) \subset F_{j-1}^{\prime} / F_{j-2} \subset \operatorname{Ker}(\bar{x})
$$

and $\operatorname{dim} F_{j-1}^{\prime} / F_{j-2}=p_{j-1}$. The new flag satisfies $x F_{i}^{\prime} \subset F_{i-1}^{\prime}$ for all $i$ and it has the same flag type as $F$. Since there are infinitely many
choices of $F_{j-1}^{\prime}$, we have infinitely many such $F^{\prime}$. This contradicts the fact that $x$ has only finite polarizations. Hence, $\operatorname{rank}(\bar{x})=p_{j-1}$. Then the flag $F^{\prime}$ in our Lemma satisfies the desired properties. When $p_{j-1}>$ $p_{j}$, we see that $\operatorname{dim} \operatorname{Ker}(\bar{x})=p_{j-1}$ by a similar way. Then the latter argument is the same as when $p_{j-1}<p_{j}$.
(ii): According to the proof of Theorem 4.5. we construct a flag $F$ such that $x F_{i} \subset F_{i-1}$. Since $q=0$ and $p_{k}$ is odd, we have the case ( $A$ ) in the last step. As a consequence, we have two choices of the flags. One of them is $F$ and another one is $F^{\prime}$.
Q.E.D.

Let $F$ be the flag in Claim 6.5.1, (i) or (ii). In the claim, we have constructed another flag $F^{\prime}$. Let $G$ be the complex Lie group $S p(V)$ or $S O(V)$ according as $V$ is a C-vector with a non-degenerate skewsymmetric form or with a non-degenerate symmetric form. Let $P \subset G$ (resp. $P^{\prime} \subset G$ ) be the stabilizer group of the flag $F$ (resp. $F^{\prime}$ ). Then $P$ and $P^{\prime}$ are both polarizations of $x \in \mathfrak{g}$. Let $\mathcal{O} \subset \mathfrak{g}$ be the nilpotent orbit containing $x$. Let us consider two Springer maps

$$
T^{*}(G / P) \xrightarrow{\mu} \overline{\mathcal{O}} \stackrel{\mu^{\prime}}{\leftarrow} T^{*}\left(G / P^{\prime}\right) .
$$

Note that $T^{*}(G / P)\left(\right.$ resp. $\left.T^{*}\left(G / P^{\prime}\right)\right)$ is embedded in $G / P \times \overline{\mathcal{O}}$ (resp. $\left.G / P^{\prime} \times \overline{\mathcal{O}}\right)$. The variety $G / P\left(\right.$ resp. $\left.G / P^{\prime}\right)$ is identified with the set of parabolic subgroups of $G$ which are conjugate to $P$ (resp. $P^{\prime}$ ). Assume that

$$
\mu^{-1}(x)=\left\{\left(P_{i}, x\right)\right\}_{1 \leq i \leq m},
$$

where $\operatorname{deg}(\mu)=m$ and $P_{1}=P$. Fix stabilized flags $F^{(i)}$ of $P_{i}$. Here $F^{(1)}=F$. For each $F^{(i)}$, we make a flag $\left(F^{(i)}\right)^{\prime}$ by Claim 6.5.1. Thus we have $\operatorname{deg}\left(\mu^{\prime}\right)=m$. An element $y \in \mathcal{O}$ can be written as $y=g x g^{-1}$ for some $g \in G$. Then we have

$$
\mu^{-1}(y)=\left\{\left(\left[g\left(F^{(i)}\right)\right], y\right)\right\}_{1 \leq i \leq m}
$$

and

$$
\left(\mu^{\prime}\right)^{-1}(y)=\left\{\left(\left[g\left(\left(F^{(i)}\right)^{\prime}\right)\right], y\right)\right\}_{1 \leq i \leq m}
$$

Here we identify a flag with the parabolic subgroup stabilizing it. We define the flag $\bar{F}$ in the following manner. If $F$ is the flag in Claim 6.5.1, ( $i$ ), then $\bar{F}$ is the flag obtained from $F$ by deleting subspaces $F_{j-1}$ and $F_{2 k+2-j}$. Finally, if $F$ is the flag in Claim 6.5.1, (ii), then $\bar{F}$ is the flag obtained from $F$ by deleting $F_{k}$. Note that $\bar{F}$ is also obtained from $F^{\prime}$ by the same manner. Let $\bar{P} \subset G$ be the stabilizer group of the flag $\bar{F}$. We then have two projections

$$
G / P \xrightarrow{p} G / \bar{P} \stackrel{p^{\prime}}{\leftarrow} G / P^{\prime} .
$$

By two projections

$$
G / P \times \overline{\mathcal{O}} \xrightarrow{p \times i d} G / \bar{P} \times \overline{\mathcal{O}} \stackrel{p^{\prime} \times i d}{\leftarrow} G / P^{\prime} \times \overline{\mathcal{O}},
$$

$T^{*}(G / P)$ and $T^{*}\left(G / P^{\prime}\right)$ have the same image $X$ in $G / \bar{P} \times \overline{\mathcal{O}}$. Since $p$ and $p^{\prime}$ are both proper maps, $X$ is a closed subvariety of $G / \bar{P} \times \overline{\mathcal{O}}$. The following diagram has been obtained as a consequence:

$$
T^{*}(G / P) \rightarrow X \leftarrow T^{*}\left(G / P^{\prime}\right)
$$

Claim 6.5.2. When $F$ is the flag in Claim 6.5.1, (i), the diagram

$$
T^{*}(G / P) \stackrel{f}{\rightarrow} X \stackrel{f^{\prime}}{\leftarrow} T^{*}\left(G / P^{\prime}\right)
$$

is locally a trivial family of Mukai flops of type A. When $F$ is the flag in Claim 6.5.1, (ii), the diagram is locally a trivial family of Mukai flops of type D.

Proof. Consider the situation in Claim 6.5.1, (i). A point of $G / \bar{P}$ corresponds to an isotropic flag $\bar{F}$ of $V$ of type $\left(p_{1}, \ldots, p_{j-1}+p_{j}, \ldots, p_{k}, q, p_{k}, \ldots, p_{j-1}+p_{j}, \ldots, p_{1}\right)$. Let

$$
0 \subset \overline{\mathcal{F}}_{1} \subset \ldots \subset \overline{\mathcal{F}}_{2 k-1}=\left(\mathcal{O}_{G / \bar{P}}\right)^{n}
$$

be the universal subundles on $G / \bar{P}$. Let

$$
W \subset \underline{\operatorname{End}}\left(\overline{\mathcal{F}}_{j-1} / \overline{\mathcal{F}}_{j-2}\right)
$$

be the subvariety consisting of the points $([\bar{F}], \bar{x})$ where $\bar{x} \in \operatorname{End}\left(\bar{F}_{j-1} / \bar{F}_{j-2}\right)$, $\bar{x}^{2}=0$ and $\operatorname{rank}(\bar{x}) \leq \min \left(p_{j}, p_{j-1}\right)$. If we put $m:=p_{j-1}+p_{j}$ and $r:=\min \left(p_{j}, p_{j-1}\right)$, then

$$
W \rightarrow G / \bar{P}
$$

is an $\overline{\mathcal{O}}_{\left[2^{r}, 1^{m-2 r}\right]}$ bundle over $G / \bar{P}$. Let us recall the definition of $X$.

$$
X \subset G / \bar{P} \times \overline{\mathcal{O}}
$$

consists of the points $([\bar{F}], x)$ such that $x \bar{F}_{i} \subset \bar{F}_{i-1}$ for all $i \neq j-1,2 k-j$ and $x \bar{F}_{i} \subset \bar{F}_{i}$ for $i=j-1,2 k-j$. Moreover, the induced endomorphism $\bar{x} \in \operatorname{End}\left(\bar{F}_{j-1} / \bar{F}_{j-2}\right)$ satisfies $\bar{x}^{2}=0$ and $\operatorname{rank}(\bar{x}) \leq \min \left(p_{j-1}, p_{j}\right)$. Let

$$
\phi: X \rightarrow W
$$

be the projection defined by $\phi([\bar{F}], x)=([\bar{F}], \bar{x})$, where $\bar{x} \in \operatorname{End}\left(\bar{F}_{j-1} / \bar{F}_{j-2}\right)$ is the induced endomorphism by $x$. It can be checked that $\phi$ is an affine
bundle. Since $W$ is an $\overline{\mathcal{O}}_{\left[2^{r}, 1^{m-2 r}\right]}$ bundle over $G / \bar{P}$, there exists a family of Mukai flops of type $A$ :

$$
Y \rightarrow W \leftarrow Y^{\prime}
$$

parametrized by $G / \bar{P}$. The diagram

$$
T^{*}(G / P) \rightarrow X \leftarrow T^{*}\left(G / P^{\prime}\right)
$$

coincides with the pull back of the previous diagram by $\phi: X \rightarrow W$. Since $\phi$ is an affine bundle, this diagram is locally a trivial family of Mukai flops of type $A$.

Next consider the situation in Claim 6.5.1, (ii). A point of $G / \bar{P}$ corresponds to an isotropic flag $\bar{F}$ of $V$ of type $\left(p_{1}, \ldots, 2 p_{k}, \ldots, p_{1}\right)$. Let

$$
0 \subset \overline{\mathcal{F}}_{1} \subset \ldots \subset \overline{\mathcal{F}}_{2 k-1}=\left(\mathcal{O}_{G / \bar{P}}\right)^{n}
$$

be the universal subundles on $G / \bar{P}$. Let

$$
W \subset \underline{\operatorname{End}}\left(\overline{\mathcal{F}}_{k} / \overline{\mathcal{F}}_{k-1}\right)
$$

be the subvariety consisting of the points $([\bar{F}], \bar{x})$ where

$$
\bar{x} \in \overline{\mathcal{O}}_{\left[2^{p_{k}-1}, 1^{2}\right]} \subset \mathfrak{s o}\left(\bar{F}_{k} / \bar{F}_{k-1}\right) .
$$

$W \rightarrow G / \bar{P}$ is an $\overline{\mathcal{O}}_{\left[2^{p_{k}-1}, 1^{2}\right]}$ bundle over $G / \bar{P}$. Let us recall the definition of $X$.

$$
X \subset G / \bar{P} \times \overline{\mathcal{O}}
$$

consists of the points $([\bar{F}], x)$ such that $x \bar{F}_{i} \subset \bar{F}_{i-1}$ for all $i \neq k$ and $x \bar{F}_{k} \subset \bar{F}_{k}$. Moreover, the induced endomorphism $\bar{x} \in \mathfrak{s o}\left(\bar{F}_{k} / \bar{F}_{k-1}\right)$ is contained in $\overline{\mathcal{O}}_{\left[2^{p_{k}-1}, 1^{2}\right]}$. Let

$$
\phi: X \rightarrow W
$$

be the projection defined by $\phi([\bar{F}], x)=([\bar{F}], \bar{x})$, where $\bar{x} \in \mathfrak{s o}\left(\bar{F}_{k} / \bar{F}_{k-1}\right)$ is the induced endomorphism by $x$. It can be checked that $\phi$ is an affine bundle. Since $W$ is an $\overline{\mathcal{O}}_{\left[2^{p_{k}-1}, 1^{2}\right]}$ bundle over $G / \bar{P}$, there exists a family of Mukai flops of type $D$ :

$$
Y \rightarrow W \leftarrow Y^{\prime}
$$

parametrized by $G / \bar{P}$. The diagram

$$
T^{*}(G / P) \rightarrow X \leftarrow T^{*}\left(G / P^{\prime}\right)
$$

coincides with the pull back of the previous diagram by $\phi: X \rightarrow W$. Since $\phi$ is an affine bundle, this diagram is locally a trivial family of Mukai flops of type D. Q.E.D.

Now let us return to the general situation. Let $\mathcal{D}$ be a marked Dynkin diagram and let $\overline{\mathcal{D}}$ be the diagram obtained from $\mathcal{D}$ by a contraction. Let $P$ and $\bar{P}$ be parabolic subgroups of $G$ corresponding to $\mathcal{D}$ and $\overline{\mathcal{D}}$ respectively. One can assume that $\bar{P}$ contains $P$. Let $\mathcal{O}$ be the Richardson orbit of $P$ and let $\nu$ be the compoed map

$$
T^{*}(G / P) \rightarrow G / P \times \overline{\mathcal{O}} \rightarrow G / \bar{P} \times \overline{\mathcal{O}}
$$

We put $X:=\operatorname{Im}(\nu)$. As above, $\mu: T^{*}(G / P) \rightarrow \overline{\mathcal{O}}$ is the Springer map.
Proposition 6.6. Let $\mathfrak{g}$ be a complex simple Lie algebra. Assume that no marked Dynkin diagram is adjacent to $\mathcal{D}$ by means of $\overline{\mathcal{D}}$. If $\operatorname{deg}(\mu)=1$, then $\nu: T^{*}(G / P) \rightarrow X$ is a divisorial birational contraction map.

Proof. As in the proof of Proposition 6.4, (iii), we construct an $\overline{\mathcal{O}^{\prime}}$ bundle $W$ over $G / \bar{P}$ and define a map $f: X \rightarrow W$. There is a family of Springer maps

$$
Y \xrightarrow{\sigma} W \rightarrow G / \bar{P} .
$$

By pulling back $Y \xrightarrow{\sigma} W$ by $f: X \rightarrow W$, we have the $\nu: T^{*}(G / P) \rightarrow X$. Since $\operatorname{deg} \mu=1, \nu$ is a birational map. Hence $\sigma$ should be a birational map. Hence $\sigma: Y \rightarrow W$ is a family of Springer resolutions. By the assumption, there are no marked Dynkin diagrams adjacent to $\mathcal{D}$ by means of $\mathcal{D}$. Now Proposition 5.1 shows that the Springer resolution is divisorial. Therefore, $\nu$ is also divisorial. Q.E.D.

Now let us prove Theorem 6.1. By Proposition 6.4, (iii), $Y_{P}:=$ $T^{*}(G / P)$ all give symplectic resolutions of $\overline{\mathcal{O}}$ for $P \sim P_{0}$. Hence the first statement of (i) has been proved. Moreover, $\left\{Y_{P}\right\}$ are connected by Mukai flops, which is nothing but (v). Let us consider $\cup_{P \sim P_{0}} \overline{\operatorname{Amp}}\left(Y_{P} / \overline{\mathcal{O}}\right)$ in $N^{1}\left(Y_{P_{0}} / \overline{\mathcal{O}}\right)$. Then (iv) follows from Proposition 6.4, (iii) and Proposition 6.6. For an $\overline{\mathcal{O}}$-movable divisor $D$ on $Y_{P_{0}}$, a $K_{Y_{P_{0}}}+D$-extremal contraction is a small birational map. Therefore, the corresponding codimension 1 face of $\overline{\operatorname{Amp}}\left(Y_{P_{0}} / \overline{\mathcal{O}}\right)$ becomes a codimension 1 face of another $\overline{\operatorname{Amp}}\left(Y_{P} / \overline{\mathcal{O}}\right)$. For this small birational map, there exists a flop. Replace $D$ by its proper transform and continue the same. We shall prove that this procedure ends in finite times. Suppose to the contrary. Since the flops occur between finite number of varieties $\left\{Y_{P}\right\}$, a variety, say $Y_{P_{1}}$, appears at least twice in the sequence of flops:

$$
Y_{P_{1}}--\rightarrow Y_{P_{2}}--\rightarrow \ldots--\rightarrow Y_{P_{1}} .
$$

For the first flop

$$
Y_{P_{1}} \xrightarrow{\nu_{1}} X_{1} \leftarrow Y_{P_{2}},
$$

take a discrete valation $v$ of the function field $K\left(Y_{P_{1}}\right)$ in such a way that its center is contained in the exceptional locus $\operatorname{Exc}\left(\nu_{1}\right)$ of $\nu_{1}$. Let $D_{i} \subset Y_{P_{i}}$ be the proper transforms of $D$. Then we have inequalities for discrepancies (cf. [KMM], Proposition 5-1-11):

$$
a\left(v, D_{1}\right)<a\left(v, D_{2}\right) \leq \ldots \leq a\left(v, D_{1}\right)
$$

Here the first inequality is a strict one since the center of $v$ is contained in $\operatorname{Exc}\left(\nu_{1}\right)$. This is absurd. Hence the procedure ends in finite times, which implies that $D \in \overline{\operatorname{Amp}}\left(Y_{P} / \overline{\mathcal{O}}\right)$ for some $P$. Therefore, (iii) has been proved. The second statement of (i) immediately follows from (iii).

Example 6.7. Assume that $\mathfrak{g}=\mathfrak{s l}(6)$. The marked Dynkin diagram D

gives a parabolic subgroup $P_{1,2,3} \subset S L(6)$ of flag type $(1,2,3)$. We put $Y_{1,2,3}:=T^{*}\left(G / P_{1,2,3}\right)$. There are 5 other marked Dynkin diagrams which are equivalent to $\mathcal{D}$ :


Five parabolic subgroups $P_{1,3,2}, P_{3,1,2}, P_{3,2,1}, P_{2,3,1}, P_{2,1,3}$ correspond to the marked Dynkin diagrams above respectively. We put $Y_{i, j, k}:=$
$T^{*}\left(S L(6) / P_{i, j, k}\right)$. Let $\mathcal{O}$ be the Richardson orbit of these parabolic subgroups. Then $\overline{\operatorname{Mov}}\left(Y_{1,2,3} / \overline{\mathcal{O}}\right) \cong \mathbf{R}^{2}$, which is divided into six chambers by the ample cones of $Y_{i, j, k}$ in the following way:


Example 6.8. Assume that $\mathfrak{g}=\mathfrak{s o}(10)$. The marked Dynkin diagram

gives a parabolic subgroup $P_{3,2,2,3}^{+}$of flag type (3,2,2,3). There are three marked Dynkin diagrams equivalent to this marked diagram:




Three parabolic subgroups $P_{2,3,3,2}^{+}, P_{2,3,3,2}^{-}, P_{3,2,2,3}^{-}$correspond to these marked Dynkin diagrams respectively. Note that there are exactly two conjugacy classes of parabolic subgroups with the same flag type (cf. Example 4.2). We put $Y_{i, j}^{+}:=T^{*}\left(S O(10) / P_{i, j, j, i}^{+}\right)$and put $Y_{i, j}^{-}:=T^{*}\left(S O(10) / P_{i, j, j, i}^{-}\right)$. Let $\mathcal{O}$ be the Richardson orbit of these parabolic subgroups. Then $\overline{\operatorname{Mov}}\left(Y_{3,2}^{+} / \overline{\mathcal{O}}\right)$ is divided into four chambers by the ample cones of $Y_{3,2}^{+}, Y_{2,3}^{+}, Y_{2,3}^{-}, Y_{3,2}^{-}$in the following way:


Example 6.9. Assume that $\mathfrak{g}$ is of type $E_{6}$. Consider the nilpotent orbit $\mathcal{O}:=\mathcal{O}_{A_{3}}$ (cf. [C-M], p.129). This is the unique orbit with dimension 52. By a dimension count, we see that $\mathcal{O}$ is the Richardson orbit of the parabolic subgroup $P_{1} \subset G$ associated with the marked Dynkin diagram


Since $\pi_{1}(\mathcal{O})=1([C-M], p .129]$, the Springer map $\nu_{1}: T^{*}\left(G / P_{1}\right) \rightarrow$ $\overline{\mathcal{O}}$ has degree 1. The following marked Dynkin diagrams are equivalent to the diagram above:



Denote by $P_{2}, P_{3}, P_{4}$ respectively the parabolic subgroups corresponding to the diagrams above. We put $Y_{i}:=T^{*}\left(G / P_{i}\right)$ for $i=1,2,3,4$. Then $\overline{\operatorname{Mov}}\left(Y_{1} / \overline{\mathcal{O}}\right)$ is divided into four chambers by the ample cones of $Y_{i}$ :

$Y_{1}$ and $Y_{2}$ are connected by a Mukai flop of type $D_{5}$ (cf. Proposition 6.4, (iii)). $Y_{2}$ and $Y_{3}$ are connected by a Mukai flop of type $A_{5,1}$ (for the notation, see Example 5.3). $Y_{3}$ and $Y_{4}$ are connected by a Mukai flop of type $D_{5}$.

Derived categories: Two smooth quasi-projective varieties $Y$ and $Y^{\prime}$ are called $D$-equivalent if there is an equivalence between the bounded derived categories of coherent sheaves $D^{b}(\operatorname{Coh}(Y))$ and $D^{b}\left(\operatorname{Coh}\left(Y^{\prime}\right)\right)$. On the other hand, if we can take common resolutions $\mu: Z \rightarrow Y$ and $\mu^{\prime}: Z \rightarrow Y^{\prime}$ in such a way that $\mu^{*}\left(K_{Y}\right)=\mu^{\prime *}\left(K_{Y^{\prime}}\right)$, then we say that $Y$ and $Y^{\prime}$ are $K$-equivalent. The following conjecture is posed by Kawamata [Ka 2].

Conjecture 1. If $Y$ and $Y^{\prime}$ are $K$-equivalent, then they are $D$ equivalent.

Assume that $Y$ and $Y^{\prime}$ are two different symplectic resolutions of a nilpotent orbit closure $\overline{\mathcal{O}}$ in a complex simple Lie algebra. Since $\overline{\mathcal{O}}$ admits a good $\mathbf{C}^{*}$-action, the conjecture is true as a special case of a result recently proved by Bezrukavnikov and Kaledin [K]. It would be
interesting to know whether the equivalence in Conjecture is realized as Fourier-Mukai functors associated with suitable objects of $D^{b}(Y \times$ $Y^{\prime}$ ). Actually, for the Mukai flop of type $A_{n, 1}$ (cf. Example 5.3), the Fourier-Mukai functor induced from the fiber product $Y \times_{\overline{\mathcal{O}}} Y^{\prime}$ gives an equivalence [Na 2]. However, the same functor is no more an equivalence for the Mukai flop of type $A_{n, k}$ with $k>1$ ([Na 3]).

## §7. Deformations of nilpotent orbits

Let $x \in \mathfrak{g}$ be a nilpotent element of a Lie algebra attached to a complex simple Lie group $G$. Let $\mathcal{O}$ be the nilpotent orbit containing $x$. In this section, by using an idea of Borho and Kraft [B-K], we shall construct a morphism $f: \mathcal{S} \rightarrow \mathfrak{k}$ such that
(i) $f^{-1}(0)=\overline{\mathcal{O}}$ for $0 \in \mathfrak{k}$,
(ii) for any Springer resolution $T^{*}(G / P) \rightarrow \overline{\mathcal{O}}$, there is a smooth morphism $\tau_{P}: \mathcal{E}_{P} \rightarrow \mathfrak{k}$ with $\left(\tau_{P}\right)^{-1}(0)=T^{*}(G / P)$ such that there is a proper birational morphism

$$
\nu_{P}: \mathcal{E}_{P} \rightarrow \mathcal{S}
$$

and
(iii) the induced map $\left(\nu_{P}\right)_{t}:\left(\tau_{P}\right)^{-1}(t) \rightarrow f^{-1}(t)$ is a resolution for every $t \in \mathfrak{k}$ and it is an isomorphism for a general point $t \in \mathfrak{k}$.

As a corollary of this construction, we can verify Conjecture 2 in [F$\mathrm{N}]$ for the closure of a nilpotent orbit of a simple Lie algebra. Conjecture 2 has already been proved for $\mathfrak{s l}(n)$ in $[\mathrm{F}-\mathrm{N}]$, Theorem 4.4 in a very explicit form. Note that, a weaker version of this conjecture has been proved by $\mathrm{Fu}[\mathrm{Fu} 2]$ for the closure of a nilpotent orbit of a classical simple Lie algebra.

The Lie algebra $\mathfrak{g}$ becomes a $G$-variety via the adjoint action. Let $Z \subset \mathfrak{g}$ be a closed subvariety. For $m \in \mathbf{N}$, put

$$
Z^{(m)}:=\{x \in Z ; \operatorname{dim} G x=m\} .
$$

$Z^{(m)}$ becomes a locally closed subset of $Z$. We put $m(Z):=\max \{m ; m=$ $\operatorname{dim} G x, \exists x \in Z\}$. Then $Z^{m(Z)}$ is an open subset of $Z$, which will be denoted by $Z^{\mathrm{reg}}$. A sheet of $Z$ is an irreducible component of some $Z^{(m)}$. A sheet of $\mathfrak{g}$ is called a Dixmier sheet if it contains a semi-simple element of $\mathfrak{g}$. We fix a maximal torus $H$ of $G$. In the remainder, all parabolic subgroups are assumed to contain $H$. Denote by $\mathfrak{h}$ the Lie algebra of $H$.

Let $P \subset G$ be a parabolic subgroup and let $\mathfrak{p}$ be its Lie algebra. Let $\mathfrak{m}(P)$ be the Levi factor of $\mathfrak{p}$ such that $\mathfrak{h} \subset \mathfrak{m}(P)$. We put $\mathfrak{k}(P):=\mathfrak{g}^{\mathfrak{m}(P)}$
where

$$
\mathfrak{g}^{\mathfrak{m}(P)}:=\{x \in \mathfrak{g} ;[x, y]=0, \forall y \in \mathfrak{m}(P)\}
$$

Let $\mathfrak{r}(P)$ be the radical of $\mathfrak{p}$.
Theorem 7.1. $G \mathfrak{r}(P)=\overline{G \mathfrak{k}(P)}$ and $G \mathfrak{r}(P)^{\mathrm{reg}}\left(=\overline{G \mathfrak{k}(P)}^{\mathrm{reg}}\right)$ is a Dixmier sheet.

Proof. See [B-K], Satz 5.6.
Every element $x$ of $\mathfrak{g}$ can be uniquely written as $x=x_{n}+x_{s}$ with $x_{n}$ nilpotent and with $x_{s}$ semi-simple such that $\left[x_{n}, x_{s}\right]=0$. Let $W$ be the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The set of semi-simple orbits is identified with $\mathfrak{h} / W$. Let $\mathfrak{g} \rightarrow \mathfrak{h} / W$ be the map defined as $x \rightarrow\left[\mathcal{O}_{x_{s}}\right]$. There is a direct sum decomposition

$$
\mathfrak{r}(P)=\mathfrak{k}(P) \oplus \mathfrak{n}(P),\left(x \rightarrow x_{1}+x_{2}\right)
$$

where $\mathfrak{n}(P)$ is the nil-radical of $\mathfrak{p}$ (cf. [Slo], 4.3). We have a well-defined map

$$
G \times^{P} \mathfrak{r}(P) \rightarrow \mathfrak{k}(P)
$$

by sending $[g, x] \in G \times{ }^{P} \mathfrak{r}(P)$ to $x_{1} \in \mathfrak{k}(P)$ and there is a commutative diagram

$$
\begin{array}{cc}
G \times{ }^{P} \mathfrak{r}(P) \rightarrow & G \mathfrak{r}(P) \\
\downarrow & \downarrow \\
\mathfrak{k}(P) \rightarrow \mathfrak{h} / W
\end{array}
$$

by [Slo], 4.3.
Lemma 7.2. The induced map

$$
G \times^{P} \mathfrak{r}(P) \xrightarrow{\mu_{P}} \mathfrak{k}(P) \times_{\mathfrak{h} / W} G \mathfrak{r}(P)
$$

is a birational map.
Proof. Let $h \in \mathfrak{k}(P)^{\text {reg }}$ and denote by $\bar{h} \in \mathfrak{h} / W$ its image by the map $\mathfrak{k}(P) \rightarrow \mathfrak{h} / W$. Then the fiber of the map $G \mathfrak{r}(P) \rightarrow \mathfrak{h} / W$ over $\bar{h}$ coincides with a semi-simple orbit $\mathcal{O}_{h}$ of $\mathfrak{g}$ containing $h$. In fact, by Theorem 7.1, the fiber actally contains this orbit. The fiber is closed in $\mathfrak{g}$ because $G \mathfrak{r}(P)$ is closed subset of $\mathfrak{g}$ by Theorem 7.1. Note that a semi-simple orbit of $\mathfrak{g}$ is also closed. Hence if the fiber and $\mathcal{O}_{h}$ does not coincide, then the fiber contains an orbit with larger dimension than $\operatorname{dim} \mathcal{O}_{h}$. This contradicts the fact that $\overline{G \mathfrak{k}(P)^{\text {reg }}}=G \mathfrak{r}(P)$. Take a point
$\left(h, h^{\prime}\right) \in \mathfrak{k}(P)^{\text {reg }} \times_{\mathfrak{h} / W} G \mathfrak{r}(P)$. Then $h^{\prime}$ is a semi-simple element $G$ conjugate to $h$. Fix an element $g_{0} \in G$ such that $h^{\prime}=g_{0} h\left(g_{0}\right)^{-1}$. We have

$$
\left(\mu_{P}\right)^{-1}\left(h, h^{\prime}\right)=\left\{[g, x] \in G \times^{P} \mathfrak{r}(P) ; x_{1}=h, g x g^{-1}=h^{\prime}\right\} .
$$

Since $x=p x_{1} p^{-1}$ for some $p \in P$ and conversely $\left(p x_{1} p^{-1}\right)_{1}=x_{1}$ for any $p \in P$ (cf. [Slo], Lemma 2, p.48), we have

$$
\begin{gathered}
\left(\mu_{P}\right)^{-1}\left(h, h^{\prime}\right)=\left\{\left[g, p h p^{-1}\right] \in G \times^{P} \mathfrak{r}(P) ; g \in G, p \in P,(g p) h(g p)^{-1}=h^{\prime}\right\} \\
=\left\{[g p, h] \in G \times^{P} \mathfrak{r}(P) ; g \in G, p \in P,(g p) h(g p)^{-1}=h^{\prime}\right\}= \\
\left\{[g, h] \in G \times^{P} \mathfrak{r}(P) ; g h g^{-1}=h^{\prime}\right\}= \\
\left\{\left[g_{0} g^{\prime}, h\right] \in G \times{ }^{P} \mathfrak{r}(P) ; g^{\prime} \in Z_{G}(h)\right\}=g_{0}\left(Z_{G}(h) / Z_{P}(h)\right)
\end{gathered}
$$

Here $Z_{G}(h)$ (resp. $\left.Z_{P}(h)\right)$ is the centralizer of $h$ in $G($ resp. $P)$. By [Ko], 3.2, Lemma $5, Z_{G}(h)$ is connected. Moreover, since $\mathfrak{g}^{h} \subset$ $\mathfrak{p}, \operatorname{Lie}\left(Z_{G}(h)\right)=\operatorname{Lie}\left(Z_{P}(h)\right)$. Therefore, $Z_{G}(h) / Z_{P}(h)=\{1\}$, and $\left(\mu_{P}\right)^{-1}\left(h, h^{\prime}\right)$ consists of one point.

Lemma 7.3. The map

$$
G \times^{P} \mathfrak{r}(P) \rightarrow G \mathfrak{r}(P)
$$

is a proper map.
Proof. As a vector subbundle, we have a closed immersion

$$
G \times{ }^{P} \mathfrak{r}(P) \rightarrow G / P \times \mathfrak{g}
$$

This map factors through $G / P \times G \mathfrak{r}(P)$, and hence we have a closed immersion

$$
G \times{ }^{P} \mathfrak{r}(P) \rightarrow G / P \times G \mathfrak{r}(P)
$$

Our map is the composition of this closed immersion and the projection

$$
G / P \times G \mathfrak{r}(P) \rightarrow G \mathfrak{r}(P)
$$

Since $G / P$ is compact, this projection is a proper map.
Lemma 7.4. Let $\mathfrak{g}$ be a nilpotent orbit of a complex simple Lie algebra and denote by $\mathcal{O}$ a nilpotent orbit. Then the polarizations of $\mathcal{O}$ giving Springer resolutions of $\overline{\mathcal{O}}$ all have conjugate Levi factors.

Proof. This follows from Theorem 6.1 and Proposition 6.4.

Lemma 7.5. Let $\mathcal{O}$ be the same as the previous lemma. Let $P$ and $P^{\prime}$ be polarizations of $\mathcal{O}$. Assume that they both give Springer resolutions of $\overline{\mathcal{O}}$. Then $\mathfrak{k}(P)$ and $\mathfrak{k}\left(P^{\prime}\right)$ are conjugate to each other.

Proof. Let $M_{P}$ and $M_{P^{\prime}}$ be Levi factors of $P$ and $P^{\prime}$ respectively. Then $M_{P}$ and $M_{P^{\prime}}$ are conjugate by the previous lemma. Hence their centralizers are also conjugate. The Lie algebras of these centralizers are $\mathfrak{k}(P)$ and $\mathfrak{k}\left(P^{\prime}\right)$.

Corollary 7.6. For $P, P^{\prime}$ which give Springer resolutions of $\overline{\mathcal{O}}$, we have $G \mathfrak{r}(P)=G \mathfrak{r}\left(P^{\prime}\right)$.

Proof. By Theorem 7.1, $G \mathfrak{r}(P)=\overline{G \mathfrak{k}(P)}$ and $G \mathfrak{r}\left(P^{\prime}\right)=\overline{G \mathfrak{k}\left(P^{\prime}\right)}$. Since $G \mathfrak{k}(P)=G \mathfrak{k}\left(P^{\prime}\right)$, we have the result.

Lemma 7.7. The image of the composed map

$$
G \mathfrak{r}(P) \rightarrow \mathfrak{g} \rightarrow \mathfrak{h} / W
$$

coincides with $\mathfrak{k}(P) / W_{P}$, where

$$
W_{P}=\{w \in W ; w(\mathfrak{k}(P))=\mathfrak{k}(P)\} .
$$

Proof. By definition, $\mathfrak{k}(P) / W_{P} \subset \mathfrak{h} / W$, which is a closed subset. Since $G \mathfrak{r}(P)=\overline{G \mathfrak{k}(P)}$, we only have to prove that the image of $G \mathfrak{k}(P)$ by the map $\mathfrak{g} \rightarrow \mathfrak{h} / W$ coincides with $\mathfrak{k}(P) / W_{P}$. Every element of $G \mathfrak{k}(P)$ is semi-simple, and the map $G \mathfrak{k}(P) \rightarrow \mathfrak{h} / W$ sends an element of $G \mathfrak{k}(P)$ to its (semi-simple) orbit. Hence the image coincides with $\mathfrak{k}(P) / W_{P}$.

Corollary 7.8. $\mathfrak{k}(P)$ and $\mathfrak{k}\left(P^{\prime}\right)$ are $W$-conjugate in $\mathfrak{h}$.
Proof. Let $q: \mathfrak{h} \rightarrow \mathfrak{h} / W$ be the quotient map. Since $G \mathfrak{r}(P)=$ $G \mathfrak{r}\left(P^{\prime}\right), q(\mathfrak{k}(P))=q\left(\mathfrak{k}\left(P^{\prime}\right)\right)$ by the previous lemma. Put $\mathfrak{k}_{\pi}:=q(\mathfrak{k}(P))$. Then $\mathfrak{k}(P)$ and $\mathfrak{k}\left(P^{\prime}\right)$ are both irreducible components of $q^{-1}\left(\mathfrak{k}_{\pi}\right)$. Hence, $\mathfrak{k}(P)$ and $\mathfrak{k}\left(P^{\prime}\right)$ are $W$-conjugate in $\mathfrak{h}$.

We fix a polarization of $P_{0}$ of $\mathcal{O}$ which gives a Springer resolution of $\overline{\mathcal{O}}$. Let $P$ be another such polarization. By the corollary above, $\mathfrak{k}(P)$ and $\mathfrak{k}\left(P_{0}\right)$ are finite coverings of $\mathfrak{k}_{\pi}$, and there is a $\mathfrak{k}_{\pi}$-isomorphism $\mathfrak{k}(P) \cong$ $\mathfrak{k}\left(P_{0}\right)$. We fix such an isomorphism. Then it induces an isomorphism

$$
\mathfrak{k}(P) \times_{\mathfrak{h} / W} G \mathfrak{r}(P) \xrightarrow{\iota_{P}} \mathfrak{k}\left(P_{0}\right) \times_{\mathfrak{h} / W} G \mathfrak{r}\left(P_{0}\right) .
$$

We put $\nu_{P}:=\iota_{P} \circ \mu_{P}$, and

$$
\mathcal{S}:=\mathfrak{k}\left(P_{0}\right) \times_{\mathfrak{h} / W} G \mathfrak{r}\left(P_{0}\right) .
$$

Denote by $f$ the first projection $\mathcal{S} \rightarrow \mathfrak{k}\left(P_{0}\right)$. Then $f^{-1}(0)=\overline{\mathcal{O}}$, and for each polarization $P$ of $x$,

$$
G \times{ }^{P} \mathfrak{r}(P) \xrightarrow{\nu_{P}} \mathcal{S} \rightarrow \mathfrak{k}\left(P_{0}\right)
$$

gives a simultaneous resolution of $f$. This simultaneous resolution coincides with the Springer resolution $T^{*}(G / P) \rightarrow \overline{\mathcal{O}}$ over $0 \in \mathfrak{k}\left(P_{0}\right)$.

The following conjecture is posed in [F-N].
Conjecture 2. Let $Z$ be a normal symplectic singularity. Then for any two symplectic resolutions $f_{i}: X_{i} \rightarrow Z, i=1,2$, there are flat deformations $\mathcal{X}_{i} \xrightarrow{F_{i}} \mathcal{Z} \rightarrow T$ such that, for $t \in T-\{0\}, F_{i, t}: \mathcal{X}_{i, t} \rightarrow \mathcal{Z}_{t}$ are isomorphisms.

Theorem 7.9. Let $\mathfrak{g}$ be a complex simple Lie algebra. Assume a nilpotent orbit closure $\overline{\mathcal{O}} \subset \mathfrak{g}$ admits a Springer resolution. Then the conjecture holds for the normalization $\tilde{\mathcal{O}}$ of $\overline{\mathcal{O}}$.

Proof. By Theorem 6.1, all symplectic resolutions of $\tilde{\mathcal{O}}$ are realized as Springer resolutions. Take a general curve $T \subset \mathfrak{k}\left(P_{0}\right)$ passing through $0 \in \mathfrak{k}\left(P_{0}\right)$, and pull back the family

$$
G \times^{P} \mathfrak{r}(P) \xrightarrow{\nu_{P}} \mathcal{S} \rightarrow \mathfrak{k}\left(P_{0}\right)
$$

by $T \rightarrow \mathfrak{k}\left(P_{0}\right)$. Put $\overline{\mathcal{Z}}:=\mathcal{S} \times_{\mathfrak{k}\left(P_{0}\right)} T$. Then, for each $P$, we have a simultaneous resolution of $\overline{\mathcal{Z}} \rightarrow T$ :

$$
\mathcal{X}_{P} \rightarrow \overline{\mathcal{Z}} \rightarrow T
$$

Let $\mathcal{Z}$ be the normalization of $\overline{\mathcal{Z}}$. Then the map $\mathcal{X}_{P} \rightarrow \overline{\mathcal{Z}}$ factors through Z. Now

$$
\mathcal{X}_{P} \rightarrow \mathcal{Z} \rightarrow T
$$

gives a desired deformation of the Springer resolution $T^{*}(G / P) \rightarrow \tilde{\mathcal{O}}$.
Example 7.10. Our abstract construction coincides with the explicit construction in $[F-N]$, Theorem 4.4 in the case where $\mathfrak{g}=\mathfrak{s l}(n)$. Let us briefly observe the correspondence between two constructions. Assume that $\mathcal{O}_{x} \subset \mathfrak{s l}(n)$ is the orbit containing an nilpotent element $x$ of type $\mathbf{d}:=\left[d_{1}, \ldots, d_{k}\right]$. Let $\left[s_{1}, \ldots, s_{m}\right]$ be the dual partition of $\mathbf{d}$ (cf. Notation (1)). By Theorem 4.4, the polarizations of $x$ have the flag type $\left(s_{\sigma(1)}, \cdots, s_{\sigma(m)}\right)$ with $\sigma \in \Sigma_{m}$. We denote them by $P_{\sigma}$. We put $P_{0}:=P_{\text {id }}$. Define $F_{\sigma}:=S L(n) / P_{\sigma}$. Let

$$
\tau_{1} \subset \cdots \subset \tau_{m-1} \subset \mathbb{C}^{n} \otimes_{\mathbb{C}} \mathcal{O}_{F_{\sigma}}
$$

be the universal subbundles on $F_{\sigma}$. A point of $T^{*} F_{\sigma}$ is expressed as a pair $(p, \phi)$ of $p \in F_{\sigma}$ and $\phi \in \operatorname{End}\left(\mathbb{C}^{n}\right)$ such that

$$
\phi\left(\mathbb{C}^{n}\right) \subset \tau_{m-1}(p), \cdots, \phi\left(\tau_{2}(p)\right) \subset \tau_{1}(p), \phi\left(\tau_{1}(p)\right)=0
$$

The Springer resolution

$$
s_{\sigma}: T^{*} F_{\sigma} \rightarrow \overline{\mathcal{O}}
$$

is defined as $s_{\sigma}((p, \phi)):=\phi . \quad$ In $[F-N]$, Theorem 4.4, we have next defined a vector bundle $\mathcal{E}_{\sigma}$ with an exact sequence

$$
0 \rightarrow T^{*} F_{\sigma} \rightarrow \mathcal{E}_{\sigma} \xrightarrow{\eta_{\sigma}} \mathcal{O}_{F_{\sigma}}^{m-1} \rightarrow 0
$$

For $p \in F_{\sigma}$, we can choose a basis of $\mathbf{C}^{n}$ such that $T^{*} F_{\sigma}(p)$ consists of the matrices of the following form

$$
\left(\begin{array}{cccc}
0 & * & \cdots & * \\
0 & 0 & \cdots & * \\
\cdots & & & \cdots \\
0 & 0 & \cdots & 0
\end{array}\right)
$$

Then $\mathcal{E}_{\sigma}(p)$ is the vector subspace of $\mathfrak{s l}(n)$ consisting of the matrices $A$ of the following form

$$
\left(\begin{array}{cccc}
a_{\sigma(1)} & * & \cdots & * \\
0 & a_{\sigma(2)} & \cdots & * \\
\cdots & & & \cdots \\
0 & 0 & \cdots & a_{\sigma(m)}
\end{array}\right)
$$

where $a_{i}:=a_{i} I_{s_{i}}$ and $I_{s_{i}}$ is the identity matrix of the size $s_{i} \times s_{i}$. Since $A \in \mathfrak{s l}(n), \Sigma_{i} s_{i} a_{i}=0$. Here we define the map $\eta_{\sigma}(p): \mathcal{E}_{\sigma}(p) \rightarrow \mathbb{C}^{\oplus m-1}$ as $\eta_{\sigma}(p)(A):=\left(a_{1}, a_{2}, \cdots, a_{m-1}\right)$. This vector bundle $\mathcal{E}_{\sigma}$ is nothing but our $S L(n) \times{ }^{P_{\sigma}} \mathfrak{r}\left(P_{\sigma}\right)$. Moreover, the map

$$
\eta_{\sigma}: \mathcal{E}_{\sigma} \rightarrow \mathbb{C}^{m-1}
$$

coincides with the map

$$
S L(n) \times{ }^{P_{\sigma}} \mathfrak{r}\left(P_{\sigma}\right) \rightarrow \mathfrak{k}\left(P_{0}\right),
$$

where we identify $\mathfrak{k}\left(P_{\sigma}\right)$ with $\mathfrak{k}\left(P_{0}\right)$ by an $\mathfrak{k}_{\pi}$-isomorphism. Finally, in [ $F$ $N]$, Theorem 4.4 we have defined $\bar{N} \subset \mathfrak{s l}(n)$ to be the set of all matrices which is conjugate to a matrix of the following form:

$$
\left(\begin{array}{cccc}
b_{1} & * & \cdots & * \\
0 & b_{2} & \cdots & * \\
\cdots & & & \cdots \\
0 & 0 & \cdots & b_{m}
\end{array}\right),
$$

where $b_{i}=b_{i} I_{s_{i}}$ and $I_{s_{i}}$ is the identity matrix of order $s_{i}$. Furthermore the zero trace condition $\sum_{i} s_{i} b_{i}=0$ was required. For $A \in \bar{N}$, let $\phi_{A}(x):=\operatorname{det}(x I-A)$ be the characteristic polynomial of $A$. Let $\phi_{i}(A)$ be the coefficient of $x^{n-i}$ in $\phi(A)$. Here the characteristic map ch: $\bar{N} \rightarrow$ $\mathbb{C}^{n-1}$ has been defined as $\operatorname{ch}(A):=\left(\phi_{2}(A), \ldots, \phi_{n}(A)\right)$. This $\bar{N}$ is nothing but our $S L(n) \mathfrak{r}\left(P_{\sigma}\right)$. As is proved in Corollary 7.6, this is independent of the choice of $P_{\sigma}$. The characteristic map ch above coincides with the composed map

$$
S L(n) \mathfrak{r}\left(P_{\sigma}\right) \subset \mathfrak{s l}(n) \rightarrow \mathfrak{h} / W
$$

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# The moduli stack of rank-two Gieseker bundles with fixed determinant on a nodal curve 

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## §1. Introduction

Let $\left\{Y_{t}\right\}$ be a family of smooth curves degenerating to a nodal curve $X_{0}$. It is an interesting problem to consider how the moduli spaces of vector bundles on $Y_{t}$ degenerate. Since the moduli space of vector bundles on the nodal curve $X_{0}$ is not compact, we need to find a good compactification. One way to compactify it is to add torsion-free sheaves. Another way, which is originally due to Gieseker [G] and developed by Nagaraj-Seshadri [NS] and Kausz [K2], is to add those vector bundles on a certain semistalbe model of $X_{0}$, which let us call Gieseker vector bundles. In these works they consider moduli spaces of vector bundles with fixed degree. In this paper we'd like to consider moduli spaces of vector bundles with fixed determinant.(See [Sun] for related results.)

This paper is heavily based on the work of Kausz [K1] [K2]. So, let me here explain his results briefly. In [K1], Kausz introduced a concept of generalized isomorphisms and showed that a projective variety $K G l_{n}$ that is a compactification of $G l_{n}$ is the fine moduli space of generalized isomorphisms. Then in [K2] he showed that the normalization of the moduli space of Gieseker vector bundles on $X_{0}$ is a $K G l_{n}$-bundle over the moduli space of vector bundles on the normalization $\tilde{X}_{0}$ of $X_{0}$. The purpose of this paper is to show that with the techniques invented by Kausz, we can also describe the structure of the moduli space of Gieseker vector bundles of rank 2 with fixed determinant on an irreducible nodal curve $X_{0}$.

The contents of the sections are as follows. In section 2 we introduce basic definitions. In section 3 we define $\theta$-determinant generalized isomorphisms (only for rank 2 case), and see that the equivalence

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classes of $\theta$-determinant generalized isomorphisms form a projective variety $K S L_{2}$. In section 4 we define the moduli stack of Gieseker- $S L_{2^{-}}$ bundles. In section 5 we investigate the local structure of this stack. In section 6 first using the results in section 5 we see that the moduli stack of Gieseker- $S L_{2}$-bundles on $X_{0}$ is a union of two closed substacks. Then we describe the structure of each substack. Our main theorems (Theorem 6.4 and Theorem 6.5) say that one of the two closed substacks is a $K S L_{2}$-bundle over the moduli stack of vector bundles with fixed determinant on the normalization $\widetilde{X}_{0}$ of $X_{0}$, and that the other is non-reduced and its induced reduced substack is a $\overline{P G l_{2}}\left(\simeq \mathbb{P}^{3}\right)$-bundle over the moduli stack of vector bundles with fixed (but different from the former one) determinant on $\widetilde{X}_{0}$.

The moduli stack of Gieseker- $S L_{2}$-bundles treated in this paper is not semistable. In a forthcoming paper, the author constructs its semistable model as a certain moduli stack.

## §2. Preliminaries and Notations

In this section, we explain some notions and fix some notations that are used in this paper. Most of them are cited directly from [K2, §3].

Throughout this paper, $B:=\operatorname{Spec} \mathbb{C}[[t]], B_{0} \hookrightarrow B$ is the closed point and $B_{\eta}$ is the generic point. $\pi: \mathcal{X} \rightarrow B$ is a stable curve of genus $g \geq 2$ over $B$ such that the generic fiber $X_{\eta}$ is smooth, the special fiber $X_{0}$ is an irreducible curve with only one node $Q$. We assume that $\mathcal{X}$ is regular and $\pi: \mathcal{X} \rightarrow B$ is induced by an analytic family $\mathcal{X}^{a n} \rightarrow B^{a n}$, where $B^{a n}$ is a small open neighborhood of $0 \in \mathbb{C}$. We fix a $\mathbb{C}[[t]]$-algebra isomorphism

$$
\widehat{\mathcal{O}}_{\mathcal{X}, Q} \simeq \mathbb{C}[[u, v, t]] /(u v-t)
$$

that is induced by an analytic isomorphism

$$
\mathcal{O}_{\mathcal{X}^{a n}, Q}^{a n} \simeq \mathbb{C}\{\{u, v, t\}\} /(u v-t)
$$

$\mathfrak{n}: \widetilde{X}_{0} \rightarrow X_{0}$ denotes the normalization and put $\left\{P_{1}, P_{2}\right\}:=\mathfrak{n}^{-1}(Q)$.
2.1. Let $R:=R_{1} \cup \cdots \cup R_{l}(l \geq 1)$ be a chain of rational curves, where $R_{i} \simeq \mathbb{P}^{1}$, and $R_{i} \cap R_{j} \neq \emptyset$ if and only if $|i-j| \leq 1$. Let $a, b$ be closed points of $R_{1}, R_{l}$ respectively such that if $l=1$ then $a \neq b$, and if $l>1$ then $a \neq R_{1} \cap R_{2}$ and $b \neq R_{l-1} \cap R_{l}$. Let $X_{l}$ be the nodal curve that is obtained by identifying the pair of points $\left(P_{1}, P_{2}\right)$ on $\widetilde{X}_{0}$ with $(a, b)$ on $R$. We have the natural morphism $q: \widetilde{X}_{0} \sqcup R \rightarrow X_{l}$. By abuse of notation, the points $q\left(P_{1}\right), q\left(P_{2}\right)$ on $X_{l}$ are also denoted by $P_{1}, P_{2}$ respectively, and $R, R_{i}, \widetilde{X}_{0}$ also denote their isomorphic image in
$X_{l}$ by $q$. By collapsing $R$ to the singular point $Q$ on $X_{0}$, we have the morphism $k: X_{l} \rightarrow X_{0}$. Throughout this paper, we fix this $X_{l}$ and the morphism $k$. By convention we let $k$ also denote id : $X_{0} \rightarrow X_{0}$.

Definition 2.2. (i) Let $T$ be a $B$-scheme and let $f: T \rightarrow B$ denote the structure morphism. A modification of $\mathcal{X}$ over $T$ is a commutative diagram

such that $\mathcal{Y}$ is flat, proper and of finite presentation over $T$, and that for any field $K$ and any morphism $\operatorname{Spec} K \rightarrow T$ if $f(\operatorname{Spec} K)$ is $B_{\eta}$ then $h \times \operatorname{id}_{\text {Spec } K}: \mathcal{Y} \times_{T} \operatorname{Spec} K \rightarrow \mathcal{X} \times_{B} \operatorname{Spec} K$ is an isomorphism, and if $f(\operatorname{Spec} K)$ is $B_{0}$ then for some $l \geq 0$, there is an isomorphism $g: X_{l} \times$ Spec $K \rightarrow \mathcal{Y} \times_{T}$ Spec $K$ satisfying $\left(h \times \operatorname{id}_{\text {Spec } K}\right) \circ g=k \times \mathrm{id}_{\text {Spec } K}$. (ii) Let $T$ be a $B_{0}$-scheme. A modification of $X_{0}$ over $T$ is a modification of $\mathcal{X}$ over $T$, where $T$ is regarded as a $B$-scheme by $B_{0} \hookrightarrow B$.
(iii) If $K$ is a field and Spec $K \rightarrow B_{0}$ is a morphism and $Y \xrightarrow{h} X_{0} \times \operatorname{Spec} K$ is a modification of $X_{0}$ over $\operatorname{Spec} K$, then $l$ that appears in (i) is called the length of the modification.

Definition 2.3. Let $K$ be a field over $\mathbb{C}$.
(i) $h:=k \times$ id : $X_{l} \times \operatorname{Spec} K \rightarrow X_{0} \times \operatorname{Spec} K$ is a modification of length $l \geq 0$ of $X_{0}$ over Spec $K$. A vector bundle $E$ on $X_{l} \times \operatorname{Spec} K$ is said to be admissible if either (a) or (b) below holds;
(a) $l=0$
(b) $l \geq 1$ and $\left.E\right|_{R_{i}}$ is isomorphic to $\mathcal{O}_{R_{i}}(1)^{m} \oplus \mathcal{O}_{R_{i}}^{\mathrm{rank} E-m}$ with $0<m \leq \operatorname{rank} E$ for $1 \leq i \leq l$ and $\mathrm{H}^{0}\left(R,\left(\left.E\right|_{R}\right)\left(-P_{1}-P_{2}\right)\right)=0$.
(ii) Let $h: Y \rightarrow X_{0} \times$ Spec $K$ be a modification of $X_{0}$ over Spec $K$ and let $g: X_{l} \times \operatorname{Spec} K \rightarrow Y$ be as in (i) of Definittion 2.2. A vector bundle $E$ on $Y$ is said to be admissible if $g^{*} E$ is admissible.
(iii) Let $f: T \rightarrow B$ be a morphism and let $h: \mathcal{Y} \rightarrow \mathcal{X} \times{ }_{B} T$ be a modification of $\mathcal{X}$ over $T$. A vector bundle $\mathcal{E}$ on $\mathcal{Y}$ is said to be admissible if for any Spec $K \rightarrow T$, where $K$ is a field, such that $f(\operatorname{Spec} K)=B_{0}$, the pullback of $\mathcal{E}$ to $\mathcal{Y} \times_{T} \operatorname{Spec} K$ is admissible.
§3. $K S L_{2}$
Definition 3.1. Let $S$ be a scheme. If we are given 2 -bundles $\mathcal{V}_{1}$ and $\mathcal{V}_{2}$ on $S$, and an isomorphism $\theta: \bigwedge^{2} \mathcal{V}_{1} \rightarrow \bigwedge^{2} \mathcal{V}_{2}$, then a $\theta$-determinant generalized isomorphism from $\mathcal{V}_{1}$ to $\mathcal{V}_{2}$ is the following data.
(i) 2-bundles $\mathcal{U}_{i}(i=1,2)$ on $S$;
(ii) bf-morphisms of rank one (cf. [K1, Definition 5.1])

$$
g_{i}:=\left(\mathcal{M}_{i}, \mu_{i}, \mathcal{U}_{i} \xrightarrow{g_{i}^{\sharp}} \mathcal{V}_{i}, \mathcal{M}_{i} \otimes \mathcal{U}_{i} \stackrel{g_{i}^{b}}{\leftrightarrows} \mathcal{V}_{i}\right),
$$

$(i=1,2)$ from $\mathcal{U}_{i}$ to $\mathcal{V}_{i} ;$
(iii) an isomorphism $v: \mathcal{M}_{1} \rightarrow \mathcal{M}_{2}$ such that $v\left(\mu_{1}\right)=\mu_{2}$;
(iv) an isomorphism $\xi: \mathcal{U}_{1} \rightarrow \mathcal{U}_{2}$, where we require them to satisfy the conditions (a) and (b) below.
(a) For $\forall s \in S$, we have

$$
\xi[s]\left(\operatorname{Ker} g_{1}^{\sharp}[s]\right) \cap \operatorname{Ker} g_{2}^{\sharp}[s]=\{o\},
$$

where ? $[s]$ means the restriction of ? to the fiber over $s$.
(b) The diagram

$$
\begin{gather*}
\mathcal{M}_{1} \otimes \Lambda^{2} \mathcal{U}_{1} \xrightarrow{v \otimes \wedge^{2} \xi} \mathcal{M}_{2} \otimes \bigwedge^{2} \mathcal{U}_{2} \\
\wedge^{-2} g_{1} \uparrow  \tag{3.1}\\
\Lambda^{2} \mathcal{V}_{1} \xrightarrow[\theta]{ } \quad \wedge^{\wedge^{-2} g_{2}}
\end{gather*}
$$

commutes. (See [K1, Proposition 6.1] for the definition of $\wedge^{-2} g_{i}$.)
Definition 3.2. Keep the notation in Definition 3.1. Let $\Phi^{(l)}:=$ $\left(\mathcal{U}_{i}^{(l)}, g_{i}^{(l)}, \mathcal{M}_{1}^{(l)} \xrightarrow{v^{(l)}} \mathcal{M}_{2}^{(l)}, \mathcal{U}_{1}^{(l)} \xrightarrow{\xi^{(l)}} \mathcal{U}_{2}^{(l)}\right)(l=1,2)$ be a $\theta$-determinant generalized isomorphism from $\mathcal{V}_{1}$ to $\mathcal{V}_{2}$, where $g_{i}^{(l)}$ is the tuple

$$
\left(\mathcal{M}_{i}^{(l)}, \mu_{i}^{(l)}, \mathcal{U}_{i}^{(l)} \xrightarrow{g_{i}^{\sharp(l)}} \mathcal{V}_{i}^{(l)}, \mathcal{M}_{i}^{(l)} \otimes \mathcal{U}_{i}^{(l)} \stackrel{g_{i}^{b(l)}}{\rightleftarrows} \mathcal{V}_{i}^{(l)}\right)
$$

An equivalence from $\Phi^{(1)}$ to $\Phi^{(2)}$ consists of isomorphisms $\mathcal{M}_{j}^{(1)} \simeq \mathcal{M}_{j}^{(2)}$ and $\mathcal{U}_{j}^{(1)} \simeq \mathcal{U}_{j}^{(2)}(j=1,2)$ that are compatible with $v^{(l)}, \xi^{(l)}$ and $g_{i}^{(l)}$.

Definition 3.3. Keep the notation in Definition 3.1. $\mathcal{K} \mathcal{S} \mathcal{L}_{2}\left(\mathcal{V}_{1}, \mathcal{V}_{2}\right)$ is the functor from the category of $S$-schemes to the category of sets that associates to an $S$-scheme $T \xrightarrow{\phi} S$ the set of equivalence classes of $\phi^{*}(\theta)$-determinant generalized isomorphisms from $\phi^{*} \mathcal{V}_{1}$ to $\phi^{*} \mathcal{V}_{2}$.

Then, as in [K1], we have
Proposition 3.4. The functor $\mathcal{K} \mathcal{S} \mathcal{L}_{2}\left(\mathcal{V}_{1}, \mathcal{V}_{2}\right)$ is representable by $a$ projective $S$-scheme $K S L_{2}\left(\mathcal{V}_{1}, \mathcal{V}_{2}\right)$.

Sketch of Proof. We may construct $K S L_{2}\left(\mathcal{V}_{1}, \mathcal{V}_{2}\right)$ locally over $S$. Therefore we may assume that $\mathcal{V}_{1}=\mathcal{V}_{2}=\mathcal{O}_{S}^{\oplus}$ and that $\theta: \bigwedge^{2} \mathcal{V}_{1}(=$ $\left.\mathcal{O}_{S}\right) \rightarrow \bigwedge^{2} \mathcal{V}_{2}\left(=\mathcal{O}_{S}\right)$ is the identity. We let $\mathbb{P}$ denote the $S$-scheme $\operatorname{Proj} \mathcal{O}_{S}\left[x_{11}, x_{12}, x_{21}, x_{22}, x_{00}\right]$ and $K S L_{2}\left(\mathcal{V}_{1}, \mathcal{V}_{2}\right)$ be the closed subscheme of $\mathbb{P}$ defined by $x_{11} x_{22}-x_{12} x_{21}-x_{00}^{2}=0$. Put $\mathbb{B}:=K S L_{2}\left(\mathcal{V}_{1}, \mathcal{V}_{2}\right) \cap$ $\left\{x_{00}=0\right\}$. Let $\pi$ be the projection $K S L_{2}\left(\mathcal{V}_{1}, \mathcal{V}_{2}\right) \rightarrow S$. Let $\mathbf{x}$ : $\pi^{*} \mathcal{V}_{1} \rightarrow \pi^{*} \mathcal{V}_{2} \otimes \mathcal{O}_{K S L_{2}}(\mathbb{B})$ be given by the matrix $\left(x_{i j} / x_{00}\right)_{1 \leq i, j \leq 2}$. Put $\mathcal{U}_{1}:=\mathbf{x}^{-1}\left(\pi^{*} \mathcal{V}_{2}\right)$ and $\mathcal{U}_{2}:=\mathbf{x}\left(\mathcal{U}_{1}\right) \subset \pi^{*} \mathcal{V}_{2}$. We have natural morphisms $\pi^{*} \mathcal{V}_{i} \hookrightarrow \mathcal{U}_{i} \otimes \mathcal{O}(\mathbb{B})(i=1,2)$. These data give us $\pi^{*}(\theta)$-determinat generalized isomorphism from $\pi^{*} \mathcal{V}_{1}$ to $\pi^{*} \mathcal{V}_{2}$, and one can check that it represents the functor $\mathcal{K} \mathcal{S} \mathcal{L}_{2}\left(\mathcal{V}_{1}, \mathcal{V}_{2}\right)$.
Q.E.D.

## §4. Gieseker $S L_{2}$-bundles

In the rest of this paper, we fix a line bundle $\mathcal{P}$ on $\mathcal{X}$, of degree $d$ on the fibers over $B$. Put $\mathcal{P}_{0}:=\left.\mathcal{P}\right|_{X_{0}}$.

Definition 4.1. Let $S$ be a $B$-scheme. A Gieseker- $S L_{2}$-bundle with determinat $\mathcal{P}$ on $\mathcal{X}$ over $S$, or a Gieseker- $S L_{2}$-bundle on $(\mathcal{X} ; \mathcal{P})$ over $S$, is a triple $\left(h: \mathcal{Y} \rightarrow \mathcal{X} \times_{B} S, \mathcal{E}, \delta: \operatorname{det} \mathcal{E} \rightarrow\left(p r_{1} \circ h\right)^{*} \mathcal{P}\right)$, where $h: \mathcal{Y} \rightarrow \mathcal{X} \times_{B} S$ is a modification, $\mathcal{E}$ is an admissible 2-bundle on $\mathcal{Y}$ of degree $d$ on the fibers over $S$, and $\delta$ is a morphism of $\mathcal{O}_{\mathcal{Y}}$-modules such that its restriction to every fiber of $\mathcal{Y} / S$ is nonzero.
$G S L_{2} B(\mathcal{X} / B ; \mathcal{P})$ denotes the $B$-groupoid that associates to an affine $B$-scheme $S$ the groupoid consisting of all the Gieseker- $S L_{2}$-bundles on $(\mathcal{X} ; \mathcal{P})$ over $S . G S L_{2} B\left(X_{0} / B_{0} ; \mathcal{P}_{0}\right)$, or simply $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)$, donotes the $B_{0}$-groupoid that is the restriction of $G S L_{2} B(\mathcal{X} / B ; \mathcal{P})$ to the category of affine $B_{0}$-schemes.

Proposition 4.2. $G S L_{2} B(\mathcal{X} / B ; \mathcal{P})$ and $G S L_{2} B\left(X_{0} / B_{0} ; \mathcal{P}_{0}\right)$ are algebraic stacks.

Remark 4.3. Let $\left\{\varphi_{\lambda \mu}: T_{\mu} \rightarrow T_{\lambda}\right\}_{\lambda \prec \mu}$ be a projective system of affine $B_{0}$-schemes and let $T \xrightarrow{\varphi_{\lambda}} T_{\lambda}$ be a projective limit. By [EGAIV, $\S 8$ and (11.2.6)], we know that $\mathcal{G}:=G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)$ satisfies the conditions (i) and (ii) below;
(i) For any object $x \in \mathcal{G}(T)$, there exist $\lambda$ and an object $x_{\lambda} \in \mathcal{G}\left(T_{\lambda}\right)$ such that $\varphi_{\lambda}^{*}\left(x_{\lambda}\right) \simeq x$;
(ii) Take $\lambda_{0}$ and $x_{\lambda_{0}}, y_{\lambda_{0}} \in \mathcal{G}\left(T_{\lambda_{0}}\right)$. Then the map

$$
\lim _{\longrightarrow} \operatorname{Hom}_{\mathcal{G}\left(T_{\mu}\right)}\left(\varphi_{\lambda_{0} \mu}^{*} x_{\lambda_{0}}, \varphi_{\lambda_{0} \mu}^{*} y_{\lambda_{0}}\right) \rightarrow \operatorname{Hom}_{\mathcal{G}(T)}\left(\varphi_{\lambda_{0}}^{*} x_{\lambda_{0}}, \varphi_{\lambda_{0}}^{*} y_{\lambda_{0}}\right)
$$

is bijective. By this fact, in many proofs we can assume that $T$ is of finite type over $B_{0}$.

## §5. Local Structure

In this section, we investigate the local structure of the algebraic stack of Gieseker- $S L_{2}$-bundles.

Let $K$ be a field extension of $\mathbb{C}$. Let $\left(Y \xrightarrow{h} X_{0} \times_{B_{0}} \operatorname{Spec} K, \mathcal{E}, \operatorname{det} \mathcal{E} \xrightarrow{\delta}\right.$ $\left.\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right)$ be a Gieseker-S $L_{2}$-bundle over Spec $K$.

Lemma 5.1. There are three possibilities:
(Type 0) $Y$ is a modification of lengh 0, i.e. $h$ is an isomorphism.
(Type 1) $Y$ is a modification of lengh 1 , moreover if $R$ is $\mathbb{P}^{1}$ of $Y$ colapsing to the singular point of $X_{0} \times_{B_{0}} \operatorname{Spec} K$, then $\left.\operatorname{deg} \mathcal{E}\right|_{R}=2$.
(Type 2) $Y$ is a modification of lengh 2 , moreover if $R_{i}(i=1,2)$ is $\mathbb{P}^{1}$ of $Y$ colapsing to the singular point of $X_{0} \times{ }_{B_{0}} \operatorname{Spec} K$, then $\left.\operatorname{deg} \mathcal{E}\right|_{R_{i}}=$ 1 for $i=1$ and 2 .

Proof. We have only to exclude the possibility that $Y$ is a modification of lengh 1 and $\left.\operatorname{deg} \mathcal{E}\right|_{R}=1$. Suppose that we had such a Gieseker$S L_{2}$-bundle. Then $\left.\delta\right|_{R}$ is zero since $\left.\operatorname{deg} \mathcal{E}\right|_{R}=1>\left.\operatorname{deg}\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right|_{R}=0$. Hence $\left.\delta\right|_{\widetilde{X}_{0} \times_{B_{0}} \operatorname{Spec} K}$ factors as

$$
\begin{aligned}
\left.\operatorname{det} \mathcal{E}\right|_{\widetilde{X}_{0} \times_{B_{0}} \operatorname{Spec} K} & \left.\rightarrow\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right|_{\tilde{X}_{0} \times_{B_{0}} \operatorname{Spec} K}\left(-P_{1}-P_{2}\right) \\
& \left.\hookrightarrow\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right|_{\widetilde{X}_{0} \times_{B_{0}} \operatorname{Spec} K}
\end{aligned}
$$

Since
$\left.\operatorname{deg} \mathcal{E}\right|_{\widetilde{X}_{0} \times_{B_{0}} \operatorname{Spec} K}=\operatorname{deg} \mathcal{E}-1>\left.\operatorname{deg}\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right|_{\widetilde{X}_{0} \times_{B_{0}} \operatorname{Spec} K}\left(-P_{1}-P_{2}\right)$,
we have $\left.\delta\right|_{\tilde{X}_{0} \times_{B_{0} \operatorname{Spec} K}}=0$, which implies $\delta=0$. This contradicts the definition of a Gieseker- $S L_{2}$-bundle.
Q.E.D.

Notation 5.2. Let $h: Y \rightarrow X_{0} \times$ Spec $K$ be a modification of length $l \geq 0$ and let $g: X_{l} \times \operatorname{Spec} K \rightarrow Y$ be as in (i) of Definition 2.2. Recall from the paragraph 2.1 that if $l \geq 1$ then we have $P_{1}, P_{2}$ on $X_{l}$. From now on, for $l \geq 1$ the points $g\left(P_{1}\right), g\left(P_{2}\right)$ on $Y$ are also denoted by $P_{1}, P_{2}$. If $l=2$, then the point $g\left(R_{1} \cap R_{2}\right)$ on $Y$ is denoted by $P_{0}$. Moreover if $l=0$, then the point $g(Q)$ on $Y$ is denoted by $P_{0}$. The reason why we use this notation will be clear in Proposition 6.1.

In order to investigate the local structure of $G S L_{2} B(\mathcal{X} / B ; \mathcal{P})$, we introduce several deformation functors. Let $\mathcal{A}$ be the category of artinian local $\mathbb{C}[[t]]$-algebra with residue field $\mathbb{C}$. Throughout this section, we fix an object $\mathbb{E}_{0}:=\left(Y \xrightarrow{h_{0}} X_{0}, E_{0}\right.$, $\left.\operatorname{det} E_{0} \xrightarrow{\delta_{0}} h_{0}^{*} \mathcal{P}_{0}\right)$ of $G S L_{2} B(\mathcal{X} / B ; \mathcal{P})\left(B_{0}\right)$. Put $L_{0}:=\left(\operatorname{det} E_{0}\right)^{\vee} \otimes h_{0}^{*} \mathcal{P}_{0}$, and let $\sigma_{0}$ be the global section of $L_{0}$ corresponding to $\delta_{0}$. Let $\mathbb{L}_{0}$ denote the triple $\left(Y \xrightarrow{h_{0}} X_{0}, L_{0}, \sigma_{0}\right)$.

Definition 5.3. Three funtors $\mathcal{G}, \mathcal{F}$ and $\mathcal{M}$ from $\mathcal{A}$ to the category of sets are defined as follows. For $A \in \mathcal{A}$,

$$
\mathcal{G}(A):=\left\{\begin{array}{l}
\mathbb{E}:=\left(\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_{B} \operatorname{Spec} A, \mathcal{E}, \operatorname{det} \mathcal{E} \xrightarrow{\delta}\left(p r_{1} \circ h\right)^{*} \mathcal{P}\right) \\
\in G S L_{2} B(\mathcal{X} / B ; \mathcal{P})(\operatorname{Spec} A) \\
\text { with isomorphism } \mathbb{E} \times{ }_{\text {Spec } A} B_{0} \xrightarrow{\alpha} \mathbb{E}_{0} .
\end{array}\right\} / \sim_{\mathcal{G}},
$$

$\mathcal{F}(A):=$
$\mathcal{M}(A):=$

$$
\left\{\begin{array}{l|l}
\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_{B} \operatorname{Spec} A & \\
\text { with isomorphism } & \mathcal{Y} \xrightarrow{h} \mathcal{X} \times_{B} \operatorname{Spec} A \text { is } \\
\begin{array}{l}
\left.\mathcal{Y} \xrightarrow{h} \mathcal{X} \times{ }_{B} \operatorname{Spec} A\right) \times \times_{\operatorname{Spec} A} B_{0} \\
\xrightarrow{\gamma}\left(Y \xrightarrow{h_{0}} X_{0}\right)
\end{array} & \begin{array}{l}
\text { aver } \operatorname{Spec} A
\end{array}
\end{array}\right\} / \sim \mathcal{M}
$$

where the equivalence relations $\sim_{\mathcal{G}}, \sim_{\mathcal{F}}$ and $\sim_{\mathcal{M}}$ are as below.

- $(\mathbb{E}, \alpha) \sim\left(\mathbb{E}^{\prime}, \alpha^{\prime}\right)$ if and only if there is an isomorphism $\mathbb{E} \xrightarrow{a} \mathbb{E}^{\prime}$ such that $\alpha=\alpha^{\prime} \circ\left(a \times_{\operatorname{Spec} A} B_{0}\right)$.
- $(\mathbb{L}, \sigma) \sim\left(\mathbb{L}^{\prime}, \sigma^{\prime}\right)$ if and only if there is an isomorphism $\mathbb{L} \xrightarrow{b} \mathbb{L}^{\prime}$ such that $\beta=\beta^{\prime} \circ\left(b \times_{\operatorname{Spec} A} B_{0}\right)$.
- $\left(\mathcal{Y} \xrightarrow{h} \mathcal{X} \times{ }_{B} \operatorname{Spec} A, \gamma\right) \sim\left(\mathcal{Y}^{\prime} \xrightarrow{h^{\prime}} \mathcal{X} \times{ }_{B} \operatorname{Spec} A, \gamma^{\prime}\right)$ if and only if there is an isomorphism $\left(\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_{B} \operatorname{Spec} A\right) \xrightarrow{c}\left(\mathcal{Y}^{\prime} \xrightarrow{h^{\prime}}\right.$ $\left.\mathcal{X} \times{ }_{B} \operatorname{Spec} A\right)$ such that $\gamma=\gamma^{\prime} \circ\left(c \times_{\text {Spec } A} B_{0}\right)$.

Lemma 5.4. $\mathcal{G}, \mathcal{F}$ and $\mathcal{M}$ satisfy the Schlessinger's condition (i.e. $\left(H_{1}\right)\left(H_{2}\right)$ and $\left(H_{3}\right)$ in Theorem2.11 of $\left.[\mathrm{Sch}]\right)$. Therefore they have a hull.

Proof. We omit the proof.
Q.E.D.

We have the natural morphism $\Phi: \mathcal{F} \rightarrow \mathcal{M}$ of functors. Using the notation in Definition5.3, by associating $\left(\mathcal{Y} \xrightarrow{h} \mathcal{X} \times{ }_{B} \operatorname{Spec} A,(\operatorname{det} \mathcal{E})^{\vee} \otimes\right.$ $\left.\left(p r_{1} \circ h\right)^{*} \mathcal{P}, \sigma\right) \in \mathcal{F}(A)$ to $(\mathbb{E}, \alpha) \in \mathcal{G}(A)$ (where $\sigma$ is the one determined by $\delta$ ), we have the natural morphism $\Psi: \mathcal{G} \rightarrow \mathcal{F}$.

Lemma 5.5. $\Psi: \mathcal{G} \rightarrow \mathcal{F}$ is smooth.
Proof. Left to the reader.
Q.E.D.

Let $\widehat{\mathcal{A}}$ be the category of complete noetherian local $\mathbb{C}[[t]]$-algebras $A$ such that $A / \mathfrak{m}^{n}$ is in $\mathcal{A}$ for all $n \in \mathbb{N}$. For $R \in \widehat{\mathcal{A}}$, we set $h_{R}(A):=$ $\operatorname{Hom}(R, A)$ to define a functor $h_{R}$ on $\mathcal{A}$.

Theorem 5.6. Let $h_{R} \rightarrow \mathcal{F}$ be a hull of $\mathcal{F}$.
(0) If $\mathbb{E}_{0}$ is of Type 0 , then we have an isomorphism $R \simeq \mathbb{C}[[t]]$ of $\mathbb{C}[[t]]$-algebras.
(1) If $\mathbb{E}_{0}$ is of Type 1 , then we have an isomorphism $R \simeq \mathbb{C}\left[\left[t, t_{1}\right]\right] /\left(t-t_{1}^{2}\right)$ of $\mathbb{C}[[t]]$-algebras.
(2) If $\mathbb{E}_{0}$ is of Type 2 , then we have an isomorphism $R \simeq \mathbb{C}\left[\left[t, t_{0}, t_{1}\right]\right] /\left(t-t_{0} t_{1}^{2}\right)$ of $\mathbb{C}[[t]]$-algebras .

Corollary 5.7. The algebraic $B$-stack $G S L_{2} B(\mathcal{X} / B ; \mathcal{P})$ is regular.
Proof. This follows from Lemma 5.5 and Theorem 5.6. Q.E.D.
The rest of this section is devoted to the proof of Theorem 5.6.
Proof of (0) of Theorem 5.6. It suffices to prove that for any $A \in$ $\mathcal{A}, \mathcal{F}(A)$ is a set consisting of one element. Since $\mathbb{E}_{0}$ is of Type 0 , we may assume that $\mathbb{L}_{0}=\left(X_{0} \xrightarrow{i d} X_{0}, \mathcal{O}_{X_{0}}, 1\right)$. For $A \in \mathcal{A}, \mathbb{L}:=\left(\mathcal{X} \times_{B}\right.$ $\left.\operatorname{Spec} A \xrightarrow{i d} \mathcal{X} \times{ }_{B} \operatorname{Spec} A, \mathcal{O}_{\mathcal{X} \times{ }_{B} \operatorname{Spec} A}, 1\right)$ with the canonical isomorphism $\mathbb{L} \times{ }_{\text {Spec } A} B_{0} \xrightarrow{\beta} \mathbb{L}_{0}$ gives an element of $\mathcal{F}(A)$. Take an element $\left(\mathbb{L}^{\prime}, \beta^{\prime}\right)$ of $\mathcal{F}(A)$, where $\mathbb{L}^{\prime}=\left(\mathcal{Y} \xrightarrow{h^{\prime}} \mathcal{X} \times{ }_{B} \operatorname{Spec} A, \mathcal{L}, \sigma\right)$ and $\beta^{\prime}: \mathbb{L}^{\prime} \times{ }_{\operatorname{Spec} A} B_{0} \xrightarrow{\sim}$ $\mathbb{L}_{0}$. Let us prove that $(\mathbb{L}, \beta) \sim_{\mathcal{F}}\left(\mathbb{L}^{\prime}, \beta^{\prime}\right)$. Since $h^{\prime}$ is an isomorphism and $\sigma$ is a nowhere-vanishing section of $\mathcal{L}$, we may assume that $\mathbb{L}^{\prime}=$ $\left(\mathcal{X} \times{ }_{B} \operatorname{Spec} A \xrightarrow{i d} \mathcal{X} \times_{B} \operatorname{Spec} A, \mathcal{O}_{\mathcal{X} \times{ }_{B} \operatorname{Spec} A}, 1\right)$. Then $\beta^{\prime}$ must be the canonical isomorphism. Thus $(\mathbb{L}, \beta) \sim_{\mathcal{F}}\left(\mathbb{L}^{\prime}, \beta^{\prime}\right)$.
Q.E.D.

We shall give a proof of only (2) of Theorem 5.6 because (1) of Theorem 5.6 is proved similarly. In the rest of the proof of Theorem 5.6 , we assume that $\mathbb{E}_{0}$ is of Type 2 . Put $W:=\operatorname{Spec} \mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right]$ and let $f: W \rightarrow B$ be given by $f^{*}(t)=t_{0} t_{1} t_{2}$. By $[\mathrm{G}, \S 4]$, there exists a modification $\mathcal{Y} \xrightarrow{h} \mathcal{X} \times_{B} W$ of $\mathcal{X} / B$ over $W$ that gives a hull of $\mathcal{M}$.

Since $Y \xrightarrow{h_{0}} X_{0}$ is a modification of length $2, Y$ is a union of $\widetilde{X}_{0}$ and a chain $R_{1} \cup R_{2}$ of $\mathbb{P}^{1}$ with $\left\{P_{i}\right\}=\widetilde{X}_{0} \cap R_{i}$ and $\left\{P_{0}\right\}:=R_{1} \cap R_{2}$. (Recall the notation 5.2.) Moreover we can find an isomorphism
( $\boldsymbol{~}$ )

$$
\widehat{\mathcal{O}}_{\mathcal{Y}, P_{i}} \simeq \mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}, x_{i}, y_{i}\right]\right] /\left(x_{i} y_{i}-t_{i}\right),
$$

of $\mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right]$-algebra $(0 \leq i \leq 2)$ such that for $i=1,2$ on the closed fiber, the branch $x_{i}=0$ corresponds to $\widetilde{X_{0}}$. We fix ( $\left.\boldsymbol{\uparrow}\right)$ and injective morphisms
$\mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}, x_{i}, y_{i}\right]\right] /\left(x_{i} y_{i}-t_{i}\right) \hookrightarrow \mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right]\left(\left(x_{i}\right)\right) \oplus \mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right]\left(\left(y_{i}\right)\right)$, given by $x_{i} \mapsto\left(x_{i}, t_{i} / y_{i}\right)$ and $y_{i} \mapsto\left(t_{i} / x_{i}, y_{i}\right)$.

If $A$ is an artinian local $\mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right]$-algebra with residue field $\mathbb{C}$, the pull-back of the versal deformation by Spec $A \rightarrow W$ gives an infinitesimal deformation $\mathcal{Y}_{A} \xrightarrow{h_{A}} \mathcal{X} \times{ }_{B} \operatorname{Spec} A$ of $Y \xrightarrow{h_{0}} X_{0}$. Let $j_{A}^{(i)}$ be the natural $\operatorname{morphism} j_{A}^{(i)}: \operatorname{Spec} \widehat{\mathcal{O}_{A}, P_{i}} \rightarrow \mathcal{Y}_{A}(i=1,2) . \operatorname{Put} U_{A}:=\mathcal{Y}_{A} \backslash\left\{P_{1}, P_{2}\right\}$. The base change of (5.1) gives rise to the isomorphism

$$
\begin{equation*}
\mathrm{H}^{0}\left(\operatorname{Spec} \widehat{O}_{\mathcal{Y}_{A}, P_{i}} \backslash\left\{P_{i}\right\}, \mathcal{O}\right) \simeq A\left(\left(x_{i}\right)\right) \oplus A\left(\left(y_{i}\right)\right) \tag{A}
\end{equation*}
$$

By the definition of $L_{0}$, we have $\left.\operatorname{deg} L_{0}\right|_{\tilde{X}_{0}}=2$ and $\left.\operatorname{deg} L_{0}\right|_{R_{i}}=-1$ $(i=1,2)$. The nonzero section $\sigma_{0}$ vanishes on $R_{1} \cup R_{2}$, and gives an isomorphism $\mathcal{O}_{\widetilde{X}_{0}} \xrightarrow{\sim}\left(\left.L_{0}\right|_{\tilde{X}_{0}}\right)\left(-P_{1}-P_{2}\right)$. Therefore $L_{0}$ is obtained by gluing at $P_{1}$ and $P_{2}$ the two line bundles $\mathcal{O}_{\tilde{X}_{0}}\left(P_{1}+P_{2}\right)$ on $\widetilde{X}_{0}$ and $\mathcal{O}_{R_{1} \cup R_{2}}\left(-P_{1}-P_{2}\right)$ on $R_{1} \cup R_{2}$. By this, we have the trivializations $\varphi_{\mathbb{C}}^{(i)}: j_{\mathbb{C}}^{(i) *} L_{0} \xrightarrow{\sim} \widehat{\mathcal{O}}_{Y_{0}, P_{i}}(i=1,2)$ and $\psi_{\mathbb{C}}:\left.L_{0}\right|_{U_{\mathbb{C}}} \xrightarrow{\sim} \mathcal{O}_{U_{\mathbb{C}}}$ such that on Spec $\widehat{\mathcal{O}}_{Y_{0}, P_{i}} \backslash\left\{P_{i}\right\}$, the morphism $\psi_{\mathbb{C}} \circ \varphi_{\mathbb{C}}^{(i)-1}$ is given by $\left(a_{i} x_{i}, \frac{1}{y_{i}}\right)$-multiplication for some nonzero complex number $a_{i}$, where $\mathbb{C}$ is considered as a $\mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right]$-algebra by $\mathbb{C} \simeq \mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right] /\left(t_{0}, t_{1}, t_{2}\right)$. By replacing the isomorphisms ( $\mathbf{~})$ if necessary, we may assume that $a_{1}=a_{2}=1$. Moreover replacing $\varphi_{\mathbb{C}}^{(i)}$ and $\psi_{\mathbb{C}}$ if necessary, we may assume that $\varphi_{\mathbb{C}}^{(i)}\left(j_{\mathbb{C}}^{(i) *} \sigma_{0}\right)=y_{i}$ and

$$
\psi_{\mathbb{C}}\left(\left.\sigma_{0}\right|_{U_{\mathbb{C}}}\right)= \begin{cases}1 & \text { on } \widetilde{X}_{0} \backslash\left\{P_{1}, P_{2}\right\}  \tag{5.2}\\ 0 & \text { on } R_{1} \cup R_{2} \backslash\left\{P_{1}, P_{2}\right\}\end{cases}
$$

Put $R:=\mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right] /\left(t_{1}-t_{2}\right)$ and let $\mathfrak{m}$ be its maximal ideal. For $\forall k>0$, we let $\mathcal{L}_{R / \mathfrak{m}^{k}}$ be a line bundle on $\mathcal{Y}_{R / \mathfrak{m}^{k}}$ (the pull-back by Spec $R / \mathfrak{m}^{k} \rightarrow W$ of the versal deformation) that has the trivializations $\varphi_{R / \mathfrak{m}^{k}}^{(i)}: j_{R / \mathfrak{m}^{k}}^{(i) *} \mathcal{L}_{R / \mathfrak{m}^{k}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{Y}_{R / \mathfrak{m}^{k}}, P_{i}}$ and $\psi_{R / \mathfrak{m}^{k}}:\left.\mathcal{L}_{R / \mathfrak{m}^{k}}\right|_{U_{R / \mathfrak{m}^{k}}} \xrightarrow{\sim}$
$\mathcal{O}_{U_{R / \mathfrak{m}^{k}}}$ such that $\psi_{R / \mathfrak{m}^{k}} \circ \varphi_{R / \mathfrak{m}^{k}}^{(i)-1}$ on $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{Y}_{R / \mathfrak{m}^{k}, P_{i}}} \backslash\left\{P_{i}\right\}$ is given by $\left(x_{i}, \frac{1}{y_{i}}\right)$-multiplication. Let $\sigma_{R / \mathfrak{m}^{k}}$ be the global section of $\mathcal{L}_{R / \mathfrak{m}^{k}}$ such that $\varphi_{R / \mathfrak{m}^{k}}^{(i)}\left(j_{R / \mathfrak{m}^{k}}^{(i) *} \sigma_{R / \mathfrak{m}^{k}}\right)=y_{i}$ and

$$
\psi_{R / \mathfrak{m}^{k}}\left(\left.\sigma_{R / \mathfrak{m}^{k}}\right|_{U_{R / \mathfrak{m}^{k}}}\right)= \begin{cases}1 & \text { on } \widetilde{X}_{0} \backslash\left\{P_{1}, P_{2}\right\}  \tag{5.3}\\ t_{1}=t_{2} & \text { on } R_{1} \cup R_{2} \backslash\left\{P_{1}, P_{2}\right\}\end{cases}
$$

(note that, as a topological space, $U_{R / \mathfrak{m}^{k}}$ is a disjoint union of $\widetilde{X}_{0} \backslash$ $\left\{P_{1}, P_{2}\right\}$ and $\left.R_{1} \cup R_{2} \backslash\left\{P_{1}, P_{2}\right\}\right)$. These data give us the formal object $\left(\mathcal{L}_{\infty}, \sigma_{\infty}\right)$, where $\mathcal{L}_{\infty}$ is a line bundle on $\mathcal{Y} \times{ }_{W} \operatorname{Spf} R$, and thus an element $\hat{\xi}=\left(\xi_{k}\right) \in \lim _{\longleftrightarrow} \mathcal{F}\left(R / \mathfrak{m}^{k}\right)$, in other words, a morphism of functors $\Upsilon: h_{R} \rightarrow \mathcal{F}$.

The following proposition completes the proof of Theorem 5.6 (2).
Proposition 5.8. $\Upsilon: h_{R} \rightarrow \mathcal{F}$ is a hull of $\mathcal{F}$.
Proof. We will apply Propositon 7.1. Lemma 5.4 implies (a). Since $\mathcal{M}\left(\mathbb{C}[[t]] /\left(t^{2}\right)\right)=\phi$ and we have a morphism of functors $\mathcal{F} \xrightarrow{\Phi} \mathcal{M},(\mathrm{b})$ also holds. $R$ satisfies (i). Let us see that (ii) holds. Let $\varphi$ be the natural injective morphism

$$
\begin{equation*}
\operatorname{Hom}_{l o c} \mathbb{C}[[t]]-a l g(R, \mathbb{C}[\epsilon]) \hookrightarrow \operatorname{Hom}_{l o c} \mathbb{C}[[t]]-a l g\left(\mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right], \mathbb{C}[\epsilon]\right) \tag{5.4}
\end{equation*}
$$

where $\epsilon^{2}=0$ and $\mathbb{C}[\epsilon]$ is considered as a $\mathbb{C}[[t]]$-algebra by $t \mapsto 0$. We have the commutative diagram

$$
\begin{array}{ccc}
\operatorname{Hom}_{l o c} \mathbb{C}[[t]]-a l g  \tag{5.5}\\
\varphi & \\
\varphi \downarrow \\
& & \mathcal{F}(\mathbb{C}[\epsilon]) \\
\operatorname{Hom}_{l o c} \mathbb{C}[[t]]-\text { alg } \\
\left(\mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right], \mathbb{C}[\epsilon]\right) & \downarrow \Phi \\
& & \mathcal{M}(\mathbb{C}[\epsilon]),
\end{array}
$$

where the morphism in the bottom row is an isomorphism. Consider the category $\mathcal{K}$ whose objects are triples $\left(\mathcal{L}, \sigma, \beta:\left.\mathcal{L}\right|_{X_{2}} \xrightarrow{\sim} L_{0}\right)$, where $\mathcal{L}$ is a line bundle on $X_{2} \times_{\text {Spec } \mathbb{C}} \operatorname{Spec} \mathbb{C}[\epsilon], \sigma$ is a global section of $\mathcal{L}$ and $\beta$ is an isomorphism with $\beta\left(\left.\sigma\right|_{X_{2}}\right)=\sigma_{0}$, and whose morphisms from $\left(\mathcal{L}, \sigma, \beta:\left.\mathcal{L}\right|_{X_{2}} \xrightarrow{\sim} L_{0}\right)$ to $\left(\mathcal{L}^{\prime}, \sigma^{\prime}, \beta^{\prime}:\left.\mathcal{L}^{\prime}\right|_{X_{2}} \xrightarrow{\sim} L_{0}\right)$, are pairs $\left(f: X_{2} \times \operatorname{Spec} \mathbb{C}[\epsilon] \rightarrow X_{2} \times \operatorname{Spec} \mathbb{C}[\epsilon], \mathcal{L} \xrightarrow{\tau} f^{*} \mathcal{L}^{\prime}\right)$, where $f$ is $\mathbb{C}[\epsilon]-$ isomorphism with $\left.f\right|_{X_{2}}=\operatorname{id}_{X_{2}}$ and $\left(h_{0} \times \operatorname{id}_{\text {Spec } \mathbb{C}[\epsilon]}\right) \circ f=h_{0} \times \operatorname{id}_{\text {Spec } \mathbb{C}[\epsilon]}$, and $\tau$ is an isomorphism with $\tau(\sigma)=f^{*} \sigma^{\prime}$. Then $\operatorname{Ker} \Phi$ is isomorphic to the set of isomorphism classes of the category $\mathcal{K}$.

Claim 5.8.1. Every object in the category $\mathcal{K}$ is isomorphic to the trivial one, i.e., $\Phi$ is injective.

Proof of Claim 5.8.1. In the proof we will use Čech cohomologies involving formal neighborhoods. See Proposition 7.3 for the justification of this calculation.

Take an object $\left(\mathcal{L}, \sigma, \beta:\left.\mathcal{L}\right|_{X_{2}} \xrightarrow{\sim} L_{0}\right)$ of the categry $\mathcal{K}$. Since we have the trivial extension of $L_{0}$ over $X_{2} \times \operatorname{Spec} \mathbb{C}[\epsilon]$, the equivalence classes of extensions of $L_{0}$ over $X_{2} \times \operatorname{Spec} \mathbb{C}[\epsilon]$ are classified by $\mathrm{H}^{1}\left(X_{2}, \mathcal{O}_{X_{2}}\right)$. Let $\mathrm{H}^{1}\left(X_{2}, \mathcal{O}_{X_{2}}\right) \xrightarrow{\mathrm{H}_{\sigma}^{1}} \mathrm{H}^{1}\left(X_{2}, L_{0}\right)$ be the morphism induced by the global section $\sigma_{0}$. For $a \in \mathrm{H}^{1}\left(X_{2}, \mathcal{O}_{X_{2}}\right)$, let $\mathcal{L}^{a}$ be the corresponding extension of $L_{0}$. Then $\mathrm{H}_{\sigma}^{1}(a)$ is the obstruction for the existence of a lifting of $\sigma_{0}$ to $\mathcal{L}^{a}$. Hence $\mathcal{L}$ corresponds to a cohomology class in $\operatorname{Ker} \mathrm{H}_{\sigma}^{1}$. Note that $\mathrm{H}_{\sigma}^{1}$ factors as

$$
\mathrm{H}^{1}\left(X_{2}, \mathcal{O}_{X_{2}}\right) \xrightarrow{\mathrm{H}_{\sigma}^{1(1)}} \mathrm{H}^{1}\left(\widetilde{X}_{0}, \mathcal{O}_{\widetilde{X}_{0}}\right) \xrightarrow{\mathrm{H}_{\sigma}^{1(2)}} \mathrm{H}^{1}\left(X_{2}, L_{0}\right),
$$

and $\mathrm{H}_{\sigma}^{1(2)}$ is injective because of the long exact sequence of cohomologies of the exact sequence $0 \rightarrow \mathcal{O}_{\widetilde{X}_{0}} \rightarrow L_{0} \rightarrow \mathcal{O}_{R_{1} \cup R_{2}}\left(-P_{1}-P_{2}\right) \rightarrow 0$. Thus we have $\operatorname{Ker} \mathrm{H}_{\sigma}^{1}=\operatorname{Ker}_{\sigma}^{1(1)}$. On the other hand the exact sequence $0 \rightarrow \mathcal{O}_{R_{1} \cup R_{2}}\left(-P_{1}-P_{2}\right) \rightarrow \mathcal{O}_{X_{2}} \rightarrow \mathcal{O}_{\widetilde{X}_{0}} \rightarrow 0$ gives rise to the exact sequence $0 \rightarrow \mathrm{H}^{1}\left(R_{1} \cup R_{2}, \mathcal{O}_{R_{1} \cup R_{2}}\left(-P_{1}-P_{2}\right)\right) \rightarrow \mathrm{H}^{1}\left(\mathcal{O}_{X_{2}}\right) \rightarrow \mathrm{H}^{1}\left(\mathcal{O}_{\tilde{X}_{0}}\right) \rightarrow 0$. Therefore

$$
\begin{equation*}
\operatorname{Ker} \mathrm{H}_{\sigma}^{1} \simeq \mathrm{H}^{1}\left(R_{1} \cup R_{2}, \mathcal{O}_{R_{1} \cup R_{2}}\left(-P_{1}-P_{2}\right)\right) \tag{5.6}
\end{equation*}
$$

It is easily seen that we have an isomorphism

$$
\begin{equation*}
\mathrm{H}^{1}\left(R_{1} \cup R_{2}, \mathcal{O}_{R_{1} \cup R_{2}}\left(-P_{1}-P_{2}\right)\right) \simeq \mathbb{C} \tag{5.7}
\end{equation*}
$$

by which the cohomology class $\left[\alpha_{1}\left(x_{1}\right), \alpha_{2}\left(x_{2}\right)\right]\left(\alpha_{i}\left(x_{i}\right) \in \mathbb{C}\left[\left[x_{i}\right]\right]\right)$ corresponds to $\alpha_{1}(0)-\alpha_{2}(0)$. So we have $\mathcal{L}=\mathcal{L}^{a}$ for some $a=\left[\left(a_{1}, a_{2}\right)\right] \in$ $\mathrm{H}^{1}\left(R_{1} \cup R_{2}, \mathcal{O}_{R_{1} \cup R_{2}}\left(-P_{1}-P_{2}\right)\right) \subset \mathrm{H}^{1}\left(X_{2}, \mathcal{O}_{X_{2}}\right)$ with $a_{i} \in \mathbb{C} \subset \mathbb{C}\left(\left(x_{i}\right)\right)$. Let $f: X_{2} \times \operatorname{Spec} \mathbb{C}[\epsilon] \rightarrow X_{2} \times \operatorname{Spec} \mathbb{C}[\epsilon]$ be the $\mathbb{C}[\epsilon]$-automorphism with $\left.f\right|_{X_{2}}=\mathrm{id}_{X_{2}}$ constructed as follows. We choose a coordinate $z_{i}$ of $R_{i}\left(\simeq \mathbb{P}^{1}\right)$ so that $P_{i}=\left\{z_{i}=0\right\}$ and $P_{0}=\left\{z_{i}=\infty\right\}$. (Then at $P_{i}$ we have $z_{i}=x_{i} A\left(x_{i}\right)$ with $A\left(x_{i}\right) \in \mathbb{C}\left[\left[x_{i}\right]\right]$ and $A(0) \neq 0$.) Let $\left.f\right|_{R_{1} \cup R_{2}}:\left(R_{1} \cup R_{2}\right) \times \operatorname{Spec} \mathbb{C}[\epsilon] \rightarrow\left(R_{1} \cup R_{2}\right) \times \operatorname{Spec} \mathbb{C}[\epsilon]$ be the $\mathbb{C}[\epsilon]-$ automorphism given by $\left(\left.f\right|_{R_{1} \cup R_{2}}\right)^{\#}\left(z_{i}\right)=\left(1-a_{i} \cdot \epsilon\right) z_{i} .\left.\quad f\right|_{R_{1} \cup R_{2}}$ and id : $\widetilde{X_{0}} \times \operatorname{Spec} \mathbb{C}[\epsilon] \rightarrow \widetilde{X_{0}} \times \operatorname{Spec} \mathbb{C}[\epsilon]$ coincide along $\left\{P_{i}\right\} \times \operatorname{Spec} \mathbb{C}[\epsilon]$ and give rise to a $\mathbb{C}[\epsilon]$-automorphism $f$ of $X_{2} \times \operatorname{Spec} \mathbb{C}[\epsilon]$ with $\left.f\right|_{X_{2}}=\operatorname{id}_{X_{2}}$.

Claim 5.8.2. $f^{*} \mathcal{L}$ is a trivial extension over $X_{2} \times \operatorname{Spec} \mathbb{C}[\epsilon]$ of $L_{0}$.
Proof of Claim 5.8.2. To prove this claim, we shall use the following general fact.

Fact. Let $Z$ be a $\mathbb{C}$-scheme. Then $\mathrm{H}^{0}\left(T_{Z}\right)$ classifies automorphisms of $Z \times \operatorname{Spec} \mathbb{C}[\epsilon]$ over Spec $\mathbb{C}[\epsilon]$ which are identity over Spec $\mathbb{C}$. Moreover, let $\mathcal{U}=\left\{U_{i}\right\}$ be an affine open covering of $Z$ and $M_{0}$ a line bundle on $Z$ defined by a cocycle $\left\{\xi_{i j}\right\} \in \mathrm{Z}^{1}\left(\mathcal{U}, \mathcal{O}_{Z}^{\times}\right)$. Let $f$ be a $\mathbb{C}[\epsilon]$-automorphism of $Z \times \operatorname{Spec} \mathbb{C}[\epsilon]$ determined by a derivation $\partial \in \mathrm{H}^{0}\left(T_{Z}\right)$. If a line bundle $\mathcal{M}$ on $Z \times \operatorname{Spec} \mathbb{C}[\epsilon]$ is an extension of $M_{0}$ determined by $\mu \in \mathrm{H}^{1}\left(Z, \mathcal{O}_{Z}\right)$ and if $\mu^{\prime} \in \mathrm{H}^{1}\left(Z, \mathcal{O}_{Z}\right)$ is the cohomology class corresponding to $f^{*} \mathcal{M}$, then the cohomology class $\mu^{\prime}-\mu \in \mathrm{H}^{1}\left(Z, \mathcal{O}_{Z}\right)$ is given by the cocycle $\left\{\partial \xi_{i j} / \xi_{i j}\right\}$.

Around $P_{i}(i=1,2), f^{\#}: \mathbb{C}[\epsilon]\left[\left[x_{i}, y_{i}\right]\right] /\left(x_{i} y_{i}\right) \rightarrow \mathbb{C}[\epsilon]\left[\left[x_{i}, y_{i}\right]\right] /\left(x_{i} y_{i}\right)$ is given by $x_{i} \mapsto x_{i}+x_{i} B_{i}\left(x_{i}\right) \cdot \epsilon$ and $y_{i} \mapsto y_{i}$, where $B_{i}\left(x_{i}\right) \in \mathbb{C}\left[\left[x_{i}\right]\right]$ with $B_{i}(0)=-a_{i}$. Thus derivation $\partial$ corresponding to $f$ is written as $\partial\left(A\left(x_{i}, y_{i}\right)\right)=x_{i} B_{i}\left(x_{i}\right) \frac{\partial}{\partial x_{i}} A\left(x_{i}, 0\right)$. Hence $\partial\left(x_{i}, 1 / y_{i}\right) /\left(x_{i}, 1 / y_{i}\right)=$ $\left(B_{i}\left(x_{i}\right), 0\right)$. Therefore by the above fact, the difference between $f^{*} \mathcal{L}$ and $\mathcal{L}$ is given by the cohomology class $b=\left[\left(B_{1}\left(x_{1}\right), B_{2}\left(x_{2}\right)\right)\right] \in \mathrm{H}^{1}\left(R_{1} \cup\right.$ $\left.R_{2}, \mathcal{O}\left(-P_{1}-P_{2}\right)\right) \subset \mathrm{H}^{1}\left(X_{2}, \mathcal{O}_{X_{2}}\right)$. We have $f^{*} \mathcal{L}=\mathcal{L}^{a+b}$. By the isomorphism (5.7), $a+b \in \mathrm{H}^{1}\left(R_{1} \cup R_{2}, \mathcal{O}\left(-P_{1}-P_{2}\right)\right)$ corresponds to $a_{1}-a_{2}+\left(B_{1}(0)-B_{2}(0)\right)=0$. Therefore $f^{*} \mathcal{L}$ is a trivial extension of $L_{0}$. This completes the proof of Claim 5.8.2.
Q.E.D.

By Claim 5.8.2, we may assume that $\mathcal{L}$ itself has an isomorphism $\tau: \mathcal{L} \simeq L_{0} \otimes \mathbb{C}[\epsilon]$ such that $\left(\left.\tau\right|_{X_{2}}\right)=\beta$. We have $\tau(\sigma)=(1+c \cdot \epsilon)\left(\sigma_{0} \otimes 1\right)$. Replacing $\tau$ by $(1-c \cdot \epsilon) \tau$, we obtain an isomorphism in $\mathcal{K}$ between $\left(\mathcal{L}, \sigma, \beta:\left.\mathcal{L}\right|_{X_{2}} \simeq L_{0}\right)$ and the trivial deformation. This completes the proof of Claim 5.8.1.
Q.E.D.

Since $\phi$ and $\Phi$ are injective, the next claim completes the proof of Proposition 5.8.

Claim 5.8.3. $\operatorname{Im} \varphi \rightarrow \operatorname{Im} \Phi$ is bijective.
Proof of Claim 5.8.3. We have only to prove the surjectivity of $\operatorname{Im} \varphi \rightarrow \operatorname{Im} \Phi$. Let $g: \operatorname{Spec} \mathbb{C}[\epsilon] \rightarrow W=\operatorname{Spec} \mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right]$ be the morphism given by $g^{\#}\left(t_{j}\right)=a_{j} \cdot \epsilon\left(a_{j} \in \mathbb{C}\right)$. Assume that the pullback $\mathcal{Y} \times_{W}$ Spec $\mathbb{C}[\epsilon]$ by $g$ of the versal family $\mathcal{Y} / W$ is in $\operatorname{Im} \Phi$. This means that there is a line bundle $\mathcal{L}$ with a section $\sigma$ on $\mathcal{Y} \times_{W} \operatorname{Spec} \mathbb{C}[\epsilon]$ that is an extension of the line bundle $L_{0}$ and its global section $\sigma_{0}$ on $\mathcal{Y} \times_{W} \operatorname{Spec} \mathbb{C} \simeq X_{2}$. At $P_{i} \in \mathcal{Y} \times_{W} \operatorname{Spec} \mathbb{C}[\epsilon]$, we have the isomorphism

$$
\widehat{\mathcal{O}}_{\mathcal{Y}_{\times_{W} \operatorname{Spec}} \mathbb{C}[\epsilon], P_{i}} \simeq \mathbb{C}[\epsilon]\left[\left[x_{i}, y_{i}\right]\right] /\left(x_{i} y_{i}-a_{i} \epsilon_{i}\right)
$$

induced by ( $\boldsymbol{\uparrow})$. Let $\mathcal{L}^{\prime}$ be the extension of $L_{0}$ to $\mathcal{Y} \times{ }_{W}$ Spec $\mathbb{C}[\epsilon]$ given by the Čech cocycle $\left\{\left(x_{i}, 1 / y_{i}\right)\right\}_{i=1,2}$, where $\left(x_{i}, 1 / y_{i}\right) \in \mathbb{C}[\epsilon]\left(\left(x_{i}\right)\right) \oplus$ $\mathbb{C}[\epsilon]\left(\left(y_{i}\right)\right) \simeq \mathrm{H}^{0}\left(\operatorname{Spec} \widehat{\mathcal{O}}_{\mathcal{Y} \times{ }_{W} \operatorname{Spec} \mathbb{C}[\epsilon], P_{i}}-\left\{P_{i}\right\}, \mathcal{O}\right)$. Let $o(\mathcal{L})$ and $o\left(\mathcal{L}^{\prime}\right) \in$
$\mathrm{H}^{1}\left(X_{2}, L_{0}\right)$ be the obstructions for the existence of a lifting of $\sigma_{0}$ of $L_{0}$ to $\mathcal{L}$ and $\mathcal{L}^{\prime}$ respectively. It is easy to see that $o(\mathcal{L})-o\left(\mathcal{L}^{\prime}\right)=\sigma_{0} \cdot \xi$, where $\xi$ is an element of $\mathrm{H}^{1}\left(X_{2}, L_{0}\right)$ corresponding to the difference of $\mathcal{L}$ and $\mathcal{L}^{\prime}$. By assumption we have $o(\mathcal{L})=0$. As in the proof of Claim 5.8.1, we have the exact sequence

$$
\begin{equation*}
\mathrm{H}^{1}\left(X_{2}, \mathcal{O}_{X_{2}}\right) \xrightarrow{\mathrm{H}_{\sigma}^{1}} \mathrm{H}^{1}\left(X_{2}, L_{0}\right) \xrightarrow{\text { restr. }} \mathrm{H}^{1}\left(R_{1} \cup R_{2}, \mathcal{O}\left(-P_{1}-P_{2}\right)\right) \rightarrow 0 \tag{5.8}
\end{equation*}
$$

By a concrete calculation, we find that $\left.o\left(\mathcal{L}^{\prime}\right)\right|_{R_{1} \cup R_{2}}$ is represented by the Čech cocycle $\left(a_{1}, a_{2}\right)$, where $a_{i} \in \mathbb{C}$ is considered as an element of $\mathrm{H}^{0}\left(\operatorname{Spec} \widehat{\mathcal{O}}_{R_{i}, P_{i}}-\left\{P_{i}\right\}, \mathcal{O}\left(-P_{i}\right)\right)$. By the isomorphism (5.7), the cohomology class $\left[\left(a_{1}, a_{2}\right)\right]$ corresponds to $a_{1}-a_{2} .\left.o\left(\mathcal{L}^{\prime}\right)\right|_{R_{1} \cup R_{2}}=0$ implies $a_{1}=$ $a_{2}$. Thus $g$ factors as Spec $\mathbb{C}[\epsilon] \rightarrow \operatorname{Spec} R \hookrightarrow W=\operatorname{Spec} \mathbb{C}\left[\left[t_{0}, t_{1}, t_{2}\right]\right]$. This completes the proof of Claim 5.8.3.
Q.E.D.

We complete the proof of Proposition 5.8.
Q.E.D.

Now for the fixed $\mathbb{E}_{0}:=\left(Y \xrightarrow{h_{0}} X_{0}, E_{0}, \operatorname{det} E_{0} \xrightarrow{\delta_{0}} h_{0}^{*} \mathcal{P}_{0}\right)$, assume that $Y \xrightarrow{h_{0}} X_{0}$ is of type 1 or 2 . For $A \in \mathcal{A}$ and an element $\mathbb{L}=(\mathcal{Y} \xrightarrow{h}$ $\left.\mathcal{X} \times{ }_{B} \operatorname{Spec} A, \mathcal{L}, \sigma\right)$ in $\mathcal{F}(A)$, let $\mathcal{Z}_{i}$ be the closed subscheme of $\mathcal{Y}$ whose support is $\left\{P_{i}\right\}$ and whose defining ideal is the first Fitting ideal of $\Omega_{\mathcal{Y} / \text { Spec } A}$ at $P_{i}$. Then $\left.\left(\operatorname{pr}_{2} \circ h\right)\right|_{\mathcal{Z}_{i}}: \mathcal{Z}_{i} \rightarrow \operatorname{Spec} A$ is a closed immersion, and it is an isomorphism if and only if the infinitesimal deformation of the node $P_{i}$ is a trivial deformation. Moreover we have

Corollary 5.9. $\left(\operatorname{pr}_{2} \circ h\right) \mid \mathcal{Z}_{i}: \mathcal{Z}_{i} \rightarrow \operatorname{Spec} A(i=1,2)$ define the same closed subscheme of $\operatorname{Spec} A$.

Proof. $\mathcal{Y} \xrightarrow{h} \mathcal{X} \times{ }_{B} \operatorname{Spec} A$ is isomorphic to the pull-back of the versal deformation by some morphism $g: \operatorname{Spec} A \rightarrow \operatorname{Spec} R$, where $R=\mathbb{C}\left[\left[t, t_{1}, t_{2}\right]\right] /\left(t-t_{1} t_{2}, t_{1}-t_{2}\right)$ if $Y \xrightarrow{h_{0}} X_{0}$ is of type 1 , and $R=$ $\mathbb{C}\left[\left[t, t_{0}, t_{1}, t_{2}\right]\right] /\left(t-t_{0} t_{1} t_{2}, t_{1}-t_{2}\right)$ if $Y \xrightarrow{h_{0}} X_{0}$ is of type 2 . The closed subscheme $\left.\left(p r_{2} \circ h\right)\right|_{\mathcal{Z}_{i}}: \mathcal{Z}_{i} \rightarrow \operatorname{Spec} A$ is defined by $g^{*}\left(t_{i}\right)$. Since $t_{1}=t_{2}$ in $R$, we have $g^{*}\left(t_{1}\right)=g^{*}\left(t_{2}\right)$.
Q.E.D.

We here prepare one proposition that is used in the next section.

Proposition 5.10. For $A \in \mathcal{A}$, let

be an object of $\mathcal{M}(A)$. Let $\iota_{0}: \widetilde{X}_{0} \rightarrow Y$ denote the unique morphism satisfying $h_{0} \circ \iota_{0}=\mathfrak{n}$. Assume that $\left.\left(\operatorname{pr}_{2} \circ h\right)\right|_{\mathcal{Z}_{i}}: \mathcal{Z}_{i} \rightarrow \operatorname{Spec} A(i=1,2)$ are isomorphisms, or equivalently that the infinitesimal deformations of the nodes $P_{i}$ are trivial. Then there exists a unique closed immersion $\iota: \widetilde{X}_{0} \times_{\text {Spec } \mathbb{C}} \operatorname{Spec} A \rightarrow \mathcal{Y}$ such that $\left.\iota\right|_{\tilde{X}_{0}}=g \circ \iota_{0}$ and that the closed subscheme of $\mathcal{Y}$ determined by $\left.\iota\right|_{\left\{P_{i}\right\} \times \operatorname{Spec} A}:\left\{P_{i}\right\} \times \operatorname{Spec} A \rightarrow \mathcal{Y}$ is $\mathcal{Z}_{i}$ for $i=1,2$.

Proof. Fix two distinct points $a_{1}, a_{2}$ on $\mathbb{P}^{1}$. Let $\alpha_{i}$ denote the section $\left(a_{1}, \mathrm{id}_{\text {Spec } \mathbb{C}[[u]]}\right): \operatorname{Spec} \mathbb{C}[[u]] \rightarrow \mathbb{P}^{1} \times \operatorname{Spec} \mathbb{C}[[u]]$ of the projection $\operatorname{pr}_{2}: \mathbb{P}^{1} \times \operatorname{Spec} \mathbb{C}[[u]] \rightarrow \operatorname{Spec} \mathbb{C}[[u]]$. Let $\beta: G \rightarrow \mathbb{P}^{1} \times \operatorname{Spec} \mathbb{C}[[u]]$ be the blowing-up at $\left(a_{1}, 0\right)$. Let $s_{i}: \operatorname{Spec} \mathbb{C}[[u]] \rightarrow G$ be the section of $G \xrightarrow{\mathrm{pr}_{2} \circ \beta} \operatorname{Spec} \mathbb{C}[[u]]$ with $\beta \circ s_{i}=\alpha_{i}$. The section $\left(P_{i}, \mathrm{id}\right): \operatorname{Spec} \mathbb{C}[[u]] \rightarrow$ $\widetilde{X}_{0} \times \operatorname{Spec} \mathbb{C}[[u]]$ is denoted by $s_{i}^{\prime}(i=1,2)$. Let $q: \mathbb{P}^{1} \rightarrow X_{0}$ be the composite $\mathbb{P}^{1} \xrightarrow{p r} \operatorname{Spec} \mathbb{C} \rightarrow\{Q\} \subset X_{0}$. (Recall that $Q$ is the unique node of $X_{0}$.) Put $m:=\left(q \times \operatorname{id}_{\operatorname{Spec} \mathbb{C}[[u]]}\right) \circ \beta$. If $\mathcal{Y}^{*}$ denotes the flat family over Spec $\mathbb{C}[[u]]$ that is constructed from $\widetilde{X}_{0} \times \operatorname{Spec} \mathbb{C}[[u]]$ and $G$ by gluing the sections $s_{i}$ and $s_{i}^{\prime}(i=1,2)$, then we have a morphism $h: \mathcal{Y}^{*} \rightarrow$ $X_{0} \times \operatorname{Spec} \mathbb{C}[[u]]$ because $m \circ s_{i}=(\mathfrak{n} \times 1) \circ s_{i}^{\prime}$. Regard $\mathbb{C}[[u]]$ as a $\mathbb{C}[[t]]$ algebra by $\mathbb{C}[[t]] /(t) \hookrightarrow \mathbb{C}[[u]]$. Applying Proposition 7.1 , it is easily seen that $h: \mathcal{Y}^{*} \rightarrow X_{0} \times \operatorname{Spec} \mathbb{C}[[u]]$ is a versal family of the deformation of the modification $h_{0}: Y \rightarrow X_{0}$ with the singularities $P_{1}, P_{2}$ nondeformed. This implies the existence of $\iota: \widetilde{X}_{0} \times \operatorname{Spec} A \rightarrow \mathcal{Y}$ in the proposition. The uniqueness follows since infinitesimal automorphism of $\widetilde{X}_{0}$ with $\left\{P_{i}\right\}$ fixed are trivial because the 2-pointed curve $\left(\widetilde{X}_{0} ; P_{1}, P_{2}\right)$ is stable.
Q.E.D.

## §6. Global Structure

Proposition 6.1. Let $T$ be a $B_{0}$-scheme.
Let $\left(h: \mathcal{Y} \rightarrow X_{0} \times_{B_{0}} T, \mathcal{E}, \delta: \wedge^{2} \mathcal{E} \rightarrow\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right)$ be a Gieseker-S $L_{2}-$ bundle on $\left(X_{0}, \mathcal{P}_{0}\right)$ over $T$. Then there are closed subsets $\Pi_{i}(i=0,1,2)$
of $\mathcal{Y}$ such that $\Pi_{i} \times_{T} \operatorname{Spec} \kappa(t)=\left\{P_{i}\right\}$ for every $t \in T$. (Here recall the notation 5.2.)

Proof. We may assume that $T$ is reduced irreducible and of finite type over $B_{0}$. We may also assume that $T$ is normal. (In fact, if $T$ is not normal, let $g: T^{\prime} \rightarrow T$ be the normalization and $\left(h^{\prime}: \mathcal{Y}^{\prime}(:=\right.$ $\left.\left.\mathcal{Y} \times_{T} T^{\prime}\right) \rightarrow X_{0} \times_{B_{0}} T^{\prime}, \mathcal{E}^{\prime}, \delta^{\prime}\right)$ be the base-change by $g$ of $(h: \mathcal{Y} \rightarrow$ $\left.X_{0} \times{ }_{B_{0}} T, \mathcal{E}, \delta: \wedge^{2} \mathcal{E} \rightarrow\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right)$. If $\Pi_{i}^{\prime}$ is a closed subset of $\mathcal{Y}^{\prime}$ such that $\Pi_{i}^{\prime} \times_{T^{\prime}} \operatorname{Spec} \kappa\left(t^{\prime}\right)=\left\{P_{i}\right\}$ for every $t^{\prime} \in T^{\prime}$, then let $\Pi_{i}$ be the image of $\Pi_{i}^{\prime}$ to $\mathcal{Y}$.) For $0 \leq l \leq 2, T_{l}$ is defined to be the subset of $T$ that consists of all points $t \in T$ such that $h \times_{T}$ id : $\mathcal{Y} \times_{T} \operatorname{Spec} \kappa(t) \rightarrow X_{0} \times{ }_{B_{0}} \operatorname{Spec} \kappa(t)$ is of length $l$. It is easy to see that $\bigcup_{l \leq m} T_{l}$ is open in $T$. Let $\eta$ be the generic point of $T$. We have an isomorphism

$$
\begin{equation*}
\mathcal{Y} \times_{T} \operatorname{Spec} \kappa(\eta) \simeq X_{l} \times_{B_{0}} \operatorname{Spec} \kappa(\eta) \tag{6.1}
\end{equation*}
$$

over $X_{0} \times_{B_{0}}$ Spec $\kappa(\eta)$ for some $0 \leq l \leq 2$. Let $\sigma_{i}: \operatorname{Spec} \kappa(\eta) \rightarrow$ $\underline{\mathcal{Y} \times_{T}} \operatorname{Spec} \kappa(\eta)$ be the morphism that maps $\eta$ to $P_{i}$, and put $\Pi_{i}:=$ $\overline{\sigma_{i}(\eta)} \subset \mathcal{Y}$, which is given the reduced scheme structure, where $i=0$ if $l=0, i \in\{1,2\}$ if $l=1$, and $i \in\{0,1,2\}$ if $l=2$. Since the isomorphism (6.1) extends over a nonempty open subset $U \subset T$, we have $\Pi_{i} \times_{T} \operatorname{Spec} \kappa(t)=\left\{P_{i}\right\}$ for $\forall t \in U$. If $\mathcal{Z}$ is the closed subscheme of $\mathcal{Y}$ defined by the first Fitting ideal of $\Omega_{\mathcal{Y} / T}$, then $\Pi_{i} \subset \mathcal{Z}$. By this, we know that $\Pi_{i} \rightarrow T$ is a finite birational morphism, hence an isomorphism because of the assumption that $T$ is normal.

Claim 6.1.1. $\Pi_{i} \cap \Pi_{j}=\emptyset$ for $i \neq j$.
Proof of Claim 6.1.1. Suppose that $\Pi_{i} \cap \Pi_{j} \neq \emptyset$ for $i \neq j$. Take a $\mathbb{C}$-valued point $t_{0} \in T$ such that $\Pi_{i} \times_{T} \operatorname{Spec} \kappa\left(t_{0}\right) \cap \Pi_{j} \times_{T} \operatorname{Spec} \kappa\left(t_{0}\right) \neq$ $\emptyset$. Then we can find a morphism $V:=\operatorname{Spec} \mathbb{C}[[v]] \xrightarrow{\alpha} T$ such that $\alpha($ the closed point of $V)=t_{0}$ and $\alpha$ (the generic point of $\left.V\right) \in U$. The base-changes of $\Pi_{i}$ and $\Pi_{j}$ give us two sections $\sigma_{i}, \sigma_{j}: V \rightarrow \mathcal{Y} \times_{T}$ $V$ such that $\sigma_{i}($ the closed point of $V)=\sigma_{j}($ the closed point of $V)$ and $\sigma_{i}($ the generic point of $V)=\left\{P_{i}\right\}$ and $\sigma_{j}($ the generic point of $V)=\left\{P_{j}\right\}$. Taking into account the fact that both $\sigma_{i}$ and $\sigma_{j}$ factor through $\mathcal{Z} \times{ }_{T}$ $V \hookrightarrow \mathcal{Y} \times_{T} V$, we know $\sigma_{i}$ and $\sigma_{j}$ coincide on the closed subscheme $\operatorname{Spec} \mathbb{C}[[v]] /\left(v^{N}\right) \subset V$ for $\forall N>0$. Then we have $\sigma_{i}(V)=\sigma_{j}(V)$. This is a contradiction.
Q.E.D.

Claim 6.1.2. If $\Pi_{i} \times_{T} \operatorname{Spec} \kappa(t)=\left\{P_{m}\right\}$ for $m=1$ or 2 and $t \in T$, then $i=m$.

Proof of Claim 6.1.2. If $\Pi_{i} \times_{T} \operatorname{Spec}(t)=\left\{P_{m}\right\}$ for some $t \in T$, then it holds for some $\mathbb{C}$-valued point $t_{0} \in T$. So we may assume that $t \in T$
is a $\mathbb{C}$-valued point. Take $V:=\operatorname{Spec} \mathbb{C}[[v]] \xrightarrow{\alpha} T$ as in the proof of Claim 6.1.1. Let $\left(\mathcal{Y}_{V} \xrightarrow{h_{V}} X_{0} \times V, \mathcal{E}_{V}, \wedge^{2} \mathcal{E}_{V} \xrightarrow{\delta_{V}}\left(p r_{1} \circ h_{V}\right)^{*} \mathcal{P}_{0}\right)$ be the pullback by $\alpha$ of the given Gieseker- $S L_{2}$-bundle over $T$. Put $\mathcal{Z}_{V}:=\mathcal{Z} \times{ }_{T} V$ and $\Pi_{i V}:=\Pi_{i} \times_{T} V$. Put $V_{N}:=\operatorname{Spec} \mathbb{C}[[v]] /\left(v^{N+1}\right)(\hookrightarrow V)$. $\mathcal{Z}_{V}$ is a disjoint union of the closed subschemes $\mathcal{Z}_{V}^{(i)}$ such that for $\forall N>0$ the support of $\mathcal{Z}_{V_{N}}^{(k)}\left(:=\mathcal{Z}_{V} \times_{V} V_{N}\right)$ is $\left\{P_{k}\right\}$, where $k \in\{1,2\}$ if the length of the modification $\mathcal{Y}_{V} \times_{V} V_{0}$ is 1 , and $k \in\{0,1,2\}$ if the length is 2. Since $\Pi_{i V} \rightarrow V$ is an isomorphism, we have $\Pi_{i V_{N}} \xrightarrow{\sim} \mathcal{Z}_{V_{N}}^{(m)} \xrightarrow{\sim} V_{N}$ for $\forall N>0$, hence $\Pi_{i V}=\mathcal{Z}_{V}^{(m)}$. By this, we know that the deformation of the singularity of $\mathcal{Y}_{V} \times_{V} V_{N}$ at $P_{m}$ is trivial. By Corollary 5.9, the deformation of the singularity of $\mathcal{Y}_{V} \times{ }_{V} V_{N}$ at $P_{3-m}$ is also trivial. Then by Proposition 5.10 and algebraization (cf. (5.1.8) of [EGAIII]), we have the closed immersion $g: \widetilde{X}_{0} \times V \hookrightarrow \mathcal{Y}_{V}$ with $h_{V} \circ g=\mathfrak{n} \times \mathrm{id}_{V}$ such that $g\left(\left\{P_{j}\right\} \times V\right)=\mathcal{Z}_{V}^{(j)}(j=1,2)$. Therefore $\mathcal{Z}_{V}^{(m)} \times_{V} \operatorname{Spec} \mathbb{C}((v))=\left\{P_{m}\right\}$. Since $\Pi_{i V}=\mathcal{Z}_{V}^{(m)}, \Pi_{i V} \times_{V} \operatorname{Spec} \mathbb{C}((v))=\left\{P_{m}\right\}$. This implies $i=m$ since $\alpha($ the generic point of $V) \in U$.
Q.E.D.

With these claims prepared, we will prove the proposition.
Case (i). $T=T_{2}$ : In this case, the above claims imply that $\Pi_{0}, \Pi_{1}, \Pi_{2}$ have the desired property.

Case (ii). $T_{0}=\emptyset$ and $T_{1} \neq \emptyset$ : In this case we have $\Pi_{1}, \Pi_{2}$. Using the above claims plus a similar argument as in the proof of Claim 6.1.2, one can check that $\Pi_{j} \times_{T} \operatorname{Spec} \kappa(t)=\left\{P_{j}\right\}$ for $\forall t \in T, j=1,2$. On $\mathcal{Y} \times_{T} T_{2}$, by Case(i) we have the desired $\Pi_{0} \subset \mathcal{Y} \times_{T} T_{2}$. These $\Pi_{0}, \Pi_{1}, \Pi_{2}$ are what we want.

Case (iii). $T_{0} \neq \emptyset:$ We have $\Pi_{0}$. By Claim 6.1.2, we have $\Pi_{0} \times_{T}$ Spec $\kappa(t)=\left\{P_{0}\right\}$ for $\forall t \in T$. Therefore $T_{1}=\emptyset$. On $\mathcal{Y} \times_{T} T_{2}$, by Case(i) we have $\Pi_{1}, \Pi_{2} \subset \mathcal{Y} \times_{T} T_{2}$ having the desired property. These $\Pi_{0}, \Pi_{1}, \Pi_{2}$ are what we want.

This is the end of the proof of Proposition 6.1.
Q.E.D.
6.2. By this proposition, if we are given a Gieseker $S L_{2}$-bundle $\left(h: \mathcal{Y} \rightarrow X_{0} \times_{B_{0}} T, \mathcal{E}, \wedge^{2} \mathcal{E} \xrightarrow{\delta}\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right)$ on $\left(X_{0}, \mathcal{P}_{0}\right)$ over $T$, the locus of $\mathcal{Y}$ where the morphism $p r_{2} \circ h$ is not smooth is the disjoint union of three closed subsets $\Pi_{0}, \Pi_{1}, \Pi_{2}$. Instead of reduced scheme structure, let us now endow each $\Pi_{i}$ with the scheme structure defined by the first Fitting ideal of $\Omega_{\mathcal{Y} / T}$.

Then $\left.\left(p r_{2} \circ h\right)\right|_{\Pi_{i}}: \Pi_{i} \rightarrow T$ is a closed immersion. Let $\mathcal{I}_{i}\left(\subset \mathcal{O}_{T}\right)$ be its defining ideal. By Corollary 5.9, we have $\mathcal{I}_{1}=\mathcal{I}_{2}$. Moreover, by the
description of the versal family, we have $\mathcal{I}_{0} \mathcal{I}_{1} \mathcal{I}_{2}\left(=\mathcal{I}_{0} \mathcal{I}_{1}^{2}=\mathcal{I}_{0} \mathcal{I}_{2}^{2}\right)=0$. Using these ideals, we shall define closed substacks of $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)$.

Definition 6.3. We define closed substacks $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)^{(0)}$, $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)^{(1)}$ and $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)_{\text {red }}^{(1)}$ of the stack $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)$ as follows:

For an affine $B_{0}$-scheme $T$, an object $\left(h: \mathcal{Y} \rightarrow X_{0} \times_{B_{0}} T, \mathcal{E}, \wedge^{2} \mathcal{E} \xrightarrow{\delta}\right.$ $\left.\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right)$ of $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)(T)$ is in $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)^{(0)}(T)$ [resp. $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)^{(1)}(T)$ or $\left.G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)_{\text {red }}^{(1)}(T)\right]$ if and only if $\mathcal{I}_{0}=0$ [resp. $\mathcal{I}_{1}^{2}=0$ or $\left.\mathcal{I}_{1}=0\right]$.

Put $\widetilde{\mathcal{P}}_{0}:=\mathfrak{n}^{*} \mathcal{P}_{0}$. Let $\mathcal{S U}_{2}\left(\widetilde{X}_{0}, \widetilde{\mathcal{P}}_{0}\right)$ be the moduli stack of 2-bundles on $\widetilde{X}_{0}$ with determinant $\widetilde{\mathcal{P}}_{0}$. More precisely, for an affine $B_{0}$-scheme $T$, objects of the groupoid $\mathcal{S U}_{2}\left(\widetilde{X}_{0}, \widetilde{\mathcal{P}}_{0}\right)(T)$ are 2-bundles $\mathcal{F}$ on $\widetilde{X}_{0} \times{ }_{B_{0}} T$ together with an isomorphism $\wedge^{2} \mathcal{F} \rightarrow p r_{1}^{*} \widetilde{\mathcal{P}}_{0}$.

Put $\sigma_{i}:=\left(P_{i}\right.$, id $): \mathcal{S U}_{2}\left(\widetilde{X}_{0}, \widetilde{\mathcal{P}}_{0}\right) \rightarrow \widetilde{X}_{0} \times \mathcal{S U}_{2}\left(\widetilde{X}_{0}, \widetilde{\mathcal{P}}_{0}\right), i=1,2$. On $\widetilde{X}_{0} \times \mathcal{S U}_{2}\left(\widetilde{X}_{0}, \widetilde{\mathcal{P}}_{0}\right)$, we have the universal 2-bundle $\mathcal{F}_{\text {univ }}$ together with the isomorphism $\wedge^{2} \mathcal{F}_{\text {univ }} \rightarrow p r_{1}^{*} \widetilde{\mathcal{P}}_{0}$. Note that we have the canonical isomorphism $\sigma_{1}^{*} p r_{1}^{*} \widetilde{\mathcal{P}}_{0} \simeq \sigma_{2}^{*} p r_{1}^{*} \widetilde{\mathcal{P}}_{0}$ and thus the canonical isomorphism $\theta: \wedge^{2} \sigma_{1}^{*} \mathcal{F}_{\text {univ }} \xrightarrow{\sim} \wedge^{2} \sigma_{2}^{*} \mathcal{F}_{\text {univ }}$. This allows us to consider the stack $K S L_{2}\left(\sigma_{1}^{*} \mathcal{F}_{\text {univ }}, \sigma_{2}^{*} \mathcal{F}_{\text {univ }}\right)$.

Theorem 6.4. We have an isomorphism of $B_{0}$-stacks

$$
\begin{equation*}
G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)^{(0)} \simeq K S L_{2}\left(\sigma_{1}^{*} \mathcal{F}_{\text {univ }}, \sigma_{2}^{*} \mathcal{F}_{\text {univ }}\right) \tag{6.2}
\end{equation*}
$$

Next consider the moduli stack $\mathcal{S U}_{2}\left(\widetilde{X}_{0}, \widetilde{\mathcal{P}}_{0} \otimes \mathcal{O}_{\widetilde{X}_{0}}\left(-P_{1}+P_{2}\right)\right)$. On $\widetilde{X}_{0} \times \mathcal{S U}_{2}\left(\widetilde{X}_{0}, \widetilde{\mathcal{P}}_{0} \otimes \mathcal{O}_{\widetilde{X}_{0}}\left(-P_{1}+P_{2}\right)\right)$, we have the universal 2-bundle $\mathcal{W}_{\text {univ }}$ with the isomorphism $\wedge^{2} \mathcal{W}_{\text {univ }} \simeq \operatorname{pr}_{1}^{*}\left(\widetilde{\mathcal{P}}_{0} \otimes \mathcal{O}_{\widetilde{X}_{0}}\left(-P_{1}+P_{2}\right)\right)$. Let $\tau_{i}$ denote the morphism $\left(P_{i}\right.$, id $): \mathcal{S U} \mathcal{U}_{2}\left(\widetilde{X}_{0}, \widetilde{\mathcal{P}}_{0} \otimes \mathcal{O}_{\widetilde{X}_{0}}\left(-P_{1}+P_{2}\right)\right) \rightarrow$ $\widetilde{X}_{0} \times \mathcal{S U}_{2}\left(\widetilde{X}_{0}, \widetilde{\mathcal{P}}_{0} \otimes \mathcal{O}_{\widetilde{X}_{0}}\left(-P_{1}+P_{2}\right)\right)$.

Theorem 6.5. We have an isomorphism of $B_{0}$-stacks

$$
\begin{equation*}
G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)_{r e d}^{(1)} \simeq \overline{P G l}\left(\tau_{1}^{*} \mathcal{W}_{\text {univ }}, \tau_{2}^{*} \mathcal{W}_{\text {univ }}\right) \tag{6.3}
\end{equation*}
$$

(See $\S 8$ and $\S 9$ of $[\mathrm{K} 1]$ for the definition of $\overline{P G l}$.)
Remark 6.6. For $i=1,2$, let $X_{i}^{\prime}$ be the nodal curve obtained from $\widetilde{X_{0}}$ and $R_{1} \cup \cdots \cup R_{i}$ by identifying the point $P_{2} \in \widetilde{X_{0}}$ and $b \in$ $R_{1} \cup \cdots \cup R_{i}$ (here we are using the notation in the paragraph 2.1). Let $g: X_{i}^{\prime} \rightarrow X_{i}$ be the morphism that glues the points $P_{1}$ and $a$. In the proof of Theorem 6.5, for a bundle $E$ on $X_{i}$, we first consider $g^{*} E$ plus
the associated gluing data between $\left.g^{*} E\right|_{P_{1}}$ and $\left.g^{*} E\right|_{a}$. At this point, the symmetry in $P_{1}$ and $P_{2}$ breaks. This is why the statement of Theorem 6.5 is not symmetric with respect to $P_{1}$ and $P_{2}$.

The rest of this section is devoted to the proof of the above two theorems.

Definition 6.7. Let

be a modification of the two-pointed curve $\left(\widetilde{X}_{0} ; P_{1}, P_{2}\right)$ over $T$ (cf. Definition 4.4 of [K2]). It is said to be bi-simple if and only if for any $t: \operatorname{Spec} \kappa(t) \rightarrow T$ either (i) or (ii) below holds.
(i) $\widetilde{\mathcal{Y}} \times{ }_{T} \operatorname{Spec} \kappa(t) \xrightarrow{\tilde{h} \times t} \widetilde{X}_{0} \times_{B_{0}} \operatorname{Spec} \kappa(t)$ is an isomorphism.
(ii) Both $(\tilde{h} \times t)^{-1}\left(P_{1}\right)$ and $(\tilde{h} \times t)^{-1}\left(P_{2}\right)$ are isomorphic to $\mathbb{P}_{\kappa(t)}^{1}$.

Definition 6.8. The $B_{0}$-groupoid $G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)}$ of Gieseker- $S L_{2}$-bundle data is defined as follows.

For an affine $B_{0}$-scheme $T$, an object of $G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)}(T)$ is the following collection of data.
(i) A bi-simple modification $\left(\widetilde{\mathcal{Y}}, s_{1}, s_{2}, \tilde{h}\right)$ of $\left(\widetilde{X}_{0}, P_{1}, P_{2}\right)$ over $T$,

(ii) A 2-bundle $\widetilde{\mathcal{E}}$ on $\widetilde{\mathcal{Y}}$,
(iii) An isomorphism $\xi: s_{1}^{*} \widetilde{\mathcal{E}} \xrightarrow{\sim} s_{2}^{*} \widetilde{\mathcal{E}}$,
(iv) An isomorphism

$$
\eta: \mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right) \xrightarrow{\sim}\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0}
$$

Furthermore, we require that they satisfy the following condition.
(a) The pair $\left(\widetilde{\mathcal{E}}, \xi: s_{1}^{*} \widetilde{\mathcal{E}} \xrightarrow{\sim} s_{2}^{*} \widetilde{\mathcal{E}}\right)$ is admissible for $\left(\widetilde{\mathcal{Y}}, s_{1}, s_{2}, \tilde{h}\right)$ in the sense of Definition 4.5 of [K2].
(b) The diagram ( 8 )

$$
\begin{aligned}
& s_{1}^{*}\left(\mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right)\right) \xrightarrow{s_{1}^{*}(\eta)} s_{1}^{*}\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes s_{1}^{*}\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0} \\
& \dagger_{s_{1}^{*}(\mu)} \\
& \stackrel{\mathcal{O}}{\boldsymbol{o}_{2}^{*}(\mu)} \\
& s_{2}^{*}\left(\mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right)\right) \xrightarrow{s_{2}^{*}(\eta)} s_{2}^{*}\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes s_{2}^{*}\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0}
\end{aligned}
$$ commutes, where $\mu$ is the section of $\mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right)$ such that its image by the canonical injection $\mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+\right.$ $\left.P_{2}\right) \hookrightarrow\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right)$ is the pull-back by $p r_{1} \circ \tilde{h}$ of the canonical section of $\mathcal{O}_{\tilde{X}_{0}}\left(P_{1}+P_{2}\right)$. (This is the section defined in Construction 5.2 in [K2].)

Morphisms of the groupoid $G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)}(T)$ are defined obviously.

Lemma 6.9. Let $\left(\widetilde{\mathcal{Y}}, s_{1}, s_{2}, \tilde{h}\right)$ be a bi-simple modification of the 2-pointed curve $\left(\widetilde{X}_{0}, P_{1}, P_{2}\right)$ over an affine $B_{0}$-scheme $T$. Let $\mathcal{L}$ be a line bundle on $\widetilde{\mathcal{Y}}$ and $\lambda$ a section of $\mathcal{L}$ such that $\left.\lambda\right|_{\tilde{\mathcal{Y}} \times_{T} \operatorname{Spec} \kappa(t)} \neq 0$ for $\forall t \in T$. Put $\mathcal{M}:=\mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right)$ and $\mu$ the canonical section defined in Definition 6.8(b). Assume that we have $\left.\left.\mathcal{L}\right|_{\tilde{\mathcal{Y}} \times_{T} \operatorname{Spec} \kappa(t)} \simeq \mathcal{M}\right|_{\tilde{\mathcal{Y}} \times_{T} \operatorname{Spec} \kappa(t)}$ for $\forall t \in T$. Then there is a unique isomorphism $\mathcal{L} \simeq \mathcal{M}$ in which $\lambda$ and $\mu$ corrspond.

Proof. We may assume that T is of finite type over $B_{0}$. We have that $\mathcal{O}_{T} \xrightarrow{\sim}\left(p r_{2} \circ \tilde{h}\right)_{*} \mathcal{O}_{\tilde{\mathcal{Y}}}$ (cf. Lemma 3.9 of [K2]). If we have two isomorphisms $\alpha_{i}: \mathcal{L} \rightarrow \mathcal{M}(i=1,2)$ with $\alpha_{i}(\lambda)=\mu$, then $\alpha_{2} \circ \alpha_{1}^{-1}$ is $\left(p r_{2} \circ \tilde{h}\right)^{*}(a)$-multiplication for some $a \in \mathcal{O}_{T}$. We have the commutative diagram

$$
\begin{align*}
& \mathcal{O}_{T}=\left(p r_{2} \circ \tilde{h}\right)_{*} \mathcal{O}_{\tilde{\mathcal{Y}}} \xrightarrow{\left(p r_{2} \circ \tilde{h}\right)_{*}(\mu)}\left(p r_{2} \circ \tilde{h}\right)_{*} \mathcal{M} \\
& \| \downarrow \times a  \tag{6.4}\\
&\left(p r_{2} \circ \tilde{h}\right)_{*} \mathcal{O}_{\tilde{\mathcal{Y}}} \xrightarrow{\left(p r_{2} \circ \tilde{h}\right)_{*}(\mu)}\left(p r_{2} \circ \tilde{h}\right)_{*} \mathcal{M}
\end{align*}
$$

Since $\operatorname{dim} \mathrm{H}^{0}\left(\left.\mathcal{M}\right|_{\tilde{\mathcal{Y}} \times_{T} \operatorname{Spec} \kappa(t)}\right)=1$ and $\left.\mu\right|_{\tilde{\mathcal{Y}} \times_{T} \operatorname{Spec} \kappa(t)} \neq 0$ for $\forall t \in T$, $\left(p r_{2} \circ \tilde{h}\right)_{*}(\mu)$ is an isomorphism by base-change theorem. So $a=1$, which proves the uniqueness. By the uniqueness it suffices to prove the lemma locally on $T$. Moreover it suffices to prove that for any closed point $t \in T$, there is a Zariski open neighborhood $U \subset T$ such that $\left.\left.\mathcal{L}\right|_{\left(p r_{2} \circ \tilde{h}\right)^{-1}(U)} \simeq \mathcal{M}\right|_{\left(p r_{2} \circ \tilde{h}\right)^{-1}(U)}$. In fact, if so, we can adjust the
isomorphism so that $\lambda$ and $\mu$ correspond because $\left(p r_{2} \circ \tilde{h}\right)_{*}(\lambda): \mathcal{O}_{T} \rightarrow$ $\left(p r_{2} \circ \tilde{h}\right)_{*} \mathcal{L}$ and $\left(p r_{2} \circ \tilde{h}\right)_{*}(\mu): \mathcal{O}_{T} \rightarrow\left(p r_{2} \circ \tilde{h}\right)_{*} \mathcal{M}$ are isomorphisms.

Claim 6.9.1. If $T=\operatorname{Spec} A$, where $(A, \mathfrak{m})$ is an artinian local $\mathbb{C}$ algebra with $\mathbb{C} \xrightarrow{\sim} A / \mathfrak{m}$, then the lemma holds.

Proof of Claim 6.9.1. We prove the claim by induction on $l=$ $\operatorname{dim}_{\mathbb{C}} A$. If $l=1$, by assumption the calim is true. If $l \geq 1$, let $I \subset A$ be an ideal of length one and put $T^{\prime}:=\operatorname{Spec} A / I$ and $\widetilde{\mathcal{Y}^{\prime}}:=\widetilde{\mathcal{Y}} \times{ }_{T} T^{\prime}$. By induction we have $\left.\left.\mathcal{L}\right|_{\tilde{\mathcal{Y}}^{\prime}} \simeq \mathcal{M}\right|_{\tilde{\mathcal{Y}}^{\prime}}$ in which $\left.\lambda\right|_{\tilde{\mathcal{Y}}^{\prime}}$ and $\left.\mu\right|_{\tilde{\mathcal{Y}}^{\prime}}$ correpond. $\mathcal{L}$ and $\mathcal{M}$ are two extensions over $\tilde{\mathcal{Y}}$ of the line bunle $\left.\mathcal{L}\right|_{\tilde{\mathcal{Y}}^{\prime}}\left(\left.\simeq \mathcal{M}\right|_{\tilde{\mathcal{Y}}^{\prime}}\right)$ on $\widetilde{\mathcal{Y}}^{\prime}$. We can express their difference by an element $e \in \mathrm{H}^{1}\left(\widetilde{\mathcal{Y}} \times_{T}\right.$ $\operatorname{Spec} A / \mathfrak{m}, \mathcal{O})$. Since the sections $\left.\lambda\right|_{\tilde{\mathcal{Y}}^{\prime}}$ and $\left.\mu\right|_{\tilde{\mathcal{Y}}^{\prime}}$ extend over $\widetilde{\mathcal{Y}}^{\prime}$, we have $e \cdot\left(\left.\mu\right|_{\tilde{\mathcal{Y}} \times{ }_{T} \operatorname{Spec} A / \mathfrak{m}}\right)=0$ in $\mathrm{H}^{1}\left(\widetilde{\mathcal{Y}} \times_{T} \operatorname{Spec} A / \mathfrak{m},\left.\mathcal{M}\right|_{\tilde{\mathcal{Y}} \times{ }_{T} \operatorname{Spec} A / \mathfrak{m}}\right)$. Since $\mathrm{H}^{1}\left(\widetilde{\mathcal{Y}} \times_{T} \operatorname{Spec} A / \mathfrak{m}, \mathcal{O}\right) \xrightarrow{\mu} \mathrm{H}^{1}\left(\widetilde{\mathcal{Y}} \times_{T} \operatorname{Spec} A / \mathfrak{m},\left.\mathcal{M}\right|_{\tilde{\mathcal{Y}}_{\times_{T}} \operatorname{Spec} A / \mathfrak{m}}\right)$ is bijective, we have $e=0$, by which we have $\mathcal{L} \simeq \mathcal{M}$. Adjusting this so that $\lambda$ and $\mu$ correspond, we prove the claim.
Q.E.D.

Take a closed point $t \in T$. Since we have $\left(p r_{2} \circ \tilde{h}\right)_{*} \mathcal{H o m}_{\mathcal{O}_{\tilde{\mathcal{V}}}}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}_{T}}$ $\widehat{\mathcal{O}}_{T, t} \xrightarrow{\sim} \underset{\rightleftarrows}{\lim } \operatorname{Hom}\left(\mathcal{L} \otimes_{\mathcal{O}_{T}} \mathcal{O}_{T} / \mathfrak{m}^{n}, \mathcal{M} \otimes_{\mathcal{O}_{T}} \mathcal{O}_{T} / \mathfrak{m}^{n}\right)$, the above claim implies that we can find $\varphi \in\left(p r_{2} \circ \tilde{h}\right)_{*} \mathcal{H o m}_{\mathcal{O}_{\tilde{\mathcal{V}}}}(\mathcal{L}, \mathcal{M}) \otimes_{\mathcal{O}_{T}} \widehat{\mathcal{O}}_{T, t}$ such that $\varphi \otimes_{\mathcal{O}_{T}} \mathcal{O}_{T, t} / \mathfrak{m}:\left.\left.\mathcal{L}\right|_{\tilde{\mathcal{Y}} \times_{T} \operatorname{Spec} \mathcal{O}_{T, t} / \mathfrak{m}} \rightarrow \mathcal{M}\right|_{\tilde{\mathcal{Y}} \times_{T} \operatorname{Spec} \mathcal{O}_{T, t} / \mathfrak{m}}$ is an isomorphism. Extending $\varphi$ over some Zariski open neighborhood of $t$, we complete the proof of the lemma.
Q.E.D.

Proposition 6.10. We have an isomorphism of $B_{0}$-groupoids

$$
G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)^{(0)} \xrightarrow{\sim} G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)} .
$$

Proof. It suffices to establish the isomorphism over the full subcategory of affine schemes of finite type over $B_{0}$.

Construction of $\Phi: G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)^{(0)} \rightarrow G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)}$.
Let $T$ be an affine scheme of finite type over $B_{0}$ and let $(\mathcal{Y} \xrightarrow{h}$ $\left.X_{0} \times T, \mathcal{E}, \wedge^{2} \mathcal{E} \xrightarrow{\delta}\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right)$ be an object of $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)^{(0)}(T)$. Let $\Pi_{0} \subset \mathcal{Y}$ be as in the paragraph 6.2. In our situation $\Pi_{0}$ is a section over $T$. Let $\widetilde{\mathcal{Y}} \xrightarrow{g} \mathcal{Y}$ be the blowing-up along $\Pi_{0}$. It is easily checked that $\widetilde{\mathcal{Y}}$ is flat over $T$ and that there is a unique morphism $\tilde{h}: \widetilde{\mathcal{Y}} \rightarrow \widetilde{X}_{0} \times T$
satisfying $\left(\mathfrak{n} \times \mathrm{id}_{T}\right) \circ \tilde{h}=h \circ g$.

$g^{-1}\left(\Pi_{0}\right)$ consists of the disjoint two sections $s_{1}$ and $s_{2}$ over $T$ such that $\tilde{h} \circ s_{i}=\left(P_{i}, \mathrm{id}_{T}\right)$. Then $\left(\tilde{h}: \widetilde{\mathcal{Y}} \rightarrow \widetilde{X}_{0} \times T, s_{1}, s_{2}\right)$ is a bi-simple modification of $\left(\widetilde{X}_{0}, P_{1}, P_{2}\right)$ over $T$. Put $\widetilde{\mathcal{E}}:=g^{*} \mathcal{E}$. Let $\xi$ be the composite of natural isomorphisms $s_{1}^{*} \widetilde{\mathcal{E}} \simeq s_{1}^{*} g^{*} \mathcal{E} \simeq \Pi_{0}^{*} \mathcal{E} \simeq s_{2}^{*} g^{*} \mathcal{E} \simeq s_{2}^{*} \widetilde{\mathcal{E}}$. We have $g^{*}(\delta): \wedge^{2} \widetilde{\mathcal{E}} \simeq g^{*} \wedge^{2} \mathcal{E} \rightarrow g^{*}\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0} \simeq\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0}$, which induces a morphism $\lambda: \mathcal{O}_{\tilde{\mathcal{Y}}} \rightarrow\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0}$. By Lemma 6.9 , there is a unique isomorphism $\eta: \mathcal{M}:=\mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right) \xrightarrow{\sim}$ $\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0}$ such that $\lambda$ corresponds to the canonical section $\mu$ of $\mathcal{M}$. Since $\lambda$ is a pull-back of the morphism on $\mathcal{Y}$, the diagram $(\Omega)$ in Definition 6.8 commutes. These data define an object of $G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)}(T)$.

Construction of $\Psi: G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)} \rightarrow G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)^{(0)}$.
Given an object of $G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)}(T), \quad\left(\left(\widetilde{\mathcal{Y}} \xrightarrow{\tilde{h}} \widetilde{X}_{0} \times\right.\right.$ $\left.T, s_{1}, s_{2}\right), \widetilde{\mathcal{E}}, s_{1}^{*} \widetilde{\mathcal{E}} \xrightarrow{\xi} s_{2}^{*} \widetilde{\mathcal{E}}, \eta: \mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right) \xrightarrow{\sim}$ $\left.\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0}\right)$, let $g: \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ be a cokernel of $T \xrightarrow[s_{2}]{\stackrel{s_{1}}{\longrightarrow}} \widetilde{\mathcal{Y}}$. There is a unique $T$-morphism $h: \mathcal{Y} \rightarrow X_{0} \times T$ with $h \circ g=\left(\mathfrak{n} \times \mathrm{id}_{T}\right) \circ \tilde{h}$. Put $\Pi_{0}:=g \circ s_{i}$ and $\mathcal{E}:=\operatorname{Ker}\left(g_{*} \widetilde{\mathcal{E}} \xrightarrow{s_{2}^{\#}-\xi \circ s_{1}^{\#}} s_{2}^{*} \widetilde{\mathcal{E}}\right)$. The commutativity of the diagram $(\Omega)$ in Definition 6.8 induces a morphism $\mathcal{O} \rightarrow\left(\wedge^{2} \mathcal{E}\right)^{\vee} \otimes\left(p r_{1} \circ\right.$ $h)^{*} \mathcal{P}_{0}$, which gives $\delta: \wedge^{2} \mathcal{E} \rightarrow\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}$.

We can see that the construction $\Phi$ and $\Psi$ commute with isomorphisms and base changes and that they are inverses to each other.
Q.E.D.

Proposition 6.11. We have an isomorphism of $B_{0}$-groupoids

$$
G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)} \simeq K S L\left(\sigma_{1}^{*} \mathcal{F}_{\text {univ }}, \sigma_{2}^{*} \mathcal{F}_{\text {univ }}\right)
$$

Lemma 6.12. Let $S$ be a locally noetherian scheme, and let $\pi$ : $\mathcal{C} \rightarrow S$ be a proper, flat morphism with connected geometric fibers of dimension one, and let $s: S \rightarrow \mathcal{C}$ be a section over $S$ such that $s(S)$ is in the smooth locus of $\pi$. Let $\left(\mathcal{C}^{\prime}, f, \pi^{\prime}, s^{\prime}\right)$ be a simple modification of $(\mathcal{C}, \pi, s)$ (cf. Definition 5.1 of [K2]). Let $n \geq 1$ and $1 \leq d \leq n$,
and let $\mathcal{E}^{\prime}$ be a rank $n$ vector bundle on $\mathcal{C}^{\prime}$ that is admissible of degree $d$ for $\left(\mathcal{C}^{\prime}, f, \pi, s^{\prime}\right)$ (cf. Definition 7.1 of $[\mathrm{K} 2]$ ). Put $\mathcal{N}:=\left(f^{*} \mathcal{O}_{\mathcal{C}}(s)\right)\left(-s^{\prime}\right)$, $N:=s^{*} \mathcal{N}$ and $\mathcal{F}:=\left(f_{*} \mathcal{E}^{\prime}\left(-s^{\prime}\right)\right)(s)$. Let $g: s^{\prime *} \mathcal{E}^{\prime} \xrightarrow{\otimes N} s^{*} \mathcal{F}$ be the bfmorphism of rank $n-d$ constructed in $\S 7$ of $[\mathrm{K} 2]$. Then there exists a unique isomorphism $\rho: \wedge^{n} \mathcal{E}^{\prime} \xrightarrow{\sim} f^{*}\left(\wedge^{n} \mathcal{F}\right) \otimes \mathcal{N}^{-d}$ such that $s^{\prime *}(\rho)=\wedge^{n} g$ (See §6 of [K2] for the definition of $\wedge^{n} g$ ).

Proof. By Lemma 7.6 of [K2], for $\forall x \in \mathcal{C}$ there exists an open neighborhood $U$ of $x$ such that we have an isomorphims $\alpha:\left.\mathcal{E}^{\prime}\right|_{f^{-1}(U)} \rightarrow$ $\left(\mathcal{N}^{-1}\right)^{\oplus d} \oplus \mathcal{O}_{f^{-1}(U)}^{\oplus n-d}$. Let $\beta$ be the composite of isomorphims
$\left.\mathcal{F}\right|_{U}=\left.f_{*}\left(\mathcal{E}^{\prime} \otimes \mathcal{N}\right)\right|_{U} \rightarrow \mathcal{O}^{\oplus d} \oplus\left(\left.f_{*}(\mathcal{N})\right|_{U}\right)^{\oplus n-d} \underset{\longleftrightarrow}{\left(1^{\oplus d}, f_{*}(\nu)^{\oplus n-d}\right)} \mathcal{O}_{U}^{\oplus d} \oplus \mathcal{O}_{U}^{\oplus n-d}$,
where $\nu$ is the canonical section of $\mathcal{N}$. We define the morphism $\left.\rho\right|_{f^{-1}(U)}$ : $\left.\left.\wedge^{n} \mathcal{E}^{\prime}\right|_{f^{-1}(U)} \rightarrow f^{*}\left(\wedge^{n} \mathcal{F}\right) \otimes \mathcal{N}^{-d}\right|_{f^{-1}(U)}$ by $\left(\wedge^{n} \alpha\right) \circ\left(f^{*}\left(\wedge^{n} \beta\right) \otimes \operatorname{id}_{\mathcal{N}^{-d}}\right)^{-1}$. One can check that $\left.\rho\right|_{f^{-1}(U)}$ is independent of the choice of $\alpha$. Therefore we have globally an isomorphism $\rho: \wedge^{n} \mathcal{E}^{\prime} \xrightarrow{\sim} f^{*}\left(\wedge^{n} \mathcal{F}\right) \otimes \mathcal{N}^{-d} . s^{\prime *}(\rho)=$ $\wedge^{n} g$ follows from Lemma 7.5 of [K2]. The uniqueness follows from the isomorphism $\pi_{*}^{\prime}\left(\mathcal{O}_{\mathcal{C}^{\prime}}\right) \xrightarrow{\sim} s^{\prime *} \mathcal{O}_{\mathcal{C}^{\prime}}$.
Q.E.D.

Proof of Proposition 6.11.
Construction of $G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)} \rightarrow K S L_{2}\left(\sigma_{1}^{*} \mathcal{F}_{\text {univ }}, \sigma_{2}^{*} \mathcal{F}_{\text {univ }}\right)$.
Let $T$ be an affine $B_{0}$-scheme. As in the proof of Proposition 6.10, we may assume that T is of finite type over $B_{0}$. Let $\left(\left(\tilde{\mathcal{Y}} \xrightarrow{\tilde{h}} \widetilde{X}_{0} \times\right.\right.$ $\left.T, s_{1}, s_{2}\right), \widetilde{\mathcal{E}}, s_{1}^{*} \widetilde{\mathcal{E}} \xrightarrow{\xi} s_{2}^{*} \widetilde{\mathcal{E}}, \eta: \mathcal{M}:=\mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right) \xrightarrow{\sim}$ $\left.\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0}\right)$ be an object of $G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)}(T)$. Let $\mu$ be the canonical section of $\mathcal{M}$. Put $\mathcal{F}:=\tilde{h}_{*}(\widetilde{\mathcal{E}} \otimes \mathcal{M}), M_{i}:=s_{i}^{*} \mathcal{M}$ and $\mu_{i}:=s_{i}^{*}(\mu)$. Then by $\S 7$ of [K2], we have bf-morphisms of rank one $\alpha_{i}: s_{i}^{*} \widetilde{\mathcal{E}} \xrightarrow{\otimes M_{i}}\left(P_{i}, \mathrm{id}_{T}\right)^{*} \mathcal{F}$. Taking $\xi$ into account, we have diagram $(\diamond)$,


$$
\left(P_{1}, \operatorname{id}_{T}\right)^{*} \mathcal{F} \quad\left(P_{2}, \mathrm{id}_{T}\right)^{*} \mathcal{F}
$$

By Lemma 6.12, we have the natural isomorphism $\wedge^{2} \mathcal{F} \simeq \tilde{h}_{*}\left(\wedge^{2} \widetilde{\mathcal{E}} \otimes \mathcal{M}\right)$. Combining this with $\tilde{h}_{*}(\eta)$, we have the isomorphism $\zeta: \wedge^{2} \mathcal{F} \xrightarrow{\sim} p r_{1}^{*} \widetilde{\mathcal{P}}_{0}$
such that $\left(P_{i}, \mathrm{id}_{T}\right)^{*}(\zeta)$ is the composite

$$
\begin{aligned}
\wedge^{2}\left(P_{i}, \mathrm{id}\right)^{*} \mathcal{F} & \xrightarrow{\wedge^{-2} \alpha_{i}}\left(\wedge^{2} s_{i}^{*} \widetilde{\mathcal{E}}\right) \otimes M_{1} \\
& \xrightarrow{s_{i}^{*}\left(\eta \otimes \mathrm{id}_{\left.\wedge^{2} \tilde{\mathcal{E}}\right)}\right.} s_{i}^{*}\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0}=\left(P_{i}, \mathrm{id}_{T}\right)^{*} p r_{1}^{*} \widetilde{\mathcal{P}}_{0}
\end{aligned}
$$

There is a unique isomorphism $v: M_{1} \rightarrow M_{2}$ such that the composite

$$
\begin{aligned}
& \left(P_{1}, \mathrm{id}_{T}\right)^{*} p r_{1}^{*} \widetilde{\mathcal{P}}_{0} \xrightarrow{\left(P_{1}, \mathrm{id}_{T}\right)^{*}(\zeta)^{-1}}\left(P_{1}, \mathrm{id}_{T}\right)^{*} \wedge^{2} \mathcal{F} \\
& \xrightarrow{\wedge^{-2} \alpha_{1}} \wedge^{2} s_{1}^{*} \widetilde{\mathcal{E}} \otimes M_{1} \xrightarrow{\wedge^{2} \xi \otimes v} \wedge^{2} s_{2}^{*} \widetilde{\mathcal{E}} \otimes M_{2} \\
& \xrightarrow{\left(\wedge^{-2} \alpha_{2}\right)^{-1}} \wedge^{2}\left(P_{2}, \mathrm{id}_{T}\right)^{*} \mathcal{F} \xrightarrow{\left(P_{2}, \mathrm{id}_{T}\right)^{*}(\zeta)}\left(P_{2}, \mathrm{id}_{T}\right)^{*} p r_{1}^{*} \widetilde{\mathcal{P}}_{0}
\end{aligned}
$$

is the canonical morphism. Moreover by the diagram ( $\Omega$ ), we have $v\left(\mu_{1}\right)=\mu_{2}$. The admissibility of the pair $(\widetilde{\mathcal{E}}, \xi)$ implies that

$$
\cap_{i=1}^{2} \operatorname{Ker}\left(s_{i}^{*} \mathcal{E}_{i} \rightarrow\left(P_{i}, \operatorname{id}_{T}\right)^{*} \mathcal{F}\right)=\{o\}
$$

Therefore these data give an object of $K S L_{2}\left(\sigma_{1}^{*} \mathcal{F}_{\text {univ }}, \sigma_{2}^{*} \mathcal{F}_{\text {univ }}\right)(T)$. Construction of $K S L_{2}\left(\sigma_{1}^{*} \mathcal{F}_{\text {univ }}, \sigma_{2}^{*} \mathcal{F}_{\text {univ }}\right) \rightarrow G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)}$.

Take an object of $K S L_{2}\left(\sigma_{1}^{*} \mathcal{F}_{\text {univ }}, \sigma_{2}^{*} \mathcal{F}_{\text {univ }}\right)(T)$, that is, the following data:

- a 2-bundle $\mathcal{F}$ on $\widetilde{X}_{0} \times T$;
- an isomorphism $\zeta: \wedge^{2} \mathcal{F} \xrightarrow{\sim} p r_{1}^{*} \widetilde{\mathcal{P}}_{0}$;
- two line bundles $M_{1}, M_{2}$ on $T$ and their sections $\mu_{1}, \mu_{2}$;
- an isomorphism $v: M_{1} \xrightarrow{\sim} M_{2}$ satisfying $v\left(\mu_{1}\right)=\mu_{2}$;
- two line bundles $E_{1}, E_{2}$ on $T$;
- an isomorphism $\xi: E_{1} \xrightarrow{\sim} \underset{\otimes M_{i}}{ } E_{2} ;$
- two bf-morphisms $\alpha_{i}: E_{i} \xrightarrow{\curvearrowleft}\left(P_{i}, \mathrm{id}_{T}\right)^{*} \mathcal{F}, i=1,2$,
such that

$$
\begin{equation*}
\xi\left(\operatorname{Ker}\left(E_{1}[t] \rightarrow\left(P_{1}, \operatorname{id}_{T}\right)^{*} \mathcal{F}[t]\right)\right) \cap \operatorname{Ker}\left(E_{2}[t] \rightarrow\left(P_{2}, \operatorname{id}_{T}\right)^{*} \mathcal{F}[t]\right)=\{o\} \tag{6.6}
\end{equation*}
$$

and the diagram

$$
\begin{array}{cc}
\wedge^{2} E_{1} \otimes M_{1} & \stackrel{\wedge^{2} \xi \otimes v}{ } \\
\wedge^{2} \alpha_{1} \uparrow & \wedge^{2} E_{2} \otimes M_{2}  \tag{6.7}\\
\wedge^{2}\left(P_{1}, \mathrm{id}_{T}\right)^{*} \mathcal{F} & \uparrow \wedge^{2} \alpha_{2} \\
\left(P_{1}, \mathrm{id}_{T}\right)^{*}(\zeta) \mid & \wedge^{2}\left(P_{2}, \mathrm{id}_{T}\right)^{*} \mathcal{F} \\
\left(P_{1}, \mathrm{id}_{T}\right)^{*} p r_{1}^{*} \widetilde{\mathcal{P}}_{0} \xrightarrow{\text { canonical isom. }} & \\
& \\
& \\
& \left.P_{2}, \mathrm{id}_{T}\right)^{*} p r_{1}^{*} \widetilde{\mathcal{P}}_{0}
\end{array}
$$

commutes. Making use of the bf-morphisms $\alpha_{1}$ and $\alpha_{2}$, we obtain a bi-simple modification $\left(\tilde{h}: \widetilde{\mathcal{Y}} \rightarrow \widetilde{X}_{0} \times T, s_{1}, s_{2}\right)$

plus isomorphisms $\epsilon_{i}: s_{i}^{*} \mathcal{M} \xrightarrow{\sim} M_{i}$ with $\epsilon_{i}(\mu)=\mu_{i}$ by $\S 5$ of [K2], where $\mathcal{M}:=\mathcal{O}\left(-s_{1}-s_{2}\right) \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \mathcal{O}\left(P_{1}+P_{2}\right)$ and $\mu$ is the canonical section of $\mathcal{M}$. By Construction 7.1 in [K2], there exists a 2-bundle $\widetilde{\mathcal{E}}$ on $\widetilde{\mathcal{Y}}$ together with isomorphisms $\tilde{h}_{*}(\widetilde{\mathcal{E}} \otimes \mathcal{M}) \simeq \mathcal{F}$ and $s_{i}^{*} \widetilde{\mathcal{E}} \simeq E_{i}$ such that they give rise to the bf-morphisms $\alpha_{i} . \widetilde{\mathcal{E}}$ and the isomorphisms are unique up to unique isomorphism by Lemma 7.7 of [K2]. By Lemma 6.12, we have a unique isomorphism $\beta: \mathcal{M} \xrightarrow{\sim}\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes \wedge^{2} \tilde{h}^{*} \mathcal{F}$ such that $s_{i}^{*}(\beta) \otimes 1_{\wedge^{2} \widetilde{\mathcal{E}}}=1_{\mathcal{M}} \otimes\left(\wedge^{2} \alpha_{i}\right)$. Put $\eta:=\left(1_{\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee}} \otimes \tilde{h}^{*}(\zeta)\right) \circ \beta: \mathcal{M} \xrightarrow{\sim}$ $\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes\left(p r_{1} \circ \tilde{h}\right)^{*} \widetilde{\mathcal{P}}_{0} . \xi: E_{1} \xrightarrow{\sim} E_{2}$ induces the isomorphism $s_{1}^{*} \widetilde{\mathcal{E}} \xrightarrow{\sim} s_{2}^{*} \widetilde{\mathcal{E}}$, which, by abuse of notation, we denote also by $\xi$. Then by ( 6.6 ), ( $\widetilde{\mathcal{E}}, \xi)$ is admissible for the above bi-simple modification. The diagram (6.7) implies the commutativity of the diagram $(\Omega)$ in Definition 6.8. Thus these data give an object of $G S L_{2} B D\left(\widetilde{X}_{0}, P_{1}, P_{2} ; \widetilde{\mathcal{P}}_{0}\right)^{(0)}(T)$.

One can check that the above constructions commute with isomorphisms and base-changes and that they are inverses to each other.
Q.E.D.

Sketch of proof of Theorem 6.5. The proof is analogous to that of Theorem 6.4. Here we shall deal with only one direction, that is, $\Phi$ : $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)_{\text {red }}^{(1)} \rightarrow \overline{P G l}\left(\tau_{1}^{*} \mathcal{W}_{\text {univ }}, \tau_{2}^{*} \mathcal{W}_{\text {univ }}\right)$ and leave the construction of its inverse to the reader.

Let $T$ be an affine scheme of finite type over $B_{0}$. Take an object $\left(h: \mathcal{Y} \rightarrow X_{0} \times T, \mathcal{E}, \delta: \wedge^{2} \mathcal{E} \rightarrow\left(p r_{1} \circ h\right)^{*} \mathcal{P}_{0}\right)$ of $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)_{\text {red }}^{(1)}(T)$. By the definition of $G S L_{2} B\left(X_{0} ; \mathcal{P}_{0}\right)_{\text {red }}^{(1)},\left.\left(p r_{2} \circ h\right)\right|_{\Pi_{1}}: \Pi_{1} \rightarrow T$ is an isomorphism, where $\Pi_{i}$ is the one described in the paragraph 6.2. Let $g: \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ be the blowing-up along $\Pi_{1}$. Let $\tilde{h}: \widetilde{\mathcal{Y}} \rightarrow \widetilde{X}_{0} \times T$ be such that $\left(\mathfrak{n} \times \mathrm{id}_{T}\right) \circ \tilde{h}=h \circ g \cdot g^{-1}\left(\Pi_{1}\right)$ consists of two sections of $\widetilde{\mathcal{Y}} \rightarrow T$, and let $\tilde{s}$ be one of them such that $\tilde{h} \circ \tilde{s}=\left(P_{2}, \mathrm{id}_{T}\right)$. Put $\widetilde{\mathcal{E}}:=g^{*} \mathcal{E}$. Then by Proposition 8.6 of [K2], we have a family of nodal curves $\mathcal{Z} \xrightarrow{\pi^{\prime}} T$ over $T$ with a section $s^{\prime}$ and a $T$-morphism $f_{2}: \widetilde{\mathcal{Y}} \rightarrow \mathcal{Z}$ and $f_{1}: \mathcal{Z} \rightarrow \widetilde{X}_{0} \times T$ such that
$f_{2}$ is a simple modification of $\left(\mathcal{Z}, \pi^{\prime}, s^{\prime}\right)$ and that $f_{1}$ is a simple modification of $\left(\widetilde{X}_{0} \times T, p r_{2},\left(P_{2}, \mathrm{id}_{T}\right)\right)$ and that $\widetilde{\mathcal{E}}$ is admissible of degree one for $\left(\mathcal{Z}, \pi^{\prime}, s^{\prime}\right)$ and that $\mathcal{F}_{1}:=\left(f_{2 *} \widetilde{E}(-\tilde{s})\right)\left(s^{\prime}\right)$ is admissible of degree two for $\left(\widetilde{X}_{0} \times T, p r_{2},\left(P_{2}, \mathrm{id}_{T}\right)\right)$ (See Definition 7.1 of [K2]). Then Constructions 9.1 and 9.2 of [K2] give a 2-bundle $\mathcal{F}:=f_{1 *}\left(\mathcal{F}_{1}\left(-s^{\prime}\right)\right)\left(\left(P_{2}, \mathrm{id}_{T}\right)(T)\right)$

## $\otimes\left(L_{1} \ni \lambda_{1}\right)$

on $\widetilde{X}_{0} \times T$ and bf-morphisms $g_{1}:\left(P_{1}, \mathrm{id}_{T}\right)^{*} \mathcal{F} \quad \curvearrowleft \quad F_{1}$ and $g_{0}:$ $\otimes\left(L_{0} \ni \lambda_{0}\right)$
$F_{1} \quad \curvearrowleft \quad\left(P_{2}, \mathrm{id}_{T}\right)^{*} \mathcal{F}$, where $F_{1}:=s^{* *} \mathcal{F}_{1}$. Since $\left.\left(p r_{2} \circ h\right)\right|_{\Pi_{2}}: \Pi_{2} \rightarrow T$ is also an isomorphism by Corollary 5.9, we have $\lambda_{0}=0$. Let $\mathcal{M}$ be the line bundle $\left(p r_{1} \circ f_{1} \circ f_{2}\right)^{*} \mathcal{O}_{\widetilde{X_{0}}}\left(P_{1}+P_{2}\right) \otimes f_{2}^{*} \mathcal{O}_{\mathcal{Z}}\left(-s^{\prime}\right) \otimes \mathcal{O}_{\mathcal{Y}}(-\tilde{s})$. If $\mathbf{1} \in p r_{1}^{*} \mathcal{O}_{\widetilde{X_{0}}}\left(P_{1}+P_{2}\right)$ denotes the canonical section, then $f_{1}^{*} \mathbf{1} \in$ $f_{1}^{*} p r_{1}^{*} \mathcal{O}_{\widetilde{X}_{0}}\left(P_{1}+P_{2}\right)$ vanishes along $s^{\prime}$ (in fact it vanishes on $f_{1}^{-1}\left(\left\{P_{2}\right\} \times\right.$ $T)$ ), thus it defines a section $\mu^{\prime}$ of $f_{1}^{*} p r_{1}^{*} \mathcal{O}_{\widetilde{X}_{0}}\left(P_{1}+P_{2}\right) \otimes \mathcal{O}_{\mathcal{Z}}\left(-s^{\prime}\right)$. The section $f_{2}^{*} \mu^{\prime} \in f_{2}^{*} f_{1}^{*} p r_{1}^{*} \mathcal{O}_{\widetilde{X}_{0}}\left(P_{1}+P_{2}\right) \otimes f_{2}^{*} \mathcal{O}_{\mathcal{Z}}\left(-s^{\prime}\right)$ vanishes on $\tilde{s}$, thus it defines a section $\mu$ of $\mathcal{M}$. Let $\lambda$ be the global section of $\left(\wedge^{2} \mathcal{E}\right)^{\vee} \otimes h^{*} \mathcal{P}_{0}$ induced by $\delta$. Then just as Lemma 6.9 we have the unique isomorphism

$$
\begin{equation*}
\left(\wedge^{2} \widetilde{\mathcal{E}}\right)^{\vee} \otimes \tilde{h}^{*} \widetilde{\mathcal{P}}_{0} \simeq \mathcal{M} \tag{6.8}
\end{equation*}
$$

in which $g^{\#}(\lambda)$ and $\mu$ correspond. The isomorphism (6.8) and Lemma 6.12 give the isomorphism $\wedge^{2} \mathcal{F} \simeq p r_{1}^{*}\left(\widetilde{\mathcal{P}}_{0} \otimes \mathcal{O}\left(-P_{1}+P_{2}\right)\right)$. These data determine an object of $\overline{P G l}\left(\tau_{1}^{*} \mathcal{W}_{\text {univ }}, \tau_{2}^{*} \mathcal{W}_{\text {univ }}\right)(T)$.
Q.E.D.

## §7. Appendix

In this appendix, we gather several propositions that are used in this paper.

Proposition 7.1. Let $k$ be a field. Let $\Lambda$ be a complete noetherian local $k$-algebra with maximal ideal $\mu$ such that $k \xrightarrow{\sim} \Lambda / \mu$. Let $\mathcal{A}$ [resp. $\widehat{\mathcal{A}}]$ be the category of artinian local $\Lambda$-algebras [resp. complete noetherian local $\Lambda$-algebras] having residue field $k$. Let $F$ be a functor from $\mathcal{A}$ to the category of sets. Assume that
(a) F has a hull,
(b) for any ideal $J$ of $\Lambda$ with $\mu \supset J \supset \mu^{2}$ and $\operatorname{dim}_{k} \mu / J=1$, we have $F(\Lambda / J)=\phi$.
Assume that we are given $S \in \widehat{\mathcal{A}}$ and $h_{S}(:=\operatorname{Hom}(S,-)) \xrightarrow{u} F$ such that
(i) as a $k$-algebra, $S$ is a ring of formal power series over $k$,
(ii) $\quad h_{S}((\Lambda / \mu)[\epsilon]) \simeq F((\Lambda / \mu)[\epsilon])$, where $(\Lambda / \mu)[\epsilon]$ is the $\Lambda / \mu$-algebra with $\epsilon^{2}=0$.

Then $h_{S} \xrightarrow{u} F$ is a hull.
Proof. By (a) we can find a hull $h_{R} \xrightarrow{v} F$, where $R \in \widehat{\mathcal{A}}$. Then we have a morphism $\varphi: R \rightarrow S$ in $\widehat{\mathcal{A}}$ such that $v \circ h_{\varphi}=u$, where $h_{\varphi}: h_{S} \rightarrow$ $h_{R}$ is the morphism induced by $\varphi$. Let $\mathfrak{m}, \mathfrak{n}$ be the maximal ideals of $R$ and $S$, respectively. By (ii), the morphism $\mathfrak{m} / \mu R+\mathfrak{m}^{2} \xrightarrow{\bar{\varphi}} \mathfrak{n} / \mu S+$ $\mathfrak{n}^{2}$ induced by $\varphi$ is an isomorphism. By (b) $\operatorname{Hom}_{l o c \Lambda-a l g}(R, \Lambda / J)=$ $\operatorname{Hom}_{l o c \Lambda-a l g}(S, \Lambda / J)=\phi$, for any ideal $J$ of $\Lambda$ as in (b). This implies that $\mu R \subset \mathfrak{m}^{2}$ and $\mu S \subset \mathfrak{n}^{2}$. (In fact, suppose that $\mu R \nsubseteq \mathfrak{m}^{2}$ for example. Then we can find an element $x \in \mu$ such that the image of $x$ in $R / \mathfrak{m}^{2}$ is not zero. So we can find an ideal $\mathfrak{m} \supset \mathfrak{a} \supset \mathfrak{m}^{2}$ with $\operatorname{dim}_{k} \mathfrak{m} / \mathfrak{a}=1$ such that the image of $x$ in $R / \mathfrak{a}$ is not zero. $\Lambda \rightarrow R / \mathfrak{a}$ is surjective, hence $R / \mathfrak{a}$ is of the form $\Lambda / J$ with an ideal $J$ of $\Lambda$ as in (b). This contradicts $\operatorname{Hom}(R, \Lambda / J)=\emptyset$.) Therefore $\varphi$ induces the isomorphism $\bar{\varphi}: \mathfrak{m} / \mathfrak{m}^{2} \rightarrow$ $\mathfrak{n} / \mathfrak{n}^{2}$. By (i), there exists a local $k$-algebra homomorphism $\theta: S \rightarrow R$ such that the induced morphism $\bar{\theta}: \mathfrak{n} / \mathfrak{n}^{2} \rightarrow \mathfrak{m} / \mathfrak{m}^{2}$ is $\bar{\varphi}^{-1}$. Then $\theta$ is surjective and $\varphi \circ \theta$ is bijective. Hence $\varphi$ is an isomorphism. Q.E.D.

Lemma 7.2. Let $R$ be a commutative ring with identity. Let $f \in R$ and let $\widehat{R}$ be the $(f)$-adic completion of $R$. Then for any $R$-module $M$, the complex

$$
\begin{equation*}
M \xrightarrow{\alpha}\left(M \otimes_{R} \widehat{R}\right) \oplus M_{f} \xrightarrow{\beta} M \otimes_{R} \widehat{R}_{f} \tag{7.1}
\end{equation*}
$$

is exact, where $\alpha(m):=(m \otimes 1, m)$ and $\beta\left(\left(m \otimes a, \frac{m^{\prime}}{f^{N}}\right)\right):=m \otimes a-m^{\prime} \otimes \frac{1}{f^{N}}$. Moreover either if $M$ is $f$-regular, or if $\widehat{R}$ is flat over $R$, then $\alpha$ is injective.

Proof. Put $T:=\operatorname{Ker}\left(M \rightarrow M_{f}\right), M_{f} / M:=\operatorname{Coker}\left(M \rightarrow M_{f}\right)$ and $\bar{M}:=M / T$. Then we have the natural morphism $M_{f} / M \simeq \bar{M}_{f} / \bar{M}$. Since $\bar{M}$ is $f$-regular, by Lemme 3(a) in [BL], we have $\operatorname{Tor}_{1}^{R}\left(\widehat{R}, \bar{M}_{f} / \bar{M}\right)=$ 0 . Hence we have the exact sequence

$$
\begin{equation*}
0 \rightarrow(M / T) \otimes_{R} \widehat{R} \rightarrow M_{f} \otimes_{R} \widehat{R} \rightarrow\left(M_{f} / M\right) \otimes_{R} \widehat{R} \rightarrow 0 \tag{7.2}
\end{equation*}
$$

Consider the commutative diagram
in which both top and bottom rows are exact. By Lemme 1 in [BL], $\varphi$ and $\psi$ are bijective. This and diagram chasing prove the first part of the
lemma. If $M$ is $f$-regular or if $\widehat{R}$ is $R$-flat, then $\theta$ in the above diagram is injective. This implies the latter part of the lemma. Q.E.D.

Proposition 7.3. Let $k$ be a field. Let $X$ be a scheme of finite type over $k$ and of pure dimension one. Let $P_{i}(1 \leq i \leq N)$ be closed points of $X$ such that $U:=X-\left\{P_{1}, \ldots, P_{N}\right\}$ is affine. Let $\mathfrak{m}_{P_{i}}$ and $\widehat{\mathfrak{m}}_{P_{i}}$ denote the maximal ideal of $\mathcal{O}_{X, P_{i}}$ and $\widehat{\mathcal{O}}_{X, P_{i}}$ respectively. Put $Q_{i}:=\mathrm{H}^{0}\left(\operatorname{Spec} \mathcal{O}_{X, P_{i}}-\left\{\mathfrak{m}_{P_{i}}\right\}, \mathcal{O}\right)$ and $\widehat{Q}_{i}:=\mathrm{H}^{0}\left(\operatorname{Spec} \widehat{\mathcal{O}}_{X, P_{i}}-\left\{\widehat{\mathfrak{m}}_{P_{i}}\right\}, \mathcal{O}\right)$. For a quasi-coherent sheaf $\mathcal{F}$ on $X$, we have natural morphisms $\rho_{i}$ : $\mathrm{H}^{0}(U, \mathcal{F}) \rightarrow \mathcal{F}_{P_{i}} \otimes_{\mathcal{O}_{P_{i}}} \widehat{Q}_{i}$ and $\gamma_{i}: \mathcal{F}_{P_{i}} \otimes_{\mathcal{O}_{P_{i}}} \widehat{\mathcal{O}}_{P_{i}} \rightarrow \mathcal{F}_{P_{i}} \otimes_{\mathcal{O}_{P_{i}}} \widehat{Q}_{P_{i}}$. We define the complex $C^{\bullet}(\mathcal{F})$ of $k$-vector spaces as follows. $C^{0}(\mathcal{F}):=$ $\mathrm{H}^{0}(U, \mathcal{F}) \oplus \oplus_{i=1}^{N}\left(\mathcal{F}_{P_{i}} \otimes_{\mathcal{O}_{P_{i}}} \widehat{\mathcal{O}}_{P_{i}}\right)$ and $C^{1}(\mathcal{F}):=\oplus_{i=1}^{N}\left(\mathcal{F}_{P_{i}} \otimes_{\mathcal{O}_{P_{i}}} \widehat{Q}_{i}\right)$ and $C^{m}(\mathcal{F})=0$ for $m \neq 0,1 . d^{0}: C^{0}(\mathcal{F}) \rightarrow C^{1}(\mathcal{F})$ maps $\left(s_{U},\left(s_{i}\right)_{i=1}^{N}\right) \in$ $C^{0}(\mathcal{F})$ to $\left(\rho_{i}\left(s_{U}\right)-\gamma_{i}\left(s_{i}\right)\right)_{i=1}^{N}$ and $d^{m}=0$ for $m \neq 0$. Then we have
(1) If $0 \rightarrow \mathcal{F}^{\prime} \rightarrow \mathcal{F} \rightarrow \mathcal{F}^{\prime \prime} \rightarrow 0$ is an exact sequence of quasi-coherent sheaves, we have a long exact sequence of cohomologies $\mathrm{H}^{m}\left(C^{\bullet}(-)\right)$.
(2) For a quasi-coherent sheaf $\mathcal{F}$, we have an isomorphism $\mathrm{H}^{m}(X, \mathcal{F}) \rightarrow$ $\mathrm{H}^{m}\left(C^{\bullet}(\mathcal{F})\right)$ that is functorial in $\mathcal{F}$ and compatible with long exact sequences.

Proof. (1) follows easily from the flatness of $\widehat{\mathcal{O}}_{P_{i}}$ and $Q_{i}$ over $\mathcal{O}_{P_{i}}$. We have the natural morphism $\mathrm{H}^{0}(X, \mathcal{F}) \rightarrow \mathrm{H}^{0}\left(C^{\bullet}(\mathcal{F})\right)$. Let $f_{i} \in \mathfrak{m}_{P_{i}}$ be such that $\mathcal{O}_{X, P_{i}} /\left(f_{i}\right)$ is artinian. Then we have the isomorphisms $Q_{i} \simeq$ $\left(\mathcal{O}_{X, P_{i}}\right)_{f_{i}}$ and $\widehat{Q}_{i} \simeq\left(\widehat{\mathcal{O}}_{X, P_{i}}\right)_{f_{i}}$. Then applying Lemma 7.2, we know that $\mathrm{H}^{0}(X, \mathcal{F}) \rightarrow \mathrm{H}^{0}\left(C^{\bullet}(\mathcal{F})\right)$ is bijective. To establish an isomorphism $\mathrm{H}^{1}(X, \mathcal{F}) \rightarrow \mathrm{H}^{1}\left(C^{\bullet}(\mathcal{F})\right)$, it suffices to prove $H^{1}\left(C^{\bullet}(\mathcal{F})\right)=0$ if $\mathcal{F}$ is a flasque quasi-coherent sheaf. Suppose $\mathcal{F}$ is flasque. Then the maps $\mathcal{F}_{P_{i}} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_{P_{i}}} Q_{i}$ are surjective, hence so are $\gamma_{i}(1 \leq i \leq N)$. Hence $\mathrm{H}^{1}\left(C^{\bullet}(\mathcal{F})\right)=0$.
Q.E.D.

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# Vector bundles on curves and theta functions 

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#### Abstract

. This is a survey lecture on the "theta map" from the moduli space of $S L_{r}$-bundles on a curve $C$ to the projective space of $r$-th order theta functions on $J C$. Some recent results and a few open problems about that map are discussed.


## Introduction

These notes survey the relation between the moduli spaces of vector bundles on a curve $C$ and the spaces of (classical) theta functions on the Jacobian $J$ of $C$. The connection appears when one tries to describe the moduli space $\mathcal{M}_{r}$ of rank $r$ vector bundles with trivial determinant as a projective variety in an explicit way (as opposed to the somewhat non-constructive way provided by GIT). The Picard group of the moduli space is infinite cyclic, generated by the determinant line bundle $\mathcal{L}$; thus the natural maps from $\mathcal{M}_{r}$ to projective spaces are those defined by the linear systems $\left|\mathcal{L}^{k}\right|$, and in the first instance the $\operatorname{map} \varphi_{\mathcal{L}}: \mathcal{M}_{r} \rightarrow|\mathcal{L}|^{*}$. The key point is that this map can be identified with the theta map

$$
\theta: \mathcal{M}_{r} \rightarrow|r \Theta|
$$

which associates to a general bundle $E \in \mathcal{M}_{r}$ its theta divisor $\Theta_{E}$, an element of the linear system $|r \Theta|$ on $J$ - we will recall the precise definitions below. This description turns out to be sufficiently manageable to get some information on the behaviour of this map, at least when $r$ or $g$ are small.

We will describe the results which have been obtained so far - most of them fairly recently. Thus these notes can be viewed as a sequel to [B2], though with a more precise focus on the theta map. For the convenience of the reader we have made this paper independent of [B2], by recalling in $\S 1$ the necessary definitions. Then we discuss the indeterminacy locus of $\theta(\S 2)$, the case $r=2(\S 3)$, the case $g=2(\S 4)$, and the
higher rank case ( $\S 5$ ). Finally, as in [B2] we will propose a small list of questions and conjectures related to the topic (§6).

## §1. The moduli space $\mathcal{M}_{r}$ and the theta map

(1.1) Throughout this paper $C$ will be a complex curve of genus $g \geq 2$. We denote by $J$ its Jacobian variety, and by $J^{k}$ the variety (isomorphic to $J=J^{0}$ ) parametrizing line bundles of degree $k$ on $C$.

For $r \geq 2$, we denote by $\mathcal{M}_{r}$ the moduli space of semi-stable vector bundles of rank $r$ and trivial determinant on $C$. It is a normal, projective, unirational variety, of dimension $\left(r^{2}-1\right)(g-1)$. The points of $\mathcal{M}_{r}$ correspond to isomorphism classes of vector bundles with trivial determinant which are direct sums of stable vector bundles of degree zero. The singular locus consists precisely of those bundles which are decomposable (with the exception of $\mathcal{M}_{2}$ in genus 2 , which is smooth). The corresponding singularities are rational Gorenstein - that is, reasonably mild.
(1.2) The Picard group of $\mathcal{M}_{r}$ has been thoroughly studied in $[\mathrm{D}-\mathrm{N}]$; let us recall the main results. Fix some $L \in J^{g-1}$, and consider the reduced subvariety

$$
\Delta_{L}:=\left\{E \in \mathcal{M}_{r} \mid H^{0}(C, E \otimes L) \neq 0\right\}
$$

Then $\Delta_{L}$ is a Cartier divisor in $\mathcal{M}_{r}$; the line bundle $\mathcal{L}:=\mathcal{O}_{\mathcal{M}_{r}}\left(\Delta_{L}\right)$, called the determinant bundle, is independent of the choice of $L$ and generates $\operatorname{Pic}\left(\mathcal{M}_{r}\right)$. The canonical bundle of $\mathcal{M}_{r}$ is $\mathcal{L}^{-2 r}$.
(1.3) To study the rational map $\varphi_{\mathcal{L}}: \mathcal{M}_{r} \rightarrow|\mathcal{L}|^{*}$ associated to the determinant line bundle, the following construction is crucial. For a vector bundle $E \in \mathcal{M}_{r}$, consider the locus

$$
\Theta_{E}:=\left\{L \in J^{g-1} \mid H^{0}(C, E \otimes L) \neq 0\right\}
$$

Since $\chi(E \otimes L)=0$ for $L$ in $J^{g-1}$, it is readily seen that $\Theta_{E}$ is in a natural way a divisor in $J^{g-1}$ - unless it is equal to $J^{g-1}$. The latter case (which may occur only for special bundles) is a serious source of trouble - see $\S 2$ below. In the former case we say that $E$ admits a theta divisor; this divisor belongs to the linear system $|r \Theta|$, where $\Theta$ is the canonical Theta divisor in $J^{g-1}$. In this way we get a rational map

$$
\theta: \mathcal{M}_{r} \rightarrow|r \Theta|
$$

Proposition 1.4. [BNR] There is a canonical isomorphism $|\mathcal{L}|^{*} \xrightarrow{\sim}|r \Theta|$ which identifies $\varphi_{\mathcal{L}}$ to $\theta$.

As a consequence, the base locus of $|\mathcal{L}|$ is the locus of bundles $E$ in $\mathcal{M}_{r}$ such that $H^{0}(C, E \otimes L) \neq 0$ for all $L \in J^{g-1}$. This is also the indeterminacy locus of $\theta$ (because $|\mathcal{L}|$ cannot have a fixed component).
(1.5) The $r$-torsion subgroup $J[r]$ of $J$ acts on $\mathcal{M}_{r}$ by tensor product; it also acts on $|r \Theta|$ by translation, and the map $\theta$ is equivariant with respect to these actions. In particular, the image of $\theta$ is $J[r]$-invariant.
(1.6) The case when $\theta$ is a morphism is much easier to analyze: we know then that it is finite (since $|\mathcal{L}|$ is ample, $\theta$ cannot contract any curve), we know its degree by the Verlinde formula, etc. Unfortunately there are few cases where this is known to happen:

Proposition 1.6. The base locus of $|\mathcal{L}|$ is empty in the following cases:
a) $r=2$;
b) $r=3, g=2$ or 3 ;
c) $r=3, C$ is generic.

All these results except the case $r=g=3$ are due to Raynaud $[R]$. While a) and the first part of b) are easy, c) and the second part of b) are much more involved. We will discuss the latter in $\S 5$ below. The proof of c) is reduced, through a degeneration argument, to an analogous statement for torsion-free sheaves on a rational curve with $g$ nodes.

## §2. Base locus

(2.1) Recall that the slope of a vector bundle $E$ of rank $r$ and degree $d$ is the rational number $\mu=d / r$. It is convenient to extend the definition of the theta divisor to vector bundles $E$ with integral slope $\mu$, by putting $\Theta_{E}:=\left\{L \in J^{g-1-\mu} \mid H^{0}(C, E \otimes L) \neq 0\right\}$. If $\delta$ is a line bundle such that $\delta^{\otimes r} \cong \operatorname{det} E$, the vector bundle $E_{0}:=E \otimes \delta^{-1}$ has trivial determinant and $\Theta_{E_{0}} \subset J^{g-1}$ is the translate by $\delta$ of $\Theta_{E} \subset J^{g-1-\mu}$.
(2.2) We have the following relations between stability and existence of the theta divisor:
(2.2 a) If $E$ admits a theta divisor, it is semi-stable;
(2.2 b) If moreover $\Theta_{E}$ is a prime divisor, $E$ is stable.

Indeed let $F$ be a proper subbundle of $E$. If $\mu(F)>\mu(E)$, the RiemannRoch theorem implies $H^{0}(C, F \otimes L) \neq 0$, and therefore $H^{0}(C, E \otimes L) \neq 0$, for all $L$ in $J^{g-1-\mu}$. If $\mu(F)=\mu(E)$, one has $\Theta_{E}=\Theta_{F}+\Theta_{E / F}$, so that $\Theta_{E}$ is not prime.
(2.3) The converse of these assertions do not hold. We will see in (2.6) examples of stable bundles with a reducible theta divisor. The first examples of stable bundles with no theta divisor are due to Raynaud $[R]$.

They are restrictions of projectively flat vector bundles on $J$. Choose a theta divisor $\Theta$ on $J$. The line bundle $\mathcal{O}_{J}(n \Theta)$ is invariant under the $n$-torsion subgroup $J[n]$ of $J$. The action of $J[n]$ does not lift to $\mathcal{O}_{J}(n \Theta)$, but it does lift to the vector bundle $H^{0}\left(J, \mathcal{O}_{J}(n \Theta)\right)^{*} \otimes_{\mathbb{C}} \mathcal{O}_{J}(n \Theta)$. Thus this vector bundle is the pull back under the multiplication $n_{J}: J \rightarrow J$ of a vector bundle $E_{n}$ on $J$. Restricting $E_{n}$ to the curve $C$ embedded in $J$ by an Abel-Jacobi mapping gives the Raynaud bundle $R_{n}$. It is well defined up to a twist by an element of $J$, has rank $n^{g}$ and slope $\frac{g}{n}$. It has the property that $H^{0}\left(C, R_{n} \otimes \alpha\right) \neq 0$ for all $\alpha \in J$. Thus if $n \mid g$ $R_{n}$ has integral slope and no theta divisor. More generally, Schneider has shown that a general vector bundle on $C$ of rank $n^{g}$, slope $g-1$ and containing $R_{n}$ is still stable [S2]. This gives a very large dimension for the base locus of $|\mathcal{L}|$, approximately $\left(1-\frac{1}{n}\right) \operatorname{dim} \mathcal{M}_{r}$ if $r=n^{g}$. Some related results are discussed in [A].
(2.4) Another series of examples have been constructed by Popa $[\mathrm{P}]$. Let $L$ be a line bundle on $C$ spanned by its global sections. The evaluation bundle $E_{L}$ is defined by the exact sequence

$$
0 \rightarrow E_{L}^{*} \longrightarrow H^{0}(L) \otimes_{\mathbb{C}} \mathcal{O}_{C} \xrightarrow{e v} L \rightarrow 0 ;
$$

it has the same degree as $L$ and $\operatorname{rank} h^{0}(L)-1$. In particular, if we choose $\operatorname{deg} L=g+r$ with $r \geq g+2, E_{L}$ has rank $r$ and slope $\mu=1+\frac{g}{r}$. Then, for all $p$ such that $2 \leq p \leq r-2$ and $p \mu \in \mathbb{Z}$, the vector bundle $\boldsymbol{\Lambda}^{p} E_{L}$ does not admit a theta divisor (see [S1]). For instance, when $r=2 g$, $\Lambda^{2} E_{L}$ gives a base point of $|\mathcal{L}|$ in $\mathcal{M}_{g(2 g-1)}$.
(2.5) An interesting limit case of this construction is when $\mu=2$; this occurs when $L=K_{C}$, or $r=g$. The first case has been studied in [FMP]. It turns out that the vector bundle $\boldsymbol{\Lambda}^{p} E_{K}$ has a theta divisor, equal to $C_{g-p-1}-C_{p}$ (here $C_{k}$ denotes the locus of effective divisor classes in $J^{k}$ ). While the proof is elementary for $p=1$, it is extremely involved for the higher exterior powers: it requires going to the moduli space of curves and computing various divisor classes in the Picard group of this moduli space. It remains a challenge to find a direct proof.
(2.6) The case deg $L=2 g$ is treated in [B4], building on the results of [FMP]. Here again $\boldsymbol{\Lambda}^{p} E_{L}$ admits a theta divisor, at least if $L$ is general enough; it has two components, namely $C_{g-p-1}-C_{p}$ and the translate of $C_{g-p}-C_{p-1}$ by the class $\left[K \otimes L^{-1}\right]$. These are the first examples defined on a general curve of stable bundles with a reducible theta divisor.
(2.7) Since $|\mathcal{L}|$ has usually a large base locus, it is natural to look at the systems $\left|\mathcal{L}^{k}\right|$ to improve the situation. There has been much progress on this question in the recent years:

Proposition 2.7. (i) $[\mathrm{P}-\mathrm{R}]\left|\mathcal{L}^{k}\right|$ is base point free on $\mathcal{M}_{r}$ for $k \geq\left[\frac{r^{2}}{4}\right]$.
(ii) $[\mathrm{E}-\mathrm{P}]$ For $k \geq r^{2}+r$, the linear system $\left|\mathcal{L}^{k}\right|$ defines an injective morphism of $\mathcal{M}_{r}$ into $\left|\mathcal{L}^{k}\right|^{*}$, which is an embedding on the stable locus.

On the other hand Popa $[\mathrm{P}]$ has observed that one should not be too optimistic, at least if one believes in the strange duality conjecture (see [B2]): this conjecture implies that for $n \mid g$ the Raynaud bundle $R_{n}$, twisted by an appropriate line bundle, is a base point of $\left|\mathcal{L}^{k}\right|$ when $k \leq n\left(1-\frac{n}{g}\right)$.

## §3. Rank 2

(3.1) In rank 2 the situation is now well understood. As pointed out in (1.6), $\theta: \mathcal{M}_{2} \rightarrow|2 \Theta|$ is a finite morphism. In genus $2, \theta$ is actually an isomorphism onto $\mathbb{P}^{3}$ [N-R1]. If $C$ is hyperelliptic of genus $g \geq 3$, it follows from [D-R] and [B1] that $\theta$ factors through the involution $\iota^{*}$ induced by the hyperelliptic involution and embeds $\mathcal{M}_{2} /\left\langle\iota^{*}\right\rangle$ into $|2 \Theta|$; moreover the image admits an explicit geometric description [D-R].
(3.2) In the non-hyperelliptic case, after much effort we have now a complete answer, which is certainly one of the highlights of the subject:

Theorem 3.2. If $C$ is not hyperelliptic, $\theta: \mathcal{M}_{2} \hookrightarrow|2 \Theta|$ is an embedding.
The fact that $\theta$ embeds the stable locus of $\mathcal{M}_{2}$ is proved in [B-V1], and the remaining part in [vG-I]. Both parts are highly nontrivial, and involve some beautiful geometric constructions.
(3.3) Thus we can identify $\mathcal{M}_{2}$ with a subvariety of $|2 \Theta| \cong \mathbb{P}^{2^{g}-1}$, canonically associated to $C$, of dimension $3 g-3$ (1.1). This variety is invariant under the natural action of $J[2]$ on $|2 \Theta|$ (1.5). Its degree can be computed from the Verlinde formula (see e.g. [Z], Thm. 1(iii)):

$$
\operatorname{deg} \mathcal{M}_{2}=(3 g-3)!2^{g}(2 \pi)^{2-2 g} \zeta(2 g-2),
$$

which gives $\operatorname{deg} \mathcal{M}_{2}=1$ for $g=2,4$ for $g=3,96$ for $g=4$, etc.
The singular locus Sing $\mathcal{M}_{2}$ is the locus of decomposable bundles in $\mathcal{M}_{2}$ (1.1), which are of the form $\alpha \oplus \alpha^{-1}$, for $\alpha \in J$; the map $\alpha \mapsto \alpha \oplus \alpha^{-1}$ identifies Sing $\mathcal{M}_{2}$ to the Kummer variety $\mathcal{K}$ of $J$ - that is, the quotient of $J$ by the involution $\alpha \mapsto \alpha^{-1}$. The restriction of $\theta$ to $\mathcal{K}=\operatorname{Sing} \mathcal{M}_{2}$ is the classical embedding of $\mathcal{K}$ in $|2 \Theta|$, deduced from the map $\alpha \mapsto \Theta_{\alpha}+\Theta_{-\alpha}$ from $J$ to $|2 \Theta|$.
(3.4) The case $g=3$, which had been treated previously in [ N R2], is particularly interesting: we obtain a hypersurface in $|2 \Theta|$, of
degree 4 , which is $J[2]$-invariant and singular along the Kummer variety. Now Coble shows in [C2] that there is a unique such quartic (the $J[2]$ invariance is actually superfluous, see [B5]). Thus in genus 3, the theta map identifies $\mathcal{M}_{2}$ with the Coble quartic hypersurface.

Coble gives an explicit equation for this hypersurface, which we now express in modern terms. Recall that Mumford's theory of the Heisenberg group allows us to find canonical coordinates $\left(X_{v}\right)_{v \in V}$ in the projective space $|2 \Theta|$, where $V$ is a 3 -dimensional vector space over $\mathbb{F}_{2}$. Then Coble equation reads:

$$
\alpha \sum_{u \in V} X_{u}^{4}+\sum_{\ell=\{u, v\}} \alpha_{d(\ell)} X_{u}^{2} X_{v}^{2}+\sum_{P=\{t, u, v, w\}} \alpha_{d(P)} X_{t} X_{u} X_{v} X_{w}=0
$$

where the second sum (resp. the third) is taken over the set of affine lines (resp. planes) in $V$, and $d(\ell) \in \mathbb{P}(V)$ (resp. $\left.d(P) \in \mathbb{P}\left(V^{*}\right)\right)$ denotes the direction of the line $\ell$ (resp. of the plane $P$ ).

In many ways the Coble quartic $\mathcal{Q} \subset \mathbb{P}^{7}$ can be seen as an analogue of the Kummer quartic surface in $\mathbb{P}^{3}$. Pauly has proved that $\mathcal{Q}$ shares a famous property of the Kummer surface, the self-duality : the dual hypersurface $\mathcal{Q}^{*} \subset\left(\mathbb{P}^{7}\right)^{*}$ is isomorphic to $\mathcal{Q}[\mathrm{Pa}]$. The proof is geometric, and includes several beautiful geometric constructions along the way.
(3.5) In genus $4, \mathcal{M}_{2}$ is a variety of dimension 9 and degree 96 in $\mathbb{P}^{15}$. Oxbury and Pauly have observed that there exists a unique $J[2]$ invariant quartic hypersurface singular along $\mathcal{M}_{2}$ [O-P]. A geometric interpretation of this quartic is not known.
(3.6) In arbitrary genus, the quartic hypersurfaces in $|2 \Theta|$ containing $\mathcal{M}_{2}$ have been studied in [vG] and [vG-P]. Here is one sample of their results:

Proposition 3.6. Assume that $C$ has no vanishing thetanull. A J[2]invariant quartic form $F$ on $|2 \Theta|$ vanishes on $\mathcal{M}_{2}$ if and only if the hypersurface $F=0$ is singular along $\mathcal{K}$.
(Note that though the action of $J[2]$ on $|2 \Theta|$ does not come from a linear action, it does lift to the space of quartic forms on $|2 \Theta|$. Requiring the invariance of $F$ is stronger than the invariance of the corresponding hypersurface.)

Van Geemen and Previato also describe the quartics containing $\mathcal{M}_{2}$ in terms of the Prym varieties associated to $C$ - this is related to the Schottky-Jung configuration studied by Mumford.

## §4. Genus 2

(4.1) Going to higher rank, it is natural to look first at the genus 2 case. There a curious numerical coincidence occurs, namely

$$
\operatorname{dim} \mathcal{M}_{r}=\operatorname{dim}|r \Theta|=r^{2}-1
$$

Recall that $\theta$ is a finite morphism for $r=2,3$ (1.6). However already for $r=4$ it is only a rational map: the Raynaud bundle $R_{2}$ has rank 4 and slope 1 (2.3), so once twisted by appropriate line bundles of degree -1 it provides finitely many (actually 16) base points of $|\mathcal{L}|$.

We have seen that $\theta$ is an isomorphism in rank 2 . In rank 3 there is again a beautiful story, surprisingly analogous to the rank 2 , genus 3 case. Indeed the Coble quartic has a companion, the Coble cubic : this is the unique cubic hypersurface $\mathcal{C} \subset|3 \Theta|^{*}$ singular along $J^{1}$ embedded in $|3 \Theta|^{*}$ by the linear system $|3 \Theta|$ (this is implicit in Coble [C1]; see [B5] for a modern explanation).

Theorem 4.2. The map $\theta: \mathcal{M}_{3} \rightarrow|3 \Theta|$ is a double covering; the corresponding involution of $\mathcal{M}_{3}$ is $E \mapsto \iota^{*} E^{*}$, where $\iota$ is the hyperelliptic involution. The branch locus $\mathcal{S} \subset|3 \Theta|$ of $\theta$ is a sextic hypersurface, which is the dual of the Coble cubic $\mathcal{C} \subset|3 \Theta|^{*}$.

This is fairly straightforward (see [O]) except for the duality statement, which was conjectured by Dolgachev and proved in [O] (a different proof appears in $[\mathrm{N}]$ ).
(4.3) Like for the Coble quartic we get an explicit equation for $\mathcal{C}$ by choosing a level 3 structure on $C$, which provides canonical coordinates $\left(X_{v}\right)_{v \in V}$ on $|3 \Theta|^{*}$, where $V$ is a 2 -dimensional vector space over $\mathbb{F}_{3}$. Then from [C1] we get the following equation for $\mathcal{C}$ :

$$
\alpha_{0} \sum_{v \in V} X_{v}^{3}+6 \sum_{\ell=\{u, v, w\}} \alpha_{d(\ell)} X_{u} X_{v} X_{w}=0
$$

where the second sum is taken over the set of affine lines in $V$, and $d(\ell) \in \mathbb{P}(V)$ is the direction of the line $\ell$. The 5 coefficients $\left(\alpha_{i}\right)$ satisfy the Burkhardt equation

$$
\alpha_{0}^{4}-\alpha_{0} \sum_{p \in \mathbb{P}(V)} \alpha_{p}^{3}+3 \prod_{p \in \mathbb{P}(V)} \alpha_{p}=0
$$

(see $[\mathrm{H}], 5.3$ ).
(4.4) In rank $r \geq 4$ we start getting base points, and this causes a lot of trouble - since $\theta$ is only rational, we cannot compute its degree using intersection theory. However we still have:

Proposition 4.5. [B6] The rational map $\theta: \mathcal{M}_{r} \rightarrow|r \Theta|$ is generically finite (or, equivalently, dominant).

The idea is to prove the finiteness of $\theta^{-1}(\Theta+\Delta)$, where $\Delta$ is a general element of $|(r-1) \Theta|$. Any decomposable bundle in that fibre must be of the form $\mathcal{O}_{C} \oplus F$ for some $F \in \mathcal{M}_{r-1}$ with $\Theta_{F}=\Delta$; reasoning by induction on $r$ we can assume that there are finitely many such $F$. Thus the whole point is to control the stable bundles $E$ with $\Theta_{E}=\Theta+\Delta$. Now the condition $\Theta_{E} \supset \Theta$ means by definition $H^{0}(C, E(p)) \neq 0$ for all $p \in C$, or equivalently $H^{0}\left(C, E^{\prime}(-p)\right) \neq 0$ for all $p \in C$, where $E^{\prime}:=E^{*} \otimes K_{C}^{-1}$ is the Serre dual of $E$. Since $h^{0}\left(E^{\prime}\right)=r$ by stability of $E$, this implies that the global sections of $E^{\prime}$ generate a subbundle of rank $<r$. A precise analysis of this situation allows us to prove that there are only finitely many such bundles $E$ with $\Theta_{E}=\Theta+\Delta$.
(4.6) The map $\theta$ is no longer finite in rank $r \geq 4$, in fact it admits some fibres of dimension $\geq\left[\frac{r}{2}\right]-1[\mathrm{~B} 6]$. When $r$ is even, this is seen by restricting $\theta$ to the moduli space of symplectic bundles: the corresponding moduli space has dimension $\frac{1}{2} r(r+1)$, but its image under $\theta$ lands in the subspace $|r \Theta|^{+}$of $|r \Theta|$ corresponding to even theta functions of order $r$, which has dimension $\frac{r^{2}}{2}+1$. For $r$ odd one considers bundles of the form $\mathcal{O}_{C} \oplus F$ with $F$ symplectic.
(4.7) It would be interesting to find the degree of $\theta$, which is unknown already in genus 4 . For trivial reasons it has to grow exponentially with $g$ (see [B6], 2.3). Brivio and Verra have found a nice geometric interpretation of the generic fibre of $\theta$ which might lead at least to a good estimate for $\operatorname{deg} \theta[\mathrm{B}-\mathrm{V} 2]$.

## §5. Higher rank and genus

Not much is known here. We already mentioned the following result proved in [B6]:

Proposition 5.1. In genus 3 the map $\theta: \mathcal{M}_{3} \rightarrow|3 \Theta|$ is a finite morphism.

The proof is rather roundabout, and gives actually a more interesting result: the complete list of stable vector bundles $E$ of rank 3 and degree 0 such that $\Theta_{E} \supset \Theta$. It turns out that each bundle in this list admits a theta divisor. Since $\Theta_{E}=J$ implies $\Theta_{E} \supset \Theta$, Proposition 5.1 follows.
(5.2) The idea for establishing that list is to translate the problem into a classical question of projective geometry. Similarly to the genus 2
case, the condition $\Theta_{E} \supset \Theta$ means $H^{0}(E(p+q)) \neq 0$ for all $p, q$ in $C$, or equivalently $H^{0}\left(E^{\prime}(-p-q)\right) \neq 0$, where $E^{\prime}:=E^{*} \otimes K_{C}^{-1}$ is the Serre dual of $E$. One checks that stability implies $h^{0}\left(E^{\prime}\right)=6$ and $h^{0}\left(E^{\prime}(-p)\right)=3$ for $p$ general in $C$. This gives a family of 2-planes in $\mathbb{P}\left(H^{0}\left(E^{\prime}\right)\right) \cong \mathbb{P}^{5}$, parametrized by $C$, such that any two planes of the family intersect. It turns out that the maximal such families have been classified in a beautiful paper by Morin $[\mathrm{M}]$ : there are three families given by linear algebra (like the 2-planes contained in a given hyperplane), and three coming from geometry: the 2 -planes contained in a smooth quadric, the tangent planes to the Veronese surface, and the planes intersecting the Veronese surface along a conic. Translating back this result in terms of vector bundles gives the list we were looking for.
(5.3) This list also shows that $\theta^{-1}\left(\Theta+\Theta_{F}\right)=\left\{\mathcal{O}_{C} \oplus F\right\}$ for $F$ general in $\mathcal{M}_{2}$. This might indicate that $\theta$ has degree one; it would follow if we could prove the injectivity of its tangent map at $\mathcal{O}_{C} \oplus E$ for some $E$ in $\mathcal{M}_{2}$, perhaps in the spirit of [vG-I].

## §6. Questions and conjectures

The list of results ends at this point, but let me finish with a (small) list of open problems. About the general behaviour of the theta map, the most optimistic statement would be:

Speculation 6.1. For $g \geq 3, \theta$ is generically injective if $C$ is not hyperelliptic, and generically two-to-one onto its image if $C$ is hyperelliptic.

Note that in the hyperelliptic case $\theta$ factors as in thm. 4.2 through the non-trivial involution $E \mapsto \iota^{*} E^{*}$. Admittedly the evidence for 6.1 is very weak: the only case where it is known is in rank 2.

As for base points, Proposition 1.6 leads naturally to:
Conjecture 6.2. Every bundle $E \in \mathcal{M}_{3}$ has a theta divisor.
(6.3) There exists an integer $r(C)$ such that $\theta$ is a morphism for $r<r(C)$ but only a rational map for $r \geq r(C)$ (observe that if $E \in \mathcal{M}_{r}$ has no theta divisor, so does $E \oplus F$ for any $F$ in $\mathcal{M}_{s}, s \geq 1$ ). We know very little about this integer: we have $r(C)=4$ for $g=2,4 \leq r(C) \leq 8$ for $g=3$, and $r(C) \leq \frac{1}{2}(g+1)(g+2)[\mathrm{A}]$.

Questions 6.4. a) Does $r(C)$ depend only on $g$ ?
b) Put $r(g):=\min r(C)$ for all curves $C$ of genus $g$. Is $r(g)$ an increasing function of $g$ ?

The next question does not involve directly the theta map, but it is related to several questions about the existence of theta divisors.

Conjecture 6.5. Let $\pi: C^{\prime} \rightarrow C$ be a finite morphism between smooth projective curves of genus $\geq 2$. The direct image $\pi_{*} L$ of a general vector bundle $L$ on $C^{\prime}$ is stable.

One reduces readily to the case when $L$ is a line bundle. The problem depends in a crucial way on the degree of $L$ : one can prove for instance that $\pi_{*} L$ is stable (for $L$ generic) if $|\chi(L)|<g+\frac{g^{2}}{r}$, where $r$ is the degree of $\pi$ and $g$ the genus of $C$ (see [B3]).

One of the relations between this conjecture and the existence of theta divisors is the following: the conjecture for a general line bundle $L$ of degree $d$ is implied by the existence of a vector bundle $E$ of rank $r$ and degree $g\left(C^{\prime}\right)-1-d$ such that $\pi^{*} E$ admits a prime theta divisor. Indeed we have $\Theta_{\pi_{*} L \otimes E}=\left(\pi^{*}\right)^{-1}\left(\Theta_{L \otimes \pi^{*} E}\right)$; if $\Theta_{\pi^{*} E}$ is prime, so is $\Theta_{\pi_{*} L \otimes E}$ for general $L$, and as in (2.2) this implies that $\pi_{*} L$ is stable.

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# On the finiteness of abelian varieties with bounded modular height 

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#### Abstract

. In this paper, we propose a definition of modular heights of abelian varieties defined over a field of finite type over $\mathbb{Q}$, and prove its bounding property, that is, the finiteness of abelian varieties with bounded modular height.


## § Introduction

The modular heights of abelian varieties and their bounding property played a crucial role in Faltings' first proof [2] of the Mordell conjecture. Although many important results concerning finiteness properties over number fields (conjectures of Tate, Shafarevich and Mordell among others) are now available over arbitrary fields of finite type over $\mathbb{Q}$, a similar generalization of the aforementioned theory of Faltings does not seem to have been explicitly formulated. In this paper, we propose a definition of the modular heights of abelian varieties and prove the finiteness of abelian varieties with bounded modular height over a general field of finite type over $\mathbb{Q}$.

Let $K$ be a field of finite type over $\mathbb{Q}$. In order to properly define the height function over $K$, we have to fix a polarization of $K$ (see [9]). A polarization of $K$ is, by definition, a collection of data $\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$, where

- $B$ is a normal and projective scheme over $\operatorname{Spec}(\mathbb{Z})$ such that its function field is isomorphic to $K$;
- $d=\operatorname{tr} . \operatorname{deg}_{\mathbb{Q}}(K)$ and $\bar{H}_{1}, \ldots, \bar{H}_{d}$ are nef $C^{\infty}$-hermitian line bundles on $B$.
Let $A$ be an abelian variety over $K$. By use of the Néron model of $A$ over $B$ defined in codimension one (see Section 1.1), the Hodge sheaf
$\lambda(A / K ; B)$ attached to $A$ is canonically defined as a reflexive sheaf of rank one on $B$. Moreover it carries a locally integrable singular hermitian metric $\|\cdot\|_{\text {Fal }}$ induced by Faltings' metric on the good reduction part of the Néron model of $A$. The arithmetic first Chern class

$$
\widehat{c}_{1}\left(\lambda(A / K ; B),\|\cdot\|_{\text {Fal }}\right)
$$

is represented by a pair of a Weil divisor and a locally integrable function. We define the modular height $h(A)$ of $A$ as the arithmetic intersection number of $\widehat{c}_{1}\left(\lambda(A / K ; B),\|\cdot\|_{\text {Fal }}\right)$ with $\bar{H}_{1}, \ldots, \bar{H}_{d}$ :

$$
h(A)=\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \cdot \widehat{c}_{1}\left(\lambda(A / K ; B),\|\cdot\|_{\text {Fal }}\right)\right) .
$$

The main objective of the present paper is to show the following finiteness result:

Theorem A (cf. Theorem 6.1). Assume that the arithmetic divisors $\bar{H}_{1}, \ldots, \bar{H}_{d}$ are big. Then, for an arbitrary fixed real number $c$, the set of $K$-isomorphism classes of the abelian varieties over $K$ with $h(A) \leq c$ is finite.

This theorem can be viewed as an Arakelov geometric analogue of a result of Moret-Bailly [8], where the ground field $K$ is replaced by a function field over a finite field and the height is defined by means of the ordinary intersection theory.

In our proof, we have to look at the compactified moduli space of abelian varieties and the local behavior of Faltings' metric around the boundary. We do not need, however, strong assertions due to FaltingsChai [4]; basic facts stated in [12] together with a lemma of Gabber (Lemma 1.2.2) are sufficient for our purpose.

The present paper is organized as follows. Basic notions and facts are prepared in Section 1. In Section 2, we study height functions of singular hermitian line bundles with logarithmic singularities. In Section 3, we observe some properties of the Faltings modular height. The proof of the bounding property is done from Section 4 through Section 6 .

Finally we would like to express hearty thanks to the referee for a lot of comments to improve the paper.

## §1. Preliminaries

### 1.1. Néron model

Let $B$ be a noetherian normal integral scheme and $K$ its function field. Let $A$ be an abelian variety over $K$. A smooth group scheme $\mathcal{A} \rightarrow$
$B$ is called a Néron model of $A$ over $B$ if the following two conditions are satisfied:
(a) The generic fiber $\mathcal{A} \times{ }_{B} \operatorname{Spec}(K)$ of $\mathcal{A} \rightarrow B$ is isomorphic to $A$ over $K$;
(b) (Universal property) If $\mathcal{X} \rightarrow B$ is a smooth $B$-scheme and $X$ its generic fiber, then any $K$-morphism $X \rightarrow A$ uniquely extends to a $B$-morphism $\mathcal{X} \rightarrow \mathcal{A}$.
If $B$ is a Dedekind scheme, then there exists a Néron model of $A$ over $B$ (cf. [1]). When $B$ has higher dimension, we still have a partial Néron mode $\mathcal{A}$ of $A$ defined over a big open subset $U \subseteq B$ (i.e. $B \backslash U$ is of codimension $\geq 2$ ), and we call $\mathcal{A}$ a Néron model over $B$ in codimension one:

Proposition 1.1.1. There exists a Néron model of $A$ over a big open set $U$ of $B$.

Proof. Let us begin with the following lemma:
Lemma 1.1.2. Let $S$ be a noetherian normal integral scheme and $K$ its function field. Let $A$ be an abelian variety over $K$, and let $\mathcal{A} \rightarrow S$ be a smooth group scheme over $S$ such that, for each point $x$ of codimension one in $S$, the restriction of $\mathcal{A} \rightarrow S$ to $\mathcal{A} \times{ }_{S} \operatorname{Spec}\left(\mathcal{O}_{S, x}\right)$ is a Néron model of $A$ over $\operatorname{Spec}\left(\mathcal{O}_{S, x}\right)$. If $\mathcal{X} \rightarrow S$ is a smooth $S$-scheme and $X$ its generic fiber, then any $K$-morphism $X \rightarrow A$ uniquely extends to an $S$-morphism $\mathcal{X} \rightarrow \mathcal{A}$.

Proof. This follows from the universal property of Néron models and Weil's extension theorem (cf. [1, Theorem 1 in 4.4]).

Let us go back to the proof of Proposition 1.1.1. First of all, we choose a non-empty Zariski open set $U_{0}$ of $B$ and an abelian scheme $\mathcal{A}_{0} \rightarrow U_{0}$ whose generic fiber is $A$. Let $x_{1}, \ldots, x_{l}$ be points of codimension one in $B \backslash U_{0}$. Then there are open neighborhoods $U_{1}, \ldots, U_{l}$ of $x_{1}, \ldots, x_{l}$ respectively, and smooth group schemes $\mathcal{A}_{i}$ over $U_{i}$ of finite type with the following properties:
(i) $x_{j} \notin U_{i}$ for all $i \neq j$.
(ii) The restriction of $\mathcal{A}_{i} \rightarrow U_{i}$ to $\mathcal{A} \times_{U_{i}} \operatorname{Spec}\left(\mathcal{O}_{B, x_{i}}\right)$ is a Néron model of $A$ over $\operatorname{Spec}\left(\mathcal{O}_{B, x_{i}}\right)$ for all $i$.
(iii) $\mathcal{A}_{i} \rightarrow U_{i}$ is an abelian scheme over $U_{i} \backslash \overline{\left\{x_{i}\right\}}$ for all $i$.

For each $i=0, \ldots, l$, let $A_{i}$ be the generic fiber of $\mathcal{A}_{i} \rightarrow U_{i}$ and $\phi_{i}$ : $A \rightarrow A_{i}$ an isomorphism over $K$. Note that $x_{1}, \ldots, x_{l} \notin U_{i} \cap U_{j}$ for $i \neq j$. Thus, by Lemma 1.1.2, the isomorphism $\phi_{j} \circ \phi_{i}^{-1}: A_{i} \rightarrow A_{j}$ over $K$ extends uniquely to an isomorphism $\psi_{j i}:\left.\left.\mathcal{A}_{i}\right|_{U_{i} \cap U_{j}} \rightarrow \mathcal{A}_{j}\right|_{U_{i} \cap U_{j}}$ over $U_{i} \cap U_{j}$. Clearly, $\psi_{k j} \circ \psi_{j i}=\psi_{k i}$. Thus, if we set $U=U_{0} \cup U_{1} \cup \cdots \cup U_{l}$,
then we can construct a smooth group scheme $\mathcal{A}$ over $U$ of finite type such that $\left.\mathcal{A}\right|_{U_{i}}$ is isomorphic to $\mathcal{A}_{i}$ over $U_{i}$. The universal property of $\mathcal{A} \rightarrow U$ is obvious by Lemma 1.1.2.

### 1.2. Semiabelian reduction

Let $B$ be a noetherian normal integral scheme and $K$ the function field of $B$. Let $A$ be an abelian variety over $K$. We say $A$ has semiabelian reduction over $B$ in codimension one if there are a big open set $U$ of $B$ (i.e. $\operatorname{codim}(B \backslash U) \geq 2$ ) and a semiabelian scheme $\mathcal{A} \rightarrow U$ such that the generic fiber of $\mathcal{A} \rightarrow U$ is isomorphic to $A$.

Proposition 1.2.1. Let $B, K$ and $A$ be same as above. Let $m$ be a positive integer which has a factorization $m=m_{1} m_{2}$ with $m_{1}, m_{2} \geq 3$ and $m_{1}$ and $m_{2}$ relatively prime (for example $m=12=3 \cdot 4$ ). If $A[m](\bar{K}) \subseteq A(K)$, then $A$ has semiabelian reduction in codimension one over $B$.

Proof. Let $x$ be a point of codimension one in $B$. Then there is $m_{i}$ which is not divisible by the characteristic of the residue field of $\mathcal{O}_{B, x}$. Moreover, $A\left[m_{i}\right](\bar{K}) \subseteq A(K)$. Thus, by [11, exposé 1, Corollaire 5.18], $A$ has semiabelian reduction at $x$.

Let $U_{0}$ be a non-empty Zariski open subset of $B$ over which we can take an abelian scheme $\mathcal{A}_{0} \rightarrow U_{0}$ whose generic fiber is $A$. Let $x_{1}, \ldots, x_{l}$ be points of codimension one in $B \backslash U_{0}$. Then there are open neighborhoods $U_{1}, \ldots, U_{l}$ of $x_{1}, \ldots, x_{l}$, and semiabelian schemes $\mathcal{A}_{i}$ over $U_{i}$ with the following properties:
(i) $x_{j} \notin U_{i}$ for all $i \neq j$.
(ii) $\mathcal{A}_{i} \rightarrow U_{i}$ is an abelian scheme over $U_{i} \backslash \overline{\left\{x_{i}\right\}}$.

Thus, as in Proposition 1.1.1, if we set $U=U_{0} \cup U_{1} \cup \cdots \cup U_{l}$, then we have our desired semiabelian scheme $\mathcal{A} \rightarrow U$.

Lemma 1.2.2 (Gabber's lemma). Let $U$ be a dense Zariski open set of an integral, normal and excellent scheme $S$ and $A$ an abelian scheme over $U$. Then there is a proper, surjective and generically finite morphism $\pi: S^{\prime} \rightarrow S$ of integral, normal and excellent schemes such that the abelian scheme $A \times_{U} \pi^{-1}(U)$ over $\pi^{-1}(U)$ extends to a semiabelian scheme over $S^{\prime}$

Proof. In [12, Théorème and Proposition 4.10 in Exposé V], the existence of $\pi: S^{\prime} \rightarrow S$ and the extension of the abelian scheme is proved under the assumption $\pi: S^{\prime} \rightarrow S$ is proper and surjective. Let $S_{\eta}^{\prime}$ be the generic fiber of $\pi$. Let $z$ be the closed point of $S_{\eta}^{\prime}$ and $Z$ the
closure of $z$ in $S^{\prime}$. Moreover, let $S_{1}$ be the normalization of $Z$. Then $\pi_{1}: S_{1} \rightarrow Z \rightarrow S$ is our desired morphism.

### 1.3. The Hodge sheaf of an abelian variety

Let $G \rightarrow S$ be a smooth group scheme over $S$. Then the Hodge line bundle $\lambda_{G / S}$ of $G \rightarrow S$ is given by

$$
\lambda_{G / S}=\operatorname{det}\left(\epsilon^{*}\left(\Omega_{G / S}\right)\right)
$$

where $\epsilon: S \rightarrow G$ is the identity of the group scheme $G \rightarrow S$.
Let $B$ be a noetherian, normal, integral scheme and $K$ its function field. Let $A$ be an abelian variety over $K$ and let $\mathcal{A} \rightarrow U$ be the Néron model over $B$ in codimension one (see Section 1.1). The Hodge sheaf $\lambda(A / K ; B)$ of $A$ with respect to $B$ is defined by

$$
\lambda(A / K ; B)=\iota_{*}\left(\lambda_{\mathcal{A} / U}\right)
$$

where $\iota: U \rightarrow B$ be the natural inclusion map. Note that $\lambda(A / K ; B)$ is a reflexive sheaf of rank one on $B$.

From now on, we assume that the characteristic of $K$ is zero. Let $\phi: A \rightarrow A^{\prime}$ be an isogeny of abelian varieties over $K$. Since there is an injective homomorphism

$$
\phi^{*}: \lambda\left(A^{\prime} / K ; B\right) \rightarrow \lambda(A / K ; B),
$$

we can find an effective Weil divisor $D_{\phi}$ such that

$$
c_{1}\left(\lambda\left(A^{\prime} / K ; B\right)\right)+D_{\phi}=c_{1}(\lambda(A / K ; B)) .
$$

The ideal sheaf $\mathcal{O}_{B}\left(-D_{\phi}\right)$ is denoted by $\mathcal{I}_{\phi}$.
Lemma 1.3.3. Let $\phi^{\vee}: A^{\wedge} \rightarrow A^{\vee}$ be the dual of $\phi: A \rightarrow A^{\prime}$. We assume that $B$ is the spectrum of a discrete valuation ring $R$ and that $A, A^{\prime}$ have semiabelian reduction over $B$. Then $\mathcal{I}_{\phi} \cdot \mathcal{I}_{\phi^{\vee}}=\operatorname{deg}(\phi) R$.

Proof. Let $R^{\prime}$ be an extension of $R$ such that $R^{\prime}$ is a complete discrete valuation ring and the residue field of $R^{\prime}$ is algebraically closed (cf. [7, Theorem 29.1]). Then, by [12, Exposé VII, Théroèm 2.1.1], $\left(\mathcal{I}_{\phi} \cdot \mathcal{I}_{\phi} \vee\right) R^{\prime}=\operatorname{deg}(\phi) R^{\prime}$. Here $R^{\prime}$ is faithfully flat over $R$. Thus $\mathcal{I}_{\phi} \cdot \mathcal{I}_{\phi} \vee=$ $\operatorname{deg}(\phi) R$.

### 1.4. Locally integrable hermitian metric

Let $M$ be a complex manifold and $L$ a line bundle on $M$. A singular hermitian metric $\|\cdot\|$ of $L$ is a $C^{\infty}$-hermitian metric of $\left.L\right|_{U}$, where $U$ is a certain dense Zariski open subset of $M$. If $\|\cdot\|_{0}$ is an arbitrary $C^{\infty}$-hermitian metric of $L$ and $\sigma \neq 0$ is a local section of $L$ around $x$, the ratio $\mu=\|\sigma\| /\|\sigma\|_{0}$ of the two norms is independent of $\sigma$, and hence $\mu$ is a positive $C^{\infty}$-function defined on $U$. A locally integrable hermitian metric (or $L_{\text {loc }}^{1}$-hermitian metric) is a singular hermitian metric such that the function $\log (\mu)$ on $U$ extends to a locally integrable function on $M$ (of course this definition does not depend on the choice of the $C^{\infty}$-hermitian metric $\left.\|\cdot\|_{0}\right)$.

Lemma 1.4.1. Let $M$ be a complex manifold and $(L,\|\cdot\|)$ a hermitian line bundle on $M$. Let s be a non-zero meromorphic section of $L$ over $M$. Then the hermitian metric $\|\cdot\|$ is locally integrable if and only if so is $\log \|s\|$.

Proof. Let $\|\cdot\|_{0}$ be a $C^{\infty}$-hermitian metric of $L$. Then

$$
\log \|s\|=\log \left(\|\cdot\| /\|\cdot\|_{0}\right)+\log \|s\|_{0}
$$

Note that $\log \|s\|_{0}$ is locally integrable. Thus $\log \|s\|$ is locally integrable if and only if so is $\log \left(\|\cdot\| /\|\cdot\|_{0}\right)$.

Lemma 1.4.2. Let $f: Y \rightarrow X$ be a surjective, proper and generically finite morphism of non-singular varieties over $\mathbb{C}$. Let $(L,\|\cdot\|)$ be a singular hermitian line bundle on $X$. Assume that there are a nonempty Zariski open set $U$ of $X$ and a hermitian line bundle $\left(L^{\prime},\|\cdot\|^{\prime}\right)$ on $Y$ such that $\left(L^{\prime},\|\cdot\|^{\prime}\right)$ is isometric to $f^{*}(L,\|\cdot\|)$ over $f^{-1}(U)$. If $\|\cdot\|^{\prime}$ is locally integrable, then so is $\|\cdot\|$.

Proof. Shrinking $U$ if necessarily, we may assume that $f$ is étale over $U$. We set $V=f^{-1}(U)$. Let $s$ be a non-zero rational section of $L$. Note that there is a divisor $D$ on $Y$ such that $L^{\prime}=f^{*}(L) \otimes \mathcal{O}_{Y}(D)$ and $\operatorname{Supp}(D) \subseteq Y \backslash V$. Thus $f^{*}(s)$ gives rise to a rational section $s^{\prime}$ of $L^{\prime}$. Then $\log \left\|s^{\prime}\right\|^{\prime}$ is locally integrable by Lemma 1.4.1. Since $\left.f^{*}(\log \|s\|)\right|_{V}=\log \|\left. s^{\prime}| |^{\prime}\right|_{V}$, we can see that $f^{*}(\log \|s\|)$ is locally integrable. Let $\left[f^{*}(\log \|s\|)\right]$ be a current associated to the locally integrable function $f^{*}(\log \|s\|)$. Then, by [5, Proposition 1.2.5], there is a locally integrable function $g$ on $X$ with $f_{*}\left[f^{*}(\log \|s\|)\right]=[g]$. Since $f$ is étale over $U$, we can easily see that

$$
\left(\left.f\right|_{V}\right)_{*}\left[\left(\left.f\right|_{V}\right)^{*}\left(\left.\log \|s\|\right|_{U}\right)\right]=\operatorname{deg}(f)\left[\left.\log \|s\|\right|_{U}\right]
$$

Thus $g=\operatorname{deg}(f) \log \|s\|$ almost everywhere over $U$. Therefore so is over $X$ because $U$ is a non-empty Zariski open set of $X$. Hence $\log \|s\|$ is locally integrable on $X$.

### 1.5. Hermitian metric with logarithmic singularities

Let $X$ be a normal variety over $\mathbb{C}$ and $Y$ a proper closed subscheme of $X$. Let $(L,\|\cdot\|)$ be a hermitian line bundle on $X$. We say that $(L,\|\cdot\|)$ is a $C^{\infty}$-hermitian line bundle with logarithmic singularities along $Y$ if the following conditions are satisfied:
(1) $\|\cdot\|$ is $C^{\infty}$ over $X \backslash Y$.
(2) Let $\|\cdot\|_{0}$ be a $C^{\infty}$-hermitian metric of $L$. For each $x \in Y$, let $f_{1}, \ldots, f_{m}$ be a system of local equations of $Y$ around $x$, i.e., $Y$ is given by $\left\{z \in X \mid f_{1}(z)=\cdots=f_{m}(z)=0\right\}$ around $x$. Then there are positive constants $C$ and $r$ such that

$$
\max \left\{\frac{\|\cdot\|}{\|\cdot\|_{0}}, \frac{\|\cdot\|_{0}}{\|\cdot\|}\right\} \leq C\left(-\sum_{i=1}^{m} \log \left|f_{i}\right|\right)^{r}
$$

around $x$.
Note that the above definition does not depend on the choice of the system of local equations $f_{1}, \ldots, f_{m}$. Moreover it is easy to see that if $(L,\|\cdot\|)$ is a $C^{\infty}$-hermitian line bundle with logarithmic singularities along $Y$, then $\|\cdot\|$ is locally integrable.

Lemma 1.5.1. Let $\pi: X^{\prime} \rightarrow X$ be a proper morphism of normal varieties over $\mathbb{C}$ and $Y$ a proper closed subscheme of $X$. Let $(L,\|\cdot\|)$ be a hermitian line bundle on $X$ such that $\|\cdot\|$ is $C^{\infty}$ over $X \backslash Y$. If $\pi\left(X^{\prime}\right) \nsubseteq Y$ and $(L,\|\cdot\|)$ has logarithmic singularities along $Y$, then so does $\pi^{*}(L,\|\cdot\|)$ along $\pi^{-1}(Y)$. Moreover, if $\pi$ is surjective and $\pi^{*}(L,\|\cdot\|)$ has logarithmic singularities along $\pi^{-1}(Y)$, then so does $(L,\|\cdot\|)$ along $Y$.

Proof. Let $\left\{f_{1}, \ldots, f_{m}\right\}$ be a system of local equations of $Y$. Then $\left\{\pi^{*}\left(f_{1}\right), \ldots, \pi^{*}\left(f_{m}\right)\right\}$ is a system of local equation of $\pi^{-1}(Y)$. Thus our assertion is obvious.

### 1.6. Faltings' metric

Let $X$ be a normal variety over $\mathbb{C}$ and let $f: A \rightarrow X$ be a $g$ dimensional semiabelian scheme over $X$. We assume that there is a non-empty Zariski open set $U$ of $X$ such that $f$ is an abelian scheme over $U$. Let $\lambda_{A / X}$ be the Hodge line bundle of $A \rightarrow X$, i.e.,

$$
\lambda_{A / X}=\operatorname{det}\left(\epsilon^{*}\left(\Omega_{A / X}\right)\right)
$$

where $\epsilon: X \rightarrow A$ is the identity of the semiabelian scheme $A \rightarrow X$. Via the natural isomorphism $\rho: \lambda_{A_{x}} \xrightarrow{\sim} f_{x_{*}}\left(\operatorname{det}\left(\Omega_{A_{x}}\right)\right)$ at each $x \in U$, we define Faltings' metric $\|\cdot\|_{\text {Fal }}$ of $\lambda_{A / X}$ by

$$
\left(\|\alpha\|_{\mathrm{Fal}, x}\right)^{2}=\left(\frac{\sqrt{-1}}{2}\right)^{g} \int_{A_{x}} \rho(\alpha) \wedge \overline{\rho(\alpha)} .
$$

Faltings' metric is a $C^{\infty}$-hermitian metric on $U$ and hence it is a singular hermitian metric on $X$. Furthermore this metric is known to have logarithmic singularities along the boundary $X \backslash U$ (cf. [12, Théorèm 3.2 in Exposé I]) and in particular a locally integrable hermitian metric.

Lemma 1.6.1. Let $X$ be a smooth variety over $\mathbb{C}$ and $X_{0}$ a nonempty Zariski open set of $X$. Let $A_{0} \rightarrow X_{0}$ be an abelian scheme over $X_{0}$. Let $\lambda$ be a line bundle on $X$ such that $\left.\lambda\right|_{X_{0}}$ coincides with the Hodge line bundle $\lambda_{A_{0} / X_{0}}$ of $A_{0} \rightarrow X_{0}$. Then Faltings' metric $\|\cdot\|_{\text {Fal }}$ of $\lambda_{A_{0} / X_{0}}$ over $X_{0}$ extends to a locally integrable metric of $\lambda$ over $X$.

Proof. By virtue of Lemma 1.2.2 (Gabber's lemma), there is a proper, surjective and generically finite morphism $\pi: X^{\prime} \rightarrow X$ of smooth varieties over $\mathbb{C}$ such that the abelian scheme $A_{0} \times_{X_{0}} \pi^{-1}\left(X_{0}\right)$ over $\pi^{-1}\left(X_{0}\right)$ extends to a semiabelian scheme $f^{\prime}: A^{\prime} \rightarrow X^{\prime}$. Let $\lambda_{A^{\prime} / X^{\prime}}$ be the Hodge line bundle of $A^{\prime} \rightarrow X^{\prime}$ and $\|\cdot\|_{\text {Fal }}^{\prime}$ Faltings' metric of $\lambda_{A^{\prime} / X^{\prime}}$. Then $\left.\left(\lambda_{A^{\prime} / X^{\prime}},\|\cdot\|_{\text {Fal }}^{\prime}\right)\right|_{X_{0}^{\prime}}$ is isometric to $\pi_{0}^{*}\left(\lambda_{A_{0} / X_{0}},\|\cdot\|_{\text {Fal }}\right)$, where $X_{0}^{\prime}=\pi^{-1}\left(X_{0}\right)$ and $\pi_{0}=\left.\pi\right|_{X_{0}^{\prime}}$. Therefore, by Lemma 1.4.2, $\|\cdot\|_{\text {Fal }}$ extends to a locally integrable metric of $\lambda$ over $X$.

### 1.7. The moduli of abelian varieties

In order to deal with the bounding property of the modular height, we need a reasonable compactification of the moduli space of polarized abelian varieties. For simplicity, an abelian variety with a polarization of degree $l^{2}$ is called an $l$-polarized abelian variety.

Theorem 1.7.1. Let $g, l$ and $m$ be positive integers with $m \geq 3$. Let $\mathbb{A}_{g, l, m, \mathbb{Q}}$ be the moduli space of $g$-dimensional and l-polarized abelian varieties over $\mathbb{Q}$ with level $m$ structure. Then there exist
(a) normal and projective arithmetic varieties $\mathbb{A}_{g, l, m}^{*}$ and $Y^{*}$ (i.e., $\mathbb{A}_{g, l, m}^{*}$ and $Y^{*}$ are normal and integral schemes flat and projective over $\mathbb{Z}$ ),
(b) a surjective and generically finite morphism $f: Y^{*} \rightarrow \mathbb{A}_{g, l, m}^{*}$,
(c) a positive integer n,
(d) a line bundle $L$ on $\mathbb{A}_{g, l, m}^{*}$, and
(e) a semiabelian scheme $G \rightarrow Y^{*}$
with the following properties:
(1) $\mathbb{A}_{g, l, m, \mathbb{Q}}$ is a Zariski open set of $\mathbb{A}_{g, l, m, \mathbb{Q}}^{*}=\mathbb{A}_{g, l, m}^{*} \times \mathbb{Z} \operatorname{Spec}(\mathbb{Q})$ and $L$ is very ample on $\mathbb{A}_{g, l, m}^{*}$.
(2) Let $\lambda_{G / Y^{*}}$ be the Hodge line bundle of the semiabelian scheme $G \rightarrow Y^{*}$. Then $f^{*}(L)=\lambda_{G / Y^{*}}^{\otimes n}$ on $Y_{\mathbb{Q}}^{*}=Y^{*} \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$.
(3) Let $U_{\mathbb{Q}} \rightarrow \mathbb{A}_{g, l, m, \mathbb{Q}}$ be the universal $g$-dimensional and l-polarized abelian scheme with level $m$ structure. Let $Y_{\mathbb{Q}}$ be the pull-back of $\mathbb{A}_{g, l, m, \mathbb{Q}}$ by $f_{\mathbb{Q}}: Y_{\mathbb{Q}}^{*} \rightarrow \mathbb{A}_{g, l, m, \mathbb{Q}}^{*}$, i.e., $Y_{\mathbb{Q}}=\left(f_{\mathbb{Q}}\right)^{-1}\left(\mathbb{A}_{g, l, m, \mathbb{Q}}\right)$. Then $G_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}^{*}$ is an extension of the abelian scheme $U_{\mathbb{Q}} \times_{\mathbb{A}_{g, l, m, \mathbb{Q}}}$ $Y_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$. (Note that $\left.G\right|_{Y_{\mathbb{Q}}} \rightarrow Y_{\mathbb{Q}}$ is naturally a $g$-dimensional and l-polarized abelian scheme with level $m$ structure.)
(4) $\quad L$ has a metric $\|\cdot\|$ over $\mathbb{A}_{g, l, m, \mathbb{Q}}(\mathbb{C})$ such that $f^{*}((L,\|\cdot\|))$ is isometric to $\left(\lambda_{G / Y^{*}},\|\cdot\|_{\text {Fal }}\right)^{\otimes n}$ over $Y_{\mathbb{Q}}(\mathbb{C})$. Moreover, $\|\cdot\|$ has logarithmic singularities along $\mathbb{A}_{g, l, m, \mathbb{Q}}^{*}(\mathbb{C}) \backslash \mathbb{A}_{g, l, m, \mathbb{Q}}(\mathbb{C})$.
Proof. Let $U_{\mathbb{Q}} \rightarrow \mathbb{A}_{g, l, m, \mathbb{Q}}$ be the universal $l$-polarized abelian scheme with level $m$ structure. By [12, Théorème 2.2 in Exposé IV], there are a normal and projective variety $\mathbb{A}_{g, l, m, \mathbb{Q}}^{*}$, a positive integer $n$ and a very ample line bundle $L_{\mathbb{Q}}$ on $\mathbb{A}_{g, l, m, \mathbb{Q}}^{*}$ with the following properties:
(i) $\mathbb{A}_{g, l, m, \mathbb{Q}}$ is an Zariski open set of $\mathbb{A}_{g, l, m, \mathbb{Q}}^{*}$.
(ii) By Gabber's lemma (cf. Lemma 1.2.2), there is a surjective and generically finite morphism $h_{\mathbb{Q}}: S_{\mathbb{Q}}^{\prime} \rightarrow \mathbb{A}_{g, l, m, \mathbb{Q}}^{*}$ of normal and projective varieties over $\mathbb{Q}$ such that the abelian scheme $U_{\mathbb{Q}} \times_{\mathbb{A}_{g, l, m, \mathbb{Q}}} h_{\mathbb{Q}}^{-1}\left(\mathbb{A}_{g, l, m, \mathbb{Q}}\right) \rightarrow h_{\mathbb{Q}}^{-1}\left(\mathbb{A}_{g, l, m, \mathbb{Q}}\right)$ extends to a semiabelian scheme $G_{\mathbb{Q}}^{\prime} \rightarrow S_{\mathbb{Q}}^{\prime}$. Then $h_{\mathbb{Q}}^{*}\left(L_{\mathbb{Q}}\right)=\lambda_{G_{\mathbb{Q}}^{\prime} / S_{\mathbb{Q}}^{\prime}}^{\otimes n}$.
Since $L_{\mathbb{Q}}$ is very ample, there is an embedding $\mathbb{A}_{g, l, m, \mathbb{Q}}^{*} \hookrightarrow \mathbb{P}_{\mathbb{Q}}^{N}$ in terms of $L_{\mathbb{Q}}$. Let $\mathbb{A}_{g, l, m}^{*}$ be the closure of the image of

$$
\mathbb{A}_{g, l, m, \mathbb{Q}}^{*} \hookrightarrow \mathbb{P}_{\mathbb{Q}}^{N} \rightarrow \mathbb{P}_{\mathbb{Z}}^{N}
$$

Let $L$ be the pull-back of $\mathcal{O}_{\mathbb{P}_{\mathbb{Z}}^{N}}(1)$ by the embedding $\mathbb{A}_{g, l, m}^{*} \hookrightarrow \mathbb{P}_{\mathbb{Z}}^{N}$. We have obvious isomorphisms $\mathbb{A}_{g, l, m, \mathbb{Q}}^{*} \simeq \mathbb{A}_{g, l, m}^{*} \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$ and $L_{\mathbb{Q}} \simeq$ $\left.L\right|_{\mathbb{A}_{g, l, m, \mathbb{Q}}^{*}}$. Let $S^{\prime}$ denote the normalization of $\mathbb{A}_{g, l, m}^{*}$ in the function field of $S_{\mathbb{Q}}^{\prime}$. There exists an open subset $S_{0}^{\prime}$ of $S^{\prime}$ such that $G^{\prime}$ is an abelian scheme over $S_{0}^{\prime}$ and $G^{\prime} \times S^{\prime} S_{0}^{\prime} \rightarrow S_{0}^{\prime}$ coincides with the abelian scheme $U_{\mathbb{Q}} \times_{\mathbb{A}_{g, l, m, \mathbb{Q}}} h_{\mathbb{Q}}^{-1}\left(\mathbb{A}_{g, l, m, \mathbb{Q}}\right) \rightarrow h_{\mathbb{Q}}^{-1}\left(\mathbb{A}_{g, l, m, \mathbb{Q}}\right)$ over $\mathbb{Q}$. Thus, using Gabber's lemma again, there are a surjective and generically finite morphism of normal and projective arithmetic varieties $h_{2}: Y^{*} \rightarrow S^{\prime}$ and a semiabelian scheme $G \rightarrow Y^{*}$ such that $G \rightarrow Y^{*}$ is an extension of $G^{\prime} \times_{S^{\prime}} h_{2}^{-1}\left(S_{0}^{\prime}\right) \rightarrow h_{2}^{-1}\left(S_{0}^{\prime}\right)$. Thus, over $Y_{\mathbb{Q}}^{*}=Y^{*} \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$, the
semiabelian variety $G$ is equal to $G_{\mathbb{Q}}^{\prime} \times_{S_{\mathbb{Q}}^{\prime}} Y_{\mathbb{Q}}^{*} \rightarrow Y_{\mathbb{Q}}^{*}$ by the uniqueness of semiabelian extensions. Thus, if we set $f=h \cdot h_{1}$, then $f^{*}(L)=\lambda_{G / Y^{*}}^{\otimes n}$ over $Y_{\mathbb{Q}}^{*}$.

Finally, since $\left.L_{\mathbb{Q}}\right|_{\mathbb{A}_{g, l, m, \mathbb{Q}}}=\lambda_{U_{\mathbb{Q}} / \mathbb{A}_{g, l, m, \mathbb{Q}}}^{\otimes n}$, if we give $L_{\mathbb{Q}}$ a metric arising from Faltings' metric of $\lambda_{U_{\mathbb{Q}} / \mathbb{A}_{g, l, m, \mathbb{Q}}}$, then assertion of (4) follows from Lemma 1.5.1 and [12, Théorèm 3.2 in Exposé I].

### 1.8. Arakelov geometry

In this paper, an arithmetic variety means an integral scheme flat and quasi-projective over $\mathbb{Z}$. If it is smooth over $\mathbb{Q}$, then it is said to be generically smooth.

Let $X$ be a generically smooth arithmetic variety. A pair $(Z, g)$ is called an arithmetic cycle of codimension $p$ if $Z$ is a cycle of codimension $p$ and $g$ is a current of type $(p-1, p-1)$ on $X(\mathbb{C})$. We denote by $\widehat{Z}^{p}(X)$ the set of all arithmetic cycles on $X$. We set

$$
\widehat{\mathrm{CH}}^{p}(X)=\widehat{Z}^{p}(X) / \sim,
$$

where $\sim$ is the arithmetic linear equivalence.
Let $\bar{L}=(L,\|\cdot\|)$ be a $C^{\infty}$ _hermitian line bundle on $X$. Then a homomorphism

$$
\widehat{c}_{1}(\bar{L}) \cdot: \widehat{\mathrm{CH}}^{p}(X) \rightarrow \widehat{\mathrm{CH}}^{p+1}(X)
$$

is define by

$$
\widehat{c}_{1}(\bar{L}) \cdot(Z, g)=\left(\operatorname{div}(s) \text { on } Z,\left[-\log \left(\|s\|_{Z}^{2}\right)\right]+c_{1}(\bar{L}) \wedge g\right)
$$

where $s$ is a rational section of $\left.L\right|_{Z}$ and $\left[-\log \left(\|s\|_{Z}^{2}\right)\right]$ is a current given by $\phi \mapsto-\int_{Z(\mathbb{C})} \log \left(\|s\|_{Z}^{2}\right) \phi$.

When $X$ is projective, we can define the canonical arithmetic degree map

$$
\widehat{\operatorname{deg}}: \widehat{\mathrm{CH}}^{\operatorname{dim} X}(X) \rightarrow \mathbb{R}
$$

given by

$$
\widehat{\operatorname{deg}}\left(\sum_{P} n_{P} P, g\right)=\sum_{P} n_{P} \log (\#(\kappa(P)))+\frac{1}{2} \int_{X(\mathbb{C})} g
$$

Thus, if $C^{\infty}$-hermitian line bundles $\bar{L}_{1}, \ldots, \bar{L}_{\operatorname{dim} X}$ are given, then we can get the number

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{L}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{L}_{\operatorname{dim} X}\right)\right)
$$

which is called the arithmetic intersection number of $\bar{L}_{1}, \ldots, \bar{L}_{\operatorname{dim} X}$.
Let $X$ be a projective arithmetic variety. Note that $X$ is not necessarily generically smooth. Let $\bar{L}_{1}, \ldots, \bar{L}_{\text {dim } X}$ be $C^{\infty}$-hermitian line bundles on $X$. By [6], we can find a generic resolution of singularities $\mu: Y \rightarrow X$, i.e., $\mu: Y \rightarrow X$ is a projective and birational morphism such that $Y$ is a generically smooth projective arithmetic variety. Then we can see that the arithmetic intersection number

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\mu^{*}\left(\bar{L}_{1}\right)\right) \cdots \widehat{c}_{1}\left(\mu^{*}\left(\bar{L}_{\operatorname{dim} X}\right)\right)\right)
$$

does not depend on the choice of the generic resolution of singularities $\mu: Y \rightarrow X$. Thus we denote this number by

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{L}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{L}_{\operatorname{dim} X}\right)\right) .
$$

Let $\bar{L}_{1}, \ldots, \bar{L}_{l}$ be $C^{\infty}$-hermitian line bundles on a projective arithmetic variety $X$. Let $V$ be an $l$-dimensional integral closed subscheme on $X$. Then $\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{L}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{L}_{l}\right) \mid V\right)$ is defined by

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\left.\bar{L}_{1}\right|_{V}\right) \cdots \widehat{c}_{1}\left(\left.\bar{L}_{l}\right|_{V}\right)\right) .
$$

Moreover, for an $l$-dimensional cycle $Z=\sum_{i} n_{i} V_{i}$ on $X$,

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{L}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{L}_{l}\right) \mid Z\right)
$$

is defined by

$$
\sum_{i} n_{i} \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{L}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{L}_{l}\right) \mid V_{i}\right) .
$$

### 1.9. Notions concerning the positivity of $\mathbb{Q}$-line bundles on an arithmetic variety

Let $X$ be a projective arithmetic variety and $\bar{L}$ a $C^{\infty}$-hermitian $\mathbb{Q}$ line bundle on $X$. Let us introduce several kinds of the positivity of $C^{\infty}$-hermitian $\mathbb{Q}$-line bundles.

- ample: $\bar{L}$ is ample if $L$ is ample on $X, c_{1}(\bar{L})$ is positive form on $X(\mathbb{C})$, and there is a positive number $n$ such that $L^{\otimes n}$ is generated by the set $\left\{s \in H^{0}\left(X, L^{\otimes n}\right) \mid\|s\|_{\text {sup }}<1\right\}$.
- nef: $\bar{L}$ is nef if $c_{1}(\bar{L})$ is a semipositive form on $X(\mathbb{C})$ and, for all one-dimensional integral closed subschemes $\Gamma$ of $X, \widehat{\operatorname{deg}}\left(\widehat{c}_{1}(\bar{L}) \mid \Gamma\right) \geq 0$.
- big: $\bar{L}$ is $b i g$ if $\mathrm{rk}_{\mathbb{Z}} H^{0}\left(X, L^{\otimes m}\right)=O\left(m^{\operatorname{dim} X_{Q}}\right)$ and there is a nonzero section $s$ of $H^{0}\left(X, L^{\otimes n}\right)$ with $\|s\|_{\text {sup }}<1$ for some positive integer $n$.
- $\mathbb{Q}$-effective: $\bar{L}$ is $\mathbb{Q}$-effective if there is a positive integer $n$ and a non-zero $s \in H^{0}\left(X, L^{\otimes n}\right)$ with $\|s\|_{\text {sup }} \leq 1$.
- pseudo-effective: $\bar{L}$ is pseudo-effective if there are (1) a sequence $\left\{\bar{L}_{n}\right\}_{n=1}^{\infty}$ of $\mathbb{Q}$-effective $C^{\infty}$-hermitian $\mathbb{Q}$-line bundles, (2) $C^{\infty}$-hermitian $\mathbb{Q}$-line bundles $\bar{E}_{1}, \ldots, \bar{E}_{r}$ and (3) sequences

$$
\left\{a_{1, n}\right\}_{n=1}^{\infty}, \ldots,\left\{a_{r, n}\right\}_{n=1}^{\infty}
$$

of rational numbers such that

$$
\widehat{c}_{1}(\bar{L})=\widehat{c}_{1}\left(\bar{L}_{n}\right)+\sum_{i=1}^{r} a_{i, n} \widehat{c}_{1}\left(\bar{E}_{i}\right)
$$

in $\widehat{\mathrm{CH}}^{1}(X) \otimes \mathbb{Q}$ and $\lim _{n \rightarrow \infty} a_{i, n}=0$ for all $i$. If $\bar{L}_{1} \otimes \bar{L}_{2}^{\otimes-1}$ is pseudoeffective for $C^{\infty}$-hermitian $\mathbb{Q}$-line bundles $\bar{L}_{1}, \bar{L}_{2}$ on $X$, then we denote this by $\bar{L}_{1} \succsim \bar{L}_{2}$.

### 1.10. Polarization of a finitely generated field over $\mathbb{Q}$

Let $K$ be a field of finite type over the rational number field $\mathbb{Q}$ with $d=\operatorname{tr} . \operatorname{deg}_{\mathbb{Q}}(K)$. A polarization $\bar{B}$ of $K$ is a collection of data $\bar{B}=\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$, where
(1) $B$ is a normal and projective arithmetic variety whose function field is isomorphic to $K$;
(2) $\bar{H}_{1}, \ldots, \bar{H}_{d}$ are nef $C^{\infty}$-hermitian line bundles on $B$.

Here $\operatorname{deg}(\bar{B})$ is given by

$$
\int_{B(\mathbb{C})} c_{1}\left(\bar{H}_{1}\right) \wedge \cdots \wedge c_{1}\left(\bar{H}_{d}\right) .
$$

Namely,

$$
\operatorname{deg}(\bar{B})= \begin{cases}{[K: \mathbb{Q}]} & \text { if } d=0, \\ \operatorname{deg}\left(\left(H_{1}\right)_{\mathbb{Q}} \cdots\left(H_{d}\right) \mathbb{Q}\right) \text { on } B \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q}) & \text { if } d>0 .\end{cases}
$$

If $B$ is generically smooth, then the polarization $\bar{B}$ is said to be generically smooth. Moreover, we say the polarization $\bar{B}=\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$ is fine (resp. strictly fine) if there are (a) a generically finite morphism $\pi: B^{\prime} \rightarrow B$ of normal projective arithmetic varieties, (b) a generically finite morphism $\mu: B^{\prime} \rightarrow\left(\mathbb{P}_{\mathbb{Z}}^{1}\right)^{d}$ and (c) ample $C^{\infty}$-hermitian $\mathbb{Q}$-line bundles $\bar{L}_{1}, \ldots, \bar{L}_{d}$ on $\mathbb{P}_{\mathbb{Z}}^{1}$ such that $\pi^{*}\left(\bar{H}_{i}\right) \otimes \mu^{*}\left(p_{i}^{*}\left(\bar{L}_{i}\right)\right)^{\otimes-1}$ is pseudoeffective (resp. $\mathbb{Q}$-effective) for every $i$, where $p_{i}:\left(\mathbb{P}_{\mathbb{Z}}^{1}\right)^{d} \rightarrow \mathbb{P}_{\mathbb{Z}}^{1}$ is the projection to the $i$-th factor. Note that if $\bar{H}_{1}, \ldots, \bar{H}_{d}$ are big, then the polarization $\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$ is strictly fine. Moreover, if $\bar{B}$ is fine, then $\operatorname{deg}(\bar{B})>0$.

Proposition 1.10.1. Let $\bar{B}=\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$ be a strictly fine polarization of $K$. Then, for all $h$, the number of prime divisors $\Gamma$ on B with

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \mid \Gamma\right) \leq h
$$

is finite.
Proof. Let us begin with the following lemma.
Lemma 1.10.2. Let $\pi: X^{\prime} \rightarrow X$ be a generically finite morphism of normal and projective arithmetic varieties. Let $\bar{H}_{1}, \ldots, \bar{H}_{d}$ be nef $C^{\infty}{ }_{-}$ hermitian line bundles on $X$, where $d=\operatorname{dim} X_{\mathbb{Q}}$. Then the following two statements are equivalent:
(1) For all $h$, the number of prime divisors $\Gamma$ on $X$ with

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \mid \Gamma\right) \leq h
$$

is finite
(2) For all $h^{\prime}$, the number of prime divisors $\Gamma^{\prime}$ on $X^{\prime}$ with

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{d}\right)\right) \mid \Gamma^{\prime}\right) \leq h^{\prime}
$$

is finite.
Proof. Let $X_{0}$ be the maximal Zariski open set of $X$ such that $X_{0}$ is regular and $\pi$ is finite over $X_{0}$. Then $\operatorname{codim}\left(X \backslash X_{0}\right) \geq 2$. We set $X_{0}^{\prime}=\pi^{-1}\left(X_{0}\right)$ and $\pi_{0}=\left.\pi\right|_{X_{0}^{\prime}}$. Let $\operatorname{Div}(X)$ and $\operatorname{Div}\left(X^{\prime}\right)$ be the groups of Weil divisors on $X$ and $X^{\prime}$ respectively. Define the homomorphism $\pi^{\star}: \operatorname{Div}(X) \rightarrow \operatorname{Div}\left(X^{\prime}\right)$ as the composition of natural homomorphisms:

$$
\operatorname{Div}(X) \rightarrow \operatorname{Div}\left(X_{0}\right) \xrightarrow{\pi_{0}^{*}} \operatorname{Div}\left(X_{0}^{\prime}\right) \rightarrow \operatorname{Div}\left(X^{\prime}\right),
$$

where $\operatorname{Div}(X) \rightarrow \operatorname{Div}\left(X_{0}\right)$ is the restriction map and $\operatorname{Div}\left(X_{0}^{\prime}\right) \rightarrow \operatorname{Div}\left(X^{\prime}\right)$ is defined by taking the Zariski closure of divisors. Note that $\pi_{*} \pi^{\star}(D)=$ $\operatorname{deg}(\pi) D$ for all $D \in \operatorname{Div}(X)$.

First we assume (1). Note that the number of prime divisors in $X^{\prime} \backslash X_{0}^{\prime}$ is finite, so that it is sufficient to show that the number of prime divisors $\Gamma^{\prime}$ on $X^{\prime}$ with $\Gamma^{\prime} \nsubseteq X^{\prime} \backslash X_{0}^{\prime}$ and

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{d}\right)\right) \mid \Gamma^{\prime}\right) \leq h^{\prime}
$$

is finite. By the projection formula,

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{d}\right)\right) \mid \Gamma^{\prime}\right)=\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \mid \pi_{*}\left(\Gamma^{\prime}\right)\right)
$$

Thus, by (1), the number of $\left(\pi_{*}\left(\Gamma^{\prime}\right)\right)_{\text {red }}$ is finite. On the other hand, the number of prime divisors in $\pi^{-1}\left(\pi_{*}(\Gamma)_{\mathrm{red}}\right)$ is finite. Hence we get (2).

Next we assume (2). Let $\Gamma$ be a prime divisor on $X$ with

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \mid \Gamma\right) \leq h .
$$

Then

$$
\begin{aligned}
& \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{d}\right)\right) \mid \pi^{\star}(\Gamma)\right) \\
&=\operatorname{deg}(\pi) \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \mid \Gamma\right) \leq \operatorname{deg}(\pi) h
\end{aligned}
$$

Thus, by (2), the number of $\pi^{\star}(\Gamma)$ 's is finite. Therefore we get (1).
Let us go back to the proof of Proposition 1.10.1. We use the notation in the above definition of strict finiteness. By Lemma 1.10.2, it is sufficient to show that the number of prime divisors $\Gamma^{\prime}$ on $B^{\prime}$ with

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{d}\right)\right) \mid \Gamma^{\prime}\right) \leq h
$$

is finite for all $h$.
There are $\mathbb{Q}$-effective $C^{\infty}$-hermitian line bundles $\bar{Q}_{1}, \ldots, \bar{Q}_{d}$ on $B^{\prime}$ with

$$
\pi^{*}\left(\bar{H}_{i}\right)=\mu^{*}\left(p_{i}^{*}\left(\bar{L}_{i}\right)\right) \otimes \bar{Q}_{i}
$$

for all $i$. Note that

$$
\begin{aligned}
& \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{d}\right)\right) \mid \Gamma^{\prime}\right) \\
&= \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\mu^{*}\left(p_{1}^{*}\left(\bar{L}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\mu^{*}\left(p_{d}^{*}\left(\bar{L}_{d}\right)\right)\right) \mid \Gamma^{\prime}\right)+ \\
& \sum_{i=1}^{d} \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\mu^{*}\left(p_{1}^{*}\left(\bar{L}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\mu^{*}\left(p_{i-1}^{*}\left(\bar{L}_{i-1}\right)\right)\right) \cdot \widehat{c}_{1}\left(\bar{Q}_{i}\right)\right. \\
&\left.\widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{i+1}\right)\right) \cdots \widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{d}\right)\right) \mid \Gamma^{\prime}\right) .
\end{aligned}
$$

Moreover, since $\bar{Q}_{i}$ is $\mathbb{Q}$-effective, the number of prime divisors $\Gamma^{\prime}$ with

$$
\begin{aligned}
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\mu^{*}\left(p_{1}^{*}\left(\bar{L}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\mu^{*}\left(p_{i-1}^{*}\left(\bar{L}_{i-1}\right)\right)\right) \cdot \widehat{c}_{1}\left(\bar{Q}_{i}\right) .\right. \\
\left.\widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{i+1}\right)\right) \cdots \widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{d}\right)\right) \mid \Gamma^{\prime}\right)<0
\end{aligned}
$$

is finite for every $i$. Thus we have

$$
\begin{aligned}
& \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(\pi^{*}\left(\bar{H}_{d}\right)\right) \mid \Gamma^{\prime}\right) \\
& \geq \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\mu^{*}\left(p_{1}^{*}\left(\bar{L}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\mu^{*}\left(p_{d}^{*}\left(\bar{L}_{d}\right)\right)\right) \mid \Gamma^{\prime}\right)
\end{aligned}
$$

except finitely many $\Gamma^{\prime}$. On the other hand, by [10, Proposition 5.1.1], the number of prime divisors $\Gamma^{\prime \prime}$ on $\left(\mathbb{P}_{\mathbb{Z}}^{1}\right)^{d}$ with

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(p_{1}^{*}\left(\bar{L}_{1}\right)\right) \cdots \widehat{c}_{1}\left(p_{d}^{*}\left(\bar{L}_{d}\right)\right) \mid \Gamma^{\prime \prime}\right) \leq h
$$

is finite. This completes the proof.
Remark 1.10.3. Let $X$ be a normal and projective arithmetic variety of dimension $n$. Let $\bar{H}_{1}, \ldots, \bar{H}_{n-2}$ be nef $C^{\infty}$-hermitian line bundles on $X$ and $\bar{L}$ a $C^{\infty}$-hermitian line bundle on $X$. If $\bar{L}$ is pseudo-effective, then we can expect the number of prime divisors $\Gamma$ on $X$ with

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{n-2}\right) \cdot \widehat{c}_{1}(\bar{L}) \mid \Gamma\right)<0
$$

to be finite. If it is true, then Proposition 1.10.1 holds under the weaker assumption that the polarization is fine.

## §2. Height functions in terms of hermitian line bundles with logarithmic singularities

Let $K$ be a finitely generated field over $\mathbb{Q}$ with $d=\operatorname{tr} . \operatorname{deg}_{\mathbb{Q}}(K)$. Let $\bar{B}=\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$ be a fine polarization of $K$. Let $X$ be a projective variety over $K$ and $L$ an ample line bundle on $X$. Moreover, let $Y$ be a proper closed subset of $X$. Let $(\mathcal{X}, \overline{\mathcal{L}})$ be a pair of a projective arithmetic variety $\mathcal{X}$ and a hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ with the following properties:
(1) There is a morphism $f: \mathcal{X} \rightarrow B$ whose generic fiber is $X$.
(2) The restriction of $\mathcal{L}$ to the generic fiber of $f$ coincides with $L$.
(3) $\mathcal{L}$ is ample with respect to the morphism $f: \mathcal{X} \rightarrow B$.
(4) Let $\mathcal{Y}$ be a closed set of $\mathcal{X}$ such that $\mathcal{Y}$ gives rise to $Y$ on the generic fiber of $\mathcal{X} \rightarrow B$. Then the hermitian metric of $\overline{\mathcal{L}}$ has logarithmic singularities along $\mathcal{Y}(\mathbb{C})$.
For $x \in X(\bar{K}) \backslash Y(\bar{K})$, we denote by $\Delta_{x}$ the Zariski closure of the image of $\operatorname{Spec}(\bar{K}) \rightarrow X \rightarrow \mathcal{X}$. The height of $x$ with respect to $\bar{B}$ and $\overline{\mathcal{L}}$ is defined by

$$
h \frac{\bar{B}}{\mathcal{L}}(x)=\frac{\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\left.f^{*}\left(\bar{H}_{1}\right)\right|_{\Delta_{x}}\right) \cdots \widehat{c}_{1}\left(\left.f^{*}\left(\bar{H}_{d}\right)\right|_{\Delta_{x}}\right) \cdot \widehat{c_{1}}\left(\left.\overline{\mathcal{L}}\right|_{\Delta_{x}}\right)\right)}{[K(x): K]}
$$

Note that since $\left.\overline{\mathcal{L}}\right|_{\Delta_{x}}$ has logarithmic singularities along $\mathcal{Y}(\mathbb{C}) \cap \Delta_{x}(\mathbb{C})$, the number

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\left.f^{*}\left(\bar{H}_{1}\right)\right|_{\Delta_{x}}\right) \cdots \widehat{c}_{1}\left(\left.f^{*}\left(\bar{H}_{d}\right)\right|_{\Delta_{x}}\right) \cdot \widehat{c_{1}}\left(\left.\overline{\mathcal{L}}\right|_{\Delta_{x}}\right)\right)
$$

is well defined. Then we have the following proposition.
Proposition 2.1. (1) Given a positive integer e, there exists a constant $C$ such that

$$
\#\left\{x \in X(\bar{K}) \backslash Y(\bar{K}) \left\lvert\, h \frac{\bar{B}}{\overline{\mathcal{L}}}(x) \leq h\right.,[K(x): K] \leq e\right\} \leq C \cdot h^{d+1}
$$

for $h \gg 0$.
(2) There is a constant $C^{\prime}$ such that $h \frac{\bar{B}}{L}(x) \geq C^{\prime}$ for all $x \in X(\bar{K}) \backslash$ $Y(\bar{K})$.

Proof. We denote by $\|\cdot\|$ the hermitian metric of $\overline{\mathcal{L}}$. Let $\bar{Q}$ be an ample $C^{\infty}$-hermitian line bundle on $B$. Then

$$
h \frac{\overline{\mathcal{L}} \otimes f^{*}\left(\bar{Q}^{\otimes n}\right)}{}(x)=h \frac{\bar{B}}{\mathcal{L}}(x)+n \widehat{\operatorname{deg}}\left(\widehat{c}_{1}(\bar{Q}) \cdot \widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right)\right) .
$$

and we may assume that $\mathcal{L}$ is ample on $\mathcal{X}$ without loss of generality. Replacing $\overline{\mathcal{L}}$ by a suitable $\overline{\mathcal{L}}^{\otimes n}$, we may furthermore assume that $\mathcal{I}_{\mathcal{Y}} \otimes \mathcal{L}$ is generated by global sections, where $\mathcal{I}_{\mathcal{Y}}$ is the defining ideal sheaf of $\mathcal{Y}$. Let $s_{1}, \ldots, s_{r}$ be generators of $H^{0}\left(\mathcal{X}, \mathcal{I}_{\mathcal{Y}} \otimes \mathcal{L}\right)$. We may view $s_{1}, \ldots, s_{r}$ as global sections of $H^{0}(\mathcal{X}, \mathcal{L})$. Then $\mathcal{Y}=\left\{x \in \mathcal{X} \mid s_{1}(x)=\cdots=\right.$ $\left.s_{r}(x)=0\right\}$. Here we choose a $C^{\infty}$-hermitian metric $\|\cdot\|_{0}$ of $\mathcal{L}$ such that $\left\|s_{i}\right\|_{0}<1$ for all $i=1, \ldots, r$. We denote $\left(\mathcal{L},\|\cdot\|_{0}\right)$ by $\overline{\mathcal{L}}^{0}$.

We claim

$$
\begin{aligned}
& {[K(x)}: K] h \frac{\overline{\overline{\mathcal{L}}^{0}}}{}(x) \\
& \quad \geq-\int_{\Delta_{x}(\mathbb{C})} \log \left(\max _{i}\left\{\left\|s_{i}\right\|_{0}\right\}\right) c_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \wedge \cdots \wedge c_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right) .
\end{aligned}
$$

Indeed we can find $s_{j}$ with $\left.s_{j}\right|_{\Delta_{x}} \neq 0$, so that

$$
\begin{aligned}
{[K(x): K] h_{\overline{\mathcal{L}}^{0}}(x) } & =\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right) \mid \operatorname{div}\left(\left.s_{j}\right|_{\Delta_{x}}\right)\right) \\
& -\int_{\Delta_{x}(\mathbb{C})} \log \left(\left\|s_{j}\right\|_{0}\right) c_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \wedge \cdots \wedge c_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right) .
\end{aligned}
$$

Then our claim follows from the following two inequalities:

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right) \mid \operatorname{div}\left(\left.s_{j}\right|_{\Delta_{x}}\right)\right) \geq 0
$$

and

$$
\left\|s_{j}\right\|_{0} \leq \max _{i}\left\{\left\|s_{i}\right\|_{0}\right\}<1
$$

Set $g=\|\cdot\| /\|\cdot\|_{0}$. Since $\|\cdot\|$ has logarithmic singularities, there are positive constants $a, b$ such that

$$
|\log (g)| \leq a+b \log \left(-\log \left(\max _{i}\left\{\left\|s_{i}\right\|_{0}\right\}\right)\right)
$$

Moreover

$$
\begin{aligned}
\left\lvert\, h \frac{\bar{B}}{\mathcal{L}}(x)\right. & -h \overline{\overline{\mathcal{L}}}^{0}(x) \mid \\
& \leq \frac{1}{[K(x): K]} \int_{\Delta_{x}(\mathbb{C})}|\log (g)| c_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \wedge \cdots \wedge c_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right)
\end{aligned}
$$

Note that

$$
\int_{\Delta_{x}(\mathbb{C})} c_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \wedge \cdots \wedge c_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right)=[K(x): K] \operatorname{deg}(\bar{B}),
$$

where $\operatorname{deg}(\bar{B})=\int_{B(\mathbb{C})} c_{1}\left(\bar{H}_{1}\right) \wedge \cdots \wedge c_{1}\left(\bar{H}_{d}\right)$ as in Section 1.10. Thus

$$
\begin{aligned}
& \frac{\left|h \frac{\bar{B}}{\mathcal{L}}(x)-h \overline{\overline{\mathcal{L}}}^{0}(x)\right|}{\operatorname{deg}(\bar{B})} \leq a+ \\
& b \int_{\Delta_{x}(\mathbb{C})} \log \left(-\log \left(\max _{i}\left\{\left\|s_{i}\right\|_{0}\right\}\right)\right) \frac{c_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \wedge \cdots \wedge c_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right)}{[K(x): K] \operatorname{deg}(\bar{B})} .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \int_{\Delta_{x}(\mathbb{C})} \log \left(-\log \left(\max _{i}\left\{\left\|s_{i}\right\|_{0}\right\}\right)\right) \frac{c_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \wedge \cdots \wedge c_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right)}{[K(x): K] \operatorname{deg}(\bar{B})} \\
& \leq \log \left(\int_{\Delta_{x}(\mathbb{C})}-\log \left(\max _{i}\left\{\left\|s_{i}\right\|_{0}\right\}\right) \frac{c_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \wedge \cdots \wedge c_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right)}{[K(x): K] \operatorname{deg}(\bar{B})}\right) .
\end{aligned}
$$

Hence we obtain

$$
\frac{\left|h \frac{\bar{B}}{\mathcal{B}}(x)-h \overline{\overline{\mathcal{L}}}^{0}(x)\right|}{\operatorname{deg}(\bar{B})} \leq a+b \log \left(\frac{h \frac{\overline{\mathcal{L}}^{0}}{}(x)}{\operatorname{deg}(\bar{B})}\right) .
$$

Note that there is a real number $t_{0}$ such that $a+b \log (t) \leq t / 2$ for all $t \geq t_{0}$. Thus

$$
h \frac{\bar{B}}{\overline{\mathcal{L}}^{0}}(x) \leq \max \left\{\operatorname{deg}(\bar{B}) t_{0}, 2 h \frac{\overline{\mathcal{L}}}{}(x)\right\}
$$

Therefore, if $h \geq \operatorname{deg}(\bar{B}) t_{0} / 2$, then $h \frac{\overline{\mathcal{L}}}{}(x) \leq h$ implies $h \overline{\overline{\mathcal{L}}}^{\overline{\mathcal{L}}}(x) \leq 2 h$. Hence we get the first assertion by virtue of [10, Theorem 6.4.1].

Next let us check the second assertion. Since

$$
\begin{aligned}
&\left\|s_{i}\right\|=g\left\|s_{i}\right\|_{0} \leq \exp (a)\left\|s_{i}\right\|_{0}\left(-\log \left(\max _{j}\left\{\left\|s_{j}\right\|_{0}\right\}\right)\right)^{b} \\
& \leq \exp (a)\left\|s_{i}\right\|_{0}\left(-\log \left(\left\|s_{i}\right\|_{0}\right)\right)^{b}
\end{aligned}
$$

and the function $t(-\log (t))^{b}$ is bounded from above for $0<t \leq 1$, there is a constant $C$ such that $\left\|s_{i}\right\| \leq C$ for all $i$. Thus, if we choose $s_{i}$ with $\left.s_{i}\right|_{\Delta_{x}} \neq 0$, then

$$
\begin{aligned}
{[K(x): K] h \frac{\bar{B}}{\mathcal{L}}(x)=} & \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right) \mid \operatorname{div}\left(\left.s_{i}\right|_{\Delta_{x}}\right)\right) \\
& -\int_{\Delta_{x}(\mathbb{C})} \log \left(\left\|s_{j}\right\|\right) c_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \wedge \cdots \wedge c_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right) \\
\geq- & \log (C) \int_{\Delta_{x}(\mathbb{C})} c_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \wedge \cdots \wedge c_{1}\left(f^{*}\left(\bar{H}_{d}\right)\right) \\
=- & \log (C) \operatorname{deg}(\bar{B})[K(x): K] .
\end{aligned}
$$

Thus we get (2).

## §3. The Faltings modular height

Let $K$ be a field of finite type over $\mathbb{Q}$ with $d=\operatorname{tr} . \operatorname{deg}_{\mathbb{Q}}(K)$ and let $\bar{B}=\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$ be a generically smooth polarization of $K$. Let $A$ be a $g$-dimensional abelian variety over $K$. Let $\lambda(A / K ; B)$ be the Hodge sheaf of $A$ with respect to $B$ (cf. Section 1.3). Note that $\lambda(A / K ; B)$ is invertible over $B_{\mathbb{Q}}$ because $B_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$. Let $\|\cdot\|_{\text {Fal }}$ be Faltings' metric of $\lambda(A / K ; B)$ over $B(\mathbb{C})$. Here we set

$$
\bar{\lambda}^{\text {Fal }}(A / K ; B)=\left(\lambda(A / K ; B),\|\cdot\|_{\text {Fal }}\right),
$$

which is called the metrized Hodge sheaf of $A$ with respect to $B$. In the case where a Néron model $\mathcal{A} \rightarrow U$ over $B$ in codimension one is specified, $\bar{\lambda}^{\text {Fal }}(A / K ; B)$ is often denoted by $\bar{\lambda}^{\text {Fal }}(\mathcal{A} / U)$. By Lemma 1.6.1, the metric of $\bar{\lambda}^{\mathrm{Fal}}(A / K ; B)$ is locally integrable. The Faltings modular height of $A$ with respect to the polarization $\bar{B}$ is defined by

$$
h_{\mathrm{Fal}}^{\bar{B}}(A)=\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \cdot \widehat{c}_{1}\left(\bar{\lambda}^{\mathrm{Fal}}(A / K ; B)\right) .\right.
$$

Even if we do not assume that $\bar{B}$ is generically smooth, we can define the Faltings modular height with respect to $\bar{B}$ as follows: Let $\mu: B^{\prime} \rightarrow B$ be a generic resolution of singularities of $B$. We set $\bar{B}^{\prime}=\left(B^{\prime} ; \mu^{*}\left(\bar{H}_{1}\right), \ldots, \mu^{*}\left(\bar{H}_{d}\right)\right)$. Then, by (1) of the following Proposition 3.1, $h_{\mathrm{Fal}}^{\bar{B}^{\prime}}(A)$ does not depend on the choice of the generic resolution $\underline{\mu}: B^{\prime} \rightarrow B$, so that $h_{\text {Fal }}^{\bar{B}}(A)$ is defined to be $h_{\text {Fal }}^{\bar{B}^{\prime}}(A)$. In the following, $\bar{B}$ is always assumed to be generically smooth.

Proposition 3.1. Let $\pi: X^{\prime} \rightarrow X$ be a generically finite morphism of normal and projective generically smooth arithmetic varieties. Let $K$ and $K^{\prime}$ be the function field of $X$ and $X^{\prime}$ respectively. Let $A$ be an abelian variety over $K$. Then there is an effective divisor $E$ on $X$ which has the following two properties:

$$
\begin{array}{r}
\pi_{*} \widehat{c}_{1}\left(\bar{\lambda}^{\text {Fal }}\left(A \times_{K} \operatorname{Spec}\left(K^{\prime}\right) / K^{\prime} ; X^{\prime}\right)\right)+(E, 0)  \tag{1}\\
=\operatorname{deg}(\pi) \widehat{c}_{1}\left(\bar{\lambda}^{\text {Fal }}(A / K ; X)\right)
\end{array}
$$

Further, if $\pi$ is birational, then $E=0$.
(2) For a scheme $S$, we denote by $S^{(1)}$ the set of points of codimension one in $S$. Then
$\left\{x \in X^{(1)} \mid A\right.$ has semiabelian reduction at $\left.x\right\}$

$$
\subseteq(X \backslash \operatorname{Supp}(E))^{(1)}
$$

Moreover, if $A \times{ }_{K} \operatorname{Spec}\left(K^{\prime}\right)$ has semiabelian reduction over $X^{\prime}$ in codimension one, then

$$
\begin{array}{r}
\left\{x \in X^{(1)} \mid \text { A has semiabelian reduction at } x\right\} \\
\\
=(X \backslash \operatorname{Supp}(E))^{(1)} .
\end{array}
$$

Proof. (1) Let $X_{0}$ be the maximal Zariski open set of $X$ such that $X_{0}$ is regular and $\pi$ is finite over $X_{0}$. Then $\operatorname{codim}\left(X \backslash X_{0}\right) \geq 2$. We set $X_{0}^{\prime}=\pi^{-1}\left(X_{0}\right)$ and $\pi_{0}=\left.\pi\right|_{X_{0}^{\prime}}$. Let $\operatorname{Div}(X)$ and $\operatorname{Div}\left(X^{\prime}\right)$ be the groups of Weil divisors on $X$ and $X^{\prime}$ respectively. A homomorphism $\pi^{\star}: \operatorname{Div}(X) \rightarrow \operatorname{Div}\left(X^{\prime}\right)$ is defined as the composition of the natural homomorphisms:

$$
\operatorname{Div}(X) \rightarrow \operatorname{Div}\left(X_{0}\right) \xrightarrow{\pi_{0}^{*}} \operatorname{Div}\left(X_{0}^{\prime}\right) \rightarrow \operatorname{Div}\left(X^{\prime}\right),
$$

where $\operatorname{Div}(X) \rightarrow \operatorname{Div}\left(X_{0}\right)$ is the restriction map and $\operatorname{Div}\left(X_{0}^{\prime}\right) \rightarrow \operatorname{Div}\left(X^{\prime}\right)$ is defined by taking the Zariski closure of divisors. Note that $\pi_{*} \pi^{\star}(D)=$ $\operatorname{deg}(\pi) D$ for all $D \in \operatorname{Div}(X)$.

Let $X_{1}$ (resp. $X_{1}^{\prime}$ ) be a Zariski open set of $X$ (resp. $X^{\prime}$ ) such that $\operatorname{codim}\left(X \backslash X_{1}\right) \geq 2$ (resp. $\operatorname{codim}\left(X^{\prime} \backslash X_{1}^{\prime}\right) \geq 2$ ) and that the Néron model $G$ (resp. $G^{\prime}$ ) exists over $X_{1}$ (resp. $X_{1}^{\prime}$ ). Clearly we may assume that $X_{1} \subseteq X_{0}$ and $\pi^{-1}\left(X_{1}\right) \subseteq X_{1}^{\prime}$. We set $U^{\prime}=\pi^{-1}\left(X_{1}\right)$ and $G_{U^{\prime}}^{\prime}=G_{U^{\prime}}^{\prime}$. Since $G_{U^{\prime}}^{\prime}$ is the Néron model of $A \times_{K} \operatorname{Spec}\left(K^{\prime}\right)$ over
$U^{\prime}$, there is a homomorphism $G \times_{X_{1}} U^{\prime} \rightarrow G_{U^{\prime}}^{\prime}$ over $U^{\prime}$. Thus we get a homomorphism

$$
\begin{equation*}
\alpha:\left.\epsilon^{\prime *}\left(\bigwedge^{g} \Omega_{G_{U^{\prime}}^{\prime} / U^{\prime}}\right) \rightarrow \pi^{*} \epsilon^{*}\left(\bigwedge^{g} \Omega_{G / X_{1}}\right)\right|_{U^{\prime}} \tag{3.1.1}
\end{equation*}
$$

where $\epsilon$ and $\epsilon^{\prime}$ are the zero sections of $G$ and $G^{\prime}$ respectively.
Let $s$ be a non-zero rational section of $\lambda(A / K ; X)$. Then

$$
\widehat{c}_{1}\left(\bar{\lambda}^{\text {Fal }}(A / K ; X)\right)=\left(\operatorname{div}(s),-\log \|s\|_{\text {Fal }}\right)
$$

Moreover, since $\pi^{*}(s)$ gives rise to a non-zero rational section of $\lambda\left(A \times_{K}\right.$ $\left.\operatorname{Spec}\left(K^{\prime}\right) / K^{\prime} ; X^{\prime}\right)$,

$$
\widehat{c}_{1}\left(\bar{\lambda}^{\text {Fal }}\left(A \times_{K} \operatorname{Spec}\left(K^{\prime}\right) / K^{\prime} ; X^{\prime}\right)\right)=\left(\operatorname{div}\left(\pi^{*}(s)\right),-\pi^{*}\left(\log \|s\|_{\text {Fal }}\right)\right),
$$

where $\pi^{*}\left(\log \|s\|_{\text {Fal }}\right)$ is the pull-back of $\log \|s\|_{\text {Fal }}$ by $\pi$ as a function on a dense open set of $X(\mathbb{C})$. Let $\Gamma_{1}, \ldots, \Gamma_{r}$ be all prime divisors in $X^{\prime} \backslash U^{\prime}$. Note that $\pi_{*}\left(\Gamma_{i}\right)=0$ for all $i$. Then, since (3.1.1) is injective, there is an effective divisor $E^{\prime}$ and integers $a_{1}, \ldots, a_{r}$ such that

$$
\operatorname{div}\left(\pi^{*}(s)\right)+E^{\prime}=\pi^{\star}(\operatorname{div}(s))+\sum_{i=1}^{r} a_{i} \Gamma_{i}
$$

Note that $E^{\prime}=\sum_{x^{\prime}} \operatorname{length}_{\mathcal{O}_{X^{\prime}, x^{\prime}}}\left(\operatorname{Coker}(\alpha)_{x^{\prime}}\right) \overline{\left\{x^{\prime}\right\}}$, where $x^{\prime}$ 's run over all points of codimension one in $U^{\prime}$. Thus, since

$$
\pi_{*}\left(\pi^{\star}(\operatorname{div}(s)),-\pi^{*}\left(\log \|s\|_{\text {Fal }}\right)\right)=\operatorname{deg}(\pi)\left(\operatorname{div}(s),-\log \|s\|_{\text {Fal }}\right),
$$

we have

$$
\begin{aligned}
\pi_{*} \widehat{c}_{1}\left(\bar{\lambda}^{\mathrm{Fal}}\left(A \times_{K} \operatorname{Spec}\left(K^{\prime}\right) / K^{\prime} ; X^{\prime}\right)\right)+ & \left(\pi_{*}\left(E^{\prime}\right), 0\right) \\
& =\operatorname{deg}(\pi) \widehat{c}_{1}\left(\bar{\lambda}^{\mathrm{Fal}}(A / K ; X)\right)
\end{aligned}
$$

yielding the first assertion of (1). If $\pi$ is birational, then $U^{\prime} \rightarrow X_{1}$ is an isomorphism, so that $E^{\prime}=0$.
(2) Assume that there is an open neighborhood $U$ of $x$ such that $\left.G^{\mathrm{o}}\right|_{U}$ is semiabelian. Then $\left.G^{\mathrm{o}}\right|_{U} \times_{U} \pi^{-1}(U)$ is semiabelian so that it is isomorphic to $\left(\left.G^{\prime}\right|_{\pi^{-1}(U)}\right)^{\circ}$. This shows that $x \notin E_{\text {red }}$. Conversely, if $x \notin E_{\text {red }}$ and $A \times_{K} \operatorname{Spec}\left(K^{\prime}\right)$ has semiabelian reduction in codimension
one, then there exists an open neighborhood $U \subset X_{1}$ of $x$ such that the homomorphism

$$
\alpha:\left.\epsilon^{\prime *}\left(\bigwedge^{g} \Omega_{G_{U^{\prime}}^{\prime} / U^{\prime}}\right) \rightarrow \pi^{*} \epsilon^{*}\left(\bigwedge^{g} \Omega_{G / X_{1}}\right)\right|_{U^{\prime}}
$$

is an isomorphism over $\pi^{-1}(U)$. Thus the natural homomorphism

$$
\left.\epsilon^{\prime *}\left(\Omega_{G_{U^{\prime}}^{\prime} / U^{\prime}}\right) \rightarrow \pi^{*} \epsilon^{*}\left(\Omega_{G / X_{1}}\right)\right|_{U^{\prime}}
$$

must be an isomorphism over $\pi^{-1}(U)$ and so is the morphism

$$
G^{\mathrm{o}} \times_{X_{1}} U^{\prime} \rightarrow\left(G_{U^{\prime}}^{\prime}\right)^{\mathrm{o}}
$$

over $\pi^{-1}(U)$, which means that $G^{\circ}$ is semiabelian over $U$.
Proposition 3.2. Let $\phi: A \rightarrow A^{\prime}$ be an isogeny of abelian varieties over $K$. Then

$$
\begin{aligned}
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right)\right. & \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \cdot \widehat{c}_{1}\left(\bar{\lambda}^{\mathrm{Fal}}\left(A^{\prime} / K ; B\right)\right) \\
& -\widehat{\operatorname{deg}}\left(\widehat { c } _ { 1 } ( \overline { H } _ { 1 } ) \cdots \widehat { c } _ { 1 } ( \overline { H } _ { d } ) \cdot \widehat { c } _ { 1 } \left(\left(^{\mathrm{Fal}}(A / K ; B)\right)\right.\right. \\
& =\frac{1}{2} \log (\operatorname{deg}(\phi)) \operatorname{deg}(\bar{B})-\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \mid D_{\phi}\right)
\end{aligned}
$$

where $D_{\phi}$ is an effective divisor given in Section 1.3 and

$$
\operatorname{deg}(\bar{B})=\int_{B(\mathbb{C})} c_{1}\left(\bar{H}_{1}\right) \wedge \cdots \wedge c_{1}\left(\bar{H}_{d}\right)
$$

as in Section 1.10.
Proof. This follows from the fact that

$$
\bar{\lambda}^{\text {Fal }}\left(A^{\prime} / K ; B\right) \otimes\left(\mathcal{O}_{B}\left(D_{\phi}\right), \operatorname{deg}(\phi)|\cdot|_{\text {can }}\right)
$$

is isometric to $\bar{\lambda}^{\text {Fal }}(A / K ; B)$.
Proposition 3.3. If an abelian variety $A$ over $K$ has semiabelian reduction in codimension one over $B$. Then

$$
\begin{aligned}
& \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \cdot \widehat{c}_{1}\left(\bar{\lambda}^{\mathrm{Fal}}(A / K ; B)\right)\right. \\
&=\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \cdot \widehat{c}_{1}\left(\bar{\lambda}^{\mathrm{Fal}}\left(A^{\vee} / K ; B\right)\right)\right.
\end{aligned}
$$

where $A^{\vee}$ is the dual abelian variety of $A$.

Proof. Let $\phi: A \rightarrow A^{\vee}$ be an isogeny over $K$ in terms of ample line bundle on $A$. Let $\phi^{\vee}: A \rightarrow A^{\vee}$ be the dual of $\phi$. Then, by Proposition 3.2,

$$
\begin{aligned}
& 2 \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \cdot \widehat{c}_{1}\left(\bar{\lambda}^{\mathrm{Fal}}\left(A^{\vee} / K ; B\right)\right)\right. \\
& \quad-2 \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \cdot \widehat{c}_{1}\left(\bar{\lambda}^{\mathrm{Fal}}(A / K ; B)\right)\right. \\
& \quad=\log (\operatorname{deg}(\phi)) \operatorname{deg}(\bar{B})-\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \mid D_{\phi}+D_{\phi^{\vee}}\right)
\end{aligned}
$$

On the other hand, by Lemma 1.3.3, $\mathcal{I}_{\phi} \cdot \mathcal{I}_{\phi} \vee=\operatorname{deg}(\phi) \mathcal{O}_{B} .\left(\mathcal{O}_{B}\left(D_{\phi}+\right.\right.$ $\left.\left.D_{\phi^{\vee}}\right),|\cdot|_{\text {can }}\right)$ is thus isometric to $\left(\mathcal{O}_{B}, \operatorname{deg}(\phi)^{-2}|\cdot|_{\text {can }}\right)$, proving the assertion.

Let $A$ be an abelian variety over a finite extension field $K^{\prime}$ of $K$. Let $m$ be a positive integer such that $m$ has a decomposition $m=m_{1} m_{2}$ with $\left(m_{1}, m_{2}\right)=1$ and $m_{1}, m_{2} \geq 3$. Let us consider a natural homomorphism

$$
\rho(A, m): \operatorname{Gal}(\bar{K} / K) \rightarrow \operatorname{Aut}(A[m](\bar{K})) \simeq \operatorname{Aut}\left((\mathbb{Z} / m \mathbb{Z})^{2 g}\right)
$$

Then there is a Galois extension $K(A, m)$ of $K^{\prime}$ with $\operatorname{Ker} \rho(A, m)=$ $\operatorname{Gal}(\bar{K} / K(A, m))$. Note that

$$
\operatorname{Gal}\left(K(A, m) / K^{\prime}\right)=\operatorname{Gal}(\bar{K} / K) / \operatorname{Ker} \rho(A, m) \hookrightarrow \operatorname{Aut}\left((\mathbb{Z} / m \mathbb{Z})^{2 g}\right)
$$

Let $B^{\prime \prime}$ be a generically smooth, normal and projective arithmetic variety with the following properties:
(i) The function field $K^{\prime \prime}$ of $B^{\prime \prime}$ is an extension of $K(A, m)$.
(ii) The natural rational map $f: B^{\prime \prime} \rightarrow B$ induced by $K \hookrightarrow K^{\prime \prime}$ is actually a morphism.
Then we have the following.

## Proposition 3.4. <br> (1) The number

$$
\frac{\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\lambda\left(A \times_{K^{\prime}} \operatorname{Spec}\left(K^{\prime \prime}\right) / K^{\prime \prime} ; B^{\prime \prime}\right) \cdot \widehat{c}_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right) \cdots \widehat{c}_{1}\left(f^{*}\left(\bar{H}_{1}\right)\right)\right)\right.}{\left[K^{\prime \prime}: K\right]}
$$

does not depend on the choice of $m$ and $B^{\prime \prime}$, so that we denote it by $h_{\text {mod }}^{\bar{B}}(A)$.
(2) $h_{\mathrm{mod}}^{\bar{B}}(A) \leq h_{\mathrm{Fal}}^{\bar{B}}(A)$.

Proof. These are consequences of Proposition 1.2.1, Proposition 3.1 and the projection formula.

Proposition 3.5 (Additivity of heights). For abelian varieties $A, A^{\prime}$ over $K$, we have

$$
\begin{aligned}
h_{\mathrm{Fal}}^{\bar{B}}\left(A \times_{K} A^{\prime}\right) & =h_{\mathrm{Fal}}^{\bar{B}}(A)+h_{\mathrm{Fal}}^{\bar{B}}\left(A^{\prime}\right), \\
h_{\mathrm{mod}}^{\bar{B}}\left(A \times_{K} A^{\prime}\right) & =h_{\mathrm{mod}}^{\bar{B}}(A)+h_{\mathrm{mod}}^{\bar{B}}\left(A^{\prime}\right) .
\end{aligned}
$$

Proof. Let $\mathcal{A}$ and $\mathcal{A}^{\prime}$ be the Néron models of $A$ and $A^{\prime}$ over $B_{0}$, where $B_{0}$ is a big open set of $B$. Then $\mathcal{A} \times{ }_{B_{0}} \mathcal{A}^{\prime}$ is the Néron model of $A \times_{K} A^{\prime}$ over $B_{0}$. Thus

$$
\widehat{c}_{1}\left(\bar{\lambda}_{\mathcal{A} \times B_{0} \mathcal{A}^{\prime} / B_{0}}^{\mathrm{Fal}}\right)=\widehat{c}_{1}\left(\bar{\lambda}_{\mathcal{A} / B_{0}}^{\mathrm{Fal}}\right)+\widehat{c}_{1}\left(\bar{\lambda}_{\mathcal{A}^{\prime} / B_{0}}^{\mathrm{Fal}}\right) .
$$

Hence we get our lemma.

## $\S 4$. Weak finiteness

Let us fix positive integers $g, l$ and $m$ such that $m$ has a decomposition $m=m_{1} m_{2}$ with $\left(m_{1}, m_{2}\right)=1$ and $m_{1}, m_{2} \geq 3$. Let $\mathbb{A}_{g, l, m, \mathbb{Q}}$, $f: Y \rightarrow \mathbb{A}_{g, l, m}^{*}, \bar{L}, n$ and $G \rightarrow Y$ be as in Theorem 1.7.1.

Let $K$ be a field of finite type over $\mathbb{Q}$ with $d=\operatorname{tr} \cdot \operatorname{deg}_{\mathbb{Q}}(K)$ and let $\bar{B}=\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$ be a generically smooth polarization of $K$.

Let $A$ be a $g$-dimensional and $l$-polarized abelian variety over a finite extension $K^{\prime}$ of $K$ with level $m$ structure. The abelian variety $A$ naturally induces a morphism $x_{A}: \operatorname{Spec}\left(K^{\prime}\right) \rightarrow \mathbb{A}_{g, l, m}^{*}$, which in turn induces $y_{A}: \operatorname{Spec}\left(K^{\prime}\right) \rightarrow \mathbb{A}_{g, l, m}^{*} \times \mathbb{Z} \operatorname{Spec}(K)$. Let $\Delta_{A}$ be the closure of the image of $y_{A}$ in $\mathbb{A}_{g, l, m}^{*} \times_{\mathbb{Z}} B$. Let $p: \mathbb{A}_{g, l, m}^{*} \times{ }_{\mathbb{Z}} B \rightarrow \mathbb{A}_{g, l, m}^{*}$ and $q: \mathbb{A}_{g, l, m}^{*} \times_{\mathbb{Z}} B \rightarrow B$ be the projections to the first factor and the second factor respectively. The number

$$
h \frac{\bar{B}}{L}(A)=\frac{\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\left.q^{*}\left(\bar{H}_{1}\right)\right|_{\Delta_{A}}\right) \cdots \widehat{c}_{1}\left(\left.q^{*}\left(\bar{H}_{d}\right)\right|_{\Delta_{A}}\right) \cdot \widehat{c}_{1}\left(\left.p^{*}(\bar{L})\right|_{\Delta_{A}}\right)\right)}{\operatorname{deg}\left(\Delta_{A} \rightarrow B\right)}
$$

is the height of $y_{A} \in\left(\mathbb{A}_{g, l, m}^{*} \times \mathbb{Z} \operatorname{Spec}(K)\right)(\bar{K})$ with respect to $\bar{L}$ and $\bar{B}$, of which the behavior is controlled by the following proposition.

Proposition 4.1. There is a constant $N(g, l, m)$ depending only on $g, l, m$ such that

$$
\left|h \frac{\bar{B}}{\bar{L}}(A)-n h_{\bmod }^{\bar{B}}(A)\right| \leq \log (N(g, l, m)) \operatorname{deg}(\bar{B})
$$

for every g-dimensional and l-polarized abelian variety $A$ over $\bar{K}$ with level $m$ structure, where

$$
\operatorname{deg}(\bar{B})=\int_{B(\mathbb{C})} c_{1}\left(\bar{H}_{1}\right) \wedge \cdots c_{1}\left(\bar{H}_{d}\right)
$$

Proof. Let $A$ be a $g$-dimensional and $l$-polarized abelian variety over $\bar{K}$ with level $m$ structure. Let $K^{\prime}$ be the minimal finite extension of $K$ such that $A$, the polarization of $A$, the level $m$ structure of $A$ are defined over $K^{\prime}$. Let $x_{A}: \operatorname{Spec}\left(K^{\prime}\right) \rightarrow \mathbb{A}_{g, l, m}^{*}$ be the morphism induced by $A$. Moreover let $y_{A}: \operatorname{Spec}\left(K^{\prime}\right) \rightarrow \mathbb{A}_{g, l, m}^{*} \times_{\mathbb{Z}} B$ be the induced morphism by $x_{A}$.

Let $\operatorname{Spec}\left(K_{1}\right)$ be a closed point of $Y \times_{\mathbb{A}_{g, l, m}^{*}} \operatorname{Spec}\left(K^{\prime}\right)$. Then we have the following commutative diagram:


Here, two $l$-polarized abelian varieties $A \times{ }_{K^{\prime}} \operatorname{Spec}\left(K_{1}\right)$ and $G \times{ }_{Y} \operatorname{Spec}\left(K_{1}\right)$ with level $m$ structures gives rise to the same $K_{1}$-valued point of $\mathbb{A}_{g, l, m}^{*}$. Thus $A \times{ }_{K^{\prime}} \operatorname{Spec}\left(K_{1}\right)$ is isomorphic to $G \times_{Y} \operatorname{Spec}\left(K_{1}\right)$ over $K_{1}$ as lpolarized abelian varieties with level $m$ structures because $m \geq 3$. The above commutative diagram induces to the commutative diagram:


Let $B_{1}$ be a generic resolution of singularities of the normalization of $B$ in $K_{1}$. Note that a generic resolution of singularities (a resolution of singularities over $\mathbb{Q}$ ) exists by Hironaka's theorem [6]. Then we have rational maps $B_{1} \rightarrow Y \times_{\mathbb{Z}} B$ and $B_{1} \rightarrow \Delta_{A}$ such that a composition $B_{1} \rightarrow \Delta_{A} \rightarrow \mathbb{A}_{g, l, m}^{*} \times_{\mathbb{Z}} B$ of rational maps is equal to $B_{1} \rightarrow Y \times_{\mathbb{Z}} B \rightarrow$ $\mathbb{A}_{g, m}^{*} \times_{\mathbb{Z}} B$. Thus there are a birational morphism $B_{2} \rightarrow B_{1}$ of projective and generically smooth arithmetic varieties, a morphism $B_{2} \rightarrow \Delta_{A}$ and a morphism $B_{2} \rightarrow Y \times_{\mathbb{Z}} B$ with the following commutative diagram:


Then

$$
\begin{aligned}
& h \frac{\bar{B}}{L}(A)=\frac{\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\iota^{*}\left(p^{*}(\bar{L})\right)\right) \cdot \widehat{c}_{1}\left(\iota^{*}\left(q^{*}\left(\bar{H}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\iota^{*}\left(q^{*}\left(\bar{H}_{1}\right)\right)\right)\right)}{\operatorname{deg}\left(\Delta_{A} \rightarrow B\right)} \\
= & \frac{\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\alpha^{*}\left(\iota^{*}\left(p^{*}(\bar{L})\right)\right)\right) \cdot \widehat{c}_{1}\left(\alpha^{*}\left(\iota^{*}\left(q^{*}\left(\bar{H}_{1}\right)\right)\right)\right) \cdots \widehat{c}_{1}\left(\alpha^{*}\left(\iota^{*}\left(q^{*}\left(\bar{H}_{1}\right)\right)\right)\right)\right)}{\operatorname{deg}\left(B_{2} \rightarrow B\right)} \\
= & \frac{\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\beta^{*}\left((f \times \mathrm{id})^{*}\left(p^{*}(\bar{L})\right)\right)\right) \cdot \widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right)\right)}{\operatorname{deg}\left(B_{2} \rightarrow B\right)} .
\end{aligned}
$$

On the other hand, since $f^{*}(L)=\lambda_{G / Y}^{\otimes n}$ over $Y \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$, there is an integer $N$ depending only on $g, l$ and $m$ such that

$$
N f^{*}(L) \subseteq \lambda_{G / Y}^{\otimes n} \subseteq(1 / N) f^{*}(L)
$$

on $Y$. Thus

$$
N \beta^{*}(f \times \mathrm{id})^{*}(L) \subseteq\left(\lambda_{G \times_{\mathbb{Z}} B / Y \times_{\mathbb{Z}} B}\right)^{\otimes n} \subseteq(1 / N) \beta^{*}(f \times \mathrm{id})^{*}(L) .
$$

Therefore

$$
\begin{aligned}
& -\frac{\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right) \mid(N)\right)}{\operatorname{deg}\left(B_{2} \rightarrow B\right)}+h \frac{\bar{B}}{L}(A) \\
& \leq \frac{\left.n \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{\lambda}_{G \times_{Y} B_{2} / B_{2}}^{\mathrm{Fal}}\right)\right) \cdot \widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right)\right)}{\operatorname{deg}\left(B_{2} \rightarrow B\right)} \\
& \quad \leq \frac{\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right) \mid(N)\right)}{\operatorname{deg}\left(B_{2} \rightarrow B\right)}+h \frac{\bar{B}}{L}(A)
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right) \cdots \widehat{c}_{1}\left(\gamma^{*}\left(\pi_{1}^{*}\left(\bar{H}_{1}\right)\right)\right) \mid(N)\right) \\
&=\log (N) \operatorname{deg}\left(B_{2} \rightarrow B\right) \operatorname{deg}(\bar{B})
\end{aligned}
$$

By Proposition 1.2.1, we can see that $A \times{ }_{K^{\prime}} \operatorname{Spec}\left(K_{1}\right)$ has semiabelian reduction in codimension one over $B_{1}$. On the other hand, by Proposition 3.1,

$$
\gamma_{*}\left(\widehat{c}_{1}\left(\bar{\lambda}_{G \times_{Y} B_{2} / B_{2}}^{\text {Faa }}\right)\right)=\widehat{c}_{1}\left(\bar{\lambda}^{\text {Fal }}\left(A \times_{K^{\prime}} \operatorname{Spec}\left(K_{1}\right) / K_{1} ; B_{1}\right)\right)
$$

Therefore we get

$$
\left|h \frac{\bar{B}}{L}(A)-n h_{\bmod }^{\bar{B}}(A)\right| \leq \log (N) \operatorname{deg}(\bar{B}) .
$$

Corollary 4.2. Let $l$ and e be positive integers and let $K$ be a field finitely generated over $\mathbb{Q}$. Put $d=\operatorname{tr} . \operatorname{deg}_{\mathbb{Q}}(K)$ and fix a generically smooth and fine polarization $\bar{B}=\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$ of $K$. Then
(1) There exists a constant $C=C(\bar{B}, l, g)$ such that $h_{\bmod }^{\bar{B}}(A) \geq C$ for an arbitrary $l$-polarized abelian variety $A$ of dimension $g$ over $\bar{K}$.
(2) There exists a constant $C^{\prime}=C^{\prime}(\bar{B}, l, e, g)$ such that the set

$$
\left\{A \times_{K^{\prime}} \operatorname{Spec}(\bar{K}) \left\lvert\, \begin{array}{l}
A \text { is a g-dimensional and } \\
\text { l-polarized abelian variety } \\
\text { over a finite extension } \\
K^{\prime} \text { of } K \text { with }\left[K^{\prime}: K\right] \leq e \\
\text { and } h_{\mathrm{mod}}^{\bar{B}}(A) \leq h .
\end{array}\right.\right\} / \simeq \bar{K}
$$

has cardinality $\leq C^{\prime} \cdot h^{d+1}$ for $h \gg 0$.
Proof. Let us fix a positive number $m$ such that $m$ has a decomposition $m=m_{1} m_{2}$ with $\left(m_{1}, m_{2}\right)=1$ and $m_{1}, m_{2} \geq 3$. Then any $l$-polarized abelian variety over $\bar{K}$ has a level $m$ structure. Thus (1) is a consequence of Proposition 2.1 and Proposition 4.1.

Let $A$ be an $l$-polarized abelian variety over a finite extension $K^{\prime}$ of $K$. Let $K^{\prime \prime}$ be the minimal extension of $K^{\prime}$ such that $A[m](\bar{K}) \subseteq A\left(K^{\prime \prime}\right)$. Then $\left[K^{\prime \prime}: K^{\prime}\right] \leq \#\left(\operatorname{Aut}(\mathbb{Z} / m \mathbb{Z})^{2 g}\right)$. Thus, by using Proposition 2.1 and Proposition 4.1, we get (2).

## §5. Galois descent

Let $A$ be a $g$-dimensional abelian variety over a field $k$. Let $m$ be a positive integer prime to the characteristic of $k$. Note that a level $m$ structure $\alpha$ of $A$ over a finite extension $k^{\prime}$ of $k$ is an isomorphism $\alpha:(\mathbb{Z} / m \mathbb{Z})^{2 g} \rightarrow A[m]\left(k^{\prime}\right)$. If $k^{\prime}$ is a finite Galois extension over $k$, then we have a homomorphism

$$
\epsilon\left(k^{\prime} / k, A, \alpha\right): \operatorname{Gal}\left(k^{\prime} / k\right) \rightarrow \operatorname{Aut}\left((\mathbb{Z} / m \mathbb{Z})^{2 g}\right)
$$

given by $\epsilon\left(k^{\prime} / k, A, \alpha\right)(\sigma)=\alpha^{-1} \cdot \sigma_{A} \cdot \alpha$, where

$$
\sigma_{A}: A \times_{k} \operatorname{Spec}\left(k^{\prime}\right) \xrightarrow{\mathrm{id}_{A} \times\left(\sigma^{-1}\right)^{a}} A \times_{k} \operatorname{Spec}\left(k^{\prime}\right)
$$

is the natural morphism arising from $\sigma$. Note that $(\sigma \cdot \tau)_{A}=\sigma_{A} \cdot \tau_{A}$.
Lemma 5.1. Let $k$ be a field, $k^{\prime}$ a finite Galois extension of $k$ and $m \geq 3$ an integer prime to the characteristic of $k$. Let $(A, \xi)$ and $\left(A^{\prime}, \xi^{\prime}\right)$
be two polarized abelian varieties over $k$ and let $\alpha, \alpha^{\prime}$ be level $m$ structures of $A, A^{\prime}$ defined over $k^{\prime}$. If a $k^{\prime}$-isomorphism

$$
\phi:(A, \xi) \times_{k} \operatorname{Spec}\left(k^{\prime}\right) \rightarrow\left(A^{\prime}, \xi^{\prime}\right) \times_{k} \operatorname{Spec}\left(k^{\prime}\right)
$$

as polarized abelian varieties over $k^{\prime}$ satisfies
(a) $\phi \cdot \alpha=\alpha^{\prime}$ and
(b) $\epsilon\left(k^{\prime} / k, A, \alpha\right)=\epsilon\left(k^{\prime} / k, A^{\prime}, \alpha^{\prime}\right)$,
then $\phi$ descends to an isomorphism $(A, \xi) \rightarrow\left(A^{\prime}, \xi^{\prime}\right)$ over $k$.
Proof. For $\sigma \in \operatorname{Gal}\left(k^{\prime} / k\right)$, let us consider a morphism

$$
\phi_{\sigma}=\sigma_{A^{\prime}}^{-1} \cdot \phi \cdot \sigma_{A}: A \times_{k} \operatorname{Spec}\left(k^{\prime}\right) \rightarrow A^{\prime} \times_{k} \operatorname{Spec}\left(k^{\prime}\right) .
$$

First of all, $\phi_{\sigma}$ is a morphism over $k^{\prime}$. We claim that $\phi_{\sigma} \cdot \alpha=\alpha^{\prime}$. Indeed, since $\alpha^{-1} \cdot \sigma_{A} \alpha=\alpha^{\prime-1} \cdot \sigma_{A^{\prime}} \cdot \alpha^{\prime}$, we have

$$
\phi_{\sigma} \cdot \alpha=\sigma_{A^{\prime}}^{-1} \cdot \phi \cdot \alpha \cdot \alpha^{-1} \cdot \sigma_{A} \cdot \alpha=\sigma_{A^{\prime}}^{-1} \cdot \alpha^{\prime} \cdot \alpha^{\prime-1} \cdot \sigma_{A^{\prime}} \cdot \alpha^{\prime}=\alpha^{\prime}
$$

Thus $\phi_{\sigma}$ preserves the level structures of $A \times_{k} \operatorname{Spec}\left(k^{\prime}\right)$ and $A^{\prime} \times_{k}$ $\operatorname{Spec}\left(k^{\prime}\right)$. Hence, since $m \geq 3$ and $\phi_{\sigma} \cdot \phi^{-1}$ preserve the polarization $\xi$ of $A$ over $k^{\prime}\left(\right.$ hence $\left(\phi_{\sigma} \cdot \phi^{-1}\right)^{N}=$ id for $N \gg 1$ ), by virtue of Serre's theorem, we have $\phi_{\sigma}=\phi$, that is,

$$
\phi \cdot \sigma_{A}=\sigma_{A^{\prime}} \cdot \phi
$$

for all $\sigma \in \operatorname{Gal}\left(k^{\prime} / k\right)$. Therefore $\phi$ descends to an isomorphism $(A, \xi) \rightarrow$ ( $\left.A^{\prime}, \xi^{\prime}\right)$ over $k$.

Proposition 5.2. Let $B$ be an irreducible normal scheme of finite type over $\mathbb{Z}$ and let $K$ denote its function field. Fix a polarized abelian variety $\left(C, \xi_{C}\right)$ of dimension $g$ defined over $\bar{K}$. Then the set

$$
\mathcal{S}=\left\{\begin{array}{l|l}
(A, \xi) & \begin{array}{l}
(A, \xi) \text { is a polarized abelian variety over } K \text { with } \\
(A, \xi) \times_{K} \operatorname{Spec}(\bar{K}) \simeq\left(C, \xi_{C}\right) \text { and } A \text { has semiabelian } \\
\text { reduction over } B \text { in codimension one. }
\end{array}
\end{array}\right\}
$$

modulo $K$-isomorphisms is finite.
Proof. For $(A, \xi) \in \mathcal{S}$, let $B_{A}$ be a big open set of $B$ over which we have a semiabelian extension $\mathcal{X}_{A} \rightarrow B_{A}$ of $A$. Let $B R(A)$ denote the set of points $x$ of codimension one in $B_{A}$ such that the fiber of $\mathcal{X}_{A}$ over $x$ is not an abelian variety.

Claim 5.2.1. For any $(A, \xi),\left(A^{\prime}, \xi^{\prime}\right) \in \mathcal{S}, B R(A)=B R\left(A^{\prime}\right)$.

Since $A \times_{K} \operatorname{Spec}(\bar{K}) \simeq A^{\prime} \times{ }_{K} \operatorname{Spec}(\bar{K})$, there is a finite extension $K^{\prime}$ of $K$ with $A \times_{K} \operatorname{Spec}\left(K^{\prime}\right) \simeq A^{\prime} \times_{K} \operatorname{Spec}\left(K^{\prime}\right)$. Let $\pi: B^{\prime} \rightarrow B$ be the normalization of $B$ in $K^{\prime}$. Then $\mathcal{X}_{A} \times_{B_{A}} \pi^{-1}\left(B_{A}\right)$ is isomorphic to $\mathcal{X}_{A^{\prime}} \times_{B_{A^{\prime}}} \pi^{-1}\left(B_{A^{\prime}}\right)$ over $\pi^{-1}\left(B_{A} \cap B_{A^{\prime}}\right)$, so that $\pi^{-1}(B R(A))=$ $\pi^{-1}\left(B R\left(A^{\prime}\right)\right)$, yielding the claim.

Let us fix a positive integer $m \geq 3$ and $\left(A_{0}, \xi_{0}\right) \in \mathcal{S}$. We set

$$
U=B \backslash\left(\left(B \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Z} / m \mathbb{Z})\right) \cup \operatorname{Sing}(B) \cup \bigcup_{x \in B R\left(A_{0}\right)} \overline{\{x\}}\right)
$$

Then $U$ is regular and of finite type over $\mathbb{Z}$. The characteristic of the residue field of any point of $U$ is prime to $m$. For $(A, \xi) \in \mathcal{S}$, let $U_{A}$ be the maximal Zariski open set of $U$ over which $\mathcal{X}_{A}$ is an abelian scheme. By the above claim, $\operatorname{codim}\left(U \backslash U_{A}\right) \geq 2$.

Claim 5.2.2. There exists a finite Galois extension $K^{\prime}$ of $K$ such that every m-torsion point of $A$ is defined over $K^{\prime}$ whenever $(A, \xi) \in \mathcal{S}$.

For $(A, \xi) \in \mathcal{S}$, let $K_{A}$ be the finite extension of $K$ obtaining by adding all $m$-torsion points of $A$ to $K$. Let $V_{A}$ be the normalization of $U$ in $K_{A}$. It is well-known that $V_{A}$ is étale over $U_{A}$. Moreover, by virtue of the purity of branch loci (cf. SGA 1, Exposé X, Thérème 3.1), $V_{A}$ is étale over $U$ because $\operatorname{codim}\left(U \backslash U_{A}\right) \geq 2$. Let $M$ be the union of the finite extensions $K^{\prime}$ of $K$ such that the normalization of $U$ in $K^{\prime}$ is étale over $U$. By construction, $M$ is a Galois extension of $K$. Since $K_{A} \subseteq M$, we have a continuous homomorphism

$$
\rho_{A}: \operatorname{Gal}(M / K) \rightarrow \operatorname{Aut}(A[m](\bar{K})) \simeq \operatorname{Aut}\left((\mathbb{Z} / m \mathbb{Z})^{2 g}\right)
$$

such that $\operatorname{ker}\left(\rho_{A}\right)=\operatorname{Gal}\left(M / K_{A}\right)$. Since $\operatorname{Gal}(M / K)=\pi_{1}(U)$, by [3, Hermite-Minkowski theorem in Chapter VI], we have only finitely many continuous homomorphisms

$$
\rho: \operatorname{Gal}(M / K) \rightarrow \operatorname{Aut}\left((\mathbb{Z} / m \mathbb{Z})^{2 g}\right)
$$

Thus there are only finitely many choices of Galois subgroups $\operatorname{Gal}\left(M / K_{A}\right) \subseteq$ $\operatorname{Gal}(M / K)$ and of subfields $K_{A} \subseteq M$. This shows our claim.

Claim 5.2.3. For any $(A, \xi),\left(A^{\prime}, \xi^{\prime}\right) \in \mathcal{S},(A, \xi) \times_{K} \operatorname{Spec}\left(K^{\prime}\right) \simeq$ $\left(A^{\prime}, \xi^{\prime}\right) \times{ }_{K} \operatorname{Spec}\left(K^{\prime}\right)$.

There is a finite Galois extension $K^{\prime \prime}$ of $K^{\prime}$ such that an isomorphism

$$
\phi:(A, \xi) \times_{K} \operatorname{Spec}\left(K^{\prime \prime}\right) \rightarrow\left(A^{\prime}, \xi^{\prime}\right) \times_{K} \operatorname{Spec}\left(K^{\prime \prime}\right)
$$

is given over $K^{\prime \prime}$. Let $\alpha$ be a level $m$ structure of $A$ over $K^{\prime \prime}$ and $\alpha^{\prime}=\phi \cdot \alpha$. Then $\epsilon\left(K^{\prime \prime} / K^{\prime}, A \times_{K} \operatorname{Spec}\left(K^{\prime}\right), \alpha\right)=\epsilon\left(K^{\prime \prime} / K^{\prime}, A^{\prime} \times_{K} \operatorname{Spec}\left(K^{\prime}\right), \alpha^{\prime}\right)=1$ because all $m$-torsion points of $A$ and $A^{\prime}$ are defined over $K^{\prime}$. Thus $A \times_{K}$ $\operatorname{Spec}\left(K^{\prime \prime}\right) \rightarrow A^{\prime} \times_{K} \operatorname{Spec}\left(K^{\prime \prime}\right)$ descends to an isomorphism $(A, \xi) \times_{K}$ $\operatorname{Spec}\left(K^{\prime}\right) \rightarrow\left(A^{\prime}, \xi^{\prime}\right) \times_{K} \operatorname{Spec}\left(K^{\prime}\right)$ by Lemma 5.1.

Finally, let us see the number of isomorphism classes in $\mathcal{S}$ is finite. Fix $\left(A_{0}, \xi_{0}\right) \in \mathcal{S}$ and a level $m$ structure $\alpha_{0}$ of $A_{0}$ over $K^{\prime}$. Let $\phi_{A}$ : $\left(A_{0}, \xi_{0}\right) \times_{K} \operatorname{Spec}\left(K^{\prime}\right) \rightarrow(A, \xi) \times_{K} \operatorname{Spec}\left(K^{\prime}\right)$ be an isomorphism over $K^{\prime}$. We set $\alpha_{A}=\phi_{A} \cdot \alpha_{0}$ and $\phi_{A^{\prime}}^{A}=\phi_{A^{\prime}} \cdot \phi_{A}^{-1}: A \times_{K} \operatorname{Spec}\left(K^{\prime}\right) \rightarrow$ $A^{\prime} \times_{K} \operatorname{Spec}\left(K^{\prime}\right)$ for $(A, \xi),\left(A^{\prime}, \xi^{\prime}\right) \in \mathcal{S}$. Then $\alpha_{A^{\prime}}=\phi_{A^{\prime}}^{A} \cdot \alpha_{A}$. Here let us consider a map

$$
\gamma: \mathcal{S} \rightarrow \operatorname{Hom}\left(\operatorname{Gal}\left(K^{\prime} / K\right), \operatorname{Aut}\left((\mathbb{Z} / m \mathbb{Z})^{2 g}\right)\right)
$$

given by $\gamma(A)=\epsilon\left(K^{\prime} / K, A, \alpha_{A}\right)$. By Lemma 5.1, if $\gamma(A)=\gamma\left(A^{\prime}\right)$, then $(A, \xi) \simeq\left(A^{\prime}, \xi^{\prime}\right)$ over $K$, while $\operatorname{Hom}\left(\operatorname{Gal}\left(K^{\prime} / K\right), \operatorname{Aut}\left((\mathbb{Z} / m \mathbb{Z})^{2 g}\right)\right)$ is a finite group. This completes the proof.

## $\S 6$. Strong finiteness

In this section, we give the proof of the main result of this paper.
Theorem 6.1. Let $K$ be a finitely generated field over $\mathbb{Q}$ with $d=$ tr. $\operatorname{deg}_{\mathbb{Q}}(K)$. Let $\bar{B}=\left(B ; \bar{H}_{1}, \ldots, \bar{H}_{d}\right)$ be a strictly fine polarization of $K$. Then, for any numbers $c$, the number of isomorphism classes of abelian varieties defined over $K$ with $h_{\mathrm{Fal}}^{\bar{B}}(A) \leq c$ is finite.

Proof. Considering a generic resolution of singularities $\mu: B^{\prime} \rightarrow B$, we may assume that $\bar{B}$ is generically smooth.

Let us consider the following two sets:

$$
\begin{aligned}
& \mathcal{S}_{0}(c)=\left\{(A, \xi) \left\lvert\, \begin{array}{l}
(A, \xi) \text { is a principally polarized abelian variety } \\
\text { over } K \text { with } h_{\mathrm{mod}}^{\bar{B}}(A) \leq 8 c
\end{array}\right.\right\} \\
& \mathcal{S}(c)=\left\{A \mid A \text { is an abelian variety over } K \text { with } h_{\mathrm{Fal}}^{\bar{B}}(A) \leq c\right\}
\end{aligned}
$$

By Corollary 4.2, $\left\{(A, \xi) \times \operatorname{Spec}(\bar{K}) \mid(A, \xi) \in \mathcal{S}_{0}(c)\right\} / \simeq_{\bar{K}}$ is finite. If $A$ is an abelian variety over $K$, then $\left(A \times A^{\vee}\right)^{4}$ is a principally polarized abelian variety over $K$ (Zarhin's trick; see [12, Exposé VIII, Proposition 1]). By Proposition 3.3 and Proposition 3.5,

$$
h_{\mathrm{mod}}^{\bar{B}}\left(\left(A \times A^{\vee}\right)^{4}\right)=8 h_{\mathrm{mod}}^{\bar{B}}(A) .
$$

Thus, if $A \in \mathcal{S}(c)$, then $\left(A \times A^{\vee}\right)^{4} \in \mathcal{S}_{0}(c)$. Here the number of isomorphism classes of direct factors of $\left(A \times A^{\vee}\right)^{4} \times{ }_{K} \operatorname{Spec}(\bar{K})$ is finite (cf. [12, Exposé VIII, Proposition 2]). Thus $\left\{A \times_{K} \operatorname{Spec}(\bar{K}) \mid A \in \mathcal{S}(c)\right\} / \simeq_{\bar{K}}$ is finite. In particular, there is a constant $C$ such that $C \leq h_{\bmod }^{\bar{B}}(A)$ for all $A \in \mathcal{S}(c)$.

Let $K_{A}$ be the minimal finite extension of $K$ such that $A[12](\bar{K}) \subseteq$ $A\left(K_{A}\right)$. Then $\left[K_{A}: K\right] \leq \# \operatorname{Aut}\left((\mathbb{Z} / 12 \mathbb{Z})^{2 g}\right)$. Let $B_{A}$ be a generic resolution of singularities of the normalization of $B$ in $K_{A}$. By Proposition 1.2.1, $A \times_{K} \operatorname{Spec}\left(K_{A}\right)$ has semiabelian reduction over $B_{A}$ in codimension one. Thus, by Proposition 3.1, there is an effective divisor $E_{A}$ on $B$ with

$$
h_{\mathrm{Fal}}^{\bar{B}}(A)-h_{\mathrm{mod}}^{\bar{B}}(A)=\frac{\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \mid E_{A}\right)}{\left[K_{A}: K\right]}
$$

Here $h_{\text {mod }}^{\bar{B}}(A) \geq C$ for all $A \in \mathcal{S}(c)$. Thus we can find a constant $C^{\prime}$ such that

$$
\widehat{\operatorname{deg}}\left(\widehat{c}_{1}\left(\bar{H}_{1}\right) \cdots \widehat{c}_{1}\left(\bar{H}_{d}\right) \mid E_{A}\right) \leq C^{\prime}
$$

for all $A \in \mathcal{S}(c)$. Therefore, by virtue of Proposition 1.10.1, there is a reduced effective divisor $D$ on $B$ such that, for all $A \in \mathcal{S}(c), A$ has semiabelian reduction over $B \backslash D$ in codimension one. Hence, by Proposition 5.2 , we have our assertion.

Remark 6.2. If the problem in Remark 1.10 .3 is true, then Theorem 6.1 holds even if the polarization $\bar{B}$ is fine.

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# Moduli of regular holonomic $\mathcal{D}_{X}$-modules with natural parabolic stability 

Nitin Nitsure<br>Dedicated to Professor Masaki Maruyama on his 60th birthday


#### Abstract

. In this paper we re-visit the moduli problem for regular holonomic $\mathcal{D}$-modules with normal crossing singularities, and give a new definition of semistability, more general than the older notion, construct the moduli scheme, and describe its points. Independently of this, we introduce another natural parabolic notion of stability for such $\mathcal{D}$-modules, and construct the moduli in a special case.


## §1. Introduction

Let $X$ be a smooth complex projective variety. In his path-breaking paper [13], Simpson constructed a moduli scheme for pairs $(E, \nabla)$ consisting of an algebraic vector bundle $E$ on $X$ with an algebraic integrable connection $\nabla$. These objects $(E, \nabla)$ are exactly the $\mathcal{O}_{X}$-coherent regular holonomic $\mathcal{D}_{X}$-modules, where $\mathcal{D}_{X}$ is the sheaf of linear partial differential operators on $X$ in the algebraic category. Using the methods introduced by Simpson, the present author embarked on a program to solve the moduli problem for general regular holonomic $\mathcal{D}_{X}$-modules on $X$. To begin with, the general case differs from the case considered by Simpson in the following two important respects.
(1) Firstly, such a $\mathcal{D}_{X}$-module need not be $\mathcal{O}_{X}$-coherent, so we have to replace it by an $\mathcal{O}_{X}$-coherent description in order to apply the usual Hilbert schemes and GIT-based machinery of moduli theory.
(2) Secondly, the resulting $\mathcal{O}_{X}$-coherent objects (which now replace the original $\mathcal{D}_{X}$-modules) need to have a reasonable intrinsic notion of stability (or semi-stability) defined on them, which will correspond to GIT

[^0]stability (or semi-stability) in the course of moduli construction. Simpson could do without explicitly introducing any notion of semi-stability in the case of an integrable connection $(E, \nabla)$, as the Hilbert polynomial of the underlying vector bundle $E$ automatically equals that of a trivial vector bundle of the same rank.

In a sequence of three papers [10], [12] (jointly with Claude Sabbah) and [11], the above programme was carried out for regular holonomic $\mathcal{D}_{X^{-}}$ modules with singularities over a normal crossing divisor $Y$ in $X$. We fix the normal-crossing divisor $Y$ in $X$, and take the resulting stratification of $X$ by locally closed smooth subvarieties $Z_{i} \subset X$ defined as the locus where exactly $d-i$ branches of $Y$ intersect for $i<d$, and with $Z_{d}=$ $X-Y$. This gives rise to a conical Lagrangian subvariety $C_{Y, X}$ of $T^{*} X$ which is the union of the co-normal bundles $N_{Z_{i}, X}^{*} \subset T^{*} X$ over $i \leq d$. We require that the characteristic variety $\operatorname{car}(M)$ of our regular holonomic $\mathcal{D}_{X}$-modules $M$ should be contained in $C_{Y, X}$. A moduli for the corresponding perverse sheaves (which are exactly those perverse sheaves on $X$ which are cohomologically constructible with respect to the stratification $Z_{i}$ ) was also constructed in the above papers, and the Riemann-Hilbert correspondence was shown to define an analytic morphism at the level of moduli.
However, the results so far had the following drawbacks.
(A) Instead of a moduli for the $\mathcal{D}_{X}$-modules themselves, the above constructions give a moduli for so-called 'pre- $\mathcal{D}$-modules' which are $\mathcal{O}_{X^{-}}$ coherent avatars of $\mathcal{D}_{X}$-modules (can be thought of as $\mathcal{D}_{X}$-modules with level structures). For example, instead of meromorphic connections we make a moduli for logarithmic connections. There is more than one pre- $\mathcal{D}$-module structure on any $\mathcal{D}$-module.
(B) The semistability condition on the pre- $\mathcal{D}$-modules in [12] and [11] is too restrictive, which though an open condition, leaves out a large class of $\mathcal{D}$-modules (which should ideally form some other irreducible components in a larger moduli).
(C) The link between meromorphic connections and Higgs bundles, established by Biquard [2] as a generalisation of the Narasimhan-Seshadri-Donaldson-Hitchin-Corlette-Simpson correspondence is neglected in the earlier moduli construction. Recall that the theorem of Biquard [2] makes a correspondence between parabolic stable logarithmic connections and parabolic stable Higgs bundles, when singular set $Y$ is a smooth divisor.

The present article presents an improved moduli construction, to take care of the above points. We succeed in fully overcoming drawbacks (A) and (B) of the earlier constructions, and partly overcome (C).
The improvement in points $(A)$ and $(B)$ is made by paying attention to the relationship between the topology of the normal bundles of the components of the divisor on one hand and residual eigenvalues on the other hand. This has nothing to do with parabolic stability. Independently of this, to take care of point (C) we present another variation on the moduli construction for regular meromorphic connections via a natural parabolic definition of semi-stability for Deligne lattices (which explains the title of this article).
This article is arranged as follows. In Section 2, we set up preliminaries involving $\mathcal{D}$-modules and some topological properties.
In Section 3, we define the natural parabolic structure on a Deligne connection (which is the logarithmic connection corresponding to a regular meromorphic connection with real parts of residual eigenvalues in $[0,1$ ), which was constructed by Deligne in early 1970's), and in section 4 we give the construction of a moduli for parabolic stable Deligne connections. Our construction presently works only under an extra assumption on the logarithmic tangent bundle of $(X, Y)$, which we expect to remove in the future.

In Section 5, we treat the case of regular holonomic $\mathcal{D}$-modules whose singularity locus $Y$ is a smooth divisor. We show how to improve our earlier constructions so as to overcome the limitations (A) and (B) discussed above. In Section 6, this is done in the general case where $Y$ is normal crossing.

## §2. Preliminaries

## $\mathcal{D}$-modules

Let $X$ be a nonsingular complex projective variety, with $\mathcal{D}_{X}$ the sheaf of linear partial differential operators acting on $\mathcal{O}_{X}$ (in the algebraic category).

Recall that $\mathcal{D}_{X}$ is generated as a sheaf of $\mathbb{C}$-algebras by $\mathcal{O}_{X}$ and $T_{X}$ where $\mathcal{O}_{X}$ are the scalar operators on $\mathcal{O}_{X}$ while the tangent sheaf $T_{X}$ acts on $\mathcal{O}_{X}$ by differentiation. We have the relations $\xi f-f \xi=\xi(f)$ and $\xi \eta-\eta \xi=[\xi, \eta]$ for $f \in \mathcal{O}_{X}$ and $\xi, \eta \in T_{X}$. The inclusion $\mathcal{O}_{X} \subset \mathcal{D}_{X}$ makes $\mathcal{D}_{X}$ a left-right $\mathcal{O}_{X}$-bimodule. For any $i \geq 0$, let $F^{i} \mathcal{D}_{X} \subset \mathcal{D}_{X}$
be the left $\mathcal{O}_{X}$-submodule generated by the image of $\otimes_{\mathbb{C}}^{i} T_{X} \rightarrow \mathcal{D}_{X}$. Then each $F^{i} \mathcal{D}_{X}$ a left-right sub- $\mathcal{O}_{X}$-bimodule, which is bi-coherent as an $\mathcal{O}_{X}$-module. We have $F^{0} \mathcal{D}_{X}=\mathcal{O}_{X}, F^{1} \mathcal{D}_{X}=\mathcal{O}_{X} \oplus T_{X}$ as left- $\mathcal{O}_{X^{-}}$ module, $F^{i} \mathcal{D}_{X} \cdot F^{j} \mathcal{D}_{X}=F^{i+j} \mathcal{D}_{X}$, and $\bigcup_{i \geq 0} F^{j} \mathcal{D}_{X}=\mathcal{D}_{X}$. Moreover, $\left[F^{i} \mathcal{D}_{X}, F^{j} \mathcal{D}_{X}\right] \subset F^{i+j-1} \mathcal{D}_{X}$, and the associated graded object is the graded $\mathcal{O}_{X}$-algebra $S y m_{\mathcal{O}_{X}}^{\bullet} T_{X}$.
By the phrase ' $\mathcal{D}_{X}$-module' we will mean a left- $\mathcal{D}_{X}$-module which is $\mathcal{O}_{X}$-quasi-coherent, unless otherwise indicated.
A $\mathcal{D}_{X}$-module $M$ is $\mathcal{D}_{X}$-coherent if and only if locally there exists a filtration $F^{i} M$ by $\mathcal{O}_{X}$-coherent $\mathcal{O}_{X}$-submodules which is $F^{i} D_{X}$-good: $F^{i} \mathcal{D}_{X} F^{j} M \subset F^{i+j} M, F^{i} M=0$ for $i \ll 0$ and locally $\exists k$ such that $F^{i} \mathcal{D}_{X} F^{k} M=F^{k+i} M$ for all $i \geq 0$.
The characteristic variety $\operatorname{car}(M) \subset T^{*} X$ is the set-theoretic support of $\operatorname{gr}(M)$ as a $\operatorname{Sym}_{\mathcal{O}_{X}} T_{X}$-module. This is well-defined.
A $\mathcal{D}_{X}$-coherent module $M$ is said to be holonomic if $\operatorname{dim}(\operatorname{car}(M))=$ $\operatorname{dim}(X)$ or if $M=0$, and regular holonomic if moreover local filtrations can be so chosen that $\operatorname{car}(M)$ is a reduced subscheme.

## Universal degree of line bundles

Definition 2.1. Let $X$ be a path-connected topological space, and $L$ a complex line bundle on $X$, with first Chern class $c_{1}(L) \in H^{2}(X ; \mathbb{Z})$. Then we will call the non-negative integer $d$ which generates the image of the group homomorphism $c_{1}(L) \cap-: H_{2}(X ; \mathbb{Z}) \rightarrow H_{0}(X ; \mathbb{Z})=\mathbb{Z}$ as the universal topological degree of $L$.
The following lemma explains the above terminology.
Lemma 2.2. The universal topological degree of $L$ as defined above is the greatest common divisor of the degrees of all pull-backs of $L$ to connected oriented compact hausdorff topological 2-manifolds under continuous maps.

The above follows immediately from the following elementary lemma.
Lemma 2.3. Let $X$ be a topological space, and let $c \in H_{2}(X)$ be a singular cohomology element with coefficients $\mathbb{Z}$. Then there exist oriented connected compact hausdorff topological manifolds $Y_{1}, \ldots, Y_{n}$ and continuous maps $f_{i}: Y_{i} \rightarrow X$ such that $c=\sum_{i} f_{i_{*}}\left[Y_{i}\right]$ where $\left[Y_{i}\right] \in H_{2}\left(Y_{i}\right)$ denotes the fundamental class of $Y_{i}$.

Lemma 2.4. (1) Let $S$ be a hausdorff, path connected topological space, and let $N$ be a complex line bundle on $S$. Let $x \in S$, and let
$\widehat{x} \in N_{x}-0$ where $N_{x}$ is the fiber of $N$ over $x$. Let $N-S$ denote the complement of the zero section of $N$. Let $\tau \in \pi_{1}(N-S, \widehat{x})$ denote the image of the positive generator of $\pi_{1}\left(N_{x}-0, \widehat{x}\right)$ under the homomorphism induced by inclusion $N_{x}-0 \hookrightarrow N-S$. Then $\tau$ lies in the center of the group $\pi_{1}(N-S, \widehat{x})$.
(2) Let $C$ be a compact real oriented surface of genus $g \geq 0$, and let $N$ be a complex line bundle on $C$ of degree $d$. Let $x_{0} \in C$, and let $\widehat{x_{0}}$ be a non-zero point in the fiber of $N$ over $x_{0}$. If $d=0$ then $N$ is topologically trivial, and so $\pi_{1}\left(N-C, \widehat{x_{0}}\right) \cong \pi_{1}\left(C, x_{0}\right) \times \pi_{1}\left(\mathbb{C}^{\times}\right)$. More generally for arbitrary $d$, the fundamental group $\pi_{1}\left(N-C, \widehat{x_{0}}\right)$ has the following description in terms of generators and relations. Let $\tau \in \pi_{1}\left(N-C, \widehat{x_{0}}\right)$ denote the positive loop in $N_{x_{0}}-0$ with base point $\widehat{x_{0}}$. Let $x_{1} \neq x_{0}$ be another point on $C$, so that the fundamental group $\pi_{1}\left(C-x_{1}, x_{0}\right)$ is the free group $F\left\langle a_{i}, b_{i}\right\rangle$ on certain generators $a_{i}, b_{i}$ with $1 \leq i \leq g$ where $g \geq 0$ is the genus of $C$, and $\pi_{1}\left(C, x_{0}\right)$ is the quotient of $F\left\langle a_{i}, b_{i}\right\rangle$ by the relation $\prod_{i} a_{i} b_{i} a_{i}^{-1} b_{i}^{-1}=1$. Then $\pi_{1}\left(N-C, \widehat{x_{0}}\right)$ is the quotient of the free group $F\left\langle a_{i}, b_{i}, \tau\right\rangle$ by the relation

$$
\tau^{d}=\prod_{i} a_{i} b_{i} a_{i}^{-1} b_{i}^{-1}
$$

(3) Let $\mu_{d}(\mathbb{C})$ denote the group of $d$ th roots of 1 in $\mathbb{C}$ (in particular we take $\left.\mu_{0}(\mathbb{C})=\mathbb{C}^{\times}\right)$. Then under any multiplicative character $\rho$ : $\pi_{1}(N-C) \rightarrow \mathbb{C}^{\times}$, the image $\rho(\tau)$ lies in $\mu_{d}(\mathbb{C})$, moreover, the following sequence of abelian groups is short exact

$$
1 \rightarrow \operatorname{Hom}\left(\pi_{1}(C), \mathbb{C}^{\times}\right) \rightarrow \operatorname{Hom}\left(\pi_{1}(N-C), \mathbb{C}^{\times}\right) \rightarrow \mu_{d}(\mathbb{C}) \rightarrow 1
$$

where the first map is induced by the projection $N-C \rightarrow C$ and the second map is given by $\rho \mapsto \rho(\tau)$.
The above sequence admits a splitting, hence for all d there is an isomorphism

$$
\operatorname{Hom}\left(\pi_{1}(N-C), \mathbb{C}^{\times}\right) \cong \operatorname{Hom}\left(\pi_{1}(C), \mathbb{C}^{\times}\right) \oplus \mu_{d}(\mathbb{C})
$$

Proof. (1) See Lemma 2.2.(1) in [11].
(2) This follows by considering the projection $p: N-C \rightarrow C$ and applying the Van Kampen theorem to the union $N-C=p^{-1}(U) \cup$ $p^{-1}\left(C-x_{1}\right)$ where $U$ is an open disc around $x_{1}$ which contains $x_{0}$. Note that $N$ is trivial over $U$ and over $V=C-x_{1}$, and with respect to any choice of trivializations, the transition function $g_{U, V}: U-x_{1} \rightarrow \mathbb{C}^{\times}$ has winding number equal to $d$. This fact gives rise to the relation $\tau^{d}=\prod_{i} a_{i} b_{i} a_{i}^{-1} b_{i}^{-1}$.
(3) This follows by the description of $\pi_{1}(N-C)$ given in (2). Another proof can be given by a combination of the Mayer-Vietoris sequence and the universal coefficient theorem for coefficients $\mathbb{C}^{\times}$. Yet another proof follows from the Gysin sequence.

## §3. Parabolic connections

Seshadri's original introduction of parabolic structures on vector bundles on curves, starting from a unitary monodromy representation of the fundamental group of the complement of a finite set of points, should rightly be viewed as a precursor of the general construction of a Deligne lattice. Following this, it has been a common observation (independently due to many mathematicians) that there is a natural parabolic structure on any Deligne construction where parabolic weights are real parts of residual eigenvalues, and for this parabolic structure the parabolic degree is zero. In particular, any such parabolic logarithmic connection is automatically par- $\mu$-semistable.
Drawing inspiration from this, we take parabolic Gieseker semi-stability as the condition for moduli construction. This is somewhat more restrictive than par- $\mu$-semistability, but better suited for GIT methods. We combine the moduli construction methods invented by Simpson [13] and Maruyama-Yokogawa [9] to construct a moduli space.

## Price to pay

In comparison with the moduli construction of [10], the parabolic construction given here has the restriction that now we have to fix eigenvalues of residue. However, this is not so severe a restriction, for this is automatic along all those components $Y_{a}$ for which the universal topological degree (see Definition 2.1) $\delta\left(N_{a}\right)$ of the normal bundle $N_{a}=N_{Y_{a}, X}$ is non-zero, as the local loop $\tau_{a}$ around $Y_{a}$ maps to $\delta\left(N_{a}\right)$ th roots of 1 under any $\pi_{1}(X-Y) \rightarrow \mathbb{C}^{\times}$as a consequence of Lemmas 2.4 and 2.2.
Next, note that the moduli construction is for connections that are parabolic Gieseker semistable, which need not coincide with par- $\mu$-semistable connections. However, in case all components of the divisor have selfintersection zero (for example, when $X$ is a curve) or if the residue is nilpotent, then par-Gieseker semistability coincides with par- $\mu$ semistability.

So far, our construction works only for parabolic stable bundles in the special case where the vector bundle $T_{X}\langle\log Y\rangle$ is generated by global sections. We would like to see this restriction removed in the future. ${ }^{1}$

## Parabolic sheaves and parabolic bundles on $(X, Y)$

Let $X$ be a non-singular variety and let $Y \subset X$ be a normal crossing divisors, whose irreducible components $Y_{a}$ are smooth. Our notion of a parabolic bundles on $(X, Y)$ and parabolic Hilbert polynomials is a slight generalisation of the notions introduced by Maruyama and Yokogawa. The difference is that for us the parabolic structure lives on the normalisation of $Y$, rather than on $Y$ itself as originally in [9].

Definition 3.1. Let $X$ and $Y$ be as above. A parabolic sheaf on $(X, Y)$ consists of the following data.
(1) A coherent sheaf of $\mathcal{O}_{X^{-}}$-modules $\mathcal{E}$ on $X$, called the underlying $\mathcal{O}_{X^{-}}$ module of the parabolic sheaf.
(2) For each irreducible component $Y_{a}$ a strictly decreasing filtration (of length $p(a)$ which depends on $a$ ) of the restriction $\left.\mathcal{E}\right|_{Y_{a}}$ by coherent subsheaves

$$
\left.\mathcal{E}\right|_{Y_{a}}=F_{a, 1}(\mathcal{E}) \supset \ldots \supset F_{a, p(a)}(\mathcal{E}) \supset 0
$$

These filtrations are called the quasi-parabolic structure on $\mathcal{E}$.
(3) For each component $Y_{a}$ a sequence of real numbers $0 \leq \alpha_{a, 1}<\ldots<$ $\alpha_{a, p(a)}<1$, called the parabolic weights.
For simplicity of notation, a parabolic sheaf will be denoted just by $\mathcal{E}$.
If the underlying $\mathcal{O}_{X}$-module $\mathcal{E}$ is locally free and moreover if each $F_{a, i}(\mathcal{E})$ is a vector subbundle of $\left.\mathcal{E}\right|_{Y_{a}}$, then $\mathcal{E}$ is called a parabolic bundle on $(X, Y)$.

Remark 3.2. It is usual to combine (2) and (3) and define a decreasing filtration of $\left.\mathcal{E}\right|_{Y_{a}}$ by subsheaves $F_{a, \alpha}(\mathcal{E})$ indexed by $\alpha \in[0,1)$ which is left-continuous and has finitely many jumps which take place at the $\alpha_{a, i}$.

Definition 3.3. Let $\mathcal{E}$ be a parabolic sheaf and let $\mathcal{E}^{\prime} \subset \mathcal{E}$ be a coherent subsheaf. The induced parabolic structure on $\mathcal{E}^{\prime}$ is defined as follows. For any $Y_{a}$ and $\alpha \in[0,1)$ we define $F_{a, \alpha}\left(E^{\prime}\right)$ to be the inverse image of $F_{a, \alpha}(E)$ under the map $\left.\left.\mathcal{E}^{\prime}\right|_{Y_{a}} \rightarrow \mathcal{E}\right|_{Y_{a}}$ which is induced

[^1]by $\mathcal{E}^{\prime} \hookrightarrow \mathcal{E}$. By Remark 3.2, this indeed defines a parabolic structure on $\mathcal{E}^{\prime}$.

## Parabolic Hilbert polynomial

Let $H$ be a very ample divisor on $X$, and $\mathcal{O}_{X}(H)$ the corresponding line bundle. For any coherent sheaf $\mathcal{F}$ on $X$, by $\chi(\mathcal{F}, m)$ we mean the Euler characteristic of $\mathcal{F}(m H)$ on $X$. Inspired by Maruyama-Yokogawa (but with the difference that our formulation requires $Y$ to be normal crossing, and pays attention to individual components $Y_{a}$ ), we define the parabolic Hilbert polynomial of a parabolic sheaf as follows.

Definition 3.4. Let $\mathcal{E}$ be a parabolic sheaf. The parabolic Hilbert polynomial of $\mathcal{E}$ is defined by the formula

$$
\operatorname{par} \chi(\mathcal{E}, m)=\chi(\mathcal{E}(-Y), m)+\sum_{a} \sum_{i} \alpha_{a, i} \chi\left(F_{a, i}(\mathcal{E}) / F_{a, i+1}(\mathcal{E}), m\right)
$$

When there is only one parabolic weight equal to 0 , then note that $\operatorname{par} \chi(\mathcal{E}, m)=\chi(\mathcal{E}(-Y), m)$.

Remark 3.5. If we index the filtration by $\alpha$ varying over $[0,1)$ and define the graded object $g r_{a, \alpha}(\mathcal{E})$ as

$$
g r_{a, \alpha}(\mathcal{E})=F_{a, \alpha}(\mathcal{E}) / F_{a, \alpha+\epsilon}(\mathcal{E})
$$

where $\epsilon>0$ is sufficiently small, then the above formula becomes

$$
\operatorname{par} \chi(\mathcal{E}, m)=\chi(\mathcal{E}(-Y), m)+\sum_{a} \int_{0}^{1} \chi\left(g r_{a, \alpha}(\mathcal{E}), m\right) \alpha d \alpha
$$

where $\chi\left(g r_{a, \alpha}(\mathcal{E}), m\right)$ is regarded as a distribution based at the point $\alpha$.

## Residues and Chern classes for logarithmic connection

Let $\left(x_{1}, \ldots, x_{d}\right)$ be local coordinates on $X$, with $Y$ locally defined by $x_{1} x_{2} \cdots x_{m}=0$. Then $\Omega_{X}^{1}\langle\log Y\rangle$ is locally free with basis $d x_{1} / x_{1}, \ldots$, $d x_{m} / x_{m}, d x_{m+1}, \ldots, d x_{d}$. Let $\tilde{Y} \rightarrow Y$ denote the normalisation of $Y$. The Poincaré residue map

$$
\text { res }:\left.\Omega_{X}^{1}\langle\log Y\rangle\right|_{Y} \rightarrow \mathcal{O}_{\tilde{Y}}
$$

is defined by $d x_{a} / x_{a} \mapsto 1$ for $a \leq m$ and $d x_{b} \mapsto 0$.
The link between residues and Chern classes originates from the following basic fact.

Lemma 3.6. The Poincare residue map fits in a short exact sequence

$$
0 \rightarrow \Omega_{X}^{1} \hookrightarrow \Omega_{X}^{1}\langle\log Y\rangle \xrightarrow{\text { res }} \mathcal{O}_{\tilde{Y}} \rightarrow 0
$$

Under the connecting map $H^{0}\left(\tilde{Y}, \mathcal{O}_{\tilde{Y}}\right) \rightarrow H^{1}\left(X, \Omega_{X}^{1}\right)$ we have $1 \mapsto-[Y]$.
The composite map $\left.\left.E \xrightarrow{\nabla} \Omega_{X}^{1}\langle\log Y\rangle \otimes E\right|_{Y} \xrightarrow{\text { res }} E\right|_{\tilde{Y}}$, is $\mathcal{O}_{X}$-linear for any a logarithmic connection $(E, \nabla)$ (even though $\nabla$ is not so). Pullback to $\tilde{Y}$ defines a section $\operatorname{res}(\nabla) \in \operatorname{End}\left(\left.E\right|_{\tilde{Y}}\right)$, called the residue of $(E, \nabla)$. This has the following description in terms of local coordinates. Let $e_{i}$ be a local basis for $E$, and $Y$ locally defined by $x_{1} x_{2} \cdots x_{m}=0$ as above. Then we can write

$$
\nabla\left(e_{i}\right)=\sum_{j}\left(\sum_{a \leq m} R_{i, a}^{j} \frac{d x_{a}}{x_{a}}+\sum_{b>m} \Gamma_{i, b}^{j} d x_{b}\right) \otimes e_{j}
$$

where $R_{i, a}^{j}$ and $\Gamma_{i, b}^{j}$ are local sections of $\mathcal{O}_{X}$. The matrices $\left(\left.R_{i, a}^{j}\right|_{Y_{a}}\right)$ define $\operatorname{res}(\nabla)$ on $\left.E\right|_{Y_{a}}$, where $Y_{a}$ is locally defined by $x_{a}=0$.

Lemma 3.7. Let $X$ be a non-singular variety (not necessarily compact), and let $Y \subset X$ be a normal-crossing divisor on $X$ whose irreducible components $Y_{a}$ are smooth. $(E, \nabla)$ a vector bundle with a logarithmic connection on $(X, Y)$. Then the following holds:
(1) For each component $Y_{a}$, the corresponding residue $R_{a} \in \operatorname{End}\left(\left.E\right|_{Y_{a}}\right)$ has a constant conjugacy class, in particular, the characteristic polynomial of $R_{a}$ has constant coefficients.
(2) Around any point $x \in X$ which lies on a $k$-fold intersection $Y_{1} \bigcap \ldots$ $\bigcap Y_{k}$ of components of $Y$, there exists a holomorphic trivialization of $E$ with respect to which the all corresponding residues $R_{1}, \ldots R_{k}$ have constant matrices, and these commute. In particular, for any non-negative integers $q_{1}, \ldots, q_{k}$, the function $\operatorname{Tr}\left(R_{1}^{q_{1}} \cdots R_{k}^{q_{k}}\right)$ is locally constant on $Y_{1} \bigcap \ldots \bigcap Y_{k}$.

Proof. (1) Let $Y_{a}^{\prime}$ be the open subscheme of $Y_{a}$ defined as $Y_{a}-$ $\bigcup_{b \neq a} Y_{b}$. Let $\left(x_{1}, \ldots, x_{d}\right)$ be étale local coordinates on a neighbourhood $V$ in $X$, with $Y_{a}$ locally defined by $x_{1}=0$, and $Y$ locally defined by $x_{1} \cdots x_{m}=0$, so $T_{X}\langle\log Y\rangle$ has local basis $x_{1} \partial_{1}, \ldots, x_{m} \partial_{m}, \partial_{m+1} \ldots, \partial_{d}$ where $\partial_{i}=\partial / \partial x_{i}$. The restriction $\left.E\right|_{Y_{a}^{\prime} \cap V}$ has an integrable connection $D$ defined by $D_{\partial_{i}} u=\nabla_{\partial_{i}} u$ for all $i \geq 2$. It follows from the facts that $\nabla$ is integrable and $\left[x_{1} \partial_{1}, \partial_{i}\right]=0$, that $\left[\nabla_{x_{1} \partial_{1}}, \nabla_{\partial_{i}}\right]=0$ for all $i \geq 2$.

Hence with respect to the connection on $\left.\operatorname{End}(E)\right|_{Y_{a}^{\prime} \cap V}$ induced by the connection $D$ on $\left.E\right|_{Y_{a}^{\prime} \cap V}$, the residue is a flat section of $\left.\operatorname{End}(E)\right|_{Y_{a}^{\prime} \cap V}$. The result follows as $Y_{a}^{\prime}$ is dense in $Y_{a}$.
(2) This follows by an argument similar to the above by considering a flat holomorphic basis for the integrable connection induced on the restriction of $E$ to $Y_{1} \bigcap \ldots \bigcap Y_{k} \cap V$.
Q.E.D.

## Connection with Newton classes $N_{p}(E)$

It is a well-known fact (see [Esnault-Viehweg 1986]) that for any logarithmic connection, Residue $\mapsto$ Atiyah obstruction $\mapsto$ Newton classes.
For $p \geq 0$, by definition the $p$ th complexified Newton class of a vector bundle $E$ is the element of $H^{2 p}\left(X^{a n} ; \mathbb{C}\right)$ given by $N_{p}(E)=\sum_{1 \leq i \leq r}\left(\gamma_{i}\right)^{p}$ where $r=\operatorname{rank}(E)$ and $\gamma_{i}$ are the complexified Chern roots of $\bar{E}$.
Let $Y=Y_{1} \cup \ldots \cup Y_{m}$ with irreducible components $Y_{a}$ smooth, crossing normally. Let

$$
R_{a}=\left.\operatorname{res}(\nabla)\right|_{Y_{a}} \in H^{0}\left(Y_{a}, \operatorname{End}\left(\left.E\right|_{Y_{a}}\right)\right)
$$

Then as a consequence of Lemma 3.6, we have

$$
N_{p}(E)=(-1)^{p} \sum_{q_{1}+\ldots+q_{m}=p} \operatorname{Tr}\left(R_{1}^{q_{1}} \cdots R_{m}^{q_{m}}\right)\left[Y_{1}\right]^{q_{1}} \cdots\left[Y_{m}\right]^{q_{m}}
$$

where $\left[Y_{a}\right]=c_{1}\left(\mathcal{O}_{X}\left(Y_{a}\right)\right) \in H^{2}\left(X^{a n} ; \mathbb{C}\right)$. Consequently the Chern character $\operatorname{ch}(E)=r+N_{1}+N_{2} / 2+N_{3} / 3!+\ldots$ is determined by the residue.

## Natural parabolic structure on logarithmic connections

Definition 3.8. We will say that a logarithmic connection $(E, \nabla)$ on $(X, Y)$ is a Deligne connection if the real parts of all eigenvalues of the residue $R_{a} \in \operatorname{End}\left(\left.E\right|_{Y_{a}}\right)$ over each irreducible component $Y_{a}$ of $Y$ all lie in the interval $[0,1)$. Let $0 \leq \alpha_{a, 1}<\ldots<\alpha_{a, p}<1$ be the distinct real parts of the residual eigenvalues along $Y_{a}$ (it is possible that two distinct eigenvalues have the same real part). Then $\left.E\right|_{Y_{a}}$ gets a direct sum decomposition

$$
\left.E\right|_{Y_{a}}=E_{a, 1} \oplus \ldots \oplus E_{a, p}
$$

where $E_{a, i}$ is the direct sum of all generalised eigensubbundles $\left.V_{\lambda} \subset E\right|_{Y_{a}}$ of $R_{a}$ corresponding to all eigenvalues $\lambda$ with real part $\operatorname{Re}(\lambda)=\alpha_{a, i}$. This in particular allows us to define a decreasing filtration

$$
\left.E\right|_{Y_{a}}=F_{a, 1} \supset \ldots \supset F_{a, p} \supset 0
$$

where $F_{a, i}=\oplus_{j \geq i} E_{a, j}$ which is a vector subbundle of $\left.E\right|_{Y_{a}}$. The filtration $F_{a, i}$ with weights $\alpha_{a, i}$ is the natural parabolic structure on a Deligne connection $E$.

## Parabolic Hilbert polynomial of a Deligne connection

Let $H$ be a very ample divisor on $X$, and $\mathcal{O}_{X}(H)$ the corresponding line bundle. For any coherent sheaf $\mathcal{F}$ on $X$, by $\chi(\mathcal{F}, m)$ we mean the Euler characteristic of $\mathcal{F}(m H)$ on $X$. Generalising Maruyama-Yokogawa, we define the parabolic Hilbert polynomial of a Deligne connection as follows.

Proposition 3.9. Let $E$ be a Deligne connection and let $\left.E\right|_{Y_{a}}=$ $E_{a, 1} \oplus \ldots \oplus E_{a, p}$ be the direct sum decomposition as in Definition 3.8 indexed by the real parts of eigenvalues of residues. Then the parabolic Hilbert polynomial of $E$ satisfies the following equality.

$$
\operatorname{par} \chi(E, m)=\chi(E(-Y), m)+\sum \alpha_{a, i} \chi\left(E_{a, i}, m\right)
$$

Proposition 3.10. The parabolic Hilbert polynomial of a Deligne connection has the form

$$
\operatorname{par} \chi(E, m)=\frac{r(E)[H]^{d}}{d!} m^{d}+\frac{r(E)\left(c_{1}(X) / 2-[Y]\right)[H]^{d-1}}{(d-1)!} m^{d-1}+\ldots
$$

where the remaining terms are of degrees $\leq d-2$ in $m$.
In particular, the parabolic degree of any such $E$ is zero and so any such $E$ is necessarily parabolic $\mu$-semistable.

Proof. By Riemann-Roch theorem, we get $\chi(E(-Y), m)=$

$$
\frac{r(E)[H]^{d}}{d!} m^{d}+\frac{\left(r(E) c_{1}(X) / 2+c_{1}(E(-Y))\right)[H]^{d-1}}{(d-1)!} m^{d-1}+\ldots
$$

(lower order terms in $m$ ), and

$$
\sum \alpha_{a, i} \chi\left(E_{a, i}, m\right)=\sum \alpha_{a, i} \frac{r\left(E_{a, i}\right)\left[Y_{a}\right][H]^{d-1}}{(d-1)!} m^{d-1}+\ldots
$$

By the relationship between residue and Chern classes described earlier,

$$
\sum \alpha_{a, i} r\left(E_{a, i}\right)\left[Y_{a}\right]=\sum \operatorname{tr}\left(R_{a}\right)\left[Y_{a}\right]=-c_{1}(E)
$$

Substituting in the above equation gives

$$
\sum \alpha_{a, i} \chi\left(E_{a, i}, m\right)=-\frac{c_{1}(E)[H]^{d-1}}{(d-1)!} m^{d-1}+\ldots
$$

Hence we get the desired equality for $\operatorname{par} \chi(E, m)$.
Q.E.D.

Remark 3.11. In contrast to the coefficients of degrees $d$ and $d-1$, the coefficients of $\operatorname{par} \chi(E, m)$ in degrees $\leq d-2$ can depend on data involving residual eigenvalues and their intersection multiplicities. So parabolic Gieseker semistability is not automatic except in special cases - say when each $\left[Y_{a}\right]\left[Y_{b}\right]=0$ or when along each $Y_{a}$ there is exactly one parabolic weight $\boldsymbol{\operatorname { R e }}\left(\lambda_{a, i}\right)$. In particular if all residues $R_{a}$ are nilpotent, then $\operatorname{par} \chi(E, m)=r(E) \cdot \chi\left(\mathcal{O}_{X}(-Y), m\right)$ and so parabolic Gieseker semistability is automatic when all local monodromies are unipotent.

Definition 3.12. We say that a Deligne connection $E$ is parabolic semistable if for each nonzero $\nabla$-invariant vector subbundle $F$ with $0 \neq F \neq E$, we have

$$
\operatorname{par} \chi(F, m) / r(F) \leq \operatorname{par} \chi(E, m) / r(E)
$$

where $r(F)$ and $r(E)$ are the ranks of the respective vector bundles, and $F$ is given the induced parabolic structure. If strict inequality always holds, we say that $E$ is parabolic stable.

## Strong local freeness

Lemma 3.13. Let $E$ be a Deligne connection on $(X, Y)$, and let $\mathcal{F} \subset E$ be an $\mathcal{O}_{X}$-coherent sub- $\mathcal{D}_{X}\langle\log Y\rangle$-module. If $E / \mathcal{F}$ is torsionfree then $E / \mathcal{F}$ is locally free, that is, $\mathcal{F}$ is a vector subbundle of $E$.

Proof. Let $\mathcal{M}$ be the local system on $X-Y$ defined by $\left.E\right|_{X-Y}$. As $\left.\mathcal{F}\right|_{X-Y}$ is an $\mathcal{O}_{X-Y}$-coherent $\mathcal{D}_{X-Y}$-module, it is locally free, and its local integrable sections define a sub local system $\mathcal{L} \subset \mathcal{M}$. The Deligne construction applied to this inclusion of local systems gives a vector subbundle $V \subset E$, with $\left.V\right|_{X-Y}=\left.\mathcal{F}\right|_{X-Y}$. The composite $\mathcal{F} \rightarrow$ $E \rightarrow E / V$ is zero on $X-Y$ hence identically zero, as $E / V$ is locally free hence torsion-free. So $\mathcal{F} \subset V$. As $\left.V\right|_{X-Y}=\left.\mathcal{F}\right|_{X-Y}$, it follows that $V / \mathcal{F}$ is a torsion sheaf, while the inclusion $V / \mathcal{F} \subset E / \mathcal{F}$ into $E / \mathcal{F}$ together with the hypothesis that $E / \mathcal{F}$ is torsion-free, shows that $V / \mathcal{F}$ is torsion-free. It follows that $V / \mathcal{F}=0$, and so $\mathcal{F}=V$.
Q.E.D.

Because of the following lemma, we do not need to impose any condition on $\nabla$-invariant $\mathcal{O}$-coherent subsheaves other than vector subbundles.

Lemma 3.14. Let $E$ be a Deligne connection on $(X, Y)$, and let $\mathcal{F} \subset E$ be a non-zero $\mathcal{O}_{X}$-coherent sub- $\mathcal{D}_{X}\langle\log Y\rangle$-module. Let $\overline{\mathcal{F}} \subset E$ be the inverse image of the torsion subsheaf $(E / \mathcal{F})_{\text {tors }}$ of $E / \mathcal{F}$ under the quotient map $E \rightarrow E / \mathcal{F}$. Then $\overline{\mathcal{F}}$ is a vector subbundle of $E$ which is invariant under $\nabla$, and given the induced parabolic structures on $\mathcal{F}$
and $\overline{\mathcal{F}}$, the normalised parabolic Hilbert polynomials satisfy

$$
\operatorname{par} \chi(\mathcal{F}, m) / r(\mathcal{F}) \leq \operatorname{par} \chi(\overline{\mathcal{F}}, m) / r(\overline{\mathcal{F}})
$$

Equality holds (if and) only if $\mathcal{F}=\overline{\mathcal{F}}$, that is, (if and) only if $\mathcal{F} \subset E$ is a vector subbundle.

Proof. By Lemma 3.13, $\overline{\mathcal{F}}$ is a vector subbundle of $E$ which is invariant under $\nabla$. Moreover, $\left.\mathcal{F}\right|_{X-Y}=\left.\overline{\mathcal{F}}\right|_{X-Y}$ which in particular means $r(\mathcal{F})=r(\overline{\mathcal{F}})$. The parabolic filtration on $\left.\mathcal{F}\right|_{Y_{a}}$ is induced from that on $\left.\overline{\mathcal{F}}\right|_{Y_{a}}$, which implies that we have inclusion of corresponding graded pieces $g r_{a, \alpha}(\mathcal{F}) \subset g r_{a, \alpha}(\overline{\mathcal{F}})$. These inclusions, along with the inclusion $\mathcal{F} \subset \overline{\mathcal{F}}$ give the inequality between parabolic Hilbert polynomials $\operatorname{par} \chi(\mathcal{F}, m) \leq \operatorname{par} \chi(\overline{\mathcal{F}}, m)$. Now the result follows by dividing by $r(\mathcal{F})=r(\overline{\mathcal{F}})$.

Remark 3.15. Sub-connections of $E$ have only finitely many possible Hilbert polynomials and parabolic Hilbert polynomials. This is because the residual eigenvalues (with multiplicities) of a sub-connections of $E$ come from that of $E$, and these determine the Hilbert polynomials and parabolic Hilbert polynomials.

Lemma 3.16. In any family $E$ of Deligne connections parametrised by a scheme $S$, the conditions of parabolic semistability and parabolic stability define open subschemes of $S$.

Proof. Let $\pi: Q \rightarrow S$ be the relative quot scheme of $\mathcal{O}$-coherent quotients of $E$ having any one of the possible Hilbert polynomials for quotients modulo sub-connections. There are only finitely many such polynomials by Remark 3.15 , so $Q$ is proper over $S$. The condition that the kernel of the quotient is $\nabla$-invariant is a closed condition, defining a closed subscheme $Q^{\prime} \subset Q$. Note that $Q^{\prime}$ has a closed subschemes $Q_{1} \subset Q_{2} \subset Q^{\prime}$ such that $S-\pi\left(Q_{2}\right)$ is the stable locus and $S-\pi\left(Q_{1}\right)$ is the semi-stable locus in $S$, hence these are open in $S$.
Q.E.D.

Lemma 3.17. Let $Y \subset X$ be a smooth divisor such that $[Y]^{2}=$ $0 \in H^{4}\left(X^{a n} ; \mathbb{C}\right)$. Then for any logarithmic connection $E$ which is a Deligne extension with natural parabolic structure, the parabolic Hilbert polynomial equals $r(E) \chi\left(\mathcal{O}_{X}(-Y), m\right)$.

Proof. Note that $[Y]=\sum_{a}\left[Y_{a}\right]$ and by assumption, $\left[Y_{a}\right]\left[Y_{b}\right]=0$ for all $a, b$. Hence we have $\operatorname{ch}\left(\mathcal{O}_{X}\left(-Y_{a}\right)\right)=1-\left[Y_{a}\right]$, and so

$$
\operatorname{ch}\left(\mathcal{O}_{Y_{a}}\right)=\operatorname{ch}\left(\mathcal{O}_{X}\right)-\operatorname{ch}\left(\mathcal{O}_{X}\left(-Y_{a}\right)\right)=1-\operatorname{ch}\left(\mathcal{O}_{X}\left(-Y_{a}\right)\right)=\left[Y_{a}\right]
$$

Note that $c_{1}(E)=-\sum_{a} \operatorname{trace}\left(R_{a}\right)\left[Y_{a}\right]$ where $R_{a}=\operatorname{trace}\left(\operatorname{res}_{Y_{a}}(E)\right)$. As $[Y]^{2}=0$, the complex Newton classes $N_{p}(E)$ vanish for $p \geq 2$. Hence the Chern character of $E(-Y)$ is given by

$$
\operatorname{ch}(E(-Y))=r(E)-\sum_{a}\left(r(E)+\operatorname{trace}\left(R_{a}\right)\right)\left[Y_{a}\right]
$$

Consider any piece $E_{a, i}$ in the direct sum decomposition $\left.E\right|_{Y_{a}}=E_{a, 1} \oplus$ $\ldots \oplus E_{a, p}$. Again by $\left[Y_{a}\right]^{2}=0$, the Newton classes $N_{p}\left(E_{a, i}\right)$ vanish for $p \geq 1$, so $E_{a, i}$ has Chern character

$$
\operatorname{ch}\left(E_{a, i}\right)=r\left(E_{a, i}\right) \operatorname{ch}\left(\mathcal{O}_{Y_{a}}\right)=r\left(E_{a, i}\right)\left[Y_{a}\right]
$$

Moreover, note that $\operatorname{trace}\left(R_{a}\right)=\sum_{i} r\left(E_{a, i}\right) \alpha_{a, i}$. As $\sum_{i} r\left(E_{a, i}\right)=r(E)$, this gives

$$
\operatorname{ch}(E(-Y))+\sum_{a, i} \alpha_{a, i} \operatorname{ch}\left(E_{a, i}\right)=r(E)(1-[Y])=r(E) \operatorname{ch}\left(\mathcal{O}_{X}(-Y)\right)
$$

Multiplying both sides of the above equation by $\operatorname{ch}\left(\mathcal{O}_{X}(m H)\right) t d(X)$ (where $t d(X)$ denotes the Todd class of $X$ ) and integrating over $X$, the result follows by the Hirzebruch-Riemann-Roch theorem. Q.E.D.

## §4. Moduli of parabolic stable connections

Given $(X, Y)$ with $Y=\cup_{a} Y_{a}$, and rank $n$, we fix
(1) along each $Y_{a}$, eigenvalues (with multiplicities) for residues $R_{a}$ with real parts in $[0,1)$ (these are constant by Lemma 3.7),
(2) along each connected component of intersection $Y_{a_{1}} \cap \ldots \cap Y_{a_{r}}$, fix ranks of intersections of the generalised eigen-subbundles of residues $R_{a_{i}}$ (these ranks are constant by Lemma 3.7).
As explained earlier, this fixes all the Newton classes of any $E$ with the above data. Let $P(m)$ denote the resulting Hilbert polynomial.

Proposition 4.1. Let the rank $r$ and residue data be fixed. There exists an integer $N_{0}$ such that for any Deligne connection of rank $r$ with the above data, the following holds for any $N \geq N_{0}$ :
(1) The bundle $E(N)$ is generated by global sections, all its higher cohomologies vanish, and all the higher cohomologies of the line bundle $\operatorname{det}(E(N))$ vanish.
(2) The higher cohomologies of the restrictions $\left.E\right|_{Y_{a}}$ also vanish.

Combining Simpson's method for $\Lambda$-modules with that of Bhosle, Maruyama, Yokogawa for parabolic bundles converts the moduli problem into a quotient problem and into a GIT problem.
By boundedness there exists $N$ such that $E(m)$ has all higher cohomologies zero for $m \geq N$, and is generated by global sections.

## Locally universal family

Fix the rank $r$ and the residual eigenvalues with multiplicities, together with ranks of intersections of generalised eigen-subbundles of the Deligne connection, as explained above. Let $\mathcal{D}_{X}\langle\log Y\rangle \subset \mathcal{D}_{X}$ consists of operators which preserve $I_{Y} \subset \mathcal{O}_{X}$. By intersection with the filtration on $\mathcal{D}_{X}$, this acquires an exhaustive filtration

$$
0 \subset \mathcal{O}_{X}=F^{0} \mathcal{D}_{X}\langle\log Y\rangle \subset F^{1} \mathcal{D}_{X}\langle\log Y\rangle \subset \ldots
$$

where each $F^{i} \mathcal{D}_{X}\langle\log Y\rangle$ is an $\mathcal{O}_{X}$-bimodule (with commutating left and right structures), which is $\mathcal{O}_{X}$-bi-coherent. A logarithmic connections is the same as a left $\mathcal{D}_{X}\langle\log Y\rangle$-module which is coherent and locally free over $\mathcal{O}_{X}$. For any integer $i$, we denote $F^{i} \mathcal{D}_{X}\langle\log Y\rangle$ simply by $F^{i}$. To keep clear whether the left or the right $\mathcal{O}_{X}$-module structure is used for the tensor product, we will use the notation

$$
F^{i}(m, n)=\mathcal{O}_{X}(m) \otimes_{\mathcal{O}_{X}} F^{i} \mathcal{D}_{X}\langle\log Y\rangle \otimes_{\mathcal{O}_{X}} \mathcal{O}_{X}(n)
$$

Following ideas of Simpson, we construct a locally universal family for Deligne connections of rank $r$ with given residual data, parametrised by a scheme $C$, defined as a locally closed subscheme of a certain Quot scheme $Q$. We define $Q$ as the Quot scheme which parametrises left-$\mathcal{O}_{X}$-linear epimorphisms

$$
q: F^{r+1}(0,-N)^{\oplus P(N)}=F^{r+1} \mathcal{D}_{X}\langle\log Y\rangle \otimes_{\mathcal{O}_{X}} \mathcal{O}_{X}(-N)^{\oplus P(N)} \rightarrow \mathcal{F}
$$

such that $\mathcal{F}$ is $\mathcal{O}_{X}$-coherent with Hilbert polynomial $P(m)$, and where $N$ is so chosen (using Proposition 4.1) that for $m \geq N$, the sheaves $E(m)$ and $E_{a, \alpha}(m)$ are generated by global sections and their higher $H^{i}$ 's vanish, and also the same holds for $I_{Y}$ times or $I_{x}$ times the sheaves for $x \in X$ and the same holds for the $\mathcal{D}_{X}\langle\log Y\rangle$-subquotient bundles of $E$ and $E_{a, \alpha}$.
Let $C \subset Q=$ Quot $_{F^{r+1}(0,-N)^{\oplus n} / X}^{P(m)}$ be the locally closed subscheme, defined by the following conditions (where $n=P(N)$ ):
(i) $\mathcal{F}$ is locally free (this is an open condition on $Q$ ).
(ii) Consider the bi- $\mathcal{O}_{X}$-module homomorphism

$$
\mathcal{O}_{X}(-N)^{\oplus n} \rightarrow F^{r+1}(0,-N)^{\oplus n}: a \mapsto 1 \otimes a
$$

We impose the condition that the composite map

$$
p: \mathcal{O}_{X}(-N)^{\oplus n} \rightarrow F^{r+1}(0,-N)^{\oplus n} \rightarrow \mathcal{F}
$$

is surjective, and on applying $\mathcal{O}_{X}(N) \otimes \mathcal{O}_{X}$ - followed by $H^{0}(X,-)$ it induces an isomorphism $\mathbb{C}^{n} \rightarrow H^{0}(\mathcal{F}(N))$ (this is an open condition).
(iii) $F^{r+1} \otimes \mathcal{O}_{X}(-N)^{n} \rightarrow \mathcal{F}$ factors via the surjection $1_{F^{r+1}} \otimes p: F^{r+1} \otimes$ $\mathcal{O}_{X}(-N)^{n} \rightarrow F^{r+1} \otimes \mathcal{F}$, giving a (uniquely determined) map $\mu: F^{r+1} \otimes$ $\mathcal{F} \rightarrow \mathcal{F}$, and the product $F^{i} \otimes F^{j} \rightarrow F^{i+j}$ for $i+j \leq r+1$ is respected (closed condition, makes $\mathcal{F}$ a $\mathcal{D}_{X}\langle\log Y\rangle$-module.)
(iv) Residual eigenvalues are correct (closed condition).

The scheme $C$ in invariant under the action of $S L_{n}$ and a good quotient $C / / S L_{n}$, if it exists, is the coarse moduli for Deligne connections with given rank and residual eigenvalues. Let $C^{\text {par } s s} \subset C$ be the open subscheme consisting of parabolic semistable bundles. Then we show that a good quotient $C^{\text {par ss }} / / S L_{n}$ exists.

## Parabolic polarisation and GIT quotient

We have a natural embedding of $C$ into a product of Quot schemes $Q_{0} \times \prod_{a, i} Q_{a, i}$ for quotients of the type $F^{r+1}(0,-N)^{\oplus n} \rightarrow E$ and of the type $\left.F^{r+1}(0,-N)^{\oplus n} \rightarrow E\right|_{Y} / F_{a, i+1}$ where $F_{a, i}=\oplus_{j \geq i} E_{a, i}$ is the parabolic filtration.
We now construct an analog of the Gieseker space (originally due to Gieseker, defined in $[\mathrm{Ge}]$ ), for our extra requirement that we need to encode not just a vector bundle $E$ but the logarithmic connection on it.
The space $Z$ we construct will be the total space of a projective fibration $Z \rightarrow A$, where $A$ is a union of certain finitely many components of the Picard scheme of $X$, to define which we first need the following lemma.

Lemma 4.2. There exists an integer $c \geq 0$ such that for each $0 \leq$ $i \leq r+1$ and for each $N$, the sheaf $F^{i}(N+c,-N)$ is generated by its global sections.

Proof. The graded pieces of of the filtration

$$
0 \subset \mathcal{O}_{X}=F^{0}(N+c,-N) \subset F^{1}(N+c,-N) \subset \ldots \subset F^{r+1}(N+c,-N)
$$

are $S y m^{i}\left(T_{X}\langle\log Y\rangle\right)(c)$, which are independent of $N$. The result follows.
Q.E.D.

Note that any quotient $F^{r+1}(0,-N)^{\oplus P(N)} \rightarrow E$, which represents a $\mathbb{C}$-valued point of $Q_{0}$, gives a quotient

$$
F^{r+1}(N+c,-N)^{\oplus P(N)} \rightarrow E(N+c)
$$

Let $A \subset P i c_{X / S}$ be the open and closed subscheme which parametrises all line bundles on $X$ whose first Chern class is equal to that of $\operatorname{det}(V)$ where $V$ is any vector bundle on $X$ with Hilbert polynomial $P(N+c)$. By choosing a $\mathbb{C}$-rational point $x \in X$ as base point, we get a unique Poincaré line bundle $\mathcal{L}$ on $X \times A$, trivialised on $x \times A$. Let $\left(p_{A}\right)_{*}(\mathcal{L})$ denote its direct image on $A$ under the projection $p_{A}: X \times A \rightarrow A$. The sheaf $\left(p_{A}\right)_{*}(\mathcal{L})$ will be a vector bundle by our choice of $A$, because all the higher cohomologies $H^{i}(X, L)$ for $i \geq 1$ vanish for line bundles $L$ represented by points of $A$. Let $Z$ be the projective scheme over $A$ defined as

$$
\begin{aligned}
& Z=\mathbf{P}\left[\operatorname{hom}\left(H^{0}\left(X, \wedge^{r}\left[F^{r+1}(N+c,-N)^{\oplus P(N)}\right]\right) \otimes_{\mathbb{C}} \mathcal{O}_{A},\left(p_{A}\right)_{*} \mathcal{L}\right)^{\vee}\right] \\
& =\operatorname{Proj} \operatorname{Sym}_{\mathcal{O}_{A}}\left[H^{0}\left(X, \wedge^{r}\left[F^{r+1}(N+c,-N)^{\oplus P(N)}\right]\right) \otimes_{\mathbb{C}}\left(p_{A *} \mathcal{L}\right)^{\vee}\right]
\end{aligned}
$$

The $\mathbb{C}$-valued points of $Z$ over a point $a \in A$ are represented by equivalence classes of pairs

$$
\left(L, \phi: H^{0}\left(X, \wedge^{r}\left[F^{r+1}(N+c,-N)^{\oplus P(N)}\right]\right) \rightarrow H^{0}(X, L)\right)
$$

where $L$ is a line bundle representing the point $a$, and $\phi$ is a non-zero linear map.
There is a natural linear representation of $S L_{n}$, where $n=P(N)$, on the vector space $H^{0}\left(X, \wedge^{r}\left[F^{r+1}(N+c,-N)^{\oplus P(N)}\right]\right)$, which induces an action of $S L_{n}$ on the pair $\left(Z, \mathcal{O}_{Z}(1)\right)$.
Recall that if $\mathcal{F}$ is a coherent torsion-free sheaf of rank 1 on a nonsingular variety $X$, then there exists (up to isomorphism) a unique pair ( $L, i$ ) consisting of an invertible sheaf $L$ on $X$ and a homomorphism $i$ : $\mathcal{F} \rightarrow L$ which is an isomorphism outside a closed subset of codimension $\geq 2$. The first Chern class of $L$ is equal to that of $\mathcal{F}$.
We define a morphism $\psi: Q_{0} \rightarrow Z$, by defining a natural transformation on functors of points as follows. For any $\mathbb{C}$-scheme $S$, we have a set map
$Q_{0}(S) \rightarrow Z(S)$ sending an $S$-valued point $F^{r+1}(0,-N)^{\oplus P(N)} \boxtimes \mathcal{O}_{S} \rightarrow E$ of $Q_{0}$ to the $S$-valued point of $Z$ represented by a pair $(L, \phi)$ where $L$ is the line bundle $\operatorname{det}(E(N+c))$ corresponding to the torsion free sheaf $\wedge^{r}(E(N+c)$ ) of generic rank 1 , and $\phi$ is the composite map $H^{0}\left(X, \wedge^{r}\left[F^{r+1}(N+c,-N)^{\oplus P(N)}\right]\right) \otimes_{\mathbb{C}} \mathcal{O}_{S} \rightarrow \pi_{S *} \wedge^{r}(E(N+c))$
$\rightarrow \pi_{S *} \operatorname{det}(E(N+c))$.
This map is clearly $S L_{n}$-equivariant, where $n=P(N)$.
Proposition 4.3. For all $N$ sufficiently large, the morphism $\psi$ : $C \rightarrow Z$ is an isomorphism of $C$ with a locally closed subscheme of $Z$.

Proof (Sketch) The scheme $C$ represents the functor which associates to each $S$ a pair $(E, u)$ where $E$ is a family of Deligne connection on $X \times S$ with prescribed residual eigenvalues data, and $u: \mathcal{O}_{S}^{P(N)} \rightarrow p_{X *} E$ is an isomorphism of vector bundles on $S$. By the use of flattening stratification, it can be seen that $Z$ has a locally closed subscheme $C^{\prime}$ which represents this same functor, such that $\psi$ induces natural isomorphism of $h_{C}$ with $h_{C^{\prime}}$. (Such a use of the flattening stratification was originally made by Grothendieck to construct Hilbert and Quot schemes as locally closed subschemes of Grassmannians. Also, Maruyama-Yokogawa use it to prove the analogous proposition in [9]). Now the result follows by Yoneda lemma.
Q.E.D.

We now define certain Grassmannians $G r_{a, i}$ as follows. Let $P_{a, i}(m)$ denote the Hilbert polynomial of $\left.E\right|_{Y} / F_{a, i}(E)$. Let $G r_{a, i}$ be the Grassmannian of quotients of $H^{0}\left(X, F^{r+1}(N+c,-N)^{\oplus P(N)}\right)$ of dimension $P_{a, i}(N+c)$. We have morphisms $\psi_{a, i}: C \rightarrow G r_{a, i}$ which at the level of functors of points sends a quotients $q_{S}: F^{r+1}(0,-N)^{\oplus P(N)} \boxtimes \mathcal{O}_{S} \rightarrow E$ to the quotient

$$
\psi_{a, i}\left(q_{S}\right): H^{0}\left(X, F^{r+1}(N+c,-N)^{\oplus P(N)}\right) \otimes_{\mathbb{C}} \mathcal{O}_{S} \rightarrow \pi_{S *}\left(\left.E\right|_{Y} / F_{a, i}(E)\right)(N+c)
$$

Together, these define a morphism

$$
\theta=\left(\psi, \psi_{a, i}\right): C \rightarrow Z \times \prod_{a, i} G r_{a, i}
$$

which is a locally closed embedding, as its composition with the projection on $Z$ is $\psi$ which is a locally closed embedding by Proposition 4.3. (Even though the morphism $\psi: C \rightarrow Z$ by itself is a locally closed embedding, we need the other factors $G r_{a, i}$ to get the polarisation right
in order that parabolic semi-stability of a Deligne connection will correspond to GIT semistability of points of $\left.Z \times \prod_{a, i} G r_{a, i}\right)$.

Remark 4.4. Unlike in the moduli construction of [1] or [9], we do not have properness of the morphism $C^{\text {par ss }} \rightarrow\left(Z \times \prod_{a, i} G r_{a, i}\right)^{s s}$.

## Linearisation of action

For simplicity, we assume each $\alpha_{a, i}$ is rational. This entails no loss of generality, as originally shown in the paper of Mehta-Seshadri [M-S]. Let $L_{a, i}$ denote the positive generator of $\operatorname{Pic}\left(G r_{a, i}\right)$. With respect to these line bundles on the factors, give $Z \times \prod_{a, i} G r_{a, i}$ the polarisation

$$
\left(P(N+c) / r, \epsilon_{a, i} / \delta\right)
$$

where $\epsilon_{a, i}=\alpha_{a, i+1}-\alpha_{a, i}$ for $i=1, \ldots \ell(a)-1$ and $\epsilon_{a, \ell(a)}=1-\alpha_{a, \ell(a)}$, and $\delta=\operatorname{dim} H^{0}(X, D)$. (Except for the presence of the constants $c$ and $\delta$, the above is the same as the corresponding polarisation in MaruyamaYokogawa [9] page 94.) In terms of line bundles, our choice of a very ample line bundle $\mathcal{L}$ on $Z \times \prod_{a, i} G r_{a, i}$ is any line bundle of the form

$$
\mathcal{L}=\pi_{Z}^{*} \mathcal{O}_{Z}(m \cdot P(N+c) / r) \otimes\left(\otimes_{a, i} \pi_{a, i}^{*} L_{a, i}^{m \cdot \epsilon_{a, i} / \delta}\right)
$$

where $m \geq 1$ is any positive integer which clears all the denominators in the above formula.

The natural action of $S L_{n}$ (where $n=P(N)$ ) on $Z \times \prod_{a, i} G r_{a, i}$ lifts to the very ample line bundle $\mathcal{L}$. Hence we get open subschemes

$$
\left(Z \times \prod_{a, i} G r_{a, i}\right)^{s} \subset\left(Z \times \prod_{a, i} G r_{a, i}\right)^{s s} \subset Z \times \prod_{a, i} G r_{a, i}
$$

of GIT-stable and GIT-semistable points.
At this point, we would ideally like to prove the following:
4.5. When $N$ is sufficiently large, a Deligne connection $E$ is parabolic stable (or parabolic semistable) if and only if for any point $q$ : $F^{r+1}(0,-N)^{\oplus n} \rightarrow E$ of $C$, the image $\theta(q)$ in $Z \times \prod_{a, i} G r_{a, i}$ is GIT-stable (or GIT-semistable) with respect to the polarisation $\left(P(N+c) / r, \epsilon_{a, i} / \delta\right)$.
However, so far we can not prove this in general, but only under the additional assumption that we can choose $c$ to be 0 , which is the case when $T_{X}\langle\log Y\rangle$ is itself generated by global sections. (We expect to be able to eventually remove this assumption. One possible method may be via the 'triples' introduced by Inaba, Iwasaki and Saito [6].)

To prove the above, we calculate the Mumford weights $\mu(\psi(q), \lambda, \mathcal{L})$ corresponding to limits of orbits under 1-parameter subgroups $\lambda$ of $S L_{n}$.

## Calculation of Mumford weights

Lemma 4.6. Let $E$ be a logarithmic connection on $(X, Y)$ of rank $r$, and let $\mathcal{F} \subset E$ be a coherent sub- $\mathcal{O}_{X}$-module. Let $E^{\prime} \subset E$ be the $\mathcal{O}_{X}$ saturation of the image of $F^{r+1} \otimes \mathcal{F} \rightarrow E$. Then $E^{\prime}$ is a sub $\mathcal{D}_{X}\langle\log Y\rangle-$ module of $E$.

Proof. This is just Simpson [13] Lemma 3.2.
Q.E.D.

Remark 4.7. Let $V$ be a finite dimensional vector space over $\mathbb{C}$ together with a given linear action of $S L(V)$ on a projective variety $\left(Y, \mathcal{O}_{Y}(1)\right)$. Consider any subspace $V^{\prime} \subset V$. and let $\operatorname{dim}\left(V^{\prime}\right)=n^{\prime}$. Choose a direct sum decomposition $V=V^{\prime} \oplus V^{\prime \prime}$. Let $\operatorname{dim}(V)=n$ and $\operatorname{dim}\left(V^{\prime}\right)=n^{\prime}$ so $\operatorname{dim}\left(V^{\prime \prime}\right)=n-n^{\prime}$. Let $\lambda: \mathbf{G}_{m} \rightarrow S L(V)$ be the 1-parameter sub group, defined by $\lambda(t)=\left(t^{n^{\prime}-n} 1_{V^{\prime}}, t^{n^{\prime}} 1_{V^{\prime \prime}}\right)$. Then any point $y \in Y$ the limit $y_{0}=\lim _{t \rightarrow 0} \lambda(t) y$ is independent of the choice of the supplement $V^{\prime \prime}$. Moreover, the Mumford weight $\mu\left(y, \lambda, \mathcal{O}_{Y}(1)\right)$ (which is by definition the weight of the character by which $\lambda$ acts on the fiber of $\mathcal{O}_{Y}(1)$ at $\left.y_{0}\right)$ is also independent of the choice of the supplement $V^{\prime \prime}$.
Let $q: F^{r+1}(0,-N) \otimes V \rightarrow E$ be a point of $C$, where $V=\mathbb{C}^{n}$ with $n=P(N)$. Let $\psi(q) \in Z(\mathbb{C})$ be its image, which is represented by the linear map $H^{0}\left(X, \wedge^{r}(D \otimes V)\right) \rightarrow H^{0}(X, L)$ where $D=F^{r+1}(N+c,-N)$ and $L=\operatorname{det}(E(N+c))$. Let $M^{\prime} \subset E$ be the $\mathcal{O}_{X}$-submodule which is the image of $F^{r+1}(0,-N) \otimes V^{\prime} \rightarrow E$ under $q$, and let $E^{\prime} \subset E$ be the $\mathcal{O}_{X}$-saturation of $M^{\prime}$. By Lemmas 4.6 and $3.13, E^{\prime}$ is a vector subbundle which is a sub-Deligne connection of $E$. Let $r^{\prime}=\operatorname{rank}\left(E^{\prime}\right)$.
The decomposition $V=V^{\prime} \oplus V^{\prime \prime}$ gives a decomposition

$$
\wedge^{r}(D \otimes V)=\oplus_{i+j=n} \wedge^{i}\left(D \otimes V^{\prime}\right) \otimes \wedge^{j}\left(D \otimes V^{\prime \prime}\right)
$$

Then as in the theory of Gieseker, the limit $\lim _{t \rightarrow 0} \lambda(t) \psi(q)$ is represented by the point of $Z$ given by the composite $\mathbb{C}$-linear map

$$
\begin{aligned}
H^{0}\left(X, \wedge^{r}(D \otimes V)\right) & \rightarrow H^{0}\left(X, \wedge^{r^{\prime}}\left(D \otimes V^{\prime}\right) \otimes \wedge^{r-r^{\prime}}\left(D \otimes V^{\prime \prime}\right)\right) \\
& \rightarrow H^{0}\left(X, \operatorname{det}\left(E^{\prime}(N+c)\right) \otimes \operatorname{det}\left(\left(E / E^{\prime}\right)(N+c)\right)\right)
\end{aligned}
$$

As $\lambda(t)$ acts by $t^{n^{\prime}-n}$ on $V^{\prime}$ and $t^{n^{\prime}}$ on $V^{\prime \prime}$, it acts by the characters $t^{-r^{\prime}\left(n-n^{\prime}\right)}$ and $t^{\left(r-r^{\prime}\right) n^{\prime}}$ respectively on $\wedge^{r^{\prime}}\left(D \otimes V^{\prime}\right)$ and $\wedge^{r^{\prime}}\left(D \otimes V^{\prime \prime}\right)$,
and hence it acts by the character $t^{r^{\prime}\left(n-n^{\prime}\right)-\left(r-r^{\prime}\right) n^{\prime}}$ on the vector space $H^{0}\left(X, \wedge^{r^{\prime}}\left(D \otimes V^{\prime}\right) \otimes \wedge^{r-r^{\prime}}\left(D \otimes V^{\prime \prime}\right)\right)$. This implies the following:
4.8. Under the action of $\lambda$, the point $\psi(q)$ of $Z$ has the Mumford weight

$$
\mu\left(\psi(q), \lambda, \mathcal{O}_{Z}(1)\right)=r^{\prime} n-r n^{\prime}
$$

Next, we consider the point $\theta(q) \in Z \times \prod_{a, i} G r_{a, i}$. For any $\psi_{a, i}: C \rightarrow$ $G r_{a, i}$, recall that $\psi_{a, i}(q)$ is represented by the quotient
$\psi_{a, i}(q): H^{0}\left(X, F^{r+1}(N+c,-N) \otimes V\right) \rightarrow H^{0}\left(X,\left(\left.E\right|_{Y} / F_{a, i}(E)\right)(N+c)\right)$
where $V=\mathbb{C}^{n}$ where $n=P(N)$. Let $V=V^{\prime} \oplus V^{\prime \prime}$ and $\lambda$ the corresponding 1-parameter subgroup as above. Then we get $\lim _{t \rightarrow 0} \lambda(t) \psi_{a, i}(q)$ to be represented by the quotient which is the composite

$$
\begin{array}{r}
H^{0}(X, D \otimes V) \xrightarrow{\sim} H^{0}\left(X, D \otimes V^{\prime}\right) \oplus H^{0}\left(X, D \otimes V^{\prime \prime}\right) \\
\rightarrow \\
\psi_{a, i} H^{0}\left(X, D \otimes V^{\prime}\right) \oplus \frac{H^{0}\left(X,\left(\left.E\right|_{Y} / F_{a, i}(E)\right)(N+c)\right)}{\psi_{a, i} H^{0}\left(X, D \otimes V^{\prime}\right)}
\end{array}
$$

This implies the following:
4.9. If $m_{a, i}^{\prime}=\operatorname{dim}\left(\psi_{a, i} H^{0}\left(X, D \otimes V^{\prime}\right)\right)$, this gives the value for the Mumford weight

$$
\mu\left(\psi_{a, i}(q), \lambda, L_{a, i}\right)=\delta \cdot\left(m_{a, i}^{\prime} P(N)-P_{a, i}(N+c) n^{\prime}\right)
$$

where $\delta=h^{0}(X, D)$ and $P_{a, i}$ is the Hilbert polynomial of $\left.E\right|_{Y} / F_{a, i}(E)$.
Remark 4.10. In particular, if $E^{\prime} \subset E$ is a sub-connection of $E$, and $V^{\prime}=H^{0}\left(X, E^{\prime}(N)\right)$, then for any 1-parameter subgroup $\lambda$ of $S L_{n}$ defined by choosing a splitting $V=V^{\prime} \oplus V^{\prime \prime}$ (where the choice of $V^{\prime \prime}$ is arbitrary), the corresponding Mumford weight is $P(N)$ times
$\frac{r\left(E^{\prime}\right)}{r(E)} \operatorname{par} \chi(E, N)-\operatorname{par} \chi\left(E^{\prime}, N\right)-h^{0}\left(E^{\prime}(N)\right) \frac{P(N+c)}{P(N)}+h^{0}\left(E^{\prime}(N+c)\right)$
In particular, when $c=0$, the value of the Mumford weight takes the simple form

$$
P(N)\left(\frac{r\left(E^{\prime}\right)}{r(E)} \operatorname{par} \chi(E, N)-\operatorname{par} \chi\left(E^{\prime}, N\right)\right)
$$

Armed with the above calculations 4.8 and 4.9 of Mumford weights, in the case where we can take $c=0$, the rest of the proof of the statement 4.5 is the same as that of the corresponding Proposition 3.4 of Maruyama-Yokogawa [9].

The open subscheme $\left(Z \times \prod_{a, i} G r_{a, i}\right)^{s} \subset Z \times \prod_{a, i} G r_{a, i}$ has a geometric quotient $M=\left(Z \times \prod_{a, i} G r_{a, i}\right)^{s} / S L_{n}$. This is the moduli space for parabolic stable Deligne connections.

## §5. Regular holonomic modules when $Y$ is smooth

In this section we consider regular holonomic $\mathcal{D}$-modules whose characteristic variety is contained in $(X-Y) \bigcup N_{Y, X}^{*}$ where $Y$ is a smooth divisor. These are not $\mathcal{O}_{X}$-coherent in general, so the concept of a pre- $\mathcal{D}$-module was introduced in the paper [12] to give an $\mathcal{O}_{X}$-coherent description of these $\mathcal{D}$-modules, much as logarithmic connections give an $\mathcal{O}_{X}$-coherent description of meromorphic connections. We now show how to re-define the concept of a pre- $\mathcal{D}$-module and its semi-stability, so as to take care of the relationship with the topology of $N_{Y, X}$, which makes the resulting notion of semi-stability (whether ordinary or parabolic) much more inclusive.
In this new construction, we have to fix the residual eigenvalues with their intersection multiplicities. However, by Lemmas 2.2 and 2.4, this is automatic whenever the universal topological degrees of the various normal bundles are non-zero, which is generally the case in higher dimensions. Hence this is a very mild restriction.

## The sheaf of rings $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$

As $\mathcal{D}_{X}\langle\log Y\rangle$ preserves $I_{Y}$, we have $I_{Y} \mathcal{D}_{X}\langle\log Y\rangle=\mathcal{D}_{X}\langle\log Y\rangle I_{Y}$, which is therefore a 2-sided ideal. The quotient ring is the restriction $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$, which can be regarded as $\mathcal{O}$-module restriction of both the left and the right $\mathcal{O}_{X}$-module $\mathcal{D}_{X}\langle\log Y\rangle$ to $Y$.
Over $Y$, we have a short-exact sequence

$$
\left.0 \rightarrow \mathcal{O}_{Y} \rightarrow T_{X}\langle\log Y\rangle\right|_{Y} \rightarrow T_{Y} \rightarrow 0
$$

The image of $1 \in H^{0}\left(Y, \mathcal{O}_{Y}\right)$ defines a section $\mathbf{e} \in H^{0}\left(Y,\left.T_{X}\langle\log Y\rangle\right|_{Y}\right)$, which has the following description in local coordinates: if $\left(x_{1}, \ldots, x_{d}\right)$ are (analytic or étale) local coordinates on $X$, with $Y$ locally defined by $x_{1}=0$, then $\mathbf{e}$ is locally defined as the restriction to $Y$ of the logarithmic vector field $x_{1}\left(\partial / \partial x_{1}\right)$.

Definition 5.1. The Euler operator is the section of $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$ over $Y$ which is the image of the above section $\mathbf{e} \in H^{0}\left(Y,\left.T_{X}\langle\log Y\rangle\right|_{Y}\right)$ under the map induced by the inclusion $T_{X}\langle\log Y\rangle \hookrightarrow \mathcal{D}_{X}\langle\log Y\rangle$. We denote the Euler operator again by e.

Remark 5.2. The Euler operator $\mathbf{e}$ is central in $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$. This fact is easy to verify, and it has the following topological analog: in the space $N_{Y, X}-Y$ (or more generally, in the open subset $L-Y$ of the total space any complex line bundle $L$ on a base $Y$ (complement of the zero section) the fundamental loop in a fiber defines a central element of $\pi_{1}(L-Y)$.

Modules over $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$
For any $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-module $F$, the induced endomorphism $\mathbf{e}: F \rightarrow F$ will be called the residue endomorphism and it will be denoted by $\operatorname{res}(F)$.

Remark 5.3. (1) As $\mathbf{e}$ is central, $\operatorname{res}(F)$ is $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-linear, so its eigenvalues are constants (even without compactness of $Y$ ). (2) If $E$ is a logarithmic connection on $(X, Y)$, then the restriction $F=\left.E\right|_{Y}$ is naturally a $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-module, and the corresponding residues are the same.

Lemma 5.4. Suppose that $Y$ is smooth, with connected components $Y_{a}$. Let $F$ be an $\mathcal{O}_{Y}$-coherent $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-module. Then the following holds.
(1) $F$ is locally free as an $\mathcal{O}_{Y}$-module (but the rank $r_{a}$ of $\left.F\right|_{Y_{a}}$ could vary from component to component).
(2) Let $\alpha_{a, i}$ be the eigenvalues of $\operatorname{res}(F)$, and let $F=\oplus F_{a, i}$ be the decomposition into generalised eigensubbundles, with ranks $r_{a, i}$. Then the Chern character of $F_{a, i}$ is given by

$$
\operatorname{ch}\left(F_{a, i}\right)=r_{a, i} \exp \left(-\alpha_{a, i} c_{1}\left(N_{Y}\right)\right)
$$

In particular, the Hilbert polynomial of $F_{a, i}$ is of the form $\chi\left(F_{a, i}, m\right)=$ $r_{a, i} f_{a, i}(m)$ where the polynomial $f_{a, i}(m)$ is independent of $F_{a, i}$.
(3) Each $F_{a, i}$ is a semi-stable $\mathcal{D}_{X}\langle\log Y\rangle$-module (in both Gieseker and $\mu$ sense).

Proof. (1) Choose a local trivialization for $N_{Y}$. Then locally we get a flat connection on $\left.F\right|_{Y}$, showing it is locally free since it is given to be $\mathcal{O}_{Y}$-coherent.
(2) The pull-back of $F$ to the total space of $N_{Y}$ becomes a logarithmic connection on $\left(N_{Y}, Y\right)$. Now the earlier result about logarithmic connections applies to give the Chern character of the pull-back of $F$ on the total space of $N_{Y}$, and the result follows by restricting to the zero section of $N_{Y}$.
(3) Any $\mathcal{O}_{Y_{a}}$-coherent $\mathcal{D}_{X}\langle\log Y\rangle$-submodule of $F_{a, i}$ will be a vector subbundle, with the same residual eigenvalue. Hence the result follows from the description of its Hilbert polynomial given by (2).
Q.E.D.

## Pre-D-modules on $(X, Y)$, semistability

The following is our modified definition of a pre- $\mathcal{D}$-module. Here the modification is minor, the main modification is in the definition of semistability (Definition 5.7).

Definition 5.5. Let $Y$ be a smooth divisor on the A pre- $\mathcal{D}$ module on $(X, Y)$ is a tuple $(E, F, s, t)$ where
(i) $E$ is a Deligne connection on $(X, Y)$,
(ii) $F$ is an $\mathcal{O}_{Y}$-coherent $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-module on $Y$ such that any eigenvalue of $\operatorname{res}(F)$ has real part in $[0,1)$.
(iii) $s:\left.F \rightarrow E\right|_{Y}$ and $t:\left.E\right|_{Y} \rightarrow F$ are $\mathcal{D}_{X}\langle\log Y\rangle$-linear with $s t=\operatorname{res}(E)$ and $t s=\operatorname{res}(F)$.

Remark 5.6. It follows from the above that the $\mathcal{D}_{X}\langle\log Y\rangle$-linear homomorphisms $s$ and $t$ are isomorphisms on generalised eigensubbundles of res for all eigenvalues $\lambda \neq 0$.

Definition 5.7. We call pre- $\mathcal{D}$-module ( $E, F, s, t$ ) semistable (respectively, parabolic semistable) if the Deligne connection $E$ is semistable (respectively, parabolic semistable) as a $\mathcal{D}_{X}\langle\log Y\rangle$-module as defined in [10] (respectively, with its natural parabolic structure as defined earlier).

Remark 5.8. The difference in this new definition and the old one of [12] is that now we do not put any semistability condition on the $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-module $F$, while earlier we had demanded that $F$ should be semistable as a $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-module.

## Relation with $\mathcal{D}$-modules

We now describe how a pre- $\mathcal{D}$-module gives rise to a $\mathcal{D}$-module. Let $(E, F, s, t)$ be a pre- $\mathcal{D}$-module on $(X, Y)$. Then we get a $\mathcal{D}_{X}\langle\log Y\rangle$ submodule

$$
E \oplus_{s} F \subset E \oplus F
$$

which by definition consists of all local sections $(u, v)$ with $\left.u\right|_{Y}=s(v)$. As $s$ is $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-linear, this condition indeed defines a $\mathcal{D}_{X}\langle\log Y\rangle$ submodule of $E \oplus F$.

Now let $M_{0}=E$ and $M_{1}=\mathcal{O}_{X}(Y) \otimes_{\mathcal{O}_{X}}\left(E \oplus_{s} F\right)$. As $\mathcal{O}_{X}(Y)$ is naturally a left $\mathcal{D}_{X}\langle\log Y\rangle$-module, the tensor product $M_{1}=\mathcal{O}_{X}(Y) \otimes_{\mathcal{O}_{X}}\left(E \oplus_{s} F\right)$ has a left $\mathcal{D}_{X}\langle\log Y\rangle$-module structure given by putting

$$
\xi(p \otimes q)=\xi(p) \otimes q+p \otimes \xi(q)
$$

where $\xi, p$, and $q$ are local sections respectively of $T_{X}\langle\log Y\rangle, \mathcal{O}_{X}(Y)$ and $E \oplus_{s} F$. We have a $\mathcal{D}_{X}\langle\log Y\rangle$-linear inclusion $M_{0} \hookrightarrow M_{1}$ defined by

$$
u \mapsto x^{-1} \otimes(x u, 0)
$$

We now define a $\mathbb{C}$-linear sheaf homomorphism $\nabla: M_{0} \rightarrow \Omega_{X}^{1} \otimes M_{1}$ by putting

$$
\nabla_{\eta}(u)=x^{-1} \otimes((x \eta)(u), \eta(x) t(u \mid Y))
$$

for any local section $\eta$ of the tangent sheaf $T_{X}$. This is compatible with given $\mathcal{D}_{X}\langle\log Y\rangle$-structures, in the sense that if $\eta$ is a section of $T_{X}\langle\log Y\rangle \subset T_{X}$, then $\nabla_{\eta}(u)$ equals the image of $\eta(u) \in M_{0}$ under the inclusion $M_{0} \hookrightarrow M_{1}$.
Finally, we define $M$ to be the left $\mathcal{D}_{X}$ module which is the quotient of $\mathcal{D}_{X} \otimes_{\mathcal{D}_{X}\langle\log Y\rangle} M_{1}$ by its submodule generated by all elements of the type $\eta \otimes u-1 \otimes \eta(u)$.

Proposition 5.9. Given any pre-D-module $(E, F, s, t)$, the associated $\mathcal{D}_{X}$-module $M$ is regular holonomic, with characteristic variety contained in $C_{X, Y}$. The following properties hold.
(i) The module $M$ is a non-singular connection if and only if $F=0$.
(ii) The module $M$ is a meromorphic connection if and only if $E \neq 0$ and $s:\left.F \rightarrow E\right|_{Y}$ is an isomorphism.
(iii) The module $M$ is set-theoretically supported on $Y$ if and only if $E=0$.
Moreover, given any regular holonomic $\mathcal{D}_{X}$-module $N$ with characteristic variety contained in $C_{X, Y}$, there exists (up to unique isomorphism) a unique pre- $\mathcal{D}$-module $(E, F, s, t)$ such that the associated $\mathcal{D}_{X}$-module $M$ is isomorphic to $N$.

Proof (Sketch, also see [11]) The module $M$ has a filtration $F^{i} M$ which is 'good' with respect to the filtration $F^{i} \mathcal{D}_{X}$ of $\mathcal{D}_{X}$, defined by $F^{0} M=M_{0}, F^{1} M=M_{1}$, and $F^{i} M=\left(F^{i-1} \mathcal{D}_{X}\right) M_{1}$ for $i \geq 2$. The associated graded module over $\operatorname{Sym}^{*}\left(T_{X}\right)$ shows that $M$ is regular holonomic with the characteristic variety contained in $C_{X, Y}$. The
statements (i), (ii), (iii) are clear from the construction of $M$. The backward passage from a $\mathcal{D}_{X}$-module to a pre- $\mathcal{D}$-module is via the existence of a V-filtration (which due to Malgrange [7] and Verdier [V]) $\ldots V^{i}(M) \subset V^{i+1}(M) \ldots$ on $M$, which is a certain filtration by $\mathcal{O}_{X^{-}}$ coherent $\mathcal{D}_{X}\langle\log Y\rangle$-modules. In terms of $V$-filtration, we put $E=$ $V^{0}(M), F=V^{1}(M) / V^{0}(M)$, and define $s$ and $t$ as the maps locally induced by $x$ and $\partial / \partial x$, where $x$ is a local defining equation for $Y$. The construction of a $V$-filtration depends on the choice of a fundamental domain in $\mathbb{C}$ for the exponential map $z \mapsto \exp (2 \pi i z)$. If we define the fundamental domain by the condition that $0 \leq \operatorname{Re}(z)<1$, then both $E=V^{0}(M)$ and $F=V^{1}(M) / V^{0}(M)$ will have all real parts of residual eigenvalues in $[0,1)$ as desired, showing that $(E, F, s, t)$ is a pre- $\mathcal{D}$-module as in Definition 5.5.
Q.E.D.

Remark 5.10 Infinitesimal rigidity As a consequence of infinitesimal rigidity for Deligne construction (see [10]), a pre- $\mathcal{D}$-module ( $E, F, s, t$ ) does not admit any nontrivial infinitesimal deformation such that the associated $\mathcal{D}_{X}$-module is constant.

## Moduli construction

A family $(E, F, s, t)_{S}$ of pre- $\mathcal{D}$-modules parametrised by a complex scheme $S$ will consist of a family of Deligne connections $E$ parametrised by $S$, a vector bundle $F$ on $Y \times S$ equipped with a structure of a $\left.\mathcal{D}_{X \times S / S}\langle Y \times S\rangle\right|_{Y \times S}$-module with all real parts of residual eigenvalues in $[0,1)$, and $\left.\mathcal{D}_{X \times S / S}\langle Y \times S\rangle\right|_{Y \times S}$-linear homomorphisms $s:\left.F \rightarrow E\right|_{Y \times S}$ and $t:\left.E\right|_{Y \times S} \rightarrow F$ with $s t=\operatorname{res}(E)$ and $t s=\operatorname{res}(F)$.
We now fix the ranks of $E$ and $F$, and the residual eigenvalues of $E$ with their multiplicities (in other words, the characteristic polynomial of $\operatorname{res}(E)$ is fixed). This automatically fixes the characteristic polynomial of $\operatorname{res}(F)$ by Remark5.6. Note that by [10], there exists a quasiprojective scheme $S$ over $\mathbb{C}$ parametrising a locally universal family of semistable logarithmic connections, together with the action of a reductive algebraic group $G$ such that a good quotient $S / / G$ exists in the sense of GIT, and is the moduli of semistable logarithmic connections. The scheme $U$ has a closed subscheme $R$ where the the characteristic polynomial of $\operatorname{res}(E)$ is the given one, and this subscheme is invariant under the action $G$. Hence a good quotient $R / / G$ exists, and it is the moduli of Deligne connections. Similarly, by the theory of $\Lambda$-modules developed in [13], there exists a quasi-projective scheme $R^{\prime}$ over $\mathbb{C}$ parametrising a locally universal family of semistable $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-modules on which the action of the Euler operator $\mathbf{e}$ is nilpotent, together with the action of a reductive algebraic group $G^{\prime}$ such that a good quotient $R^{\prime} / / G$ exists in
the sense of GIT, and is the moduli of semistable $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-modules with given residual eigenvalues. The extra data $(s, t)$ is parametrised by a scheme $V$ which is affine over $R \times R^{\prime}$, such that there is a natural lift of $G \times G^{\prime}$-action to $U$. By Ramanathan's lemma, a good quotient $\mathcal{M}$ exists. By its construction, $\mathcal{M}$ is coarse moduli for semistable pre- $\mathcal{D}$ modules ( $E, F, s, t$ ) with given ranks and residues. By construction, this is a quasi-projective scheme over $\mathbb{C}$.
If a moduli for parabolic semistable Deligne constructions is obtained as a good GIT quotient as above, then a similar construction will give the moduli for parabolic semistable pre- $\mathcal{D}$-modules $(E, F, s, t)$ with given ranks and residues.

## Points of the moduli

Proposition 5.11. The closed points of the moduli scheme of semistable pre-D-modules bijectively correspond to isomorphism classes of $\mathcal{D}_{X}$-modules of the form

$$
j!_{+}\left(\left.E\right|_{X-Y}\right) \oplus i_{+}(W)
$$

where $E$ is a polystable Deligne connection on $(X, Y), j_{!_{+}}\left(\left.E\right|_{X-Y}\right)$ is the minimal prolongation of the $\mathcal{D}_{X-Y}$-module $\left.E\right|_{X-Y}$ to a $\mathcal{D}_{X}$-module, $i: Y \rightarrow X$ is the closed embedding of $Y$ in $X, W$ is a semisimple nonsingular integrable connection on $Y$, and $i_{+}(W)$ is the $D_{X}$-module which is the direct image (in the sense of $\mathcal{D}$-modules) of the $\mathcal{D}_{Y}$-module $W$.

Remark 5.12. Let $E^{\prime}$ be kernel of $E \rightarrow \operatorname{coker}(\operatorname{res}(E))$, which is an elementary transform of $E$ in the sense of Maruyama [8]. Then $j_{!_{+}}\left(\left.E\right|_{X-Y}\right)$ is the $\mathcal{D}_{X}$-sub-module of $j_{*}\left(\left.E\right|_{X-Y}\right)$ generated by $\mathcal{O}_{X}(Y) \otimes$ $E^{\prime}$.

Proof of 5.11: Any Deligne connection $E$ gives rise to a pre- $\mathcal{D}$ module

$$
P_{E}=\left(E, \operatorname{im}(\operatorname{res}(E)), \theta_{E}, \operatorname{res}(E)\right)
$$

where $\left.\operatorname{im}(\operatorname{res}(E)) \subset E\right|_{Y}$ is naturally a sub- $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-module with inclusion $\theta_{E}:\left.\operatorname{im}(\operatorname{res}(E)) \hookrightarrow E\right|_{Y}$, and $\operatorname{res}(E):\left.E\right|_{Y} \rightarrow \operatorname{im}(\operatorname{res}(E))$ is the map induced by $\operatorname{res}(E):\left.\left.E\right|_{Y} \rightarrow E\right|_{Y}$.
Similarly, an integrable connection $W$ on $Y$ corresponds to a pre- $\mathcal{D}$ module

$$
Q_{W}=(0, W, 0,0)
$$

In terms of the correspondence between pre- $\mathcal{D}$-modules and $\mathcal{D}$-modules, to prove the above proposition we have to show that the closed points
of $\mathcal{M}$ are in bijection with the isomorphism classes of pre- $\mathcal{D}$-modules of the type

$$
P_{E} \oplus Q_{W}
$$

where $E$ is a polystable Deligne connection with the given rank and residual eigenvalues, and $W$ is an integrable connection on $Y$.
Note that closed orbits in $R$ and $R^{\prime}$ exactly correspond to polystable modules, which in the case of (non-singular) connections on $Y$ means those which are semi-simple (monodromy is completely reducible). So the set of isomorphism classes of pre- $\mathcal{D}$-modules of the type $P_{E} \oplus Q_{W}$, where $E$ is polystable and $W$ semisimple, injects into the set of closed points of the moduli. It remains to show that these are all the points.

For this, given any pre- $\mathcal{D}$-module $(E, F, s, t)$, let $W^{\prime}=\operatorname{ker}(s)$ with inclusion $\alpha: W^{\prime} \hookrightarrow F$ and $W^{\prime \prime}=\operatorname{im}(s) / \operatorname{im}(\operatorname{res}(E))$ with quotient map $\beta: \operatorname{im}(s) \rightarrow W^{\prime \prime}$. These vector bundles on $Y$ which are $\mathcal{D}_{Y}$-modules, and the maps $\alpha$ and $\beta$ are $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-linear. We have the following commutative diagram with short-exact columns.

$$
\begin{array}{ccccc}
0 & \rightarrow & W^{\prime} & \rightarrow & 0 \\
\downarrow & & \alpha \downarrow & & \downarrow \\
\left.E\right|_{Y} & \xrightarrow{\rightarrow} & F & \xrightarrow{t} & \left.E\right|_{Y} \\
\| & & s \downarrow & & \| \\
\left.E\right|_{Y} & \xrightarrow{\operatorname{res}(E)} & \operatorname{im}(s) & \hookrightarrow & \left.E\right|_{Y}
\end{array}
$$

This allows us to construct a family of pre-D-modules parametrised by the affine line $\mathbf{A}_{\mathbb{C}}^{1}$, which outside the origin is the constant family corresponding to ( $E, F, s, t$ ) and at origin restricts to $\left(0, W^{\prime}, 0,0\right) \oplus$ $(E, \operatorname{im}(s), \nu, \operatorname{res}(E))$ where $\nu:\left.\operatorname{im}(s) \hookrightarrow E\right|_{Y}$ is the inclusion.
Next, we have the commutative diagram with short-exact columns

$$
\begin{array}{ccccc}
\left.E\right|_{Y} & \xrightarrow[\operatorname{res}(E)]{\rightarrow} & \operatorname{im}(\operatorname{res}(E)) & \hookrightarrow & \left.E\right|_{Y} \\
\| & & \downarrow & & \downarrow \\
\left.E\right|_{Y} & \xrightarrow{s} & \operatorname{im}(s) & \hookrightarrow & \left.E\right|_{Y} \\
\downarrow & & \beta \downarrow & & \downarrow \\
0 & \rightarrow & W^{\prime \prime} & \rightarrow & 0
\end{array}
$$

Therefore there exists a family of pre- $\mathcal{D}$-modules parametrised by $\mathbf{A}_{\mathbb{C}}^{1}$, which outside the origin is the constant family $(E, \operatorname{im}(s), \nu, \operatorname{res}(E))$ and at origin restricts to $\left(E, \operatorname{im}(\operatorname{res}(E)), \theta_{E}, \operatorname{res}(E)\right) \oplus\left(0, W^{\prime \prime}, 0,0\right)$ where by definition $\theta_{E}:\left.\operatorname{im}(\operatorname{res}(E)) \hookrightarrow E\right|_{Y}$ is the inclusion.

Hence by the separatedness of the quasi-projective moduli scheme $\mathcal{M}$, it follows that the original semistable pre- $\mathcal{D}$-module $(E, F, s, t)$ and the semistable pre- $\mathcal{D}$-module $\left(E, \operatorname{im}(\operatorname{res}(E)), \theta_{E}, \operatorname{res}(E)\right) \oplus\left(0, W^{\prime} \oplus W^{\prime \prime}, 0,0\right)$ are represented by the same point of $\mathcal{M}$.
Next, let $0 \subset E_{1} \subset \ldots \subset E_{\ell}=E$ be an $S$-filtration of $E$, that is, the $E_{i}$ are sub Deligne connections which have the same reduced Hilbert polynomial as $E$. Let $\operatorname{Gr}(E)$ be the associated graded Deligne connection. Hence there exists a 1-parameter family of logarithmic connections which is generically $E$ and restricts to $\operatorname{Gr}(E)$ at the origin. This gives a 1-parameter family of pre- $\mathcal{D}$-modules which is generically $(E, \operatorname{im}(\operatorname{res}(E)), \theta, \operatorname{res}(E))$ and restricts at the origin to the object
$\left(\operatorname{Gr}(E), \operatorname{Gr}\left(\operatorname{im}(\operatorname{res}(E)), \operatorname{Gr}\left(\theta_{E}\right), \operatorname{Gr}(\operatorname{res}(E))\right)\right.$. In turn the latter deforms to a pre- $\mathcal{D}$-module

$$
\left(\operatorname{Gr}(E), \operatorname{im}(\operatorname{res}(\operatorname{Gr}(E))), \theta_{\operatorname{Gr}(E)}, \operatorname{res}(\operatorname{Gr}(E))\right) \oplus\left(0, W^{\prime \prime \prime}, 0,0\right)
$$

where by definition $\theta_{\operatorname{Gr}(E)}:\left.\operatorname{im}(\operatorname{res}(\operatorname{Gr}(E))) \hookrightarrow \operatorname{Gr}(E)\right|_{Y}$ is the inclusion (which does not equal $\operatorname{Gr}\left(\theta_{E}\right)$ in general).
If $W$ is the semisimplification of $W^{\prime} \oplus W^{\prime \prime} \oplus W^{\prime \prime \prime}$, then it follows that ( $E, F, s, t$ ) is represented by the closed point corresponding to the polystable pre- $\mathcal{D}$-module $P_{\operatorname{Gr}(E)} \oplus Q_{W}$. This completes the proof of the proposition.
Q.E.D.

## §6. Case of a normal crossing divisor

In this last section, we consider the general case where $Y$ is a normal crossing divisor. To carry out the general theory, it is not necessary to assume that the irreducible components of $Y$ are smooth. We can in that case set up a system of finite étale Galois covers of the normalisations of the closures of the strata defined by $Y$ as in [11], and carry out the discussion below. However, just to keep the notation simple in this article, we make the assumption that $Y$ is the union of two smooth divisors $Y=Y_{1} \bigcup Y_{2}$, which intersect transversely along $Z=Y_{1} \bigcap Y_{2}$ which is a smooth connected codimension 2 subvariety of $X$. This simplified situation is already adequate to exhibit the changes we make in the definition of pre- $\mathcal{D}$-modules and in the definition of their semistability.
Let $\left.\mathbf{e}_{1} \in H^{0}\left(Y_{1}, \mathcal{D}_{X}\langle\log Y\rangle\right)\right|_{Y_{1}}$ and $\left.\mathbf{e}_{2} \in H^{0}\left(Y_{2}, \mathcal{D}_{X}\langle\log Y\rangle\right)\right|_{Y_{2}}$ denote the respective Euler operators. Any point of $Z$ has an analytic or étale coordinate neighbourhood in $X$ with coordinates $x_{1}, \ldots, x_{d}(d=$ $\operatorname{dim}(X))$ in which $Y$ is defined by $x_{1} x_{2}=0$. Then $\mathbf{e}_{1}$ is locally defined by $x_{1}\left(\partial / \partial x_{1}\right)$ and $\mathbf{e}_{2}$ is defined by $x_{2}\left(\partial / \partial x_{2}\right)$.

We are now ready to state the new definition of a pre- $\mathcal{D}$-module and its semi-stability, in the above simple set-up.

Definition 6.1. A pre- $\mathcal{D}$-module on $(X, Y)$ consists of the following data, satisfying certain conditions.
(1) A $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y \text {-module }} E$ on $(X, Y), \mathcal{D}_{X}\langle\log Y\rangle$-modules $F_{1}$ and $F_{2}$ on $Y_{1}$ and $Y_{2}$, and a $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y}$-module $G$ on $(X, Y)$ such that $E$ is a vector bundle on $X, F_{1}$ and $F_{2}$ are vector bundles on $Y_{1}$ and $Y_{2}$, and $G$ is a vector bundle on $Z$.
(2) For $a=1$, 2, we are given $\left.\mathcal{D}_{X}\langle\log Y\rangle\right)\left.\right|_{Y_{a}}$-linear maps $s_{a}:\left.F_{a} \rightarrow E\right|_{Y_{a}}$ and $t_{a}:\left.E\right|_{Y_{a}} \rightarrow F_{a}$.
(3) For $a=1,2$, we are given $\left.\mathcal{D}_{X}\langle\log Y\rangle\right)\left.\right|_{Z}$-linear maps $s_{b}^{\prime}:\left.G \rightarrow F_{a}\right|_{Z}$ and $t_{b}^{\prime}:\left.F_{a}\right|_{Z} \rightarrow G$.
The above data should satisfy the following conditions:
(4) Consider the endomorphism $\operatorname{res}_{a}(E)$ of $\left.E\right|_{Y_{a}}$ induced by $\mathbf{e}_{a}$, the endomorphism $\operatorname{res}_{a}\left(F_{a}\right)$ of $F_{a}$ induced by $\mathbf{e}_{a}$, the endomorphism $\operatorname{res}_{b}\left(F_{a}\right)$ of $\left.F_{a}\right|_{Z}$ induced by $\mathbf{e}_{b}$ for $a \neq b$, and the endomorphism of $\operatorname{res}_{a}(G)$ of $G$ induced by $\mathbf{e}_{a}$, for $a=1,2$. All the eigenvalues of these generalised residue endomorphisms should have their real parts in $[0,1$ ). (In particular, $E$ is a Deligne connection.)
(5) We should have $s_{a} t_{a}=\operatorname{res}_{a}(E), t_{a} s_{a}=\operatorname{res}_{a}\left(F_{a}\right), s_{b}^{\prime} t_{b}^{\prime}=\operatorname{res}_{b}\left(F_{a}\right)$ for $a \neq b$, and $t_{b}^{\prime} s_{b}^{\prime}=\operatorname{res}_{b}(G)$.
(6) The following commutativity conditions should hold over $Z$ for $a \neq b$ :

$$
\begin{align*}
\left(\left.s_{b}\right|_{Z}\right) \circ s_{a}^{\prime} & =\left(\left.s_{a}\right|_{Z}\right) \circ s_{b}^{\prime}:\left.G \rightarrow E\right|_{Z}  \tag{1}\\
t_{b}^{\prime} \circ\left(\left.t_{a}\right|_{Z}\right) & =t_{a}^{\prime} \circ\left(t_{b} \mid Z_{Z}\right):\left.E\right|_{Z} \rightarrow G  \tag{2}\\
s_{a}^{\prime} \circ t_{b}^{\prime} & =\left(\left.t_{b}\right|_{Z}\right) \circ\left(\left.s_{a}\right|_{Z}\right):\left.\left.F_{a}\right|_{Z} \rightarrow F_{b}\right|_{Z} \tag{3}
\end{align*}
$$

We say that a pre-D-module as defined above is semi-stable if the following two conditions hold:
(1) The logarithmic connection $E$ is semistable.
(2) The generalised eigensubbundle $F_{a, 0} \subset F_{a}$ for eigenvalue 0 of $\operatorname{res}_{a}\left(F_{a}\right)$ is a semistable $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Y_{a}}$-module for $a=1,2$.

Remark 6.2. The intersection $G_{0}$ of the generalised eigensubbundles of $G$ for eigenvalue 0 under $\operatorname{res}_{a}(G)$ and $\operatorname{res}_{b}(G)$ (which is indeed a vector subbundle by integrability) is automatically a semistable $\left.\mathcal{D}_{X}\langle\log Y\rangle\right|_{Z}$-module.

## Relation with $\mathcal{D}$-modules

A regular holonomic $\mathcal{D}$-module $M$ on $X$ whose characteristic variety is contained in $C_{X, Y}$ corresponds uniquely to a pre- $\mathcal{D}$-module as defined above. Over a polydisk, this is essentially the content of the main theorem of the paper of Galligo, Granger, Maisonobe [5]. This is best expressed in terms of a $V$-filtration on $M$ (see Section 4.3, page 16 [11]), which actually produces a pre- $\mathcal{D}$-module as defined above (only the new definition of a pre- $\mathcal{D}$-module was missing earlier!).
Again, this bijective correspondence is infinitesimally rigid as shown in [11].

## Construction of moduli

We fix the ranks of $E, F_{a}, G$ and also the characteristic polynomials of the various residue endomorphisms $\operatorname{res}_{a}(E), \operatorname{res}_{a}\left(F_{a}\right), \operatorname{res}_{b}\left(F_{a}\right)$, and $\operatorname{res}_{a}(G)$, and the dimensions of intersections of their generalised eigensubbundles. In particular, this fixes the Hilbert polynomials of the bundles $E, F_{a}, G$, and those of the generalised eigensubbundles of the various residues.

By the theory of Simpson of moduli for $\Lambda$-modules together with the Lemma 3.13 above, there exist the following:
(1) A quasi-projective scheme $R$ together with action of a reductive algebraic group $H$ and a locally universal family of $\mathcal{O}_{X}$-locally free semistable $\mathcal{D}_{X}\langle\log Y\rangle$-modules $E$ with prescribed residual eigenvalues for $\operatorname{res}_{a}(E)$ over $Y_{a}$ and prescribed ranks for intersection of generalised eigensubbundles of $\operatorname{res}_{a}(E)$ and $\operatorname{res}_{b}(E)$ over $Z$, with lift of $H$-action to the family, such that a good quotient $R / / H$ exists, which is the moduli of semistable Deligne connections on $(X, Y)$ for the prescribed residual data.
(2) For $a=1,2$, a quasi-projective scheme $R_{a}^{\prime}$ with action of a reductive algebraic group $H_{a}^{\prime}$, with a locally family of $\mathcal{O}_{Y_{a}}$-locally free semistable $\mathcal{D}_{X}\langle\log Y\rangle$-modules $E$ with $\operatorname{res}_{a}\left(F_{a}\right)$ nilpotent, and prescribed residual eigenvalues for $\operatorname{res}_{b}\left(F_{a}\right)$ over $Z$ and prescribed ranks for intersection of generalised eigensubbundles of $\operatorname{res}_{a}\left(F_{b, 0}\right)$ and $\operatorname{res}_{b}\left(F_{a, 0}\right)$ over $Z$, with lift of $H$-action to the family, such that a good quotient $R_{a}^{\prime} / / H_{a}$ exists, which is the moduli of $\mathcal{O}_{Y_{a}}$-locally free semistable $\mathcal{D}_{X}\langle\log Y\rangle$-modules for the given residual data.
(3) A quasi-projective scheme $R^{\prime \prime}$ together with action of reductive algebraic groups $H^{\prime \prime}$ and a locally universal family of $\mathcal{O}_{Z}$-locally free semistable $\mathcal{D}_{X}\langle\log Y\rangle$-modules $G_{0}$ such that $\operatorname{res}_{a}(G)$ is nilpotent for $a=1,2$, with lift of $H^{\prime \prime}$-action to the family, such that a good quotient $R^{\prime \prime} / / H^{\prime \prime}$
exists, which is the moduli of $\mathcal{O}_{Z}$-locally free semistable $\mathcal{D}_{X}\langle\log Y\rangle$ modules $G_{0}$ for the given residual data.
Over the product $U=R \times R_{1}^{\prime} \times R_{2}^{\prime} \times R^{\prime \prime}$, we have an scheme $V$ with an affine morphism to $U$ and a natural lift of the action of $K=H \times$ $H_{1}^{\prime} \times H_{2}^{\prime} \times H^{\prime \prime}$, such that $V$ parametrises a locally universal family of semistable pre- $\mathcal{D}$-modules with the prescribed residual data. Then $V / / K($ a good quotient which exists by Ramanathan's lemma) is the desired moduli scheme $\mathcal{M}$ for semi-stable pre- $\mathcal{D}$-modules. It is quasiprojective over $\mathbb{C}$ by its construction.
A description of the points of $\mathcal{M}$, analogous to that of Proposition 5.11 is possible also in the general case, and a description of the tangent space to the moduli functor in terms of certain hypercohomologies can be given.

Remark 6.3. As over any perverse sheaf we have at most one semistable pre- $\mathcal{D}$-module in the new sense, which moreover is infinitesimally rigid as before, the Riemann-Hilbert morphism from $\mathcal{M}$ to the moduli of perverse sheaf is an open embedding. It can fail to be surjective as semistability is not automatic for pre- $\mathcal{D}$-modules.

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# The cohomology groups of stable quasi-abelian schemes and degenerations associated with the $E_{8}$-lattice 

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#### Abstract

. We study certain degenerate abelian schemes $\left(Q_{0}, L_{0}\right)$ that are GIT-stable in the sense that their SL-orbits are closed in the semistable locus. We prove the vanishing of the cohomology groups $H^{q}\left(Q_{0}, L_{0}^{n}\right)=$ 0 for $q, n>0$ for a naturally defined ample invertible sheaf $L_{0}$ on $Q_{0}$. When $n=1$, this implies that $H^{0}\left(Q_{0}, L_{0}\right)$, the space of global sections, is an irreducible module of the noncommutative Heisenberg group of $\left(Q_{0}, L_{0}\right)$.


## §1. Introduction

In 1970's Namikawa [Nw76] and Nakamura [Nr75] studied the problem of compactifying the moduli $A_{g}$ of abelian varieties over $\mathbf{C}$, and their papers introduced a certain class of degenerate abelian varieties. In 1990's in their joint work [AN99] Alexeev and Nakamura again discussed the same problem of compactifying $A_{g}$ over any field in an algebraic manner, and the objects they studied are nearly the same as those studied by Namikawa and Nakamura in 1970's.

After their joint work [AN99] Alexeev and Nakamura independently constructed respectively reasonable compactifications, using almost the same class of degenerate abelian varieties or schemes as above. Alexeev's moduli $\bar{A}_{g}$ [A02] is a coarse moduli of a certain kind of principally polarized reduced, possibly degenerate, abelian varieties with (continuous) group action. On the other hand Nakamura's moduli [Nr99] is a fine moduli $S Q_{g, K}$ of polarized, possibly nonreduced, possibly degenerate, abelian schemes which are GIT-stable in the sense that their

[^2]SL-orbits are closed in the semistable locus, though their stabilizer subgroups of SL could be of infinite order. The moduli $S Q_{g, K}$ compactifies the moduli scheme $A_{g, K}$ of abelian varieties with certain noncommutative level $K$-structures (to be more precise, of abelian varieties, each with a very ample invertible sheaf linearized with regards to the Heisenberg group $G(K)$ ) where $K$ is a certain symplectic, sufficiently large finite abelian group. We note that both $\overline{A_{g}}$ and $S Q_{g, K}$ are projective over $\mathbf{Z}$ or $\mathbf{Z}\left[\zeta_{N}, \frac{1}{N}\right]$ respectively where $N=\sqrt{|K|}$. Since $S Q_{g, K}$ is a fine moduli, there is a universal family over $S Q_{g, K}$ of polarized generalized abelian schemes of dimension $g$ so that any fibre of the family over a geometric point of $S Q_{g, K}$ represents an isomorphism class corresponding to the geometric point. We call any fibre of the family a projectively stable quasi-abelian scheme, or simply a PSQAS. We note that a PSQAS is singular if and only if the PSQAS lies over the boundary $S Q_{g, K} \backslash A_{g, K}$.

The purpose of this article is first of all to prove the vanishing of certain cohomology groups of PSQASes. This solves a conjecture raised by [Nr99, section 9] in the affirmative. The second purpose of this article is to study PSQASes associated with the $E_{8}$ lattice. The structures of some of PSQASes over the boundary of $S Q_{g, K}$ are very complicated when they are nonreduced. Any even unimodular definite lattice provides us with a nonreduced PSQAS. Since there are at least $8 \cdot 10^{7}$ inequivalent even unimodular definite lattice for $g=32$, there could be a lot of nonreduced PSQASes. The first nontrivial example of a nonreduced PSQAS is provided by $E_{8}$ [AN99], which we will study in detail in the second half of the article. As a matter of fact, this detailed study of the $E_{8}$-case removes the last psychological obstacle for our complete computation of the cohomology groups of PSQASes in the general case.

The article is organized as follows. The first two sections 2 and 3 recall the basic facts about Delaunay decompositions and degenerating families of abelian varieties. We construct a particular class of degenerating families $(Q, L)$ of polarized abelian varieties over complete discrete valuation rings, whose closed fibres $\left(Q_{0}, L_{0}\right)$ are nothing but the PSQASes mentioned above. The sections 4,5 and 6 are devoted to studying closed fibres $\left(Q_{0}, L_{0}\right)$ of the families $(Q, L)$, in particular, their cohomology groups $H^{q}\left(Q_{0}, L_{0}^{n}\right)$ in the general case including the case where $Q_{0}$ is nonreduced. In the section 5 , the following Theorem 1 is proved, while in the section 4 an outline of the proof is explained. A key result for proving Theorem 1 is proved in the section 6 .

Theorem 1. Let $\left(Q_{0}, L_{0}\right)$ be a PSQAS. Then $H^{q}\left(Q_{0}, L_{0}^{n}\right)=0$ for $q>0$ and $n>0$.

An important corollary to it is the following

Theorem 2. Let $K$ be a finite symplectic abelian group of order $N^{2}$. Let $k$ be any field over $\mathbf{Z}\left[\zeta_{N}, \frac{1}{N}\right]$. Let $G(K)$ be a noncommutative finite Heisenberg group, namely a central extension of $K$ by the group $\mu_{N}$ of $N$-th roots of unity. Let $\left(Q_{0}, L_{0}\right)$ be a PSQAS over $k$ with a level $G(K)$-structure in the sense of $[\mathrm{Nr} 99]$. Then $H^{0}\left(Q_{0}, L_{0}\right)$ is an irreducible $G(K)$-module of weight one.

Let $L$ be the natural polarization of the universal family of PSQASes over $S Q_{g, K}$. By Theorem 1 the 0 -th direct images of $L^{n}(n \geq 1)$ are locally free sheaves over $S Q_{g, K}$, whose determinant bundles are expected to give rise to the most natural ample invertible sheaves of $S Q_{g, K}$.

The second half of the article starting from the section 7 is devoted to studying a PSQAS $Q_{0}$ associated with $E_{8}$. Among other things the nilradical of $O_{0, Q_{0}}$ is calculated completely in the section 11. This calculation helps us to get convinced that nilpotent elements of $O_{0, Q_{0}}$ have large support and that therefore the cohomology groups $H^{q}\left(Q_{0}, L_{0}^{n}\right)$ will behave in the same manner as those of nonsingular abelian varieties. This was psychologically a key step to the proof of Theorem 1.

We would like to thank Professor K. Shinoda for his many advices on $E_{8}$ during the preparation of the article.

## $\S 2 . \quad$ Basic facts about Delaunay decompositions

Let $\mathbf{Z}$ be the set of integers, $\mathbf{Z}_{0}$ the set of nonnegative integers, $\mathbf{Q}$ the set of rational numbers, $\mathbf{R}$ the set of real numbers, and $\mathbf{R}_{0}$ the set of nonnegative real numbers. Let $X$ be a lattice of rank $g, B$ an integral positive definite symmetric bilinear form on $X \times X$. Let $X_{\mathbf{Q}}=X \otimes_{\mathbf{Z}} \mathbf{Q}$ and $X_{\mathbf{R}}=X \otimes_{\mathbf{Z}} \mathbf{R}$. The bilinear form $B$ determines a distance $\left\|\|_{B}\right.$ on $X_{\mathbf{R}}$ by $\|x\|_{B}:=\sqrt{B(x, x)}\left(x \in X_{\mathbf{R}}\right)$. For an arbitrary $\alpha \in X_{\mathbf{R}}$ we say that a lattice element $a \in X$ is $\alpha$-nearest if

$$
\|a-\alpha\|_{B}=\min \left\{\|b-\alpha\|_{B} ; b \in X\right\}
$$

We define a (closed) $B$-Delaunay cell $\sigma$ (or simply a Delaunay cell if $B$ is understood) to be the closed convex hull of all lattice elements which are $\alpha$-nearest for some $\alpha \in X_{\mathbf{R}}$ for a fixed $\alpha$. Note that for a given Delaunay cell $\sigma, \alpha \in \sigma$ is uniquely determined by $\sigma$, which we call the hole of $\sigma$ and denote by $\alpha(\sigma)$. All the $B$-Delaunay cells constitute a locally finite decomposition of $X_{\mathbf{R}}$ into infinitely many bounded convex polyhedra, which we call the Delaunay decomposition $\mathrm{Del}_{B}$.

Definition 2.1. In what follows we fix the bilinear form $B$, so we denote $B(x, y)$ simply by $(x, y), B(x, x)$ by $x^{2}$ and the norm $\|x\|_{B}$ by $\|x\|$ if no confusion is possible. Let Del $:=\operatorname{Del}_{B}$ be the Delaunay
decomposition on $X_{\mathbf{R}}$ defined by the distance $\|x\|:=\sqrt{B(x, x)}$. For any subset $T$ of $X_{\mathbf{R}}$ let $\operatorname{Del}(T)$ be the set of all Delaunay cells containing $T$, and $\operatorname{Star}(T)$ the union of all $\sigma \in \operatorname{Del}(T)$. In particular, for any $c \in X, \operatorname{Del}(c)$ is the set of all the Delaunay cells containing $c \in X$ and $\operatorname{Star}(c)$ is the union of all $\sigma \in \operatorname{Del}(c)$. We note $\operatorname{Del}(c)=c+\operatorname{Del}(0)$, the translate of $\operatorname{Del}(0)$ by $c$. We denote by $\operatorname{Del}^{(k)}$ the set of Delaunay cells $\sigma \in \operatorname{Del}$ such that $\operatorname{dim} \sigma=k$. Let $\operatorname{Del}^{(k)}(T)=\operatorname{Del}(T) \cap \operatorname{Del}^{(k)}$. For a $\sigma \in \operatorname{Del}$, we define $\operatorname{Del}_{\sigma}$ to be the set of all faces of $\sigma$ and $\operatorname{Del}_{\sigma}^{(k)}:=$ $\operatorname{Del}^{(k)} \cap \operatorname{Del}_{\sigma}$. For $\tau \in \operatorname{Del}$, we define $\operatorname{Del}_{\sigma}(\tau):=\operatorname{Del}_{\sigma} \cap \operatorname{Del}(\tau)$ and $\operatorname{Del}_{\sigma}^{(k)}(\tau):=\operatorname{Del}^{(k)} \cap \operatorname{Del}_{\sigma}(\tau)$.

Definition 2.2. Let $D$ be a subset of $X_{\mathbf{R}}$. If $D$ contains the origin 0 , we define $C(0, D)$ to be the cone over $\mathbf{R}_{0}$ generated by $D$, and define $\operatorname{Semi}(0, D)$ to be the cone over $\mathbf{Z}_{0}$ generated by $D \cap X$. For any subset $S$ of $D$ we define $X(S)$ to be the subgroup of $X$ generated by $s-t$, $(\forall s, t \in S)$. We denote $X(S) \otimes \mathbf{R}$ by $X(S)_{\mathbf{R}}$. We also define

$$
\begin{aligned}
C(s, D): & =s+C(0, D-s) \quad(\text { for } s \in D) \\
C(S, D): & =\bigcup_{a \in X(S), s \in S \cap X}(a+C(s, D)) \\
& =X(S)+C\left(s_{0}, D\right) \quad\left(\forall s_{0} \in S\right) .
\end{aligned}
$$

If $S$ is a one-codimensional face of a $g$-dimensional convex polytope $D$ of $X_{\mathbf{R}}$, then $S$ spans a hyperplane of $X_{\mathbf{R}}$, which we denote by $H(S)$, and $C(S, D)$ is a closed half space of $X_{\mathbf{R}}$ containing $D$ bounded by $H(S)$.

In order to make this article as self-contained as possible. we give proofs for basic facts about Delaunay/Voronoi decompositions. See also [Nr99].

Definition 2.3. The Voronoi cell $V(0)$ at 0 is defined to be

$$
V(0)=\left\{\alpha \in X_{\mathbf{R}} ;\|y-\alpha\| \geq\|\alpha\| \text { for any } y \in X\right\}
$$

Lemma 2.4. For any $x \in X$ the following are equivalent:
(i) $\quad x \in 2 V(0) \cap X$, namely, $(y, y) \geq(x, y)$ for any $y \in X$,
(ii) $x \in \operatorname{Star}(0) \cap X$, namely, there is $\sigma \in \operatorname{Del}(0)$ such that $x \in$ $\sigma \cap X$.

Proof. Assume (i). Then $\|y-(x / 2)\| \geq\|x / 2\|$ for any $y \in X$, where the minimum of $\|y-(x / 2)\|$ is attained at $y=0$ and $x$. Hence (ii) follows.

Conversely if there is a Delaunay cell $\sigma \in \operatorname{Del}(0)$ such that $x \in \sigma \cap X$, then there is an $\alpha \in X_{\mathbf{R}}$ such that $\|y-\alpha\|^{2} \geq\|\alpha\|^{2}$ and $\|x-\alpha\|^{2}=\|\alpha\|^{2}$.

Hence $\alpha \in V(0)$. By the first inequality we have $\|-y+x-\alpha\|^{2} \geq\left\|\alpha^{2}\right\|$ for any $y$, from which it follows that $\|y\|^{2} \geq 2(x-\alpha, y)$, namely, $x-\alpha \in V(0)$. Hence $x=\alpha+(x-\alpha) \in 2 V(0)$. This proves (i). This proves the lemma.
Q.E.D.

Lemma 2.5. Let $a_{i} \in \operatorname{Star}(0)(1 \leq i \leq n)$. Assume that there is $z(\neq 0) \in X$ such that $a_{1}+\cdots+a_{n}=m z$. Then $n \geq m$, equality holding if and only if $(z, z)=\left(a_{i}, z\right)$ for any $i$.

Proof. Since $a_{i} \in \operatorname{Star}(0)$, we have $y^{2} \geq\left(a_{i}, y\right)$ for any $y \in X$ by Lemma 2.4. In particular, $z^{2} \geq\left(a_{i}, z\right)$. It follows that $n z^{2} \geq\left(a_{1}+\cdots+\right.$ $\left.a_{n}, z\right)=m z^{2}$. Hence $n \geq m$. If $n=m$, then any inequality in the above is equality. This proves the lemma.
Q.E.D.

Definition 2.6. We say that $x_{1}, \cdots, x_{m} \in X\left(x_{i} \neq x_{j}\right)$ are cellmates if there is a Delaunay cell $\sigma \in$ Del that contains all of $x_{i}$. We say that $x_{1}, \cdots, x_{m} \in \operatorname{Star}(0)$ are cellmates at 0 if there is a Delaunay cell $\sigma \in \operatorname{Del}(0)$ that contains all of $x_{i}$.

Lemma 2.7. Let $\sigma$ be a Delaunay cell and $z(\neq 0) \in X$. Then
(i) $\quad \sigma \cap(m z+\sigma)=\emptyset$ for $m \geq 2$.
(ii) $\operatorname{Star}(0) \cap(m z+\operatorname{Star}(0))=\emptyset$ if $m \geq 3$.

Proof. Suppose that $c \in \sigma \cap X$ and $d=c+m z \in \sigma$ for some nonzero $z \in X$. Since $c$ and $d$ are cellmates, we have $c-d \in \operatorname{Star}(0)$. Hence $m z \in \operatorname{Star}(0)$. It follows from Lemma 2.5 that $m=1$. This proves (i).

Next we prove (ii). Suppose $\operatorname{Star}(0) \cap(m z+\operatorname{Star}(0)) \neq \emptyset$. Then there are $a, b$ and $z \in X$ such that $a-b=m z$ and $a, b \in \operatorname{Star}(0)$. Then by Lemma 2.5 we have $m \leq 2$. This proves the assertion. Q.E.D..

Lemma 2.8. (i) Let $\sigma \in \operatorname{Del}(0)$ and $b \in C(0, \sigma) \cap X$. If $b \notin \sigma \cap X$, then there is $a \in \sigma \cap X$ such that $(b-a, a)>0$.
(ii) If $x \notin \operatorname{Star}(0) \cap X$, then there exists $a \in \operatorname{Star}(0) \cap X$ such that $\|x\|^{2}>\|x-a\|^{2}+\|a\|^{2}$.
Proof. We prove (i). Let $b \in C(0, \sigma) \cap X$ and $\alpha(\sigma)$ the hole of $\sigma$. We assume $(b, a) \leq(a, a)$ for any $a \in \sigma \cap X$. Then we prove $b \in \sigma \cap X$. For this let $b=\sum_{i=1}^{r} r_{i} a_{i}$ for $a_{i} \in \sigma \cap X$ and some $r_{i} \geq 0$. We see

$$
(b, b)=\sum_{i=1}^{r} r_{i}\left(b, a_{i}\right) \leq \sum_{i=1}^{r} r_{i}\left(a_{i}, a_{i}\right)=2 \sum_{i=1}^{r} r_{i}\left(\alpha(\sigma), a_{i}\right)=2(\alpha(\sigma), b)
$$

whence $(b, b)=2(\alpha(\sigma), b)$. It follows $b \in \sigma \cap X$.
We shall prove (ii). Let $x \in X$. Since $\operatorname{Star}(0)$ contains an open neighborhood of the origin in $X_{\mathbf{R}}$, there is $\sigma \in \operatorname{Del}(0)$ such that $x \in$
$C(0, \sigma) \cap X \backslash \sigma$. By (i) there exists $a \in \sigma \cap X$ such that $(x-a, a)>0$. Hence $\|x\|^{2}>\|x-a\|^{2}+\|a\|^{2}$.

## Definition 2.9. We set

$$
\begin{aligned}
v(x) & =\min \left\{\frac{1}{2} \sum_{i=1}^{m}\left(x_{i}, x_{i}\right) ; x=x_{1}+\cdots+x_{m}, x_{i} \in X, m \geq 1\right\} \\
v(x, c) & =v(x)+(x, c)
\end{aligned}
$$

Lemma 2.10. Let $\sigma \in \operatorname{Del}(0)$ and $\alpha(\sigma) \in \sigma$ the hole of $\sigma$. Then $v(x) \geq(x, \alpha(\sigma))$ for any $x \in X$, equality holding iff $x \in \operatorname{Semi}(0, \sigma)$.

Proof. Choose $x_{i} \in X$ such that $x=x_{1}+\cdots+x_{m}$ and $v(x)=$ $\frac{1}{2} \sum_{i=1}^{m}\left(x_{i}, x_{i}\right)$. Then

$$
\sum_{i=1}^{m}\left(x_{i}, x_{i}\right) \geq 2 \sum_{i=1}^{m}\left(x_{i}, \alpha(\sigma)\right)=2(x, \alpha(\sigma))
$$

This proves $v(x) \geq(x, \alpha(\sigma))$. If $v(x)=(x, \alpha(\sigma))$, then we have $\left(x_{i}, x_{i}\right)=2\left(x_{i}, \alpha(\sigma)\right)$ for any $i$. The equality $\left(x_{i}, x_{i}\right)=2\left(x_{i}, \alpha(\sigma)\right)$ implies that $x_{i} \in \sigma \cap X$. This proves $x \in \operatorname{Semi}(0, \sigma)$.
Q.E.D.

## §3. Degenerating families of abelian varieties - general case

Let $R$ be a complete discrete valuation ring, $q$ a uniformizing parameter of $R, k(0)=R / q R$ and $k(\eta)$ the fraction field of $R, 0$ the closed point and $\eta$ the generic point of Spec $R$. The purpose of this section is to recall the (simplified) Mumford construction over $R$ [AN99]. See also [M72].

Let $X$ be a free Z-module of rank $g$ and $a(x) \in k(\eta)^{\times}:=k(\eta) \backslash\{0\}$ for any $x \in X$.

Definition 3.1. Let $b(x, y):=a(x+y) a(x)^{-1} a(y)^{-1}$. If the following conditions are satisfied, $\{a(x) ; x \in X\}$ is called a (Faltings-Chai's) degeneration data :
(i) $a(0)=1$,
(ii) $b(x, y)$ is a (multiplicatively) bilinear form on $X \times X$ with values in $k(\eta)^{\times}$,
(iii) $\quad B(x, y):=\operatorname{val}_{q} b(x, y)$, a positive definite symmetric bilinear form of $X \times X$.
Definition 3.2. Let $\{a(x) ; x \in X\}$ be a degeneration data and $A(x)=\operatorname{val}_{q} a(x)$. Let $\vartheta$ be an indeterminate over $R, R[\vartheta][X]$ the group algebra over $R[\vartheta]$ of the additive group $X\left(\simeq \mathbf{Z}^{g}\right)$. The algebra $R[\vartheta][X]$
is regarded as a graded algebra by $\operatorname{setting} \operatorname{deg}(\vartheta)=1$ and $\operatorname{deg}(a)=0$ for any $a \in R[X]$.

We define a graded subalgebra $\widetilde{R}$ of $R[\vartheta][X]$ by

$$
\widetilde{R}:=R\left[a(x) w^{x} \vartheta ; x \in X\right]=R\left[\xi_{x} \vartheta ; x \in X\right], \quad \xi_{x}:=q^{A(x)} w^{x}
$$

Let $\widetilde{Q}:=\operatorname{Proj}(\widetilde{R})$ Let $Y$ be a sublattice of $X$ of finite index. Then $Y$ acts on $\widetilde{Q}$ by

$$
S_{y}^{*}\left(a(x) w^{x} \vartheta\right)=a(x+y) w^{x+y} \vartheta \quad \text { for } y \in Y
$$

The invertible sheaf $O_{\widetilde{Q}}(1)$ is kept invariant by the action of $Y$.
Let $\widetilde{Q}_{\text {for }}$ be the formal completion of $\widetilde{Q}$ along $\widetilde{Q}_{0}:=\operatorname{Proj}(\widetilde{R} / q \widetilde{R})$. The induced action of $Y$ on $\widetilde{Q}_{\text {for }}$, which we denote also by $S_{y}$, is free. The invertible sheaf $O_{\widetilde{Q}_{\text {for }}}(1)$ descends to an invertible sheaf $L_{\text {for }}$ on the formal quotient $\widetilde{Q}_{\text {for }} / Y$. This turns out to be ample on $\widetilde{Q}_{\text {for }} / Y$. In fact, it is very ample on $\widetilde{Q}_{\text {for }} / n Y$ for any $n \geq 3$. See [Nr99, Theorem 6.2].

Hence by the algebrization theorem of Grothendieck we have
Theorem 3.3. There is a projective $R$-scheme $Q$ with an ample invertible sheaf $L$ such that the formal completion of $(Q, L)$ along the closed fibre is isomorphic to the pair $\left(\widetilde{Q}_{\text {for }} / Y, O_{\widetilde{Q}_{\text {for }}}(1) / Y\right)$. The generic fibre $\left(Q_{\eta}, L_{\eta}\right)$ is a polarized abelian scheme by enlarging $k(\eta)$ if necessary.

Proof. The last assertion about the generic fibre follows from [M72]. We omit the details because they are more or less well known. See also [AN99, Remark 3.10].
Q.E.D.

Proposition 3.4. Let $(\widetilde{Q}, \widetilde{L})=\left(\operatorname{Proj} \widetilde{R}, O_{\operatorname{Proj} \widetilde{R}}(1)\right)$. Then
(i) $\widetilde{Q}$ is covered with open affine subschemes $W(c):=\operatorname{Spec} S(c)$ where

$$
S(c):=R\left[\xi_{x, c} ; x \in X\right] \quad(c \in X), \quad \xi_{x, c}:=\xi_{x+c} / \xi_{c}
$$

(ii) The coordinate ring $S(c)$ of $W(c)$ is an $R$-algebra of finite type generated by $\xi_{x, c}(x \in \operatorname{Star}(0) \cap X)$. All the ring $S(c)$ are isomorphic to each other as $R$-algebras. The isomorphism $\phi_{c, d}: S(d) \rightarrow S(c)$ is given by $\phi_{c, d}\left(\xi_{x+d} / \xi_{d}\right)=\xi_{x+c} / \xi_{c}$ for any $x \in X$.

Remark 3.5. For a given abelian scheme $G$ over $R$ with $G_{0}$ a split torus over $k(0)$, we can construct a degeneration data $\{a(x) ; x \in X\}$
by taking a finite base change when necessary. Let $G_{\text {for }}$ be the formal completion of $G$ along the closed fibre $G_{0}$. Then $G_{\text {for }}$ is proved to be isomorphic to a formal split torus $\mathbf{G}_{m, \text { for }}^{g}$ over $R$. In that case, $X$ is the character group of $G_{\text {for }}$ while $Y$ is the character group of the formal completion of the dual abelian scheme of $G$. Letting $A(x)=\operatorname{val}_{q} a(x)$, we see $A(x+y)-A(x)-A(y)=B(x, y)$. Hence $A(x)-\frac{1}{2} B(x, x)$ is linear in $x$, which we can write as $\frac{1}{2} r$ for some $r \in \operatorname{Hom}(X, \mathbf{Z})$. By furthermore taking pull back of the family by replacing $R$ by $R[s]$ with $s^{2}=q$ if necessary, we may assume $B(x, x)$ and $r(x)$ are even-integers for any $x \in X$. Then by choosing $u^{x}=w^{x} s^{r(x)}$ instead of $w^{x}$ (the coordinates of the formal torus $\mathbf{G}_{m, \text { for }}^{g}$ ), we may assume $A(x)=\frac{1}{2} B(x, x)$ and it is integer-valued on $X$. This assumption is harmless for our study of the closed fibres $\left(Q_{0}, L_{0}\right)$ because the closed fibres are unchanged by the pull back and we study only cohomology groups of the closed fibres. Therefore in what follows we assume
(ii) $\quad A(x)=\frac{1}{2} B(x, x), r(x)=0$.

Definition 3.6. With the notation in Definition 2.9, we define

$$
\begin{gathered}
\xi(x, c)=q^{v(x, c)} w^{x}=q^{v(x)+(x, c)} w^{x} \in \Gamma\left(W(c), O_{\widetilde{Q}}\right) \\
\bar{\xi}(x, c):=\xi(x, c) \otimes k(0), \quad \xi(x):=\xi(x, 0) \in \Gamma\left(W(0), O_{\widetilde{Q}}\right) .
\end{gathered}
$$

We define $R(c)=S(c) \otimes k(0)$ and $U(c)=W(c) \otimes k(0)=\operatorname{Spec} R(c)$. We also set $\bar{\xi}(x):=\xi(x) \otimes k(0)$. It is clear that

$$
\Gamma\left(U(c), O_{U(c)}\right)=R(c)=\oplus_{x \in X} k(0) \cdot \bar{\xi}(x, c)
$$

With the above notation, $\phi_{c, d}(\xi(x, d))=\xi(x, c)$ for any $x \in X$.
Lemma 3.7. Let $\bar{\xi}(x):=\xi(x) \otimes k(0) \in S(0) \otimes k(0)(x \in X)$.
(i) If $x \notin \operatorname{Star}(0) \cap X$, then $\bar{\xi}(x)=0$.
(ii) If $x_{1}, \cdots, x_{m} \in \operatorname{Star}(0)$ are not cellmates at 0 , then the product $\bar{\xi}\left(x_{1}\right) \cdots \bar{\xi}\left(x_{m}\right)$ is either zero or nilpotent.

Proof. By Lemma 2.8 (ii) $\xi_{x}$ is divisible by $q \xi_{x-a} \xi_{a}$ in $S(0)$, which proves (i). Next we prove (ii). Let $x=x_{1}+\cdots+x_{m}$. Choose $\sigma \in \operatorname{Del}(0)$ such that $x \in C(0, \sigma)$, and let $\alpha(\sigma) \in \sigma$ be the hole of $\sigma$. Then there exist some positive integers $n \in \mathbf{Z}_{+}, n_{i} \in \mathbf{Z}_{+}$and $a_{i} \in \sigma \cap X$ such that
$n x=n_{1} a_{1}+\cdots+n_{r} a_{r}$. We have

$$
\begin{aligned}
n \sum_{i=1}^{m}\left(x_{i}, x_{i}\right) & \geq 2 n\left(\alpha(\sigma), \sum_{i=1}^{m} x_{i}\right)=2(\alpha(\sigma), n x) \\
& =2 \sum_{i=1}^{r} n_{i}\left(\alpha(\sigma), a_{i}\right)=\sum_{i=1}^{r} n_{i}\left(a_{i}, a_{i}\right)
\end{aligned}
$$

Since $x_{i}$ are not cellmates at 0 , there is at least an $x_{i}$ such that $x_{i} \notin \sigma$, hence $\left(x_{i}, x_{i}\right)>2\left(\alpha(\sigma), x_{i}\right)$. Therefore the above inequality is strict. This proves (ii).
Q.E.D.

Lemma 3.8. $U\left(c_{0}\right) \cap U\left(c_{1}\right) \cap \cdots \cap U\left(c_{q}\right) \neq \emptyset$ iff $c_{0}, c_{1}, \cdots, c_{q}$ are cellmates.

Proof. If $c_{0}, c_{1}, \cdots, c_{q}$ are cellmates, then it is clear that $U\left(c_{0}\right) \cap$ $U\left(c_{1}\right) \cap \cdots \cap U\left(c_{q}\right) \neq \emptyset$. We shall prove the converse. We suppose that $U\left(c_{0}\right) \cap U\left(c_{1}\right) \cap \cdots \cap U\left(c_{q}\right) \neq \emptyset$ and that $c_{0}, c_{1}, \cdots, c_{q}$ are not cellmates to derive a contradiction. We may assume $c_{0}=0$ without loss of generality. We note any $\bar{\xi}_{c_{i}}$ is invertible on $U\left(c_{0}\right) \cap U\left(c_{1}\right) \cap \cdots \cap U\left(c_{q}\right)$. If there is some $c_{i}(i>0)$ such that $c_{i} \notin \operatorname{Star}(0)$, then $\bar{\xi}_{c_{i}}=0$ by Corollary 3.7, a contradiction. If $c_{i} \in \operatorname{Star}(0)$ for any $i>0$, the product $\bar{\xi}_{c_{1}} \ldots \bar{\xi}_{c_{q}}$ is zero or nilpotent by Corollary 3.7, which contradicts that $\bar{\xi}_{c_{i}}$ is invertible on the nonempty set $U\left(c_{0}\right) \cap U\left(c_{1}\right) \cap \cdots \cap U\left(c_{q}\right)$. This proves the lemma.
Q.E.D.

From Lemma 2.7 (ii) and Lemma 3.8 we infer
Corollary 3.9. (i) $U(c)(c \in X)$ is an affine covering of $\widetilde{Q}_{0}$.
(ii) If $Y \subset m X$ for some $m \geq 3$, then $U(c) \cap U(c+y)=\emptyset$ for nonzero $y \in Y$, and $U(c)(c \in X / Y)$ is an affine covering of $Q_{0}$.
Lemma 3.8 gives a direct proof of the following
Theorem 3.10. Let $\mathbf{G}_{m}^{g}:=\operatorname{Spec} k(0)\left[w^{x} ; x \in X\right]$. Then there is a natural action of $\mathbf{G}_{m}^{g}$ on $\widetilde{Q}_{0}$. For any Delaunay cell $\sigma$ we define

$$
\begin{aligned}
& V(\sigma):=\operatorname{Proj} k(0)\left[\bar{\xi}_{a} ; a \in \sigma \cap X\right] \\
& O(\sigma):=\operatorname{Spec} k(0)\left[\bar{\xi}_{a} / \bar{\xi}_{b} ; a, b \in \sigma \cap X\right] .
\end{aligned}
$$

Then
(i) $\quad \underset{\sim}{\sim}(\sigma)$ is the unique closed $\mathbf{G}_{m}^{g}$-orbit in $\bigcap_{c \in \sigma \cap X} U(c)_{\text {red }}$,
(ii) $\widetilde{Q}_{0, \text { red }}=\bigcup_{\sigma \in \operatorname{Del}} O(\sigma)$ with $O(\sigma) \cap O(\tau)=\emptyset$ for $\sigma \neq \tau$ and $\sigma, \tau \in \mathrm{Del}$.
(iii) $\quad V(\sigma)$ is naturally a closed reduced subscheme of $\widetilde{Q}_{0}$ of $\operatorname{dim} V(\sigma)$ $=\operatorname{dim} \sigma$, which is the closure of $O(\sigma)$.
(iv) Let $\tau, \sigma \in$ Del. Then $V(\tau) \subset V(\sigma)$ iff $\tau \subset \sigma$.

Proof. We may assume $c_{0}=0 \in \sigma \cap X$ without loss of generality. First we note $\mathbf{G}_{m}^{g}$ acts on $\widetilde{Q}_{0}$ by $S_{a}^{*}\left(q^{A} w^{x}\right)=a^{x} q^{A} w^{x}$ for any $T$-valued point $a \in \mathbf{G}_{m}^{g}(T)$. By the definition

$$
\Gamma\left(O_{O(\sigma)}\right)=k(0)\left[\bar{\xi}_{a} / \bar{\xi}_{b} ; a, b \in \sigma \cap X\right] .
$$

By Lemma 3.8

$$
\begin{aligned}
\bigcap_{c \in \sigma \cap X} U(c)_{\mathrm{red}} & =\operatorname{Spec} k(0)\left[\bar{\xi}_{x} / \bar{\xi}_{b} ; x \in X, b \in \sigma \cap X\right] / \sqrt{(0)} \\
& =\operatorname{Spec} \Gamma\left(O_{O(\sigma)}\right)\left[\bar{\xi}_{x} ; x \in \operatorname{Star}(\sigma) \cap X\right] / \sqrt{(0)} \\
& =\operatorname{Spec} \Gamma\left(O_{O(\sigma)}\right)\left[\bar{\xi}_{x} ; x \in(\operatorname{Star}(\sigma) \backslash \sigma) \cap X\right] / \sqrt{(0)}
\end{aligned}
$$

Its unique closed orbit is given by the equations

$$
\bar{\xi}_{x}=0 \quad(\forall x \in(\operatorname{Star}(\sigma) \backslash \sigma) \cap X)
$$

Thus the assertions (i) and (ii) are clear from the above description. The assertion (iii) except its reducedness is clear from the definition of Proj.

We prove that $V(\sigma)$ is a reduced subscheme of $\widetilde{Q}_{0}$. Because the affine coordinate ring $\Gamma\left(O_{V(\sigma) \cap U(0)}\right)$ of $V(\sigma) \cap U(0)$ is $k(0)\left[\bar{\xi}_{x} ; x \in \sigma \cap X\right]$. Any nontrivial monomial of weight $x \in X$ in it is a product of $\bar{\xi}_{x_{i}}$ with cellmates $x_{i} \in \sigma \cap X$. By Corollary 3.7 it is $q^{(x, \alpha(\sigma))} w^{x}$, whence $\Gamma\left(O_{V(\sigma) \cap U(0)}\right)$ has no nilpotent elements.

Next we prove (iv). Let $\left\{c_{0}=0, c_{1}, \cdots, c_{q}\right\}=\tau \cap X$. Let $U(\tau):=$ $\bigcap_{c \in \tau \cap X} U(c)$. Suppose $\tau \subset \sigma$. First we note $V(\sigma) \cap U(\tau)=V(\sigma)_{\text {red }} \cap$ $U(\tau)=V(\sigma)_{\text {red }} \cap U(\tau)_{\text {red }}$. We also see

$$
\Gamma\left(O_{U(\tau)_{\mathrm{red}}}\right)=\Gamma\left(O_{O(\tau)}\right)\left[\bar{\xi}_{x} ; x \in(\operatorname{Star}(\tau) \backslash \tau) \cap X\right] / \sqrt{(0)}
$$

The closed subscheme $V(\sigma) \cap U(\tau)$ of $U(\tau)$ is defined by the ideal $\left(\bar{\xi}_{x} ; x \in(\operatorname{Star}(\tau) \backslash \sigma) \cap X\right)$, while $O(\tau)$ is defined by the ideal $\left(\bar{\xi}_{x} ; x \in\right.$ (Star $(\tau) \backslash \tau) \cap X)$ By the assumption $\tau \subset \sigma, V(\sigma) \cap U(\tau)$ contains $O(\tau)$, whence $V(\sigma) \supset V(\tau)$.

Next we assume $\tau \not \subset \sigma$ to prove $V(\tau) \not \subset V(\sigma)$. Then there is $a \in$ $\tau \cap X$ such that $a \notin \sigma$. Then $V(\tau) \cap U(a)=\operatorname{Spec} k(0)\left[\bar{\xi}_{x} / \bar{\xi}_{a}, x \in \tau \cap X\right]$. Let $p_{a}$ be a closed point of $U(a)$ defined by $\bar{\xi}_{x} / \bar{\xi}_{a}=0$ for any $x(\neq a) \in$ $X$. Hence $p_{a} \notin U(x)$ for any $x \neq a$. Since $V(\sigma)$ is covered with $U(b)$ $(b \in \sigma \cap X)$, this shows that $p_{a} \notin V(\sigma)$. This implies $V(\tau) \not \subset V(\sigma)$. This completes the proof of (iv), hence of the lemma.
Q.E.D.

## $\S 4$. Outline of the proof of Theorem 1

The purpose of this section is not to give a proof of Theorem 1 (Theorem 5.17), but to explain the outline of it.

For simplicity we assume $Y \subset m X$ for some $m \geq 3$.
Under the assumption $S_{y}(U(c)) \cap U(c)=\emptyset$ for any $c \in X$ and $y \in Y \backslash\{0\}$ and $U(c)(c \in X / Y)$ is an affine covering of $Q_{0}$ in view of Corollary 3.9. Therefore the cohomology groups $H^{q}\left(Q_{0}, L_{0}^{n}\right)$ are computed by using the Čech cohomology relative to the covering $U(c)$ $(c \in X / Y)$.

### 4.1. The particular case where $Q_{0}$ is reduced

First we consider the particular case when $k(0) \subset R$ and $(\widetilde{Q}, \widetilde{L})$ is the pull back of a normal torus embedding locally of finite type over $k(0)$ by the inclusion of Spec $R$ into Spec $k(0)[q]$. Then $(Q, L)=(P, L)$ with the notation of [Nr99]. We recall the proof of $H^{q}\left(Q_{0}, L_{0}^{n}\right)=0$ for $q, n>0$ from [Nr99].

First we have an exact sequence of $O_{Q_{0}}$-modules

$$
\begin{equation*}
0 \rightarrow O_{Q_{0}} \rightarrow \oplus O_{V\left(\sigma_{g}\right)} \xrightarrow{\partial_{g}} \cdots \xrightarrow{\partial_{2}} \oplus O_{V\left(\sigma_{1}\right)} \xrightarrow{\partial_{1}} \oplus O_{V\left(\sigma_{0}\right)} \rightarrow 0 \tag{1}
\end{equation*}
$$

where $\sigma_{i}$ ranges over the set of all $i$-dimensional Delaunay cells mod $Y$. The homomorphism $\partial_{i}: \oplus O_{V\left(\sigma_{i}\right)} \rightarrow \oplus O_{V\left(\sigma_{i-1}\right)}$ in the above is defined by

$$
\partial_{i}\left(\bigoplus_{\sigma \in \operatorname{Del}^{(i)}} \phi_{\sigma}\right)=\bigoplus_{\tau \in \operatorname{Del}^{(i-1)}} \sum_{\tau \subset \sigma}[\sigma: \tau] \phi_{\sigma}
$$

where the summation $\sum_{\tau \subset \sigma}$ runs over the set of all $i$-dimensional Delaunay cells $\sigma$ containing a fixed $\tau$ as a face of codimension one, and any Delaunay cell $\sigma$ is oriented and $[\sigma: \tau](= \pm 1)$ is the incidence number of $\sigma$ relative to $\tau$. Then by tensoring (1) with $L_{0}^{n}$ we have an exact sequence

$$
0 \rightarrow L_{0}^{n} \otimes O_{P_{0}} \rightarrow \oplus L_{0}^{n} \otimes O_{V\left(\sigma_{g}\right)} \xrightarrow{\partial_{g}} \cdots \xrightarrow{\partial_{1}} \oplus L_{0}^{n} \otimes O_{V\left(\sigma_{0}\right)} \rightarrow 0
$$

Now the proof of $H^{q}\left(Q_{0}, L_{0}^{n}\right)=0$ goes as follows.
(i) Since $V(\sigma)$ is a normal torus embedding with $L_{0}$ ample, we have

$$
H^{q}\left(V(\sigma), L_{0}^{n}\right)= \begin{cases}\bigoplus_{\frac{x}{n} \in \sigma \cap \frac{x}{n}} k(0) \cdot[x] & \text { if } q=0 \\ 0 & \text { if } q>0\end{cases}
$$

where $[x]$ is a certain monomial in $\widetilde{R} / q \widetilde{R}$ of weight $x$.
(ii) By (i) $H^{*}\left(P_{0}, L_{0}^{n}\right)$ is the cohomology of the complex

$$
0 \rightarrow \oplus \Gamma\left(V\left(\sigma_{g}\right), L_{0}^{n}\right) \xrightarrow{H^{0}\left(\partial_{g}\right)} \ldots \xrightarrow{H^{0}\left(\partial_{1}\right)} \oplus \Gamma\left(V\left(\sigma_{0}\right), L_{0}^{n}\right) \rightarrow 0 .
$$

(iii) By (i) and (ii)

$$
H^{q}\left(Q_{0}, L_{0}^{n}\right) \simeq \bigoplus_{\frac{x}{n} \in \frac{X}{n} \bmod Y} H^{q}\left(\operatorname{Star}\left(\frac{x}{n}\right)^{0}, k(0)\right)=0 \text { for } q, n>0
$$

where $\operatorname{Star}(a)$ denotes the union of $\sigma \in \operatorname{Del}(a)$, and $\operatorname{Star}(a)^{0}$ denotes the relative interior of $\operatorname{Star}(a)$. The subset $\operatorname{Star}(a)^{0}$ of $X_{\mathbf{R}}$ is connected and contractible.

### 4.2. The general case

In the case where $Q_{0}$ is possibly nonreduced or $(Q, L)$ may not come from a torus embedding, we have no exact sequences like (1). Nevertheless we can imitate the above proof of $H^{q}\left(Q_{0}, L_{0}^{n}\right)=0$.

We will construct a double complex $\left({ }_{n} C^{\cdot}, \Delta_{n}^{*}\right)$ for each positive integer $n$ such that

$$
\begin{gathered}
{ }_{n} C^{\cdot}=\bigoplus_{n} C^{p}, \quad{ }_{n} C^{p}=\bigoplus_{k+q=p}{ }_{n} F^{k, q}, \quad \Delta_{n}^{p}=\bigoplus_{k+q=p}\left(\partial^{k, q}+(-1)^{q} \delta_{n}^{k, q}\right), \\
{ }_{n} F^{k, q}=\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)} \bmod Y}{ }_{n} F_{\sigma}^{k, q}=\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)} \bmod Y}\left(\bigoplus_{x \in X}{ }_{n} F_{\sigma}^{k, q}[x]\right)
\end{gathered}
$$

where ${ }_{n} F_{\sigma}^{k, q}[x]$ is the weight $x$-part of ${ }_{n} F_{\sigma}^{k, q}$. We see

$$
\Delta_{n}^{p+1} \cdot \Delta_{n}^{p}=0, \quad \partial^{k+1, q} \cdot \partial^{k, q}=0, \quad \delta_{n}^{k, q+1} \cdot \delta_{n}^{k, q}=0
$$

Then our new proof goes as follows.
(a)

$$
{ }^{\prime \prime} E_{2}^{k, q}=\left\{\begin{array}{cl}
H^{q}\left(Q_{0}, L_{0}^{n}\right) & \text { if } k=0 \\
0 & \text { if } k>0
\end{array}\right.
$$

(b)

$$
H^{q}\left({ }_{n} F_{\sigma}^{k \cdot \cdot}, \delta_{n}^{k, \cdot}\right)= \begin{cases}\bigoplus_{\frac{x}{n} \in \sigma \cap \frac{X}{n}} k(0) \cdot[x] & \text { if } q=0 \\ 0 & \text { if } q>0\end{cases}
$$

where $[x]$ is a certain monomial in $\widetilde{R} / q \widetilde{R}$ of weight $x$.
(c) By (b)

$$
\begin{aligned}
{ }^{\prime} E_{1}^{k, q} & =H^{q}\left({ }_{n} F^{k, \cdot}, \delta_{n}^{k, \cdot}\right)=\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)} \bmod Y} H^{q}\left({ }_{n} F_{\sigma}^{k \cdot \cdot}, \delta_{n}^{k, \cdot}\right) \\
& =\left\{\begin{array}{cl}
\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)} \bmod Y}\left(\bigoplus_{\frac{x}{n} \in \sigma \cap \frac{X}{n}} k(0) \cdot[x]\right) & \text { if } q=0 \\
0 & \text { if } q>0
\end{array}\right.
\end{aligned}
$$

(d) $\mathrm{By}(\mathrm{c})$

$$
E_{2}^{k, q}=\left\{\begin{array}{cl}
\bigoplus_{\frac{x}{n} \in \frac{X}{n} \bmod Y} H^{k}\left(\operatorname{Star}\left(\frac{x}{n}\right)^{0}, k(0)\right) & \text { if } q=0 \\
0 & \text { if } q>0
\end{array}\right.
$$

(e) By (a) and (d)

$$
\begin{aligned}
H^{q}\left(Q_{0}, L_{0}^{n}\right) & ={ }^{\prime \prime} E_{2}^{0, q}=\mathbf{H}^{q}\left({ }_{n} C^{\cdot}, \Delta_{n}^{*}\right)=^{\prime} E_{2}^{q, 0} \\
& =\bigoplus_{\frac{x}{n} \in \frac{X}{n} \bmod Y} H^{q}\left(\operatorname{Star}\left(\frac{x}{n}\right)^{0}, k(0)\right)=0 \text { if } q>0
\end{aligned}
$$

The hardest in the above is the part (b), which is an alternative for the part (i) in the first particular case. The assertion (b) is proved by using Lemma 4.3 (or Theorem 5.15)

$$
H^{q}\left({ }_{n} F_{\sigma}^{k, \cdot}[x], \delta_{n}^{k, \cdot}\right)=H^{q}\left(\Delta(\sigma), B_{\Delta(\sigma)}\left(\frac{x}{n}\right)\right)= \begin{cases}k(0) & \text { if } q=0, \frac{x}{n} \in \sigma \\ 0 & \text { otherwise }\end{cases}
$$

where ${ }_{n} F_{\sigma}^{k, q}[x]$ is the weight $x$-part of ${ }_{n} F_{\sigma}^{k, q}$. See also Theorem 6.11.
Lemma 4.3. Let $\sigma \in \operatorname{Del}^{(g-k)}$. Let $\Delta(\sigma)$ be the abstract simplex with vertices $\sigma \cap X$. Then there is a subset $\left.B_{\Delta(\sigma)}\left(\frac{x}{n}\right)\right)$ of $\Delta(\sigma)$ such that

$$
H^{q}\left({ }_{n} F_{\sigma}^{k, \cdot}[x], \delta_{n}^{k, \cdot}\right)=H^{q}\left(\Delta(\sigma), B_{\Delta(\sigma)}\left(\frac{x}{n}\right)\right)
$$

Moreover
(i) if $B_{\Delta(\sigma)}\left(\frac{x}{n}\right)$ is nonempty, then it is connected and contractible.
(ii) $\quad B_{\Delta(\sigma)}\left(\frac{x}{n}\right)$ is empty iff $\frac{x}{n} \in \sigma$.

This lemma is obtained by combining Lemma 6.10 and Theorem 6.11.

## §5. Proof of Theorem 1

The purpose of this section is to prove Theorem 1 (Theorem 5.17). For simplicity we first assume

$$
Y \subset m X \text { for some } m \geq 3
$$

In what follows we denote $\bar{\xi}(x, c)$ by $\xi(x, c)$ if no confusion is possible.
Definition 5.1. Let $c \in X$. Let $R(c)=S(c) \otimes k(0)=\Gamma\left(O_{U(c)}\right)$. For a Delaunay cell $\sigma$ containing $c$, we define $k(0)$-modules

$$
\begin{aligned}
F_{\sigma}(c) & =\bigoplus_{x \in C(0, \sigma-c) \cap X} k(0) \cdot \xi(x, c) \\
F^{k}(c) & =\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)}(c)} F_{\sigma}(c)
\end{aligned}
$$

It should be mentioned that $F_{\sigma}(c)$ is not an $R(c)$-module in general, though $F^{k}(c)$ is an $R(c)$-module. Nevertheless we imitate the way of computing $H^{q}\left(P_{0}, L_{0}^{n}\right)$ in [Nr99, Theorem 3.9] and construct, by replacing $O_{P_{0}}$-modules $L_{0}^{n} \otimes O_{V(\sigma) \cap U(c)}$ [ibid.] by analogous $k(0)$-modules, a double complex $F^{k, q}$ whose first row $F^{k, 0}(c)$ at $c$ is a resolution of $R(c)$ $(c \in X)$.

Any $\phi_{\sigma} \in F_{\sigma}(c)$ is written

$$
\phi_{\sigma}=\sum_{x \in C(0, \sigma-c) \cap X} a_{\sigma}(x, c) \xi(x, c), \quad\left(a_{\sigma}(x, c) \in k(0)\right)
$$

Then we define

$$
\operatorname{res}_{\tau}^{\sigma}\left(\phi_{\sigma}\right)=\sum_{x \in C(0, \tau-c) \cap X} a_{\sigma}(x, c) \xi(x, c)
$$

We also define $\partial^{k}: F^{k}(c) \rightarrow F^{k+1}(c)$ by

$$
\partial^{k}\left(\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)}(c)} \phi_{\sigma}\right)=\bigoplus_{\tau \in \operatorname{Del}^{(g-k-1)}(c)} \sum_{\tau \subset \sigma}[\sigma: \tau] \operatorname{res}_{\tau}^{\sigma}\left(\phi_{\sigma}\right)
$$

where $\phi_{\sigma} \in F_{\sigma}(c)$, and the summation in RHS ranges over the set of all $\sigma$ containing a fixed $\tau$ as a face of codimension one.

Lemma 5.2. There is an exact sequence of $k(0)$-modules

$$
0 \rightarrow R(c) \rightarrow F^{0}(c) \xrightarrow{\partial^{0}} F^{1}(c) \xrightarrow{\partial^{1}} \cdots \xrightarrow{\partial^{g-2}} F^{g-1}(c) \xrightarrow{\partial^{g-1}} F^{g}(c) \rightarrow 0
$$

where $F^{g}(c)=k(0) \cdot \xi(0, c)$.

Proof. Let $f \in F^{0}(c)$. Then $f$ is written as

$$
f=\sum_{\sigma \in \operatorname{Del}^{(g)}(c)}\left(\sum_{x \in C(0, \sigma-c) \cap X} a_{\sigma}(x, c) \xi(x, c)\right)
$$

Then we see that $f \in \operatorname{Ker}\left(\partial^{0}\right)$ if and only if $a_{\sigma}(x, c)=a_{\sigma^{\prime}}(x, c)$ for any adjacent pair $\sigma, \sigma^{\prime} \in \operatorname{Del}^{(g)}(c)$ and any $x \in C\left(0,\left(\sigma \cap \sigma^{\prime}\right)-c\right) \cap X$. It follows that $R(c)=\operatorname{Ker}\left(\partial^{0}\right)$. We denote $R(c)_{x}=k(0) \xi(x, c)$.

The exactness of the rest of the sequence is proved as follows. Now we choose and fix any $x \in X$ for all. For $\sigma \in \operatorname{Del}^{(g-k)}(c)$ we define

$$
F_{\sigma}(c)_{x}:=\left\{\begin{array}{cl}
k(0) \cdot \xi(x, c) & \text { if } x \in C(0, \sigma-c) \cap X \\
0 & \text { (otherwise) }
\end{array}\right.
$$

and

$$
F^{k}(c)_{x}:=\bigoplus_{\substack{\sigma \in \operatorname{Del}(g-k) \\ x \in C(c) \\ x \in C-c)}} F_{\sigma}(c)_{x} .
$$

Note that $\partial^{g-k}\left(F^{k}(c)_{x}\right) \subset F^{k+1}(c)_{x}$. Now we define the complex $\left(F^{\cdot}(c)_{x}, \partial_{\mid F^{\cdot}(c)_{x}}\right)$ by

$$
F^{0}(c)_{x} \xrightarrow{\partial^{0}} F^{1}(c)_{x} \xrightarrow{\partial^{1}} \cdots \xrightarrow{\partial^{g-2}} F^{g-1}(c)_{x} \xrightarrow{\partial^{g-1}} F^{g}(c)_{x} \rightarrow 0 .
$$

It remains to prove the exactness of the complex $\left(F^{*}(c)_{x}, \partial_{\mid F^{*}(c)_{x}}\right)$ for each $x \in X$.

There is a Delaunay cell $\sigma \in \operatorname{Del}(c)$ such that the relative interior of $C(0, \sigma-c)$ contains $x$. The Delaunay cell $\sigma$ is uniquely determined by the given $x$, which we denote $\sigma_{\min }(x, c)$. We note that for $\sigma \in \operatorname{Del}(c), x \in$ $C(0, \sigma-c)$ if and only if $\sigma_{\min }(x, c) \subset \sigma$. Let $\operatorname{Del}(x, c)$ be the set of Delaunay cells $\sigma \in \operatorname{Del}(c)$ such that $\sigma_{\min }(x) \subset \sigma$, and $\operatorname{Del}^{(k)}(x, c)=\operatorname{Del}(x, c) \cap$ $\operatorname{Del}^{(k)}$. Let $\operatorname{Star}(x, c)$ be the union of $\sigma \in \operatorname{Del}(x, c), \sigma_{\text {min }}(x, c)^{\perp}$ the affine linear subspace of $X_{\mathbf{R}}$ passing through $x$, perpendicular to $\sigma_{\min }(x, c)$. Let $\operatorname{Star}^{\perp}(x, c)$ be the intersection $\operatorname{Star}(x, c) \cap \sigma_{\text {min }}(x, c)^{\perp}, \partial \operatorname{Star}^{\perp}(x, c)$ the boundary of $\operatorname{Star}^{\perp}(x, c)$. We note $\operatorname{Star}(x, c)=\operatorname{Star}\left(\sigma_{\min }(x, c)\right)$. Let $\mathbf{B}$ be a closed ball of dimension $g-\operatorname{dim} \sigma_{\min }(x, c), \partial \mathbf{B}$ its boundary. Since $\left(\operatorname{Star}^{\perp}(x, c), \partial \operatorname{Star}^{\perp}(x, c)\right)$ is homeomorphic to $(\mathbf{B}, \partial \mathbf{B})$, we have an isomorphism

$$
H_{q}\left(\operatorname{Star}^{\perp}(x, c), \partial \operatorname{Star}^{\perp}(x, c), k(0)\right)=\left\{\begin{array}{cl}
k(0) & \text { if } q=g-\operatorname{dim} \sigma_{\min }(x, c) \\
0 & \text { (otherwise) }
\end{array}\right.
$$

For the chosen and fixed $x$ and $c$, we introduce a new complex $(G ., \delta$.) by

$$
\begin{gathered}
G_{q}:=\bigoplus_{\sigma \in \operatorname{Del}^{(q)}(x, c)} k(0) \cdot \sigma \\
\delta_{q}\left(\bigoplus_{\sigma \in \operatorname{Del}^{(q)}(x, c)} a_{\sigma} \sigma\right)=\bigoplus_{\tau \in \operatorname{Del}^{(q-1)}(x, c)}\left(\sum_{\tau \subset \sigma}[\sigma: \tau] a_{\sigma}\right) \tau .
\end{gathered}
$$

When $\sigma$ ranges over $\operatorname{Del}(x, c), \sigma \cap \sigma_{\min }(x, c)^{\perp}$ gives a cell decomposition of $\operatorname{Star}^{\perp}(x, c)$. Since $(G ., \delta$.) is the relative chain complex of

$$
\left(\operatorname{Star}^{\perp}(x, c), \partial \operatorname{Star}^{\perp}(x, c)\right)
$$

with coefficients in $k(0)$ whose degree is shifted by $\operatorname{dim} \sigma_{\min }(x, c)$, we have an isomorphism

$$
\begin{aligned}
\mathbf{H}_{q}\left(G ., \delta_{.}\right) & \simeq H_{q-\operatorname{dim} \sigma_{\min (x, c)}\left(\operatorname{Star}^{\perp}(x, c), \partial \operatorname{Star}^{\perp}(x, c), k(0)\right)} \\
& =\left\{\begin{array}{cl}
k(0) & \text { if } q=g \\
0 & \text { (otherwise) }
\end{array}\right.
\end{aligned}
$$

Suppose $\sigma \in \operatorname{Del}^{(q)}$. By the definition of $G$.,

$$
\begin{aligned}
F_{\sigma}^{g-q}(c)_{x}=k(0) \xi(x, c) & \Longleftrightarrow x \in C(0, \sigma-c) \cap X \\
& \Longleftrightarrow \sigma_{\min }(x, c) \subset \sigma \\
& \Longleftrightarrow \sigma \in \operatorname{Del}^{(q)}(x, c) \Longleftrightarrow k(0) \cdot \sigma \subset G_{q}
\end{aligned}
$$

Hence $\left(G_{q}, \delta_{q}\right)=\left(F^{g-q}(c)_{x}, \partial^{g-q}\right)$. It follows

$$
\mathbf{H}^{q}\left(F^{\cdot}(c)_{x}, \partial \cdot\right)=\mathbf{H}_{g-q}\left(G ., \delta_{.}\right)=\left\{\begin{array}{cl}
k(0) & \text { if } q=0 \\
0 & \text { if } q>0
\end{array}\right.
$$

This proves the exactness of $\left(F^{\cdot}(c)_{x}, \partial^{\cdot}\right)$ except at $q=0$, which completes the proof of the lemma. We note $H^{0}\left(F^{\cdot}(c)_{x}, \partial^{\cdot}\right)=R(c)_{x}:=$ $k(0) \xi(x, c)$.
Q.E.D.

Definition 5.3. Let $\mathbf{c}=\left(c_{0}, c_{1}, \cdots, c_{q}\right)\left(c_{i} \neq c_{j}\right)$ be an ordered set of cellmates, and $|\mathbf{c}|=\left\{c_{0}, c_{1}, \cdots, c_{q}\right\}$ an unordered set of cellmates. Then we define

$$
\begin{gathered}
U(\mathbf{c})=U\left(c_{0}, c_{1}, \cdots, c_{q}\right):=U\left(c_{0}\right) \cap U\left(c_{1}\right) \cap U\left(c_{2}\right) \cap \cdots \cap U\left(c_{q}\right) \\
R(\mathbf{c})=R\left(c_{0}, c_{1}, \cdots, c_{q}\right):=\Gamma\left(U(\mathbf{c}), O_{U(\mathbf{c})}\right)
\end{gathered}
$$

and

$$
C^{q}:=\bigoplus_{\substack{\left(c_{0}, c_{1}, \ldots, c_{q}\right) \\ c_{j}: \text { cellmates }}} R\left(c_{0}, c_{1}, \cdots, c_{q}\right)
$$

We denote the set $\left\{c_{0}, c_{1}, \cdots, c_{q}\right\}$ by $|\mathbf{c}|$. Let $X(\mathbf{c}):=X(|\mathbf{c}|)=$ $\mathbf{Z}\left(c_{1}-c_{0}\right)+\cdots+\mathbf{Z}\left(c_{q}-c_{0}\right)$ and we define

$$
k(0)[X(\mathbf{c})]=k(0)\left[\left(\frac{\bar{\xi}_{c_{1}}}{\bar{\xi}_{c_{0}}}\right)^{ \pm 1}, \cdots,\left(\frac{\bar{\xi}_{c_{q}}}{\bar{\xi}_{c_{0}}}\right)^{ \pm 1}\right] \quad(\text { resp. } 0)
$$

if $c_{0}, c_{1}, \cdots, c_{q}$ are cellmates (resp. if $c_{0}, c_{1}, \cdots, c_{q}$ are not cellmates).
Remark 5.4. We denote the set $\left\{c_{0}, c_{1}, \cdots, c_{q}\right\}$ by $|\mathbf{c}|$. Lemma 3.8 shows that $U(\mathbf{c}) \neq \emptyset$ iff $c_{0}, c_{1}, \cdots, c_{q}$ are cellmates. Hence if $c_{j}$ are cellmates and if $|\mathbf{c}|=\sigma \cap X$ for some $\sigma \in$ Del, then by Theorem 3.10, $O(\sigma)$ is the unique closed $\mathbf{G}_{m}^{g}$-orbit in $U(\mathbf{c})_{\text {red }}$ with $\Gamma\left(O_{O(\sigma)}\right)=k(0)[X(\mathbf{c})]$. If $c_{0}, c_{1}, \cdots, c_{q}$ are not cellmates, then the product $f:=\prod_{j=1}^{q}\left(\bar{\xi}_{c_{j}} / \bar{\xi}_{c_{0}}\right)$ is nilpotent. This contradicts that $f$ has the inverse in $k(0)[X(\mathbf{c})]$. This is why we define $k(0)[X(\mathbf{c})]:=0$ in the case. We also note that $\operatorname{dim} \sigma \geq \operatorname{rank} X(\mathbf{c})$ if $|\mathbf{c}| \subset \sigma \in$ Del, where equality may not be true in general.

Lemma 5.5. Let $\tau$ be a Delaunay cell and $\alpha(\tau) \in \tau$ the hole of $\tau$. Let $\mathbf{c}=\left(c_{0}, c_{1}, \cdots, c_{q}\right)$. Assume $|\mathbf{c}| \subset \tau$. Then

$$
k(0)[X(\mathbf{c})]=k(0)\left[q^{(a, \alpha(\tau))} w^{a} ; a \in X(\mathbf{c})\right] .
$$

Proof. By the assumption, $\left\|c_{0}-\alpha(\tau)\right\|=\left\|c_{j}-\alpha(\tau)\right\|$, whence $c_{j}^{2}-$ $2\left(c_{j}, \alpha(\tau)\right)=c_{0}^{2}-2\left(c_{0}, \alpha(\tau)\right)$ for any $j$. Hence $\bar{\xi}_{c_{j}} / \bar{\xi}_{c_{0}}=q^{\left(c_{j}-c_{0}, \alpha(\tau)\right)} w^{c_{j}-c_{0}}$.
Q.E.D.

Lemma 5.6. Let $\mathbf{c}=\left(c_{0}, c_{1}, \cdots, c_{q}\right)$ with $c_{i}$ cellmates, $\operatorname{Star}(\mathbf{c}):=$ $\operatorname{Star}(|\mathbf{c}|)$. Let $\sigma \in \operatorname{Del}$ and $C\left(c_{0}, \sigma\right)^{0}$ the relative interior of $C\left(c_{0}, \sigma\right)$. For any class $(x \bmod X(\mathbf{c}))$
(i) there is $x^{\prime} \in C\left(0, \operatorname{Star}(\mathbf{c})-c_{0}\right)^{0}$ such that $x^{\prime} \equiv x \bmod X(\mathbf{c})$.
(ii) If $x^{\prime}+c_{0} \in C\left(c_{0}, \operatorname{Star}(\mathbf{c})\right)^{0}$ and $x^{\prime} \equiv x \bmod X(\mathbf{c})$, then there is the unique Delaunay cell $\sigma$ such that $|\mathbf{c}| \subset \sigma$ and $x^{\prime}+c_{0} \in$ $C\left(c_{0}, \sigma\right)^{0}$.
(iii) The above Delaunay cell $\sigma$ is uniquely determined by the given class $x \bmod X(\mathbf{c})$, independent of the choice of $x^{\prime}$ with $x^{\prime}+$ $c_{0} \in C\left(c_{0}, \sigma\right)^{0}$.
We denote by $\sigma_{\min }(x, \mathbf{c})$ the unique Delaunay cell satisfying the condition (ii).

Proof. We recall $\operatorname{Star}\left(c_{j}\right)$ is the union of all the Delaunay cells containing $c_{j}$, which is bounded convex. Hence $\operatorname{Star}(\mathbf{c})=\bigcap_{j=0}^{q} \operatorname{Star}\left(c_{j}\right)$ is a bounded convex subset of $X_{\mathbf{R}}$. Therefore $C\left(c_{0}, \operatorname{Star}(\mathbf{c})\right)$ is a convex closed subset of $X_{\mathbf{R}}$ given by finitely many (affine-)linear inequalities:

$$
C\left(c_{0}, \operatorname{Star}(\mathbf{c})\right)=\left\{x \in X_{\mathbf{R}} ; F_{j}(x) \geq 0(j=1, \cdots, N)\right\}
$$

where $F_{j}\left(c_{0}\right)=0, F_{j}\left(c_{k}\right) \geq 0(\forall j, k)$. We note $F_{j}\left(c_{k}\right)>0(\exists k \geq 1)$ for each $j$ because $\operatorname{Star}(\mathbf{c})$ is bounded with $\operatorname{dim} \operatorname{Star}(\mathbf{c})=g$. Since $F_{j}(x)$ is linear in $x-c_{0}, F_{j}(x)=\left(A_{j}, x-c_{0}\right)$ for some $A_{j} \in X_{\mathbf{R}}$. For $x \in X$, we set

$$
x_{N}=x+N\left(c_{1}-c_{0}\right)+N\left(c_{2}-c_{0}\right)+\cdots+N\left(c_{q}-c_{0}\right)
$$

If $N$ is large enough, then

$$
F_{j}\left(x_{N}+c_{0}\right)=\left(A_{j}, x_{N}\right)=\left(A_{j}, x\right)+N \cdot\left(F_{j}\left(c_{1}\right)+\cdots+F_{j}\left(c_{q}\right)\right)>0
$$

This implies that $x_{N}+c_{0} \in C\left(c_{0}, \operatorname{Star}(\mathbf{c})\right)^{0}$. It suffices to choose $x^{\prime}=x_{N}$ for (i).

Next we prove (ii). Suppose $x^{\prime}+c_{0} \in C\left(c_{0}, \operatorname{Star}(\mathbf{c})\right)^{0}$ and $x^{\prime} \equiv x$ $\bmod X(\mathbf{c})$. Since $\operatorname{Star}(\mathbf{c})$ is the union of all the Delaunay cells $\sigma$ with $|\mathbf{c}| \subset \sigma$ and since Del is a polyhedral decomposition of $X_{\mathbf{R}}$, there is the minimal Delaunay cell $\sigma$ such that $|\mathbf{c}| \subset \sigma$ and $x^{\prime}+c_{0} \in C\left(c_{0}, \sigma\right)$. If $x^{\prime}+c_{0} \notin C\left(c_{0}, \sigma\right)^{0}$, then $x^{\prime}+c_{0} \in C\left(c_{0}, \tau\right)$ for a face $\tau$ of $\sigma$. Since $x^{\prime}+c_{0} \in C\left(c_{0}, \operatorname{Star}(\mathbf{c})\right)^{0}, \tau$ intersects $\operatorname{Star}(\mathbf{c})^{0}$, hence the relative interior $\tau^{0}$ of $\tau$ intersects the interior of $\operatorname{Star}(\mathbf{c})$. Hence $\tau \subset \operatorname{Star}(\mathbf{c})$, whence $|\mathbf{c}| \subset \tau$. This contradicts that $\sigma$ is minimal. This proves (ii).

Finally we prove (iii). Suppose $x^{\prime}+c_{0} \in C\left(c_{0}, \sigma^{\prime}\right)^{0}$ and $x^{\prime \prime}+$ $c_{0} \in C\left(c_{0}, \sigma^{\prime \prime}\right)^{0}$ and that $x^{\prime} \equiv x^{\prime \prime} \equiv x \bmod X(\mathbf{c})$. Then $x^{\prime}=x^{\prime \prime}+$ $\sum_{j=1}^{q} a_{j}\left(c_{j}-c_{0}\right)$ for some $a_{j} \in \mathbf{Z}$. Since $x^{\prime}+c_{0}+\sum_{j=1}^{q} N_{j}^{\prime}\left(c_{j}-c_{0}\right)$ (resp. $\left.x^{\prime \prime}+c_{0}+\sum_{j=1}^{q} N_{j}^{\prime \prime}\left(c_{j}-c_{0}\right)\right)$ stays inside $C\left(c_{0}, \sigma^{\prime}\right)^{0}$ (resp. $\left.C\left(c_{0}, \sigma^{\prime \prime}\right)^{0}\right)$ for any large $N_{j}^{\prime}>0$ and $N_{j}^{\prime \prime}>0, C\left(c_{0}, \sigma^{\prime}\right)^{0}$ and $C\left(c_{0}, \sigma^{\prime \prime}\right)^{0}$, two cones at $c_{0}$ of Delaunay cells, have common relative interior points. It follows $C\left(c_{0}, \sigma^{\prime}\right)=C\left(c_{0}, \sigma^{\prime \prime}\right)$ and $\operatorname{dim} \sigma^{\prime}=\operatorname{dim} \sigma^{\prime \prime}$. Since $c_{0} \in \sigma^{\prime} \subset C\left(c_{0}, \sigma^{\prime}\right)$, $c_{0} \in \sigma^{\prime \prime} \subset C\left(c_{0}, \sigma^{\prime \prime}\right)$, two Delaunay cells $\sigma^{\prime}$ and $\sigma^{\prime \prime}$ have common relative interiors. Therefore $\sigma^{\prime}=\sigma^{\prime \prime}$. It is clear that $\sigma^{\prime}$ depends only on the class $(x \bmod X(\mathbf{c}))$, and is independent of the choice of $x \in X$.
Q.E.D.

Definition 5.7. Let $\mathbf{c}=\left(c_{0}, \cdots, c_{q}\right)$ with $c_{j} \in X$ cellmates. We recall $|\mathbf{c}|=\left\{c_{0}, \cdots, c_{q}\right\}$. We define $\operatorname{Del}(\mathbf{c})$ to be $\operatorname{Del}(|\mathbf{c}|)$. Let $\operatorname{Del}^{(g-k)}(\mathbf{c})=$ $\operatorname{Del}(\mathbf{c}) \cap \operatorname{Del}^{(g-k)}$. We define $C(\mathbf{c}, \sigma):=C(|\mathbf{c}|, \sigma)$, which is the union of all the translates $C\left(c_{0}, \sigma\right)$ by $a \in X(\mathbf{c})$. See Definition 2.2. This depends only on the unordered set $\mathbf{c}$, independent of the order of $c_{j}$.

Lemma 5.8. Let $c_{0}, c_{1}, \cdots, c_{q}$ be cellmates, $\mathbf{c}=\left(c_{0}, \cdots, c_{q}\right)$ ordered cellmates, and $|\mathbf{c}|$ unordered cellmates. Let

$$
R(\mathbf{c}):=\bigoplus_{x \in X} k(0) \cdot \xi(x, \mathbf{c})
$$

for some nonzero monomials $\xi(x, \mathbf{c})$. Then
(i) If there is some $\sigma \in \operatorname{Del}(\mathbf{c})$ such that $x \in C\left(0, \sigma-c_{0}\right) \cap X$, then

$$
\xi(x, \mathbf{c})=\xi\left(x, c_{0}\right)
$$

(ii) If there are $a \in X(\mathbf{c})$ and $\sigma \in \operatorname{Del}(\mathbf{c})$ such that $x-a \in$ $C\left(0, \sigma-c_{0}\right) \cap X$,

$$
\xi(x, \mathbf{c})=q^{(a, \alpha(\sigma))} w^{a} \cdot \xi\left(x-a, c_{0}\right)
$$

(iii)

$$
R(\mathbf{c})=\bigoplus_{\substack{\sigma \in \operatorname{Del}(\mathbf{c}), x \in X / X(\mathbf{c}) \\ x+c_{0} \in C(\mathbf{c}, \sigma) \cap X}} k(0)[X(\mathbf{c})] \cdot \xi\left(x, c_{0}\right)
$$

Proof. Suppose that some $\sigma \in \operatorname{Del}(\mathbf{c})$ such that $x \in C\left(0, \sigma-c_{0}\right) \cap X$. The element $\xi(x, \mathbf{c})$ is nonzero on $U(\mathbf{c})$, hence it is nonzero on $U\left(c_{0}\right)$ because $U(\mathbf{c}) \subset U\left(c_{0}\right)$. Thus it restricts to a nonzero element of $R\left(c_{0}\right)$ of weight $x$, which is $\xi\left(x, c_{0}\right)$. Hence $\xi(x, \mathbf{c})=\xi\left(x, c_{0}\right)$. This proves (i).

Next we prove (ii). We choose $\tau \in \operatorname{Del}$ such that $|\mathbf{c}| \subset \tau$. It is clear that

$$
R(\mathbf{c}):=\bigoplus_{x \in X} k(0) \cdot \xi(x, \mathbf{c})=\bigoplus_{x \in X / X(\mathbf{c})} k(0)[X(\mathbf{c})] \cdot \xi(x, \mathbf{c})
$$

for some nonzero element $\xi(x, \mathbf{c})$ of weight $x \in X$. Suppose that $a \in$ $X(\mathbf{c}), \sigma \in \operatorname{Del}(\mathbf{c})$ and $x-a \in C\left(0, \sigma-c_{0}\right) \cap X$. Let $\zeta=q^{(a, \alpha(\sigma))} w^{a} \in$ $k(0)[X(\mathbf{c})]$. Since $\zeta$ is a unit in $k(0)[X(\mathbf{c})]$ by Lemma 5.5 , we have $\xi(x, \mathbf{c})=\xi(x-a, \mathbf{c}) \zeta$ for any $x \in X$. It is equal to $\xi\left(x-a, c_{0}\right) \zeta=$ $\xi\left(x-a, c_{0}\right) q^{(a, \alpha(\sigma))} w^{a}$ by (i). This proves (ii).

Next we prove (iii). We choose $\tau \in \operatorname{Del}(\mathbf{c})$. We choose and fix any $x \in X$ and let $\bar{x} \in X / X(\mathbf{c})$ be the class of $x$. We define

$$
R(\mathbf{c})_{\bar{x}}:=\bigoplus_{z \in x+X(\mathbf{c})} k(0) \cdot \xi(z, \mathbf{c})=k(0)[X(\mathbf{c})] \cdot \xi(x, \mathbf{c})
$$

If necessary, by multiplying $\xi(x, \mathbf{c})$ by a product of $\xi_{c_{j}} / \xi_{c_{0}}$, which is of the form $q^{(a, \alpha(\tau))} w^{a}$ for some $a \in X(\mathbf{c})$, we can choose $\xi(x, \mathbf{c})$.
$q^{(a, \alpha(\tau))} w^{a} \in R\left(c_{0}\right)$ as a generator of $k(0)[X(\mathbf{c})]$-module $k(0)[X(\mathbf{c})] \xi(x, \mathbf{c})$. Hence we may assume $\xi(x, \mathbf{c}) \in R\left(c_{0}\right)$ from the start. The element $\xi(x, \mathbf{c})$ is nonzero on $U(\mathbf{c})$, hence $\xi(x, \mathbf{c})=\xi\left(x, c_{0}\right)$ by (i). Next for $N$ large enough we choose $x_{N}$ instead of $x$ with the notation of Lemma 5.6. Then by Lemma 5.6 there is $\sigma \in$ Del such that $x_{N} \in C\left(0, \sigma-c_{0}\right)^{0} \cap X$, $|\mathbf{c}| \subset \sigma$ and

$$
R(\mathbf{c})_{\bar{x}}=k(0)[X(\mathbf{c})] \cdot \xi\left(x, c_{0}\right)=k(0)[X(\mathbf{c})] \cdot \xi\left(x_{N}, c_{0}\right)
$$

where

$$
\xi\left(x_{N}, c_{0}\right)=\xi\left(x, c_{0}\right) \cdot \prod_{j=1}^{q}\left(\frac{\xi_{c_{j}}}{\xi_{c_{0}}}\right)^{N}
$$

Hence $x+c_{0}=x_{N}+c_{0}-N \sum_{j=1}^{q}\left(c_{j}-c_{0}\right) \in C(\mathbf{c}, \sigma) \cap X$. This proves (iii).
Q.E.D.

Definition 5.9. Let $\mathbf{c}=\left(c_{0}, \cdots, c_{q}\right)$ with $c_{j} \in X$ cellmates. We define $\ell(\mathbf{c})=q$. For a Delaunay cell $\sigma \in \operatorname{Del}^{(g-k)}(\mathbf{c})$, we define

$$
\begin{gathered}
F_{\sigma}^{k, q}(\mathbf{c})=\bigoplus_{\substack{x+c_{0} \in C(\mathbf{c}, \sigma) \cap x \\
\ell(\mathbf{c})=q}} k(0) \cdot \xi(x, \mathbf{c}), \\
F^{k, q}(\mathbf{c})=\bigoplus_{\substack{|\mathbf{c}| \subset \in \in \operatorname{Del}^{(g-k)} \\
\ell(\mathbf{c})=q}} F_{\sigma}^{k, q}(\mathbf{c})=\bigoplus_{\substack{\sigma \in \operatorname{Del}^{(g-k)}(\mathbf{c}) \\
\ell(\mathbf{c})=q}} F_{\sigma}^{k, q}(\mathbf{c}), \\
F^{k, q}=\bigoplus_{\substack{\mathbf{c}: \text { cellmates } \\
\ell(\mathbf{c})=q}} F^{k, q}(\mathbf{c})=\bigoplus_{\substack{\mathbf{c}: \text { cellmates } \\
\ell(\mathbf{c})=q}}\left(\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)}(\mathbf{c})} F_{\sigma}^{k, q}(\mathbf{c})\right) .
\end{gathered}
$$

where $F^{k, 0}(c)=F^{k}(c)$ for $c \in X$.
The definition of $F_{\sigma}^{k, q}(\mathbf{c})$ is independent of the choice of $c_{0} \in|\mathbf{c}|$. We note that if $\mathbf{c}=\left(c_{0}, c_{1}, \cdots, c_{q}\right)$ are not cellmates or if $\mathbf{c}=\left(c_{0}, c_{1}, \cdots, c_{q}\right)$ are cellmates but $|\mathbf{c}| \not \subset \sigma$, then $F^{k, q}(\mathbf{c})=0$. For $\sigma \in \operatorname{Del}^{(g-k)}$ we also define

$$
F_{\sigma}^{k}=\bigoplus_{q=0}^{\infty} F_{\sigma}^{k, q}, \quad F_{\sigma}^{k, q}=\bigoplus_{|\mathbf{c}| \subset \sigma, \ell(\mathbf{c})=q} F_{\sigma}^{k, q}(\mathbf{c})
$$

Finally we define $\partial^{k, q}: F^{k, q}(\mathbf{c}) \rightarrow F^{k+1, q}(\mathbf{c})$ by

$$
\partial^{k, q}\left(\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)}(\mathbf{c})} \phi_{\sigma}\right)=\bigoplus_{\tau \in \operatorname{Del}^{(g-k-1)}(\mathbf{c})|\mathbf{c}| \subset \tau \subset \sigma}[\sigma: \tau] \operatorname{res}_{\tau}^{\sigma}\left(\phi_{\sigma}\right)
$$

where $\phi_{\sigma} \in F_{\sigma}^{k, q}(\mathbf{c})$, and the summation in RHS ranges over the set of all $\sigma$ containing a fixed $\tau$ as a face of codimension one. We note $\partial^{k+1, q} \cdot \partial^{k, q}=0$.

Lemma 5.10. Suppose $q \geq 1$ and that $c_{0}, \cdots, c_{q-1}, c_{q}$ are cellmates. Let $\mathbf{c}^{\prime}=\left(c_{0}, \cdots, c_{q-1}\right), \mathbf{c}=\left(c_{0}, \cdots, c_{q}\right)$ and $\sigma \in \operatorname{Del}^{(g-k)}(\mathbf{c})$. Let

$$
\begin{aligned}
F_{\sigma}^{k, q-1}\left(\mathbf{c}^{\prime}\right) & =\bigoplus_{x+c_{0} \in C\left(\mathbf{c}^{\prime}, \sigma\right) \cap X} k(0) \cdot \xi\left(x, \mathbf{c}^{\prime}\right) \\
F_{\sigma}^{k, q}(\mathbf{c}) & =\bigoplus_{x+c_{0} \in C(\mathbf{c}, \sigma) \cap X} k(0) \cdot \xi(x, \mathbf{c})
\end{aligned}
$$

Then $\xi\left(x, \mathbf{c}^{\prime}\right)=\xi(x, \mathbf{c})$.
Proof. It is clear from $\sigma \in \operatorname{Del}(\mathbf{c})$ that $\sigma \in \operatorname{Del}\left(\mathbf{c}^{\prime}\right)$. If $x \in C(0, \sigma-$ $\left.c_{0}\right) \cap X$, then $\xi\left(x, \mathbf{c}^{\prime}\right)=\xi(x, \mathbf{c})=\xi\left(x, c_{0}\right)$ by Lemma 5.8. Otherwise we choose $a \in X\left(\mathbf{c}^{\prime}\right)$ such that $x-a \in C\left(0, \sigma-c_{0}\right) \cap X$. Then $\xi\left(x-a, \mathbf{c}^{\prime}\right)=$ $\xi(x-a, \mathbf{c})=\xi\left(x-a, c_{0}\right)$. Let $\zeta=q^{(a, \alpha(\sigma))} w^{a}$ for the hole $\alpha(\sigma) \in \sigma$. Since $\zeta$ is a unit in both $R\left(\mathbf{c}^{\prime}\right)$ and $R(\mathbf{c})$, by the definition of generators $\xi\left(x, \mathbf{c}^{\prime}\right)$ and $\xi(x, \mathbf{c})$ we have $\xi\left(x, \mathbf{c}^{\prime}\right)=\xi\left(x-a, \mathbf{c}^{\prime}\right) \zeta$ and $\xi(x, \mathbf{c})=\xi(x-a, \mathbf{c}) \zeta$. It follows $\xi\left(x, \mathbf{c}^{\prime}\right)=\xi(x, \mathbf{c})$.
Q.E.D.

Lemma 5.11. Let $\mathbf{c}=\left(c_{0}, \cdots, c_{q}\right)$ be cellmates with $\ell(\mathbf{c})=q$. Then the following sequence of $k(0)[X(\mathbf{c})]$-modules is exact,

$$
0 \rightarrow R(\mathbf{c}) \rightarrow F^{0, q}(\mathbf{c}) \xrightarrow{\partial^{0, q}} F^{1, q}(\mathbf{c}) \rightarrow \cdots \rightarrow F^{g-1, q}(\mathbf{c}) \xrightarrow{\partial^{g-1, q}} F^{g, q}(\mathbf{c}) \rightarrow 0
$$

Proof. The proof is similar to that of Lemma 5.2. Imitating the proof of Lemma 5.2, for each class $\bar{x} \in X / X(\mathbf{c})$, we choose by Lemma 5.2 a Delaunay cell $\sigma_{\min }(x, \mathbf{c}) \in \operatorname{Del}(\mathbf{c})$ such that $x+c_{0} \in C\left(c_{0}, \sigma_{\min }(x, \mathbf{c})\right)^{0}$ and $x \in \bar{x}+X(\mathbf{c})$, which is uniquely determined by $\bar{x}$. In what follows, for each $\bar{x}$ we choose and fix the pair $\left(x, \sigma_{\min }(x, \mathbf{c})\right)$ such that $x+c_{0} \in$ $C\left(c_{0}, \sigma_{\min }(x, \mathbf{c})\right)^{0}$ and $x \in \bar{x}+X(\mathbf{c})$. Let $g-k=\operatorname{dim} \sigma_{\min }(x, \mathbf{c})$. We note $\sigma \in \operatorname{Del}(\mathbf{c})$ iff $\sigma_{\min }(x, \mathbf{c}) \subset \sigma$. For any $\sigma \in \operatorname{Del}(\mathbf{c})$, we have $x \in$ $C\left(0, \sigma-c_{0}\right)$ because $x \in C\left(0, \sigma_{\min }(x, \mathbf{c})-c_{0}\right)$. In what follow, for any $\sigma \in$ $\operatorname{Del}(\mathbf{c})$ we choose the same $\xi\left(x, c_{0}\right)$ as a common generator of $k(0)[X(\mathbf{c})]$ modules $F_{\sigma}^{k}(\mathbf{c})_{x}$ and $R(\mathbf{c})$.

For a fixed $x \in X$ (or a fixed class $x \in X / X(\mathbf{c})$ ) we define

$$
F_{\sigma}^{k}(\mathbf{c})_{x}:=\left\{\begin{array}{cl}
k(0)[X(\mathbf{c})] \cdot \xi\left(x, c_{0}\right) & \text { if } x \in C\left(0, \sigma-c_{0}\right) \cap X \\
0 & \text { (otherwise) }
\end{array}\right.
$$

and

$$
F^{k, q}(\mathbf{c})_{x}:=\bigoplus_{\substack{\sigma \in \operatorname{Del}(g-k)(\mathbf{c}) \\ x+c_{0} \in C(\mathbf{c}, \sigma) \cap X}} F_{\sigma}^{k}(\mathbf{c})_{x} .
$$

We also denote $R(\mathbf{c})_{\bar{x}}$ by $R(\mathbf{c})_{x}$. We define $\partial^{k, q}: F^{k, q}(\mathbf{c})_{x} \rightarrow$ $F^{k+1, q}(\mathbf{c})_{x}$ by restriction of $\partial^{k, q}$ in Definition 5.9. Thus we have a complex of $k(0)[X(\mathbf{c})]$-modules with coboundary operators $\partial^{k, q}$

$$
F^{0, q}(\mathbf{c})_{x} \xrightarrow{\partial^{0, q}} F^{1, q}(\mathbf{c})_{x} \xrightarrow{\partial^{1, q}} \cdots \xrightarrow{\partial^{g-2, q}} F^{g-1, q}(\mathbf{c})_{x} \xrightarrow{\partial^{g-1, q}} F^{g, q}(\mathbf{c})_{x} \rightarrow 0 .
$$

The exactness of the sequence as well as $R(\mathbf{c}) \simeq \operatorname{Ker}\left(\partial^{0, q}\right)$ is proved in a manner entirely analogous to Lemma 5.2.
Q.E.D.

Definition 5.12. Let $\theta_{c d}$ be the one cocycle associated with $L_{0}$ :

$$
\theta_{c d}=\xi_{d} / \xi_{c}
$$

In order to compute $H^{q}\left(Q_{0}, L_{0}^{n}\right)$ we define a complex ${ }_{n} R$ by

$$
{ }_{n} R^{q}=\bigoplus_{\ell(\mathbf{c})=q} R(\mathbf{c})
$$

where $f\left(c_{0}, \cdots, c_{q}\right) \in R(\mathbf{c})$ and $g\left(d_{0}, \cdots, d_{q}\right) \in R(\mathbf{d})$ are identified iff

$$
|\mathbf{c}|=|\mathbf{d}|, \quad \xi_{c_{0}}^{n} f\left(c_{0}, \cdots, c_{q}\right)=\xi_{d_{0}}^{n} g\left(d_{0}, \cdots, d_{q}\right) .
$$

We define the twisted coboundary operator $\delta_{n}^{q}:{ }_{n} R^{q} \rightarrow{ }_{n} R^{q+1}$ by

$$
\begin{aligned}
\xi_{c_{0}}^{n} g\left(c_{0}, c_{1}, \cdots, c_{q+1}\right)=\xi_{c_{1}}^{n} f & \left(c_{1}, c_{2}, \cdots, c_{q+1}\right) \\
& +\sum_{j=1}^{q+1}(-1)^{j} \xi_{c_{0}}^{n} f\left(c_{0}, \cdots, \hat{c}_{j}, \cdots, c_{q+1}\right)
\end{aligned}
$$

where $f=\sum f\left(c_{0}, c_{1}, \cdots, c_{q}\right) \in{ }_{n} R^{q}, g=\delta_{n}^{q} f \in{ }_{n} R^{q+1}$.
Definition 5.13. Now we define ${ }_{n} F^{k, q}$ and the twisted coboundary operator $\delta_{n}^{k, q}:{ }_{n} F^{k, q} \rightarrow{ }_{n} F^{k, q+1}$ so that the definitions of $\delta_{n}^{k, q}$ for ${ }_{n} R^{q}$ and ${ }_{n} F^{k, q}$ are compatible. Let $\mathbf{c}=\left(c_{0}, \ldots, c_{q}\right)$ be ordered cellmates, ${ }_{n} F^{k, q}(\mathbf{c})=F^{k, q}(\mathbf{c})$. We define

$$
{ }_{n} F^{k, q}=\bigoplus_{\ell(\mathbf{c})=q}{ }_{n} F^{k, q}(\mathbf{c})
$$

where $f\left(c_{0}, \cdots, c_{q}\right) \in{ }_{n} F^{k, q}(\mathbf{c})$ and $g\left(d_{0}, \cdots, d_{q}\right) \in{ }_{n} F^{k, q}(\mathbf{d})$ are identified iff

$$
|\mathbf{c}|=|\mathbf{d}|, \quad \xi_{c_{0}}^{n} f\left(c_{0}, \cdots, c_{q}\right)=\xi_{d_{0}}^{n} g\left(d_{0}, \cdots, d_{q}\right) .
$$

For $f \in{ }_{n} F^{k, q}$, we define $\delta_{n}^{k, q}:{ }_{n} F^{k, q} \rightarrow{ }_{n} F^{k, q+1}$ as follows.
Let $f=\bigoplus f\left(c_{0}, c_{1}, \cdots, c_{q}\right) \in{ }_{n} F^{k, q}$ and $g=\delta_{n}^{k, q} f \in{ }_{n} F^{k, q+1}$. Then

$$
\begin{aligned}
\xi_{c_{0}}^{n} g\left(c_{0}, c_{1}, \cdots, c_{q+1}\right)=\xi_{c_{1}}^{n} f & \left(c_{1}, c_{2}, \cdots, c_{q+1}\right) \\
& +\sum_{j=1}^{q+1}(-1)^{j} \xi_{c_{0}}^{n} f\left(c_{0}, \cdots, \hat{c_{j}}, \cdots, c_{q+1}\right)
\end{aligned}
$$

If $\mathbf{c}=\left(c_{0}, c_{1}, \cdots, c_{q}\right)$ are not cellmates, then we have ${ }_{n} F^{k, q}(\mathbf{c})=0$ and $f\left(c_{0}, c_{1}, \cdots, c_{q}\right)=0$ by definition. We note $\delta_{n}^{k, q} \cdot \delta_{n}^{k, q-1}=0$. Since we have $\delta_{n}^{k, q}\left({ }_{n} F_{\sigma}^{k, q}\right) \subset{ }_{n} F_{\sigma}^{k, q+1}$, we have a complex

$$
{ }_{n} F_{\sigma}^{k, 0} \xrightarrow{\delta_{\xrightarrow{k, 0}}}{ }_{n} F_{\sigma}^{k, 1} \xrightarrow{\delta_{n}^{k, 1}} \cdots \xrightarrow{\delta_{n}^{k, q-1}}{ }_{n} F_{\sigma}^{k, q} \xrightarrow{\delta_{n}^{k, q}}{ }_{n} F_{\sigma}^{k, q+1} \rightarrow \cdots .
$$

Definition 5.14. For each positive integer $n$, we define a double complex $\left({ }_{n} C^{\cdot}, \Delta_{n}^{*}\right)$ by

$$
\begin{gathered}
{ }_{n} C^{\cdot}=\bigoplus_{n} C^{p}, \quad{ }_{n} C^{p}=\bigoplus_{k+q=p}{ }_{n} F^{k, q}, \quad \Delta_{n}^{p}=\bigoplus_{k+q=p}\left(\partial^{k, q}+(-1)^{q} \delta_{n}^{k, q}\right), \\
{ }_{n} F^{k, q}=\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)} \bmod Y}{ }_{n} F_{\sigma}^{k, q}=\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)} \bmod Y}\left(\bigoplus_{x \in X}{ }_{n} F_{\sigma}^{k, q}[x]\right) \\
{ }_{n} F_{\sigma}^{k, q}[x]=\bigoplus_{\substack{|\mathbf{c}| \subset \sigma \\
\ell(\mathbf{c})=q}}{ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x]
\end{gathered}
$$

where ${ }_{n} F_{\sigma}^{k, q}[x]$ is the weight $x$-part of ${ }_{n} F_{\sigma}^{k, q}$, and $\partial^{k, q}$ on ${ }_{n} F_{\sigma}^{k, q}$ is defined to be $\partial^{k, q}$ on $F_{\sigma}^{k, q}$. We easily check

$$
\begin{gathered}
\Delta_{n}^{p+1} \cdot \Delta_{n}^{p}=0 \\
\partial^{k+1, q} \cdot \partial^{k, q}=0, \quad \delta_{n}^{k, q+1} \cdot \delta_{n}^{k, q}=0, \\
\delta_{n}^{k+1, q} \cdot \partial^{k, q}=\partial^{k, q+1} \cdot \delta_{n}^{k, q} \\
\partial^{k, q}\left({ }_{n} F^{k, q}\right) \subset{ }_{n} F^{k+1, q}, \quad \delta_{n}^{k, q}\left({ }_{n} F^{k, q}\right) \subset{ }_{n} F^{k, q+1} .
\end{gathered}
$$

We also check that $\delta_{n}^{k+1, q} \cdot \operatorname{res}_{\tau}^{\sigma}=\operatorname{res}_{\tau}^{\sigma} \cdot \delta_{n}^{k, q}$.
The following theorem will be proved in the section 6 .
Theorem 5.15. For any $\sigma \in \operatorname{Del}^{(g-k)}$, there is a natural isomorphism

$$
\mathbf{H}^{q}\left({ }_{n} F_{\sigma}^{k, \cdot}, \delta_{n}^{k, \cdot}\right)=\left\{\begin{array}{cc}
\bigoplus_{\frac{x}{n} \in \sigma \cap \frac{X}{n} \bmod Y} k(0) \cdot[x] & \text { if } q=0 \\
0 & \text { if } q>0
\end{array}\right.
$$

where $[x]$ denotes the monomial generator $\xi_{c}^{n} \xi(x-n c, c)$ of weight $x$, which is independent of the choice of $c \in \sigma \cap X$.

Remark 5.16. When $Q_{0}$ is reduced, the cohomology group in Theorem 5.15 coincides with $H^{q}\left(V(\sigma), L_{0}^{n} \otimes O_{V(\sigma)}\right)$. However there might be no subscheme of $Q_{0}$ which properly corresponds to $\sigma$ when $Q_{0}$ is nonreduced.

Theorem 5.17. Let $\left(Q_{0}, L_{0}\right)$ be a PSQAS with a level $G(K)$-structure, the closed fibre of $(Q, L)$. Then
(i) $\quad H^{q}\left(Q_{0}, L_{0}^{n}\right)=0$ for $q \geq 1$ and $n \geq 1$.
(ii) $\quad \operatorname{dim} H^{0}\left(Q_{0}, L_{0}^{n}\right)=n^{g} \sqrt{|K|}$ for $n \geq 1$.

Proof. We note that the assertion (i) is always true for any PSQASes.
We prove (i). First we consider the case where $\left(Q_{0}, L_{0}\right)$ is totally degenerate, in which case $\sqrt{|K|}=|X / Y|$ by [Nr99, Lemma 5.12, Lemma 7.11]. We use the complex $\left({ }_{n} C^{\cdot}, \Delta_{n}^{\prime}\right)$ to prove $H^{q}\left(Q_{0}, L_{0}^{n}\right)=0$.

First we compute the spectral sequences for the above complex. By Theorem 5.15

$$
E_{1}^{k, q}=\left\{\begin{array}{cl}
\bigoplus_{\sigma \in \operatorname{Del}^{(g-k)}}^{\bmod Y} \\
0 & \left.\bigoplus_{\frac{x}{n} \in \sigma \cap \frac{x}{n}} k(0) \cdot[x]\right) \\
\text { if } q=0 \\
& \text { if } q>0
\end{array}\right.
$$

It follows ' $E_{2}^{k, q}=0$ for $q>0$.
In view of Lemma 5.2 and Lemma 5.11

$$
{ }^{\prime \prime} E_{1}^{k, q}=\left\{\begin{array}{cl}
{ }_{n} R^{q} & \text { if } k=0 \\
0 & \text { if } k>0
\end{array}\right.
$$

Therefore we have

$$
\begin{aligned}
{ }^{\prime \prime} E_{2}^{k, q} & =\left\{\begin{array}{cl}
H^{q}\left({ }_{n} R, \delta_{n}\right) & \text { if } k=0 \\
0 & \text { if } k>0
\end{array}\right. \\
& =\left\{\begin{array}{cl}
H^{q}\left(Q_{0}, L_{0}^{n}\right) & \text { if } k=0 \\
0 & \text { if } k>0
\end{array}\right.
\end{aligned}
$$

because $U(\mathbf{c})$ is affine for any cellmates $\mathbf{c}$.
Since the spectral sequences degenerate at $E_{2}$-terms, we see

$$
H^{q}\left(Q_{0}, L_{0}^{n}\right)={ }^{\prime \prime} E_{2}^{0, q}=\mathbf{H}^{q}\left({ }_{n} C^{\cdot}, \Delta_{n}^{\cdot}\right)=^{\prime} E_{2}^{q, 0}
$$

Since the coboundary operator of the complex $\left({ }^{\prime} E_{1}^{;, 0}, \delta_{n}^{, 0}\right)$ is (regarded as) homogeneous (see the proof of Theorem 6.11), it suffices to
compute the weight $x$-part of the cohomology ${ }^{\prime} E_{2}^{q, 0}[x]$ of the complex. Let $\operatorname{Star}\left(\frac{x}{n}\right)$ be the union of $\sigma \in \operatorname{Del}$ such that $\frac{x}{n} \in \sigma$ and $\operatorname{Star}\left(\frac{x}{n}\right)^{0}$ the relative interior of $\operatorname{Star}\left(\frac{x}{n}\right)$. We see $H^{0}\left(\operatorname{Star}\left(\frac{x}{n}\right)^{0}, k(0)\right)=k(0)$ and $H^{q}\left(\operatorname{Star}\left(\frac{x}{n}\right)^{0}, k(0)\right)=0$ for $q>0$. It is also easy to see that the weight $x$-part of the complex $\left({ }^{\prime} E_{1}^{, 0}, \delta_{\dot{n}}{ }^{, 0}\right)$ is isomorphic to the cochain complex of $\operatorname{Star}\left(\frac{x}{n}\right)^{0}$ indexed by Delaunay cells. Hence for $q>0$

$$
\begin{gathered}
' E_{2}^{q, 0}[x]=H^{q}\left(\operatorname{Star}\left(\frac{x}{n}\right)^{0}, k(0)\right)=0 \quad(\forall x), \\
H^{q}\left(Q_{0}, L_{0}^{n}\right)={ }^{\prime \prime} E_{2}^{0, q}=^{\prime} E_{2}^{q, 0}=\bigoplus_{\frac{x}{n} \in \frac{X}{n} \bmod Y}{ }^{\prime} E_{2}^{q, 0}[x]=0 .
\end{gathered}
$$

Since $Q$ is flat over $R$, we have $\operatorname{dim} H^{0}\left(Q_{0}, L_{0}^{n}\right)=\operatorname{dim} H^{0}\left(Q_{\eta}, L_{\eta}^{n}\right)=$ $n^{g}|X / Y|$ where $\left(Q_{\eta}, L_{\eta}^{n}\right)$ is the generic fibre of $\left(Q, L^{n}\right)$. This completes the proof in the totally-degenerate case when $Y \subset m X$ for some $m \geq 3$.

Next we consider the case where $Y$ is not a subgroup of $m X$ for any $m \geq 3$. We note that $\left(Q_{0}, L_{0}\right)$ has an étale covering $\left(Q_{0}^{\prime}, L_{0}^{\prime}\right)=$ $\left(\widetilde{Q}_{0}, \widetilde{L}_{0}\right) / Y^{\prime}$ where we choose $Y^{\prime}=3 Y$. The second PSQAS $\left(Q_{0}^{\prime}, L_{0}^{\prime}\right)$ satisfies the assumption $Y^{\prime}=3 Y \subset 3 X$, from which we infer that $\operatorname{dim} H^{q}\left(Q_{0}^{\prime},\left(L_{0}^{\prime}\right)^{n}\right)=0$ for any $q>0$. Since $H^{q}\left(Q_{0}, L_{0}^{n}\right)$ is a direct summand of $H^{q}\left(Q_{0}^{\prime},\left(L_{0}^{\prime}\right)^{n}\right)=0$, we have $H^{q}\left(Q_{0}, L_{0}^{n}\right)=0$ for $q>0$. Once we prove $H^{q}\left(Q_{0}, L_{0}^{n}\right)=0$ for $q>0$, then since $Q$ is flat over $R$, we have $\operatorname{dim} H^{0}\left(Q_{0}, L_{0}^{n}\right)=\operatorname{dim} H^{0}\left(Q_{\eta}, L_{\eta}^{n}\right)=n^{g}|X / Y|=n^{g} \sqrt{|K|}$. Thus we complete the proof of the theorem in the totally degenerate case. The vanishing in the partially degenerate case follows easily from it by the standard argument. See [Nr99, Theorem 4.10].
Q.E.D.

The following is a corollary to Theorem 5.17.
Theorem 5.18. Let $k(0)$ be a field of characteristic prime to $|K|$, and $\left(Q_{0}, L_{0}\right)$ be a PSQAS over $k(0)$ with a level $G(K)$-structure. Then
(i) $\operatorname{dim} H^{0}\left(Q_{0}, L_{0}\right)=\sqrt{|K|}$
(ii) $H^{0}\left(Q_{0}, L_{0}\right)$ is an irreducible $G(K)$-module of weight one.

Proof. Since $H^{q}\left(Q_{0}, L_{0}\right)=0$ for $q>0$ by Theorem 5.17 , we see $H^{0}\left(Q_{0}, L_{0}\right)=\Gamma(Q, L) \otimes k(0)$. Therefore $\Gamma(Q, L) \otimes k(0)$ is an irreducible $G(K)$-module of weight one in view of [Nr99, Lemma 5.12]. This proves the theorem.
Q.E.D.

Corollary 5.19. Let $K$ be a finite symplectic abelian group and $\pi:(Q, L) \rightarrow S Q_{g, K}$ the universal family of PSQASes over $S Q_{g, K}$. Then $\pi_{*}\left(L^{n}\right)$ is locally free for any $n>0$.

Proof. Since $S Q_{g, K}$ is reduced by the definition of [Nr99, § 12], $\pi_{*}\left(L^{n}\right)$ is locally free by Theorem 5.17 and [M74, Corollary 2, p. 51].
Q.E.D.

## §6. Proof of Theorem 5.15

Lemma 6.1. Let $\sigma \in \operatorname{Del}^{(g)}$ and $\mathbf{c}=\left(c_{0}, \cdots, c_{q}\right)$ cellmates such that $|\mathbf{c}| \subset \sigma$. Suppose $0 \in|\mathbf{c}|$. Let $f_{j}(1 \leq j \leq N)$ be linear functions on $X_{\mathbf{R}}$ such that $C(0, \sigma)=\left\{x \in X_{\mathbf{R}} ; f_{j}(x) \geq 0(1 \leq j \leq N)\right\}, f_{j}\left(c_{k}\right)=$ $0(\forall j \leq n, \forall k)$ and $f_{j}\left(c_{k_{j}}\right)>0\left(\forall j>n, \exists k_{j}\right)$. Then we have

$$
C(\mathbf{c}, \sigma)=\left\{x \in X_{\mathbf{R}} ; f_{j}(x) \geq 0(\forall j \leq n)\right\}
$$

Proof. First we note that $f_{j}(1 \leq j \leq n)$ is the set of all $f_{j}$ whose restriction to $|\mathbf{c}|$ is identically zero. Let $S=\left\{x \in X_{\mathbf{R}} ; f_{j}(x) \geq 0(\forall j \leq\right.$ $n)\}$. Let $a \in X(\mathbf{c})_{\mathbf{R}}$ and $x \in C(0, \sigma)$. Then since $f_{j}$ is linear, $f_{j}(x+a)=$ $f_{j}(x)+f_{j}(a)=f_{j}(x) \geq 0$ for $j \leq n$. Therefore $C(\mathbf{c}, \sigma) \subset S$. We shall prove the converse. Let $\langle\mathbf{c}\rangle$ be the convex closure of $|\mathbf{c}|$. By the choice of $f_{k}(1 \leq k \leq N)$ there is an $a \in\langle\mathbf{c}\rangle$ such that $f_{j}(a)>0$ for any $j \geq n+1$. Hence if $x \in S$, then $f_{j}(x+A a)=f_{j}(x)+A f_{j}(a)>0$ for a large $A>0$. Hence $x+A a \in C(0, \sigma)$. Since $A a=A(a-0), a \in\langle\mathbf{c}\rangle$ and $0 \in|\mathbf{c}|$, we see $A a \in X(\mathbf{c})_{\mathbf{R}}$. This proves $x \in X(\mathbf{c})_{\mathbf{R}}+C(0, \sigma)=C(\mathbf{c}, \sigma)$. Q.E.D.

Lemma 6.2. Let $\sigma \in \mathrm{Del}^{(g)}, \mathbf{c}=\left(c_{0}, \cdots, c_{q}\right)$ cellmates such that $|\mathbf{c}| \subset \sigma$, and $\tau(\mathbf{c})$ the minimal Delaunay cell containing $|\mathbf{c}|$. Then $C(\mathbf{c}, \sigma)=C(\tau(\mathbf{c}), \sigma)$.

Proof. It should be cautioned that $X(\mathbf{c}) \neq X(\tau(\mathbf{c}))$ in general. We may assume $c_{0}=0$ without loss of generality. Then by Lemma 6.1 $C(\mathbf{c}, \sigma)=\left\{x \in X_{\mathbf{R}} ; f_{j}(x) \geq 0(\forall j \leq n)\right\}$. Let $H$ be a hyperplane of $X_{\mathbf{R}}$ defined by $f_{j}=0$ for some $j(1 \leq j \leq n)$. Then $H \cap \sigma$ is a face of $\sigma$. Since $|\mathbf{c}| \subset H \cap \sigma, \tau(\mathbf{c}) \subset H \cap \sigma$ by the definition of $\tau(\mathbf{c})$. Hence $f_{j}=0$ on $\tau(\mathbf{c})$, hence $f_{j}=0$ on $X(\tau(\mathbf{c}))$. It follows that $X(\tau(\mathbf{c})) \subset C(\mathbf{c}, \sigma)$. This proves the lemma.
Q.E.D.

Lemma 6.3. Let $\sigma \in \operatorname{Del}^{(g)}$ and $\tau$ and $\tau^{\prime}$ faces of $\sigma$ with $\tau \cap \tau^{\prime} \neq \emptyset$. Then $C(\tau, \sigma) \cap C\left(\tau^{\prime}, \sigma\right)=C\left(\tau \cap \tau^{\prime}, \sigma\right)$.

Proof. We may assume $0 \in \tau \cap \tau^{\prime}$ without loss of generality. It suffices to prove $C(\tau, \sigma) \cap C\left(\tau^{\prime}, \sigma\right) \subset C\left(\tau \cap \tau^{\prime}, \sigma\right)$. By the proof of Lemma 6.1 we have linear functions $f_{j}(1 \leq j \leq N)$ such that

$$
\begin{aligned}
C(0, \sigma) & =\left\{x \in X_{\mathbf{R}} ; f_{j}(x) \geq 0(1 \leq j \leq N)\right\} \\
C(0, \tau) & =\left\{x \in C(0, \sigma) ; f_{j}(x)=0(1 \leq j \leq n)\right\} \\
C\left(0, \tau^{\prime}\right) & =\left\{x \in C(0, \sigma) ; f_{j}(x)=0(1 \leq j \leq k \text { and } n+1 \leq j \leq m)\right\}
\end{aligned}
$$

It follows $C\left(0, \tau \cap \tau^{\prime}\right)=\left\{x \in C(0, \sigma) ; f_{j}(x)=0(\forall j \leq m)\right\}$. Hence

$$
C\left(\tau \cap \tau^{\prime}, \sigma\right)=\left\{x \in X_{\mathbf{R}} ; f_{j}(x) \geq 0(\forall j \leq m)\right\}
$$

By Lemma 6.1 we see

$$
\begin{aligned}
C(\tau, \sigma) & =\left\{x \in X_{\mathbf{R}} ; f_{j}(x) \geq 0(1 \leq j \leq n)\right\} \\
C\left(\tau^{\prime}, \sigma\right) & =\left\{x \in X_{\mathbf{R}} ; f_{j}(x) \geq 0(1 \leq j \leq k \text { and } n+1 \leq j \leq m)\right\}
\end{aligned}
$$

It follows that

$$
C(\tau, \sigma) \cap C\left(\tau^{\prime}, \sigma\right)=\left\{x \in X_{\mathbf{R}} ; f_{j}(x) \geq 0(1 \leq j \leq m)\right\}
$$

This completes the proof.
Q.E.D.

Example 6.4. Let $g=2$ and $B(x, x)=x_{1}^{2}+x_{2}^{2}$ for $x=x_{1} e_{1}+x_{2} e_{2} \in$ $X$. Let $\sigma=\left\langle 0, e_{1}, e_{1}+e_{2}, e_{2}\right\rangle, \tau=\{0\}$ and $\tau^{\prime}=\left\{e_{1}+e_{2}\right\}$. In this case,

$$
\begin{aligned}
C(\tau, \sigma) & =\left\{x_{1} e_{1}+x_{2} e_{2} ; x_{1}, x_{2} \geq 0\right\} \\
C\left(\tau^{\prime}, \sigma\right) & =\left\{x_{1} e_{1}+x_{2} e_{2} ; x_{1}, x_{2} \leq 1\right\}
\end{aligned}
$$

Hence $C(\tau, \sigma) \cap C\left(\tau^{\prime}, \sigma\right)=\sigma \neq \emptyset$, while $\tau \cap \tau^{\prime}=\emptyset$.
Next let $\rho=\left\langle 0, e_{1}\right\rangle$ and $\rho^{\prime}=\left\langle e_{2}, e_{1}+e_{2}\right\rangle$. We note $\rho \cap \rho^{\prime}=\emptyset$. Then

$$
\begin{aligned}
C(\rho, \sigma)= & \left\{x_{1} e_{1}+x_{2} e_{2} ; x_{2} \geq 0\right\}, C\left(\rho^{\prime}, \sigma\right)=\left\{x_{1} e_{1}+x_{2} e_{2} ; x_{2} \leq 1\right\} \\
& C(\rho, \sigma) \cap C\left(\rho^{\prime}, \sigma\right)=\left\{x_{1} e_{1}+x_{2} e_{2} ; 0 \leq x_{2} \leq 1\right\}
\end{aligned}
$$

Thus Lemma 6.3 is true only when $\tau \cap \tau^{\prime}$ is nonempty.
Definition 6.5. We choose and fix $\sigma \in \operatorname{Del}^{(g)}$. For each $\rho \in$ $\operatorname{Del}_{\sigma}^{(g-1)}, C(\rho, \sigma)$ is a closed half-space of $X_{\mathbf{R}}$. Let $C(\rho, \sigma)^{c}$ be the complement of $C(\rho, \sigma)$ in $X_{\mathbf{R}}$. Let $\mathcal{H}:=\mathcal{H}(\sigma)$ be the set of all hyperplanes of $X_{\mathbf{R}}$ of the form $H(\rho):=\rho+X(\rho)_{\mathbf{R}}$ for some $\rho \in \operatorname{Del}_{\sigma}^{(g-1)}$. For any subset $\mathcal{H}^{\prime}$ of $\mathcal{H}(\sigma)$ we define

$$
D\left(\mathcal{H}^{\prime}\right)=\left(\bigcap_{H(\rho) \in \mathcal{H} \backslash \mathcal{H}^{\prime}} C(\rho, \sigma)\right) \bigcap\left(\bigcap_{H(\rho) \in \mathcal{H}^{\prime}} C(\rho, \sigma)^{c}\right)
$$

We note that the expression in RHS could be redundant because the intersection of some $C(\rho, \sigma)^{\prime}$ s could be a proper subset of another $C\left(\rho^{\prime}, \sigma\right)$. Let $\overline{D\left(\mathcal{H}^{\prime}\right)}$ be the closure of $D\left(\mathcal{H}^{\prime}\right)$ in $X_{\mathbf{R}}$ and $D\left(\mathcal{H}^{\prime}\right)^{0}$ the relative interior of $\overline{D\left(\mathcal{H}^{\prime}\right)}$. Each $D\left(\mathcal{H}^{\prime}\right)^{0}$ is an open connected domain of $X_{\mathbf{R}}$. If $\mathcal{H}^{\prime}=\emptyset$, then $D\left(\mathcal{H}^{\prime}\right)=\sigma$, while if $\mathcal{H}^{\prime}=\mathcal{H}(\sigma)$, then $D\left(\mathcal{H}^{\prime}\right)=\emptyset$.

Let $|\mathcal{H}(\sigma)|$ be the union of all $H(\rho) \in \mathcal{H}(\sigma)$. The complement of $|\mathcal{H}(\sigma)|$ in $X_{\mathbf{R}}$ is the disjoint union of $D\left(\mathcal{H}^{\prime}\right)^{0}$, while $X_{\mathbf{R}}$ is the disjoint union of $D\left(\mathcal{H}^{\prime}\right)$.

Lemma 6.6. Let $\sigma \in \operatorname{Del}^{(g)}$ and $x \in X_{\mathbf{R}}$. Let $B_{\sigma}(x)$ be the union of all faces $\tau$ of $\sigma$ such that $x \in C(\tau, \sigma)^{c}$. Then $B_{\sigma}(x)$ is the union of all $(g-1)$-dimensional faces $\rho$ of $\sigma$ such that $x \in C(\rho, \sigma)^{c}$.

Proof. Let $\tau^{*}$ be a face of $\sigma$. Then we remark that by the definition of $B_{\sigma}(x), x \in C\left(\tau^{*}, \sigma\right)^{c}$ iff $\tau^{*} \subset B_{\sigma}(x)$. Let $\tau$ be a face of $\sigma$. Then $\tau$ is the intersection of all $(g-1)$-dimensional faces of $\sigma$ containing $\tau$. By Lemma 6.3

$$
C(\tau, \sigma)=\underset{\rho \in \operatorname{Del}_{\sigma}^{(g-1)}(\tau)}{\cap} C(\rho, \sigma)
$$

Hence $x \in C(\tau, \sigma)^{c}$ iff $x \in C(\rho, \sigma)^{c}\left(\exists \rho \in \operatorname{Del}_{\sigma}^{(g-1)}(\tau)\right)$, and by the above remark, iff $\tau \subset \rho \subset B_{\sigma}(x) \quad\left(\exists \rho \in \operatorname{Del}_{\sigma}^{(g-1)}\right)$. This proves the lemma.
Q.E.D.

Lemma 6.7. Let $\sigma \in \mathrm{Del}^{(g)}$. If $x \in \sigma$, then $B_{\sigma}(x)=\emptyset$.
Proof. If $x \in \sigma$, then $x \in C(\tau, \sigma)$ for any $\tau \in \operatorname{Del}_{\sigma}$. It follows that $B_{\sigma}(x)=\emptyset$.
Q.E.D.

Lemma 6.8. Let $\sigma \in \operatorname{Del}^{(g)}$ and $x \in X_{\mathbf{R}} \backslash \sigma$. Then $B_{\sigma}(x)$ is nonempty, connected and contractible.

This is a corollary to the following more general lemma.
Lemma 6.9. Let $\Delta$ be a bounded convex polytope in $X_{\mathbf{R}}=\mathbf{R}^{g}, \mathcal{H}$ the set of one-codimensional faces of $\Delta$. For a one-codimensional face $\rho$ of $\Delta$ we define $H(\rho)$ a hyperplane of $X_{\mathbf{R}}$ spanned by $\rho, C(\rho, \Delta)$ the closed half space of $X_{\mathbf{R}}$ bounded by $H(\rho)$ containing $\Delta, C(\rho, \Delta)^{c}$ the complement of $C(\rho, \Delta)$ in $X_{\mathbf{R}}$. For any point $x$ of $X_{\mathbf{R}} \backslash \Delta$. Let $B_{\Delta}(x)$ be the union of one-codimensional faces of $\Delta$ with $x \in C(\rho, \Delta)^{c}$. Then $B_{\Delta}(x)$ is connected and contractible.

Proof. To explain our idea let us first suppose that $\Delta$ is a closed ball of dimension $g$. Let $\partial \Delta$ be the boundary of $\Delta$, and $x$ a point outside of $\Delta$. Set a source of light at $x$ and light the ball up from $x$. Let $B_{\Delta}(x)$ be the part of $\partial \Delta$ illuminated by the light. It is clear that $B_{\Delta}(x)$ is homeomorphic to a hemisphere, hence homeomorphic to a closed ball of dimension $g-1$.

Now we turn to the proof of our lemma. Let $\Delta$ be a convex polytope of dimension $g, \partial \Delta$ the boundary of it and $x$ a point outside of $\Delta$. Set a source of light at $x$ and light the polytope $\Delta$ up from $x$. Then for a one-codimensional face $\rho$ of $\Delta, x \in C(\rho, \Delta)^{c}$ iff $\rho$ is illuminated by the light whose source is set at the point $x$. Here we regard that $\rho$ is not illuminated by the light if the source of the light is set at a point $x$ on the hyperplane $H(\rho)$ spanned by $\rho$. Since $\Delta$ is convex, the part of
$\partial \Delta$ illuminated by the light is the union of $\rho$ with $x \in C(\rho, \Delta)^{c}$, that is, $B_{\Delta}(x)$. This proves that $B_{\Delta}(x)$ is homeomorphic to a hemisphere, hence it is a nonempty connected contractible subset of $\partial \Delta$. Q.E.D.

Lemma 6.10. Let $\sigma \in \operatorname{Del}^{(g)}$ and $x \in X_{\mathbf{R}}$. Let $\sharp(\sigma \cap X)=N+1$ and $\Delta(\sigma)$ an abstract $N$-dimensional simplex with vertices $\sigma \cap X$. For any subset $S$ of $\sigma \cap X$, we define $\Delta(S)$ to be the subsimplex of $\Delta(\sigma)$ spanned by $S$, and $B_{\Delta(\sigma)}(x)$ be the union of all $\Delta(S)$ such that $x \in C(S, \sigma)^{c}$ and $S \subset \sigma \cap X$. Then
(i) $\quad B_{\Delta(\sigma)}(x)$ is the union of $\Delta(\rho \cap X)$ for all $(g-1)$-dimensional faces $\rho$ of $\sigma$ such that $x \in C(\rho, \sigma)^{c}$.
(ii) $\quad B_{\Delta(\sigma)}(x)$ is nonempty, connected and contractible.

Proof. Let $\mathbf{c}$ be cellmates and $\tau(\mathbf{c})$ the minimal face of $\sigma$ such that $|\mathbf{c}| \subset \tau(\mathbf{c})$. Let $S=|\mathbf{c}|$. By Lemma 6.2, $C(S, \sigma)=C(\mathbf{c}, \sigma)=C(\tau(\mathbf{c}), \sigma)$. Hence by Lemma 6.6

$$
\begin{aligned}
\Delta(S) \subset B_{\Delta(\sigma)}(x) & \Longleftrightarrow x \in C(S, \sigma)^{c} \\
& \Longleftrightarrow x \in C(\tau(\mathbf{c}), \sigma)^{c} \\
& \Longleftrightarrow \tau(\mathbf{c}) \subset B_{\sigma}(x) \\
& \Longleftrightarrow \tau(\mathbf{c}) \subset \rho \subset B_{\sigma}(x) \quad\left(\exists \rho \in \operatorname{Del}_{\sigma}^{(g-1)}\right) \\
& \Longleftrightarrow S \subset \rho \subset B_{\sigma}(x) \quad\left(\exists \rho \in \operatorname{Del}_{\sigma}^{(g-1)}\right) \\
& \Longleftrightarrow \Delta(S) \subset \Delta(\rho \cap X) \subset B_{\Delta(\sigma)}(x) \quad\left(\exists \rho \in \operatorname{Del}_{\sigma}^{(g-1)}\right)
\end{aligned}
$$

This proves (i). Next we prove (ii). By (i) $B_{\Delta(\sigma)}(x)$ is the union of $\Delta(\rho \cap X)$ such that $\rho \subset B_{\sigma}(x)$. For simplicity we denote $\Delta(\rho \cap X)$ by $\Delta(\rho)$.

Let $\rho \in \operatorname{Del}_{\sigma}^{(g-1)}$ such that $\rho \subset B_{\sigma}(x)$. Since $\Delta(\rho)$ is an abstract simplex with vertices $\rho \cap X$, we have a natural map $\pi_{\rho}$ from $\Delta(\rho)$ onto $\rho$. Thus for any vertex $P$ of $\rho$, we have a vertex of $\Delta(\rho)$ mapped to $P$, which we denote by $\Delta(P)$. Let $\rho \cap X=\left\{P_{0}, \cdots, P_{r}\right\}$. Then the natural map $\pi_{\rho}$ from $\Delta(\rho)$ onto $\rho$ is given by

$$
\Delta(\rho) \ni t_{0} \Delta\left(P_{0}\right)+\cdots+t_{r} \Delta\left(P_{r}\right) \mapsto t_{0} P_{0}+\cdots+t_{r} P_{r} \in \sigma
$$

where $t_{0}+\cdots+t_{r}=1$. When $\rho$ ranges over the set of the faces contained in $B_{\sigma}(x)$, the natural maps $\pi_{\rho}$ glue together to give rise to a natural surjective continuous polytope map $\pi: B_{\Delta(\sigma)}(x) \rightarrow B_{\sigma}(x)$. We prove that any fibre of $\pi$ is connected and contractible. Let $\rho$ be the above Delaunay cell and $P$ any point of $\rho$. Then the inverse image $\pi^{-1}(P)$ is the intersection of $\Delta(\rho)$ with an affine linear subspace $H_{P}: t_{0} P_{0}+t_{1} P_{1}+$ $\cdots+t_{r} P_{r}=P,\left(t_{0}+\cdots+t_{r}=1\right)$ in the $\left(t_{0}, \cdots, t_{r}\right)$-space $\mathbf{R}^{r+1}$. The
simplex $\Delta(\rho)$ is just the subset of $\mathbf{R}^{r+1}$ defined by $t_{0}+\cdots+t_{r}=1$ and $0 \leq t_{j} \leq 1$ for any $j=0,1, \cdots, r$. Since $\Delta(\rho)$ is convex, the intersection $H_{P} \cap \Delta(\rho)=\pi^{-1}(P)$ is connected and contractible. Since $B_{\sigma}(x)$ is connected and contractible, so is $B_{\Delta(\sigma)}(x)$. This proves (ii). Q.E.D.

Theorem 6.11. Let $x \in X, \sigma \in \operatorname{Del}{ }^{(g-k)}$ and let ${ }_{n} F_{\sigma}^{k, \cdot}$ be the complex defined in Definition 5.13. Let $\Delta(\sigma)$ be the abstract simplex with vertices $\sigma \cap X$. Then

$$
\begin{aligned}
\mathbf{H}^{q}\left({ }_{n} F_{\sigma}^{k, \cdot}[x], \delta_{n}^{k, \cdot}\right) & \simeq \mathbf{H}^{q}\left(C^{\cdot}\left(\Delta(\sigma), B_{\Delta(\sigma)}\left(\frac{x}{n}\right)\right)\right) \\
& = \begin{cases}k(0) & \text { if } q=0 \text { and } \frac{x}{n} \in \sigma \\
0 & \text { otherwise }\end{cases}
\end{aligned}
$$

Proof. Since the coboundary operator $\delta_{n}^{k, q}$ of the complex ${ }_{n} F^{k,}$ is (regarded as) homogeneous in the sense we are going to explain, it suffices to compute the cohomology of the complex for a fixed weight $x \in X$.

Let $f \in{ }_{n} F^{k, q}$ and $g=\delta_{n}^{k, q}(f)$. Then by the definition of the coboundary operator $\delta_{n}^{k, q}$ we have the equality as

$$
\begin{aligned}
\xi_{c_{0}}^{n} g\left(c_{0}, c_{1}, \cdots, c_{q+1}\right)=\xi_{c_{1}}^{n} f & \left(c_{1}, c_{2}, \cdots, c_{q+1}\right) \\
& +\sum_{j=1}^{q+1}(-1)^{j} \xi_{c_{0}}^{n} f\left(c_{0}, \cdots, \hat{c_{j}}, \cdots, c_{q+1}\right)
\end{aligned}
$$

which is homogeneous with regard to the weights $X$.
Let $\sigma \in \operatorname{Del}{ }^{(g-k)}$. Let ${ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x]$ be the weight $x$-part of ${ }_{n} F_{\sigma}^{k, q}(\mathbf{c})$ in the above sense. For brevity we first consider the case $\frac{x}{n} \in \sigma \cap \frac{X}{n}$.

Let $\mathbf{c}=\left(c_{0}, \cdots, c_{q}\right)$ be cellmates with $|\mathbf{c}| \subset \sigma$. Then $\xi\left(x-n c_{0}, \mathbf{c}\right)=$ $\xi\left(x-n c_{0}, c_{0}\right)$ by Lemma 5.10. We see that

$$
\begin{aligned}
\frac{x}{n} \in \sigma \cap \frac{X}{n} & \Longleftrightarrow \frac{x}{n} \in C(c, \sigma) \cap \frac{X}{n} \quad(\forall c \in \sigma \cap X) \\
& \Longleftrightarrow \frac{x}{n}-c \in C(0, \sigma-c) \cap \frac{X}{n} \quad(\forall c \in \sigma \cap X) \\
& \Longleftrightarrow x-n c \in C(0, \sigma-c) \cap X \quad(\forall c \in \sigma \cap X) \\
& \Longleftrightarrow(x-n c)+c \in C(c, \sigma) \cap X \quad(\forall c \in \sigma \cap X) .
\end{aligned}
$$

If $\frac{x}{n} \in \sigma \cap \frac{X}{n}$, then $(x-n c)+c \in C(c, \sigma) \cap X \subset C(\mathbf{c}, \sigma) \cap X$. Hence

$$
{ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x]=k(0) \cdot \xi\left(x-n c_{0}, \mathbf{c}\right)=k(0) \cdot \xi\left(x-n c_{0}, c_{0}\right)
$$

by Definition 5.9. Hence we have

$$
{ }_{n} F_{\sigma}^{k, q}[x]=\bigoplus_{\substack{|\mathbf{c |}|{ }_{2} \\ \ell(\mathbf{c})=q}}{ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x]=\bigoplus_{\substack{|\mathbf{c}| \subset \sigma \\ \ell(\mathbf{c})=q}} k(0) \Delta(\mathbf{c})^{*}
$$

where $\Delta(\mathbf{c})^{*}$ is the dual cochain of an abstract $q$-simplex $\Delta(\mathbf{c})$ with vertices $|\mathbf{c}|$ and $\ell(\mathbf{c})=q$. Thus we see that the complex $\left(F_{\sigma}^{k, \cdot}[x], \delta^{k, \cdot}\right)$ is isomorphic to the standard cochain complex over $k(0)$ of an abstract $N$-simplex $\Delta(\sigma)$ with vertices $\sigma \cap X$.

Let $N=\sharp(\sigma \cap X)-1$. We note that $N$ could be different from the real dimension of $\sigma$. Since the $N$-simplex $\Delta(\sigma)$ is contractible to one point, we have

$$
\mathbf{H}^{q}\left({ }_{n} F_{\sigma}^{k, \cdot}[x], \delta^{k, \cdot}\right)= \begin{cases}k(0) \cdot[x] & \text { if } q=0 \\ 0 & \text { if } q>0\end{cases}
$$

where $[x]$ denotes the (unique) monomial generator $\xi_{c}^{n} \xi(x-n c, c)$ of weight $x$, independent of the choice of $c(c \in \sigma \cap X)$. This proves the theorem when $\frac{x}{n} \in \sigma \cap \frac{X}{n}$.

Now we consider the general case. For $\sigma \in \operatorname{Del}^{(g-k)}$ we define $H(\sigma):=\sigma+X(\sigma) \otimes \mathbf{R}$. Note that $\operatorname{dim} H(\sigma)=g-k=\operatorname{dim} \sigma$. First we prove that for any $x \in H(\sigma) \cap X$

$$
{ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x]= \begin{cases}k(0) \cdot \xi\left(x-n c_{0}, \mathbf{c}\right) & \text { if } \frac{x}{n} \in C(\mathbf{c}, \sigma) \\ 0 & \text { otherwise }\end{cases}
$$

where $\mathbf{c}=\left(c_{0}, \cdots, c_{q}\right)$. In fact, ${ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x]=k(0) \cdot \xi\left(x-n c_{0}, \mathbf{c}\right)$ iff $x-n c_{0}+c_{0} \in C(\mathbf{c}, \sigma)$ by the definition of ${ }_{n} F_{\sigma}^{k, q}$. We also see

$$
\begin{aligned}
x-n c_{0}+c_{0} \in C(\mathbf{c}, \sigma) & \Longleftrightarrow x-n c_{0} \in C\left(0, \sigma-c_{0}\right)+X(\mathbf{c})_{\mathbf{R}} \\
& \Longleftrightarrow \frac{x}{n}-c_{0} \in C\left(0, \sigma-c_{0}\right)+X(\mathbf{c})_{\mathbf{R}} \\
& \Longleftrightarrow \frac{x}{n} \in C(\mathbf{c}, \sigma)
\end{aligned}
$$

Therefore ${ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x] \simeq k(0)$ iff $\frac{x}{n} \in C(\mathbf{c}, \sigma)$.
We recall the modified generator $\xi_{c_{0}}^{n} \xi\left(x-n c_{0}, \mathbf{c}\right)=\xi_{c_{0}}^{n} \xi\left(x-n c_{0}, c_{0}\right)$ is independent of the choice of both $c_{0} \in \mathbf{c}$ and $\mathbf{c}$, and it depends only on $\sigma$ (Lemma 5.10) because $\xi_{c_{0}}^{n} \xi\left(x-n c_{0}, c_{0}\right)$ and $\xi_{c_{1}}^{n} \xi\left(x-n c_{1}, c_{1}\right)$ are identified in ${ }_{n} F^{k, q}$ by Definition 5.13.

Let $\sigma \in \operatorname{Del}^{(g-k)}$ and cellmates $\mathbf{c}$ such that $|\mathbf{c}| \subset \sigma$. Then by Lemma 6.2 we see

$$
\begin{aligned}
{ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x]=0 & \Longleftrightarrow \frac{x}{n} \in C(\mathbf{c}, \sigma)^{c} \\
& \Longleftrightarrow \frac{x}{n} \in C(\tau(\mathbf{c}), \sigma)^{c} \\
& \Longleftrightarrow \tau(\mathbf{c}) \subset B_{\sigma}\left(\frac{x}{n}\right) \Longleftrightarrow|\mathbf{c}| \subset B_{\sigma}\left(\frac{x}{n}\right)
\end{aligned}
$$

where $q=\ell(\mathbf{c}), \tau(\mathbf{c})$ is the minimal face of $\sigma$ such that $|\mathbf{c}| \subset \tau$. It follows that

$$
{ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x] \simeq k(0) \Longleftrightarrow|\mathbf{c}| \not \subset B_{\sigma}\left(\frac{x}{n}\right)
$$

Thus there is an isomorphism of $k(0)$-modules

$$
{ }_{n} F_{\sigma}^{k, q}[x]:=\bigoplus_{\substack{\ell(\mathbf{c})=q \\|\mathbf{c}| \subset \sigma}}{ }_{n} F_{\sigma}^{k, q}(\mathbf{c})[x] \simeq C^{q}\left(\Delta(\sigma), B_{\Delta(\sigma)}\left(\frac{x}{n}\right)\right)
$$

It is easy to see that this induces an isomorphism between the complex ${ }_{n} F_{\sigma}^{k, \cdot}[x]$ and the relative cochain complex $C^{\cdot}\left(\Delta(\sigma), B_{\Delta(\sigma)}\left(\frac{x}{n}\right)\right)$. By Lemma 6.10 if $B_{\Delta(\sigma)}\left(\frac{x}{n}\right)$ is nonempty, then $H^{q}\left(\Delta(\sigma), B_{\Delta(\sigma)}\left(\frac{x}{n}\right)\right)=0$ for any $q$. If $B_{\Delta(\sigma)}\left(\frac{x}{n}\right)$ is empty $\left(\Longleftrightarrow \frac{x}{n} \in \sigma\right)$, then $H^{q}\left(\Delta(\sigma), B_{\Delta(\sigma)}\left(\frac{x}{n}\right)\right)=$ $k(0)$ (resp. 0) for $q=0$ (resp. $q>0$ ). It follows that

$$
\begin{aligned}
\mathbf{H}^{q}\left({ }_{n} F_{\sigma}^{k, \cdot}[x]\right) & =\mathbf{H}^{q}\left(C^{\cdot}\left(\Delta(\sigma), B_{\Delta(\sigma)}\left(\frac{x}{n}\right)\right)\right) \\
& = \begin{cases}k(0) & \text { if } q=0 \text { and } \frac{x}{n} \in \sigma \\
0 & \text { otherwise }\end{cases}
\end{aligned}
$$

This completes the proof of Theorem 6.11, hence of Theorem 5.15.
Q.E.D.

Example 6.12. Here is an example. Let $k=k(0), g=2, X=$ $\mathbf{Z} e_{1}+\mathbf{Z} e_{2}$ and $B(x, x)=2\left(x_{1}^{2}-x_{1} x_{2}+x_{2}^{2}\right)$ for $x=x_{1} e_{1}+x_{2} e_{2} \in X$. Let

$$
\begin{gathered}
e_{1}=(1,0), e_{2}=(0,1), c_{0}=0, c_{1}=e_{1}, c_{2}=e_{1}+e_{2} \\
c_{3}=e_{2}, c_{4}=-e_{1}, c_{5}=-e_{1}-e_{2}, c_{6}=-e_{2}
\end{gathered}
$$

Let $\sigma$ (resp. $\sigma^{\prime}$ ) be the convex closure $\left\langle c_{0}, c_{1}, c_{2}\right\rangle$ (resp. $\left\langle c_{0}, c_{2}, c_{3}\right\rangle$ ). Any Delaunay two-cell is a translate by $X$ of either $\sigma$ or $\sigma^{\prime}$. $\operatorname{Star}(0)$ is the convex closure of $c_{j}(j=1, \cdots, 6)$, which is a hexagon with the six vertices $c_{j}$.

There are essentially different three cases
(i) $x \in \sigma$,
(ii) $\quad x \in C\left(c_{0}, \sigma\right) \backslash \sigma$,
(iii) $\quad x \in C\left(c_{0}, c_{1}, \sigma\right) \backslash \bigcup_{i=0,1} C\left(c_{i}, \sigma\right)$.

In the case (i) $B_{\sigma}(x)=\emptyset$. In the case (ii) $B_{\sigma}(x)=\left\langle c_{1}, c_{2}\right\rangle$. In the case (iii) $B_{\sigma}(x)=\left\langle c_{0}, c_{2}\right\rangle \cup\left\langle c_{1}, c_{2}\right\rangle$. In the cases (ii) and (iii) $B_{\sigma}(x)$ is connected and contractible.

## §7. The $E_{8}$ lattice

In this section we recall the notation for $E_{8}$ [Bourbaki, pp. 268270]. Let $\mathbf{Z}^{8}$ be the lattice of rank 8 with the standard inner product, $e_{j}$ $(1 \leq j \leq 8)$ an orthogonal basis of it, and $\left(\frac{1}{2} \mathbf{Z}\right)^{8}$ the overlattice spanned by $\frac{1}{2} e_{j} \quad(1 \leq j \leq 8)$ with inner product induced naturally from that of $\mathbf{Z}^{8}$. Then the sublattice $X$ of $\left(\frac{1}{2} \mathbf{Z}\right)^{8}$ is defined to be

$$
\left\{\sum_{i=1}^{8} x_{i} e_{i} ; 2 x_{i} \in \mathbf{Z}, x_{i}+x_{j} \in \mathbf{Z}, \sum_{i=1}^{8} x_{i} \in 2 \mathbf{Z}\right\}
$$

with bilinear form inherited from $\left(\frac{1}{2} \mathbf{Z}\right)^{8}$. This is the lattice $E_{8}$.
Let $\left\{\alpha_{j}, j=1, \cdots, 8\right\}$ be a positive root system

$$
\begin{gathered}
\alpha_{1}=\frac{1}{2}\left(e_{1}+e_{8}-\left(e_{2}+\cdots+e_{7}\right)\right) \\
\alpha_{2}=e_{1}+e_{2}, \quad \alpha_{j}=e_{j-1}-e_{j-2} \quad(3 \leq j \leq 8)
\end{gathered}
$$

The maximal root $\alpha_{0}$ of the root system is given by

$$
\alpha_{0}=e_{7}+e_{8}=2 \alpha_{1}+3 \alpha_{2}+4 \alpha_{3}+6 \alpha_{4}+5 \alpha_{5}+4 \alpha_{6}+3 \alpha_{7}+2 \alpha_{8}\left(=\omega_{8}\right)
$$

We define $m_{j}(1 \leq j \leq 8)$ to be the multiplicity of $\alpha_{j}$ in $\alpha_{0}$. Thus for instance, $m_{1}=2, m_{2}=3$ and $m_{3}=4$. The root diagram of $\alpha_{j}$ $(1 \leq j \leq 8)$ is $E_{8}$, while the root diagram of $\alpha_{j}(0 \leq j \leq 8)$ is the extended Dynkin diagram $\widetilde{E}_{8}$ given below


We also define the dual roots $\omega_{k} \in X$ by $\left(\alpha_{j}, \omega_{k}\right)=\delta_{j k}$. Hence we have

$$
\begin{gathered}
\omega_{1}=2 e_{8}, \quad \omega_{2}=\frac{1}{2}\left(e_{1}+e_{2}+\cdots+e_{7}+5 e_{8}\right) \\
\omega_{3}=\frac{1}{2}\left(-e_{1}+e_{2}+\cdots+e_{7}+7 e_{8}\right), \quad \omega_{4}=e_{3}+e_{4}+\cdots+e_{7}+5 e_{8} \\
\omega_{5}=e_{4}+\cdots+e_{7}+4 e_{8}, \quad \omega_{6}=e_{5}+\cdots+e_{7}+3 e_{8} \\
\omega_{7}=e_{6}+e_{7}+2 e_{8}, \quad \omega_{8}=e_{7}+e_{8}
\end{gathered}
$$

For any $\alpha \in X(\neq 0)$ we define a hyperplane $H_{\alpha}$ of $X \otimes \mathbf{R}$ to be $H_{\alpha}=\left\{x \in X_{\mathbf{R}} ; \alpha(x)=0\right\}$ and the linear transformation $r_{\alpha}$ of $X \otimes \mathbf{R}$ to be the reflection with regards to $H_{\alpha}$ :

$$
r_{\alpha}(x)=x-\frac{2(\alpha, x)}{(\alpha, \alpha)} \alpha
$$

If $\alpha$ is a root of $E_{8}$, then $r_{\alpha}(x)=x-(\alpha, x) \alpha$. We also define $r_{0}$ to be

$$
r_{0}(x)=x+\left(1-\left(\alpha_{0}, x\right)\right) \alpha_{0}
$$

Then $r_{0}$ is a reflection of $X_{\mathbf{R}}$ with regards to the hyperplane $H_{0}:=$ $\left\{x \in X_{\mathbf{R}} ;\left(\alpha_{0}, x\right)=1\right\}$. The seven reflections $r_{\alpha_{j}}(1 \leq j \leq 7)$ generate the Weyl group $W\left(E_{8}\right)$, while the eight reflections $r_{0}$ and $r_{\alpha_{j}}(1 \leq j \leq 7)$ generate the affine Weyl group $W\left(\widetilde{E}_{8}\right)$. The order of $W\left(E_{8}\right)$ equals $2^{14} \cdot 3^{5} \cdot 5^{2} \cdot 7$, while $W\left(\widetilde{E}_{8}\right)$ is of infinite order. We note that $r_{\alpha_{j}}$ keeps $\omega_{k} \quad(k \neq j)$ invariant because $\left(\alpha_{j}, \omega_{k}\right)=0$.


$A_{7}$


The diagram $D_{7}$ is a subdiagram of $E_{8}$ obtained by deleting $\alpha_{1}$. Therefore $W\left(D_{7}\right)$ is a subgroup of $W\left(E_{8}\right)$ naturally. Similarly since $A_{7}$ is $E_{8}$ with $\alpha_{2}$ deleted, $W\left(A_{7}\right)$ is a subgroup of $W\left(E_{8}\right)$. For a group $W$ acting on $X$, let $\operatorname{Stab}_{W}(\omega)$ (resp. $\operatorname{Stab}_{W}\left(\omega, \omega^{\prime}\right)$ ) be the stabilizer subgroup of $W$ of $\omega \in X$ (resp. of both $\omega$ and $\omega^{\prime} \in X$ ). Then $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\frac{\omega_{1}}{2}\right)=W\left(D_{7}\right)$ where $D_{7}=E_{8} \backslash\left\{\alpha_{1}\right\}$ because by [Bourbaki, Ch. 5, Prop. 2. p. 75] it is generated by the reflections $r_{\alpha}$ with roots $\alpha$ orthogonal to $\omega_{1}$. Similarly we see $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\frac{\omega_{2}}{3}\right)=W\left(A_{7}\right)$ where $A_{7}=E_{8} \backslash\left\{\alpha_{2}\right\}$.

## §8. Elements of the lattice $E_{8}$

Let $X$ be the lattice $E_{8}, a, b \in X,(a, b)$ the bilinear form of $E_{8}$ and $a^{2}=(a, a)$. We call $\sqrt{a^{2}}$ the length of $a$, which we denote $\|a\|$. An element $a \in X$ is called a root (of $E_{8}$ ) if $a^{2}=2$, equivalently, the length of $a$ equals $\sqrt{2}$.

Lemma 8.1. Any element $a \in X$ with $a^{2}=2$ is one of 240 roots:
(i) $\pm e_{i} \pm e_{j}(1 \leq i<j \leq 8)$,
(ii) $\frac{1}{2}\left(\sum_{j=1}^{8}(-1)^{\nu(j)} e_{j}\right)$ with $\sum_{j} \nu(j)$ even.

Any of them is $W\left(E_{8}\right)$-equivalent.
Proof. Any root $\alpha \in X$ with $\alpha^{2}=2$ is one of (i) and (ii). The number of these elements totals $112+128=240$, as is seen easily. Let $\alpha_{0}=e_{7}+e_{8}$ be the maximal root. Then $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\alpha_{0}\right)=W\left(E_{7}\right)$ by [Bourbaki, p. 75], whence the number of roots is equal to $\left|W\left(E_{8}\right) / W\left(E_{7}\right)\right|$ $\left(=2^{14} \cdot 3^{5} \cdot 5^{2} \cdot 7 / 2^{10} \cdot 3^{4} \cdot 5 \cdot 7=240\right)$. Hence the set of roots is transitive under $W\left(E_{8}\right)$.
Q.E.D.

Lemma 8.2. Any element $a \in X$ with $a^{2}=4$ is one of the following
(i) $\pm 2 e_{k}(1 \leq k \leq 8)$,
(ii) $\pm e_{i} \pm e_{j} \pm e_{k} \pm e_{\ell}(1 \leq i<j<k<\ell \leq 8)$,
(iii) $\pm \frac{1}{2}\left(3 e_{i}+\sum_{j \neq i}(-1)^{\nu(\bar{j})} e_{j}\right)$ with $\sum_{j \neq i} \nu(j)$ odd.

Any of them is $W\left(E_{8}\right)$-equivalent.
Proof. Let $a_{0}=2 e_{8}$. By [Bourbaki, p. 75] $\operatorname{Stab}_{W\left(E_{8}\right)}\left(a_{0}\right)=W\left(D_{7}\right)$, the subgroup of $W\left(E_{8}\right)$ generated by $r_{\alpha_{j}}(j \geq 2)$ because $\left(a_{0}, \alpha_{j}\right)=0$ for $j \neq 1$. Hence the orbit $W\left(E_{8}\right) \cdot a_{0}$ consists of 2160 elements where $2160=\left|W\left(E_{8}\right) / W\left(D_{7}\right)\right|$. Meanwhile the number of the elements of type (i), (ii) and (iii) are respectively $16,1120=2^{4} \cdot\binom{8}{4}$ and $1024=2^{7} \cdot\binom{8}{1}$ which totals 2160 . This shows that the above 2160 elements are in the single $W\left(E_{8}\right)$-orbit of $a_{0}$.
Q.E.D.

Lemma 8.3. Any element $a \in X$ with $a^{2}=6$ is one of the following
(i) $\pm e_{i} \pm e_{j} \pm 2 e_{k}$ for $i, j, k$ all distinct
(ii) $\sum_{k=1}^{6} \pm e_{i_{k}}\left(1 \leq i_{k} \leq 8\right)$ for $i_{k}$ all distinct
(iii) $\pm \frac{1}{2}\left(3 e_{i}+3 e_{j}+\sum_{k \neq i, j}(-1)^{\nu(k)} e_{k}\right)$ with $\sum_{k \neq i, j} \nu(k)$ even.

Any of them is $W\left(E_{8}\right)$-equivalent.
Proof. Let $a_{0}=e_{6}+e_{7}+2 e_{8}$. By [Bourbaki, p. 75] $\operatorname{Stab}_{W\left(E_{8}\right)}\left(a_{0}\right)=$ $W\left(A_{1} \times E_{6}\right)$, where the subgroup of $W\left(E_{6}\right)$ is generated by $r_{\alpha_{j}}(1 \leq j \leq$ $6)$ and $W\left(A_{1}\right)$ is generated by $r_{\alpha_{8}}$ because $\left(a_{0}, \alpha_{j}\right)=0$ for $j \neq 7$. Hence the orbit $W\left(E_{8}\right) \cdot a_{0}$ consists of $\left|W\left(E_{8}\right) / W\left(A_{1}\right)\right|\left|W\left(E_{6}\right)\right|=2^{14} \cdot 3^{5} \cdot 5^{2}$. $7 / 2^{8} \cdot 3^{4} \cdot 5=6720$ elements. Meanwhile the number of the elements of type (i), (ii) and (iii) are respectively $1344=2^{3} \cdot\binom{8}{1} \cdot\binom{7}{2}, 1792=2^{6} \cdot\binom{8}{6}$ and $3584=2^{7} \cdot\binom{8}{2}$ which totals 6720 . This shows that the above 6720 elements are in the single $W\left(E_{8}\right)$-orbit of $a$. Q.E.D.

Lemma 8.4. Any pair of $a, b \in X$ with $a^{2}=b^{2}=2$ and $(a, b)=0$ is $W\left(E_{8}\right)$-equivalent.

Proof. We may assume $a=e_{7}+e_{8}\left(=\alpha_{0}\right)$. Then $b \in X$ satisfying the conditions $b^{2}=2$ and $(a, b)=0$ are one of the following
(i) $\pm e_{i} \pm e_{j}$ for $i, j \in\{1,2,3,4,5,6\}$ and $i<j$,
(ii) $\pm\left(e_{7}-e_{8}\right)$,
(iii) $\frac{1}{2}\left(\sum_{j=1}^{8}(-1)^{\nu_{j}} e_{j}\right)$ with $\sum_{j} \nu_{j}$ even, $\nu_{7}+\nu_{8}=1$.

One counts the number of elements of (i), (ii) and (iii) respectively as 60,2 and 64. These total 126. Meanwhile let $\beta=-e_{7}+e_{8}$. Then $\beta$ is a root with $\left(\alpha_{0}, \beta\right)=0$ and $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\alpha_{0}\right)=W\left(E_{7}\right), \operatorname{Stab}_{W\left(E_{8}\right)}\left(\alpha_{0}, \beta\right)=$ $\operatorname{Stab}_{W\left(E_{7}\right)}(\beta)=W\left(D_{6}\right)$ by [Bourbaki, p. 75] because the subspace of $X$ orthogonal to $\alpha_{0}$ and $\beta$ is spanned by $\alpha_{j}(2 \leq j \leq 7)$. Let $F$ be the subset of roots $b$ of $E_{8}$ with $(a, b)=0$. We want to prove that there is $\sigma \in$ $W\left(E_{8}\right)$ such that $a=\sigma\left(\alpha_{0}\right)$ and $b=\sigma(\beta)$. Since $a$ is in the $W\left(E_{8}\right)$-orbit of $\alpha_{0}$ by Lemma 8.1, we may assume $a=\alpha_{0}$. We see $\mid \operatorname{Stab}_{W\left(E_{8}\right)}\left(\alpha_{0}\right)$. $\beta\left|=\left|\operatorname{Stab}_{W\left(E_{8}\right)}\left(\alpha_{0}\right) / \operatorname{Stab}_{W\left(E_{8}\right)}\left(\alpha_{0}, \beta\right)\right|=\left|W\left(E_{7}\right) / W\left(D_{6}\right)\right|=2^{10} \cdot 3^{4}\right.$. $5 \cdot 7 / 2^{5} \cdot 6!=2 \cdot 3^{2} \cdot 7=126$. It follows that the orbit $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\alpha_{0}\right) \cdot \beta$ consists of 126 elements. Hence $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\alpha_{0}\right)$ acts transitively on the set $F$. This completes the proof.
Q.E.D.

Lemma 8.5. Any pair of $a, b \in X$ with $a^{2}=4, b^{2}=2$ and $(a, b)=0$ is $W\left(E_{8}\right)$-equivalent to $a=2 e_{8}$ and $b=-e_{6}+e_{7}$.

Proof. We may assume $a=2 e_{8}$ by Lemma 8.2. Let $F$ be the set of all $b$ with $b^{2}=2$ and $(a, b)=0$. It is the set of all roots of $D_{7}$, $F=\left\{ \pm e_{i} \pm e_{j} ; 1 \leq i<j \leq 7\right\}$ where $D_{7}=E_{8} \backslash\left\{\alpha_{1}\right\}$. It follows $\operatorname{Stab}_{W\left(E_{8}\right)}\left(2 e_{8}\right)=W\left(D_{7}\right)$. Since $W\left(D_{7}\right)$ acts on $F$ transitively, so acts $\mathrm{Stab}_{W\left(E_{8}\right)}\left(2 e_{8}\right)$ on $F$. This proves the lemma.
Q.E.D.

Lemma 8.6. Any pair of $a, b \in X$ with $a^{2}=4, b^{2}=2$ and $(a, b)=1$ is $W\left(E_{8}\right)$-equivalent to $a=2 e_{8}$ and $b=\frac{1}{2}\left(\sum_{j=1}^{8} e_{j}\right)$.

Proof. We may assume $a=2 e_{8}$ by Lemma 8.2. Let $F$ be the set of all $b \in X$ with $b^{2}=2$ and $(a, b)=1$. Then $F=\left\{\frac{1}{2}\left(\sum_{j=1}^{7}(-1)^{\nu_{j}} e_{j}+\right.\right.$ $\left.e_{8}\right) ; \sum_{j=1}^{7} \nu_{j}$ even $\}$. We see $|F|=64$. Let $b=\frac{1}{2}\left(e_{1}+e_{2}+\cdots+e_{8}\right)$. Then we see $\operatorname{Stab}_{W\left(E_{8}\right)}(a)=W\left(D_{7}\right)$ and $\operatorname{Stab}_{W\left(E_{8}\right)}(a, b)=W\left(A_{6}\right)$ where $A_{6}=D_{7} \backslash\left\{\alpha_{2}\right\}$ because the subspace of $X$ orthogonal to $a$ and $b$ is spanned by $\alpha_{j}(3 \leq j \leq 8)$. It follows that the orbit $\operatorname{Stab}_{W\left(E_{8}\right)}(a) \cdot b$ consists of $\left|\operatorname{Stab}_{W\left(E_{8}\right)}(a) / \operatorname{Stab}_{W\left(E_{8}\right)}(a, b)\right|=\left|W\left(D_{7}\right) / W\left(A_{6}\right)\right|=2^{6}$. $7!/ 7!=64$ elements. This implies that the action of $\operatorname{Stab}_{W\left(E_{8}\right)}(a)$ on $F$ is transitive.
Q.E.D.

Corollary 8.7. Any pair of $a, b \in X$ with $a^{2}=4, b^{2}=2$ and $(a, b)=1$ is $W\left(E_{8}\right)$-equivalent to $a=e_{5}+e_{6}+e_{7}+e_{8}$ and $b=e_{4}+e_{8}$.

Lemma 8.8. Any pair of $a, b \in X$ with $a^{2}=4, b^{2}=2$ and $(a, b)=2$ is $W\left(E_{8}\right)$-equivalent to $a=2 e_{8}$ and $b=e_{7}+e_{8}$.

Proof. We may assume $a=2 e_{8}$ by Lemma 8.2. Let $F$ be the set of all $b \in X$ with $b^{2}=2$ and $(a, b)=2$. Then $F=\left\{ \pm e_{j}+e_{8} ; 1 \leq j \leq 7\right\}$ and $|F|=14$. Let $b=e_{7}+e_{8}$. Then $b \in F$ and $\operatorname{Stab}_{W\left(E_{8}\right)}(a)=$ $W\left(D_{7}\right), \operatorname{Stab}_{W\left(E_{8}\right)}(a, b)=W\left(D_{6}\right)$ where $D_{6}=D_{7} \backslash\left\{\alpha_{8}\right\}$. It follows that the orbit $\operatorname{Stab}_{W\left(E_{8}\right)}(a) \cdot b$ consists of $\left|\operatorname{Stab}_{W\left(E_{8}\right)}(a) / \operatorname{Stab}_{W\left(E_{8}\right)}(a, b)\right|=$ $\left|W\left(D_{7}\right) / W\left(D_{6}\right)\right|=2^{6} \cdot 7!/ 2^{5} \cdot 6!=14$ elements. This implies that the action of $\mathrm{Stab}_{W\left(E_{8}\right)}(a)$ on $F$ is transitive.
Q.E.D.

Lemma 8.9. Let $\left\{a_{k}, a_{k+1}, \cdots, a_{7}\right\}(1 \leq k \leq 7)$ be a set of roots such that $\left(a_{i}, a_{j}\right)=1$ for any $i \neq j$. Up to $W\left(E_{8}\right)$,
(i) if $k \geq 2$, it is equivalent to the set $\left\{e_{k}+e_{8}, e_{k+1}+e_{8}, \cdots, e_{7}+\right.$ $\left.e_{8}\right\}$.
(ii) if $k=1$, then it is equivalent to either $\left\{e_{1}+e_{8}, e_{2}+e_{8}, \cdots, e_{7}+\right.$ $\left.e_{8}\right\}$ or $\left\{-e_{1}+e_{8}, e_{2}+e_{8}, \cdots, e_{7}+e_{8}\right\}$.

Proof. We prove the lemma by the descending induction on $k$. The case $k=7$ follows from Lemma 8.2. Let $\beta_{j}=e_{j}+e_{8}(1 \leq j \leq 7)$. Next we consider the case $k=6$. We may assume $a_{7}=\beta_{7}$ by Lemma 8.2. Let $F$ be the set of all $a$ with $(a, a)=2$ and $\left(a, a_{7}\right)=1$. Then $|F|=56$. Then $\beta_{6} \in F$. Since $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\beta_{7}\right)=W\left(E_{7}\right)$ and $\operatorname{Stab}_{W\left(E_{7}\right)}\left(\beta_{6}\right)=W\left(E_{6}\right)$ where $E_{6}=E_{8} \backslash\left\{\alpha_{6}, \alpha_{7}\right\}$, we see $W\left(E_{7}\right) \cdot \beta_{6}=\left|W\left(E_{7}\right) / W\left(E_{6}\right)\right|=$ $2^{10} \cdot 3^{4} \cdot 5 \cdot 7 / 2^{7} \cdot 3^{4} \cdot 5=56$. This shows that $W\left(E_{7}\right)$ acts transitively on $F$. This proves the lemma for $k=6$.

Next we consider the case $k=5$. We may assume $a_{6}=\beta_{6}$ and $a_{7}=\beta_{7}$ by the induction hypothesis. There are exactly 27 roots $a$
with $\left(a, \beta_{6}\right)=\left(a, \beta_{7}\right)=1$. Meanwhile $\operatorname{Stab}_{W\left(E_{6}\right)}\left(\beta_{5}\right)=W\left(D_{5}\right)$ and $\left|W\left(E_{6}\right) / W\left(D_{5}\right)\right|=2^{7} \cdot 3^{4} \cdot 5 / 2^{4} \cdot 5!=27$ where $D_{5}=\left\{\alpha_{j} ; 1 \leq j \leq 5\right\}$. This proves the case $k=5$.

There are exactly 16 roots $a$ with $\left(a, \beta_{j}\right)=1(j=5,6,7)$. Meanwhile $\operatorname{Stab}_{W\left(D_{5}\right)}\left(\beta_{4}\right)=W\left(A_{4}\right)$ and $\mid W\left(\left(D_{5}\right) / W\left(A_{4}\right) \mid=2^{4} \cdot 5!/ 5!=16\right.$ where $A_{4}=\left\{\alpha_{j} ; 1 \leq j \leq 4\right\}$. This proves the case $k=4$. Similarly there are exactly 10 roots $a$ with $\left(a, \beta_{j}\right)=1$ for $4 \leq j \leq 7$, while $\operatorname{Stab}_{W\left(A_{4}\right)}\left(\beta_{3}\right)=W\left(A_{2} \times A_{1}\right)$ and $\left|W\left(A_{4}\right) / W\left(A_{2} \times A_{1}\right)\right|=10$ where $A_{2} \times A_{1}=\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$. This proves the case $k=3$. When $k=2$, there are exactly 6 roots $a$ with $\left(a, \beta_{j}\right)=1$ for $3 \leq j \leq 7$, and $\operatorname{Stab}_{W\left(A_{2} \times A_{1}\right)}\left(\beta_{2}\right)=W\left(A_{1}\right)$ and $\left|W\left(A_{2} \times A_{1}\right) / W\left(A_{1}\right)\right|=6$ where $A_{1}=\left\{\alpha_{1}\right\}$. Hence the case of $k=2$ is proved.

If $k=1$, we may suppose $a_{j}=\beta_{j}$ for $2 \leq j \leq 7$ by the induction hypothesis. Then there are three choices $a_{1}= \pm e_{1}+e_{8}$ and $\frac{1}{2}\left(e_{1}+\cdots+\right.$ $\left.e_{8}\right)$. Since $A_{1}=\left\{\alpha_{1}\right\}, W\left(A_{1}\right)$ is generated by $r_{\alpha_{1}}$ and $r_{\alpha_{1}}\left(-e_{1}+e_{8}\right)=$ $-e_{1}+e_{8}, r_{\alpha_{1}}\left(e_{1}+e_{8}\right)=\frac{1}{2}\left(e_{1}+\cdots+e_{8}\right)$. This shows that there are two $W\left(A_{1}\right)$-orbits. This completes the proof of the lemma.
Q.E.D.

Corollary 8.10. Any sublattice $A_{8-k}$ of $E_{8}$ is $W\left(E_{8}\right)$-equivalent to the sublattice $\left\{\alpha_{k}, \cdots, \alpha_{8},-\alpha_{0}\right\}$ if $k \geq 2$. If $k=1$ and if there is no root orthogonal to the sublattice, then it is $W\left(E_{8}\right)$-equivalent to $\left\{\alpha_{3}, \alpha_{4}, \cdots, \alpha_{8},-\alpha_{0}\right\}$. If $k=1$ and if there is a root orthogonal to the sublattice, then it is $W\left(E_{8}\right)$-equivalent to $\left\{\alpha_{2}, \alpha_{4}, \cdots, \alpha_{8},-\alpha_{0}\right\}$.

Proof. Let $X_{k}$ be the sublattice of $X=E_{8}$ isomorphic (as a lattice) to $A_{8-k}$. Hence there is a basis $b_{j}$ of $X_{k}(k \leq j \leq 7)$ such that $\left(b_{j}, b_{j+1}\right)=-1,\left(b_{j}, b_{j}\right)=2$ and $\left(b_{i}, b_{j}\right)=0$ (otherwise). Let $\gamma_{7}=-b_{7}$ and $\gamma_{j}=-\sum_{\ell=j}^{7} b_{\ell}(k \leq j \leq 7)$. We note that $b_{7}=-\gamma_{7}$ and $b_{j}=\gamma_{j+1}-\gamma_{j}(k \leq j \leq 6)$. Then we see $\left(\gamma_{i}, \gamma_{i}\right)=2$ and $\left(\gamma_{i}, \gamma_{j}\right)=1$ for any $i \neq j$. Hence if $k \geq 2$, the ordered set $\left\{\gamma_{j} ; k \leq j \leq 7\right\}$ is $W\left(E_{8}\right)$-equivalent to $\left\{e_{k}+e_{8}, e_{k+1}+e_{8}, \cdots, e_{7}+e_{8}\right\}$ by Lemma 8.9. It follows that the ordered set $\left\{b_{j} ; k \leq j \leq 7\right\}$ is $W\left(E_{8}\right)$-equivalent to $\left\{\alpha_{k+2}, \alpha_{k+3}, \cdots, \alpha_{8},-\alpha_{0}\right\}$. When $k=1$, then the ordered set $\left\{\gamma_{j} ; k \leq\right.$ $j \leq 7\}$ is $W\left(E_{8}\right)$-equivalent to either $\left\{e_{1}+e_{8}, e_{2}+e_{8}, \cdots, e_{7}+e_{8}\right\}$ or $\left\{-e_{1}+e_{8}, e_{2}+e_{8}, \cdots, e_{7}+e_{8}\right\}$ by Lemma 8.9. It follows that the ordered set $\left\{b_{j} ; 1 \leq j \leq 7\right\}$ is $W\left(E_{8}\right)$-equivalent to either $\left\{\alpha_{3}, \alpha_{4}, \cdots, \alpha_{8},-\alpha_{0}\right\}$ or $\left\{\alpha_{2}, \alpha_{4}, \cdots, \alpha_{8},-\alpha_{0}\right\}$. This proves the corollary.
Q.E.D.

Lemma 8.11. For a given set $\left\{a_{1}, a_{2}, \cdots, a_{7}\right\}$ as in Lemma 8.9 there are at most two elements $\omega \in X$ such that $\omega^{2}=4$ and $\left(\omega, a_{j}\right)=2$ for any $j \leq 7$. If $a_{j}=e_{j}+e_{8}(1 \leq j \leq 7)$, then $\omega=2 e_{8}$. If $a_{1}=-e_{1}+e_{8}$ and $a_{j}=e_{j}+e_{8}(2 \leq j \leq 7)$, then $\omega=2 e_{8}$ or $\omega=\frac{1}{2}\left(-e_{1}+e_{2}+\cdots+3 e_{8}\right)$.

Proof. It suffices to prove the lemma up to $W\left(E_{8}\right)$-equivalence. Hence by Lemma 8.9 we may assume $a_{1}= \pm e_{1}+e_{8}$ and $a_{j}=e_{j}+e_{8}$ $(j \geq 2)$. In either case $\omega=2 e_{8}$ satisfies the conditions. If $a_{1}=-e_{1}+e_{8}$ and $a_{j}=e_{j}+e_{8}(j \geq 2)$, then $\omega=\frac{1}{2}\left(-e_{1}+e_{2}+\cdots+3 e_{8}\right)$ also satisfies the conditions. Suppose $\omega$ satisfies the conditions. Let $s=\omega-a_{1}$. It follows from $(\omega, \omega)=4$ that $(s, s)=2$. Moreover $\left(s, a_{j}\right)=1$ for any $j \leq 7$, which implies $s= \pm e_{1}+e_{8}$ or $s=\frac{1}{2}\left(e_{1}+\cdots+e_{8}\right)$. Hence if $a_{1}=e_{1}+e_{8}$, then $s=-e_{1}+e_{8}$ and $\omega=2 e_{8}$. If $a_{1}=-e_{1}+e_{8}$, then $s=e_{1}+e_{8}$ or $s=\frac{1}{2}\left(e_{1}+\cdots+e_{8}\right)$. Therefore $\omega=2 e_{8}$ or $\frac{1}{2}\left(-e_{1}+e_{2}+\cdots+3 e_{8}\right)$. Q.E.D.

We note that if we let $s_{j}:=\omega-a_{j}(1 \leq j \leq 7)$ in Lemma 8.11, then $s_{j}$ satisfies $\left(s_{j}, s_{k}\right)=1+\delta_{j k}$ and $\left(s_{j}, a_{k}\right)=1-\delta_{j k}$. We call $a \in X$ primitive if $a$ is not an integral multiple of any element of $X$.

Lemma 8.12. There are 17280 primitive elements $a \in X$ with $a^{2}=8$. Any element $a \in X$ with $a^{2}=8$ is one of the following
(i) $\quad \sum_{k=1}^{4}(-1)^{\nu\left(i_{k}\right)} e_{i_{k}}+(-1)^{\nu(m)} 2 e_{m}\left(i_{k}, m\right.$ all distinct),
(ii) $\sum_{i=1}^{8}(-1)^{\nu(i)} e_{i}$ with $\sum_{i=1}^{8} \nu(i)$ odd,
(iii) $\pm \frac{1}{2}\left(\sum_{i \neq k}(-1)^{\nu(i)} e_{i}+5 e_{k}\right)$ with $\sum_{i \neq k} \nu(i)$ even,
(iv) $\frac{1}{2}\left(\sum_{i \neq j, k, \ell}(-1)^{\nu(i)} e_{i}\right)+\frac{3}{2}\left(\sum_{i=j, k, \ell}(-1)^{\nu(i)} e_{i}\right)$ with $\sum_{i=1}^{8} \nu(i)$ odd.
Any of them is $W\left(E_{8}\right)$-equivalent.
Proof. Let $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\omega_{2}\right)$ be the stabilizer subgroup of $\omega_{2}$. By [Bourbaki, p. 75] it is the subgroup of $W\left(E_{8}\right)$ generated by $r_{\alpha}(\alpha \in X)$ with $\alpha^{2}=2$ and $\left(\omega_{2}, \alpha\right)=0$. The roots orthogonal to $\omega_{2}$ is the root system $A_{7}$ spanned by $\alpha_{j}$ for $j \neq 2$. Thus $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\omega_{2}\right)$ is $W\left(A_{7}\right)$. Hence the orbit $W\left(E_{8}\right) \cdot \omega_{2}$ consists of $\left|W\left(E_{8}\right) / W\left(A_{7}\right)\right|=17280$ elements. Meanwhile if $a^{2}=8$ and $a \in X$, then either $a=2 b$ for some root $b \in X$ or $a$ is primitive. If $b$ is a root and it is not in the lattice $\mathbf{Z}^{8}$, then $b$ equals $\frac{1}{2}\left(\sum_{i=1}^{8}(-1)^{\nu(i)} e_{i}\right)$ with $\sum_{i=1}^{8} \nu(i)$ even. Hence if $a$ is primitive and $a^{2}=8$, then it is one of the elements of type (i)-(iv). The number of elements of type (i), (ii), (iii) and (iv) are respectively 8960, 128, 1024 and 7168 , which totals 17280 . This shows that the above 17280 elements are in the single $W\left(E_{8}\right)$-orbit of $\omega_{2}$.
Q.E.D.

Lemma 8.13. Any pair of $a, b \in X$ with $a^{2}=b^{2}=4$ and $(a, b)=3$ is $W\left(E_{8}\right)$-equivalent.

Proof. We may assume $a=2 e_{8}$ by Lemma 8.2. Let $F$ be the set of all $b$ with $b^{2}=4$ and $\left(2 e_{8}, b\right)=3$. Then $F=\left\{\frac{1}{2}\left(\sum_{j=1}^{7}(-1)^{\nu(j)} e_{j}+\right.\right.$ $\left.3 e_{8}\right) ; \sum_{j=1}^{7} \nu(j)$ even $\}$ and $|F|=64$. Let $b_{0}=\frac{1}{2}\left(\sum_{j=1}^{7} e_{j}+3 e_{8}\right)$. Then we see $W\left(D_{7}\right)=\operatorname{Stab}_{W\left(E_{8}\right)}\left(2 e_{8}\right)$ and $W\left(A_{6}\right)=\operatorname{Stab}_{W\left(E_{8}\right)}\left(2 e_{8}, b_{0}\right)$.

Thus the orbit $W\left(D_{7}\right) \cdot b_{0}$ consists of $\left|W\left(D_{7}\right) / W\left(A_{6}\right)\right|=2^{6} \cdot 7!/ 7!=64$ elements. This proves that $W\left(D_{7}\right)$ acts transitively on $F$. Q.E.D.

Lemma 8.14. Any pair of $a, b \in X$ with $a^{2}=4, b^{2}=8$ and $(a, b)=5$ is $W\left(E_{8}\right)$-equivalent.

Proof. We may assume $a=2 e_{8}$ by Lemma 8.2. Let $F$ be the set of all $b$ with $b^{2}=8$ and $\left(2 e_{8}, b\right)=5$. Then $F=\left\{\frac{1}{2}\left(\sum_{j=1}^{7}(-1)^{\nu(j)} e_{j}+\right.\right.$ $\left.35 e_{8}\right) ; \sum_{j=1}^{7} \nu(j)$ even $\}$ and $|F|=64$. Let $b_{0}=\frac{1}{2}\left(\sum_{j=1}^{7} e_{j}+5 e_{8}\right)$. Then we see $W\left(D_{7}\right)=\operatorname{Stab}_{W\left(E_{8}\right)}\left(2 e_{8}\right)$ and $W\left(A_{6}\right)=\operatorname{Stab}_{W\left(E_{8}\right)}\left(2 e_{8}, b_{0}\right)$. Thus the orbit $W\left(D_{7}\right) \cdot b_{0}$ consists of $\left|W\left(D_{7}\right) / W\left(A_{6}\right)\right|=2^{6} \cdot 7!/ 7!=64$ elements. This proves that $W\left(D_{7}\right)$ acts transitively on $F$. Q.E.D.

Table 1. The elements of $E_{8}$

| $a^{2}$ | $W\left(E_{8}\right)$ | number |
| :---: | :---: | :---: |
| $a^{2}=2$ (root) | transitive | 240 |
| $a^{2}=4$ | transitive | 2160 |
| $a^{2}=6$ | transitive | 6720 |
| $a^{2}=8$ (prim.) | transitive | 17280 |
| $a^{2}=8$ (not prim.) | transitive | 240 |

Table 2. The pairs of $E_{8}$ elements

| $a, b$ | $W\left(E_{8}\right)$ |
| :---: | :---: |
| $a^{2}=b^{2}=2, a b=0$ | transitive |
| $a^{2}=4, b^{2}=2, a b=k(k=0,1,2)$ | transitive |
| $a^{2}=4, b^{2}=4, a b=3$ | transitive |
| $a^{2}=4, b^{2}=8, a b=5$ | transitive |
| $A_{k} \subset E_{8}(2 \leq k \leq 6)$ | transitive |

Example 8.15. Examples of the pairs in Table 2 are given as follows. The pair $a=e_{1}+e_{2}, b=e_{3}+e_{4}$ resp. $a=e_{1}+e_{2}+e_{3}+$ $e_{4}, b=e_{4-k}+e_{5-k}$ satisfies satisfies $a^{2}=b^{2}=2$ and $a b=0$, resp. a $2=4, b^{2}=2$ and $a b=k(k=0,1,2)$.

The pair $a=e_{1}+e_{2}+e_{3}+e_{4}, b=e_{2}+e_{3}+e_{4}+e_{5}$ satisfies $a^{2}=b^{2}=4$ and $a b=3$, while the pair $a=e_{1}+e_{2}+e_{3}+e_{4}, b=2 e_{1}+e_{2}+e_{3}+e_{4}+e_{5}$ satisfies $a^{2}=4, b^{2}=8$ and $a b=5$. Similarly an example of $A_{k}$ for $2 \leq k \leq 6$ is given by the sublattice of $E_{8}$ spanned by $\alpha_{j}(9-k \leq j \leq 8)$.

However we note that for $k=7$ there are two $W\left(E_{8}\right)$-orbits of sublattices spanned either by (1) $\alpha_{j}(3 \leq j \leq 8$ and $j=0)$ or by (2) $\alpha_{j}$ ( $4 \leq j \leq 8$ and $j=0,2$ ). See Lemma 10.3.

## §9. Decorated diagrams and the Wythoff construction

The purpose of this section is to recall the notions of decorated diagrams of a Dynkin diagram from [MP92], and then the Wythoff construction, due to Coxeter, of Delaunay cells associated with decorated diagrams.

Definition 9.1. A decorated diagram $\Delta$ of $\widetilde{E}_{8}$ is by definition a decomposition of $\widetilde{E}_{8}$ into two subdiagrams $\Delta_{\text {Vor }}$ and $\Delta_{\text {Del }}$ such that
(i) $\left|\widetilde{E}_{8}\right|=|\Delta|=\left|\Delta_{\text {Vor }}\right| \cup\left|\Delta_{\text {Del }}\right|$,
(ii) $\Delta_{\text {Vor }}$ is a subdiagram of $\widetilde{E}_{8}$ with square nodes $\square$, crossed unless the square node is connected to $\Delta_{\text {Del }}$ by an edge,
(iii) $\Delta_{\text {Del }}$ is a connected subdiagram of $\widetilde{E}_{8}$ with circle nodes containing the node $\odot$
where $\left|\Delta_{A}\right|$ is the support of $\Delta_{A}$, that is, the set of nodes and edges.
Definition 9.2. We define the Voronoi cell $V(q)$ by

$$
V(q)=\left\{\alpha \in X_{\mathbf{R}} ;\|y-\alpha\| \geq\|q-\alpha\| \text { for any } y \in X\right\}
$$

for $q \in X$. A Voronoi cell $V$ is defined to be a face of $V(q)$ for some $q \in X$.

Let $H_{0}$ be the reflection hyperplane of $r_{0}$ (see section two), that is, the hyperplane of $X_{\mathbf{R}}$ defined by $H_{0}=\left\{x \in X_{\mathbf{R}} ;\left(\alpha_{0}, x\right)=1\right\}$. Define $F$ to be the closed domain

$$
F=\left\{x \in X_{\mathbf{R}} ;\left(\alpha_{j}, x\right) \geq 0(1 \leq j \leq 8),\left(\alpha_{0}, x\right) \leq 1\right\}
$$

and define $F_{0}$ to be the intersection of $F$ and $H_{0}$.
We quote a few basic facts from [MP92, pp. 5095 and section 4].
Lemma 9.3. (i) $F$ is the convex closure of the origin 0 and

$$
\frac{\omega_{i}}{m_{i}}(1 \leq i \leq 8)
$$

(ii) $F$ is a fundamental domain for $W\left(\widetilde{E}_{8}\right)$ in the sense that
(a) $X_{\mathbf{R}}$ is the union of $w F\left(w \in W\left(\widetilde{E}_{8}\right)\right)$,











Fig. 1. Decorated diagrams
(b) if $x \in F$ and $w \in W\left(\widetilde{E}_{8}\right)$, then $w x \in F \Longleftrightarrow w x=x$,
(c) if $x \in F$, then $\operatorname{Stab}_{W\left(\widetilde{E}_{8}\right)}(x)$ is generated by the reflections with regards to the walls (=one-codimensional faces) of $F$ containing $x$.
(iii) The Voronoi cell $V(0)$ is the union of $w F\left(w \in W\left(E_{8}\right)\right)$.
(iv) Any Voronoi cell $V$ is the intersection of all $V(q)$ which contains $V$.

The Wythoff construction of Delaunay cells due to Coxeter is described as follows: Let $\Delta$ be a decorated diagram of $\widetilde{E}_{8}$. Let $S_{\Delta}$ (resp. $\left.S_{\Delta}^{*}\right)$ be the set of nodes of $E_{8}$ contained in ${\underset{\sim}{~}}_{\text {Vor }}\left(\right.$ resp. $\left.\Delta_{\text {Del }} \backslash\left\{-\alpha_{0}\right\}\right)$. Let $W_{a, \Delta}$ be the reflection subgroup of $W\left(\widetilde{E}_{8}\right)$ generated by $r_{0}$ and $r_{\alpha}$ $\left(\alpha \in S_{\Delta}^{*}\right)$. Then $V_{\Delta}^{0}$ is defined to be the convex closure of $\frac{\omega_{i}}{m_{i}}\left(\alpha_{i} \in S_{\Delta}\right)$ and $V_{\Delta}$ the minimal face of $V(0)$ containing $V_{\Delta}^{0}$. Hence $V_{\Delta}$ is the intersection of all $V(q)$ such that $V_{\Delta}^{0} \subset V(q)$, while $V_{\Delta}^{0}=V_{\Delta} \cap F_{0}$. We define $D_{\Delta}$ to be the convex closure of $W_{a, \Delta}(0)$. Since any Delaunay cell is the convex closure of some points of $X$, this implies that the Delaunay cell $D_{\Delta}$ is the convex closure of all $q$ with $q \in W_{a, \Delta}(0) \cap X$.

For instance, let $\Delta=\Delta_{2}$. Then $\Delta_{\text {Vor }}$ is the disjoint union of $A_{2}$ and $A_{1}$ with square nodes, crossed or uncrossed, while $\Delta_{\text {Del }}$ is $A_{6}$ with the extreme node $\odot$. Thus $S_{\Delta}=\left\{\alpha_{1}, \alpha_{2}, \alpha_{3}\right\}$ and $S_{\Delta}^{*}=\left\{\alpha_{4}, \alpha_{5}, \alpha_{6}, \alpha_{7}, \alpha_{8}\right\}$.

The following theorem is a summary for the Wythoff construction. See [MP92, Lemma 3-Lemma 5 and (4.29)-(4.31), pp. 5108-5111].

Theorem 9.4. Let $\Delta$ be a decorated diagram of $\widetilde{E}_{8}$. Then
(i) $\quad V_{\Delta}$ is a Voronoi cell of $E_{8}$, while $D_{\Delta}$ is a Delaunay cell of $E_{8}$ dual to $V_{\Delta}$ in the sense that $D_{\Delta}$ is the convex closure of all $a \in X$ such that $\|a-y\|=\min _{b \in X}\|b-y\|$ for any $y \in V_{\Delta}$.
(ii) $V_{\Delta}$ is the intersection of all $V(q)$ with $q \in W_{a, \Delta}(0)$, while $D_{\Delta}$ is the convex closure of all $q$ with $q \in W_{a, \Delta}(0)$.
(iii) If $\Delta=\Delta_{k}$ or $\Delta_{k}^{\ell}$, then $\operatorname{dim} V_{\Delta}=k$ and $\operatorname{dim} D_{\Delta}=8-k$.
(iv) Any Delaunay cell $\sigma$ of $E_{8}$ is a $W\left(\widetilde{E}_{8}\right)$-transform of $D_{\Delta}$ for a decorated diagram $\Delta$ of $\widetilde{E}_{8}$. If $\sigma$ contains the origin, then it is a $W\left(E_{8}\right)$-transform of $D_{\Delta}$.
(v) For a subset $A$ of $X_{\mathbf{R}}$, we define

$$
\begin{aligned}
& \operatorname{Stab}_{W\left(\widetilde{E}_{8}\right)}(A)=\left\{w \in W\left(\widetilde{E}_{8}\right) ; w A \subset A\right\} \\
& \operatorname{Stab}_{W\left(E_{8}\right)}(A)=\left\{w \in W\left(E_{8}\right) ; w A \subset A\right\}
\end{aligned}
$$

Let $W_{\Delta}^{1}$ (resp. $W_{\Delta}^{2}$ ) be the subgroup of $W\left(\widetilde{E}_{8}\right)$ generated by $r_{\alpha_{j}}$ with $\alpha_{j} \in S_{\Delta}^{*}$ (resp. by $r_{\alpha_{j}}$ with $\alpha_{j}$ orthogonal to both $S_{\Delta}^{*}$ and $\alpha_{0}$ ). Then

$$
\operatorname{Stab}_{W\left(\widetilde{E}_{8}\right)}\left(D_{\Delta}\right)=W_{a, \Delta} \times W_{\Delta}^{2}, \quad \operatorname{Stab}_{W\left(E_{8}\right)}\left(D_{\Delta}\right)=W_{\Delta}^{1} \times W_{\Delta}^{2}
$$

### 9.5. Wythoff construction for $E_{8}$

In this subsection we give examples of the Wythoff construction for $E_{8}$. Let $h_{0}=2 e_{8}, h_{j}=e_{j}+e_{8}$ and $h_{15-j}=-e_{j}+e_{8}(1 \leq j \leq 7)$. We recall $\omega_{1}=2 e_{8}$ and $\omega_{2}=\frac{1}{2}\left(e_{1}+e_{2}+\cdots+e_{7}+5 e_{8}\right)$.
9.5.1. $D_{\Delta_{0}^{1}}=D\left(\frac{\omega_{1}}{2}\right)$. Let $\Delta=\Delta_{0}^{1}$. Then we see $\Delta_{\text {Vor }}=\square$ and $V_{\Delta}=\left\{\frac{\omega_{1}}{2}\right\}$. First we note $r_{0}\left(\frac{\omega_{1}}{2}\right)=\frac{\omega_{1}}{2}$, hence $r_{0} \in \operatorname{Stab}_{W\left(\widetilde{E}_{8}\right)}\left(\frac{\omega_{1}}{2}\right)$. The stabilizer subgroup $\operatorname{Stab}_{W\left(\widetilde{E}_{8}\right)}\left(\frac{\omega_{1}}{2}\right)$ is the reflection subgroup of $W\left(\widetilde{E}_{8}\right)$ generated by $r_{0}$ and $r_{\alpha}$ with $\left(\alpha, \omega_{1}\right)=0$, hence it is generated by $r_{0}$ and $r_{\alpha_{j}}(j=2, \cdots, 8)$. We note $W_{a, \Delta_{0}^{1}}=\operatorname{Stab}_{W\left(\widetilde{E}_{8}\right)}\left(\frac{\omega_{1}}{2}\right)$ and $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\frac{\omega_{1}}{2}\right)=W\left(D_{7}\right)$ where $D_{7}=E_{8} \backslash\left\{\alpha_{1}\right\}$ because it is generated by $r_{\alpha}$ with roots $\alpha$ orthogonal to $\omega_{1}$, hence it is generated by $r_{\alpha_{j}}(j=2, \cdots, 8)$. Since $\left(\alpha_{0}, h_{j}\right)=1$ for any $1 \leq j \leq 14$ and $j \neq 7,8$, we have $r_{0}\left(h_{j}\right)=h_{j}$, while $r_{0}\left(h_{7}\right)=0, r_{0}\left(h_{8}\right)=h_{0}$. Let $S=\left\{0, h_{0}, h_{j}, h_{15-j} ; 1 \leq j \leq 7\right\}$. Then $r_{0}(S)=S$.

As is well known, $W\left(D_{7}\right)$ is a semi-direct product of $(\mathbf{Z} / 2 \mathbf{Z})^{6}$ and $S_{7}$. There is a natural surjection $\pi: W\left(D_{7}\right) \rightarrow S_{7}$. Let $\sigma \in W\left(D_{7}\right)$. Then $\pi(\sigma) \in S_{7}$. Let $h_{j}=e_{j}+e_{8}(1 \leq j \leq 7)$. For $\sigma \in W\left(D_{7}\right)$, $\sigma\left(e_{8}\right)=e_{8}, \sigma\left(e_{j}\right)=(-1)^{\nu(\pi(\sigma)(j))} e_{\pi(\sigma)(j)}$ with $\sum_{j=1}^{7} \nu(\pi(\sigma)(j))$ even. For instance, for $3 \leq k \leq 8$ we have $r_{\alpha_{k}}\left(e_{k-1}\right)=e_{k-2}, r_{\alpha_{k}}\left(e_{k-2}\right)=e_{k-1}$ and $r_{\alpha_{k}}\left(e_{j}\right)=e_{j}$ (otherwise). Therefore $r_{\alpha_{k}}(S)=S$ for any $2 \leq k \leq 8$. It follows that $W\left(D_{7}\right)(S)=S$. Hence $D\left(\frac{\omega_{1}}{2}\right)$ is the convex closure of $W\left(D_{7}\right)(S)=S$. This can be shown directly as we see in Lemma 10.2.

9.5.2. $D_{\Delta_{0}^{2}}=D\left(\frac{\omega_{2}}{3}\right)$. Let $\Delta=\Delta_{0}^{2}$. Then we see $\Delta_{\text {Vor }}=\square$ and $V_{\Delta}=\left\{\frac{\omega_{2}}{3}\right\}$. The stabilizer group $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\frac{\omega_{2}}{3}\right)=W\left(A_{7}\right)$ because it is generated by $r_{\alpha}$ with $\left(\alpha, \omega_{2}\right)=0$, hence it is generated by $r_{\alpha_{j}}$ $(j=1,3, \cdots, 8)$. We also see $W_{a, \Delta}=\operatorname{Stab}_{W\left(\widetilde{E}_{8}\right)}\left(\frac{\omega_{2}}{3}\right)$ is generated by $r_{0}$ and $\operatorname{Stab}_{W\left(E_{8}\right)}\left(\frac{\omega_{2}}{3}\right)$. Let $g_{0}=\frac{1}{2}\left(e_{1}+e_{2}+\cdots+e_{8}\right)$ and $S=\left\{0, g_{0}, h_{j}(1 \leq\right.$ $j \leq 7)\}$. Then $r_{0}\left(g_{0}\right)=r_{\alpha_{k}}\left(g_{0}\right)=g_{0}(3 \leq k \leq 8)$. We also see $r_{\alpha_{1}}\left(h_{1}\right)=g_{0}, r_{\alpha_{1}}\left(g_{0}\right)=h_{1}$ and $r_{\alpha_{1}}\left(h_{j}\right)=h_{j}$ (otherwise). Though $\left\{\alpha_{k}(3 \leq k \leq 8)\right\}=A_{6}, W\left(A_{6}\right)=S_{7}$ acts on the set $\left\{h_{j}(1 \leq j \leq 7)\right\}$ as standard permutations. It follows that $D\left(\frac{\omega_{2}}{3}\right)$ is the convex closure of $0, h_{j}(1 \leq j \leq 7)$ and $g_{0}$. See Lemma 10.8.
9.5.3. $D_{\Delta_{1}^{k}}$. For $\Delta=\Delta_{1}^{1}, W_{a, \Delta}$ is generated by $r_{0}$ and $r_{\alpha_{j}}(3 \leq$ $j \leq 8)$. Hence $D_{\Delta}$ is the convex closure of 0 and $h_{j}(1 \leq j \leq 7)$. For
$\Delta=\Delta_{1}^{2}, W_{a, \Delta}$ is generated by $r_{0}$ and $r_{\alpha_{j}}(j=2,4,5, \cdots, 8)$. Hence $D_{\Delta}$ is the convex closure of $0, h_{14}$ and $h_{j}(2 \leq j \leq 7)$.
9.5.4. $D_{\Delta_{k}}$. For a fixed $k$ we let $\Delta=\Delta_{k}(2 \leq k \leq 7)$. Then $W_{a, \Delta}$ is generated by $r_{0}$ and $r_{\alpha_{j}}(j=k+2, \cdots, 8)$. Hence $D_{\Delta}$ is the convex closure of 0 and $h_{j}(k \leq j \leq 7)$.

## §10. Delaunay cells

By Theorem 9.4 any 8 -dimensional Delaunay cell is either $D\left(\frac{\omega_{1}}{2}\right)$ or $D\left(\frac{\omega_{2}}{3}\right)$ up to $W\left(\widetilde{E}_{8}\right)$ where $\frac{\omega_{1}}{2}=e_{8}$ and $\frac{\omega_{2}}{3}=\frac{1}{6}\left(e_{1}+e_{2}+\cdots+e_{7}+5 e_{8}\right)$. We recall
$\operatorname{Stab}_{W\left(E_{8}\right)}\left(D\left(\frac{\omega_{1}}{2}\right)\right)=W\left(D_{7}\right), \quad \operatorname{Stab}_{W\left(E_{8}\right)}\left(D\left(\frac{\omega_{2}}{3}\right)\right)=W\left(A_{7}\right)$.

### 10.1. The Delaunay cell $D\left(\frac{\omega_{1}}{2}\right)$

Lemma 10.2. The Delaunay cell $D\left(\frac{\omega_{1}}{2}\right)=D\left(e_{8}\right)$ is the convex closure of the origin $0, \pm e_{j}+e_{8}(1 \leq j \leq 7)$ and $2 e_{8}$. For $0<\epsilon<1$, $D\left(\frac{\epsilon \omega_{1}}{2}\right)$ consists of 0 only.

The polytope $D\left(\frac{\omega_{1}}{2}\right)$ is called a 8-cross polytope.
Proof. The cell $D\left(\frac{\omega_{1}}{2}\right)=D\left(e_{8}\right)$ is the convex closure of $a \in X$ with $\left\|a-e_{8}\right\|=1$. If $a(\neq 0) \in X$ and $\left\|a-e_{8}\right\|=1$, then writing $a=\sum_{i=1}^{8} x_{i} e_{i}$ we have $\sum_{i=1}^{7} x_{i}^{2}+\left(x_{8}-1\right)^{2}=1$. If $x_{8} \notin \mathbf{Z}$, then $x_{8}=\frac{1}{2}$, or $\frac{3}{2}$ and there are exactly three $x_{i}$ 's such that $x_{i}=\frac{1}{2}$ and otherwise $x_{j}=0$ for $j \leq 8$. But in either case there is a pair $x_{i}+x_{j} \notin \mathbf{Z}$, which is absurd. If $x_{8} \in \mathbf{Z}$, then $x_{8}=1$ or 2 . If $x_{8}=1$, then $x_{i}=1$ for a unique $i$ and $x_{j}=0$ for the other $j$. The rest is clear.
Q.E.D.

Lemma 10.3. Let $h_{0}=2 e_{8}, h_{j}=e_{j}+e_{8}$ and $h_{15-j}=-e_{j}+e_{8}$ $(1 \leq j \leq 7)$. Let $\sigma_{0}$ (resp. $\left.\sigma_{1}, \tau_{0}, \tau_{1}, \tau_{2}\right)$ be the convex closure

$$
\begin{gathered}
\sigma_{0}=\left\langle 0, h_{1}, h_{2}, \cdots, h_{7}, h_{0}\right\rangle, \quad \sigma_{1}=\left\langle 0, h_{1}, h_{2}, \cdots, h_{6}, h_{8}, h_{0}\right\rangle \\
\tau_{0}=\left\langle 0, h_{1}, h_{2}, \cdots, h_{7}\right\rangle, \quad \tau_{1}=\left\langle 0, h_{1}, h_{2}, \cdots, h_{6}, h_{8}\right\rangle \\
\tau_{2}=\left\langle h_{0}, h_{1}, h_{2}, \cdots, h_{6}, h_{7}\right\rangle .
\end{gathered}
$$

Then
(i) $\sigma_{0}$ and $\sigma_{1}$ are 8 -dimensional. They are not Delaunay cells. The Delaunay cell $D\left(\frac{\omega_{1}}{2}\right)$ is the union of $2^{6} W\left(D_{7}\right)$-transforms of $\sigma_{0}$ and $\sigma_{1}$.
(ii) Let $k \leq 7$. Any $k$-dimensional face of $D\left(\frac{\omega_{1}}{2}\right)$ is a $W\left(D_{7}\right)$ transform of a face of $\sigma_{0}$. No $k$-dimensional face of $D\left(\frac{\omega_{1}}{2}\right)$ contains both the origin and $h_{0}$. There are exactly $2^{k+1} \cdot\binom{8}{k+1}$ $k$-dimensional faces of $D\left(\frac{\omega_{1}}{2}\right)$.
(iii) Any $k$-dimensional face of $D\left(\frac{\omega_{1}}{2}\right)$ is $W\left(\widetilde{E}_{8}\right)$-equivalent to $D_{\Delta_{k}}$ for $1 \leq k \leq 6$.
(iv) Any 7-dimensional face of $D\left(\frac{\omega_{1}}{2}\right)$ is $W\left(\widetilde{E}_{8}\right)$-equivalent to either $D_{\Delta_{1}^{1}}$ or $D_{\Delta_{1}^{2}}$. $\tau_{1}\left(\right.$ resp. $\left.\tau_{0}\right)$ is a Delaunay cell and it is a face of $D\left(\frac{\omega_{1}}{2}\right), W\left(\widetilde{E}_{8}\right)$-equivalent to $D_{\Delta_{1}^{1}}\left(\right.$ resp. $\left.D_{\Delta_{1}^{2}}\right)$ and $\tau_{2}=r_{0}\left(\tau_{1}\right)$.
Proof. $W\left(D_{7}\right)\left(=\operatorname{Stab}_{W\left(E_{8}\right)}\left(\frac{\omega_{1}}{2}\right)\right)$ is a semi-direct product of $(\mathbf{Z} / 2 \mathbf{Z})^{6}$ and the symmetry group $S_{7}$, where $S_{7}$ keeps both $\sigma_{0}$ and $\sigma_{1}$ respectively invariant. Let $\pi: W\left(D_{7}\right) \rightarrow S_{7}$ be the natural surjection. If $\pi(w)$ is the identity, then $w\left(h_{j}\right)=h_{j}$ or $h_{15-j}(1 \leq j \leq 7)$ according as $\nu(j)$ even or odd. Thus if $\pi(w)$ is the identity, we define $\bar{w}(j):=j$ or $15-j$ according as $\nu(j)$ even or odd. Then we have

$$
w \cdot\left\langle 0, h_{1}, \cdots, h_{7}, h_{0}\right\rangle=\left\langle 0, h_{\bar{w}(1)}, \cdots, h_{\bar{w}(7)}, h_{0}\right\rangle
$$

For $w \in W\left(D_{7}\right)$, we have $w\left(e_{8}\right)=e_{8}, w\left(e_{j}\right)=(-1)^{\nu(\pi(w)(j))} e_{\pi(w)(j)}$ with $\sum_{j=1}^{7} \nu(\pi(w)(j))$ even. See Subsection 9.5. Then we have

$$
w \cdot\left\langle 0, h_{1}, \cdots, h_{7}, h_{0}\right\rangle=\left\langle 0, h_{k_{1}}, \cdots, h_{k_{7}}, h_{0}\right\rangle
$$

where $k_{j}=\pi(w)(j)$ or $15-\pi(w)(j)$ according as $\nu(\pi(w)(j))=0$ or 1 . Note that $\sum_{j=1}^{7} \nu(\pi(w)(j))$ is even. Hence there are exactly $2^{6} W\left(D_{7}\right)$ transforms of $\sigma_{0}$. Similarly there are exactly $2^{6} W\left(D_{7}\right)$-transforms of $\sigma_{1}$. Thus the convex closure $\left\langle 0, h_{i_{1}}, \cdots, h_{i_{7}}, h_{0}\right\rangle$ is a $W\left(D_{7}\right)$-transform of either $\sigma_{0}$ or $\sigma_{1}$ for any $i_{k} \in\{k, 15-k\}$.

Next let $z \in D\left(\frac{\omega_{1}}{2}\right)$. Since $D\left(\frac{\omega_{1}}{2}\right)$ is the convex closure of $0, h_{0}$ and $h_{j}(1 \leq j \leq 14)$, we write $z=x_{0} h_{0}+\sum_{j=1}^{14} x_{i} h_{i}$ where $x_{0}+\sum_{j=1}^{14} x_{i} \leq 1$ and $x_{j} \geq 0(0 \leq j \leq 14)$. Then we have

$$
\begin{aligned}
z=\sum_{x_{i} \geq x_{15-i}}\left(x_{i}-x_{15-i}\right) h_{i} & +\sum_{x_{i}<x_{15-i}}\left(x_{i}-x_{15-i}\right) h_{15-i} \\
& +\left(x_{0}+\sum_{i=1}^{7} \min \left(x_{i}, x_{15-i}\right)\right) h_{0}
\end{aligned}
$$

The sum of the coefficients of $h_{i}$ is equal to

$$
\left.\sum_{x_{i} \geq x_{15-i}}\left(x_{i}-x_{15-i}\right)+\sum_{x_{i}<x_{15-i}}\left(x_{i}-x_{15-i}\right)+x_{0}+\sum_{i=1}^{7} \min \left(x_{i}, x_{15-i}\right)\right)
$$

which is equal to $\left.x_{0}+\sum_{i=1}^{7} \max \left(x_{i}, x_{15-i}\right)\right)$. By our assumption on $x_{i}$ it is not greater than 1 . This implies $z \in\left\langle 0, h_{0}, h_{i_{1}}, \cdots, h_{i_{7}}\right\rangle$ for some $i_{k} \in\{k, 15-k\}(1 \leq k \leq 7)$. This proves (i).

Next we prove that the convex closure $\left\langle 0, h_{0}\right\rangle$ of 0 and $h_{0}=2 e_{8}$ intersects the interior of $D\left(\frac{\omega_{1}}{2}\right)$. To see this it suffices to prove $e_{8}:=\frac{h_{0}}{2}$ is in the interior of $D\left(\frac{\omega_{1}}{2}\right)$. In fact, we choose $x_{j}>0(0 \leq j \leq 7)$ such that $z:=\sum_{j=0}^{7} x_{j}=\frac{1}{2}$. Then we have

$$
e_{8}=\frac{1}{2} \cdot 0+\frac{1}{2} \sum_{j=1}^{7} x_{j}\left(h_{j}+h_{15-j}\right)+x_{0} h_{0}
$$

Since $0<x_{j}<1$ for any $j$ and $0<z<1, e_{8}$ is in the interior of $D\left(\frac{\omega_{1}}{2}\right)$. It follows that the line segment $\left\langle 0, h_{0}\right\rangle$ intersects the interior of $D\left(\frac{\omega_{1}}{2}\right)$. In particular, $\left\langle 0, h_{0}\right\rangle$ is not a Delaunay cell.

If any lower dimensional face of $\sigma_{0}$ contains both the origin and $h_{0}$, then it is contained in the interior of $D\left(\frac{\omega_{1}}{2}\right)$, which is impossible. Therefore no lower dimensional face of $\sigma_{0}$ contains both the origin and $h_{0}$. Hence any lower dimensional face of $D\left(\frac{\omega_{1}}{2}\right)$ is a face of the simplex either $w \cdot\left\langle 0, h_{1}, \cdots, h_{7}\right\rangle$ or $w \cdot\left\langle h_{0}, h_{1}, \cdots, h_{7}\right\rangle$ for some $w \in W\left(D_{7}\right)$. Hence any lower dimensional face of $D\left(\frac{\omega_{1}}{2}\right)$ is a $W\left(D_{7}\right)$-transform of a face of $\tau_{0}$ or $\tau_{2}$. If any $k$-dimensional face of $D\left(\frac{\omega_{1}}{2}\right)$ contains the origin, it is $\left\langle 0, h_{i_{1}}, \cdots, h_{i_{k}}\right\rangle$ where $i_{j}+i_{\ell} \neq 15$ and $i_{j} \neq 0$. There are these $2^{k}\binom{7}{k}$ faces in total. If it contains $h_{0}$, then it is $\left\langle h_{0}, h_{i_{1}}, \cdots, h_{i_{k}}\right\rangle$ where $i_{j}+i_{\ell} \neq 15$ and $i_{j} \neq 0$. There are these $2^{k}\binom{7}{k}$ faces in total. If it contain neither the origin nor $h_{0}$, then it is $\left\langle h_{i_{1}}, \cdots, h_{i_{k+1}}\right\rangle$ where $i_{j}+i_{\ell} \neq 15$ and $i_{j} \neq 0$. These total $2^{k+1}\binom{7}{k+1}$. Thus we see that there are $2^{k+1}\binom{8}{k+1}=2^{k+1}\binom{7}{k}+2^{k+1}\binom{7}{k+1} k$-dimensional faces of $D\left(\frac{\omega_{1}}{2}\right)$. This proves (ii).

Since $\tau_{2}=r_{0}\left(\tau_{1}\right), \tau_{2}$ is a $W\left(\widetilde{E}_{8}\right)$-transform of $\tau_{1}$. By (ii) any $k$ dimensional face of $D\left(\frac{\omega_{1}}{2}\right)$ is a $W\left(D_{7}\right)$-transform of a face of $\tau_{0}$ or $\tau_{2}$ for $k \leq 7$. Therefore it is a $W\left(\widetilde{E}_{8}\right)$-transform of a face of $\tau_{0}$ or $\tau_{1}$. We note that there are exactly the same number of lower-dimensional faces of $D\left(\frac{\omega_{1}}{2}\right)$ containing $h_{0}$ as those containing the origin. The assertions (iii) and (iv) follow from Subsection 9.5 and the proof of Lemma 8.9 or Corollary 8.10.
Q.E.D.

Lemma 10.4. There are exactly $2160 W\left(E_{8}\right)$-transforms of $D\left(\frac{\omega_{1}}{2}\right)$ containing the origin. Each $W\left(E_{8}\right)$-transform is of the form $D\left(\frac{a}{2}\right)$ for some $a \in X$ with $a^{2}=4$ and vice versa.

Proof. Any $W\left(E_{8}\right)$-transform of $D\left(\frac{\omega_{1}}{2}\right)$ is of the form $D\left(w \cdot \frac{\omega_{1}}{2}\right)$ $\left(w \in W\left(E_{8}\right)\right)$. Hence the number of $W\left(E_{8}\right)$-transforms of $D\left(\frac{\omega_{1}}{2}\right)$ is equal to $\left|W\left(E_{8}\right) / W\left(D_{7}\right)\right|(=2160)$, which is the number of $a \in X$ with $a^{2}=4$ by Lemma 8.2.
Q.E.D.

Proposition 10.5. There are exactly $135 W\left(E_{8}\right)$-transforms of $D\left(\frac{\omega_{1}}{2}\right)$ up to translation by $X$.

Proof. Those 2160 copies of $D\left(\frac{\omega_{1}}{2}\right)$ are of the form $D\left(\frac{a}{2}\right)$ with $a \in X$ and $a^{2}=4$ by Lemma 10.4. Since $D\left(\frac{a}{2}\right)$ has 16 vertices, there are 16 translates-by- $X$ of $D\left(\frac{a}{2}\right)$ containing the origin. Hence there are exactly $135(=2160 / 16) W\left(E_{8}\right)$-transforms of $D\left(\frac{\omega_{1}}{2}\right)$ up to translation by $X$.
Q.E.D.

Remark 10.6. $D\left(\frac{a^{\prime}}{2}\right)$ is a translate of $D\left(\frac{a}{2}\right)$ by $X$ if and only if $a-a^{\prime}=2 x$ for some root $x$. By Lemma 8.2, we assume $a=2 e_{8}$. By Lemma 8.2 we see readily $a^{\prime}= \pm 2 e_{k}$. It follows that there are precisely 16 translates $D\left(\frac{a^{\prime}}{2}\right)$ by $X$ of $D\left(\frac{2 e_{8}}{2}\right)$.

### 10.7. The Delaunay cell $D\left(\frac{\omega_{2}}{3}\right)$

Lemma 10.8. The Delaunay cell $D\left(\frac{\omega_{2}}{3}\right)$ is the convex closure of the origin $0, h_{j}=e_{j}+e_{8}(1 \leq j \leq 7)$ and $g_{0}:=\frac{1}{2}\left(e_{1}+e_{2}+\cdots+e_{8}\right)$.

Proof. $D\left(\frac{\omega_{2}}{3}\right)$ is the convex closure of $a \in X$ with $\left\|a-\frac{\omega_{2}}{3}\right\|^{2}=$ $\left\|\frac{\omega_{2}}{3}\right\|^{2}=\frac{8}{9}$. Let $a=\sum_{j=1}^{8} x_{j} e_{j}$ and suppose $\left\|a-\frac{\omega_{2}}{3}\right\|^{2}=\frac{8}{9}$. If $x_{8} \in \mathbf{Z}$, then $x_{8}=0$ or 1. If $x_{8}=0$, then $a=0$. If $x_{8}=1$, then $a=e_{j}+e_{8}$ for some $j \leq 7$. If $x_{8}$ is not an integer, then $x_{8}=\frac{1}{2}$ or $\frac{3}{2}$ and $x_{j}=\frac{1}{2}$ for $1 \leq 7$. If $x_{8}=\frac{1}{2}$, then $a=g_{0}$. If $x_{8}=\frac{3}{2}$, then no $a \in X$ is possible.
Q.E.D.

Corollary 10.9. There are exactly $\binom{8}{k} k$-dimensional faces of $D\left(\frac{\omega_{2}}{3}\right)$.
Proof. Clear because the 8 -dimensional cell $D\left(\frac{\omega_{2}}{3}\right)$ has only nine vertices.
Q.E.D.

We call $a \in X$ primitive if $a$ is not an integral multiple of any element of $X$.

Lemma 10.10. There are exactly $17280 W\left(E_{8}\right)$-transforms of $D\left(\frac{\omega_{2}}{3}\right)$ containing the origin. Each $W\left(E_{8}\right)$-transform is of the form $D\left(\frac{a}{3}\right)$ for some primitive $a \in X$ with $a^{2}=8$ and vice versa.

Proof. Any $W\left(E_{8}\right)$-transform of $D\left(\frac{\omega_{1}}{2}\right)$ is of the form $D\left(w \cdot \frac{\omega_{2}}{3}\right)$ $\left(w \in W\left(E_{8}\right)\right)$, hence of the form $D\left(\frac{a}{3}\right)$ with $a$ primitive and $a^{2}=$ 8. Therefore the number of $W\left(E_{8}\right)$-transforms of $D\left(\frac{\omega_{2}}{3}\right)$ is equal to $\left|W\left(E_{8}\right) / W\left(A_{7}\right)\right|=17280$, the number of $a \in X$ with $a^{2}=8$ by Lemma 8.12.
Q.E.D.

Proposition 10.11. There are exactly $1920 W\left(E_{8}\right)$-transforms of $D\left(\frac{\omega_{2}}{3}\right)$ up to translation by $X$.

Proof. Those 17280 copies are of the form $D\left(\frac{a}{3}\right)$ with $a \in X$ and $a^{2}=8$. Each copy has 9 vertices, hence there are exactly 1920 (= $17280 / 9) W\left(E_{8}\right)$-transforms of $D\left(\frac{\omega_{2}}{3}\right)$ up to translation by $X$. Q.E.D.

Remark 10.12. Since any vertex of $D\left(\frac{\omega_{2}}{3}\right)$ other than 0 is a root, $D\left(\frac{a^{\prime}}{3}\right)$ is a translate of $D\left(\frac{a}{3}\right)$ by $X$ if and only if $a-a^{\prime}=3 x$ for some root $x \in X$ or $x=0$. If $a-a^{\prime}=3 x \neq 0$, then $2 a a^{\prime}=-9 x^{2}+a^{2}+\left(a^{\prime}\right)^{2}=-2$. Hence $a a^{\prime}=-1$. Therefore $x$ is a root with $a x=3$. Conversely if $x$ is a root with $a x=3$, then $a^{\prime}=a-3 x$ gives a translate $D\left(\frac{a^{\prime}}{3}\right)$ of $D\left(\frac{a}{3}\right)$. By Lemma 8.12 , we may assume $a=e_{1}+e_{2}+e_{3}+e_{4}+2 e_{8}$. Suppose $x$ is a root with $a x=3$. Then by Lemma $8.12, x=e_{k}+e_{8}(1 \leq k \leq 4)$ or $x=\frac{1}{2}\left(e_{1}+e_{2}+e_{3}+e_{4} \pm e_{5} \pm e_{6} \pm e_{7}+e_{8}\right)$. Hence there are precisely 9 $(=1+4+4) X$-translates $D\left(\frac{a^{\prime}}{3}\right)$ of $D\left(\frac{a}{3}\right)$.

Thus we see the following table by applying Lemma 10.3 and Corollary 10.9 .

Table 3. The number of faces of 8-dim Delaunay cells

| $d$ | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $D\left(\frac{\omega_{1}}{2}\right)$ | 256 | 1024 | 1792 | 1792 | 1120 | 448 | 112 | 16 |
| $D\left(\frac{\omega_{2}}{3}\right)$ | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 |

### 10.13. Adjacency of 8-dimensional Delaunay cells

Lemma 10.14. No pair of $a, b \in X$ with $a^{2}=4, b^{2}=2$ and $(a, b)=0$ belong to the same 8-dimensional Delaunay cells.

Proof. By Lemma 8.5 they are equivalent to $a=2 e_{8}$ and $b=$ $-e_{6}+e_{7}$. They could belong to one of the Delaunay cells $D\left(\frac{a}{2}\right)$ with $a^{2}=4$. Since $h_{0}$ is the unique vertex of $D\left(\frac{\omega_{1}}{2}\right)$ with $h_{0}^{2}=4$, there are no vertex $z(\neq 0)$ of $D\left(\frac{\omega_{1}}{2}\right)$ with $\left(h_{0}, z\right)=0$. This proves the lemma.
Q.E.D.

Proposition 10.15. Let $a, a^{\prime}, b$ and $b^{\prime} \in X$ with $a^{2}=\left(a^{\prime}\right)^{2}=4$ and $b^{2}=\left(b^{\prime}\right)^{2}=8$.
(i) $D\left(\frac{a}{2}\right)$ and $D\left(\frac{a^{\prime}}{2}\right)$ are adjacent iff $\left(a, a^{\prime}\right)=3$.
(ii) $D\left(\frac{a}{2}\right)$ and $D\left(\frac{b}{3}\right)$ are adjacent iff $(a, b)=5$.
(iii) $D\left(\frac{b}{3}\right)$ and $D\left(\frac{b^{\prime}}{3}\right)$ are not adjacent.

Proof. By Theorem 9.4 there are precisely two $W\left(\widetilde{E}_{8}\right)$ equivalence classes of 7-dimensional Delaunay cells. By Lemma 10.3, each class is
represented by either $\left\langle 0, h_{1}, \cdots, h_{7}\right\rangle$ or $\left\langle 0, h_{1}, \cdots, h_{6}, h_{8}\right\rangle$. In the first case, the face $\left\langle 0, h_{1}, \cdots, h_{7}\right\rangle$ is a common face of $D\left(\frac{\omega_{1}}{2}\right)$ and $D\left(\frac{\omega_{2}}{3}\right)$ by Lemma 10.2 and Lemma 10.8. We have $\omega_{1} \omega_{2}=5$. Any pair $a$ and $b$ with $a^{2}=4, b^{2}=8$ and $(a, b)=5$ is unique up to $W\left(E_{8}\right)$ by Lemma 8.14. This proves (ii).

In the second case let $\alpha=\frac{1}{2}\left(e_{7}+e_{8}-\left(e_{1}+\cdots+e_{6}\right)\right)$. Then since $\left(\alpha, h_{j}\right)=\left(\alpha, h_{8}\right)=0(1 \leq j \leq 6), r_{\alpha}$ keeps the face $\tau_{1}=$ $\left\langle 0, h_{1}, \cdots, h_{6}, h_{8}\right\rangle$ invariant. Therefore $\tau_{1}$ is a common face of $D\left(\frac{\omega_{1}}{2}\right)$ and $r_{\alpha} D\left(\frac{\omega_{1}}{2}\right)=D\left(\frac{\omega}{2}\right)$ where $\omega=r_{\alpha}\left(\omega_{1}\right)=\frac{1}{2}\left(e_{1}+\cdots+e_{6}-e_{7}+3 e_{8}\right)$. We have $\omega_{1} \omega=3$. Any pair $a$ and $b$ with $a^{2}=b^{2}=4$ and $a b=3$ is unique up to $W\left(E_{8}\right)$ by Lemma 8.14. This proves (i).

There are 17280 copies of $D\left(\frac{\omega_{2}}{3}\right)$. Hence there are $8 \cdot 17280=138240$ 7 -dimensional faces of copies of $D\left(\frac{\omega_{2}}{3}\right)$. Meanwhile there are 2160 copies of $D\left(\frac{\omega_{1}}{2}\right)$, hence there are $128 \cdot 2160=2764807$-dimensional faces of copies of $D\left(\frac{\omega_{2}}{3}\right)$, the half of which are faces of copies of $D\left(\frac{\omega_{1}}{2}\right)$ and the other half of which are faces of copies of $D\left(\frac{\omega_{2}}{3}\right)$. It follows that there are no common faces of $D\left(\frac{b}{3}\right)$ and $D\left(\frac{b^{\prime}}{3}\right)$. This proves (iii). Q.E.D.

Corollary 10.16. (i) Any 8-dimensional cell adjacent to $D\left(\frac{\omega_{1}}{2}\right)$ is either $D\left(w r_{\alpha_{1}} \frac{\omega_{1}}{2}\right)$ or $D\left(w \frac{\omega_{2}}{3}\right)\left(w \in \operatorname{Stab}_{W\left(E_{8}\right)}\left(\omega_{1}\right)=W\left(D_{7}\right)\right)$. There are exactly 128 copies of $D\left(\frac{\omega_{1}}{2}\right)$ adjacent to $D\left(\frac{\omega_{1}}{2}\right)$ and exactly 128 copies of $D\left(\frac{\omega_{2}}{3}\right)$ adjacent to $D\left(\frac{\omega_{1}}{2}\right)$.
(ii) Any 8-dimensional cell adjacent to $D\left(\frac{\omega_{2}}{3}\right)$ is $D\left(w \frac{\omega_{1}}{2}\right)$ where $w \in \operatorname{Stab}_{W\left(E_{8}\right)}\left(\omega_{2}\right)=W\left(A_{7}\right)$. There are exactly 8 copies of $D\left(\frac{\omega_{1}}{2}\right)$ adjacent to $D\left(\frac{\omega_{2}}{3}\right)$.
Proof. By Lemma $10.15, D\left(\frac{\omega_{1}}{2}\right)$ is adjacent to $D\left(\frac{\omega}{2}\right)$ and $D\left(\frac{\omega_{2}}{3}\right)$ where $\omega=\frac{1}{2}\left(e_{1}+\cdots+e_{6}-e_{7}+3 e_{8}\right)=r_{\alpha}\left(\omega_{1}\right)$. Therefore any 8 dimensional Delaunay cell adjacent to $D\left(\frac{\omega_{1}}{2}\right)$ is either $D\left(w \cdot \frac{\omega}{2}\right)$ or $D(w$. $\left.\frac{\omega_{2}}{3}\right)$ for any $w \in \operatorname{Stab}_{W\left(E_{8}\right)}\left(\omega_{1}\right)=W\left(D_{7}\right)$. We note

$$
\alpha=\frac{1}{2}\left(e_{7}+e_{8}-\left(e_{1}+\cdots+e_{6}\right)\right)=r_{\alpha_{8}} r_{\alpha_{7}} \cdots r_{\alpha_{3}}\left(\alpha_{1}\right) .
$$

Let $w_{0}=r_{\alpha_{8}} r_{\alpha_{7}} \cdots r_{\alpha_{3}} \in W\left(D_{7}\right)$. Then $r_{\alpha}=w_{0} \cdot r_{\alpha_{1}} \cdot w_{0}$. Hence

$$
D\left(\frac{\omega}{2}\right)=w_{0} \cdot r_{\alpha_{1}} \cdot w_{0}\left(D\left(\frac{\omega_{1}}{2}\right)\right)=w_{0} \cdot r_{\alpha_{1}}\left(D\left(\frac{\omega_{1}}{2}\right)\right)
$$

Hence any 8-dimensional Delaunay cell adjacent to $D\left(\frac{\omega_{1}}{2}\right)$ is either $D\left(w \cdot r_{\alpha_{1}} \frac{\omega_{1}}{2}\right)$ or $D\left(w \frac{\omega_{2}}{3}\right)$ for any $w \in \operatorname{Stab}_{W\left(E_{8}\right)}\left(\omega_{1}\right)=W\left(D_{7}\right)$. The number of $D\left(w \cdot r_{\alpha_{1}} \frac{\omega_{1}}{2}\right)$ adjacent to $D\left(\frac{\omega_{1}}{2}\right)$ is equal to the number of 7 -dimensional Delaunay faces of $D\left(\frac{\omega_{1}}{2}\right), W\left(E_{8}\right)$-equivalent to $\tau_{1}$ by the proof of Proposition 10.15, hence it is equal to $2^{8} \cdot\binom{8}{7} / 8=128$ where 8
in the denominator is the number of vertices of $\tau_{1}$. Similarly the number of $D\left(w \frac{\omega_{2}}{3}\right)$ adjacent to $D\left(\frac{\omega_{1}}{2}\right)$ is equal to $2^{8} \cdot\binom{8}{7} / 8=128$. The assertion (ii) is clear.
Q.E.D.

### 10.17. Inclusion relation of Delaunay cells

Proposition 10.18. Let $a, b \in X$ with $a^{2}=4, b^{2}=8$ and let $\left\{a_{k}, a_{k+1}, \cdots, a_{7}\right\}(1 \leq k \leq 7)$ be a set of roots such that $a_{i}^{2}=2$ and $\left(a_{i}, a_{j}\right)=1$ for any $i \neq j$. Let $D$ be the convex closure of the origin and $a_{k}, \cdots, a_{7}$. Then $D$ is a Delaunay cell and
(i) $D \subset D\left(\frac{a}{2}\right)$ iff $\left(a_{i}, a\right)=2$ for any $i$.
(ii) $D \subset D\left(\frac{b}{3}\right)$ iff $\left(a_{i}, b\right)=3$ for any $i$.

Proof. Since $D$ is the convex closure of 0 and $a_{i}, D \subset D\left(\frac{a}{2}\right)$ iff 0 and $a_{i}$ are closest to the hole $\frac{a}{2}$. Hence $\left\|\frac{a}{2}\right\|=\left\|a_{i}-\frac{a}{2}\right\|$. This proves (i). The proof of (ii) is similar.
Q.E.D.

Corollary 10.19. Let $D$ be the convex closure of the origin and $a_{k}, \cdots, a_{7}$ as in Proposition 10.18. Then $D$ is the intersection of $D\left(\frac{a}{2}\right)$ and $D\left(\frac{b}{3}\right)$ for all $a$ and $b$ such that $a^{2}=4$ and $\left(a_{i}, a\right)=2$ for any $i$, or $b^{2}=6$ and $\left(a_{i}, b\right)=3$ for any $i$ respectively.

Proof. Since any Delaunay cell is the intersection of all maximal dimensional Delaunay cells containing it, Corollary follows from Proposition 10.18 .
Q.E.D.

Corollary 10.20. For a Delaunay cell $D$ of dimension $8-k$ given in 10.18, there are exactly the following number given in Table 4 of $D\left(\frac{a}{2}\right)$ 's and $D\left(\frac{b}{3}\right)$ 's containing $D$ :

Table 4. The number of 8-dim. cells containing a fixed Delaunay cell

| $k$ | 7 | 6 | 5 | 4 | 3 | 2 | $\Delta_{1}^{1}$ | $\Delta_{1}^{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $D\left(\frac{a}{2}\right)$ | 126 | 27 | 10 | 5 | 3 | 2 | 1 | 2 |
| $D\left(\frac{b}{3}\right)$ | 576 | 72 | 16 | 5 | 2 | 1 | 1 | 0 |
| total | 702 | 99 | 26 | 10 | 5 | 3 | 2 | 2 |

Proof. Suppose $k \geq 2$. Then by Lemma 8.9 we may assume $a_{i}=$ $e_{i}+e_{8}(k \leq i \leq 7)$. Let $D(k)$ be the convex closure of $a_{i}=e_{i}+e_{8}(k \leq i \leq$ 7). In view of Lemma $10.18 D(k) \subset D\left(\frac{a}{2}\right)$ iff $\left(a_{i}, a\right)=2$ for any $i$. Suppose $k=7$. Then $D(7) \subset D\left(\frac{2 e_{8}}{2}\right)$. We see $\operatorname{Stab}_{W\left(E_{8}\right)}\left(e_{7}+e_{8}\right)=W\left(E_{7}\right)$
and $\operatorname{Stab}_{W\left(E_{8}\right)}\left(e_{7}+e_{8}, 2 e_{8}\right)=W\left(D_{6}\right)$. Thus in view of Lemma 8.8 the number of $D\left(\frac{a}{2}\right)$ with $D(7) \subset D\left(\frac{a}{2}\right)$ is equal to $\left|W\left(E_{7}\right)\right| /\left|W\left(D_{6}\right)\right|=$ $2^{10} \cdot 3 \cdot 5 \cdot 7 / 2^{9} \cdot 3^{2} \cdot 5=126$. Similarly If $k=6$, then $D(6) \subset D\left(\frac{a}{2}\right)$ iff $a=2 e_{8}, \pm e_{i}+e_{6}+e_{7}+e_{8}$, or $\frac{1}{2}\left(\sum_{j=1}^{5} \pm e_{j}+e_{6}+e_{7}+3 e_{8}\right)$. Hence there are exactly $1+10+2^{4}=27$ cells $D\left(\frac{a}{2}\right)$ which contain $D(6)$. This is checked by computing $\left|W\left(E_{6}\right)\right| /\left|W\left(D_{5}\right)\right|=27$. If $k=5$, then $D(5) \subset D\left(\frac{a}{2}\right)$ iff $a=2 e_{8}, e_{5}+e_{6}+e_{7}+e_{8}$, or $\frac{1}{2}\left(\sum_{j=1}^{4} \pm e_{j}+e_{5}+e_{6}+e_{7}+3 e_{8}\right)$. Hence there are exactly $1+1+8=10$ cells $D\left(\frac{a}{2}\right)$ which contain $D(5)$. This is checked by computing $\left|W\left(D_{5}\right)\right| /\left|W\left(D_{4}\right)\right|=10$. If $2 \leq k \leq 4$, then $D(k) \subset D\left(\frac{a}{2}\right)$ iff $a=2 e_{8}$ or $\frac{1}{2}\left(\sum_{j=1}^{k-1} \pm e_{j}+e_{k}+\cdots+e_{7}+3 e_{8}\right)$. Hence there are exactly $1+2^{k-2}$ cells $D\left(\frac{a}{2}\right)$ which contain $D(k)$. This is checked by computing $\left|W\left(A_{4}\right)\right| /\left|W\left(A_{3}\right)\right|=5,\left|W\left(A_{1}\right) \times W\left(A_{2}\right)\right| /\left|W\left(A_{1}\right) \times W\left(A_{1}\right)\right|=3$ and $\left.\mid W\left(A_{1}\right)\right) \mid=2$. If $k=1$, then there is a unique $D\left(\frac{a}{2}\right)$ which contain $D$.

Next we consider $D\left(\frac{b}{3}\right)$. Let $G(k)=\operatorname{Stab}_{W\left(E_{8}\right)}\left(e_{k}+e_{8}, \cdots, e_{7}+e_{8}\right)$ and $H(k)=\operatorname{Stab}_{W\left(E_{8}\right)}\left(\omega_{2}\right) \cap G(k)$. Then though it is nontrivial, by explicit computation we see the number of $D\left(\frac{b}{3}\right)$ containing $D(k)$ is equal to $|G(k)| /|H(k)|$. We see $G(k)=W\left(E_{7}\right), W\left(E_{6}\right), W\left(D_{5}\right), W\left(A_{4}\right), W\left(A_{1} \times\right.$ $A_{2}$ ) and $W\left(A_{1}\right)$, while $H(k)=W\left(A_{k-1}\right)$ for any $k$. Hence the number of $D\left(\frac{b}{3}\right)$ containing $D(k)$ is equal to $576,72,16,5,2$ and 1 respectively. The case $k=1$ is clear from Proposition 10.15.
Q.E.D.

## §11. A PSQAS associated with $E_{8}$

Now we return to the situation in the section three. Let $B(x, y)$ be the bilinear form on the lattice $X$ in Definition 3.1. We assume that $(X, B)$ is the $E_{8}$-lattice. Let $(Q, L)$ be the flat projective $R$-scheme in Theorem 3.3, $\left(Q_{0}, L_{0}\right)$ the closed fibre of it. Let $R(c)$ be the coordinate ring of an affine chart $U(c)(c \in X / Y)$ of $Q_{0}$ in Definition 3.6. The purpose of this section is to show that there are actually nilpotent elements in $R(0)$. For this purpose we determine the function $v$ on $X$ in Definition 2.9 explicitly.

Let $D$ be a convex polytope containing the origin, $C(0, D)$ the cone over $\mathbf{R}_{0}$ generated by $D \cap X$, and $\operatorname{Semi}(0, D)$ the cone over $\mathbf{Z}_{0}$ of $D \cap X$.

Recall (and define)

$$
\begin{gathered}
h_{0}=2 e_{8}, h_{j}=e_{j}+e_{8}, h_{15-j}=-e_{j}+e_{8}(1 \leq j \leq 7) \\
g_{0}=\frac{1}{2}\left(e_{1}+e_{2}+\cdots+e_{8}\right), g_{\infty}=g_{0}+h_{0}=\frac{1}{2}\left(\sum_{j=1}^{7} e_{j}+5 e_{8}\right), \\
\sigma_{0}=\left\langle 0, h_{1}, h_{2}, \cdots, h_{7}, h_{0}\right\rangle, \quad \sigma_{1}=\left\langle 0, h_{1}, h_{2}, \cdots, h_{6}, h_{8}, h_{0}\right\rangle .
\end{gathered}
$$

Lemma 11.1. Let $h\left(\sigma_{0}\right)=\frac{1}{2}\left(\sum_{j=0}^{7} h_{j}\right)$ and $h\left(\sigma_{1}\right)=\frac{1}{2}\left(\sum_{j=1}^{6} h_{j}+\right.$ $h_{8}$ ). Then
(i) we have

$$
\begin{aligned}
& \operatorname{Semi}\left(0, \sigma_{0}\right)=\mathbf{Z}_{0} h_{1}+\cdots+\mathbf{Z}_{0} h_{6}+\mathbf{Z}_{0} h_{7}+\mathbf{Z}_{0} h_{0} \\
& \operatorname{Semi}\left(0, \sigma_{1}\right)=\mathbf{Z}_{0} h_{1}+\cdots+\mathbf{Z}_{0} h_{6}+\mathbf{Z}_{0} h_{8}+\mathbf{Z}_{0} h_{0}
\end{aligned}
$$

(ii) $h\left(\sigma_{k}\right) \in C\left(0, \sigma_{k}\right) \cap X$ but $h\left(\sigma_{k}\right) \notin \operatorname{Semi}\left(0, \sigma_{k}\right)(k=0,1)$.
(iii) $\quad C\left(0, D\left(\frac{\omega_{1}}{2}\right)\right) \cap X$ is the union of all $C\left(0, w \cdot \sigma_{0}\right) \cap X$ and $C(0, w$. $\left.\sigma_{1}\right) \cap X$ where $w$ ranges over $W\left(D_{7}\right)$.
(iv) $C\left(0, \sigma_{0}\right) \cap X$ is generated by $\operatorname{Semi}\left(0, \sigma_{0}\right)$ and $h\left(\sigma_{0}\right)$. It is the disjoint union of $\operatorname{Semi}\left(0, \sigma_{0}\right)$ and $h\left(\sigma_{0}\right)+\operatorname{Semi}\left(0, \sigma_{0}\right)$ :
$C\left(0, \sigma_{0}\right) \cap X=\operatorname{Semi}\left(0, \sigma_{0}\right) \sqcup\left(h\left(\sigma_{0}\right)+\operatorname{Semi}\left(0, \sigma_{0}\right)\right)$.
(v) $\quad C\left(0, \sigma_{1}\right) \cap X$ is generated by $\operatorname{Semi}\left(0, \sigma_{1}\right)$ and $h\left(\sigma_{1}\right)$. It is the disjoint union of $\operatorname{Semi}\left(0, \sigma_{1}\right)$ and $h\left(\sigma_{1}\right)+\operatorname{Semi}\left(0, \sigma_{1}\right)$ :
$C\left(0, \sigma_{1}\right) \cap X=\operatorname{Semi}\left(0, \sigma_{1}\right) \sqcup\left(h\left(\sigma_{1}\right)+\operatorname{Semi}\left(0, \sigma_{1}\right)\right)$.
(vi) $\quad C\left(0, w \cdot \sigma_{k}\right) \cap X=w \cdot\left(C\left(0, \sigma_{k}\right) \cap X\right)$ where $k=0,1$ and $w \in W\left(D_{7}\right)$.

Proof. By Lemma 10.2, $\sigma_{0} \cap X \subset D\left(\frac{\omega_{1}}{2}\right) \cap X=\left\{0, h_{j}(0 \leq j \leq 14)\right\}$ and $\sigma_{1} \cap X \subset D\left(\frac{\omega_{1}}{2}\right) \cap X$, which implies (i). Since $h\left(\sigma_{0}\right)=g_{0}+2 h_{0} \in X$, (ii) is clear for $\sigma_{0}$ because $h_{j}(0 \leq j \leq 7)$ are linearly independent and $\sigma_{0} \cap X=\left\{0, h_{j}(0 \leq j \leq 7)\right\}$. Since $h\left(\sigma_{1}\right)=g_{0}+h_{0}+h_{8} \in X$ (ii) is also clear for $\sigma_{1}$. (iii) follows from the fact that $D\left(\frac{\omega_{1}}{2}\right)$ is the union of $w \cdot \sigma_{0}$ and $w \cdot \sigma_{1}\left(w \in W\left(D_{7}\right)\right)$. See Lemma 10.3 (i). Next we prove (iii). Let $x \in C\left(0, \sigma_{0}\right) \cap X$. Then we write $x=\sum_{j=0}^{7} a_{j} h_{j}$ with $a_{j} \geq 0$. If $a_{j}=0$ for any $j \geq 1$, then $x=a_{0} h_{0}, a_{0} \in \mathbf{Z}_{+}$. Hence $x \in \operatorname{Semi}\left(0, \sigma_{0}\right)$. So we may assume $a_{1}>0$ (by transforming $x$ by $S_{7}$ if necessary). If $a_{1} \in \mathbf{Z}_{+}$, then $x \in \operatorname{Semi}\left(0, \sigma_{0}\right)$. So we assume $a_{1}$ is not an integer, hence $a_{1} \equiv \frac{1}{2} \bmod \mathbf{Z}$. Hence $a_{j} \equiv \frac{1}{2} \bmod \mathbf{Z}$ for any $j \geq 2$. Since $x \in X$, $\sum_{j=0}^{7} a_{j}$ is integral, hence $a_{0} \equiv \frac{1}{2} \bmod \mathbf{Z}$. Hence $a_{j} \geq \frac{1}{2}$ for any $j \geq 0$. Let $z=x-h\left(\sigma_{0}\right)$. Since $h\left(\sigma_{0}\right) \in X$, we have $z \in C\left(0, \sigma_{0}\right) \cap X$ and $z=\sum_{j=0}^{7} b_{j} h_{j}$ for some $b_{j} \in \mathbf{Z}_{0}$, namely, $z \in \operatorname{Semi}\left(0, \sigma_{0}\right)$. This proves (iv).

Next we prove (v). Let $x \in C\left(0, \sigma_{0}\right) \cap X$. Then we write $x=$ $\sum_{j=0}^{6} a_{j} h_{j}+a_{8} h_{8}$ with $a_{j} \geq 0$. If $a_{j}=0$ for any $j \geq 1$, then $x=$ $a_{0} h_{0}, a_{0} \in \mathbf{Z}_{+}$. Hence $x \in \operatorname{Semi}\left(0, \sigma_{0}\right)$. So we may assume $a_{1}>0$ (by transforming $x$ by $\operatorname{Stab}_{W\left(D_{7}\right)}\left(\sigma_{1}\right)$ if necessary). If $a_{1} \in \mathbf{Z}_{+}$, then $x \in \operatorname{Semi}\left(0, \sigma_{0}\right)$. So we assume $a_{1}$ is not an integer, hence $a_{1} \equiv \frac{1}{2}$
$\bmod \mathbf{Z}$. Hence $a_{j} \equiv \frac{1}{2} \bmod \mathbf{Z}$ for any $j \geq 2$. Since $x \in X, \sum_{j=0}^{6} a_{j}$ is integral, hence $a_{0}$ is integral. Let $z=x-h\left(\sigma_{1}\right)$. Since $h\left(\sigma_{1}\right) \in X$, we have $z \in C\left(0, \sigma_{1}\right) \cap X$ and $z=a_{0} h_{0}+\sum_{j=1}^{6} b_{j} h_{j}+b_{8} h_{8}$ for some $b_{j} \in \mathbf{Z}_{0}$. Since $a_{0} \in \mathbf{Z}_{0}$, we have $z \in \operatorname{Semi}\left(0, \sigma_{0}\right)$. This proves (v). The remaining assertions are clear.
Q.E.D.

Lemma 11.2. Let $g_{\infty}=g_{0}+h_{0}=\frac{1}{2}\left(\sum_{j=1}^{7} e_{j}+5 e_{8}\right)$. Then $C\left(0, D\left(\frac{\omega_{2}}{3}\right)\right) \cap X$ is generated by $h_{1}, h_{2}, \cdots, h_{7}, g_{0}$ and $g_{\infty}$. It is the disjoint union of $\operatorname{Semi}\left(0, D\left(\frac{\omega_{2}}{3}\right)\right), g_{\infty}+\operatorname{Semi}\left(0, D\left(\frac{\omega_{2}}{3}\right)\right)$ and $2 g_{\infty}+$ $\operatorname{Semi}\left(0, D\left(\frac{\omega_{2}}{3}\right)\right)$ :

$$
C\left(0, D\left(\frac{\omega_{2}}{3}\right)\right) \cap X=\sqcup_{k=0,1,2}\left(k g_{\infty}+\operatorname{Semi}\left(0, D\left(\frac{\omega_{2}}{3}\right)\right)\right.
$$

where we note that $g_{\infty}$ does not belong to $D\left(\frac{\omega_{2}}{3}\right) \cap X$.
Proof. First we note that $3 g_{\infty}=h_{1}+h_{2}+\cdots+h_{7}+g_{0}$ and hence $g_{\infty} \in C\left(0, D\left(\frac{\omega_{2}}{3}\right)\right) \cap X$. Let $C_{0}=\mathbf{Z}_{0} h_{1}+\cdots+\mathbf{Z}_{0} h_{7}+\mathbf{Z}_{0} g_{0}$. Then $C_{0}=\operatorname{Semi}\left(0, D\left(\frac{\omega_{2}}{3}\right)\right)$. Suppose $x \in C\left(0, D\left(\frac{\omega_{2}}{3}\right)\right) \cap X$. Then we write

$$
x=\sum_{j=1}^{7} x_{j} h_{j}+x_{0} g_{0}=\sum_{j=1}^{7}\left(x_{j}+\frac{x_{0}}{2}\right) e_{j}+\left(\sum_{j=1}^{7} x_{j}+\frac{x_{0}}{2}\right) e_{8}
$$

where $x_{j} \geq 0, x_{j}-x_{j} \in \mathbf{Z}, 2 x_{1}+x_{0} \in \mathbf{Z}$ and $7 x_{1}+2 x_{0} \in \mathbf{Z}$. It follows that $3 x_{0} \in \mathbf{Z}$ and $x_{k} \equiv x_{0} \bmod \mathbf{Z}$ for any $1 \leq k \leq 7$. Suppose $x_{0} \in \mathbf{Z}$. Then any $x_{j} \in \mathbf{Z}$ and $x \in C_{0}$. Suppose next $x_{0} \equiv \frac{1}{3} \bmod \mathbf{Z}$. Then let $z_{j}=x_{j}-\frac{1}{3}$ and $z=x-g_{\infty}$. Since $x_{j} \geq 0$ and $x_{j} \equiv \frac{1}{3} \bmod \mathbf{Z}$, we have $z_{j} \in \mathbf{Z}_{0}$. It follows $z \in C_{0}$. Suppose finally $x_{0} \equiv \frac{2}{3} \bmod \mathbf{Z}$. Then $z=x-2 g_{\infty} \in C_{0}$. This proves the lemma.
Q.E.D.

Lemma 11.3. Let $D=D\left(\frac{\omega_{1}}{2}\right)$ and let $\alpha(D)=\frac{\omega_{1}}{2}=e_{8}$ be the hole of $D$. Then
$v(x)=\left\{\begin{array}{lll}(x, \alpha(D)) & \text { if } & x \in \operatorname{Semi}(0, D) \\ \left(x-h\left(\sigma_{0}\right), \alpha(D)\right)+5 & \text { if } & x \in h\left(\sigma_{0}\right)+\operatorname{Semi}\left(0, \sigma_{0}\right) \\ \left(x-h\left(\sigma_{1}\right), \alpha(D)\right)+4 & \text { if } & x \in h\left(\sigma_{1}\right)+\operatorname{Semi}\left(0, \sigma_{1}\right) \\ \left(x-w \cdot h\left(\sigma_{0}\right), \alpha(D)\right)+5 & \text { if } & x \in w \cdot h\left(\sigma_{0}\right)+\operatorname{Semi}\left(0, w \cdot \sigma_{0}\right) \\ \left(x-w \cdot h\left(\sigma_{1}\right), \alpha(D)\right)+4 & \text { if } & x \in w \cdot h\left(\sigma_{1}\right)+\operatorname{Semi}\left(0, w \cdot \sigma_{1}\right)\end{array}\right.$
where $w \in W\left(D_{7}\right)=\operatorname{Stab}_{W\left(E_{8}\right)}\left(\frac{\omega_{1}}{2}\right)$.
Proof. If $x \in \operatorname{Semi}(0, D)$, then $v(x)=(x, \alpha(D))$ by Lemma 2.10. Next suppose $x=h\left(\sigma_{0}\right)$. Then $h\left(\sigma_{0}\right)=g_{0}+h_{0}+h_{0}$ where $g_{0}=$ $\frac{1}{2}\left(\sum_{j=1}^{8} e_{j}\right)$. Therefore $v\left(h\left(\sigma_{0}\right)\right) \leq \frac{1}{2}\left(g_{0}^{2}+2 h_{0}^{2}\right)=5$.

Meanwhile $\left(2 h\left(\sigma_{0}\right), \alpha(D)\right)=\left(\sum_{j=0}^{7} h_{j}, e_{8}\right)=9$, whence $v\left(h\left(\sigma_{0}\right)\right) \geq$ 5 by Lemma 2.10 and Lemma 11.1. This proves $v\left(h\left(\sigma_{0}\right)\right)=5$. This also proves the second equality for $x=h\left(\sigma_{0}\right)$. Next suppose $x=h\left(\sigma_{0}\right)+z$ for some $z \in \operatorname{Semi}\left(0, \sigma_{0}\right)$. Then $v(x) \leq v\left(h\left(\sigma_{0}\right)\right)+v(z)=v(z)+5$. Meanwhile $\left.v(x) \geq\left(h\left(\sigma_{0}\right)\right)+z, \alpha(D)\right)=\frac{9}{2}+v(z)$. This proves the second equality for $x=h\left(\sigma_{0}\right)+z, z \in \operatorname{Semi}\left(0, \sigma_{0}\right)$.

We see $h\left(\sigma_{1}\right)=g_{0}+h_{0}+h_{8}$ and $v\left(h\left(\sigma_{1}\right)\right) \leq 4$. On the other hand $\left(2 h\left(\sigma_{1}\right), \alpha(D)\right)=\left(\sum_{j=1}^{6} h_{j}+h_{8}, \alpha(D)\right)=\left(\sum_{j=1}^{6} h_{j}+h_{8}, e_{8}\right)=7$, whence $v\left(h\left(\sigma_{1}\right)\right) \geq 4$ by Lemma 2.10 and Lemma 11.1. This proves $v\left(h\left(\sigma_{1}\right)\right)=4$. This also proves the third equality for $x \in h\left(\sigma_{1}\right)+\operatorname{Semi}\left(0, \sigma_{1}\right)$. The remaining assertions are clear.
Q.E.D.

Lemma 11.4. Let $D=D\left(\frac{\omega_{2}}{3}\right)$ and $\alpha(D)=\frac{\omega_{2}}{3}$ the hole of $D$. Then $v\left(x+k g_{\infty}\right)=(x, \alpha(D))+3 k$ for $k=0,1,2$ and $x \in \operatorname{Semi}(0, D)$.

Proof. Let $a_{\infty}=\frac{1}{2}\left(e_{1}+e_{2}+e_{3}+e_{4}-e_{5}-e_{6}+e_{7}+e_{8}\right)$. Then $g_{\infty}=h_{5}+h_{6}+a_{\infty}$ and $v\left(g_{\infty}\right) \leq 3$. Since $\left(3 g_{\infty}, \alpha(D)\right)=8$, we have $v\left(g_{\infty}\right) \geq\left(g_{\infty}, \alpha(D)\right)=\frac{8}{3}$. This proves $v\left(g_{\infty}\right)=3$. This also proves the lemma in the case $k=1$. Similarly we see $v\left(g_{\infty}\right) \leq 6$ while $v\left(2 g_{\infty}\right) \geq$ $\left(2 g_{\infty}, \alpha(D)\right)=\frac{16}{3}$. Since $v\left(2 g_{\infty}\right)$ is an integer, we have $v\left(2 g_{\infty}\right)=6$. This also proves the lemma in the case $k=2$.
Q.E.D.

Theorem 11.5. Let $D \in \operatorname{Del}(0)$ and $\alpha(D)$ its hole. For $x \in$ $C(0, D) \cap X$ we have
$v(x)=\lceil(x, \alpha(D))\rceil:=-[-(x, \alpha(D))]$, the round-up of $(x, \alpha(D))$.
In particular, $x \in \operatorname{Semi}(0, D)$ iff $(x, \alpha(D)) \in \mathbf{Z}$.
Proof. We may assume $D$ is 8 -dimensional. If $D=D\left(\frac{\omega_{1}}{2}\right)$, then

$$
v(x)= \begin{cases}(x, \alpha(D)) & \text { if } x \in \operatorname{Semi}(0, D) \\ (x, \alpha(D))+\frac{1}{2} & \text { (otherwise) }\end{cases}
$$

This also proves the corollary when $D \in \operatorname{Del}(0)$ is an 8 -dimensional Delaunay cell $W\left(E_{8}\right)$-equivalent to $D=D\left(\frac{\omega_{1}}{2}\right)$. If $D=D\left(\frac{\omega_{2}}{3}\right)$, then

$$
v(x)= \begin{cases}(x, \alpha(D)) & \text { if } x \in \operatorname{Semi}(0, D) \\ (x, \alpha(D))+\frac{1}{3} & \text { if } x \in g_{\infty}+\operatorname{Semi}(0, D) \\ (x, \alpha(D))+\frac{2}{3} & \text { if } x \in 2 g_{\infty}+\operatorname{Semi}(0, D)\end{cases}
$$

This also proves the corollary when $D \in \operatorname{Del}(0)$ is an 8 -dimensional Delaunay cell $W\left(E_{8}\right)$-equivalent to $D=D\left(\frac{\omega_{2}}{3}\right)$. The above proof also proves the second assertion of the corollary. This completes the proof.
Q.E.D.

Theorem 11.6. Let $Q_{0}$ be the closed fibre of $Q$ and $\operatorname{rad}\left(O_{0, Q_{0}}\right)$ the radical of the algebra $O_{0, Q_{0}}$. Then $\operatorname{rad}\left(O_{0, Q_{0}}\right)$ is generated over $k(0)$ by the monomials $\bar{\xi}(x)$ with $v(x)>(x, \alpha(D))$ and $x \in C(0, D) \cap X$ for some $D \in \operatorname{Del}(0)$. It is also generated by $\bar{\xi}(x)$ with $x \in C(0, D) \cap X$ and $(x, \alpha(D))$ not integral.

Proof. Let $z \in O_{0, Q_{0}}$. We write $z$ as a $k(0)$-linear irredundant combination of $\bar{\xi}(x),(x \in X)$. Then if $z \in O_{0, Q_{0}}$ is nilpotent, each monomial component $\bar{\xi}(x)$ of $z$ is also nilpotent because the algebra $O_{0, Q_{0}}$ is $X$-graded. The monomial $\bar{\xi}(x)=q^{v(x)} w^{x} \in \operatorname{rad}\left(O_{0, Q_{0}}\right)$ iff $q^{n v(x)} w^{n x}=0$ for some positive $n$, iff $q^{6 n v(x)} w^{6 n x}=0$ for some positive $n$. We see by Lemma 2.10 that $q^{6 n v(x)} w^{6 n x}=0$ iff $6 n v(x)>v(6 n x)$. Let $D \in \operatorname{Del}(0)$ such that $x \in C(0, D) \cap X$. In the $E_{8}$-case, $6 x \in \operatorname{Semi}(0, D)$ iff $x \in C(0, D) \cap X$ because $2 x \in \operatorname{Semi}\left(0, D_{1}\right)$ iff $x \in C\left(0, D_{1}\right) \cap X$, while $3 x \in \operatorname{Semi}\left(0, D_{2}\right)$ iff $x \in C\left(0, D_{2}\right) \cap X$. It follows that $6 n v(x)>$ $v(6 n x)$ iff $6 n v(x)>(6 n x, \alpha(D))$. Thus $\bar{\xi}(x)=q^{v(x)} w^{x} \in \operatorname{rad}\left(O_{0, Q_{0}}\right)$ iff $v(x)>(x, \alpha(D))$. This proves the first part of the theorem. By Theorem $11.5 v(x)=\lceil(x, \alpha(D))\rceil$. Hence $v(x)>(x, \alpha(D))$ iff $(x, \alpha(D))$ is not an integer. This proves the second part of the theorem. Q.E.D.

Corollary 11.7. $O_{c, Q_{0}}$ is nonreduced for any $c \in X$.
Corollary 11.8. Let $f=\bar{\xi}(a)$ and $g=\bar{\xi}(b) \in O_{0, Q_{0}}$. Assume that $a, b \in C(0, D)$ for the same Delaunay cell $D \in \operatorname{Del}(0)$. If $b \in \operatorname{Semi}(0, D)$, then $f g \neq 0$ in $O_{0, Q_{0}}$.

Proof. By Theorem 11.5, $v(a)=\lceil(a, \alpha(D))\rceil$, while $v(b)=(b, \alpha(D))$ is an integer. Hence $v(a+b)=\lceil(a+b, \alpha(D))\rceil=\lceil(a, \alpha(D))\rceil+(b, \alpha(D))=$ $v(a)+v(b)$. It follows from Theorem 11.5 that $f g \neq 0$ in $O_{c, Q_{0}}$. Q.E.D.

Example 11.9. We give examples of nilpotent elements of $O_{0, Q_{0}}$. Let $D_{1}=D\left(\frac{\omega_{1}}{2}\right)$ and $D_{2}=D\left(\frac{\omega_{2}}{3}\right)$. Consider $\xi\left(h\left(\sigma_{0}\right)\right)$. Then $h\left(\sigma_{0}\right) \in$ $C\left(0, D_{1}\right) \cap X,\left(h\left(\sigma_{0}\right), \alpha\left(D_{1}\right)\right)=\frac{9}{2}$ and $v\left(h\left(\sigma_{0}\right)\right)=\left\lceil\frac{9}{2}\right\rceil=5$. Consider next $\xi\left(h\left(\sigma_{1}\right)\right)$. Then we see $h\left(\sigma_{1}\right) \in C\left(0, D_{1}\right) \cap X,\left(h\left(\sigma_{1}\right), \alpha\left(D_{1}\right)\right)=\frac{7}{2}$ and $v\left(h\left(\sigma_{1}\right)\right)=\left\lceil\frac{7}{2}\right\rceil=4$. Finally consider $\xi\left(g_{\infty}\right)$. Then we see $g_{\infty} \in$ $C\left(0, D_{2}\right) \cap X,\left(g_{\infty}, \alpha\left(D_{2}\right)\right)=\frac{8}{3}$ and $v\left(g_{\infty}\right)=\left\lceil\frac{8}{3}\right\rceil=3$. It follows from these that

$$
\left.\xi\left(h\left(\sigma_{0}\right)\right)^{2}=\xi\left(h\left(\sigma_{1}\right)\right)^{2}=\xi\left(g_{\infty}\right)\right)^{3}=0 .
$$

To be more precise, since

$$
\begin{gathered}
h\left(\sigma_{0}\right)=g_{0}+2 h_{0}, h\left(\sigma_{1}\right)=g_{0}+h_{0}+h_{8}, g_{\infty}=g_{0}+h_{0} \\
\xi\left(h\left(\sigma_{0}\right)\right)=\xi_{g_{0}} \xi_{h_{0}}^{2}, \xi\left(h\left(\sigma_{1}\right)\right)=\xi_{g_{0}} \xi_{h_{0}} \xi_{h_{8}}, \quad \xi\left(g_{\infty}\right)=\xi_{g_{0}} \xi_{h_{0}}
\end{gathered}
$$

we see

$$
\begin{gathered}
\xi\left(h\left(\sigma_{0}\right)\right)^{2}=q \cdot \xi_{h_{0}} \prod_{j=1}^{7} \xi_{h_{j}}, \quad \xi\left(h\left(\sigma_{1}\right)\right)^{2}=q \cdot \xi_{h_{8}} \prod_{j=1}^{6} \xi_{h_{j}} \\
\xi\left(g_{\infty}\right)^{3}=q \cdot \xi_{g_{0}} \prod_{j=1}^{7} \xi_{h_{j}} .
\end{gathered}
$$

We note that $h\left(\sigma_{0}\right) \in C\left(0, D_{1}\right)$ and $h_{0} \in D_{1}$, while $g_{0} \notin D_{1}$ by Proposition 10.18 (i) because $\left(g_{0}, \omega_{1}\right)=1 \neq 2$. Let $a_{0}=\frac{1}{2}\left(3 e_{8}-\right.$ $\left.e_{6}+\sum_{k \neq 6,8} e_{k}\right)$. Then $a_{0}^{2}=4,\left(a_{0}, \omega_{1}\right)=3$ and $\left(a_{0}, g_{0}\right)=2$, which implies that $D\left(\frac{a_{0}}{2}\right)$ is adjacent to $D_{1}=D\left(\frac{\omega_{1}}{2}\right)$ and $g_{0} \in D\left(\frac{a_{0}}{2}\right)$ by Proposition 10.15 (i) and Proposition 10.18 (i). In other words, though $g_{0} \notin D_{1}, g_{0}$ belongs to $D\left(\frac{a_{0}}{2}\right)$ adjacent to $D_{1}$. We also note $g_{0} \in D_{2}$, which is adjacent to $D_{1}$.

Similarly $h\left(\sigma_{1}\right) \in C\left(0, D_{1}\right)$ and $h_{0}, h_{8} \in D_{1}$, while $g_{0} \notin D_{1}$ and $g_{0} \in D\left(\frac{a_{0}}{2}\right) \cap D_{2}$ as we saw above. We see $h_{0} \notin D_{2}$ because $D_{2}$ is a convex closure of $0, g_{0}$ and $h_{j}(1 \leq j \leq 7)$, and $g_{0}^{2}=h_{j}^{2}=2$, but $h_{0}^{2}=4$. Since $h_{0} \in D_{1}, h_{0}$ belongs to a Delaunay cell $D_{1}$ adjacent to $D_{2}$. See Proposition 10.15 (ii). Finally we note that $g_{\infty} \in C\left(0, D_{2}\right), g_{0} \in D_{2}$, while $h_{0} \notin D_{2}$ but $h_{0} \in D_{1}$, which is adjacent to $D_{2}$.

Corollary 11.10. The (reduced) support of $\xi\left(h\left(\sigma_{k}\right)\right)$ (resp. $\xi\left(g_{\infty}\right)$ ) contains one of the irreducible components of $Q_{0}, V\left(D\left(\frac{\omega_{1}}{2}\right)\right) \cap U(0)$ (resp. $V\left(D\left(\frac{\omega_{2}}{3}\right) \cap U(0)\right)$.

Proof. Let $Z=V\left(D\left(\frac{\omega_{1}}{2}\right)\right) \cap U(0)$. Then $Z$ is reduced by definition, whose coordinate ring $\Gamma\left(O_{Z}\right)$ is $k(0)\left[\operatorname{Semi}\left(0, D\left(\frac{\omega_{1}}{2}\right)\right)\right]$, the ring generated by the semi-group $\operatorname{Semi}\left(0, D\left(\frac{\omega_{1}}{2}\right)\right)$. No element of this ring except 0 annihilates $\xi\left(h\left(\sigma_{k}\right)\right)$ in $R(0)$ by Theorem 11.5. Similarly the coordinate ring of $V\left(D\left(\frac{\omega_{2}}{3}\right) \cap U(0)\right)$ is $k(0)\left[\operatorname{Semi}\left(0, D\left(\frac{\omega_{2}}{3}\right)\right)\right]$, none of whose elements except 0 annihilate $\xi\left(g_{\infty}\right)$. This proves the corollary.
Q.E.D.

### 11.11. Degrees of irreducible components of $Q_{0}$

Let $D_{1}=D\left(\frac{\omega_{1}}{2}\right)$ or $D_{2}=D\left(\frac{\omega_{2}}{3}\right)$. Let $V\left(D_{k}\right)$ be the closure of $\mathbf{G}_{m}^{8}$-orbit $O\left(D_{k}\right)$ with reduced structure. By Lemma 11.1 and Theorem 11.2, at a generic point of $V(\sigma)$, we have $\operatorname{rank}_{k\left(V\left(D_{1}\right)\right)}{ }_{n} F_{D_{1}}^{0,0}=2$ and $\operatorname{rank}_{k\left(V\left(D_{2}\right)\right)}{ }_{n} F_{D_{1}}^{0,0}=3$. Thus by Proposition 10.5 and Proposition 10.11 we have an equivalence

$$
Q_{0}=2 \cdot 135[X: Y] V\left(D_{1}\right)+3 \cdot 1920[X: Y] V\left(D_{2}\right)
$$

modulo identification of the irreducible components of $Q_{0}$ of the same type. By Theorem 5.15 we have

$$
\begin{gathered}
\operatorname{dim} \mathbf{H}^{0}\left({ }_{n} F_{\sigma}^{0, \cdot}, \delta_{n}^{0, \cdot}\right)=\sharp\left(\sigma \cap \frac{X}{n}\right)=\frac{\operatorname{vol}(\sigma)}{8!} \cdot n^{8}+O\left(n^{7}\right), \\
\operatorname{dim} \mathbf{H}^{0}\left({ }_{n} F_{D_{1}}^{0, \cdot}, \delta_{n}^{0, \cdot}\right)=\frac{2^{7} \cdot 2}{8!} \cdot n^{8}+O\left(n^{7}\right), \\
\operatorname{dim} \mathbf{H}^{0}\left({ }_{n} F_{D_{2}}^{0, \cdot}, \delta_{n}^{0, \cdot}\right)=\frac{3}{8!} \cdot n^{8}+O\left(n^{7}\right) .
\end{gathered}
$$

Since $\mathbf{H}^{q}\left({ }_{n} F_{\sigma}^{k, \cdot}, \delta_{n}^{k, \cdot}\right)=0(q>0)$, we have $2 \cdot\left(L_{V\left(D_{1}\right)}^{8}\right)=\operatorname{vol}\left(D_{1}\right)=$ $2^{8}$, and $3 \cdot\left(L_{V\left(D_{2}\right)}^{8}\right)=\operatorname{vol}\left(D_{2}\right)=3$. Thus we have

$$
\begin{aligned}
L_{Q_{0}}^{8} & =\left(L^{8} Q_{0}\right)_{(Q, \partial Q)} \\
& =L^{8}\left(2 \cdot 135[X: Y] V\left(D_{1}\right)+3 \cdot 1920[X: Y] V\left(D_{2}\right)\right)_{(Q, \partial Q)} \\
& =[X: Y]\left(135 \cdot 2 \cdot\left(L_{V\left(D_{1}\right)}^{8}\right)+1920 \cdot 3 \cdot\left(L_{V\left(D_{2}\right)}^{8}\right)\right) \\
& =[X: Y]\left(135 \cdot 2^{8}+1920 \cdot 3\right)=8!\cdot[X: Y]
\end{aligned}
$$

which is compatible with $L_{Q_{0}}^{8}=L_{Q_{\eta}}^{8}=8!\cdot[X: Y]$.

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# Semi-stable extensions on arithmetic surfaces 

## Christophe Soulé

Let $S$ be a smooth projective curve over the complex numbers and $X \rightarrow S$ a semi-stable projective family of curves. Assume that both $S$ and the generic fiber of $X$ over $S$ have genus at least two. Then the sheaf of absolute differentials $\Omega_{X}^{1}$ defines a vector bundle on $X$ which is semi-stable in the sense of Mumford-Takemoto with respect to the canonical line bundle on $X$. The Bogomolov inequality

$$
c_{1}^{2}\left(\Omega_{X}^{1}\right) \leq 4 c_{2}\left(\Omega_{X}^{1}\right)
$$

leads to an upper bound for the self-intersection $c_{1}\left(\omega_{X / S}\right)^{2}$ of the relative dualizing sheaf $\omega_{X / S}$.

Assume now that $S$ is the spectrum $\operatorname{Spec}\left(\mathcal{O}_{F}\right)$ of the ring of integers in a number field $F$ and that $X \rightarrow S$ is a semi-stable (regular) curve over $S$, with generic genus at least two. In [7], Parshin asked for a similar upper bound for the arithmetic self-intersection $\hat{c}_{1}\left(\bar{\omega}_{X / S}\right)^{2}$ of the relative dualizing sheaf of $X$ over $S$, equipped with its Arakelov metric. He and Moret-Bailly [5] proved that a good upper bound for this real number $\hat{c}_{1}\left(\bar{\omega}_{X / S}\right)^{2}$ would have beautiful arithmetic consequences (including the $a b c$ conjecture).

If one tries to mimick in the arithmetic case the proof that we have just checked in the geometric case, one soon faces the difficulty that we do not know any arithmetic analog for the sheaf of absolute differentials $\Omega_{X}^{1}$. In [3], Miyaoka proposed to turn this difficulty as follows. He noticed that, in the geometric case, any general enough rank two extension $E$ of $\omega_{X / S}$ by the pull-back to $X$ of $\Omega_{S}^{1}$ is semi-stable and that it can be used instead of $\Omega_{X}^{1}$ in the argument. When $S=\operatorname{Spec}\left(\mathcal{O}_{F}\right)$ it is then natural to apply an arithmetic analog of Bogomolov inequality to a rank two extension $\bar{E}$ of $\bar{\omega}_{X / S}$ by some hermitian line bundle pulled back from $S$.

But then, a new difficulty arises. Namely, the second Chern number $\hat{c}_{2}(\bar{E})$ of $\bar{E}$ is more involved in the arithmetic case than in the geometric

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one, as it contains an archimedean summand - an integral over the set of complex points of $X$ - which is not easy to bound from above.

In this paper, which is a sequel to [8] and [9], although we are unable to prove Parshin's conjecture by Miyaoka's argument, we show that his method still provides interesting lower bounds form some successive minima of the euclidean lattice of sections of hermitian line bundles on the arithmetic surface $X$.

More precisely, we consider an hermitian line bundle $\bar{N}$ on $X$, with positive even degree on the generic fiber. We prove that, when $k$ is big enough, the logarithm $\mu_{k}$ of the $k$-th successive minimum of $H^{1}\left(X, N^{-1}\right)$, endowed with its $L^{2}$-metric, is bounded below:

$$
\begin{equation*}
\mu_{k} \geq \frac{\hat{c}_{1}(\bar{N})^{2}}{2 n d}-A \tag{*}
\end{equation*}
$$

where $n$ is the degree of $N, d=[F: \mathbb{Q}]$ and $A$ is a simple constant (Theorem 2).

This result is a complement to Theorem 4 in [9], where smaller values of $k$ were studied. The proof is similar and consists mainly in making precise Miyaoka's assertion that a general extension $E$ of $N$ by the trivial line bundle is semi-stable on $X \otimes \bar{F}$. For that, again inspired by Miyaoka, we write $E$ as an extension

$$
0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0
$$

over $X \otimes \bar{F}$, with $L=\omega_{X / S} \otimes M^{-1}$ being the Serre dual of $M$, and we show (Proposition 1) that $E$ is semi-stable on $X \otimes \bar{F}$ as soon as the boundary map

$$
\partial: H^{0}\left(X_{\bar{F}}, M\right) \rightarrow H^{1}\left(X_{\bar{F}}, L\right)
$$

is an isomorphism. Next, we give an upper bound for the dimension of a vector space $V \subset \operatorname{Ext}(L, M)$ such that, for every extension class in $V$, the corresponding map $\partial$ is singular (Proposition 2). By a standard argument it follows that, if $k$ is big enough, there exists an extension $E$ of $N$ by $\mathcal{O}_{X}$ which is semi-stable over $X \otimes \bar{F}$ and such that the $L^{2}$-norm of its extension class is bounded above by $\exp \left(\mu_{k}+A\right)$. The proof of $(*)$ (see Theorem 2) then follows from a theorem "à la Bogomolov" for semi-stable hermitian vector bundles on arithmetic surfaces, which is due to Miyaoka [4], [8] and Moriwaki [6].

The geometric aspect of our argument can also be expressed in terms of the secant variety $\Sigma_{d}$ of a smooth projective curve $C$. In [10], Voisin gave an upper bound for the dimension of projective spaces contained in
$\Sigma_{d}$, when $d$ is small enough with respect to the degree of $C$. In Theorem 1, we prove a similar result for a slightly bigger value of $d$.

When doing this work, I got help from C. Gasbarri, B. Mazur, Y. Miyaoka and especially C. Voisin, who found a gap in the proof of Proposition 2 and fixed it. I wish to express to them my gratitude, as well as to the organizers of this conference.

Notation. Given two line bundles on a scheme $X$, we denote by $L^{-1}$ the dual of $L$ and by $L M$ the tensor product of $L$ with $M$.

## §1. Semi-stable extensions on curves

## 1.1.

Let $k=\bar{k}$ be an algebraically closed field, and $C$ a smooth connected projective curve of genus $g \geq 0$ over $k$. Let $L$ and $M$ be two line bundles on $X$ and

$$
\begin{equation*}
0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0 \tag{1}
\end{equation*}
$$

a rank two extension of $M$ by $L$. Consider the associated boundary map in cohomology

$$
\partial: H^{0}(C, M) \rightarrow H^{1}(C, L)
$$

Proposition 1. Assume that $\operatorname{deg}(L) \leq \operatorname{deg}(M)$ and that
a) Either $\operatorname{deg}(L)+\operatorname{deg}(M) \geq 2 g-2$ and $\partial$ is injective;
b) $\operatorname{Or} \operatorname{deg}(L)+\operatorname{deg}(M) \leq 2 g-2$ and $\partial$ is surjective.

Then the vector bundle $E$ is semi-stable on $C$.

### 1.2. Proof.

Let us prove a) by contradiction. Let $N \subset E$ be a line bundle on $C$ such that

$$
\operatorname{deg}(N)>\frac{\operatorname{deg}(E)}{2}=\frac{\operatorname{deg}(L)+\operatorname{deg}(M)}{2} .
$$

Then $\operatorname{deg}(N)>\operatorname{deg}(L)$, therefore $N \cap L=0$, and the composite map

$$
N \rightarrow E \rightarrow M
$$

is injective. The extension

$$
0 \rightarrow L \rightarrow E^{\prime} \rightarrow N \rightarrow 0
$$

induced by (1) and this map is split. Therefore the associated boundary map

$$
H^{0}(C, N) \rightarrow H^{1}(C, L)
$$

is zero, i.e. the restriction of $\partial$ to $H^{0}(C, N) \subset H^{0}(C, M)$ vanishes.
On the other hand, since $\operatorname{deg}(L)+\operatorname{deg}(M) \geq 2 g-2$, we have

$$
\operatorname{deg}(N)>g-1
$$

hence, by Riemann-Roch, $H^{0}(C, N) \neq 0$. This contradicts the assumption that $\partial$ is injective.

To prove b) by contradiction we may consider a quotient $N$ of $E$ of degree less than $\operatorname{deg}(E) / 2$ and look at the extension

$$
0 \rightarrow N \rightarrow E^{\prime} \rightarrow M \rightarrow 0
$$

induced by the composite map $L \rightarrow E^{\prime} \rightarrow N$. Alternatively, one can deduce b) from a) by considering the Serre dual of $E$.

### 1.3. Remark.

There are cases where $E$ is semi-stable when neither a) nor b) holds.
1.3.1. For instance, when $C$ is an elliptic curve and $A \in C(k)$, let $L=\mathcal{O}(-A)$ and $M=\mathcal{O}(A)$. The group of extensions

$$
\operatorname{Ext}(M, L)=H^{1}(C, \mathcal{O}(2 A))
$$

has dimension two, when $H^{0}(C, M)$ and $H^{1}(C, L)$ have dimension one. Therefore there exists a nontrivial extension

$$
0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0
$$

such that $\partial$ vanishes. On the other hand, if $N \subset E$ has degree $\operatorname{deg}(N)>$ $\frac{\operatorname{deg}(E)}{2}=0$, it must be contained in $M$. Since $\operatorname{deg}(M)=1$ we get $N=M$ and the extension has to be trivial.
1.3.2. Another example, where $L=\mathcal{O}_{C}$ is the trivial line bundle and $M$ is the sheaf $\omega=\Omega_{C}^{1}$ of differentials on $C$, was proposed by J. Harris (I thank B. Mazur for explaining this to me). Choose a sextic $C^{\prime} \subset \mathbb{P}^{2}$ with exactly two nodal singularities, and let $C$ be the normalization of $C^{\prime}$. On this curve $C$ of genus 8 let $N$ be the pull-back of $\mathcal{O}(1)$ from $\mathbb{P}^{2}$ to $C$. One can show that there exists an extension

$$
0 \rightarrow N \rightarrow E \rightarrow \omega N^{-1} \rightarrow 0
$$

such that the boundary map

$$
H^{0}\left(C, \omega N^{-1}\right) \rightarrow H^{1}(C, N)
$$

has rank three, when $H^{1}(C, N)$ has dimension four. Furthermore $E$ is stable and has a nowhere vanishing section. Therefore $E$ is an extension

$$
0 \rightarrow \mathcal{O}_{C} \rightarrow E \rightarrow \omega \rightarrow 0
$$

with associated boundary map

$$
\partial: H^{0}(C, \omega) \rightarrow H^{1}\left(C, \mathcal{O}_{C}\right)
$$

which is neither injective nor surjective.

## §2. Projective subspaces in secant varieties

## 2.1.

Let $k$ be a field of characteristic zero, $C$ a smooth projective curve over $k$, and $C_{\bar{k}}=C \underset{k}{\otimes} \bar{k}$ its extension of scalars to the algebraic closure of $k$. We assume that $C_{\bar{k}}$ is irreducible of genus $g \geq 0$.

Consider a line bundle $N$ on $C$. Each cohomology class

$$
e \in H^{1}\left(C, N^{-1}\right)=\operatorname{Ext}\left(N, \mathcal{O}_{C}\right)
$$

classifies an extension

$$
0 \rightarrow \mathcal{O}_{C} \rightarrow E \rightarrow N \rightarrow 0
$$

Let $n=\operatorname{deg}(N)$ be the degree of $N$.

Proposition 2. Assume that the degree $n$ is even and nonnegative, and that $N$ is not trivial. Let $V \subset H^{1}\left(C, N^{-1}\right)$ be a $k$-vector space of dimension

$$
\operatorname{dim}(V) \geq n-m+g
$$

where $m$ is the integer defined by formula (3) below.
Then there exists $e \in V$ such that the corresponding vector bundle $E$ is semi-stable (over $C_{\bar{k}}$ ).

### 2.2. Proof.

Since $n$ is even, we can choose a line bundle $H$ on $C_{\bar{k}}$ such that, if $\omega=\Omega_{C}^{1}$,

$$
\operatorname{deg}(N \omega)=2 \operatorname{deg}(H)
$$

If $H^{\prime}=N \omega H^{-1}$, we get $\operatorname{deg}\left(H^{\prime}\right)=\operatorname{deg}(H)$, and

$$
N \omega=H H^{\prime}
$$

Since $\operatorname{Pic}^{0}\left(C_{\bar{k}}\right)$ is divisible, there exists a line bundle $A$ of degree zero on $C_{\bar{k}}$ such that

$$
H^{\prime}=H A^{2}
$$

Let $M=H A$ and $L=\omega M^{-1}$. We get

$$
N=H H^{\prime} \omega^{-1}=H^{2} A^{2} \omega^{-1}=M\left(\omega M^{-1}\right)^{-1}=M L^{-1}
$$

Any class $e \in H^{1}\left(C, N^{-1}\right)$ defines an extension

$$
0 \rightarrow \mathcal{O}_{C} \rightarrow E \rightarrow N \rightarrow 0
$$

over $C$ and, by tensoring by $L$, an extension

$$
\begin{equation*}
0 \rightarrow L \rightarrow E \otimes L \rightarrow M \rightarrow 0 \tag{2}
\end{equation*}
$$

over $C_{\bar{k}}$. The vector bundle $E$ is semi-stable if and only if $E \otimes L$ is semi-stable.

From now on, and till the end of $\S 2$, we assume that $k=\bar{k}$. Since $\operatorname{deg}(N) \geq 0$ we have $\operatorname{deg}(L) \leq \operatorname{deg}(M)$. Furthermore

$$
\operatorname{deg}(L)+\operatorname{deg}(M)=2 g-2
$$

Therefore, by Proposition $1, E \otimes L$ is semi-stable if and only if the boundary map

$$
\partial_{e}: H^{0}(C, M) \rightarrow H^{1}(C, L)
$$

defined by (2) is an isomorphism. Note that, by Serre duality,

$$
H^{1}(C, L)=H^{0}\left(C, \omega L^{-1}\right)^{*}=H^{0}(C, M)^{*}
$$

has the same dimension as $H^{0}(C, M)$. Let

$$
\begin{equation*}
m=\operatorname{dim}_{k} H^{0}(C, M) \tag{3}
\end{equation*}
$$

To prove Proposition 2, we now follow an argument of C.Voisin. The map $\partial_{e}$ is the cup-product by $e \in H^{1}\left(C, L M^{-1}\right)$. Therefore, by Serre duality again, the map

$$
\begin{aligned}
H^{1}\left(C, N^{-1}\right) & =H^{0}(C, N \omega)^{*} \\
& =H^{0}\left(C, M^{2}\right)^{*} \rightarrow \operatorname{Hom}\left(H^{0}(C, M) \rightarrow H^{0}(C, M)^{*}\right)
\end{aligned}
$$

which maps $e$ to $\partial_{e}$ is dual to the cup-product

$$
H^{0}(C, M)^{\otimes 2} \rightarrow H^{0}\left(C, M^{2}\right)
$$

We denote by

$$
\mu: H^{0}(C, M)^{\otimes 2} \rightarrow V^{*}
$$

the composite of this cup-product with the projection of $H^{0}\left(C, M^{2}\right)$ onto the dual of $V$. Since the cup-product is commutative, any element in $V$ defines, via $\mu$, a quadric in the projective space $\mathbf{P}\left(H^{0}(C, M)\right)$.

Arguing by contradiction, we assume that all these quadrics are singular. Consider the Zariski closure $B \subset \mathbf{P}\left(H^{0}(C, M)\right)$ of the union of the singular loci of the quadrics with singular locus of minimal dimension, and let $b$ be the dimension of $B$.

Let $\sigma \in H^{0}(C, M)$ be a representative of a generic point $[\sigma] \in B$. We claim that the map

$$
\mu_{\sigma}: H^{0}(C, M) \rightarrow V^{*}
$$

mapping $\tau \in H^{0}(C, M)$ to

$$
\mu_{\sigma}(\tau)=\mu(\sigma \otimes \tau)
$$

has rank at most $b$. Indeed, it follows from the definitions that a quadric $q \in V$ is singular at $\tau \in H^{0}(C, M)$ if and only if it lies in the subspace $Q_{\tau} \subset V$ orthogonal to the image of $\mu_{\tau}$. Generically, the singular locus of $q$ is minimal. Therefore the union all the vector spaces $Q_{\tau},[\tau] \in B$, is an open subset of $V$. Since $[\sigma]$ is generic in $B$, the dimension of $Q_{\sigma}$ is at least $\operatorname{dim}(\mathrm{V})-\mathrm{b}$, and the rank of $\mu_{\sigma}$ is at most $b$ as claimed.

This implies that the kernel $H_{\sigma} \subset H^{0}(C, M)$ of $\mu_{\sigma}$ has dimension $c \geq m-b$ (note that this dimension $c$ has a fixed value when $[\sigma]$ is generic in $B)$. Let $K \subset H^{0}\left(C, M^{2}\right)$ be the subspace orthogonal to $V$. By definition, the vector space

$$
K_{\sigma}=\sigma \cup H_{\sigma}
$$

is contained in $K$. Its dimension is $c$.

On the other hand, we can choose points $x_{0}, \ldots, x_{b}$ on $C$ and vectors $\sigma_{0}, \ldots, \sigma_{b} \in H^{0}(C, M)$ such that $\left[\sigma_{i}\right]$ lies in $B$ and

$$
\sigma_{i}\left(x_{j}\right)=\delta_{i j}
$$

for all $i$ and $j$. By moving $x_{i}$ without moving the other points, we can also assume that, for every $i$, at least one section in $H_{\sigma_{i}}$ does not vanish at $x_{i}$. As a consequence, $K_{\sigma_{i}}$ is not contain in the sum of the $K_{\sigma_{j}}$ 's, $j \neq i$, and the dimension of the sum of the subspaces $K_{\sigma_{i}}, i=0, \ldots, b$, is at least

$$
b+c \geq m
$$

Therefore $K$ has dimension at least $m$ and, since $H^{1}\left(C, N^{-1}\right)$ has dimension $n+g-1$, the dimension of $V$ is at most $n+g-m-1$. This contradicts one of our hypotheses.

### 2.3. Remark.

In the proof of Proposition 2, $N=M^{2} \omega^{-1}$, therefore

$$
\operatorname{deg}(M)=\frac{n}{2}+g-1 \geq g-1
$$

By the Riemann-Roch theorem:

$$
\chi(C, M)=\operatorname{deg}(M)-g+1=\frac{n}{2} .
$$

By Clifford's theorem

$$
\operatorname{dim}_{k} H^{1}(C, M) \leq \operatorname{Sup}(g-1,0)
$$

and $\operatorname{dim}_{k} H^{1}(C, M)=0$ whenever $\operatorname{deg}(M)>2 g-2$. Therefore

$$
\frac{n}{2} \leq m \leq \frac{n}{2}+\operatorname{Sup}(g-1,0)
$$

hence

$$
\frac{n}{2}+\operatorname{Inf}(g, 1) \leq n-m+g \leq \frac{n}{2}+g
$$

and $n-m+g=\frac{n}{2}+g$ as soon as $n>2 g-2$.

### 2.4. Secant varieties

2.4.1. The Proposition 2 can be rephrased in terms of secant varieties. Let $d \geq 1$ be an integer and

$$
n=2 d+2
$$

Let $k$ be an algebraically closed field of characteristic zero and $C$ a smooth connected projective curve over $k$, of genus $g$, say $C \subset \mathbb{P}$. Let
$N^{-1}=\omega \mathcal{O}(-1)$ be the Serre dual of the canonical sheaf on $C$, and assume that $\operatorname{deg}(N)=n$.

Consider the secant variety

$$
\Sigma_{d}=\bigcup_{Z \in X^{(d)}}\langle Z\rangle
$$

swept out by the linear spans of $d$-uples of points on $C$. Define $m$ as in (3).

Theorem 1. The secant variety $\Sigma_{d}$ does not contain any projective space $\mathbb{P}^{\delta}$ of dimension

$$
\delta \geq n-m+g-1
$$

2.4.2. Proof. Let $e \in H^{1}\left(C, N^{-1}\right), e \neq 0$, and

$$
0 \rightarrow \mathcal{O}_{C} \rightarrow E \rightarrow N \rightarrow 0
$$

the corresponding extension. the semi-stability of $E$ means that $e$ does not lie in the image of the boundary map

$$
\partial_{D}: H^{0}\left(D, N^{-1}(D)\right) \rightarrow H^{1}\left(C, N^{-1}\right)
$$

coming from

$$
0 \rightarrow N^{-1} \rightarrow N^{-1}(D) \rightarrow N^{-1}(D) / D \rightarrow 0
$$

for any effective divisor $D$ of degree less than $\frac{n}{2}$, i.e. $\operatorname{deg}(D) \leq d$. This condition happens to be equivalent to the fact that the point in $\mathbb{P}=\mathbb{P}\left(H^{1}\left(C, N^{-1}\right)\right)$ defined by $e$ does not belong to $\Sigma_{d}$. For more details see [2], p. 451, or [9], § 1.6. Therefore Theorem 1 follows from Proposition 2.
2.4.3. Using 2.3 we see that the lower bound

$$
\delta_{0}=n-m+g-1
$$

in Theorem 1 is such that

$$
\delta_{0} \geq \frac{n}{2}+\operatorname{Inf}(g-1,0)=d+\operatorname{Inf}(g, 1)
$$

and $\delta_{0}=d+g$ when $n>2 g-2$. The remark 1.3 above suggests that this bound is not optimal. According to C. Voisin, when $g>0$, Theorem 1 should remain true with $\delta \geq d([9], \S 1.3)$.

## §3. Semi-stable extensions on arithmetic surfaces

## 3.1.

Let $F$ be a number field, $\mathcal{O}_{F}$ its ring of integers and $S=\operatorname{Spec}\left(\mathcal{O}_{F}\right)$. Consider a semi-stable curve $X$ over $S$ such that $X$ is regular and its generic fiber $X_{F}$ is geometrically irreducible of genus $g \geq 0$. Let

$$
\operatorname{deg}: \operatorname{Pic}(X) \rightarrow \mathbb{Z}
$$

be the morphism sending the class of a line bundle on $X$ to the degree of its restriction to $X_{F}$.

Let $\bar{N}=(N, h)$ be an hermitian line bundle over $X$, i.e. a line bundle $N$ on $X$ together with an hermitian metric $h$ on the restriction $N_{\mathbb{C}}$ of $N$ to $X(\mathbb{C})$ which is invariant under complex conjugation. The cohomology group

$$
\Lambda=H^{1}\left(X, N^{-1}\right)
$$

is a finitely generated module over $\mathcal{O}_{F}$. For every complex embedding $\sigma: F \rightarrow \mathbb{C}$, let $X_{\sigma}=X \otimes \mathbb{C}$ be the corresponding surface and $\Lambda_{\sigma}=\Lambda \otimes \mathbb{C}$. This cohomology group

$$
\Lambda_{\sigma}=H^{1}\left(X_{\sigma}, N_{\mathbb{C}}^{-1}\right)
$$

is canonically isomorphic to the complex vector space $\mathcal{H}^{0,1}\left(X_{\sigma}, N_{\mathbb{C}}^{-1}\right)$ of harmonic differential forms of type $(0,1)$ with coefficients in the restriction $N_{\mathbb{C}}^{-1}$ of the line bundle $N^{-1}$ to $X(\mathbb{C})=\coprod_{\sigma} X_{\sigma}$. Given $\alpha \in$ $\mathcal{H}^{0,1}\left(X_{\sigma}, N_{\mathbb{C}}^{-1}\right)$, we let $\alpha^{*}$ be its transposed conjugate (the definition of which uses the metric $h$ ), and we define

$$
\|\alpha\|_{L^{2}}^{2}=\frac{i}{2 \pi} \int_{X_{\sigma}} \alpha^{*} \alpha
$$

Given $e \in \Lambda$, we let

$$
\|e\|=\operatorname{Sup}_{\sigma}\|\sigma(e)\|_{L^{2}}
$$

where $\sigma$ runs over all complex embeddings of $F$.
We are interested in (the logarithm of) the successive minima of $\Lambda$. Namely, for any positive integer $k \leq r k(\Lambda)$, we let $\mu_{k}$ be the infimum of all real numbers $\mu$ such that there exist $k$ elements $e_{1}, \ldots, e_{k}$ in $\Lambda$ which are linearly independent in

$$
\Lambda \otimes F=H^{1}\left(X_{F}, N^{-1}\right)
$$

and such that

$$
\begin{equation*}
\left\|e_{i}\right\| \leq \exp (\mu) \quad \text { for all } i=1, \ldots, k \tag{4}
\end{equation*}
$$

Let $n=\operatorname{deg}(N)$. We assume that $n>0$ and that $n$ is even. We define $m$ by the formula (3) above (with ground field $\bar{F}$ instead of $\bar{k}$ ). Finally, let

$$
d=[F: \mathbb{Q}]
$$

be the degree of $F$ over $\mathbb{Q}$.
Theorem 2. Assume that $k \geq n-m+g$. Then

$$
\mu_{k} \geq \frac{\hat{c}_{1}(\bar{N})^{2}}{2 n d}-A
$$

where

$$
A=\frac{1}{n}+\log (m(n+g-1))
$$

and $\hat{c}_{1}(\bar{N})^{2} \in \mathbb{R}$ denotes the self-intersection of the arithmetic Chern class $\hat{c}_{1}(\bar{N}) \in \widehat{\mathrm{CH}}^{1}(X)$.

### 3.2. Proof.

Let $e_{1}, \ldots, e_{k}$ be elements of $\Lambda$ which are $F$-linearly independent and such that (4) holds. Call $V \subset H^{1}\left(X_{F}, N^{-1}\right)$ the $F$-vector space spanned by $e_{1}, \ldots, e_{k}$. According to Proposition 2 there exists $e \in V$ such that the corresponding extension $E$ of $N$ by the trivial line bundle on $X_{F}$ is semi-stable on $X_{\bar{F}}$. Furthermore, using the notation of the proof of Proposition 2, $E$ is semi-stable as soon as

$$
\partial_{e}: H^{0}\left(X_{\bar{F}}, M\right) \rightarrow H^{1}\left(X_{\bar{F}}, L\right)
$$

is an isomorphism. Choosing a basis of these two vector spaces, we get a polynomial $P$ of degree $m$ on $V \otimes \bar{F}$ such that

$$
P(e)=\operatorname{det}\left(\partial_{e}\right),
$$

so that $E$ is semi-stable as soon as $P(e) \neq 0$. Therefore, by a standard argument (see [9], proof of Proposition 5), there exists $k$ integers $n_{1}, \ldots, n_{k}$, with $\left|n_{i}\right| \leq m$ for all $i$, such that

$$
\begin{equation*}
e=n_{1} e_{1}+\ldots+n_{k} e_{k} \tag{5}
\end{equation*}
$$

satisfies $P(e) \neq 0$, hence $E$ is semi-stable.

From the definition of $\mu_{k}$ and (5) we get

$$
\begin{equation*}
\|e\| \leq m k \exp \left(\mu_{k}\right) \leq m(n+g-1) \exp \left(\mu_{k}\right) \tag{6}
\end{equation*}
$$

(since $r k(\Lambda)=n+g-1$ ). According to a result of Miyaoka ([4], [8] Theorem 1) and Moriwaki [6], this implies that, for any choice of a metric on $E$ (invariant under complex conjugation), the inequality "à la Bogomolov"

$$
\begin{equation*}
\hat{c}_{1}(\bar{E})^{2} \leq 4 \hat{c}_{2}(\bar{E}) \tag{7}
\end{equation*}
$$

is satisfied in $\mathbb{R}$. Here, as in $[8] \S 2.1$, given $x \in \widehat{\mathrm{CH}}^{2}(X)$ we also denote by $x \in \mathbb{R}$ its arithmetic degree $\widehat{\operatorname{deg}}(x)$.

We now proceed in a way similar to [8], Proposition 1 and Corollary (where more details can be found). Recall that $E$ is an extension

$$
\begin{equation*}
0 \rightarrow \mathcal{O}_{X} \rightarrow E \rightarrow N \rightarrow 0 \tag{8}
\end{equation*}
$$

We endow $\mathcal{O}_{X}$ with the trivial metric and $N$ with a metric $h^{\prime}$ to be specified below. For any choice of a smooth splitting of (8) over $X(\mathbb{C})$, we get a metric on $E$, namely the orthogonal direct sum of the chosen metrics on $\mathcal{O}_{X}$ and $N$. The Cauchy-Riemann operator on $E_{\mathbb{C}}$ can be written in matrix form according to that splitting:

$$
\bar{\partial}_{E}=\left(\begin{array}{cc}
\bar{\partial} & \alpha \\
0 & \bar{\partial}_{N}
\end{array}\right)
$$

where $\alpha$ is a smooth form of type $(0,1)$ over $X(\mathbb{C})$ with coefficients in $N_{\mathbb{C}}^{-1}$. One can choose the smooth splitting of (8) over $X(\mathbb{C})$ in such a way that $\alpha$ is the harmonic representative of the restriction of $e$ to $X(\mathbb{C})$. With this choice we get

$$
\hat{c}_{1}(\bar{E})=\hat{c}_{1}\left(N, h^{\prime}\right)
$$

and

$$
2 \hat{c}_{2}(\bar{E})=\sum_{\sigma: F \rightarrow \mathbb{C}}\|\sigma(e)\|_{L^{2}}^{2}
$$

where, for every complex embedding $\sigma,\|\cdot\|_{L^{2}}^{\prime}$ is the $L^{2}$-norm on $H^{1}\left(X_{\sigma}\right.$, $N_{\mathbb{C}}^{-1}$ ) defined by $h^{\prime}$.

Now let $t=\|e\|^{2}$ and let us choose $h^{\prime}=t h$. We get

$$
\hat{c}_{1}\left(N, h^{\prime}\right)^{2}=\hat{c}_{1}(\bar{N})^{2}-n d \log (t)
$$

and

$$
\sum_{\sigma}\|\sigma(e)\|_{L^{2}}^{2}=t^{-1} \sum_{\sigma}\|\sigma(e)\|_{L^{2}}^{2} \leq d
$$

Therefore the inequality (7) reads

$$
\hat{c}_{1}(\bar{N})^{2} \leq 2 n d \log \|e\|+2 d
$$

Since, by (6),

$$
\log \|e\| \leq \mu_{k}+\log (m(n+g-1))
$$

Theorem 2 follows.

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# On the cusp form motives in genus 1 and level 1 

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#### Abstract

. We prove that the moduli space of stable $n$-pointed curves of genus 1 and the projector associated to the alternating representation of the symmetric group on $n$ letters define (for $n>1$ ) the Chow motive corresponding to cusp forms of weight $n+1$ for $\operatorname{SL}(2, \mathbb{Z})$. This provides an alternative (in level 1) to the construction of Scholl.


## §1. Introduction

In this paper we give an alternative construction of the Chow motives $S[k]$ corresponding to cusp forms of weight $k$ for $\operatorname{SL}(2, \mathbb{Z})$. The Betti cohomology related to these cusp forms was initially studied by Eichler and Shimura, after which Deligne constructed the corresponding $\ell$-adic Galois representations. Using the canonical desingularization of the fiber products of the compactified universal elliptic curve constructed by Deligne, Scholl then defined projectors such that the realizations of the associated Chow motives are these parabolic cohomology groups. The smooth projective varieties used in this construction are called Kuga-Sato varieties.

Instead of the Kuga-Sato varieties, we use the spaces $\bar{M}_{1, n}$, the Knudsen-Deligne-Mumford moduli spaces of stable $n$-pointed curves of genus 1. The symmetric group $\Sigma_{n}$ acts naturally on $\bar{M}_{1, n}$, by permuting the $n$ marked points. Let $\alpha$ denote its alternating character. Our main result is that $\bar{M}_{1, n}($ for $n>1)$ and the projector $\Pi_{\alpha}$ corresponding to $\alpha$ define the Chow motive $S[n+1]$. In other words, we have the following result.

[^3]Theorem. For $n>1$,

$$
\Pi_{\alpha}\left(H^{*}\left(\bar{M}_{1, n}, \mathbb{Q}\right)\right)=\Pi_{\alpha}\left(H^{n}\left(\bar{M}_{1, n}, \mathbb{Q}\right)\right)=H_{!}^{1}\left(M_{1,1}, \operatorname{Sym}^{n-1} R^{1} \pi_{*} \mathbb{Q}\right)
$$

Here $\pi: \mathcal{E} \rightarrow M_{1,1}$ is the universal elliptic curve and $H_{!}^{i}=\operatorname{Im}\left(H_{c}^{i} \rightarrow H^{i}\right)$ denotes the parabolic cohomology.

The cohomology $H^{*}\left(\bar{M}_{g, n}\right)$ of the moduli space of stable $n$-pointed curves of genus $g$ has been studied intensively in recent years, in particular for $n>0$ through the connection with Gromov-Witten theory. Since $\bar{M}_{g, n}$ is a smooth projective stack over $\mathbb{Z}$, these groups have arithmetic relevance as well. Getzler has initiated the study of the cohomology $H^{*}\left(M_{g, n}\right)$ of the moduli space of smooth $n$-pointed curves of genus $g$ as a representation of $\Sigma_{n}$. Through the theory of modular operads, as developed by Getzler and Kapranov, the $\Sigma_{n}$-equivariant Euler characteristics of the cohomology of the spaces $\bar{M}_{g, n}$ are expressed in the $\Sigma_{n^{-}}$ equivariant Euler characteristics of the cohomology of the spaces $M_{g, n}$. The action of $\Sigma_{n}$ is crucial here. Another central idea of Getzler is to express the $\Sigma_{n}$-equivariant Euler characteristic of $H^{*}\left(M_{g, n}\right)$ in terms of the Euler characteristics of the cohomology of irreducible symplectic local systems on $M_{g}$. Since these local systems are pulled back from the moduli space $A_{g}$ of principally polarized abelian varieties of dimension $g$, this provides a connection with genus $g$ Siegel modular forms.

In genus 1, this connection is given by Eichler-Shimura theory. In higher genus, despite the very important work of Faltings and Chai, much less is known. Van der Geer and the second author have obtained an explicit conjectural formula for the motivic Euler characteristics of these local systems in genus 2.

Our work is motivated by the desire to understand the motives underlying Siegel modular forms and the cohomology of the corresponding local systems. We expect that the results proved in this paper for genus 1 , when suitably generalized, will provide a major step towards this goal.

Unbeknownst to us, Manin had suggested in [Ma1], 0.2 and 2.5, that it would be desirable to replace the Kuga-Sato varieties by moduli spaces of curves of genus 1 with marked points and a level structure. Cf. [Ma2], 3.6.2.

In section 2 , we determine the alternating part of the $\Sigma_{n}$-equivariant Euler characteristic of $M_{1, n}$. Section 3 deals with the $\Sigma_{n}$-equivariant cohomology of $M_{0, n}$; some of the results obtained here may be of independent interest. The theory of modular operads and the results of section 3 are used in section 4 to determine the alternating part of the Euler characteristic of $\bar{M}_{1, n} \backslash M_{1, n}$. In section 5 we combine the results of sections 2 and 4 and prove our main theorem.

## §2. The contribution of the interior

In this section we determine the contribution of $M_{1, n}$, i.e., we determine

$$
\left\langle s_{1^{n}}, e_{c}^{\Sigma_{n}}\left(M_{1, n}\right)\right\rangle,
$$

the alternating part of the $\Sigma_{n}$-equivariant Euler characteristic of the compactly supported cohomology of $M_{1, n}$. Here, for a partition $\lambda$ of $n$, the notation $s_{\lambda}$ is used for the Schur function corresponding to the irreducible representation of $\Sigma_{n}$ indexed by $\lambda$, and $\langle$,$\rangle stands for the$ standard inner product on the ring of symmetric functions, for which the $s_{\lambda}$ form an orthonormal basis. We will usually not make a notational distinction between a (possibly virtual) $\Sigma_{n}$-representation $V$ and its characteristic $\operatorname{ch}_{n}(V)$, the symmetric function corresponding to it ([GK], 7.1). The Euler characteristic is taken in $K_{0}$ of a convenient category, such as the category of mixed Hodge structures or of $\ell$-adic Galois representations.

Let $E=(E, 0)$ be an elliptic curve. We may think of the points of $E^{n-1}$ as $n$-tuples

$$
\left(0, x_{2}, \ldots, x_{n}\right)
$$

(with $x_{1}=0$ ) and by doing so we find a natural action of $\Sigma_{n}$ on $E^{n-1}$ (combine the effect of a permutation $\sigma$ with a translation of each coordinate over $\left.-x_{\sigma^{-1}(1)}\right)$. We are interested in the subspace of $H^{\bullet}\left(E^{n-1}\right)=H^{\bullet}(E)^{\otimes(n-1)}$ where the induced action of $\Sigma_{n}$ is via the alternating representation.

Let $\Sigma_{n-1} \subset \Sigma_{n}$ be the subgroup permuting the last $n-1$ entries.
Lemma 1. The subspace of $H^{\bullet}(E)^{\otimes(n-1)}$ where the induced action of $\Sigma_{n-1}$ is via the alternating representation is isomorphic to

$$
\oplus_{k=0}^{n-1} \wedge^{k} H^{\mathrm{even}}(E) \otimes \operatorname{Sym}^{n-1-k} H^{1}(E)
$$

Proof. The subspace of $V=H^{\bullet}(E)^{\otimes(n-1)}$ where $\Sigma_{n-1}$ acts alternatingly is generated by sums

$$
\sum_{\sigma \in \Sigma_{n-1}}(-1)^{\operatorname{sgn}(\sigma)} \sigma^{*}(v)
$$

with $v \in V$. Clearly, we may restrict ourselves to pure tensors $v$ such that the first $k$ factors are in $H^{\text {even }}(E)$ and the remaining $n-1-k$ factors are in $H^{1}(E)$, for some $k$. Fix $k$. It suffices now to consider the action of $\Sigma_{k} \times \Sigma_{n-1-k}$ on such $v$. This leads to the claimed isomorphism. Q.E.D.

Note that only the terms with $k \leq 2$ in the direct sum above are nonzero. Thus it is concentrated in degrees $n-2, n-1$, and $n$.

Proposition 1. The subspace of $H^{\bullet}(E)^{\otimes(n-1)}$ where the induced action of $\Sigma_{n}$ is via the alternating representation is $\operatorname{Sym}^{n-1} H^{1}(E)$.

Proof. Let $\tau \in \Sigma_{n}$ be the transposition (12). We need to show that $\tau^{*}(\gamma)=-\gamma$ for all $\gamma \in \operatorname{Sym}^{n-1} H^{1}(E)$, but that none of the $\Sigma_{n-1}$ alternating vectors coming from elements of $H^{\text {even }}(E) \otimes \operatorname{Sym}^{n-2} H^{1}(E)$ and $\wedge^{2} H^{\text {even }}(E) \otimes \operatorname{Sym}^{n-3} H^{1}(E)$ have this property.

As an example, consider the case $n=2$. Note that $\tau\left(0, x_{2}\right)=$ $\left(x_{2}, 0\right)=\left(0,-x_{2}\right)$. Thus $\tau=-1_{E}$ and the $(-1)$-eigenspace of $\tau^{*}$ on $H^{\bullet}(E)$ is $H^{1}(E)$.

In the general case,

$$
\tau\left(0, x_{2}, x_{3}, \ldots, x_{n}\right)=\left(0,-x_{2}, x_{3}-x_{2}, \ldots, x_{n}-x_{2}\right)
$$

Denote by $\mathrm{pr}_{i}: E^{n-1} \rightarrow E$ the projection onto the $i$ th factor (with $2 \leq i \leq n)$ and by $\tau_{i}$ the composition $\mathrm{pr}_{i} \circ \tau$. Then

$$
\tau^{*}\left(\gamma_{2} \otimes \cdots \otimes \gamma_{n}\right)=\tau_{2}^{*}\left(\gamma_{2}\right) \cdot \ldots \cdot \tau_{n}^{*}\left(\gamma_{n}\right)
$$

For $k \geq 3$ we have $\tau_{k}=m \circ\left(\left(-\mathrm{pr}_{2}\right) \times \mathrm{pr}_{k}\right)$, where $m: E \times E \rightarrow E$ denotes the group law. Observe now that

$$
\tau_{k}^{*}(\zeta)=-\operatorname{pr}_{2}^{*}(\zeta)+\operatorname{pr}_{k}^{*}(\zeta)
$$

for $\zeta \in H^{1}(E)$ and $k \geq 3$.
Denote by $p_{i}: E^{n-1} \rightarrow E^{n-2}$ the projection forgetting the $i$ th factor $(2 \leq i \leq n)$. Let $\gamma \in \operatorname{Sym}^{n-2} H^{1}(E)$ and denote by

$$
\Gamma=\sum_{i=2}^{n}(-1)^{i} p_{i}^{*} \gamma
$$

the $\Sigma_{n-1}$-alternating vector corresponding to $1 \otimes \gamma$. Let $I$ be the ideal $\operatorname{pr}_{2}^{*}\left(H^{1}(E) \oplus H^{2}(E)\right)$. Note that $\Gamma \equiv p_{2}^{*} \gamma=1 \otimes \gamma \bmod I$. But $\tau^{*}(1 \otimes$ $\gamma) \equiv 1 \otimes \gamma \bmod I$ by the above. Thus $\tau^{*} \Gamma=-\Gamma$ implies $\gamma=0$.

This shows that the $\Sigma_{n-1}$-alternating vectors corresponding to elements of $H^{0}(E) \otimes \operatorname{Sym}^{n-2} H^{1}(E)$ are not $\Sigma_{n}$-alternating. We conclude that the alternating representation of $\Sigma_{n}$ does not occur in degree $n-2$. By duality, it does not occur in degree $n$ either.

Denote by $p_{i j}: E^{n-1} \rightarrow E^{n-3}$ the projection forgetting the $i$ th and $j$ th factors $(2 \leq i<j \leq n)$. Let $\gamma \in \operatorname{Sym}^{n-3} H^{1}(E)$ and denote by

$$
\Xi=\sum_{i=2}^{n-1} \sum_{j=i+1}^{n}(-1)^{i+j} p_{i j}^{*} \gamma \cdot\left(\operatorname{pr}_{i}^{*} p-\operatorname{pr}_{j}^{*} p\right)
$$

the $\Sigma_{n-1}$-alternating vector corresponding to $(1 \wedge p) \otimes \gamma$ (here $p$ is the class of a point). Then

$$
(-1)^{n+1} p_{n *} \Xi=p_{n *}\left(\sum_{i=2}^{n-1}(-1)^{i} p_{i n}^{*} \gamma \cdot \operatorname{pr}_{n}^{*} p\right)=\sum_{i=2}^{n-1}(-1)^{i} p_{i}^{*} \gamma
$$

the $\Sigma_{n-2}$-alternating vector in $H^{\bullet}(E)^{\otimes(n-2)}$ corresponding to $1 \otimes \gamma$. Using that $p_{n} \circ \tau=\tau \circ p_{n}$, one shows that $\tau^{*} \Xi=-\Xi$ implies $\gamma=0$. Thus the alternating representation of $\Sigma_{n}$ can occur only in $\operatorname{Sym}^{n-1} H^{1}(E)$.

To conclude, we show that these vectors are indeed $\Sigma_{n}$-alternating. Choose $\alpha$ and $\beta$ in $H^{1}(E)$ with $\alpha \cdot \beta=p$. Fix $k$ and $l$ with sum $n-1$ and let $\gamma=\gamma_{2} \otimes \cdots \otimes \gamma_{n}$ with $k$ of the factors equal to $\alpha$ and the remaining $l$ equal to $\beta$. If $\gamma_{2}=\alpha$, then $\gamma+\tau^{*} \gamma$ is a sum of $l$ terms; each term arises from $\gamma$ by replacing $\gamma_{2}$ by $p$ and one of the $\beta$ 's by 1 . If $\gamma_{2}=\beta$, then $\gamma+\tau^{*} \gamma$ is a sum of $k$ terms; each term arises from $\gamma$ by replacing $\gamma_{2}$ by $-p$ and one of the $\alpha$ 's by 1 . It is now easy to see that the symmetric tensor $\Gamma$ that is the sum of all $\gamma$ satisfies $\tau^{*} \Gamma=-\Gamma$. This finishes the proof.
Q.E.D.

We may think of the fiber of $M_{1, n}$ over $[E]$ as the open subset $D_{n}^{\circ}$ of $E^{n-1}$ where the $n$ points $0, x_{2}, \ldots, x_{n}$ are mutually distinct, i.e., the complement of the $n-1$ zero sections $x_{i}=0$ (with $2 \leq i \leq n$ ) and the diagonals $x_{i}=x_{j}$ (with $2 \leq i<j \leq n$ ). Clearly this open subset is $\Sigma_{n}$-invariant.

Lemma 2. The subspace of $H_{c}^{\bullet}\left(D_{n}^{\circ}\right)$ where the induced action of $\Sigma_{n}$ is via the alternating representation is canonically isomorphic to the corresponding subspace of $H^{\bullet}\left(E^{n-1}\right)$, thus to $\operatorname{Sym}^{n-1} H^{1}(E)$.

Proof. Write $D_{k}$ for the closed subset of $E^{n-1}$ where $\left\{0, x_{2}, \ldots, x_{n}\right\}$ has cardinality at most $k$ and $D_{k}^{\circ}=D_{k} \backslash D_{k-1}$ for its open subset where $\left\{0, x_{2}, \ldots, x_{n}\right\}$ has cardinality $k$. The subsets $D_{k}$ and $D_{k}^{\circ}$ are $\Sigma_{n^{-}}$ invariant. By induction on $k$ we show that $H_{c}^{\bullet}\left(D_{k}\right)$ does not contain a copy of the alternating representation for $k \leq n-1$. Note that $D_{1}$ is a point. We may assume $n>2$. We have exact sequences

$$
H_{c}^{i-1}\left(D_{k-1}\right) \rightarrow H_{c}^{i}\left(D_{k}^{\circ}\right) \rightarrow H_{c}^{i}\left(D_{k}\right) \rightarrow H_{c}^{i}\left(D_{k-1}\right)
$$

of $\Sigma_{n}$-representations. By induction, the outer terms do not contain alternating representations. Consider $D_{k}^{\circ}$ for $k \leq n-1$. For every connected component, there exists a transposition in $\Sigma_{n}$ acting on it as the identity. This shows that $H_{c}^{\bullet}\left(D_{k}^{\circ}\right)$ does not contain an alternating representation, and the same holds for $H_{c}^{\bullet}\left(D_{k}\right)$. The exact sequence above, with $k=n$, now gives the result.
Q.E.D.

For a variety $X$ with $\Sigma_{n^{\prime}}$-action, denote $\left\langle s_{1^{n}}, e_{c}^{\Sigma_{n}}(X)\right\rangle$ by $A_{c}(X)$. Clearly, we have

$$
A_{c}\left(E^{n-1}\right)=A_{c}\left(D_{n}^{\circ}\right)=(-1)^{n-1} \operatorname{Sym}^{n-1} H^{1}(E)
$$

Let $\pi: \mathcal{E} \rightarrow S$ be a relative elliptic curve. We may consider the $\Sigma_{n}{ }^{-}$ action on the relative spaces $\mathcal{E}^{n-1} / S$ and $\mathcal{D}_{n}^{\circ} / S$ and obtain

$$
A_{c}\left(\mathcal{E}^{n-1} / S\right)=A_{c}\left(\mathcal{D}_{n}^{\circ} / S\right)=(-1)^{n-1} \operatorname{Sym}^{n-1} R^{1} \pi_{*} \mathbb{Q}
$$

and similarly with $\mathbb{Q}_{\ell}$-coefficients. The Leray spectral sequence gives then immediately

$$
A_{c}\left(\mathcal{E}^{n-1}\right)=A_{c}\left(\mathcal{D}_{n}^{\circ}\right)=(-1)^{n-1} e_{c}\left(S, \operatorname{Sym}^{n-1} R^{1} \pi_{*} \mathbb{Q}\right) .
$$

Applying this to the universal elliptic curve, we obtain in particular

$$
A_{c}\left(M_{1, n}\right)=(-1)^{n-1} e_{c}\left(M_{1,1}, \operatorname{Sym}^{n-1} R^{1} \pi_{*} \mathbb{Q}\right)
$$

Let $n>1$. Then $H_{c}^{i}\left(M_{1,1}, \operatorname{Sym}^{n-1} R^{1} \pi_{*} \mathbb{Q}\right)=0$ when $i \neq 1$ or $n$ even. For $n$ odd,

$$
H_{c}^{1}\left(M_{1,1}, \operatorname{Sym}^{n-1} R^{1} \pi_{*} \mathbb{Q}\right)=S[n+1]+1,
$$

cf. [Ge4], Thm. 5.3 and below. Here we have written 1 for the trivial Hodge structure $\mathbb{Q}$ (or the corresponding $\ell$-adic Galois representation) and $S[n+1]$ for Getzler's $S_{n+1}$; this is an equality in the Grothendieck group of our category. We have proved the following result.

Theorem 1. The alternating part of the $\Sigma_{n}$-equivariant Euler characteristic of the compactly supported cohomology of $M_{1, n}$ is given by the following formula:

$$
A_{c}\left(M_{1, n}\right)=\left\{\begin{aligned}
-S[n+1]-1, & n>1 \text { odd } \\
0, & n \text { even }
\end{aligned}\right.
$$

Here $S[n+1]=H_{!}^{1}\left(M_{1,1}, \operatorname{Sym}^{n-1} R^{1} \pi_{*} \mathbb{Q}\right)$, the parabolic cohomology of the local system $\operatorname{Sym}^{n-1} R^{1} \pi_{*} \mathbb{Q}$, is the part of the cohomology of $M_{1, n}$ corresponding to cusp forms of weight $n+1$.

Of course $A_{c}\left(M_{1,1}\right)=L$, the Hodge structure $\mathbb{Q}(-1)$. If we formally define $S[2]=-L-1$, then the formula above holds for $n=1$ as well.

## §3. Cohomology of genus 0 moduli spaces and representations of symmetric groups

In this section we study the cohomology groups $H^{i}\left(M_{0, n}\right)$ as representations of the symmetric group $\Sigma_{n}$. One of our main tools is the following. Let $X$ be an algebraic variety, let $Y \subset X$ be a closed subvariety, and let $U=X \backslash Y$ denote the complement. Then the long exact sequence of compactly supported cohomology

$$
\cdots \rightarrow H_{c}^{k}(U) \rightarrow H_{c}^{k}(X) \rightarrow H_{c}^{k}(Y) \rightarrow H_{c}^{k+1}(U) \rightarrow \ldots
$$

is a sequence of mixed Hodge structures. See [DK], p. 282.
Lemma 3. (Getzler) The mixed Hodge structure on $H^{i}\left(M_{0, n}\right)$ is pure of weight $2 i$.

Proof. This is Lemma 3.12 in [Ge1]. We wish to give a different proof here. The case $n=3$ is trivial. For $n=4$, we use the sequence above, with $X=\mathbf{P}^{1}, Y=\{0,1, \infty\}$, and $U=M_{0,4}$. The sequence reads

$$
\begin{aligned}
& 0 \rightarrow H_{c}^{0}\left(M_{0,4}\right) \rightarrow H_{c}^{0}\left(\mathbf{P}^{1}\right) \rightarrow H_{c}^{0}(\{0,1, \infty\}) \rightarrow H_{c}^{1}\left(M_{0,4}\right) \rightarrow 0 \rightarrow 0 \rightarrow \\
& \rightarrow H_{c}^{2}\left(M_{0,4}\right) \rightarrow H_{c}^{2}\left(\mathbf{P}^{1}\right) \rightarrow 0 .
\end{aligned}
$$

Note first that $H_{c}^{0}\left(M_{0,4}\right)=0$. Clearly, $H_{c}^{1}\left(M_{0,4}\right)$ has weight 0 and $H_{c}^{2}\left(M_{0,4}\right)$ has weight 2 . The statement follows by duality:

$$
H_{c}^{k}(V)^{\vee} \cong H^{2 m-k}(V)(m)
$$

as mixed Hodge structures, for $V$ a nonsingular irreducible variety of dimension $m$.

For $n>4$ we have that $U=M_{0, n}$ is isomorphic to the complement in $X=M_{0, n-1} \times M_{0,4}$ of the disjoint union

$$
Y=\coprod_{i=4}^{n-1}\left\{x_{i}=x_{n}\right\}
$$

where we think of a $k$-pointed curve of genus 0 as given by a $k$-tuple $\left(0,1, \infty, x_{4}, \ldots, x_{k}\right)$ on $\mathbf{P}^{1}$. Thus,

$$
H_{c}^{k-1}(Y) \rightarrow H_{c}^{k}(U) \rightarrow H_{c}^{k}(X)
$$

is an exact sequence of mixed Hodge structures. By dualizing and applying a Tate twist, the same holds for

$$
H^{i}(X) \rightarrow H^{i}(U) \rightarrow H^{i-1}(Y)(-1)
$$

(with $i=2(n-3)-k$ ). By the Künneth formula and induction on $n$, the terms on the left and right have pure Hodge structures of weight $2 i$. Hence the same holds for the term in the middle.
Q.E.D.

For $k \geq 0$, denote by $\Delta_{k}$ the closed part of $\bar{M}_{0, n}$ corresponding to stable curves with at least $k$ nodes and denote by $\Delta_{k}^{\circ}$ the open part $\Delta_{k} \backslash \Delta_{k+1}$ corresponding to stable curves with exactly $k$ nodes. Put $d=n-3$. Clearly, $\Delta_{k} \neq \emptyset$ for $0 \leq k \leq d$. In general, $\Delta_{k}$ is singular, with nonsingular irreducible components, all of codimension $k$. But $\Delta_{0}=\bar{M}_{0, n}$ and $\Delta_{d}$ (a collection of points) are nonsingular. All $\Delta_{k}^{\circ}$ are nonsingular. Of course $\Delta_{0}^{\circ}=M_{0, n}$ and $\Delta_{d}^{\circ}=\Delta_{d}$. We have the long exact sequence

$$
\cdots \rightarrow H_{c}^{a-1}\left(\Delta_{k+1}\right) \rightarrow H_{c}^{a}\left(\Delta_{k}^{\circ}\right) \rightarrow H_{c}^{a}\left(\Delta_{k}\right) \rightarrow H_{c}^{a}\left(\Delta_{k+1}\right) \rightarrow \ldots
$$

of mixed Hodge structures. Since the $\Delta_{k}$ are invariant for the natural action of $\Sigma_{n}$, it is also a sequence of $\Sigma_{n}$-representations.

Lemma 4. The cohomology groups $H^{i}\left(M_{0, n}\right)$ vanish for $i>n-3$. For $0 \leq i \leq n-3$, the irreducible representations of $\Sigma_{n}$ occurring in $H^{i}\left(M_{0, n}\right)$ have Young diagrams with at most $i+1$ rows. In particular, the irreducible representations of $\Sigma_{n}$ occurring in $H^{\bullet}\left(M_{0, n}\right)$ have Young diagrams with at most $n-2$ rows.
Proof. The claimed vanishing is immediate. Let us abbreviate the rest of the statement by " $H^{i}\left(M_{0, n}\right)$ has $\leq i+1$ rows". We prove it by induction on $n$. The case $n=3$ is trivial. Assume $n>3$. Recall that $d=n-3$. We require an analysis of the boundary strata:
Claim. Assume $d-b>0$. Then $H_{c}^{a}\left(\Delta_{d-b}\right)$ has $\leq d+1+b-a$ rows.
We prove the claim by induction on $b$. We begin with the case $b=0$. Since $\Delta_{d}$ is a collection of points, $a=0$ may be assumed. Each point corresponds to a stable curve with $d$ nodes, hence with $d+1$ components. Each component has exactly three special points (nodes or marked points). Let $n_{j}$ be the number of marked points on the $j$ th component, for some numbering of the components. By permuting the $n$ marked points on the stable curve, we obtain a $\Sigma_{n}$-representation $R$, which is a direct summand of $H^{0}\left(\Delta_{d}\right)$. Note that $R$ is a subrepresentation of the induced representation

$$
\operatorname{Ind}_{\prod_{j=1}^{d+1} \Sigma_{n_{j}}}^{\Sigma_{n}} 1
$$

The induced representation has $\leq d+1$ rows, hence $R$ does. Now $H^{0}\left(\Delta_{d}\right)$ is a direct sum of representations analogous to $R$, thus it has $\leq d+1$ rows as well. This proves the claim in the case $b=0$.

Assume $b>0$. Observe that $H_{c}^{a}\left(\Delta_{k}^{\circ}\right) \cong H^{2(d-k)-a}\left(\Delta_{k}^{\circ}\right)$ as $\Sigma_{n^{-}}$ representations. Also, each connected component of $\Delta_{k}^{\circ}$ is for $k \geq 1$ a product of $k+1$ spaces $M_{0, m_{j}}$, with $m_{j}<n$. By induction on $n$ and the Künneth formula, $H_{c}^{a}\left(\Delta_{k}^{\circ}\right)$ has $\leq 2(d-k)-a+k+1=2 d-k-a+1$ rows, for $k \geq 1$. Putting $k=d-b$, we find that $H_{c}^{a}\left(\Delta_{d-b}^{\circ}\right)$ has $\leq d+1+b-a$ rows.

By induction on $b$, we have that $H_{c}^{a}\left(\Delta_{d-b+1}\right)$ has $\leq d+b-a$ rows. From the long exact sequence, we find that $H_{c}^{a}\left(\Delta_{d-b}\right)$ has $\leq d+1+b-a$ rows. This proves the claim.

In particular, $H_{c}^{a}\left(\Delta_{1}\right)$ has $\leq 2 d-a$ rows. Consider the exact sequence

$$
H_{c}^{k-1}\left(\Delta_{1}\right) \rightarrow H_{c}^{k}\left(M_{0, n}\right) \xrightarrow{\alpha} H_{c}^{k}\left(\bar{M}_{0, n}\right) .
$$

From Lemma 3 we know that $H_{c}^{k}\left(M_{0, n}\right)$ has weight $2 k-2 d$. But $H_{c}^{k}\left(\bar{M}_{0, n}\right)$ has weight $k$. Thus $\alpha=0$ for $k<2 d$. Hence $H_{c}^{k}\left(M_{0, n}\right)$ has $\leq 2 d+1-k$ rows for $k<2 d$. Thus $H^{i}\left(M_{0, n}\right)$ has $\leq i+1$ rows for $i>0$. But it is obviously true for $i=0$ as well. This finishes the proof.
Q.E.D.

## §4. The contribution of the boundary

In this section we determine the contribution of the boundary

$$
\partial M_{1, n}=\bar{M}_{1, n} \backslash M_{1, n}
$$

i.e., we determine

$$
\left\langle s_{1^{n}}, e_{c}^{\Sigma_{n}}\left(\partial M_{1, n}\right)\right\rangle .
$$

We use the main result of [Ge2]. To state it, we introduce the following notations:

$$
\begin{aligned}
& \mathbf{a}_{g}:=\sum_{n>2-2 g} \operatorname{ch}_{n}\left(e_{c}^{\Sigma_{n}}\left(M_{g, n}\right)\right), \quad \text { and } \\
& \mathbf{b}_{g}:=\sum_{n>2-2 g} \operatorname{ch}_{n}\left(e_{c}^{\Sigma_{n}}\left(\bar{M}_{g, n}\right)\right) .
\end{aligned}
$$

Here $\mathrm{ch}_{n}$ denotes the characteristic of a finite-dimensional $\Sigma_{n}$-representation ([GK], 7.1) and its extension by linearity to virtual representations. For a (formal) symmetric function $f$ (such as $\mathbf{a}_{g}$ and $\mathbf{b}_{g}$ ), we also
write

$$
\begin{aligned}
& f^{\prime}=\frac{\partial f}{\partial p_{1}}=p_{1}^{\perp} f \\
& \dot{f}=\frac{\partial f}{\partial p_{2}}=\frac{1}{2} p_{2}^{\perp} f \\
& \psi_{i}(f)=p_{i} \circ f
\end{aligned}
$$

Here $p_{i}$ is the symmetric function equal to the sum of the $i$ th powers of the variables, $p_{i}^{\perp}$ is the adjoint of multiplication with $p_{i}$ with respect to the standard inner product, and $\circ$ is the plethysm of symmetric functions ([GK], 7.2). We will denote the $i$ th complete symmetric function by $h_{i}$ and the $i$ th elementary symmetric function by $e_{i}$.

We can now state Getzler's result (Theorem 2.5 in [Ge2]):
$\mathbf{b}_{1}=\left(\mathbf{a}_{1}-\frac{1}{2} \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \left(1-\psi_{n}\left(\mathbf{a}_{0}^{\prime \prime}\right)\right)+\frac{\dot{\mathbf{a}}_{0}^{2}+\dot{\mathbf{a}}_{0}+\frac{1}{4} \psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}{1-\psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}\right) \circ\left(h_{1}+\mathbf{b}_{0}^{\prime}\right)$.
The numerator of the third term inside the big parentheses on the righthand side has been corrected here; there is a minor computational mistake in the derivation of the theorem in line 4 on page 487, which affects the result (but not Corollary 2.8).

As Getzler remarks, the term $\mathbf{a}_{1} \circ\left(h_{1}+\mathbf{b}_{0}^{\prime}\right)$ corresponds to the sum over graphs obtained by attaching a forest whose vertices have genus 0 to a vertex of genus 1 ; in particular, $\mathbf{a}_{1} \circ h_{1}=\mathbf{a}_{1}$, the contribution of smooth curves, is part of this term, corresponding to graphs consisting of a single vertex of genus 1 . The remainder of this term, corresponding to graphs where at least one vertex of genus 0 has been attached to a vertex of genus 1 , is part of the contribution of the boundary. We show that the alternating representation does not occur here. For a symmetric function

$$
f=\sum_{n=0}^{\infty} f_{n}
$$

we write

$$
\operatorname{Alt}(f)=\sum_{n=0}^{\infty}\left\langle s_{1^{n}}, f_{n}\right\rangle t^{n}
$$

Lemma 5. The alternating representation does not occur in the contribution of the part of the boundary of $M_{1, n}$ corresponding to graphs where at least one vertex of genus 0 has been attached to a vertex of genus 1. In terms of the notation introduced above:

$$
\operatorname{Alt}\left(\mathbf{a}_{1} \circ\left(h_{1}+\mathbf{b}_{0}^{\prime}\right)\right)=\operatorname{Alt}\left(\mathbf{a}_{1}\right)
$$

Proof. We choose to give a somewhat geometric proof instead of a proof using mostly the language of symmetric functions.

Observe first that a boundary stratum corresponding to a graph with a genus 1 vertex is isomorphic to a product

$$
M_{1, m} \times \prod_{i} M_{0, n_{i}}
$$

i.e., it is not necessary to take the quotient by a finite group. (The corresponding graph has no automorphisms: there is a unique shortest path from each of the $n$ legs to the vertex of genus 1 , and every vertex and every edge lie on such a path.) By the Künneth formula, the cohomology of such a product is isomorphic to the tensor product of the cohomologies of the factors.

Consider the $\Sigma_{n}$-orbit of such a stratum. The direct sum of the cohomologies of the strata in the orbit forms a $\Sigma_{n}$-representation $V$. It is induced from the cohomology of a single stratum, considered as a representation $W$ of the stabilizer $G$ in $\Sigma_{n}$ of the stratum. By Frobenius Reciprocity, $V$ contains a copy of the alternating representation if and only if $W$ contains a copy of the restriction of the alternating representation to $G$.

To each vertex of the graph, one associates the symmetric group corresponding to the legs attached to the vertex. The product over the vertices of these symmetric groups is a subgroup $H$ of $G$ and the further restriction of the alternating representation to $H$ is the tensor product over the vertices of the alternating representations of these symmetric groups.

Consider a moduli space $M_{0, k}$ corresponding to an extremal vertex of the graph corresponding to a boundary stratum as above. The symmetric group associated to this vertex is a standard subgroup $\Sigma_{k-1} \subset \Sigma_{k}$, permuting the $k-1$ legs attached to the vertex and leaving the unique half-edge fixed. By Lemma 4, the irreducible $\Sigma_{k}$-representations occurring in $H^{\bullet}\left(M_{0, k}\right)$ have Young diagrams with at most $k-2$ rows. The Young diagrams of the irreducible representations occurring in the restriction to $\Sigma_{k-1}$ also have at most $k-2$ rows, as they are obtained by removing one box. Therefore the alternating representation does not occur here. It follows that $V$ does not contain a copy of the alternating representation either.
Q.E.D.

We return to Getzler's result. We need to evaluate
$\operatorname{Alt}\left(\left(-\frac{1}{2} \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \left(1-\psi_{n}\left(\mathbf{a}_{0}^{\prime \prime}\right)\right)+\frac{\dot{\mathbf{a}}_{0}^{2}+\dot{\mathbf{a}}_{0}+\frac{1}{4} \psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}{1-\psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}\right) \circ\left(h_{1}+\mathbf{b}_{0}^{\prime}\right)\right)$.

Getzler remarks that the two terms inside the big inner parentheses may be thought of as a sum over necklaces (graphs consisting of a single circuit) and a correction term, taking into account the fact that necklaces of 1 or 2 vertices have non-trivial involutions (while those with more vertices do not). The plethysm with $h_{1}+\mathbf{b}_{0}^{\prime}$ stands again for attaching a forest whose vertices have genus 0 . We begin with the analogue of Lemma 5.

Lemma 6. The alternating representation does not occur in the contribution of the part of the boundary of $M_{1, n}$ corresponding to graphs where at least one vertex of genus 0 has been attached to a necklace. In terms of the notation introduced above:

$$
\begin{aligned}
\operatorname{Alt}( & \left.\left(-\frac{1}{2} \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \left(1-\psi_{n}\left(\mathbf{a}_{0}^{\prime \prime}\right)\right)+\frac{\dot{\mathbf{a}}_{0}^{2}+\dot{\mathbf{a}}_{0}+\frac{1}{4} \psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}{1-\psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}\right) \circ\left(h_{1}+\mathbf{b}_{0}^{\prime}\right)\right) \\
& =\operatorname{Alt}\left(-\frac{1}{2} \sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \left(1-\psi_{n}\left(\mathbf{a}_{0}^{\prime \prime}\right)\right)+\frac{\dot{\mathbf{a}}_{0}^{2}+\dot{\mathbf{a}}_{0}+\frac{1}{4} \psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}{1-\psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}\right)
\end{aligned}
$$

Proof. In this case, each boundary stratum is isomorphic to a product

$$
\left(\prod_{v \in \text { necklace }} M_{0, n(v)}\right) / I \times \prod_{v \notin \text { necklace }} M_{0, n(v)}
$$

The finite group $I$ is trivial when the necklace has at least 3 vertices. It has 2 elements when the necklace has 1 resp. 2 vertices and acts by reversing the edge in the necklace resp. by interchanging the two edges of the necklace. In particular, $I$ acts trivially on the moduli spaces corresponding to the vertices of the forest.

Just as in the proof of Lemma 5, the alternating representation does not occur in the cohomology of a moduli space $M_{0, k}$ corresponding to an extremal vertex of one of the trees of the forest. It follows that the alternating representation does not occur in the cohomology of a $\Sigma_{n}$-orbit of boundary strata as soon as the forest is nonempty. Q.E.D.

In order to determine the contribution of the part of the boundary of $M_{1, n}$ corresponding to necklaces without attached trees, we need several lemmas.

Lemma 7. The restriction of the $\Sigma_{n}$-representation $H^{\bullet}\left(M_{0, n}\right)$ to the standard subgroup $\Sigma_{n-2}$ contains the alternating representation exactly once. In terms of the notation introduced above:

$$
\operatorname{Alt}\left(\mathbf{a}_{0}^{\prime \prime}\right)=\frac{t}{1+t}
$$

Proof. The Young diagrams corresponding to the irreducible representations of $\Sigma_{n-2}$ occurring in $\mathbf{a}_{0}^{\prime \prime}$ are obtained by removing 2 boxes from a Young diagram occurring in $\mathbf{a}_{0}$. To obtain a copy of the alternating representation of $\Sigma_{n-2}$, one needs to start with a Young diagram with at least $n-2$ rows. From Lemma 4, only the top cohomology $H^{n-3}\left(M_{0, n}\right)$ can contribute. Observe that $H^{n-3}\left(M_{0, n}\right) \cong$ $H_{c}^{n-3}\left(M_{0, n}\right) \cong H_{n-3}\left(M_{0, n}\right)$ as $\Sigma_{n}$-representations. Thus

$$
\begin{aligned}
\operatorname{Alt}\left(\mathbf{a}_{0}^{\prime \prime}\right) & =\operatorname{Alt}\left(\frac{\partial^{2}}{\partial p_{1}^{2}} \sum_{n=3}^{\infty} \operatorname{ch}_{n}\left(e_{c}^{\Sigma_{n}}\left(M_{0, n}\right)\right)\right) \\
& =\operatorname{Alt}\left(\frac{\partial^{2}}{\partial p_{1}^{2}} \sum_{n=3}^{\infty}(-1)^{n-3} \operatorname{ch}_{n}\left(H_{c}^{n-3}\left(M_{0, n}\right)\right)\right)
\end{aligned}
$$

Getzler shows in [Ge1], p. 213, l. 3 that

$$
H_{c}^{n-3}\left(M_{0, n}\right) \cong \operatorname{sgn}_{n} \otimes \mathcal{L} i e((n)) .
$$

Here $\operatorname{sgn}_{n}$ denotes the alternating representation and $\mathcal{L} i e((n))$ the $\Sigma_{n^{-}}$ representation that is part of the cyclic Lie operad. Getzler and Kapranov show in [GK], Example 7.24 that
$\operatorname{Ch}(\mathcal{L} i e):=\sum_{n=3}^{\infty} \operatorname{ch}_{n}(\mathcal{L} i e((n)))=\left(1-p_{1}\right) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \left(1-p_{n}\right)+h_{1}-h_{2}$, where $\mu(n)$ is the Möbius function. Hence
$\sum_{n=3}^{\infty}(-1)^{n-3} \operatorname{ch}_{n}\left(\operatorname{sgn}_{n} \otimes \mathcal{L} i e((n))\right)=-\left(1+p_{1}\right) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \left(1+p_{n}\right)+h_{1}+e_{2}$ and

$$
\begin{aligned}
& \frac{\partial^{2}}{\partial p_{1}^{2}}\left(\sum_{n=3}^{\infty}(-1)^{n-3} \operatorname{ch}_{n}\left(\operatorname{sgn}_{n} \otimes \mathcal{L} i e((n))\right)\right) \\
& \quad=\frac{\partial}{\partial p_{1}}\left(-\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \left(1+p_{n}\right)-\left(1+p_{1}\right) \frac{1}{1+p_{1}}\right)+1 \\
& \quad=1-\frac{1}{1+p_{1}}=\frac{p_{1}}{1+p_{1}}
\end{aligned}
$$

But

$$
\operatorname{Alt}\left(\frac{p_{1}}{1+p_{1}}\right)=\frac{t}{1+t}
$$

since $\left\langle p_{1}^{n}, s_{1^{n}}\right\rangle=1$.
Q.E.D.

Lemma 8. Let $f_{n}$ be a symmetric function of degree $n$. Assume that $\left\langle s_{1^{n}}, f_{n}\right\rangle=0$. Then $\left\langle s_{1^{n k}}, p_{k} \circ f_{n}\right\rangle=0$.

Proof. Write $e(\lambda)$ resp. $o(\lambda)$ for the number of even resp. odd parts of a partition $\lambda$. For $\lambda$ a partition of $n$,

$$
o(\lambda) \equiv n \quad(\bmod 2) \quad \text { and } \quad\left\langle s_{1^{n}}, p_{\lambda}\right\rangle=(-1)^{e(\lambda)} .
$$

Here $p_{\lambda}=\prod_{i} p_{\lambda_{i}}$ is the symmetric function of degree $n$ that is the product of the power sums corresponding to the parts of $\lambda$. If $f_{n}=$ $\sum_{\lambda} a_{\lambda} p_{\lambda}$, then $p_{k} \circ f_{n}=\sum_{\lambda} a_{\lambda} p_{k \lambda}$, where $k \lambda$ is the partition of $k n$ obtained from $\lambda$ by multiplying all parts with $k$. For $k$ odd,

$$
\sum_{\lambda} a_{\lambda}(-1)^{e(\lambda)}=\sum_{\lambda} a_{\lambda}(-1)^{e(k \lambda)}
$$

whereas for $k$ even,

$$
e(k \lambda)=o(\lambda)+e(\lambda) \equiv n+e(\lambda) \quad(\bmod 2),
$$

so that

$$
\sum_{\lambda} a_{\lambda}(-1)^{e(k \lambda)}=(-1)^{n} \sum_{\lambda} a_{\lambda}(-1)^{e(\lambda)} .
$$

The result follows.
Q.E.D.

Lemma 9. The occurrence of the alternating representation observed in Lemma 7 is stable under plethysm with $p_{k}$. In terms of the notation introduced above:

$$
\operatorname{Alt}\left(\psi_{k}\left(\mathbf{a}_{0}^{\prime \prime}\right)\right)=\frac{-(-t)^{k}}{1-(-t)^{k}}
$$

Proof. From Lemma 7,

$$
\operatorname{Alt}\left(\mathbf{a}_{0}^{\prime \prime}\right)=\operatorname{Alt}\left(\frac{p_{1}}{1+p_{1}}\right)
$$

Applying Lemma 8 and using that $\left\langle s_{1^{k n}}, p_{k}^{n}\right\rangle=(-1)^{(k-1) n}$, we find

$$
\begin{aligned}
& \operatorname{Alt}\left(\psi_{k}\left(\mathbf{a}_{0}^{\prime \prime}\right)\right)=\operatorname{Alt}\left(p_{k} \circ\left(\frac{p_{1}}{1+p_{1}}\right)\right)=\operatorname{Alt}\left(\frac{p_{k}}{1+p_{k}}\right) \\
& =\operatorname{Alt}\left(\sum_{n=1}^{\infty}(-1)^{n-1} p_{k}^{n}\right)=\sum_{n=1}^{\infty}(-1)^{n-1}(-1)^{(k-1) n} t^{k n}
\end{aligned}
$$

$$
=-\sum_{n=1}^{\infty}\left((-t)^{k}\right)^{n}=\frac{-(-t)^{k}}{1-(-t)^{k}} .
$$

Q.E.D.

For two symmetric functions $f$ and $g$, we have

$$
\operatorname{Alt}(f g)=\operatorname{Alt}(f) \operatorname{Alt}(g)
$$

This follows immediately from the Littlewood-Richardson rule (cf. [FH], p. 456). One may use this identity to shorten the proof of Lemma 9. Similarly,

$$
\operatorname{Alt}\left(\log \left(1-\psi_{k}\left(\mathbf{a}_{0}^{\prime \prime}\right)\right)\right)=\log \left(1-\frac{-(-t)^{k}}{1-(-t)^{k}}\right)=-\log \left(1-(-t)^{k}\right)
$$

and

$$
\begin{gathered}
\operatorname{Alt}\left(-\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \left(1-\psi_{n}\left(\mathbf{a}_{0}^{\prime \prime}\right)\right)=\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \log \left(1-(-t)^{n}\right)\right. \\
=-\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \sum_{k=1}^{\infty} \frac{(-t)^{n k}}{k}=-\sum_{m=1}^{\infty} \frac{(-t)^{m}}{m} \sum_{d \mid m} \phi(d) \\
=-\sum_{m=1}^{\infty}(-t)^{m}=\frac{t}{1+t}
\end{gathered}
$$

It remains to evaluate the contribution of the correction term,

$$
\operatorname{Alt}\left(\frac{\dot{\mathbf{a}}_{0}^{2}+\dot{\mathbf{a}}_{0}+\frac{1}{4} \psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}{1-\psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}\right)
$$

We need one more lemma.
Lemma 10. The alternating part of the formal symmetric function

$$
\dot{\mathbf{a}}_{0}=\frac{\partial \mathbf{a}_{0}}{\partial p_{2}}
$$

is given by the following formula:

$$
\operatorname{Alt}\left(\dot{\mathbf{a}}_{0}\right)=\frac{1}{2} \frac{t}{1-t}
$$

Proof. In terms of Young diagrams, multiplication by $s_{2}$ is the operation of adding two boxes, not in the same column, and multiplication by $s_{1^{2}}$ is the operation of adding two boxes, not in the same row. Now $p_{2}=s_{2}-s_{1^{2}}$ and $\frac{\partial}{\partial p_{2}}=\frac{1}{2} p_{2}^{\perp}$, where $p_{2}^{\perp}$ is the adjoint of multiplication by $p_{2}$. Thus, to obtain a copy of the alternating representation of $\Sigma_{n-2}$ in a term of $\dot{\mathbf{a}}_{0}$, one needs to start with a Young diagram with at least $n-2$ rows, just as in the proof of Lemma 7. So

$$
\operatorname{Alt}\left(\dot{\mathbf{a}}_{0}\right)=\operatorname{Alt}\left(\frac{\partial}{\partial p_{2}} \sum_{n=3}^{\infty}(-1)^{n-3} \operatorname{ch}_{n}\left(H_{c}^{n-3}\left(M_{0, n}\right)\right)\right)
$$

We now find

$$
\frac{\partial}{\partial p_{2}}\left(\sum_{n=3}^{\infty}(-1)^{n-3} \operatorname{ch}_{n}\left(\operatorname{sgn}_{n} \otimes \mathcal{L} i e((n))\right)\right)=\frac{1}{2} \frac{1+p_{1}}{1+p_{2}}-\frac{1}{2}=\frac{1}{2} \frac{p_{1}-p_{2}}{1+p_{2}} .
$$

But

$$
\operatorname{Alt}\left(\frac{1}{2} \frac{p_{1}-p_{2}}{1+p_{2}}\right)=\frac{1}{2} \frac{t+t^{2}}{1-t^{2}}=\frac{1}{2} \frac{t}{1-t}
$$

Q.E.D.

An easy calculation combining Lemmas 9 and 10 gives

$$
\operatorname{Alt}\left(\frac{\dot{\mathbf{a}}_{0}^{2}+\dot{\mathbf{a}}_{0}+\frac{1}{4} \psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}{1-\psi_{2}\left(\mathbf{a}_{0}^{\prime \prime}\right)}\right)=\frac{1}{2} \frac{t}{1-t}
$$

The contribution from the necklaces becomes then

$$
\frac{1}{2} \frac{t}{1+t}+\frac{1}{2} \frac{t}{1-t}=\frac{t}{1-t^{2}}
$$

i.e., 1 for $n$ odd and 0 for $n$ even. We have proved the following result.

Theorem 2. The alternating part of the $\Sigma_{n}$-equivariant Euler characteristic of the cohomology of $\partial M_{1, n}$ is given by the following formula:

$$
\left\langle s_{1^{n}}, e_{c}^{\Sigma_{n}}\left(\partial M_{1, n}\right)\right\rangle= \begin{cases}1, & n \text { odd } \\ 0, & n \text { even }\end{cases}
$$

## §5. The construction of the motive

The main result of Section 4 (Theorem 2) is

$$
A_{c}\left(\partial M_{1, n}\right)= \begin{cases}1, & n \text { odd } \\ 0, & n \text { even }\end{cases}
$$

Combining this with Theorem 1, we immediately obtain

$$
A_{c}\left(\bar{M}_{1, n}\right)=\left\{\begin{aligned}
-S[n+1], & n \text { odd } \\
0, & n \text { even } .
\end{aligned}\right.
$$

Let $n>1$ be an odd integer. The pair consisting of $\bar{M}_{1, n}$ and the projector

$$
\frac{1}{n!} \sum_{\sigma \in \Sigma_{n}}(-1)^{\operatorname{sgn}(\sigma)} \sigma_{*}
$$

defines a Chow motive, since $\bar{M}_{1, n}$ is the quotient of a smooth projective variety by a finite group $[\mathrm{BP}]$. We wish to show that it is pure of degree $n$. This will conclude our construction of the motive $S[n+1]$, an alternative (in level 1 only) to Scholl's construction. The arguments below are similar to those in $[\mathrm{Sc}]$, 1.3.4.

First, $\left\langle s_{1^{n}}, H_{c}^{i}\left(M_{1, n}\right)\right\rangle=0$ for $i \neq n$. As in [De], proof of 5.3 , the degeneration of the Leray spectral sequence at $E_{2}$ due to Lieberman's trick implies that $\left\langle s_{1^{n}}, H_{c}^{i}\left(\mathcal{E}^{n-1}\right)\right\rangle=0$ when $i \neq n$ for a relative elliptic curve $\mathcal{E} \rightarrow S$ and this implies the statement.

Thus we have an exact sequence

$$
\begin{gathered}
0 \rightarrow H_{c}^{n-1}\left(\bar{M}_{1, n}\right)(\alpha) \rightarrow H_{c}^{n-1}\left(\partial M_{1, n}\right)(\alpha) \rightarrow H_{c}^{n}\left(M_{1, n}\right)(\alpha) \rightarrow \\
\rightarrow H_{c}^{n}\left(\bar{M}_{1, n}\right)(\alpha) \rightarrow H_{c}^{n}\left(\partial M_{1, n}\right)(\alpha) \rightarrow 0
\end{gathered}
$$

and isomorphisms $H^{i}\left(\bar{M}_{1, n}\right)(\alpha) \cong H^{i}\left(\partial M_{1, n}\right)(\alpha)$ for $i \notin\{n-1, n\}$. Here $V(\alpha)$ denotes the alternating part of a $\Sigma_{n}$-representation $V$. Therefore $H^{i}\left(\partial M_{1, n}\right)(\alpha)$ is pure of weight $i$ for $i>n$. But then all these spaces vanish, since $H^{i}\left(\partial M_{1, n}\right)$ has weight $\leq i$ for all $i$ and since $A_{c}\left(\partial M_{1, n}\right)$ has weight 0 . Hence $H^{i}\left(\bar{M}_{1, n}\right)(\alpha)=0$ for $i>n$ and then by duality for $i<n$ as well. This shows that $H^{n}\left(\bar{M}_{1, n}\right)(\alpha)=S[n+1]$ and concludes the alternative construction of these motives.

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## Polarized K3 surfaces of genus thirteen

## Shigeru Mukai

A smooth complete algebraic surface $S$ is of type $K 3$ if $S$ is regular and the canonical class $K_{S}$ is trivial. A primitively polarized K3 surface is a pair $(S, h)$ of a K3 surface $S$ and a primitive ample divisor class $h \in \operatorname{Pic} S$. The integer $g:=\frac{1}{2}\left(h^{2}\right)+1 \geq 2$ is called the genus of $(S, h)$. The moduli space of primitively polarized K3 surfaces of genus $g$ exists as a quasi-projective (irreducible) variety, which we denote by $\mathcal{F}_{g}$. As is well known a general polarized K3 surface of genus $2 \leq g \leq 5$ is a complete intersection of hypersurfaces in a weighted projective space: $(6) \subset \mathbf{P}(1112),(4) \subset \mathbf{P}^{3},(2) \cap(3) \subset \mathbf{P}^{4}$ and $(2) \cap(2) \cap(2) \subset \mathbf{P}^{5}$.

In connection with the classification of Fano threefolds, we have studied the system of defining equations of the projective model $S_{2 g-2} \subset$ $\mathbf{P}^{g}$ and shown that a general polarized K3 surface of genus $g$ is a complete intersection with respect to a homogeneous vector bundle $\mathcal{V}_{g-2}$ (of rank $g-2$ ) in a $g$-dimensional Grassmannian $G(n, r), g=r(n-r)$, in a unique way for the following six values of $g$ :

| $g$ | 6 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: |
| $r$ | 2 | 2 | 3 | 5 |
| $\mathcal{V}_{g-2}$ | $3 \mathcal{O}_{G}(1) \oplus \mathcal{O}_{G}(2)$ | $6 \mathcal{O}_{G}(1)$ | $\bigwedge^{2} \mathcal{E} \oplus 4 \mathcal{O}_{G}(1)$ | $\bigwedge^{4} \mathcal{E} \oplus 3 \mathcal{O}_{G}(1)$ |


| 12 | 20 |
| :---: | :---: |
| 3 | 4 |
| $3 \bigwedge^{2} \mathcal{E} \oplus \mathcal{O}_{G}(1)$ | $3 \bigwedge^{2} \mathcal{E}$ |

Here $\mathcal{E}$ is the universal quotient bundle on $G(n, r)$. See [4] and [5] for the case $g=6,8,9,10,[6, \S 5]$ for $g=20$ and $\S 3$ for $g=12$.

By this description, the moduli space $\mathcal{F}_{g}$ is birationally equivalent to the orbit space $H^{0}\left(G(n, r), \mathcal{V}_{g-2}\right) /\left(P G L(n) \times A u t_{G(n, r)} \mathcal{V}_{g-2}\right)$ and

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hence is unirational for these values of $g$. The uniqueness of the description modulo the automorphism group is essentially due to the rigidity of the vector bundle $E:=\left.\mathcal{E}\right|_{S}$. All the cohomology groups $H^{i}(s l(E))$ vanish.

A general member $(S, h) \in \mathcal{F}_{g}$ is a complete intersection with respect to the homogeneous vector bundle $8 \mathcal{U}$ in the orthogonal Grassmannian $O-G(10,5)$ in the case $g=7([4])$, and with respect to $5 \mathcal{U}$ in $O-G(9,3)$ in the case $18([6])$, where $\mathcal{U}$ is the homogeneous vector bundle on the orthogonal Grassmannian such that $H^{0}(\mathcal{U})$ is a half spinor representation $U^{16}$. Both descriptions are unique modulo the orthogonal group. Hence $\mathcal{F}_{7}$ and $\mathcal{F}_{18}$ are birationally equivalent to $G\left(8, U^{16}\right) / P S O(10)$ and $G\left(5, U^{16}\right) / S O(9)$, respectively. The unirationality of $\mathcal{F}_{11}$ is proved in [7] using a non-abelian Brill-Noether locus and the unirationality of $\mathcal{M}_{11}$, the moduli space of curves of genus 11 .

In this article, we shall study the case $g=13$ and show the following:
Theorem 1. A general member $(S, h) \in \mathcal{F}_{13}$ is isomorphic to a complete intersection with respect to the homogeneous vector bundle

$$
\mathcal{V}=\bigwedge^{2} \mathcal{E} \oplus \bigwedge^{2} \mathcal{E} \oplus \bigwedge^{3} \mathcal{F}
$$

of rank 10 in the 12-dimensional Grassmannian $G(7,3)$, where $\mathcal{F}$ is the dual of the universal subbundle.

Corollary $\mathcal{F}_{13}$ is unirational.
Remark 1. A general complete intersection $(S, h)$ with respect to the homogeneous vector bundle $\bigwedge^{4} \mathcal{F} \oplus S^{2} \mathcal{E}$ in the 10-dimensional Grassmannian $G(7,2)$ is also a primitively polarized K3 surface of genus 13. But $(S, h)$ is not a general member of $\mathcal{F}_{13}$. In fact, $S$ contains 8 mutually disjoint rational curves $R_{1}, \ldots, R_{7}$, which are of degree 3 with respect to $h$. This will be discussed elsewhere.

Unlike the known cases described above, the vector bundle $E=\left.\mathcal{E}\right|_{S}$ in the theorem is not rigid. Hence the theorem does not give a birational equivalence between $\mathcal{F}_{13}$ and an orbit space. But $E$ is semi-rigid, that is, $H^{0}(\operatorname{sl}(E))=0$ and $\operatorname{dim} H^{1}(\operatorname{sl}(E))=2$. Instead of $\mathcal{F}_{13}$ itself, the theorem gives a birational equivalence between the universal family over it and an orbit space.

Let $S \subset G(7,3)$ be a general complete intersection with respect to $\mathcal{V}$. Then $S$ is the common zero locus of the two global sections of $\bigwedge^{2} \mathcal{E}$ corresponding to general bivectors $\sigma_{1}, \sigma_{2} \in \Lambda^{2} \mathbf{C}^{7}$ and one global section of $\Lambda^{3} \mathcal{F}$ corresponding to a general $\tau \in \Lambda^{3} \mathbf{C}^{7, \vee}$. The 2-dimensional
subspace $P=\left\langle\sigma_{1}, \sigma_{2}\right\rangle \subset \bigwedge^{2} \mathbf{C}^{7}$ is uniquely determined by $S$. Let $\overline{P \wedge P}$ be the subspace of $\bigwedge^{3} \mathbf{C}^{7, \vee}$ corresponding to $P \wedge P \subset \bigwedge^{4} \mathbf{C}^{7}$. Then $\mathbf{C} \tau$ modulo $\overline{P \wedge P}$ is also uniquely determined by $S$. It is known that the natural action of $P G L(7)$ on $G\left(2, \bigwedge^{2} \mathbf{C}^{7}\right)$ has an open dense orbit (Sato-Kimura[9, p. 94]). Hence we obtain the natural birational map

$$
\begin{equation*}
\psi: \mathbf{P}_{*}\left(\bigwedge^{4} \mathbf{C}^{7} /(P \wedge P)\right) / G \cdots \rightarrow \mathcal{F}_{13} \tag{1}
\end{equation*}
$$

which is dominant by the theorem, where $G$ is the (10-dimensional) stabilizer group of the action at $P \in G\left(2, \bigwedge^{2} \mathbf{C}^{7}\right)$.

Theorem 2. For every general member $p=(S, h) \in \mathcal{F}_{13}$, the fiber of $\psi$ at $p$ is birationally equivalent to the moduli K3 surface $M_{S}(3, h, 4)$ of semi-rigid rank three vector bundles with $c_{1}=h$ and $\chi=3+4$.

As is shown in [8], $\hat{S}:=M_{S}(3, h, 4)$ carries a natural ample divisor class $\hat{h}$ of the same genus $(=13)$ and $(S, h) \mapsto(\hat{S}, \hat{h})$ induces an automorphism of $\mathcal{F}_{13}$. (In fact, this is an involution.) Hence we have

Corollary The orbit space $\mathbf{P}^{*}\left(\bigwedge^{4} \mathbf{C}^{7} /(P \wedge P)\right) / G$ is birationally equivalent to the universal family over $\mathcal{F}_{13}$, or the coarse moduli space of one pointed polarized K3 surfaces $(S, h, x)$ of genus 13 .

Remark 2. 8 Kondō[3] proves that the Kodaira dimension of $\mathcal{F}_{g}$ is non-negative for the following 17 values:

$$
g=41,42,50,52,54,56,58,60,65,66,68,73,82,84,104,118,132
$$

The Kodaira dimension of $\mathcal{F}_{m^{2}(g-1)+1}$ is non-negative for these values of $g$ and for every $m \geq 2$ since it is a finite covering of $\mathcal{F}_{g}$.

Notations and convention. Algebraic varieties and vector bundles are considered over the complex number field $\mathbf{C}$. The dual of a vector bundle (or a vector space) $E$ is denoted by $E^{\vee}$. Its Euler-Poincarè characteristic $\sum_{i}(-)^{i} h^{i}(E)$ is denoted by $\chi(E)$. The vector bundles of traceless endomorphisms of $E$ is denoted by $\operatorname{sl}(E)$. For a vector space $V, G(V, r)$ is the Grassmannian of $r$-dimensional quotient spaces of $V$ and $G(r, V)$ that of $r$-dimensional subspaces. The isomorphism class of $G(V, r)$ with $\operatorname{dim} V=n$ is denoted by $G(n, r)$. The projective spaces $G(V, 1)$ and $G(1, V)$ are denoted by $\mathbf{P}^{*}(V)$ and $\mathbf{P}_{*}(V)$, respectively. $\mathcal{O}_{G}(1)$ is the pull-back of the tautological line bundle by the Plücker embedding $G(V, r) \hookrightarrow \mathbf{P}^{*}\left(\bigwedge^{r} V\right)$.

## §1. Vanishing

We prepare the vanishing of cohomology groups of homogeneous vector bundles on the Grassmannian $G(n, r)$, which is the quotient of $S L(n)$ by a parabolic subgroup $P$. The reductive part $P_{\text {red }}$ of $P$ is the intersection of $G L(r) \times G L(n-r)$ and $S L(n)$ in $G L(n)$. We take $\left\{\left(a_{1}, \cdots, a_{r} ; a_{r+1}, \ldots, a_{n}\right) \mid \sum_{1}^{n} a_{i}=0\right\} \subset \mathbf{Z}^{n}$ as root lattice and $\mathbf{Z}^{n} / \mathbf{Z}(1,1, \ldots, 1)$ as the common weight lattice of $S L(n)$ and $P_{\text {red }}$. We take $\left\{e_{i}-e_{i+1} \mid 1 \leq i \leq n-1\right\}$ as standard root basis. The half of the sum of all positive roots is equal to

$$
\delta=(n-1, n-3, n-5, \ldots,-n+3,-n+1) / 2
$$

Let $\rho$ be an irreducible representation of $P_{\text {red }}$ and
$w \in \mathbf{Z}^{n} / \mathbf{Z}(1,1, \ldots, 1)$ its highest weight. We denote the homogeneous vector bundle on $G(n, r)$ induced from $\rho$ by $E_{w}$. $w$ is singular if a number appears more than once in $w+\delta$. If $w$ is not singular and $w+\delta=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$, then there is a unique (Grassmann) permutation $\alpha=\alpha_{w}$ such that $a_{\alpha(1)}>a_{\alpha(2)}>\cdots>a_{\alpha(n)}$. We denote the length of $\alpha_{w}$, that is, the cardinality of the set $\left\{(i, j) \mid 1 \leq i<j \leq n, a_{i}<a_{j}\right\}$, by $l(w)$.

Theorem 3 (Borel-Hirzebruch[2]). (a) If $w$ is singular, then all the cohomology groups $H^{i}\left(G(n, r), \mathcal{E}_{w}\right)$ vanish.
(b) If $w$ is not, then all the cohomology groups $H^{i}\left(G(n, r), \mathcal{E}_{w}\right)$ vanish except for one $i:=l(w)$. Moreover, $H^{l(w)}\left(G(n, r), \mathcal{E}_{\rho}\right)$ is an irreducible representation of $S L(n)$ with highest weight

$$
\left(a_{\alpha(1)}, a_{\alpha(2)}, \ldots, a_{\alpha(n)}\right)-\delta .
$$

The dimension of this unique nonzero cohomology group is equal to $\prod_{1 \leq i<j \leq n}\left|a_{i}-a_{j}\right| /(j-i)$.
$l(w)$ is called the index of the homogeneous vector bundle $E_{w}$.

Example. In the following table, - means that the weight $w$ is singular and we put $s=n-r$.

| weight $w$ | homogeneous bundle $\mathcal{E}_{w}$ | $l(w)$ | $H^{l(w)}$ |
| :--- | :--- | :---: | :---: |
| $(1,0,0, \ldots, 0,0 ; 0, \ldots, 0,0)$ | $\mathcal{E}$, universal quotient | 0 | $\mathbf{C}^{n}$ |
|  | bundle |  |  |
| $(0,0,0, \ldots,-1,0 ; 0, \ldots, 0,0)$ | $\mathcal{E}^{\vee}$ | - |  |
| $(1,1,0, \ldots, 0,0 ; 0, \ldots, 0,0)$ | $\bigwedge^{2} \mathcal{E}$ | 0 | $\bigwedge^{2} \mathbf{C}^{n}$ |
| $(1,1,1, \ldots, 1,1 ; 0, \ldots, 0,0)$ | $\mathcal{O}_{G}(1)=\operatorname{det} \mathcal{E}=\operatorname{det} \mathcal{F}$ | 0 | $\bigwedge^{r} \mathbf{C}^{n}$ |
| $(0,0,0, \ldots, 0,0 ;-1, \ldots,-1)$ |  |  |  |
| $(0,0,0, \ldots, 0,0 ; 1, \ldots, 0,0)$ | $\mathcal{F}^{\vee}$, universal subbundle | - |  |
| $(0,0,0, \ldots, 0,0 ; 0, \ldots, 0,-1)$ | $\mathcal{F}$ | 0 | $\mathbf{C}^{n, \vee}$ |
| $(1,0,0, \ldots, 0,0 ; 0, \ldots, 0,-1)$ | $T_{G(n, r)}$, tangent bundle | 0 | $s l\left(\mathbf{C}^{n}\right)$ |
| $(0,0,0, \ldots,-1 ; 1,0, \ldots, 0,0)$ | $\Omega_{G(n, r)}$, cotangent bundle | 1 | $\mathbf{C}$ |
| $(-s,-s, \ldots,-s ; r, r, \ldots, r)$ | $\mathcal{O}_{G}(-n)$, canonical bundle | $r s$ | $\mathbf{C}$ |

We apply the theorem to the 12-dimensional Grassmannian $G(7,3)$.
Lemma 4. (a) All cohomology groups of the homogeneous vector bundle $\bigwedge^{p}(2 \mathcal{E}(-1)) \otimes \bigwedge^{q}(\mathcal{F}(-1))$ on $G(7,3)$ vanish except for the following:
i) $p=q=0, \quad h^{0}\left(\mathcal{O}_{G}\right)=1$, and
ii) $p=6, q=4, \quad h^{12}\left(\mathcal{O}_{G}(-7)\right)=1$.
(b) All cohomology groups of $\mathcal{O}_{G}(1) \otimes \bigwedge^{p}(2 \mathcal{E}(-1)) \otimes \bigwedge^{q}(\mathcal{F}(-1))$ vanish except for the following:
i) $p=q=0, \quad h^{0}\left(\mathcal{O}_{G}(1)\right)=35$,
ii) $p=1, q=0, \quad h^{0}(2 \mathcal{E})=2 \cdot 7=14$, and
iii) $p=0, q=1, \quad h^{0}(\mathcal{F})=7$.
(c) All cohomology groups of $\mathcal{E} \otimes \bigwedge^{p}(2 \mathcal{E}(-1)) \otimes \bigwedge^{q}(\mathcal{F}(-1))$ vanish except for $h^{0}(\mathcal{E})=7$ with $p=q=0$.
(d) All cohomology groups of $\mathcal{F} \otimes \bigwedge^{p}(2 \mathcal{E}(-1)) \otimes \bigwedge^{q}(\mathcal{F}(-1))$ vanish except for $h^{0}(\mathcal{F})=7$ with $p=q=0$.
(e) All cohomology groups of $\bigwedge^{2} \mathcal{E} \otimes \bigwedge^{p}(2 \mathcal{E}(-1)) \otimes \bigwedge^{q}(\mathcal{F}(-1))$ vanish except for the following:
i) $p=q=0, \quad h^{0}\left(\bigwedge^{2} \mathcal{E}\right)=21$, and
ii) $p=1, q=0, \quad h^{0}\left(\bigwedge^{2} \mathcal{E} \otimes(2 \mathcal{E}(-1))\right)=2$.
(f) All cohomology groups of $\bigwedge^{3} \mathcal{F} \otimes \bigwedge^{p}(2 \mathcal{E}(-1)) \otimes \bigwedge^{q}(\mathcal{F}(-1))$ vanish except for the following:
i) $p=q=0, \quad h^{0}\left(\bigwedge^{3} \mathcal{F}\right)=35$,
ii) $p=0, q=1, \quad h^{0}\left(\bigwedge^{3} \mathcal{F} \otimes \mathcal{F}(-1)\right)=1$, and
iii) $p=2, q=0, \quad h^{1}\left(\bigwedge^{3} \mathcal{F} \otimes \bigwedge^{2}(2 \mathcal{E}(-1))\right)=3 h^{1}\left(\bigwedge^{3} \mathcal{F} \otimes\right.$ $\left.\bigwedge^{2} \mathcal{E}^{\vee}\right)=3$.
(g) All cohomology groups of $\operatorname{sl}(\mathcal{E}) \otimes \bigwedge^{p}(2 \mathcal{E}(-1)) \otimes \bigwedge^{q}(\mathcal{F}(-1))$ vanish except for $h^{6}=2$ with $p=3, q=2$.

Proof. The following table describes the decomposition of $\bigwedge^{p}(2 \mathcal{E}(-1))$ into indecomposable homogeneous vector bundles.
(2)

| $p$ | decomposition | weight $w^{\prime}$ | $w^{\prime}+\delta^{\prime}$ |
| :--- | :--- | :--- | :--- |
| 0 | $\mathcal{O}_{G}$ | $(0,0,0)$ | $(3,2,1)$ |
| 1 | $2 \mathcal{E}(-1)$ | $2(0,-1,-1)$ | $(3,1,0)$ |
| 2 | $3\left(\bigwedge^{2} \mathcal{E}\right)(-2)$ | $3(-1,-1,-2)$ | $(2,1,-1)$, |
|  | $\oplus S^{2} \mathcal{E}(-2)$ | $\oplus(0,-2,-2)$ | $(3,0,-1)$ |
| 3 | $4 \mathcal{O}_{G}(-2)$ | $4(-2,-2,-2)$ | $(1,0,-1)$, |
|  | $\oplus 2 \operatorname{sl}(\mathcal{E})(-2)$ | $\oplus 2(-1,-2,-3)$ | $(2,0,-2)$ |
| 4 | $3 \mathcal{E}(-3)$ | $3(-2,-3,-3)$ | $(1,-1,-2)$, |
|  | $\oplus\left(S^{2} \bigwedge^{2} \mathcal{E}\right)(-4)$ | $\oplus(-2,-2,-4)$ | $(1,0,-3)$ |
| 5 | $2\left(\bigwedge^{2} \mathcal{E}\right)(-4)$ | $2(-3,-3,-4)$ | $(0,-1,-3)$ |
| 6 | $\mathcal{O}_{G}(-4)$ | $(-4,-4,-4)$ | $(-1,-2,-3)$ |

Here $\delta^{\prime}=(3,2,1)$ is the head of $\delta=(3,2,1 ; 0,-1,-2,-3)$.
$\bigwedge^{q}(\mathcal{F}(-1))$ is indecomposable. The following lists its weight $w^{\prime \prime}$ and $w^{\prime \prime}+\delta^{\prime \prime}$, where $\delta^{\prime \prime}=(0,-1,-2,-3)$ is the tail of $\delta$.

| $q$ | bundle | weight $w^{\prime \prime}$ | $w^{\prime \prime}+\delta^{\prime \prime}$ |
| :--- | :--- | :--- | :--- |
| 0 | $\mathcal{O}_{G}$ | $(0,0,0,0)$ | $(0,-1,-2,-3)$ |
| 1 | $\mathcal{F}(-1)$ | $(1,1,1,0)$ | $(1,0,-1,-3)$ |
| 2 | $\left(\bigwedge^{2} \mathcal{F}\right)(-2)$ | $(2,2,1,1)$ | $(2,1,-1,-2)$ |
| 3 | $\left(\bigwedge^{3} \mathcal{F}\right)(-3)$ | $(3,2,2,2)$ | $(3,1,0,-1)$ |
| 4 | $\mathcal{O}_{G}(-3)$ | $(3,3,3,3)$ | $(3,2,1,0)$ |

We prove (a), (f) and (g) applying Theorem 3. The other cases are similar.
(a) Take $w^{\prime}$ and $w^{\prime \prime}$ from the tables (2) and (3), respectively, and combine into $w=\left(w^{\prime} ; w^{\prime \prime}\right)$. Then $w$ is singular except for the two cases

$$
w+\delta=(3,2,1 ; 0,-1,-2,-3) \quad \text { with } \quad p=q=0
$$

and

$$
w+\delta=(-1,-2,-3 ; 3,2,1,0) \quad \text { with } \quad p=6, q=4
$$

The indices $l(w)$ are equal to 0 and 12 , respectively.
(f) The homogeneous vector bundle $\bigwedge^{3} \mathcal{F} \otimes \bigwedge^{q}(\mathcal{F}(-1))$ decomposes in the following way:

| $q$ | weight $w^{\prime \prime}$ | $w^{\prime \prime}+\delta^{\prime \prime}$ |
| :--- | :--- | :--- |
| 0 | $(0,-1,-1,-1)$ | $(0,-2,-3,-4)$ |
| 1 | $(1,0,0,-1) \oplus(0,0,0,0)$ | $(1,-1,-2,-4),(0,-1,-2,-3)$ |
| 2 | $(2,1,0,0) \oplus(1,1,1,0)$ | $(2,0,-2,-3),(1,0,-1,-3)$ |
| 3 | $(3,1,1,1) \oplus(2,2,1,1)$ | $(3,0,-1,-2),(2,1,-1,-2)$ |
| 4 | $(3,2,2,2)$ | $(3,1,0,-1)$ |

Take $w^{\prime}$ and $w^{\prime \prime}$ from the table (2) and this table, respectively, and consider $w=\left(w^{\prime} ; w^{\prime \prime}\right)$. Then $w$ is singular except for the following three cases.
i) $p=q=0, w+\delta=(3,2,1 ; 0,-2,-3,-4), l(w)=0$,
ii) $p=0, q=1, w+\delta=(3,2,1 ; 0,-1,-2,-3), l(w)=0$, and
iii) $p=2, q=0, w+\delta=(2,1,-1 ; 0,-2,-3,-4), l(w)=1$.
(g) The following table shows the indecomposable components of $s l(\mathcal{E}) \otimes \bigwedge^{p}(2 \mathcal{E}(-1))$ which do not appear in that of $\bigwedge^{p}(2 \mathcal{E}(-1))$.

| $p$ | weight $w^{\prime}$ other than Table $(2)$ | $w^{\prime}+\delta^{\prime}$ |
| :--- | :--- | :--- |
| 0 | $(1,0,-1)$ | $(4,2,0)$ |
| 1 | $2(1,-1,-2) \oplus 2(0,0,-2)$ | $(4,1,-1),(3,2,-1)$ |
| 2 | $4(0,-1,-3) \oplus(1,-2,-3)$ | $(3,1,-2),(4,0,-2)$ |
| 3 | $2(0,-2,-4) \oplus 2(-1,-1,-4)$ | $(3,0,-3),(2,1,-3)$ |
|  | $\oplus 2(0,-3,-3)$ | $(3,-1,-2)$ |
| 4 | $(-1,-2,-5) \oplus 4(-1,-3,-4)$ | $(2,0,-4),(2,-1,-3)$ |
| 5 | $2(-2,-3,-5) \oplus 2(-2,-4,-4)$ | $(1,-1,-4),(1,-2,-3)$ |
| 6 | $(-3,-4,-5)$ | $(0,-2,-4)$ |

Take $w^{\prime}$ and $w^{\prime \prime}$ from the table (2) and this table, respectively, and consider $w=\left(w^{\prime} ; w^{\prime \prime}\right)$. Then $w$ is singular except for the case $w+\delta=$ $(3,0,-3 ; 2,1,-1,-2)$ with $p=3$ and $q=2$. The index is equal to 6.
Q.E.D.

Let $S \subset G(7,3)$ be a complete intersection with respect to $\mathcal{V}=$ $2 \bigwedge^{2} \mathcal{E} \oplus \bigwedge^{3} \mathcal{F}$. The Koszul complex

$$
\mathbf{K}: \mathcal{O}_{G} \longleftarrow \mathcal{V}^{\vee} \longleftarrow \bigwedge^{2} \mathcal{V}^{\vee} \longleftarrow \cdots \longleftarrow \bigwedge^{9} \mathcal{V}^{\vee} \longleftarrow \bigwedge^{10} \mathcal{V}^{\vee} \longleftarrow 0
$$

gives a resolution of the structure sheaf $\mathcal{O}_{S} . \bigwedge^{n} \mathcal{V}^{\vee}$ is isomorphic to $\bigoplus_{p+q=n} \bigwedge^{p}(2 \mathcal{E}(-1)) \otimes \bigwedge^{q}(\mathcal{F}(-1))$.

Proposition 5. (a) $H^{0}\left(S, \mathcal{O}_{S}\right)=\mathbf{C}, H^{1}\left(S, \mathcal{O}_{S}\right)=0$.
(b) The restriction map $H^{0}\left(G(7,3), \mathcal{O}_{G}(1)\right) \longrightarrow H^{0}\left(S, \mathcal{O}_{S}(1)\right)$ is surjective, $H^{0}\left(S, \mathcal{O}_{S}(1)\right)$ is of dimension 14 and $H^{1}\left(S, \mathcal{O}_{S}(1)\right)$ $=H^{2}\left(S, \mathcal{O}_{S}(1)\right)=0$.
(c) The restriction map $H^{0}(G(7,3), \mathcal{E}) \longrightarrow H^{0}(S, E)$ is an isomorphism and $H^{1}(S, E)=H^{2}(S, E)=0$.
(d) The restriction map $H^{0}(G(7,3), \mathcal{F}) \longrightarrow H^{0}(S, F)$ is an isomorphism.
(e) $\quad H^{0}\left(G(7,3), \bigwedge^{2} \mathcal{E}\right) \longrightarrow H^{0}\left(S, \bigwedge^{2} E\right)$ is surjective and the kernel is of dimension 2.
(f) $\quad H^{0}\left(G(7,3), \bigwedge^{3} \mathcal{F}\right) \longrightarrow H^{0}\left(S, \bigwedge^{3} F\right)$ is surjective and the kernel is of dimension 4 .
(g) $E$ is simple and semi-rigid, that $i s, H^{0}(s l(E))=0$ and $h^{1}(\operatorname{sl}(E))=2$.
Proof. We prove (a) and (f) as sample. Other cases are similar.
(a) The restriction map $H^{0}\left(G(7,3), \mathcal{O}_{G}\right) \longrightarrow H^{0}\left(S, \mathcal{O}_{S}\right)$ is surjective by the vanishing $H^{1}\left(\mathcal{V}^{\vee}\right)=H^{2}\left(\bigwedge^{2} \mathcal{V}^{\vee}\right)=\cdots=H^{10}\left(\bigwedge^{10} \mathcal{V}^{\vee}\right)=0$ and the exact sequence $0 \longleftarrow \mathcal{O}_{S} \longleftarrow \mathbf{K} . H^{1}\left(S, \mathcal{O}_{S}\right)$ vanishes since $H^{1}\left(\mathcal{O}_{G}\right)$ $=H^{2}\left(\mathcal{V}^{\vee}\right)=\cdots=H^{11}\left(\bigwedge^{10} \mathcal{V}^{\vee}\right)=0$.
(f) The restriction map is surjective by the vanishing $H^{n}\left(\bigwedge^{3} \mathcal{F} \otimes\right.$ $\left.\bigwedge^{n} \mathcal{V}^{\vee}\right)$ for $n=1, \ldots, 10$ and the exact sequence

$$
0 \longleftarrow \bigwedge^{3} F \longleftarrow \bigwedge^{3} \mathcal{F} \otimes \mathbf{K}
$$

The dimension of the kernel is equal to

$$
h^{0}\left(\bigwedge^{3} \mathcal{F} \otimes \mathcal{V}^{\vee}\right)+h^{1}\left(\bigwedge^{3} \mathcal{F} \otimes \bigwedge^{2} \mathcal{V}^{\vee}\right)=1+3=4
$$

since $H^{n-1}\left(\bigwedge^{3} \mathcal{F} \otimes \bigwedge^{n} \mathcal{V}^{\vee}\right)=0$ for $n=3, \ldots, 10$.
Q.E.D.

## §2. Proof of Theorems 1 and 2

Let $S$ be the zero locus $(s)_{0}$ of a general global section $s$ of the homogeneous vector bundle $\mathcal{V}=\bigwedge^{2} \mathcal{E} \oplus \bigwedge^{2} \mathcal{E} \oplus \bigwedge^{3} \mathcal{F}$ on the Grassmannian $G(7,3)$. Since $\mathcal{V}$ is generated by global sections, $S$ is smooth by [6, Theorem 1.10], the Bertini type theorem for vector bundles. Since $r(\mathcal{V})=3+3+4=\operatorname{dim} G(7,3)-2$ and

$$
\operatorname{det} \mathcal{V} \simeq \mathcal{O}_{G}(2) \otimes \mathcal{O}_{G}(2) \otimes \mathcal{O}_{G}(3) \simeq \operatorname{det} T_{G(7,3)}
$$

$S$ is of dimension two and the canonical line bundle is trivial. By (a) of Proposition 5, S is connected and regular. Hence $S$ is a K3 surface. We denote the class of hyperplane section by $h$. Then, by (b) of

Proposition 5, we have $\chi\left(\mathcal{O}_{S}(h)\right)=14$, which implies $\left(h^{2}\right)=24$ by the Riemann-Roch theorem. Hence we obtain the rational map

$$
\Psi: \mathbf{P}_{*} H^{0}(G(7,3), \mathcal{V}) \cdots \rightarrow \mathcal{F}_{13}^{\prime} \quad s \mapsto\left((s)_{0}, h\right)
$$

to the moduli space $\mathcal{F}_{13}^{\prime}$ of polarized K3 surfaces which are not necessarily primitive.

By (g) of Proposition 5, the vector bundle $E=\left.\mathcal{E}\right|_{S}$ is simple. Let $\left(S^{\prime}, h^{\prime}\right)$ be a small deformation of $(S, h)$. Then there is a vector bundle $E^{\prime}$ on $S^{\prime}$ which is a deformation of $E$ by Proposition 4.1 of [6]. $E^{\prime}$ enjoys many properties satisfied by $E: E^{\prime}$ is simple, generated by global sections, $h^{0}\left(E^{\prime}\right)=7, \bigwedge^{3} H^{0}\left(E^{\prime}\right) \longrightarrow H^{0}\left(\bigwedge^{3} E^{\prime}\right)$ is surjective, etc. Therefore, $E^{\prime}$ embeds $S^{\prime}$ into $G(7,3)$ and $S^{\prime}$ is also a complete intersection with respect to $\mathcal{V}$. Hence the rational map $\Psi$ is dominant onto an irreducible component of $\mathcal{F}_{13}^{\prime}$ and Theorem 1 follows from the following:

Proposition 6. The polarization $h$ of $(S, h)$, a complete intersection with respect to $\mathcal{V}$ in $G(7,3)$, is primitive.

In the local deformation space of $(S, h)$, the deformations $\left(S^{\prime}, h^{\prime}\right)$ 's with Picard number one form a dense subset. More precisely, it is the complement of an infinite but countable union of divisors. Hence we have

Lemma 7. There exists a smooth complete intersection $S$ with respect to $\mathcal{V}$ whose Picard number is equal to one.

Proof of Proposition 6. Since the assertion is topological it suffices to show it for one such $(S, h)$. We take $(S, h)$ as in this lemma. Assume that $h$ is not primitive. Since $\left(h^{2}\right)=24, h$ is linearly equivalent to $2 l$ for a divisor class $l$ with $\left(l^{2}\right)=6$. The Picard group Pic $S$ is generated by $l$. By the Riemann-Roch theorem and the (Kodaira) vanishing, we have $h^{0}\left(\mathcal{O}_{S}(n l)\right)=3 n^{2}+2$ for $n \geq 1$.

Claim 1. $h^{0}(E(-l))=0$.
Assume the contrary. Then $E$ contains a subsheaf isomorphic to $\mathcal{O}_{S}(n l)$ with $n \geq 1$. Since $h^{0}\left(\mathcal{O}_{S}(n l)\right) \leq h^{0}(E)=7$, we have $n=1$ and the quotient sheaf $Q=E / \mathcal{O}_{S}(l)$ is torsion free. Since $5=h^{0}\left(\mathcal{O}_{S}(l)\right)<$ $h^{0}(E)=7$, we have $H^{0}(Q) \neq 0$. Since $Q$ is of rank two and $\operatorname{det} Q \simeq$ $\mathcal{O}_{S}(l)$, we have $\operatorname{Hom}\left(Q, \mathcal{O}_{S}(l)\right) \neq 0$, which contradicts (g) of Proposition 5.

Now we consider the vector bundle $M=\left(\bigwedge^{2} E\right)(-l)$. By the claim and the Serre duality, we have $h^{2}(M)=\operatorname{dim} \operatorname{Hom}\left(M, \mathcal{O}_{S}\right)=$ $h^{0}(E(-l))=0$. Hence we have $h^{0}(M) \geq \chi(M)=4$. Take 4 linearly
independent global sections of $M$ and we consider the homomorphism $\varphi: 4 \mathcal{O}_{S} \longrightarrow M$.

Claim 2. $\varphi$ is surjective outside a finite set of points on $S$.
Let $r$ be the rank of the image of $\varphi$. Since $\operatorname{Hom}\left(\mathcal{O}_{S}(l), M\right)=$ $\left.H^{0}\left(\bigwedge^{2} E\right)(-h)\right)=H^{0}\left(E^{\vee}\right)=H^{2}(E)^{\vee}=0$ by (c) of Proposition 5, we have $r \geq 2$. Since $\operatorname{Hom}\left(M, \mathcal{O}_{S}\right)=0, r=2$ is impossible. Hence we have $r=3$. Since the image and $M$ have the same determinant line bundle $\left(\simeq \mathcal{O}_{S}(l)\right)$, the cokernel of $\varphi$ is supported by a finite set of points.

The kernel of $\varphi$ is a line bundle by the claim. It is isomorphic to $\mathcal{O}_{S}(-l)$. Hence we have the exact sequence

$$
0 \longrightarrow \mathcal{O}_{S}(-l) \longrightarrow 4 \mathcal{O}_{S} \xrightarrow{\varphi} M .
$$

Since $\chi(\operatorname{Coker} \varphi)=3<\chi(M), \varphi$ is not surjective. In fact, the cokernel is a skyscraper sheaf supported at a point. Tensoring $\mathcal{O}_{S}(l)$, we have the exact sequence

$$
0 \longrightarrow \mathcal{O}_{S} \longrightarrow 4 \mathcal{O}_{S}(l) \xrightarrow{\varphi(l)} \bigwedge^{2} E \longrightarrow \mathbf{C}(p) \longrightarrow 0
$$

$H^{0}(\varphi(l))$ is surjective since $h^{0}\left(4 \mathcal{O}_{S}(l)\right)=20$ and $h^{0}\left(\bigwedge^{2} E\right)=19$. But this contradicts (e) of Proposition 5.
Q.E.D.

Proof of Theorem 2. Let $P=\left\langle\sigma_{1}, \sigma_{2}\right\rangle$ be a general 2-dimensional subspace of $\Lambda^{2} \mathbf{C}^{7}$ and $X^{6} \subset G(7,3)$ the common zero locus of the two global sections of $\bigwedge^{2} \mathcal{E}$ corresponding to $\sigma_{1}$ and $\sigma_{2}$. A point $q$ of $\mathbf{P}_{*}\left(\bigwedge^{3} \mathbf{C}^{7, \vee} / \overline{P \wedge P}\right)$ determines a global section of $\left.\bigwedge^{3} \mathcal{F}\right|_{X}$. We denote the zero locus by $S_{q} \subset X^{6}$.


The restriction of $\mathcal{E}$ to $S_{q}$ is semi-rigid by (g) of Proposition 5. Let $\Xi^{31} \subset \mathbf{P}_{*}\left(\bigwedge^{3} \mathbf{C}^{7, \vee} / \overline{P \wedge P}\right)$ be the open subset consisting of points $q$ such that $S_{q}$ is a K3 surface and the restriction $\left.\mathcal{E}\right|_{S_{q}}$ is stable with respect to $h$.

Lemma 8. $\Xi^{31}$ is not empty.
Proof. Let $(S, h)$ be as in Lemma 7 and put $E=\left.\mathcal{E}\right|_{S}$. Then, by Proposition 6, $\operatorname{Pic} S$ is generated by $h$. Since $h^{0}\left(\mathcal{O}_{S}(h)\right)=14>h^{0}(E)=$ 7, we have $\operatorname{Hom}\left(\mathcal{O}_{S}(n h), E\right)=0$ for every integer $n \geq 1 / 3$. Since
$c_{1}(E)=h$ and since $\operatorname{Hom}\left(E, \mathcal{O}_{S}(n h)\right)=0$ for every integer $n \leq 1 / 3, E$ is stable.
Q.E.D.

The correspondence $\left.q \mapsto \mathcal{E}\right|_{S_{q}}$ induces a morphism from a general fiber of $\Xi^{31} / G \cdots \rightarrow \mathcal{F}_{13}$ at $\left[S_{q}\right]$ to the moduli space $M_{S}(3, h, 4)$ of semirigid bundles. Conversely there exists a morphism from a non-empty open subset of $M_{S}(3, h, 4)$ to the fiber since a small deformation $E^{\prime}$ of $\left.\mathcal{E}\right|_{S_{q}}$ gives an embedding of $S_{q}$ into $G(7,3)$ such that the image is a complete intersection with respect to $\mathcal{V}$.

Remark 3. By (f) of Proposition $5, H^{0}\left(X^{6},\left.\bigwedge^{3} \mathcal{F}\right|_{X}\right)$ is isomorphic to $\wedge^{3} \mathbf{C}^{7, \vee} / \overline{P \wedge P}$. Hence the rational map $\psi$ in (1) coincides with $\mathbf{P}_{*}\left(H^{0}\left(X^{6},\left.\bigwedge^{3} \mathcal{F}\right|_{X}\right)\right) / G \cdots \rightarrow \mathcal{F}_{13}$ induced by $s \mapsto(s)_{0}$.

## $\S 3$. K3 surface of genus seven and twelve

We describe two cases $g=7$ and 12 closely related with Theorems 1 and 2. The proofs are quite similar to the cases $g=13$ and 18, respectively, and we omit them.

First a polarized K3 surface of genus 7 has the following description other than that in the orthogonal Grassmannian $O-G(5,10)$ :

Theorem 9. A general polarized $K 3$ surface $(S, h)$ of genus 7 is a complete intersection with respect to the rank four homogeneous vector bundle $2 \mathcal{O}_{G}(1) \oplus \mathcal{E}(1)$ in the 6 -dimensional Grassmannian $G(5,2)$.
$S$ is the common zero locus of two hyperplane sections $H_{1}$ and $H_{2}$ of $G(5,2) \subset \mathbf{P}^{9}$ corresponding to $\sigma_{1}, \sigma_{2} \in \bigwedge^{2} \mathbf{C}^{5}$ and one global section $s$ of $\mathcal{E}(1)$. The 2-dimensional subspace $P=\left\langle\sigma_{1}, \sigma_{2}\right\rangle \subset \bigwedge^{2} \mathbf{C}^{5}$ is uniquely determined by $S$ and $X^{4}=G(5,2) \cap H_{1} \cap H_{2}$ is a quintic del Pezzo fourfold. Let $Q$ be the image of $\mathbf{C}^{5} \otimes P$ by the natural linear map $\mathbf{C}^{7} \otimes \bigwedge^{2} \mathbf{C}^{7} \longrightarrow H^{0}(\mathcal{E}(1))$. Then $Q$ is of dimension 10 and we obtain the natural rational map

$$
\begin{equation*}
\mathbf{P}_{*}\left(H^{0}(\mathcal{E}(1)) / Q\right) / G^{8}=\mathbf{P}_{*}\left(H^{0}\left(\left.\mathcal{E}(1)\right|_{X}\right)\right) / G^{8} \cdots \rightarrow \mathcal{F}_{7} \tag{6}
\end{equation*}
$$

as in the case $g=13$, where $G^{8}$ is the general stabilizer group of the action $P G L(5) \curvearrowright G\left(2, \bigwedge^{2} \mathbf{C}^{5}\right) . H^{0}(\mathcal{E}(1))$ is a 40-dimensional irreducible representation of $G L(5)$ by Theorem 3. The fiber of the map (6) at general $(S, h)$ is a surface and birationally equivalent to the moduli K3 surface $M_{S}(2, h, 3)$ of semi-rigid rank two vector bundles with $c_{1}=h$ and $\chi=2+3$.

Secondly, in the 12-dimensional Grassmannian $G(7,3)$, there is another type of K3 complete intersection other than Theorem 1.

Theorem 10. A general member $(S, h) \in \mathcal{F}_{12}$ is a complete intersection with respect to $\mathcal{V}_{10}=3 \bigwedge^{2} \mathcal{E} \oplus \mathcal{O}_{G}(1)$ in $G(7,3)$.
$S$ is the common zero locus of the three global sections of $\bigwedge^{2} \mathcal{E}$ corresponding to general bivectors $\sigma_{1}, \sigma_{2}, \sigma_{3} \in \Lambda^{2} \mathbf{C}^{7}$. The 3-dimensional subspace $N=\left\langle\sigma_{1}, \sigma_{2}, \sigma_{3}\right\rangle \subset \bigwedge^{2} \mathbf{C}^{7}$ is uniquely determined by $S$. The common zero locus $X_{N}$ of the global sections of $\bigwedge^{2} \mathcal{E}$ corresponding to $N$ is a Fano threefold and is embedded into $\mathbf{P}^{13}$ anti-canonically. $X_{N}$ 's are parameterized by an open set $\Xi^{6}$ of the orbit space $G\left(3, \bigwedge^{2} \mathbf{C}^{7}\right) / P G L(7)$. See [5] for other descriptions of $X_{N}$ 's and their moduli spaces. The moduli space $\mathcal{F}_{12}$ is birationally equivalent to a $\mathbf{P}^{13}$-bundle over this $\Xi^{6}$.

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# Rigid geometry and applications 

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#### Abstract

. In this paper we present a survey of rigid geometry. Here, special emphasis is put on the so-called "birational approach" to rigid geometry, which adopts classical methods of birational geometry to the theory of rigid spaces. The paper is divided into three parts. Part I is a general introduction to rigid geometry a la J. Tate and M. Raynaud. In Part II we are to overview the birational approach to rigid geometry, which combines the idea of Raynaud and that of O. Zariski, as one of the conceptual starting points of rigid geometry. In Part III we discuss some applications, which reveal the effectiveness of the ideas in rigid geometry that arise from our viewpoint.


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## Part I. Classical rigid analytic geometry

## §1. What is rigid geometry?

### 1.1. Introduction

It is well-known that the field $\mathbb{Q}$ of rational numbers admits for any prime number $p$ a so-called $p$-adic norm $|\cdot|_{p}$, and they together with the usual absolute value norm $|\cdot|_{\infty}$ constitute the complete list of nontrivial norms on $\mathbb{Q}$ up to equivalence. The completion of $\mathbb{Q}$ by the usual absolute value $|\cdot|_{\infty}$ yields the field $\mathbb{R}$ of real numbers, and its algebraic closure $\mathbb{C}$, the field of complex numbers. These complete fields are at the bases of real and complex analytic geometries. As the absolute value norm is merely one of infinitely many possible norms on $\mathbb{Q}$, it is only natural to imagine a similar realm of analytic geometries arising from $p$-adic norms. The completion of $\mathbb{Q}$ by the $p$-adic norm $|\cdot|_{p}$ is the field $\mathbb{Q}_{p}$ of $p$-adic numbers, and the $p$-adic counterpart of the field $\mathbb{C}$ of complex numbers, denoted by $\mathbb{C}_{p}$, is the completion of the algebraic closure of $\mathbb{Q}_{p}$. Note that it is not simply the algebraic closure $\overline{\mathbb{Q}}_{p}$, since that turns out not to be complete with respect to the unique extension of the $p$ adic norm. Assuming the existence of analytic geometry based on the complete fields $\mathbb{Q}_{p}$ and $\mathbb{C}_{p}$ corresponding to real and complex analysis, one would, thus, finally arrive at the diagram starting from $\mathbb{Q}$ as in Figure 1. The vacant slot in the diagram is actually occupied by rigid


Fig. 1. Dichotomy between real-complex world and $p$-adic world
geometry, ${ }^{1}$ which provides a systematic theory for analytic geometry over complete non-archimedean valued fields, not only $\mathbb{Q}_{p}$ and $\mathbb{C}_{p}$.

Table 1 shows points of similarity between the fields $\mathbb{C}$ and $\mathbb{C}_{p}$, which are considered to be important in the genesis of analytic geometry. As

Table 1. $\mathbb{C}$ vs $\mathbb{C}_{p}$

| $\mathbb{C}$ | $\mathbb{C}_{p}$ |
| :---: | :---: |
| Algebraically closed | Algebraically closed |
| Complete with respect <br> to absolute value $\|\cdot\|_{\infty}$ | Complete with respect <br> to $p$-adic norm $\|\cdot\|_{p}$ |

the table shows, $\mathbb{C}_{p}$ is algebraically closed ${ }^{2}$ and complete. By completeness one can speak of convergent power series and functions expressed by them, which are, as in complex analysis, the fundamental things to consider also in rigid analytic geometry.

### 1.2. Why analytic geometry?

But, already having nice analytic geometry on the real-complex side, why do we need to consider analytic geometry also on the $p$-adic side? It turns out that the reason mainly comes from number-theoretic considerations. This is best explained in the context of uniformization, which is one of the useful techniques that reveal already in complex analytic geometry the true value of analytic methods.

Let us first briefly recall complex analytic uniformization of elliptic curves:

- regarded as a compact Riemann surface, an elliptic curve over $\mathbb{C}$ is realized as a quotient $\mathbb{C} / \Lambda$, where $\Lambda$ is a lattice in $\mathbb{C}$ of the form $\Lambda=2 \pi \sqrt{-1}(\mathbb{Z}+\mathbb{Z} \cdot \tau)$ for $\tau \in \mathbb{H}=\{z \in \mathbb{C} \mid \operatorname{Im} z>0\}$;
- another way of analytic representation is provided by the quotient $\mathbb{C}^{\times} \rightarrow \mathbb{C}^{\times} / q^{\mathbb{Z}}=\mathbb{C} / \Lambda$, where $q=\exp (2 \pi \sqrt{-1} \tau)$, which factorizes the previously mentioned quotient map $\mathbb{C} \rightarrow \mathbb{C} / \Lambda$ through the exponential mapping $\exp (\cdot): \mathbb{C} \rightarrow \mathbb{C}^{\times}$.
Whereas rigid analytic geometry over $\mathbb{C}_{p}$ fails to have an analogue of the first uniformization, it actually affords that of the second, the so-called Tate's uniformization, given by a quotient of the form $\mathbb{C}_{p}^{\times} \rightarrow$

[^4]$\mathbb{C}_{p}^{\times} / q^{\mathbb{Z}}$ with $q \in \mathbb{C}_{q}^{\times},|q|_{p}<1$, of which Tate was able to give an analytic description [41]. In fact, we have the Weierstrass $\wp$-function on $\mathbb{C}_{p}^{\times}$ defined by the usual (but transcribed by the coordinate change $w=$ $e^{2 \pi \sqrt{-1} z}$ ) formula, which induces the following commutative diagram

where the dashed arrow embeds $\mathbb{C}_{p}^{\times} / q^{\mathbb{Z}}$ in $\mathbb{P}^{2}\left(\mathbb{C}_{p}\right)$ onto a cubic curve. The analytic curve $\mathbb{C}_{p}^{\times} / q^{\mathbb{Z}}$ thus obtained is called a Tate curve.

Remark 1.1. Contrary to the complex case, not all elliptic curves can be realized as Tate curves. It is known that an elliptic curve $E$ over $\mathbb{C}_{p}$ is realized as a Tate curve if and only if $|j(E)|_{p}>1$, where $j(E)$ denotes the $j$-invariant of $E$; note that the last condition is equivalent to $E$ having multiplicative reduction.

Now we return to our first question: why do we need analytic geometry on the p-adic side? Consider an elliptic curve $E$ over $\mathbb{Q}$. The complex analytic method tells us that the Riemann surface $E(\mathbb{C})$ is a complex torus, and gives us several useful analytical and topological properties. On the $p$-adic side, on the other hand, assuming that there exists a prime $p$ at which $E$ has multiplicative reduction, we know that $E\left(\mathbb{Q}_{p}\right)$ is written in the form $\mathbb{Q}_{p}^{\times} / q^{\mathbb{Z}}$ (by the $\mathbb{Q}_{p}$-rational version of Tate's uniformization). This representation of $E$ allows one to have a good grasp on rational points on $E$; for example, one is able to show at a glance that the torsion part of $E(\mathbb{Q})$ is a finite group (Nagell-Lutz Theorem; this is, however not the way they proved it).

One can therefore expect in general that for an algebraic variety $X$ over a number field, rigid analytic geometry reveals number-theoretic information hidden behind $X$, and thus compensates for properties that complex analytic geometry fails to capture. This is the reason why rigid analytic geometry is useful.

The Tate curve $\mathbb{C}_{p}^{\times} / q^{\mathbb{Z}}$ is our first example of a rigid analytic space, which will appear again and again in the sequel. Also for Tate, this curve was actually the starting point that led him to discover rigid analytic geometry. In the next section, we will overview Tate's theory of rigid analytic geometry [40], and will see at the end (in Example 2.15) how the above picture is justified.

## §2. Tate's rigid analytic geometry

### 2.1. Non-archimedean valued fields

The above-mentioned normed fields $\left(\mathbb{Q}_{p},|\cdot|_{p}\right)$ and $\left(\mathbb{C}_{p},|\cdot|_{p}\right)$ are examples of so-called complete non-archimedean valued fields with nontrivial valuation, which are one of the basic cornerstones of Tate's rigid analytic geometry.

By a non-archimedean valued field we mean a pair $(K,|\cdot|)$ consisting of a field $K$ and a non-archimedean norm $|\cdot|$, that is, a mapping $|\cdot|: K \rightarrow$ $\mathbb{R}_{\geq 0}$ such that
(1) $|x|=0 \Longleftrightarrow x=0$;
(2) $|x y|=|x||y|$;
(3) $|x+y| \leq \max \{|x|,|y|\}$,
for any $x, y \in K$. The norm $|\cdot|$ is said to be non-trivial if $\left|K^{\times}\right| \neq\{1\}$. Finally, we need to assume that $(K,|\cdot|)$ is complete, that is, $K$ is complete with respect to the norm $|\cdot|$.

Example 2.1. Let $V$ be a complete discrete valuation ring, and $K$ its field of fractions. As usual, the field $K$ comes with a discrete valuation $v: K \rightarrow \mathbb{Z} \cup\{\infty\}$, which induces the corresponding norm $|\cdot|_{v}: K \rightarrow \mathbb{R}_{\geq 0}$ by the formula $|x|_{v}=e^{-v(x)}$ for any $x \in K$, where $e$ is a real number with $e>1$. Then the pair $\left(K,|\cdot|_{v}\right)$ is a complete non-archimedean valued field with non-trivial valuation.

The $p$-adic number field $\left(\mathbb{Q}_{p},|\cdot|_{p}\right)$ together with the $p$-adic norm is an example of this kind. Another such example is provided by $\left(k((x)),|\cdot|_{x}\right)$, where $k((x))$ is the fractional field of $V=k[[x]]$, the ring of formal power series over a field $k$ endowed with the $x$-adic valuation. These are examples of complete discrete valuation fields, which the reader is invited to always bear in mind.

Similarly to the construction of $\mathbb{C}_{p}$, the completion $\mathbb{C}_{K}$ of the algebraic closure of the valuation field $K$ from Example 2.1 is algebraically closed, and the resulting pair $\left(\mathbb{C}_{K},|\cdot|_{v}\right)$ provides another example of a complete non-archimedean valued field with non-trivial valuation.

Notice that one can perform a similar construction as in Example 2.1 starting from a valuation ring $V$ of height 1 (but not necessarily discrete), that is, the fractional field $K$ has a valuation of the form $v: K \rightarrow \mathbb{R} \cup\{\infty\} .{ }^{3}$

In the sequel of this section, $K$ denotes a complete non-archimedean valued fields with non-trivial valuation.

[^5]
### 2.2. Basic idea

Tate modeled his rigid analytic geometry on the geometry of schemes in the sense that his rigid analytic spaces are constructed by gluing certain "affine" objects. As such objects are defined, like affine schemes, as certain spectra of rings of some kind, one can say that Tate's rigid analytic geometry belongs to the general trend of understanding spaces as spectra of rings, the historical origin of which can be traced back to Gelfand. A consequence of this is the seemingly strange-looking fact that Tate's rigid analytic geometry is better understood in analogy with classical algebraic geometry over a field $k$ than with complex analytic geometry.

Table 2. Comparison between algebraic geometry and rigid geometry (the italic-written items are explained in the text.)

|  | Algebraic geometry $/ k$ | Rigid geometry $/ K$ |
| :---: | :---: | :---: |
| Function <br> algebra | Finitely generated <br> algebra $A / k$ | Topologically finitely <br> generated algebra $A / K$ <br> (called: affinoid algebra) |
| Points <br> (Naive) | Maximal ideals of $A$ <br> (with Zariski topology) | Maximal ideals of $A$ <br> (with admissible <br> topology) |
| Building <br> block | Affine variety <br> (Specm $A, \mathscr{O}_{X}$ ) | Affinoid <br> (Spm $A, \mathscr{O}_{X}$ ) |

The rings that rigid analytic geometry deals with, which in algebraic geometry correspond to finitely generated algebras over $k$, are the socalled affinoid algebras, ${ }^{4}$ which are by definition topologically finitely generated algebras over $K$ (cf. Definition 2.4). Similarly to algebraic geometry, Tate's rigid analytic geometry takes the maximal ideals of $A$ as the spectrum. As a counterpart of Zariski topology, we have the socalled admissible topology, which is, however, not a topology in the naive sense, but is actually a Grothendieck topology. ${ }^{5}$ Finally, the maximal

[^6]spectrum $\operatorname{Spm} A$ together with a suitably defined structure sheaf with respect to the admissible topology provides the basic building block of general analytic spaces in a similar way that varieties in algebraic geometry are constructed by gluing affine varieties. The building block thus obtained is called an affinoid.

Remark 2.2. Notwithstanding the perfect looking comparison with algebraic geometry, there is in fact no a priori reason in rigid geometry why one should take maximal ideals as points, and one could even say that here lies a serious problem of Tate's approach. In fact, Tate's rigid analytic spaces in general are severely deficient in points, and it is for this reason that one has to use Grothendieck topology as the natural topology to think about. This mismatching of points and topology leads to several problems: for instance, points of Tate's rigid analytic spaces are not enough to detect abelian sheaves with respect to the admissible topology.

As a matter of fact, there are many more approaches to rigid geometry, including ours (which will be explained later), and one of the most important differences between these approaches lies in what to choose as points. Namely, the notion of points in rigid analytic geometry depends entirely on the way one approaches it. Thus one can say that it is only due to Tate's way of approaching rigid geometry that one takes maximal ideals as points. This means, in other words, that another choice of points would avoid Grothendieck topology. We will see that this is in fact the case. ${ }^{6}$

### 2.3. Affinoid algebras

The most important example of affinoid algebras, which plays the role of polynomial rings in algebraic geometry, is the so-called Tate algebra.

Definition 2.3 (Tate algebra).

$$
\begin{aligned}
& K\left\langle\left\langle T_{1}, \ldots, T_{n}\right\rangle\right\rangle \\
& \quad=\left\{\begin{array}{r|l}
\sum_{\nu_{1}, \ldots, \nu_{n} \geq 0} a_{\nu_{1}, \ldots, \nu_{n}} T_{1}^{\nu_{1}} \cdots T_{n}^{\nu_{n}} & \left|a_{\nu_{1}, \ldots, \nu_{n}}\right| \rightarrow 0 \text { as } \\
\in K\left[\left[T_{1}, \ldots, T_{n}\right]\right] & \nu_{1}+\cdots+\nu_{n} \rightarrow \infty
\end{array}\right\} .
\end{aligned}
$$

of this paper to show that it is by no means essential to use Grothendieck topologies in developing rigid geometry. See Remark 2.2.
${ }^{6}$ Here we would like to stress that, nevertheless, it is not our intension to defy Tate's approach; each approach has its own advantage and drawback. Rather, we believe that a good attitude is to have various approaches at one's disposal and to feel free in choosing one of them depending on the situation.

The similarity with the polynomial ring comes from the fact that the Tate algebra $K\left\langle\left\langle T_{1}, \ldots, T_{n}\right\rangle\right\rangle$ is the $K$-algebra consisting of power series converging absolutely and uniformly on the closed unit polydisk $\left\{\left(z_{1}, \ldots, z_{n}\right) \in K^{n}| | z_{i} \mid \leq 1\right.$ for $\left.1 \leq i \leq n\right\}$ in $K^{n} .{ }^{7}$ Assume for simplicity that $K$ is algebraically closed. Then the set of all maximal ideals of $K\left\langle\left\langle T_{1}, \ldots, T_{n}\right\rangle\right\rangle$ coincides with the closed unit polydisk (this follows from the weak Nullstellensatz for affinoid algebras stated below). The corresponding affinoid is, therefore, underlain by this set. Table 3 shows the dictionary for comparison between the polynomial ring and the Tate algebra.

Table 3. Polynomial ring vs Tate algebra

| Algebraic geometry $/ k=\bar{k}$ | Rigid geometry $/ K=\bar{K}$ |
| :---: | :---: |
| $k\left[X_{1}, \ldots, X_{n}\right]$ | $K\left\langle\left\langle X_{1}, \ldots, X_{n}\right\rangle\right\rangle$ |
| $k^{n}$ | $\left(z_{1}, \ldots, z_{n}\right) \in K^{n}$ |
| with $\left\|z_{i}\right\| \leq 1$ |  |

Basic properties. Here we list some basic properties of the Tate algebra; one finds more in [5, Chap. 5]:

- it is a $K$-Banach algebra endowed with the so-called Gauss norm:

$$
\left\|\sum_{\nu_{1}, \ldots, \nu_{n} \geq 0} a_{\nu_{1}, \ldots, \nu_{n}} T_{1}^{\nu_{1}} \cdots T_{n}^{\nu_{n}}\right\|=\sup _{\nu_{1}, \ldots, \nu_{n} \geq 0}\left|a_{\nu_{1}, \ldots, \nu_{n}}\right|
$$

- it is Noetherian, and every ideal is closed with respect to the topology induced by the Gauss norm.

Definition 2.4 (Affinoid algebra). An affinoid algebra is a $K$-algebra of the form

$$
A=K\left\langle\left\langle T_{1}, \ldots, T_{n}\right\rangle\right\rangle / I
$$

for some $n$, where $I$ is an ideal. This is a $K$-Banach algebra by the norm induced from the Gauss norm.

[^7]Among several basic properties of affinoid algebras, we mention the analogue of Noether's normalization theorem ([5, 6.1.2]):

- (Noether's normalization theorem for affinoid algebras) for any affinoid algebra $A$ over $K$ there exists a finite injective $K$ algebra homomorphism

$$
K\left\langle\left\langle T_{1}, \ldots, T_{d}\right\rangle\right\rangle \longrightarrow A
$$

for some $d \geq 0$.
By this we have the following property, which implies the functoriality of taking the maximal spectrum:

- (Weak Nullstellensatz for affinoid algebras) for any maximal ideal $\mathfrak{m}$ of $A$, the residue field $A / \mathfrak{m}$ is a finite extension of $K$.


### 2.4. Wobbly topology

For an affinoid algebra $A$ we set $\operatorname{Spm} A$ to be the set of all maximal ideals of $A$. For any $K$-algebra homomorphism $A \rightarrow B$ between affinoid algebras $^{8}$ we have an induced mapping $\operatorname{Spm} B \rightarrow \operatorname{Spm} A$. As usual, any element $f$ of $A$ is regarded as a function on the set $\operatorname{Spm} A$; since for any $x \in \operatorname{Spm} A$ the residue field at $x$ is a finite extension of $K$ and thus admits a unique extension of the norm $|\cdot|$, one can put $|f(x)|=|f \bmod x|$. For any $f, g \in A$ we set

$$
R(f, g)=\{x \in \operatorname{Spm} A| | f(x)|\leq|g(x)|\}
$$

As a subset of $\operatorname{Spm} A$, we have

$$
R(f, g)=\operatorname{Spm} A\langle\langle X\rangle\rangle /(g X-f),
$$

where $A\langle\langle X\rangle\rangle$ denotes the ring $A \widehat{\otimes}_{K} K\langle\langle X\rangle\rangle$. The ring $A\langle\langle X\rangle\rangle /(g X-f)$, which is again an affinoid algebra, is often abbreviated as $A\left\langle\left\langle\frac{f}{g}\right\rangle\right\rangle$.

Definition 2.5 (Wobbly topology). The wobbly topology on the set $\operatorname{Spm} A$ is the topology having $\{R(f, g)\}_{f, g \in A}$ as open basis.

Example 2.6. Suppose for simplicity that $K$ is algebraically closed, and consider $\mathbb{D}_{K}^{n}=\operatorname{Spm} K\left\langle\left\langle X_{1}, \ldots, X_{n}\right\rangle\right\rangle$, which is identified as a set with the closed unit polydisk in $K^{n}$. Then one sees easily that the wobbly topology on $\mathbb{D}_{K}^{n}$ coincides with the topology induced from the metric topology on $K^{n}$ (which is, as is well-known, totally disconnected).

[^8]Difficulties. As Example 2.6 indicates, the wobbly topology is not such a good topology; for example:

- $\operatorname{Spm} A$ with the wobbly topology is, in most cases, not quasicompact, which would be troublesome when one considers gluing;
- the presheaf $R(f, g) \mapsto A\left\langle\left\langle\frac{f}{g}\right\rangle\right\rangle$, which comes as the most natural candidate for the structure sheaf on $\operatorname{Spm} A$, is in general not a sheaf.
These difficulties come from the fact that the wobbly topology is somewhat too fine. Indeed, considering the sheafification of the above presheaf, we would get a ring of functions on $\operatorname{Spm} A$ that is much larger than $A$ itself, which contradicts our basic requirement that $A$ should be the ring of all "holomorphic" functions on $\operatorname{Spm} A$. In other words, the wobbly topology leads to a very feeble notion of analytic functions. Hence, to obtain a reasonable theory of analysis, one has to "rigidify" the notion of analytic functions, ${ }^{9}$ and, to this end, one wants to replace the wobbly topology with a more legitimate one.


### 2.5. Admissible topology

In 1961 Tate [40] overcame the above-mentioned difficulties by introducing the so-called admissible topology. The admissible topology is, in short, a Grothendieck topology that is

- weaker than the wobbly topology,
- the strongest one that makes each $R(f, g)$ quasi-compact.

The actual definition is given as follows.
Definition 2.7 (Admissible site). Let $\mathfrak{A}_{K}$ be the category of affinoid algebras over $K$ and $K$-algebra homomorphisms. For any object $A$ of $\mathfrak{A}_{K}$, we denote by $\operatorname{Spm} A$ the same object considered as an object of the opposite category $\overline{\mathfrak{A}_{K}^{\mathrm{opp}}}$. We define a Grothendieck topology on $\mathfrak{A}_{K}^{\text {opp }}$ as follows: a finite collection $\left\{\underline{\operatorname{Spm}} A_{i} \rightarrow \underline{\operatorname{Spm}} A\right\}_{i \in I}$ of morphisms in $\mathfrak{A}_{K}^{\mathrm{opp}}$ is a covering of $\operatorname{Spm} A$ if and only if
(1) each $A_{i}$ is étale over $A$ (see, for example, [17, §8.1] for the definition of étaleness);
(2) $\operatorname{Spm} A_{i} \rightarrow \operatorname{Spm} A$ is injective for each $i$ and induces an isomorphism between the residue fields at each point of $\operatorname{Spm} A_{i}$;
(3) $\operatorname{Spm} A=\bigcup_{i \in I} \operatorname{Spm} A_{i}$.

We denote the resulting site by $\mathfrak{A}_{K, \text { ad }}^{\mathrm{opp}}$.

[^9]Here is a typical example of coverings in the admissible site. Let $A$ be an affinoid algebra over $K$, and $f_{0}, \ldots, f_{n} \in A$ elements of $A$ such that $\left(f_{0}, \ldots, f_{n}\right)=A$. Set

$$
A_{i}=A\left\langle\left\langle X_{0}, \ldots, \widehat{X}_{i}, \ldots, X_{n}\right\rangle\right\rangle /\left(f_{i} X_{j}-f_{j} \mid j \neq i\right)
$$

Then the collection $\left\{\underline{\operatorname{Spm}} A_{i} \rightarrow \underline{\operatorname{Spm}} A\right\}_{0 \leq i \leq n}$ is a covering in the site $\mathfrak{A}_{K, \text { ad }}^{\text {opp }}$; for each $i$, the image of $\overline{\operatorname{Spm}} A_{i} \rightarrow \operatorname{Spm} A$ is given as $\{x \in$ $\operatorname{Spm} A\left|\left|f_{i}(x)\right| \geq\left|f_{j}(x)\right|\right.$ for $\left.j \neq i\right\}$. Let us denote this covering by $\mathscr{R}\left(f_{0}, \ldots, f_{n}\right)$.

Theorem 2.8 (Gerritzen-Grauert). Any covering family $\left\{\operatorname{Spm} A_{i} \rightarrow\right.$ $\underline{\operatorname{Spm}} A\}_{i \in I}$ in the site $\mathfrak{A}_{K, \text { ad }}^{\mathrm{opp}}$ has a refinement to a covering of the form $\overline{\mathscr{R}}\left(f_{0}, \ldots, f_{n}\right)$ for some $f_{0}, \ldots, f_{n} \in A$.

Another version of the Gerritzen-Grauert theorem will be stated in Corollary 6.21 below.

Affinoids and general rigid spaces. For an affinoid algebra $A$ over $K$, consider the presheaf $\mathscr{O}_{\mathrm{Spm} A}$ on the comma site $\left(\mathfrak{A}_{K, \mathrm{ad}}^{\mathrm{opp}}\right)_{\underline{\operatorname{Spm}} A}$ defined by

$$
\mathscr{O}_{\mathrm{Spm} A}:(\underline{\operatorname{Spm}} B \rightarrow \underline{\operatorname{Spm}} A) \mapsto B .
$$

The following theorem says that the admissible topology defined above is the good one in the sense that it gives rise to the correct notion of "holomorphic" functions.

Theorem 2.9 (Tate's acyclicity theorem). The presheaf $\mathscr{O}_{\operatorname{Spm} A}$ is a sheaf on $\left(\mathfrak{A}_{K, a d}^{\mathrm{opp}}\right)_{\underline{\operatorname{Spm}} A}$ with respect to the admissible topology.

Definition 2.10 (Tate's rigid analytic space). (1) A representable sheaf on the site $\mathfrak{A}_{K, \text { ad }}^{\mathrm{opp}}$ is called an affinoid.
(2) A map $\mathscr{Y} \hookrightarrow \mathscr{X}$ between affinoids is said to be an open immersion if, identified with a morphism in the category $\mathfrak{A}_{K}^{\text {opp }}$, it satisfies the conditions (1) and (2) in Definition 2.7.
(3) A sheaf $\mathscr{X}$ of sets on the site $\mathfrak{A}_{K, \text { ad }}^{\mathrm{opp}}$ is called a (Tate's) rigid analytic space if there exists a surjective map of sheaves

$$
\coprod_{i \in I} \mathscr{Y}_{i} \longrightarrow \mathscr{X}
$$

where $\mathscr{Y}_{i}$ for each $i \in I$ is an affinoid, such that, for each $i, j \in I$, the projection $\mathscr{Y}_{i} \times \mathscr{X} \mathscr{Y}_{j} \rightarrow \mathscr{Y}_{i}$ is isomorphic to the limit of a filtered direct system $\left\{\mathscr{U}_{\lambda} \rightarrow \mathscr{Y}_{i}\right\}_{\lambda \in \Lambda}$ of maps between affinoids such that all maps in
the commutative diagram for $\mu \leq \lambda$

are open immersions.
In other words, Tate's rigid analytic spaces are constructed by gluing affinoids. As the definition indicates, it allows non-separated or non-quasi-separated gluing.

Remark 2.11. In Tate's original approach, rigid analytic spaces are regarded as local ringed spaces with Grothendieck topology. ${ }^{10}$ For example, an affinoid is such a space isomorphic to the one given by the data $\left(\operatorname{Spm} A, \mathscr{T}_{A}, \mathscr{O}_{\operatorname{Spm} A}\right)$ consisting of the set $\operatorname{Spm} A$, the Grothendieck topology $\mathscr{T}_{A}$ (equivalent to the admissible topology in our sense), and the sheaf of rings (essentially the same as the one that we have given above). General rigid analytic spaces are obtained by gluing these spaces with respect to what is called the strong topology. This viewpoint of rigid analytic geometry is surely useful. But one has to be careful, since, as we have already seen in Remark 2.2, the point set $\operatorname{Spm} A$ is not the correct "underlying set" for the affinoid $\underline{\operatorname{Spm}} A$.

In the sequel, for brevity and conformity with the usual notation, we denote the affinoid $\underline{\mathrm{Spm}} A$ simply by $\operatorname{Spm} A$.

### 2.6. Examples

Example 2.12 (Annulus). An annulus is an affinoid that is, if $K$ is algebraically closed, supported on the set

$$
\{z \in K||a| \leq|z| \leq|b|\}
$$

with $a, b \in K$. The corresponding affinoid algebra is given by

$$
K\left\langle\left\langle\frac{a}{z}, \frac{z}{b}\right\rangle\right\rangle=K\langle\langle X, Y\rangle\rangle /\left(X Y-\frac{a}{b}\right) .
$$

Note that, since it is an affinoid, it is quasi-compact.
In general, a rigid analytic space is said to be quasi-compact if it has an admissible covering consisting of finitely many affinoids.

[^10]Example 2.13 (Affine line). An affine line $\mathbb{A}_{K}^{1, \text { an }}$ in rigid analytic geometry is realized as, for example, the limit of concentric closed disks, each of which is an affinoid:

$$
\mathbb{A}_{K}^{1, \text { an }}=\underset{n \geq 1}{\lim } \operatorname{Spm} K\left\langle\left\langle a^{n} z\right\rangle\right\rangle,
$$

where $a$ is an element of $K$ with $|a|<1$. This of course reflects the equality

$$
K=\bigcup_{n \geq 1} \mathbb{D}\left(0,|a|^{-n}\right)
$$

where $\mathbb{D}(0, r)=\{z \in K| | z \mid \leq r\}$. The affine line $\mathbb{A}_{K}^{1, \text { an }}$ is not quasicompact.

Example 2.14 (Multiplicative group). Let $a \in K$ be as above. The multiplicative group $K^{\times}$is regarded as the union of countably many annuli:

$$
\begin{aligned}
K^{\times} & =\bigcup_{n \geq 1}\left\{\left.z \in K| | a\right|^{n} \leq|z| \leq|a|^{-n}\right\} \\
& =\bigcup_{n \in \mathbb{Z}}\left\{\left.z \in K| | a\right|^{n+1} \leq|z| \leq|a|^{n}\right\}
\end{aligned}
$$

Taking up, for example, the latter description, one defines

$$
\mathbb{G}_{m, K}^{\mathrm{an}}=\bigcup_{n \in \mathbb{Z}} \operatorname{Spm} K\left\langle\left\langle\frac{a^{n+1}}{z}, \frac{z}{a^{n}}\right\rangle\right\rangle
$$

This is again a rigid analytic space that is not quasi-compact.
Example 2.15 (Tate curve). The last description of the rigid analytic multiplicative group $\mathbb{G}_{m, K}^{a n}$ allows one to display the analytic structure of the Tate curve discussed in §1.2. For $q \in K$ with $|q|<1$, the Tate curve is given by $\mathbb{G}_{m, K}^{\mathrm{an}} / q^{\mathbb{Z}}$. In order to describe an analytic covering, take $a \in K$ such that $|a|^{k}=|q|$ for some $k \geq 2$ and the analytic covering $\mathbb{G}_{m, K}^{\text {an }}=\bigcup_{n \in \mathbb{Z}} A_{n}$ considered in Example 2.14, where $A_{n}=\operatorname{Spm} K\left\langle\left\langle\frac{a^{n+1}}{z}, \frac{z}{a^{n}}\right\rangle\right\rangle$. Multiplication by $q$ maps each $A_{n}$ isomorphically onto $A_{n+k}$. Thus, $\mathbb{G}_{m, K}^{\text {an }} / q^{\mathbb{Z}}$ is written as the union of $k$ annuli, glued together by identifying the "exterior" boundary component with the "interior" boundary component of another one. In particular, it is a quasi-compact rigid analytic space.

As mentioned at the end of $\S 1$, one of Tate's goals in formulating rigid analytic geometry was to give a legitimate way of regarding $\mathbb{C}_{p}^{\times} / q^{\mathbb{Z}}$ as an "analytification" of an elliptic curve. This was done in the last example.

## §3. Raynaud's approach to rigid geometry

### 3.1. Formal models of affinoids

The moral basis of the $p$-adic counterpart of real-complex analytic geometry, leading to the saga of Tate's theory of rigid analytic geometry, was, as we have seen in $\S 1.1$, the similarity between the complex number field $\mathbb{C}$ and its $p$-adic counterpart $\mathbb{C}_{p}$, as listed in Table 1 . Now we change our view to the completely opposite direction, and rather pay attention to differences between $\mathbb{C}$ and $\mathbb{C}_{p}$. The most important difference is that $\mathbb{C}_{p}$ has, while $\mathbb{C}$ does not, the subring consisting of integral elements, that is, elements of norm $\leq 1$ (Table 4). Similarly, any affinoid algebras,

Table 4. $\mathbb{C}$ vs $\mathbb{C}_{p}$ (continued)

| $\mathbb{C}$ | $\mathbb{C}_{p}$ |
| :---: | :---: |
| $\nexists$ integer ring | $\exists$ integer ring |

unlike function algebras in real-complex analysis, have a "model" over the integer ring. This observation, however simple it might look, is the starting point of Raynaud's approach to rigid analytic geometry, which, as we will see, leads to a bold shift of viewpoint.

Situation. In the sequel of this section we work in the following situation:

- $V$ is a valuation ring of height 1 that is complete with respect to the $a$-adic topology for an element $a$ belonging to the maximal ideal $\mathfrak{m}_{V}$;
- we set $K=\operatorname{Frac}(V)$ (the field of fractions), which has the $a$ adic norm $|\cdot|$ and is complete with respect to the metric topology induced from this norm.
Note that a valuation ring $V$ of arbitrary height is $a$-adically separated if and only if $V\left[\frac{1}{a}\right]$ is a field (and hence coincides with $\operatorname{Frac}(V)$ ).

Example 3.1. The typical example is provided by a complete discrete valuation ring $V$ with $\pi$-adic topology, where $\pi$ is a generator of the maximal ideal (uniformizer). The corresponding norm $|\cdot|$ on the fractional field $K$ coincides with the one as in Example 2.1 up to equivalence.

In this situation, for any topologically finitely generated $V$-algebra $A$, we obtain an affinoid algebra

$$
A_{K}=A \otimes_{V} K
$$

over $K$. Here, a $V$-algebra $A$ is said to be topologically finitely generated if it is a quotient by an ideal of an algebra of the form $V\left\langle\left\langle X_{1}, \ldots, X_{n}\right\rangle\right\rangle$, the $a$-adic completion of the polynomial ring $V\left[X_{1}, \ldots, X_{n}\right]$.

In general, let $\mathscr{A}$ be an affinoid algebra over $K$. A formal model of $\mathscr{A}$ is a topologically finitely generated $V$-algebra $A$ such that $A_{K}=A \otimes_{V} K$ is isomorphic to $\mathscr{A}$ as a $K$-algebra. If, in addition, $A$ is flat over $V$, or what amounts to the same, $A$ is $a$-torsion free, then we say that $A$ is a distinguished (or flat) formal model of $\mathscr{A}$. For example, the algebra $V\left\langle\left\langle X_{1}, \ldots, X_{n}\right\rangle\right\rangle$ is a distinguished formal model of the Tate algebra $K\left\langle\left\langle X_{1}, \ldots, X_{n}\right\rangle\right\rangle$. Any affinoid algebra has a distinguished formal model; indeed, if it is given as $K\left\langle\left\langle X_{1}, \ldots, X_{n}\right\rangle\right\rangle / I$ with $I=\left(f_{1}, \ldots, f_{r}\right)$ finitely generated (recall that the Tate algebra is Noetherian), by multiplying each $f_{i}$ with a power of $a$, one can assume $f_{i} \in V\left\langle\left\langle X_{1}, \ldots, X_{n}\right\rangle\right\rangle$ and then $A / A_{a \text {-tor }}$, where $A=V\left\langle\left\langle X_{1}, \ldots, X_{n}\right\rangle\right\rangle /\left(f_{1}, \ldots, f_{r}\right)$, gives a desired formal model.

Remark 3.2. It is known that, if a topologically finitely generated $V$ algebra $A$ is flat, then it is actually topologically finitely presented ([6]). As it is also known that any finitely generated ideal of $V\left\langle\left\langle X_{1}, \ldots, X_{n}\right\rangle\right\rangle$ is closed with respect to the $a$-adic topology (i.e. the Artin-Rees lemma is valid for finitely generated ideals; cf. [18]), any topologically finitely generated flat $V$-algebra is complete with respect to the $a$-adic topology.

### 3.2. Raynaud's functor

Let $X=\operatorname{Spf} A$ be an affine flat formal scheme of finite type over Spf $V$, where $V$ is considered with some $a$-adic topology. Then, as we have seen, $A_{K}=A \otimes_{V} K$ is an affinoid algebra, and thus we can consider the corresponding affinoid $X_{K}=\operatorname{Spm} A_{K}$ over $K$. This correspondence $X \mapsto X_{K}$ is globalized in the following way.

Consider the localization $A_{\{f\}}$ by an element $f \in A$, that is, the $a$-adic completion of $A_{f}$. Note that we have ${ }^{11}$

$$
A_{\{f\}}=A\langle\langle X\rangle\rangle /(f X-1)
$$

Therefore, the corresponding affinoid $\operatorname{Spm}\left(A_{\{f\}}\right)_{K}$ is nothing but $R(1, f)$ (cf. §2.4), an admissible open subset of $\operatorname{Spm} A_{K}$ with respect to the admissible topology. Hence, by patching, one obtains a functor

$$
X \mapsto X_{K}
$$

[^11]from the category of coherent (= quasi-compact and quasi-separated) flat formal schemes of finite type over $V$ to the category of Tate's rigid spaces over $K$ ([37]). This functor is called the Raynaud functor, and the rigid space $X_{K}$ associated to $X$ is the Raynaud generic fiber of $X$. Put in the other way, when a rigid space $\mathscr{X}$ is isomorphic to $X_{K}$ for a flat formal scheme $X$ as above, we say that $X$ is a (distinguished) formal model of $\mathscr{X}$.

Note that, by the definition of the functor, if $X$ has an affine covering $X=\bigcup_{i \in I} U_{i}$, then we get an admissible covering $X_{K}=\bigcup_{i \in I} U_{i, K}$. Let us call this covering of $X_{K}$ the admissible covering induced from the affine covering $\left\{U_{i}\right\}_{i \in I}$ of $X$.

Example 3.3. Let $V$ be as in Example 3.1, and consider a semistable curve $E \rightarrow \operatorname{Spec} V$ such that the generic fiber $E_{\eta}$ is an elliptic curve over $K$, and that the closed fiber $E_{0}$ is the union of non-singular rational curves arranged as type $\mathrm{I}_{k}$ in Kodaira's classification. Consider the formal completion $\widehat{E}$ along the closed fiber $E_{0}$. It admits the affine covering $\widehat{E}=\bigcup_{n=1}^{k} U_{n}$, where each $U_{n}=\operatorname{Spf} V\left\langle\left\langle\frac{\pi^{n+1}}{z}, \frac{z}{\pi^{n}}\right\rangle\right\rangle$ is isomorphic to $\operatorname{Spf} V\langle\langle X, Y\rangle\rangle /(X Y-\pi)$. The corresponding rigid space $\mathscr{E}=\widehat{E}_{K}$ is the Tate curve, which has the induced admissible covering $U_{n, K}=$ $\operatorname{Spm} K\left\langle\left\langle\frac{\pi^{n+1}}{z}, \frac{z}{\pi^{n}}\right\rangle\right\rangle$. Note that this rigid analytic space $\mathscr{E}$, as well as the admissible covering, is nothing but the one we have already described in Example 2.15.

### 3.3. Zariski topology vs admissible topology

Needless to say, there may be many choices of formal models for a given rigid space, and this diversity of choice is, in fact, reflected in diversity of choice of admissible coverings of the rigid space. To see this, let us first establish a typical change of formal models.

Admissible blow-up. Let $X$ be a formal scheme of finite type over $V$. An admissible ideal is a quasi-coherent open ideal $\mathscr{J}$ of $\mathscr{O}_{X}$ of finite type. For an admissible ideal $\mathscr{J}$, the admissible blow-up along $\mathscr{J}$ is the morphism of formal schemes

$$
X^{\prime}={\underset{k \geq 0}{l} \lim }_{\vec{l}} \operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathscr{J}^{n} \otimes \mathscr{O}_{X_{k}}\right) \longrightarrow X
$$

where $X_{k}=\left(X, \mathscr{O}_{X} / a^{k+1} \mathscr{O}_{X}\right)$. If, for instance, $X=\operatorname{Spf} A$ is affine, then $\mathscr{J}$ is of the form ${ }^{12} J^{\Delta}$ for a uniquely determined finitely generated

[^12]ideal $J$ of $A$ that contains a power of $a$, and the admissible blow-up $X^{\prime} \rightarrow X$ is nothing but the $a$-adic completion of the usual blow-up $Y^{\prime}=\operatorname{Proj} \bigoplus_{n \geq 0} J^{n} \rightarrow Y=\operatorname{Spec} A$.

Example 3.4. Consider $X=\operatorname{Spf} V\langle\langle z\rangle\rangle$. The corresponding rigid space $X_{K}$ is the "closed unit disk" $\mathbb{D}_{K}^{1}=\operatorname{Spm} K\langle\langle z\rangle\rangle$. Consider the admissible ideal $J=(X, a)$ of $V\langle\langle z\rangle\rangle$. The admissible blow-up $X^{\prime}$ along $J$ is the union of two affine subsets $U=\operatorname{Spf} V\left\langle\left\langle\frac{z}{a}\right\rangle\right\rangle$ and $W=$ $\operatorname{Spf} V\left\langle\left\langle z, \frac{a}{z}\right\rangle\right\rangle=\operatorname{Spf} V\langle\langle z, w\rangle\rangle /(z w-a)$. The resulting rigid space $X_{K}^{\prime}$ is therefore covered by two admissible open subsets $U_{K}=\operatorname{Spm} K\left\langle\left\langle\frac{z}{a}\right\rangle\right\rangle$ and $W_{K}=\operatorname{Spm} K\left\langle\left\langle z, \frac{a}{z}\right\rangle\right\rangle ; U_{K}$ is again a closed disk but having a different radius equal to $|a|$, and $W_{K}$ is a closed annulus " $\{z \in K||a| \leq|z| \leq 1\}$ ". Thus the rigid space $X_{K}^{\prime}$ is isomorphic to $X_{K}$. The difference is that, while $X_{K}$ was considered as rigid space by the trivial covering (the covering by itself), $X_{K}^{\prime}$ has the non-trivial induced covering $\left\{U_{K}, W_{K}\right\}$.

As indicated in Example 3.4, whereas an admissible blow-up does not change the Raynaud generic fiber, viz. for a coherent (= quasicompact and quasi-separated) flat formal $V$-scheme of finite type $X$ and an admissible blow-up $X^{\prime} \rightarrow X$ we have $X_{K}^{\prime}=X_{K}$, it replaces the admissible covering by a refinement. Raynaud's very important insight is that this fact is the key point for comparing admissible topology and Zariski topology.

Consider, for example, the affine case $X=\operatorname{Spf} A$, and let $U$ be a quasi-compact open subset of an admissible blow-up $X^{\prime}$ of $X$ :


Then we have the open immersion

$$
U_{K} \longleftrightarrow \operatorname{Spm} A_{K}
$$

identifying $U_{K}$ with a quasi-compact open subset of $\operatorname{Spm} A_{K}$ with respect to the admissible topology. Due to the Gerritzen-Grauert theorem (Theorem 2.8), the open subsets of the form $U_{K}$ constructed as above constitute an open basis for the admissible topology. Thus one can recover the admissible topology on $X_{K}$ from the Zariski topology of formal models. The important fact is that, in order to recover the admissible topology, one has to vary the formal model.

### 3.4. Raynaud's viewpoint

The important point of the above observation is that it explains the admissible topology entirely in terms of formal models. Based on this,
we can now illustrate Raynaud's viewpoint of rigid analytic geometry; this is itemized as follows:

- from this viewpoint, rigid analytic geometry in totality is induced from a geometry of "models" (Figure 2);
- as the geometry of models, Raynaud suggests geometry of formal schemes over valuation rings.


Fig. 2. Raynaud's viewpoint

For instance, if $K$ is the fractional field of a complete discrete valuation ring $V$, then theorems in rigid analytic geometry over $K$ should follow from theorems in formal geometry over $V$, already stated in [EGA, III, §4, §5].

In practice, this program goes along the following thread. Starting from a coherent formal scheme $X$ of finite type over $V$, we obtain the rigid analytic space $\mathscr{X}=X_{K}$ over $K$, whose topology, points, and structure sheaf are characterized as follows.

- Topology: a quasi-compact admissible open subset of $\mathscr{X}$ is of the form $\mathscr{U}=U_{K}$ where $U$ is a quasi-compact open subset of an admissible blow-up $X^{\prime}$ of $X$;
- Points:

$$
\begin{aligned}
\mathscr{X}(K) & =\{\text { sections Spf } V \rightarrow X\} \\
\mathscr{U}(K) & =\{\text { sections that factors through } U\}
\end{aligned}
$$

- Structure sheaf: when $U=\operatorname{Spf} A$, then $\Gamma\left(\mathscr{U}, \mathscr{O}_{\mathscr{X}}\right)=A_{K}$.

This viewpoint culminates in the following theorem.
Theorem 3.5 (Raynaud 1972 [37]). The Raynaud functor $X \mapsto X_{K}$ gives rise to the categorical equivalence

$$
\left\{\begin{array}{l}
\text { Coherent formal } \\
\text { schemes of finite } \\
\text { type over } V
\end{array}\right\} /\left\{\begin{array}{l}
\text { Admissible } \\
\text { blow-ups }
\end{array}\right\} \stackrel{\sim}{\sim}\left\{\begin{array}{l}
\text { Coherent rigid } \\
\text { analytic spaces of } \\
\text { finite type over } K
\end{array}\right\} .
$$

Here the left-hand category is the quotient category, that is, the category consisting of the same objects as the category of coherent formal schemes of finite type over $V$ but of arrows with all admissible blow-ups inverted.

Remark 3.6. (1) The objects in the right-hand category are defined a priori by "patching affinoids" (cf. Definition 2.10). The equivalence shows that this patching turns out to be equivalent to, so to speak, "birational patching" (birational up to admissible blow-ups). This invokes a birational viewpoint in rigid geometry, which will play a very important role in our approach (to be explained) to rigid geometry.
(2) Let us briefly mention something about the proof of Theorem 3.5. There are two important ingredients:

- existence of formal birational patching,
- comparison of topologies.

The last point was already mentioned in connection with the GerritzenGrauert Theorem.

### 3.5. Significance of Raynaud's viewpoint

Perhaps the most significant aspect of Raynaud's viewpoint (and Raynaud's theorem) lies in the shift from "analysis" to "geometry". To be more precise, whereas Tate's rigid analytic geometry is motivated by "analysis" over non-archimedean fields, Raynaud's approach starts totally differently, namely from formal "geometry," and is developed entirely as a geometric theory with seemingly no flavor of analysis. Consequently, contrary to Tate's rigid analytic geometry, which aims at something similar to complex analytic geometry, Raynaud's approach forces one to think that rigid geometry is entirely not similar to complex analytic geometry.

## §4. Our approach: brief announcement

In the next part, we are to exhibit our approach to rigid geometry, which is different both from Tate's and Raynaud's approaches. Our General Policy is the following.

General Policy: rigid geometry is a hybrid of formal geometry and birational geometry.

There is little doubt that our approach has been largely influenced by Raynaud's approach. But, nevertheless, it differs much from Raynaud's in how to deal with birational geometry, on which our approach puts much more stress. For the general treatment of birational geometry, we
will take up Zariski's classical idea that deals with the so-called ZariskiRiemann spaces as its foremost objects. Schematically shown, our approach is an "amalgam" of Raynaud's approach and Zariski's classical approach to birational geometry (Figure 3).


Fig. 3. Our approach

For this reason, we will start the next part of this paper with a brief recap of birational geometry from Zariski's classical viewpoint.

## Part II. Birational approach to rigid geometry

Part II consists of two sections. In $\S 5$ we describe some birational geometry in the spirit of Zariski's classical viewpoint. What we do in this section is a preparation for the next section, $\S 6$, where we will outline our approach to rigid geometry.

## §5. Birational geometry from Zariski's viewpoint

### 5.1. Basic Question: Extension problem

Throughout this section we work in the following situation:

- $S$ : a coherent scheme,
- $\mathscr{I}=\mathscr{I}_{D}$ : a quasi-coherent ideal sheaf of finite type such that $U=S \backslash D$ is a dense open subset of $S$, where $D=V(\mathscr{I})$.

Here a scheme is said to be coherent if it is quasi-compact and quasiseparated. ${ }^{13}$ Note that, as the ideal sheaf $\mathscr{I}_{D}$ is of finite type, the open subset $U$ is quasi-compact.

The basic problem we are concerned with is of the following type.

[^13]

Fig. 4. Situation for the extension problem

Problem 5.1 (Extension problem). Let $P$ be a property of morphisms (e.g. $P=$ "flat"). Let $f_{U}: X_{U} \rightarrow U$ be a morphism of schemes of finite presentation with the property $P$. Suppose there exists at least one morphism $f: X \rightarrow S$ such that $f \times{ }_{S} U=f_{U}$. Then, can one find such an $f$ that satisfies the property $P$ ?

This problem may have a trivial solution; for instance, if $P=$ "flat", then $f=j \circ f_{U}$, where $j: U \rightarrow S$ is the open immersion, gives a solution. Such a solution is, needless to say, not the one we want to have. We like to find a "good" solution. However, if we like to clarify what "good" means, we find that the problem itself is not well-posed (or, say, not reasonable). For instance, if, trying to make the problem well-posed, we put $P=$ "proper and flat", then a moment thought immediately gives a negative answer in practically important cases (e.g. family of curves over a surface $S$ with $D$ a normal crossing divisor), and hence we find that the problem in this case is not reasonable.

### 5.2. Admissible modifications and modified extension problem

In order to make the extension problem more reasonable, one needs to allow birational changes of $S$ that preserve the dense open part $U$. Thus we are naturally led to the following notion.

Definition 5.2 ( $U$-admissible modification). (1) A $U$-admissible modification of $S$ is a diagram

such that the vertical arrow is proper and the other arrows are open immersions onto dense open subsets (hence the vertical arrow is birational).
(2) A morphism between two $U$-admissible modifications $S^{\prime} \rightarrow S$ and $S^{\prime \prime} \rightarrow S$ is an $S$-morphism $S^{\prime} \rightarrow S^{\prime \prime}$.
$U$-admissible modifications constitute the category $\mathbf{M D}_{(S, U)}$, which is cofiltered; indeed, for two $U$-admissible modifications $S^{\prime} \rightarrow S$ and $S^{\prime \prime} \rightarrow S$ one constructs the diagram in $\mathbf{M D}_{(S, U)}$

where $S^{\prime \prime \prime}$ is the closure of the image of the diagonal mapping $U \hookrightarrow$ $S^{\prime} \times{ }_{S} S^{\prime \prime} .{ }^{14}$

The following special class of $U$-admissible modifications will be of particular importance.

Definition 5.3 ( $U$-admissible blow-up). A $U$-admissible blow-up of $S$ is a blow-up $S^{\prime} \rightarrow S$ whose center is given by a quasi-coherent ideal $\mathscr{J}$ of $\mathscr{O}_{S}$ of finite type such that the corresponding closed subscheme $V(\mathscr{J})$ is set-theoretically contained in $D$, or what amounts to the same, there exists a positive integer $n$ such that $\mathscr{I}_{D}^{n} \subseteq \mathscr{J}$.

Here is an example: when $S=\operatorname{Spec} A$ is affine, and $D=V(I)$, then a $U$-admissible blow-up is given by

$$
S^{\prime}=\operatorname{Proj} \bigoplus_{n \geq 0} J^{n} \rightarrow S
$$

where $J$ is a finitely generated ideal of $A$ that contains $I^{k}$ for some $k>0$.
We denote by $\mathbf{B L}_{(S, U)}$ the full subcategory of $\mathbf{M D}(S, U)$ consisting of $U$-admissible blow-ups. To state the modified extension problem, we need yet one more concept.

Definition 5.4 (Strict transform). Let $S^{\prime} \rightarrow S$ be a $U$-admissible modification, and $f: X \rightarrow S$ an $S$-scheme. The strict transform $f^{\prime}: X^{\prime} \rightarrow$ $S^{\prime}$ of $f$ is the $S^{\prime}$-scheme defined by the commutative diagram

where the map $X^{\prime} \hookrightarrow X_{S^{\prime}}$ is the closed immersion given by dividing out $\mathscr{I}_{D}$-torsion.

[^14]Having these notions on birational changes of schemes, we can now state the desired "modified" version of our basic question that we are going to consider.

Problem 5.5 (Modified extension problem). Let $f_{U}: X_{U} \rightarrow U$ be a morphism of finite presentation that satisfies the property P. Suppose an extension $f: X \rightarrow S$ of $f_{U}$ on $S$, that is, a morphism such that $f \times{ }_{S} U=f_{U}$, is given. Then, can one find a $U$-admissible modification (resp. blow-up) $S^{\prime} \rightarrow S$ such that the strict transform $f^{\prime}: X^{\prime} \rightarrow S^{\prime}$ of $f$ satisfies $P$ ?

### 5.3. Flattening theorem

Problem 5.5 in the case $P=$ "flat" is the so-called flattening problem, and was affirmatively solved by Raynaud and Gruson [38].

Theorem 5.6 (Raynaud-Gruson 1970 [38]). Let $f: X \rightarrow S$ be a morphism of finite presentation such that $f \times_{S} U: X \times_{S} U \rightarrow U$ is flat. Then there exists a $U$-admissible blow-up $S^{\prime} \rightarrow S$ such that the strict transform $f^{\prime}: X^{\prime} \rightarrow S^{\prime}$ is flat of finite presentation.

Among many valuable corollaries of this theorem, we refer to the following one.

Corollary 5.7 ([38, (5.7.12)]). The full subcategory $\mathbf{B L}_{(S, U)}$ is cofinal in the category $\mathbf{M D}_{(S, U)}$.

Remark 5.8. Here a few remarks on the flattening theorem are in order.
(1) The theorem is entirely clear in case $S=\operatorname{Spec} V$ where $V$ is a discrete valuation ring. Indeed, in this case, one can take as $S^{\prime} \rightarrow S$ the identity map $S^{\prime}=S$, and thus the strict transform $X^{\prime}$ is the closed subscheme of $X$ given by dividing out $V$-torsions.
(2) More generally, if $S=\operatorname{Spec} V$ where $V$ is a (not necessarily discrete) valuation ring, then flatness of a similarly defined $X^{\prime}$ is clear by the same reasoning, whereas the finite presentation of $f^{\prime}$ is rather difficult to show.

### 5.4. Revival of Zariski's idea

In the rest of this section we are going to outline the proof of Theorem 5.6. The proof that we are going to present here is not the one in [38], but is done by Zariski's classical idea, which Zariski invented in order to apply it to the resolution of singularities of algebraic surfaces [44]. The keystone of Zariski's argument is the so-called Zariski-Riemann space, and the most crucial point of the proof is its quasi-compactness.

Definition 5.9 (Zariski-Riemann space; cf. [44][45]).

$$
\langle U\rangle_{\mathrm{cpt}}=\lim _{S^{\prime} \in \overleftarrow{\mathrm{BL}}_{(S, U)}} S^{\prime}
$$

where the projective limit is taken in the category of local ringed spaces.
Let us say that an ideal $\mathscr{J}$ of $\mathscr{O}_{S}$ is admissible if it is quasi-coherent of finite type and the corresponding closed subscheme $V(\mathscr{J})$ is settheoretically contained in $D$. Then $U$-admissible blow-ups are exactly the morphisms of the form $\operatorname{Proj} \bigoplus_{n>0} \mathscr{J}^{n} \rightarrow S$ by an admissible ideal $\mathscr{J}$. Hence the projective limit in Definition 5.9 is regarded as the filtered projective limit taken along the directed set of all admissible ideals with the ordering $\leq$ defined as follows: $\mathscr{J} \geq \mathscr{J}^{\prime}$ if and only if there exists an admissible ideal $\mathscr{J}^{\prime \prime}$ such that $\mathscr{J}=\mathscr{J}^{\prime} \mathscr{J}^{\prime \prime}$. This justifies Definition 5.9 , for the category of local ringed spaces is closed under filtered projective limits. Note that the Zariski-Riemann space thus defined generalizes the so-called abstract Riemann surface, the introduction of which traces back to Dedekind-Weber in the 19th century, for if $S$ is a regular curve then we have $\langle U\rangle_{\mathrm{cpt}}=S$.

Points. Let $x \in\langle U\rangle_{\mathrm{cpt}}$. The point $x$ is, by definition, a compatible system of points $\left\{x_{S^{\prime}}\right\}_{S^{\prime} \in \mathbf{B L}_{(S, U)}}$ with $x_{S^{\prime}} \in S^{\prime}$ for any $S^{\prime} \in \mathbf{B L}_{(S, U)}$.

- The topological space $\langle U\rangle_{\text {cpt }}$ contains $U$. If $x \in U$, then the corresponding points $x_{S^{\prime}}$ lie in the common $U$, and all of them are equal.
- If on the other hand $x \notin U$, then the system $\left\{x_{S^{\prime}}\right\}$ is described in terms of a valuation ring ${ }^{15}$ as follows: there exists a valuation ring $V_{x}$ (of height $\geq 1$ ) and a map $\alpha: \operatorname{Spec} V_{x} \rightarrow S$ of schemes mapping the closed point to $x_{S}$ and the generic point to a point in $U$. For any $U$-admissible blow-up $S^{\prime} \rightarrow S$, by the valuative criterion of properness, one has a unique arrow $\alpha^{\prime}: \operatorname{Spec} V_{x} \rightarrow$ $S^{\prime}$ such that the resulting triangle

commutes. The point $x_{S^{\prime}}$ is the image of the closed point by $\alpha^{\prime}$.

[^15]Local rings. The local rings of the structure sheaf $\mathscr{O}_{\langle U\rangle_{\mathrm{cpt}}}$ are best described by the following notion.

Definition 5.10. Let $A$ be a ring, and $I$ a finitely generated ideal. The ring $A$ is said to be $I$-valuative if any finitely generated ideal $J$ of $A$ that contains $I^{k}$ for some $k>0$ (called an $I$-admissible ideal) is invertible.

In case $A$ is a local ring, then $A$ is $I$-valuative if and only if $I$ is a principal ideal $I=(a)$ generated by a non-zero-divisor $a \in A$ and every $I$-admissible ideal is principal.

Proposition 5.11. (1) Let $A$ be a local ring, and $I=($ a) a principal ideal generated by a non-zero-divisor $a \in A$. Set $J=\bigcap_{n \geq 1} I^{n}$. Suppose $A$ is I-valuative. Then:
(a) $B=A\left[\frac{1}{a}\right]$ is a local ring, and $V=A / J$ is a valuation ring, which is $\bar{a}$-adically separated, where $\bar{a}=(a \bmod J)$;
(b) $A=\left\{f \in B \mid\left(f \bmod \mathfrak{m}_{B}\right) \in V\right\}$, where $\mathfrak{m}_{B}$ is the maximal ideal of $B$;
(c) $J=\mathfrak{m}_{B}$.
(2) Conversely, if $B$ is a local ring and $V$ is an $\bar{a}$-adically separated valuation ring for some non-zero $\bar{a} \in V$ such that the fractional field of $V$ coincides with the residue field of $B$, then the subring $A=\left\{f \in B \mid\left(f \bmod \mathfrak{m}_{B}\right) \in V\right\}$ is an I-valuative local ring for any finitely generated ideal $I$ such that $I V=(\bar{a})$, and $B=A\left[\frac{1}{a}\right]$.

Proposition 5.11 shows that an $I$-valuative local ring is a "composite" of a local ring and a valuation ring. The following proposition follows from basic properties of $U$-admissible blow-ups, and is easy to verify.

Proposition 5.12. For any point $x \in\langle U\rangle_{\mathrm{cpt}}$ the local ring $\mathscr{O}_{\langle U\rangle_{\mathrm{cpt}}, x}$ is an $\left(\mathscr{I}_{D} \mathscr{O}_{\langle U\rangle_{\mathrm{cpt}}, x}\right)$-valuative ring.

For $A_{x}=\mathscr{O}_{\langle U\rangle_{\mathrm{cpt}}, x}$, we set $B_{x}=A_{x}\left[\frac{1}{a}\right]$ (where $I_{x}=\mathscr{I}_{D} \mathscr{O}_{\langle U\rangle_{\mathrm{cpt}}, x}=$ (a)) and $V_{x}=A_{x} / J_{x}$, where $J_{x}=\bigcap_{n \geq 1} I_{x}^{n}$. The local ring $B_{x}$ is a local ring on $U$, and the valuation ring $V_{x}$ is the one that describes the point $x=\left\{x_{S^{\prime}}\right\}_{S^{\prime} \in \mathbf{B L}_{(S, U)}}$ as above. In other words, each local ring of $\langle U\rangle_{\mathrm{cpt}}$ is a "composite" of a valution ring and a local ring of $U$.

Note that the above description of points and the local rings is essentially due to Zariski's original description of Zariski-Riemann space, which Zariski originally introduced not by projective limit of varieties, but as a certain space of places.

Remark 5.13. Notice that the valuation rings $V_{x}$ that appear in the above context are not necessarily of height 1 even if the scheme $S$ is Noetherian (e.g. an algebraic variety over a field). This is exactly the reason why valuation rings of higher height need to be considered in Zariski's argument. See Table 5 for the classification given by Zariski [44] of possible valuations that appear on algebraic surfaces.

Table 5. Valuation rings on algebraic surfaces

| Height | Rational rank |  |
| :---: | :---: | :---: |
| 0 | 0 | trivial valuation |
| 1 | 1 | divisorial |
|  |  | non-divisorial |
|  | 2 | non-divisorial |
| 2 | 2 | composite of two divisorial valuations |

Intuitive description. Recall that the set of ideals of a valuation ring $V$ is totally ordered by the inclusion order. In particular, the spectrum Spec $V$ consists of points that are linearly configured as depicted in Figure 5. It can therefore be understood as a "long curve" ${ }^{16}$ with


Fig. 5. Spectrum of valuation ring
the extremities (0), the generic point, and $\mathfrak{m}_{V}$, the closed points. Each point is the specialization of points sitting on its left (in the figure), and the generalization of points sitting on its right. In the finite height case, the height is the number of points minus one.

It is, therefore, appropriate to say that (the image of) a map Spec $V \rightarrow$ $S$ of schemes from a valuation ring is a "long path" in $S$. Intuitively, from what we have seen above in the description of points, one can say

[^16]that the space $\langle U\rangle_{\mathrm{cpt}}$ is like a "path space." More precisely, we have a set-theoretical decomposition
$$
\langle U\rangle_{\mathrm{cpt}}=U \amalg T_{D / S}^{*},
$$
where $T_{D / S}^{*}$ is the set of all "long paths" that pass through $D$, or is, so to speak, an analogue of a tubular neighborhood ${ }^{17}$ of $D$ in $S$; see Figure 6.


Fig. 6. Set-theoretical description of $\langle U\rangle_{\mathrm{cpt}}$

### 5.5. Quasi-compactness

The space $\langle U\rangle_{\mathrm{cpt}}$, being defined as the projective limit of all $U$ admissible blow-ups, would seem fairly gigantic. The following theorem, which turns out to be ineffably important, says that it is actually not.

Theorem 5.14 (Zariski 1944). The space $\langle U\rangle_{\mathrm{cpt}}$ is quasi-compact.
This theorem played one of the most essential roles in Zariski's proofs of resolution of singularities on algebraic surfaces (cf. §5.7) and Abhyankar's proof for three-folds. Also in our proof of the flattening theorem, quite similarly, this plays a very important role. The proof of Theorem 5.14 is by no means technical, but rather, one can say, the quintessence lies in a general principle applicable to a much wider situation.

One way of proof relies on the fact that the (2-categorical) filtered projective limit of coherent topoi with coherent transition maps is again coherent [SGA4-2, Exposé VI], which confers with the well-known fact that the filtered projective limit of compact Hausdorff spaces is again compact Hausdorff. Applying Deligne's theorem on the existence of points for locally coherent topoi, one shows the theorem.

[^17]A more handy way is provided by Stone's representation theorem, which asserts that the category of coherent topological spaces ${ }^{18}$ and quasi-compact maps is categorically equivalent to the opposite category of unital distributive lattices (cf. [24]). As the latter category is closed under filtered direct limit, the theorem follows immediately (a minor point that should be confirmed here is that the direct limit taken in the category of topological spaces is equal to the one taken in the category of coherent topological spaces and quasi-compact maps).

In both proofs, the most important point is the following fact (existence of points): the filtered projective limit of non-empty coherent spaces with coherent transition maps is non-empty. An extensive use of this fact verifies the finite intersection property for open coverings, whence the quasi-compactness as desired. Notice that the above two ways of the proof are not entirely different from each other, and both arguments actually prove coherence, not only quasi-compactness.

### 5.6. Outline of the proof of Theorem 5.6

Now we can outline the proof of Theorem 5.6. The idea of the proof is the following.

Idea: reduction to the case of "long curves" Spec $V$ by means of quasi-compactness of Zariski-Riemann space.
This can be regarded as a "curve-cut" technique, which is quite often employed in algebraic geometry. In this sense, one can say that our approach is a quite geometric one.

First step. Observe first that the theorem is true for long curves $S=$ Spec $V$, where $V$ is a valuation ring. As we have mentioned in Remark 5.8, the flattening theorem in this case is not easy, whereas the "flattening part" (without finiteness property) is trivial. The proof of the finiteness part has a quite different flavor from the other part; first, using composition of valuation rings, we reduce to the case of height 1 , and then employ Gröbner basis arguments to show the finiteness. We omit the details here, and proceed to the general case, assuming the validity of the theorem in this case.

Second step. Observe next that the theorem is true for $S=\operatorname{Spec} A$, where $A$ is the local ring at a point of $\langle U\rangle_{\mathrm{cpt}}$. Here we use the fact that the ring $A$ is $I$-valuative, and the assertion follows from the previous

[^18]step, the assumption that $f$ is flat on $U$, and patching of flatness, where "patching" means composition in the sense of Proposition 5.11.

Third step. By the previous steps and the fact that the property $P=$ "flat" is locally finitely presented, one deduces that the assertion is true locally on $\langle U\rangle_{\mathrm{cpt}}$, that is, for any point $x \in\langle U\rangle_{\mathrm{cpt}}$, there exists an admissible blow-up $S^{\prime} \rightarrow S$ and a quasi-compact open subset $U_{x}$ of $S^{\prime}$ such that

- $U_{x}$ contains the image of $x$ by the projection $\langle U\rangle_{\mathrm{cpt}} \rightarrow S^{\prime}$;
- $\left.f^{\prime}\right|_{U_{x}}$ is flat and finitely presented, where $f^{\prime}: X^{\prime} \rightarrow S^{\prime}$ is the strict transform of $f$.
Here we have tacitly used the following extension of admissible ideals.
Proposition 5.15. Let $T \subset S$ be a quasi-compact open subset of $S$, and $\mathscr{J}$ an admissible ideal on $T$ (with respect to $V=U \cap T$ ). Then there exists an admissible ideal $\widetilde{\mathscr{J}}$ on $S$ such that $\left.\widetilde{\mathscr{J}}\right|_{T}=\mathscr{J}$.

Fourth step. Finally, by quasi-compactness (Theorem 5.14), the assertion follows by birational patching. More precisely, there exist finitely many points $x_{i}(i=1, \ldots, n)$ such that $\langle U\rangle_{\mathrm{cpt}}=\bigcup_{i=1}^{n} p_{i}^{-1}\left(U_{x_{i}}\right)$, where, for each $i, U_{x_{i}} \subset S_{i}$, and $p_{i}:\langle U\rangle_{\mathrm{cpt}} \rightarrow S_{i}$ is the projection map. Take $S^{\prime} \in \mathbf{B L}_{(S, U)}$ that dominates the $S_{i}$ 's. Replacing $S^{\prime}$ by the blow-up along $\mathscr{I}_{D} \mathscr{O}_{S^{\prime}}$, we may assume that $\mathscr{I}_{D} \mathscr{O}_{S^{\prime}}$ is an invertible ideal. Let $U_{i}^{\prime}$ be the pull-back of $U_{x_{i}}$ by the map $S^{\prime} \rightarrow S_{i}$. Then $S^{\prime}=\bigcup_{i=1}^{n} U_{i}^{\prime}$, and thus the strict transform $f^{\prime}: X^{\prime} \rightarrow S^{\prime}$ is flat and finitely presented.

### 5.7. Other applications

The argument of the above type, which uses the quasi-compactness of Zariski-Riemann spaces, was largely applicable to several other situations. Let us list some of them (which are, however, not new).

Resolution of singularities of quasi-excellent surfaces. This is the one to which Zariski originally applied this argument (in the case of algebraic sufaces). Similarly to the above-mentioned procedure, one first reduces the claim to the case of "long curves" to show that resolution can be done locally (local uniformization), and then patches the resulting local resolutions into a regular model by using quasi-compactness of Zariski-Riemann space. See, for example, [31, Chap. I] for more details.

Embedding theorem for algebraic spaces (cf. Nagata 1963). This asserts that a separated algebraic space of finite type over a coherent scheme can be embedded into a proper space. This was first proved by Nagata in 1963 for Noetherian separated schemes. Considering "long curves", one first observes that, locally, an appropriate embedding can be constructed (local extension lemma), and then birationally patches these locally extended data into a globally extended space, which is possible
because of the quasi-compactness of Zariski-Riemann space. Finally by the valuative criterion of properness, one shows that the resulting space is proper.

Remark 5.16. There are two remarks in order on the Nagata's embedding theorem.
(1) The theorem is true not only for schemes but also for algebraic spaces. There are several motivations for this generalization. One of them will be seen below (Theorem 6.11). Another motivation is that the embedding theorem potentially has large applications to the compactification of moduli spaces that are usually not representable by schemes; e.g. M. Rapoport's Habilitationschrift. We also remark that this generalized form of Nagata's embedding theorem has an application to trace formula; cf. Remark 8.3.
(2) One can actually simplify the proof of the embedding theorem by using ideas from rigid geometry. The details will be shown in [21].

## §6. Birational approach to rigid geometry

### 6.1. Introduction

Now we come to the stage of expounding our approach to rigid geometry. As we have briefly announced in $\S 4$ our general policy is that rigid geometry is a hybrid of formal geometry and birational geometry; here, in our approach to rigid geometry, we will see that Zariski's classical idea of birational geometry explained in $\S 5$ revives, and plays one of the most important roles. One can thus refine the picture from Figure 3 into the one from Figure 7.

As Raynaud's viewpoint of rigid geometry takes up geometry of formal schemes as the starting point, from which rigid geometry is supposed to arise in the way that birational changes by admissible blow-ups are inverted, the birational geometry on the right-hand side means, so to speak, birational geometry of formal schemes, which should be a theory of a formal analogue of Zariski-Riemann spaces. One arrives in this way at the following "central dogma," which realizes more concretely our general policy for approaching rigid geometry: ${ }^{19}$

Birational geometry of formal schemes $=$ Rigid geometry.
The analogue of Zariski-Riemann spaces in this context gives rise to the so-called Zariski-Riemann triples (Definition 6.15), which provide

[^19]

Fig. 7. Birational approach to rigid geometry
for each rigid space a topological space together with two sheaves of local rings, the integral structure sheaf and the rigid structure sheaf. We think of these objects as the most basic figure in rigid geometry in which rigid analytic and formal geometric aspects are amalgamated and crystalized in a certain canonical way. Moreover, the admissible topology of a rigid space is honestly represented by the topology of the underlying topological space of the corresponding Zariski-Riemann triple. In this sense, one can say that the Zariski-Riemann triple visualizes the rigid space (cf. Proposition 6.16).

Our basic dictionary of comparing the situation of birational geometry as in $\S 5.1$ with that of, say, $p$-adic rigid geometry is as follows:

- $S \longleftrightarrow$ formal scheme of finite type over $\operatorname{Spf} \mathbb{Z}_{p}$;
- $D \longleftrightarrow$ the closed fiber, that is, the closed subscheme defined by " $p=0$."
Note that, by means of this comparison, the notion of $U$-admissible blowups as in $\S 5.2$ precisely correspond to the admissible blow-ups introduced in $\S 3.3$. The object corresponding to the classical Zariski-Riemann space $\langle U\rangle_{\mathrm{cpt}}$ is the underlying topological space of the Zariski-Riemann triple arising from formal schemes.


### 6.2. Adequate formal schemes

We have seen in Remark 5.13 that, in Zariski's approach to birational geometry, one needs to consider valuation rings of large height in general, even when dealing with Noetherian schemes. It turns out, for
the same reason, that valuation rings of higher height have to be considered also in our situation. Indeed, even when we deal with Noetherian formal schemes to define rigid spaces, points are described by means of valuation rings. However, the valuation rings that enter in this situation may be of height greater than 1. Note that, without such valuation rings, or, as a result, without enough points, one cannot detect topology and sheaves. Hence, as such valuation rings are rarely Noetherian, one almost always has to deal with non-Noetherian formal schemes, of which we lack sufficiently practical knowledge; even in [EGA], apart from the generalities at the first set-up, most of the theorems, such as finitudes, GFGA, etc., are proven under the Noetherian hypothesis. Thus, one first has to establish a class of formal schemes that is wide enough to contain Noetherian and some other hitherto considered classes of formal schemes (such as formal spectra of $a$-adically complete valuation rings), and to generalize the necessary theorems.

The new class of adic formal schemes that we would like to offer here is that of so-called adequate formal schemes. We postpone the precise definition of them to another opportunity [21], and confine ourselves to the following rough explanation.

Basic properties.

- the definition is given ring theoretically;
- the rings are Noetherian outside the ideal of definition.

Objects. Let $\mathbf{F s}{ }^{\text {adq }}$ denote the category of adequate formal schemes. It contains as objects

- $\operatorname{Spf} V$, where $V$ is an $a$-adically complete valuation ring for some non-zero $a \in \mathfrak{m}_{V}$,
- Noetherian formal schemes.

Functoriality. The category Fs ${ }^{\text {adq }}$ has the following pleasant functoriality: it is

- closed under finite type extensions;
- closed under base change by finite type morphisms.

Figure 8 depicts the category Fs $^{\text {adq }}$ together with some subcategories, where $\mathbf{F s}{ }^{\text {Noe }}, \mathbf{F s}_{/ V}^{\mathrm{fin}}, \mathbf{F s}_{/ \mathrm{DVR}}^{\mathrm{fin}}$ denote respectively the categories of Noetherian formal schemes, of formal schemes of finite type over an $a$-adically complete valuation ring, and of formal schemes of finite type over a complete discrete valuation ring. Notice that:

- the height of valuation rings appearing in the category $\mathbf{F s}{ }^{\text {adq }}$ is arbitrary, finite or infinite;


Fig. 8. Category of adequate formal schemes

- more importantly, the category $\mathbf{F s}^{\text {adq }}$ contains all objects of the form $\operatorname{Spf} A$, where $A$ is a formal model of an affinoid algebra in Tate's theory of rigid analytic geometry.
Among several nice points of adequate formal schemes, we would like to announce that most of the important theorems, such as finitudes, GFGA comparison, GFGA existence theorems, can be proved in this category, which therefore gives generalizations of the theorems in [EGA, III]. The details will be shown in [21]. There, these theorems are stated and proved entirely by using systematically the derived categorical framework. ${ }^{20}$


### 6.3. Coherent rigid spaces

Let us denote by CFs ${ }^{\text {adq }}$ the category of coherent adequate formal schemes.

Proposition 6.1. (1) Any coherent (= quasi-compact and quasiseparated) adequate formal scheme has an ideal of definition of finite type.

[^20](2) Let $X$ be a coherent adequate formal scheme, and $\mathscr{I}$ an ideal of definition. If $\mathscr{O}_{X}$ is $\mathscr{I}$-torsion free, then $\mathscr{O}_{X}$ is coherent ${ }^{21}$ as a module over itself.

Definition 6.2 (Admissible ideal). Let $X$ be a coherent adequate formal scheme, and $\mathscr{J}$ an ideal of $\mathscr{O}_{X}$. Then $\mathscr{J}$ is said to be admissible if it is an adically quasi-coherent open ideal of finite type.

Here an $\mathscr{O}_{X}$-module $\mathscr{F}$ is said to be adically quasi-coherent if the following conditions are satisfied:
(a) $\mathscr{F}$ is complete with respect to $\mathscr{I}$-adic topology, where $\mathscr{I}$ is an ideal of definition of $X$;
(b) for any $k \geq 0$, the sheaf $\mathscr{F}_{k}=\mathscr{F} / \mathscr{I}^{k+1} \mathscr{F}$ is a quasi-coherent sheaf on the scheme $X_{k}=\left(X, \mathscr{O}_{X} / \mathscr{I}^{k+1}\right)$.

Definition 6.3 (Admissible blow-up). Let $X$ be a coherent adequate formal scheme, and $\mathscr{J}$ an admissible ideal. The admissible blowup along $\mathscr{J}$ is the morphism of formal schemes
where $X_{k}=\left(X, \mathscr{O}_{X} / \mathscr{I}^{k+1}\right)$ is the scheme defined as above.
As $X^{\prime}$ is clearly of finite type over $X, X^{\prime}$ is again a coherent adequate formal scheme. Notice that the above definition of admissible blow-ups does not depend on the choice of an ideal of definition $\mathscr{I}$.

Having obtained a nice category of formal schemes and a nice notion of admissible blow-ups, we can now define rigid spaces in our approach by applying Raynaud's idea.

Definition 6.4 (Coherent rigid spaces). The category CRf of coherent rigid spaces is defined to be the quotient category of $\mathbf{C F s}{ }^{\text {adq }}$ where all admissible blow-ups are inverted:

$$
\mathbf{C R f}=\mathbf{C F s}^{\text {adq }} /\{\text { admissible blow-ups }\}
$$

We denote the quotient functor $\mathbf{C F s}{ }^{\text {adq }} \rightarrow \mathbf{C R f}$ by

$$
X \longmapsto X^{\text {rig }} .
$$

[^21]For a coherent rigid space $\mathscr{X}$, a formal model of $\mathscr{X}$ is defined to be a coherent adequate formal scheme $X$ such that $X^{\text {rig }} \cong \mathscr{X}$. A formal model $X$ of $\mathscr{X}$ is said to be distinguished if $\mathscr{O}_{X}$ is $\mathscr{I}$-torsion free, where $\mathscr{I}$ is an ideal of definition of $X$.

### 6.4. Admissible topology

Definition 6.5. (1) A morphism $\mathscr{U} \rightarrow \mathscr{X}$ of coherent rigid spaces is said to be a (coherent) open immersion if it has as a formal model an open immersion $U \hookrightarrow X$.
(2) Let $\left\{\mathscr{U}_{\alpha} \hookrightarrow \mathscr{X}\right\}$ be a family of open immersions between coherent rigid spaces. We say that the family is a covering with respect to the admissible topology if it has a finite refinement $\left\{\mathscr{V}_{i} \hookrightarrow \mathscr{X}\right\}$ satisfying the following condition: there exist a formal model $X$ of $\mathscr{X}$ and formal models $V_{i} \hookrightarrow X$ of $\mathscr{V}_{i} \hookrightarrow \mathscr{X}$ such that $X=\bigcup V_{i}$.

The last notion gives rise to a topology on CRf, called the admissible topology. The resulting site is denoted by $\mathbf{C R f}_{\text {ad }}$.

### 6.5. General rigid spaces

The category CFs ${ }^{\text {adq }}$ is a good category that allows "formal birational patching"; the following statement is a consequence of the existence of formal birational patching of morphisms.

Proposition 6.6. Any representable presheaf on $\mathbf{C R f}_{\mathrm{ad}}$ is a sheaf.
The proposition allows a consistent definition of more general rigid spaces.

Definition 6.7 (General rigid spaces). A general rigid space is a sheaf $\mathscr{F}$ of sets on the site $\mathbf{C R f}_{\text {ad }}$ such that the following conditions are satisfied:
(1) there exists a surjective map of sheaves

$$
\coprod_{i \in I} \mathscr{Y}_{i} \longrightarrow \mathscr{F},
$$

where $\left\{\mathscr{\mathscr { T }}_{i}\right\}_{i \in I}$ is a collection of sheaves represented by coherent rigid spaces;
(2) for $i, j \in I$, the map $\mathscr{Y}_{i} \times_{\mathscr{F}} \mathscr{Y}_{j} \longrightarrow \mathscr{Y}_{i}$ is isomorphic to the direct limit of a direct system $\left\{\mathscr{U}_{\lambda} \rightarrow \mathscr{Y}_{i}\right\}_{\lambda \in \Lambda}$ of maps between coherent rigid spaces such that all maps in the commutative diagram for $\mu \leq \lambda$

are coherent open immersions.
We denote by $\mathbf{R f}$ the category of general rigid spaces. It has $\mathbf{C R f}$ as a full subcategory.

Example 6.8. Here is an example of (coherent) rigid spaces that cannot be dealt with in classical rigid geometry (by Tate). Consider the ring $\mathbb{Z}[[q]]$ of formal power series with integral coefficients. This ring is not a valuation ring, but is a complete ring with respect to the $q$-adic topology. Hence we can consider the formal scheme $S=\operatorname{Spf} \mathbb{Z}[[q]]$, which is clearly adequate, since it is Noetherian. Any adic formal scheme $X$ of finite type over $S$ therefore gives rise to a rigid space $\mathscr{X}=X^{\text {rig }}$ over $\mathscr{S}=S^{\text {rig }}$. A particularly important example of this form, which we will discuss later in $\S 7.1$, is a Tate curve over $\mathscr{S}$.

Rigid spaces of the above form over ( $\operatorname{Spf} \mathbb{Z}[[q]])^{\text {rig }}$ (or higher dimensional adic rings) enter quite naturally in discussions on compactification of moduli spaces. Although such kinds of rigid spaces are ruled out in the classical rigid geometry, they come rather naturally in our approach to rigid geometry, and this proves to be one of the advantages of our approach.

### 6.6. Fiber products

A morphism $\varphi: \mathscr{X} \rightarrow \mathscr{Y}$ of coherent rigid spaces is said to be of finite type if it has a formal model $f: X \rightarrow Y$ that is of finite type. The notion of "locally of finite type" is defined for morphisms between general rigid spaces in an obvious way. The following proposition follows from the fact that the adequateness of formal schemes is closed under base change locally of finite type (as we have mentioned in §6.2).

Proposition 6.9. Consider the diagram

$$
\mathscr{X} \xrightarrow{\varphi} \mathscr{S} \stackrel{\psi}{\leftrightarrows} \mathscr{Y}
$$

in Rf. If either one of the morphisms is locally of finite type, then the fiber product $\mathscr{X} \times{ }_{\mathscr{S}} \mathscr{Y}$ is representable in $\mathbf{R f}$.

Remark 6.10. As we will see later (Remark 6.17), for a rigid space $\mathscr{X}$, points (in a certain topos-theoretic sense) correspond to valuation rings of a certain kind; that is, points are represented by morphisms of the form ( $\operatorname{Spf} V)^{\text {rig }} \rightarrow \mathscr{X}$, where $V$ is an $a$-adically complete valuation ring. Hence, in our rigid geometry, "fibers over points" are those fiber products taken with morphisms of this kind. The importance of studying rigid spaces over rigid spaces of the form ( $\operatorname{Spf} V)^{\text {rig }}$ thus arises. Notice that, even if we work in the categories of rigid spaces coming
from Noetherian formal schemes, valuation rings $V$ of higher height are inevitable.

Technically, the importance of the last remark lies in the fact that, by considering fibers over points, one can usually reduce quite a few geometric properties of rigid spaces of finite type to those of rigid spaces of finite type over valuation rings. In case the valuation ring is of finite height, one can further reduce to the case of height 1 (by the gluing method), where one can use some extra tools, such as Noether's normalization theorem, etc.

### 6.7. Relation with algebraic spaces

Let $\operatorname{Spf} A$ be an affine adequate formal scheme, and $I$ a finitely generated ideal of definition of $A$. We set $U=\operatorname{Spec} A \backslash V(I)$, which is a Noetherian scheme (cf. $\S 6.2$ ). The precise meaning of the following somewhat vague statement will be clarified in [21].

Theorem 6.11 (GAGA functor). The GAGA functor

$$
\left\{\begin{array}{l}
\text { Separated algebraic spaces } \\
\text { of finite type } / U
\end{array}\right\} \longrightarrow \mathbf{R f}_{\mathscr{S}}, \quad X \mapsto X^{\text {an }}
$$

where $\mathscr{S}=(\operatorname{Spf} A)^{\text {rig }}$, exists.
Notice that the GAGA functor of this general form has not been defined even in the classical rigid geometry (at least in literature). There are two main ingredients for the proof. One is the embedding theorem (of Nagata) for algebraic spaces, and the other one is the following.

Theorem 6.12 (Equivalence theorem). Let $S$ be a coherent adequate formal scheme. Then the natural functor

$$
\left\{\begin{array}{l}
\text { Formal schemes } \\
\text { of finite type } / S
\end{array}\right\} /\left\{\begin{array}{l}
\text { Admissible } \\
\text { blow-ups }
\end{array}\right\} \longrightarrow\left\{\begin{array}{l}
\text { Formal alge- } \\
\text { braic spaces of } \\
\text { finite type } / S
\end{array}\right\} /\left\{\begin{array}{l}
\text { Admissible } \\
\text { blow-ups }
\end{array}\right\}
$$

is a categorical equivalence.
This follows from the following theorem.
Theorem 6.13. Let $S$ be as above, and $X \rightarrow S$ a formal algebraic space of finite type. Then there exists an admissible blow-up $X^{\prime} \rightarrow X$ such that $X^{\prime}$ is a formal scheme.

The proof of this theorem uses (again!) the technique of ZariskiRiemann spaces.

### 6.8. Tate's rigid analytic spaces

Tate's rigid analytic spaces are naturally objects of the category $\mathbf{R f}$ via Raynaud's theorem (Theorem 3.5) and obvious patching arguments, that is, we have the natural functor

$$
\left\{\begin{array}{c}
\text { Tate's rigid } \\
\text { spaces }
\end{array}\right\} \longrightarrow \mathbf{R f}
$$

which maps affinoids to affinoids. Here by an affinoid in Rf we mean a coherent rigid space of the form $(\operatorname{Spf} A)^{\text {rig. }}$.

The essential image of the above functor considered on the category of Tate's rigid analytic spaces over $K$ is the category of rigid spaces locally of finite type over $(\operatorname{Spf} V)^{\text {rig }}$, where $V$ is a complete valuation ring of height 1 , and $K$ is its fractional field. Note that this is essentially the assertion of Raynaud's theorem (Theorem 3.5).

### 6.9. Visualization

The moral basis of our (and hence Raynaud's) defining rigid spaces as "generic fibers" of formal schemes stems from the policy that rigid geometry is so to speak the birational geometry of formal schemes (cf. §6.1). It being so, one can say that the visualization of rigid spaces, which we are going to pursue below, is the way to enhance the birational geometric aspect of rigid geometry. It does this job by adopting Zariski's old idea of birational geometry, and the visualization itself is given by the so-called Zariski-Riemann triple. The pleasant thing is that the admissible topology attached to a rigid space is equivalent to the topology (in the usual sense) of the associated Zariski-Riemann space, the underlying topological space of the Zariski-Riemann triple. This is the origin of the name "visualization." As we can easily imagine, having the genuine ringed space that really represents the rigid space helps and streamlines discussions, and enables many applications.

Definition 6.14. Let $\mathscr{X}=X^{\text {rig }}$ be a coherent rigid space.
(1) Define the projective limit

$$
\langle\mathscr{X}\rangle=\lim _{X^{\prime} \rightarrow X} X^{\prime}
$$

along all admissible blow-ups of $X$ taken in the category of local ringed spaces. Note that, by the similar reasoning as in $\S 5.4$, the projective limit can be replaced by the filtered projective limit taken along the directed set of all admissible ideals, and hence is well-defined as a local ringed space. The canonical projection map $\langle\mathscr{X}\rangle \rightarrow X^{\prime}$ for any admissible
blow-up $X^{\prime}$ of $X$ is called the specialization map, and is denoted by

$$
\mathrm{sp}_{X^{\prime}}:\langle\mathscr{X}\rangle \longrightarrow X^{\prime}
$$

This is a continuous map.
(2) The structure sheaf of $\langle\mathscr{X}\rangle$, which is the direct limit of the sheaf $\mathrm{sp}_{X^{\prime}}^{-1} \mathscr{O}_{X^{\prime}}$, is called the integral structure sheaf, and is denoted by $\mathscr{O}_{\mathscr{X}}^{\mathrm{int}}$.
(3) The rigid structure sheaf $\mathscr{O}_{\mathscr{X}}$ is the sheaf on $\langle\mathscr{X}\rangle$ defined by
here we take an ideal of definition $\mathscr{I}_{X}$ of $X$ and set $\mathscr{I}=\left(\mathrm{sp}_{X}^{-1} \mathscr{I}_{X}\right) \mathscr{O}_{\mathscr{X}}^{\text {int }}$.
Here the definition of $\mathscr{O} \mathscr{X}$ calls for an explanation. It turns out that the sheaf $\mathscr{O}_{\mathscr{X}}^{\text {int }}$ of local rings is $\mathscr{I}$-valuative, and due to Proposition 5.11, one sees that the sheaf $\mathscr{O}_{\mathscr{X}}$ is also a sheaf of local rings. For example, in the $p$-adic situation, we have $\mathscr{O}_{\mathscr{X}}=\mathscr{O}_{\mathscr{X}}^{\mathrm{int}}\left[\frac{1}{p}\right]$. As this particular example indicates, it is $\mathscr{O}_{\mathscr{X}}$ that plays the role of the structure sheaves of Tate's rigid analytic geometry. In fact, when $\mathscr{X}$ comes from a rigid analytic space in the sense of Tate via the functor as in $\S 6.8, \mathscr{O}_{\mathscr{X}}$ "is" the structure sheaf of the original Tate rigid space. Thus the realization of rigid spaces as a topological space $\langle\mathscr{X}\rangle$ naturally weaves its structure sheaf $\mathscr{O}_{\mathscr{X}}$ with, one can say, its "canonical" formal model $\mathscr{O}_{\mathscr{X}}^{\mathrm{int}}$.

Definition 6.15 (Zariski-Riemann triple). We write

$$
\mathbf{Z R}(\mathscr{X})=\left(\langle\mathscr{X}\rangle, \mathscr{O}_{\mathscr{X}}^{\mathrm{int}}, \mathscr{O} \mathscr{X}\right)
$$

and call it the Zariski-Riemann triple associated to the rigid space $\mathscr{X}$.
One can, in fact, extend the above definition to general rigid spaces by gluing. It is worth remarking here that the idea of considering the triple as above, rather than merely a local ringed space, comes from the analogy between hermitian vector bundles $(\mathscr{E},|\cdot|)$ and pairs $\left(\mathscr{E}, \mathscr{E}^{\text {int }}\right)$ of vector bundle with its integral model (which is at the center of the idea of, for example, Arakelov geometry).

Be that as it may, the main motivation for introducing ZariskiRiemann triples is that they really visualize the rigid spaces, as the following proposition indicates.

Proposition 6.16. The topos associated to the topological space $\langle\mathscr{X}\rangle$ is isomorphic to the admissible topos $\mathscr{X}_{\mathrm{ad}}^{\sim}{ }^{22}$

[^22]Remark 6.17. The last proposition forces us to review the issue of points of rigid analytic spaces, which was already considered during the discussion of Tate's basic idea of approaching rigid geometry (see Remark 2.2). Even in case where $\mathscr{X}$ comes from a Tate rigid analytic space, the topological space $\langle\mathscr{X}\rangle$ was not considered by Tate, since Tate's notion of points only grasps points coming from maximal ideals of affinoid algebras, which occupies only a very small part of the space $\langle\mathscr{X}\rangle$. This is why Tate had to introduce the Grothendieck topology machinery to obtain the admissible topology.

Now, the space $\langle\mathscr{X}\rangle$ gives the correct notion of points for rigid analytic spaces; in fact, quite similarly to $\S 5.4$, points of $\langle\mathscr{X}\rangle$ are described in terms of $a$-adically complete valuation rings. It is based on this fact that we say that the Zariski-Riemann triple visualizes rigid spaces.

Remark 6.18. We would like to mention that, by using visualization, one can simplify the definition of the so-called "dagger-ring" that appears in the theory of rigid cohomology (cf. [4]). Let us give a simple example. Let $A=V\langle\langle X\rangle$, where $V$ is a complete discrete valuation ring of mixed characteristic $(0, p)$ such that the residue field $k$ is perfect, and consider $\mathbb{D}=(\operatorname{Spf} A)^{\text {rig }}$ (closed unit disk). It is a coherent open rigid subspace of the projective line $\mathbb{P}_{\mathscr{S}}^{1}=\left(\left(\mathbb{P}_{V}^{1}\right)^{\wedge}\right)^{\text {rig }}$, where $\mathscr{S}=(\operatorname{Spf} V)^{\text {rig }}$. Consider the closure $\overline{\langle\mathbb{D}\rangle}$ of $\langle\mathbb{D}\rangle$ in $\left\langle\mathbb{P}_{\mathscr{S}}^{1}\right\rangle$. Consider the sheaf $\mathscr{O}_{\mathscr{D}}^{\dagger}$ defined by the pull-back $i^{*} \mathscr{O}_{\mathbb{P}_{\mathscr{S}}^{1}}$ of $\mathscr{O}_{\mathbb{P}_{\mathscr{S}}^{1}}$ by the inclusion $i: \overline{\langle\mathbb{D}\rangle} \hookrightarrow\left\langle\underline{\left.\mathbb{P}_{\mathscr{S}}^{1}\right\rangle}\right.$. The dagger-ring in this case, usually denoted by $A_{K}^{\dagger}$, is the ring $\Gamma\left(\overline{\langle\mathbb{D}\rangle}, i^{*} \mathscr{O}_{\mathbb{P}_{\mathscr{S}}^{1}}\right)$.

### 6.10. Relation with other theories

Now let us mention something about the relation between our approach to rigid geometry and other hitherto known approaches.
(1) As we have already mentioned in $\S 6.8$, there exists a natural functor that maps Tate's rigid analytic spaces to rigid spaces in Rf.
(2) Zariski-Riemann triples are regarded as Huber's adic spaces, whence we have a natural functor

$$
\mathbf{Z R}: \mathbf{R f} \longrightarrow\left\{\begin{array}{c}
\text { Huber's adic } \\
\text { spaces }
\end{array}\right\}
$$

which is, however, not fully faithful in general; but it is fully faithful in practically important situations.
(3) Each Zariski-Riemann space $\langle\mathscr{X}\rangle$ admits, by means of maximal generalization ${ }^{23}$ of all points, a so-called separation map

$$
\operatorname{sep}_{\mathscr{X}}:\langle\mathscr{X}\rangle \longrightarrow[\mathscr{X}]
$$

where $[\mathscr{X}]$ is the set of all points of $\langle\mathscr{X}\rangle$ of height 1 . The map sep $\mathscr{X}$ is a continuous map. At least in case where $\mathscr{X}$ comes from a Tate rigid analytic space via the functor as in $\S 6.8$, the target space [ $\mathscr{X}$ ] (with more structure coming from $\mathscr{X}$ ) can naturally be regarded as a Berkovich space ([2][3]).

Figure 9 depicts the above mentioned relations.


Fig. 9. Relation with other theories (f.f. = fully faithful)

### 6.11. Formal flattening theorem

Applying Zariski's idea explained in §5, but now using the ZariskiRiemann triple introduced as above, one can show the following theorem.

Theorem 6.19 (Bosch-Raynaud, Fujiwara). Let $f: X \rightarrow S$ be a morphism of finite type between coherent adequate formal schemes. Then the following conditions are equivalent:
(1) $\quad f^{\text {rig }}: X^{\text {rig }} \rightarrow S^{\text {rig }}$ is flat, that is, $\left\langle f^{\mathrm{rig}}\right\rangle:\left\langle X^{\mathrm{rig}}\right\rangle \rightarrow\left\langle S^{\mathrm{rig}}\right\rangle$ is flat as a mapping of local ringed spaces (with the rigid structure sheaf);

[^23](2) there exists an admissible blow-up $S^{\prime} \rightarrow S$ such that the strict transform $f^{\prime}: X^{\prime} \rightarrow S^{\prime}$ is flat.

Corollary 6.20. Admissible blow-ups are cofinal in the category of formal modifications.

The following corollary is, as we have already seen in $\S 3.3$, important in Tate's rigid analytic geometry (see Theorem 2.8).

Corollary 6.21 (Gerritzen-Grauert). Let $\varphi: \mathscr{X} \rightarrow \mathscr{Y}$ be a morphism of Tate's rigid analytic spaces over a complete non-archimedean valued field $K$ with non-trivial valuation. Then the following conditions are equivalent:
(1) $\varphi$ is an open immersion;
(2) $\varphi$ is separated, étale, and injective, and induces an isomorphism between the residue fields at any point.

### 6.12. Properness in rigid geometry

Definition 6.22. (1) A morphism $\varphi: \mathscr{X} \rightarrow \mathscr{Y}$ of rigid spaces is said to be closed if the induced map $\langle\varphi\rangle:\langle\mathscr{X}\rangle \rightarrow\langle\mathscr{Y}\rangle$ of topological spaces is closed.
(2) Let $\varphi: \mathscr{X} \rightarrow \mathscr{Y}$ be a morphism locally of finite type. The morphism $\varphi$ is said to be universally closed if, for any morphism $\mathscr{Z} \rightarrow \mathscr{Y}$ of rigid spaces, the base change $\varphi_{\mathscr{X}}: \mathscr{X} \times \mathscr{Y} \not \mathscr{Z} \rightarrow \mathscr{Z}$ is closed.

Definition 6.23. A morphism $\varphi: \mathscr{X} \rightarrow \mathscr{Y}$ of rigid spaces is said to be proper if it is universally closed, separated, and of finite type.

In case $\mathscr{X}$ and $\mathscr{Y}$ are coherent, according to our general policy of regarding rigid geometry as birational geometry of formal schemes, the properness thus defined should be equivalent to that in formal geometry as follows.

Proposition 6.24. Let $\varphi: \mathscr{X} \rightarrow \mathscr{Y}$ be a morphism of coherent rigid spaces. Then the following conditions are equivalent:
(1) $\varphi$ is proper;
(2) (Raynaud properness) there exists a proper formal model $f: X \rightarrow$ $Y$ of $\varphi$.
(3) (Kiehl properness) there exist affinoid enlargements (cf. [26]) of coverings for each relatively compact affinoid open subset; that is, there exists a finite admissible covering $\mathscr{X}=\bigcup \mathscr{U}_{i}$ consisting of affinoids together with a refinement $\mathscr{X}=\bigcup \mathscr{V}_{i}$ again consisting of affinoids with $\mathscr{U}_{i} \hookrightarrow \mathscr{V}_{i}$ such that, for each $i, \overline{\left\langle\mathscr{U}_{i}\right\rangle} \subset\left\langle\mathscr{V}_{i}\right\rangle$, where the closure is taken in $\langle\mathscr{X}\rangle$.

Historically, properness in Tate's rigid geometry has been first defined by Kiehl in his work [26] on finiteness theorem; there, properness was defined by existence of enlargements according to the general idea by Cartan-Serre and Grauert for proving finiteness of cohomologies of coherent sheaves.

Whereas the implication $(3) \Rightarrow(2)$ is in general not difficult to show, the converse is a very difficult theorem; even in case where all rigid spaces are of finite type over $(\operatorname{Spf} V)^{\text {rig }}$ with $V$ being a complete discrete valuation ring, Lütkebohmert's 1990 paper [32] was the first for the proof. We claim (in [21]) that this is also valid in general. In the case over $(\operatorname{Spf} V)^{\text {rig }}$ where $V$ is an $a$-adically complete valuation ring, this amounts to showing the following statement.

Theorem 6.25. Let $f: X \rightarrow \operatorname{Spf} V$ be a morphism of adequate formal schemes of finite type, and $U \subset X$ an affine open subset such that $\bar{U}$ is proper. Then there exists an admissible blow-up $\pi: X^{\prime} \rightarrow X$ and an open subset $W \subset X^{\prime}$ such that the following conditions are satisfied:
(a) $\overline{\pi^{-1}(U)} \subseteq W$;
(b) there exists a map $W \rightarrow \operatorname{Spf} A$ to an affine adequate formal scheme that is a contraction (that is, $\left.W^{\text {rig }}=(\operatorname{Spf} A)^{\text {rig }}\right)$.

### 6.13. Cohomology theory

Let $\mathscr{X}$ be a rigid space, and $\mathscr{F}$ an abelian sheaf on the topological space $\langle\mathscr{X}\rangle$. We write

$$
\mathrm{H}^{q}(\mathscr{X}, \mathscr{F})=\mathrm{H}^{q}(\langle\mathscr{X}\rangle, \mathscr{F}) .
$$

Similarly, for a morphism $\varphi: \mathscr{X} \rightarrow \mathscr{Y}$ of rigid spaces, we write

$$
\mathrm{R}^{q} \varphi_{*} \mathscr{F}=\mathrm{R}^{q}\langle\varphi\rangle_{*} \mathscr{F} .
$$

As we have mentioned before, an affinoid is a coherent rigid space of the form $X^{\text {rig }}$, where $X$ is an affine adequate formal scheme. For a rigid space $\mathscr{X}$, by a coherent sheaf on $\mathscr{X}$, we mean a coherent $\mathscr{O}_{\mathscr{X}}$-module on $\langle\mathscr{X}\rangle$.

Theorem 6.26. For a rigid space $\mathscr{X}$, the rigid structure sheaf $\mathscr{O}_{\mathscr{X}}$ is coherent.

Thus an $\mathscr{O}_{\mathscr{X}}$-module is coherent if and only if it is finitely presented.
Definition 6.27. An affinoid $\mathscr{X}$ is said to be a Stein affinoid if one of the following equivalent conditions is satisfied:
(1) $\mathrm{H}^{1}(\mathscr{X}, \mathscr{F})=0$ for any coherent sheaf $\mathscr{F}$;
(2) $\mathrm{H}^{q}(\mathscr{X}, \mathscr{F})=0$ for $q \geq 1$ and for any coherent sheaf $\mathscr{F}$;
(3) There exists a formal model $X$ of $\mathscr{X}$ such that $X$ is affine $X=\operatorname{Spf} A$ and that $\operatorname{Spec} A \backslash V(I)$ is an affine scheme, where $I$ is an ideal of definition of $A$.
(4) There exists a distinguished formal model $X$ of $\mathscr{X}$ such that $X$ is affine $X=\operatorname{Spf} A$ and that $\operatorname{Spec} A \backslash V(I)$ is an affine scheme, where $I$ is an ideal of definition of $A$.

The equivalence of the above conditions follows from the comparison theorem for affinoids and GFGA existence theorem. It can be shown that, for any rigid space $\mathscr{X}$, any admissible covering of $\mathscr{X}$ by affinoids can be refined by an admissible covering consisting of Stein affinoids. Combined with this fact, the next theorem shows that one can compute cohomology of coherent sheaves by means of Cech calculation using admissible covering by Stein affinoids.

Theorem 6.28 (Theorem A and Theorem B). Let $\mathscr{X}$ be a Stein affinoid, and $\mathscr{F}$ a coherent sheaf on $\mathscr{X}$.
(1) If $X=\operatorname{Spf} A$ is a distinguished formal model of $\mathscr{X}$ such that Spec $A \backslash V(I)$ (where $I$ is an ideal of definition of $A$ ) is affine, then there exists a finitely presented $A$-module $M$ such that

$$
\mathrm{H}^{0}(\mathscr{X}, \mathscr{F})=\underset{n \geq 0}{\lim } \operatorname{Hom}_{A}\left(I^{n}, M\right) .
$$

(2) For $q \geq 1$, we have $\mathrm{H}^{q}(\mathscr{X}, \mathscr{F})=0$.

Finally, we mention the finiteness theorem for proper morphisms.
Theorem 6.29 (Finiteness theorem for proper morphisms of rigid spaces). Let $\varphi: \mathscr{X} \rightarrow \mathscr{Y}$ be a proper morphism between quasi-compact ${ }^{24}$ rigid spaces. Then the functor $\mathrm{R} \varphi_{*} \operatorname{maps} \mathbf{D}_{\text {coh }}^{*}(\mathscr{X})$ to $\mathbf{D}_{\text {coh }}^{*}(\mathscr{Y})$ for $*=\emptyset,+,-, \mathrm{b}$.

Here, for a rigid space $\mathscr{X}, \mathbf{D}_{\text {coh }}^{*}(\mathscr{X})$ denotes the full subcategory of the derived category of the category of $\mathscr{O} \mathscr{X}$-modules consisting of objects that have only coherent cohomologies.

Most of material presented in this part will be written in detail in the book [21] in preparation by the authors.

## Part III. Applications

[^24]We expect that rigid geometry, especially that of our approach explained in the previous part, allows diverse applications. The applications within our scope at this moment include, at least, the following things:

- arithmetic geometry of Shimura varieties: p-adic period map and local models, $p$-adic automorphic representations, etc.,
- cohomology theory of algebraic varieties: $\ell$-adic Lefschetz trace formulas, $p$-adic cohomology theory, etc.
In this final part, we discuss the applications of these things. In $\S 7$ we discuss arithmetic compactification of moduli of elliptic curves, and in $\S 8$, Lefschetz trace formulas.

Although we are not going to treat in this paper, one might moreover expect, in addition, the following applications:

- mirror symmetry (construction of mirror partner); cf. [28],
- $p$-adic Hodge theory (via theory of almost étale extensions); cf. [22],
- derived category equivalence,
- non-archimedean uniformization.

As for the last, we remark that, by means of the visualization, one can understand the known uniformization (e.g. [34], [35]) entirely as topological uniformization, that is, the uniformization by taking the universal covering. This point also streamlines the theory of orbifold uniformizations of rigid analytic curves developed in [25] (see also [10]).

## §7. Application to compactification of moduli

In this section we discuss compactification of moduli spaces. We want to show that rigid geometry is useful in the analysis of moduli object near the boundary, and thus can be applied to the construction of the compactification. As the method that we are going to take is abstract enough, it affords the construction of the compactifications not only over fields, but over $\mathbb{Z}$, that is, arithmetic compactifications.

Here, at first, we would like to remind the reader of the fact that, in the classical theory of toroidal compactifications, complex analytic methods play an important role. The important point here is that the notion of rigid spaces is much broader than that of schemes, and rigid spaces are much flexible than schemes. In fact, there are several merits of using non-scheme theoretical geometric objects, such as rigid spaces, in application to the theory of moduli; among them are:

- topological feature: admissible topology of rigid spaces is finer than Zariski topology;
- it allows, in general, "construction by infinite repetition;" in this context, non-coherent objects play an essential role.

A typical example of the methods in the second point is the theory of $p$-adic uniformization, which provides, as we have already seen in $\S 1.2$, numerous nice techniques and viewpoints already in the classical rigid geometry by Tate.

First we fix some notations that we are going to use frequently in the sequel. We often consider pairs of the form $(\bar{X}, D)$, where $\bar{X}=\operatorname{Spec} A$ is an affine scheme, and $D$ is a closed subscheme defined by a finitely generated ideal $I$ of $A$. Let $\widehat{A}$ be the $I$-adic completion of $A$. We write $\bar{X}_{/ D}=\operatorname{Spec} \widehat{A}$. It forms another pair $\left(\bar{X}_{/ D}, D\right)$ of the form as above together with the closed subscheme defined by the ideal $I \widehat{A}$, which is, by a slight abuse of notation, again denoted by $D$.

In practice, the affine scheme $\bar{X}$ in the sequel appears as a "partial compactification" of a scheme $X$ such that $\partial X=\bar{X} \backslash X=D$. In this situation, the complement $\left(\bar{X}_{/ D}\right) \backslash D\left(\cong \bar{X}_{/ D} \times \bar{X} X\right)$ is denoted by $X_{/ D}$.

For a pair $(\bar{X}, D)$ as above we denote by $\left.\widehat{\bar{X}}\right|_{D}$ the formal completion of $\bar{X}$ along $D$, i.e., $\left.\widehat{\bar{X}}\right|_{D}=\operatorname{Spf} \widehat{A}$. The canonical morphism $\gamma_{\bar{X}}:\left.\widehat{\bar{X}}\right|_{D} \rightarrow$ $\bar{X}$ is factorized into the composite

$$
\left.\widehat{\bar{X}}\right|_{D} \xrightarrow{\beta_{\bar{X}}} \bar{X}_{/ D} \xrightarrow{\alpha_{\bar{X}}} \bar{X} .
$$

Notice that the formal completion of $\left(\bar{X}_{/ D}, D\right)$ is the same as that of $(\bar{X}, D)$.

### 7.1. Analysis near cusps

In this section, we discuss the arithmetic compactification of the moduli space of elliptic curves over $\mathbb{Z}$, which is considered as one of the simplest but non-trivial examples, and then, later, indicate more general situation of Shimura varieties of PEL-type.

Let $\mathscr{M}$ be the moduli stack of elliptic curves over $\mathbb{Z}$. It is the algebraic stack characterized by the following condition: for a scheme $S$ the category of Cartesian sections $\mathscr{M}(S)$ over $S$ forms the groupoid consisting of elliptic curves over $S$, where morphisms are isomorphisms of the elliptic curves. We view $\mathscr{M}$ as a Deligne-Mumford stack, and denote by $f^{\text {univ }}: \mathscr{E}$ univ $\rightarrow \mathscr{M}$ the universal elliptic curve. We want to compactify the stack $\mathscr{M}$. To this end, first we are to analyze points near the cusps.

In order to do this, the construction of the Tate curves as in Example 3.3 provides a good picture. To apply it to our situation, we need to recast the construction in the universal form. Let $W$ be the moduli
space of maps of the form

$$
u: \mathbb{Z} \longrightarrow \mathbb{G}_{m}
$$

over $\mathbb{Z}$; that is, an affine scheme identified with $\operatorname{Spec} \mathbb{Z}\left[q, q^{-1}\right]$. Notice that the map as above is determined by its value at $u=1$, and thus, the identification of $W$ with $\operatorname{Spec} \mathbb{Z}\left[q, q^{-1}\right]$ is given by the universal map $u^{\text {univ }}: \mathbb{Z} \rightarrow \mathbb{G}_{m}$ that maps 1 to $q$. We choose a torus embedding $W \hookrightarrow$ $\bar{W}=\operatorname{Spec} \mathbb{Z}[q]$, which is seen as a partial compactification, and denote the infinity $\bar{W} \backslash W$ by $D$. The closed subscheme $D$ is defined by the equation $q=0$. We consider the pair $(\bar{W}, D)$.

By Tate construction (or the generalization due to Mumford), we have a semi-abelian scheme $\mathscr{A}$ over $\bar{W}_{/ D}$, which has the following properties:
(1) the restriction $\mathscr{A}_{W_{/ D}}$ of $\mathscr{A}$ to $W_{/ D}$ is an elliptic curve, and the restriction to $D$ is isomorphic to $\mathbb{G}_{m}$;
(2) $\left(\mathscr{A}_{W / D}\right)^{\text {an }}$ is canonically isomorphic to the quotient $\left(\mathbb{G}_{m}\right)^{\text {an }}$ by the subgroup generated by $q$ as a rigid space over $\mathscr{W}=$ $\left(\left.\widehat{\bar{W}}\right|_{D}\right)^{\text {rig }} ;$
(3) the construction is functorial in the sense as follows: for any complete valuation ring $V$ of height 1 and an adic homomorphism $\Gamma\left(\bar{W}_{/ D}, \mathscr{O}_{\bar{W}_{/ D}}\right)=\mathbb{Z}[[q]] \rightarrow V$ that maps $q$ to an element in $\mathfrak{m}_{V}$ (which we again denote by $q$ ), the base change of $\mathscr{A}$ to $V$ corresponds to the Tate curve $\left(\mathbb{G}_{m, K}\right)^{\text {an }} / q^{\mathbb{Z}}$ as in Example 2.15 , where $K$ is the field of fractions of $V$.

Moreover the following property is known for the Tate construction:
Proposition 7.1 (Uniformization theorem, converse to
Tate construction; cf. [16, Chap. II, §4]). Assume that $(S, D)=(\operatorname{Spec} V, V(I))$ is a pair of affine schemes, where $V$ is a Noetherian normal ring that is complete with respect to the I-adic topology. Let A be a semi-abelian scheme over $S$ that satisfies following conditions:
(1) the relative dimension is 1 ;
(2) the restriction of $A$ to $D$ is a split torus;
(3) the restriction of $A$ to $S \backslash D$ is an elliptic curve.

Then there exists a morphism $g: S \rightarrow \bar{W}_{/ D}$ such that $A$ is isomorphic to the pullback $g^{*} \mathscr{A}$. Moreover, the morphism $g$ is unique up to isomorphisms.

The proposition says that $\mathscr{A}$ is seen as the universal Tate curve, and $\bar{W}_{/ D}$ is the classifying space for elliptic curves with split multiplicative reductions over complete base schemes. Moreover one can drop the
assumption "normal" of $S$ when $S$ is of dimension 1. The uniformizaiton theorem, and the extension to the general 1-dimensional base schemes due to Raynaud, which we often abbreviate to "Raynaud-Tate theory", becomes very important later.

Now let us return to the moduli stack $\mathscr{M}$. We have a map

$$
\epsilon: W_{/ D} \longrightarrow \mathscr{M}
$$

defined by the elliptic curve $\mathscr{A}_{W_{/ D}}$ over $W_{/ D}$. This map sits in the following 2-commutative diagram in a suitable 2-category of spaces:


The rigid space $\mathscr{W}=\left(\left.\widehat{\bar{W}}\right|_{D}\right)^{\text {rig }}$ is considered to be the family of "punctured unit disks" over $\mathbb{Z}$, or "(deleted) tubular neighborhood" of $D$ inside $\bar{W}$. The desired compactification $\overline{\mathscr{M}}$ is obtained by patching the stack $\mathscr{M}$ and the scheme $\bar{W}_{/ D}$ along the rigid space $\mathscr{W}$. This will be made more precise in the next two sections.

### 7.2. Arithmetic compactification

The following assertion provides the model case of the arithmetic compactifications in general.

Proposition 7.2. There exists a proper smooth Deligne-Mumford stack $\mathscr{M}$ over $\mathbb{Z}$ that contains $\mathscr{M}$ as an open substack enjoying the following properties:
(1) there exists a semi-abelian scheme $\bar{f}: \overline{\mathscr{E}}^{\text {univ }} \rightarrow \overline{\mathscr{M}}$ that extends $\mathscr{E}^{\text {univ }}$;
(2) the morphism $\epsilon: W_{/ D} \rightarrow \mathscr{M}$ extends to $\bar{\epsilon}: \bar{W}{ }_{/ D} \rightarrow \overline{\mathscr{M}}$ in such a way that $\bar{\epsilon}^{*} \overline{\mathscr{E}}^{\text {univ }}=\mathscr{A}$ holds;
(3) moreover, the morphism $\bar{\epsilon}$ induces a formally étale surjective morphism on passage to the formal completions.
In fact, the rigid space $\mathscr{W}=\left(\left.\widehat{\bar{W}}{ }_{/ D}\right|_{D}\right)^{\text {rig }}=\left(\left.\widehat{\bar{W}}\right|_{D}\right)^{\text {rig }}$ is almost isomorphic to $\left(\left.\widehat{\overline{\mathscr{M}}}\right|_{\partial \mathscr{M}}\right)^{\text {rig }}$; it gives an isomorphism when we introduce level structures to make the moduli problem fine. Thus the compatification $\overline{\mathscr{M}}$ in question should be constructed as the patching of $\mathscr{M}$ and $\bar{W}_{/ D}$ along $\mathscr{W}=\left(\widehat{\bar{W}} /\left.D\right|_{D}\right)^{\text {rig }}$. This gives the strategy for the construction that is quite similar to the complex analytic case. Notice that, to carry
out this strategy, the framework of general rigid spaces (introduced in $\S 6.5)$ is necessary.

### 7.3. Construction

The construction of $\overline{\mathscr{M}}$ takes three steps. The method exhibited here follows [20]. It is influenced by M. Rapoport work on Hilbert-Blumenthal varieties [36] and G. Faltings work on Siegel modular varieties [15].

First step (Algebraization). First we are to algebraize the family $\mathscr{A}$ over $\bar{W}_{/ D}$ to a semi-abelian scheme over an affine scheme of finite type over $\mathbb{Z}$. For any $n \geq 1$, by Artin's approximation theorem, we have an affine smooth scheme $\bar{V}_{n}$ over $\mathbb{Z}$ and a closed subscheme $D_{n} \subset \bar{V}_{n}$ such that the formal completion of $\bar{V}_{n}$ along $D_{n}$ is identified with $\left.\widehat{W}\right|_{D}$ (we fix this identification). Moreover, there is a semi-abelian scheme $A_{n}$ of relative dimension 1 over $\bar{V}_{n}$ such that $A_{n}$ is an elliptic curve over $V_{n}=\bar{V}_{n} \backslash D_{n}$ and is a split torus over $D_{n}$.

The family $A_{n}$ is "very near" to $\mathscr{A}$ in the following sense: when we regard $A_{n}$ as a quotient of $\mathbb{G}_{m}$ by $q_{n}{ }^{\mathbb{Z}}$ over $\bar{V}_{n / D_{n}}$ for some $q_{n} \in$ $\Gamma\left(\left.\widehat{\bar{W}}\right|_{D}, \mathscr{O} \widehat{\left.\widehat{W}\right|_{D}}\right)$ by the uniformization theorem (Proposition 7.1 ), $q_{n} \equiv q$ $\bmod q^{n}$ holds. To achieve the last condition, one must approximate the semi-abelian scheme with a line bundle and sections, i.e. with theta functions.

When $A_{n}$ is very near to $\mathscr{A}$ in the sense as above and $n \geq 2$, the morphism

$$
\delta_{n}: \bar{V}_{n / D_{n}} \longrightarrow \bar{W}_{/ D}, \quad q \mapsto q_{n}
$$

by the universality of $\bar{W}_{/ D}$ (again by Proposition 7.1) is an isomorphism ${ }^{25}$, and that the pull back $\delta_{n}^{*} \mathscr{A}$ is isomorphic to $A_{n}$.

One sets $\bar{V}=\bar{V}_{n}, D=D_{n}$, and $A_{\bar{V}}=A_{n}$ for some $n \geq 2$, and $V=\bar{V} \backslash D$.

Second step (Openness of versality). We show that the classifying morphism $V=\bar{V} \backslash D \rightarrow \mathscr{M}$ defined by $A_{\bar{V}}$ is étale by shrinking $V$ around $D$ if necessary. For this, it suffices to show that $V_{/ D} \rightarrow \mathscr{M}$ is formally smooth at any closed point of $V_{/ D}$ (note that the residue field at any closed point is a complete discrete valuation field). To show this, one uses the infinitesimal criterion of formal smoothness, and reduces to show the following assertion.

Proposition 7.3. Let $R_{0}$ be a complete discrete valuation ring, $\pi$ a uniformizer, and $R$ a finite local algebra that is a thickening of $R_{0}$.

[^25]Assume that we are given an elliptic curve $E$ over $R\left[\frac{1}{\pi}\right]$ such that the restriction $E_{R_{0}}$ to $R_{0}$ is a Tate curve over $R_{0}$. Then, by replacing $R$ by a finite modification ( $=$ finite map that induces isomorphism outside the ideal $(\pi)$ ) if necessary, $E$ is also a Tate curve.

This is a direct consequence of the uniformization theorem (RaynaudTate theory over 1-dimensional complete rings). Roughly speaking, the point is to show the deformations of an elliptic curve with split multiplicative reduction are the same as the deformations of corresponding 1-motives obtained by the Raynaud-Tate theory. (A related work for Mumford curves is in $[11, \S 9]$.)

Third step (Patching). We construct $\overline{\mathscr{M}}$ by patching $\bar{V}$ (obtained in Step 2) and $\mathscr{M}$ along $\left(\left.\bar{V}\right|_{D}\right)^{\text {rig }}$. This is easy by using the openness of versality (Proposition 7.3). Since $\mathscr{M}$ is a Deligne-Mumford stack, there are an étale surjective morphism $P \rightarrow \mathscr{M}$ from a smooth affine scheme $P$, and a relation $R \rightarrow P \times_{\mathbb{Z}} P$ that defines $\mathscr{M}$ as a stack. Note that we have the pull back $A_{P}$ to $P$ of the universal elliptic curve. Together with $A_{\bar{V}}$ over $\bar{V}$, we have a semi-abelian scheme $A_{P} \amalg \bar{V}$ on $P \amalg \bar{V}$.

We take the normalization $\widetilde{R}$ of $(P \coprod \bar{V}) \times_{\mathbb{Z}}(P \coprod \bar{V})$ in $R$, and show that $\widetilde{R}$ defines an étale relation on $P \amalg \bar{V}$ and defines a Deligne-Mumford stack $\overline{\mathscr{M}}$. Semi-abelian scheme $A_{P} \amalg \bar{V}$ also descends to a semi-abelian scheme $\overline{\mathscr{E}}^{\text {univ }} \rightarrow \overline{\mathscr{M}}$. The point here is that one can control the situation using the semi-abelian scheme on $P \coprod \bar{V}$ and the uniformization theorem. By the construction, the properties (1)-(3) of Proposition 7.2 follow.

The properness of $\overline{\mathscr{M}}$ follows from the valuative criterion, using Grothendieck's semi-stable reduction theorem for abelian varieties. Then we finish the construction.

### 7.4. General case: Shimura varieties of PEL-type

The method of the arithmetic compactification of the moduli of elliptic curves generalizes to more general Shimura varieties.

First, we need a good model of Shimura varieties over $\mathbb{Z}$. For this purpose, we must restrict ourselves to the so-called PEL-case, which can be seen as a moduli of abelian varieties with some rigidification structures, namely a rigidification of a polarization, the endomorphism ring, the Hodge filtration, and the Betti realization ([39], [12]). For the general definition of Shimura varieties we refer to [12] and [13].

Let $L$ be a semi-simple algebra over $\mathbb{Q}$ with a positive involution *, $V$ a finite dimensional $\mathbb{Q}$-vector space that is a faithful $L$-module with a non-degenerate $\mathbb{Q}$-valued skew symmetric form $\varphi$ that satisfies
the equality

$$
\varphi(\ell x, y)=\varphi\left(x, \ell^{*} y\right), \quad \text { for } x, y \in V, \ell \in L
$$

The reductive group $G$ over $\mathbb{Q}$ is the group of $L$-linear symplectic similitudes of $V$.

Let $X$ be the set of all homomorphisms $h: \operatorname{Res}_{\mathbb{C} / \mathbb{R}} \mathbb{G}_{m} \rightarrow G_{\mathbb{R}}$ such that the $\mathbb{R}$-Hodge structure defined by $h$ on $V_{\mathbb{R}}$ has the type $\{(-1,0),(0$, $-1)\}$ and polarized by $\varphi$. The involution of $L$ is required to be positive for this structure.

Then $X$ carries a natural complex structure. Each connected component of $X$ is a hermitian symmetric domain. To simplify the situation, we assume that all $\mathbb{R}$-simple factors of the derived group $G^{\text {der }}$ are of type $A$ or $C$, and hence $G^{\text {der }}$ is simply connected. The corresponding (nonconnected) Shimura variety for $(G, X)$ over $\mathbb{C}$ is defined by

$$
S h_{K}(G, X)(\mathbb{C})=G(\mathbb{Q}) \backslash X \times G\left(\mathbb{A}_{f}\right) / K,
$$

where $K$ is a compact open subgroup of $G\left(\mathbb{A}_{f}\right)$. The space $S h_{K}(G, X)$ is a quasi-projective variety defined over an explicit number field $E$ called the reflex field of $(G, X)$.

To get an arithmetic moduli, except for several successful cases, only the case of good reduction has been considered in general. Zink [47], Langlands-Rappoport [30] have defined a smooth arithmetic model $S h_{K}(G, X)$ over $\mathscr{S}_{K}$ of $S h_{K}(G, X)(\mathbb{C})$ as the solution of a moduli problem involving abelian schemes (under the restriction on $G$ ). Here $\mathscr{O}_{E}$ is the ring of integers of $E$, and $\mathscr{S}_{K} \subset \operatorname{Spec} \mathscr{O}_{E}$ is an open set explicitly described by $K$ (for general $K$, we regard $S h_{K}(G, X)$ as a DeligneMumford stack).

By definition, there exists the universal abelian scheme

$$
f^{\text {univ }}: \mathscr{A}^{\text {univ }} \rightarrow S h_{K}(G, X)
$$

that gives Shimura's family over $\mathbb{C}$ in [39].
Proposition 7.4. Choose an admissible cone decomposition $\Sigma$ that is compatible with $K$. Then the toroidal compactification $\operatorname{Sh}_{K}(G, X)(\Sigma)$ of $S h_{K}(G, X)$ over $\mathscr{S}_{K}$ for this cone decomposition that satisfies the following properties exists:
(1) $S h_{K}(G, X)(\Sigma)$ is a proper Deligne-Mumford stack over $\mathscr{S}_{K}$ whose local structure near the boundary is described by the toroidal embeddings that correspond to cones in $\Sigma$;
(2) the geometric fiber over $\operatorname{Spec} \mathbb{C}$ is the one constructed in [1];
(3) the universal abelian scheme $f^{\text {univ }}: \mathscr{A}^{\text {univ }} \rightarrow \operatorname{Sh}_{K}(G, X)$ extends uniquely to a semiabelian scheme

$$
\bar{f}^{\text {univ }}: \overline{\mathscr{A}}^{\text {univ }} \longrightarrow S h_{K}(G, X)(\Sigma) .
$$

Note that our compactification $S h_{K}(G, X)(\Sigma)$ is, a priori, an algebraic stack or, for $K$ small enough, an algebraic space. Here we suggest how the construction will be done along the line described in subsection 7.3.

The role of $W$ in $\S 7.1$ is played by (the arithmetic model of) the mixed Shimura varieties $S h\left(P, X_{P}\right)$ associated to $\mathbb{Q}$-maximal parabolic subgroup $P$ of $G[8]$. These mixed Shimura varieties are seen as a moduli space of 1-motives with PEL-structure, and admit a fibration $S h\left(P, X_{P}\right) \rightarrow B_{P}$ by a split torus $T_{P}$. Our choice of the cone decomposition determines a torus embedding $T_{P} \hookrightarrow T_{P, \sigma}$ for a cone $\sigma$, and a partial compactification $\operatorname{Sh}\left(P, X_{P}\right)_{\sigma}$ of $S h\left(P, X_{P}\right)$ is obtained by the contracted product $S h\left(P, X_{P}\right) \wedge^{T_{P}} T_{P, \sigma}$ (that is, the fiber bundle with the fibers $T_{P, \sigma}$ associated to the torus bundle $\left.S h\left(P, X_{P}\right) \rightarrow B_{P}\right)$.

The partial compactification $\operatorname{Sh}\left(P, X_{P}\right)_{\sigma}$ plays the role of $\bar{W}$ in $\S 7.1$. Then one uses the Mumford construction of semi-abelian schemes, which is a generalization of Tate construction to higher dimensional abelian schemes, to get a semi-abelian scheme $\mathscr{A}_{\sigma}$ from the universal 1-motive on $\operatorname{Sh}\left(P, X_{P}\right)$ after completion along the closed $T_{P}$-orbit $D_{\sigma}$. Then we algebraize $\left(\mathscr{A}_{\sigma}, S h\left(P, X_{P}\right)_{\sigma / D_{\sigma}}\right)$ by using Artin's approximation theorem as in $\S 7.3$, Step 1.

The difficulty to construct $S h_{K}(G, X)(\Sigma)$, compared to the elliptic curve case, lies in the fact that the openness of versality is much harder to show. For example, the types of degenerations of abelian varieties of fixed PEL-type is much more complicated, so we must somehow control the various types of degenerations and different partial compactifications at the same time to show the openness of versality.

To check the versality, we use rigid geometry and the Raynaud-Tate theory for semiabelian schemes over one-dimensional complete rings (the argument is similar to that in $\S 7.3$, Step 2 and Step 3, but more complicated), with a closer analysis of degenerations using the uniformization theory in [16].

Recall that the use of rigid geometry in compactification problem goes back to Rapoport's fundamental and important work on HilbertBlumental varieties [36]. The use of Artin's approximation theorem goes back to Faltings work, and discussed in [16] for Siegel modular varieties. For Siegel modular varieties, there is also a method of Chai [9]. He constructs the arithmetic toroidal compactification (corresponding to
projective cone decomposition) by blowing up the minimal compactification, using the theory of algebraic theta functions ${ }^{26}$.

Remark 7.5. In [16], Kodaira-Spencer mappings are used to verify the openness of versality, so one needs to assume the smoothness of arithmetic models in principle. The method here has the advantage that it is singularity free: if a good theory of canonical arithmetic models of Shimura varieties over the ring of integers were available, our method in $\S 7.3$ also gives the arithmetic compactifications including the bad reduction cases, as long as arithmetic models of mixed Shimura varieties corresponding to parabolics are normal.

### 7.5. Applications of arithmetic compactifications

The existence of arithmetic compactification has important consequences on modular forms. Fix an admissible cone decomposition $\Sigma$ and consider the arithmetic toroidal compactification. The line bundle

$$
\omega=\operatorname{det}\left(\operatorname{Lie}\left(\overline{\mathscr{A}}^{\mathrm{univ}} / S h_{K}(G, X)(\Sigma)\right)^{\vee}\right)
$$

on $S h_{K}(G, X)(\Sigma)$ is semi-ample by a theorem of Moret-Bailly [33]. The space of sections

$$
M_{k}=\Gamma\left(S h_{K}(G, K)(\Sigma), \omega^{\otimes k}\right)
$$

is independent of $\Sigma$ and regarded as a space of geometric modular forms of weight $k$. By the properness of $S h_{K}(G, X)(\Sigma), M_{k}$ is finitely generated $\Gamma\left(C_{K}, \mathscr{O}_{C_{K}}\right)$-module, and the graded ring $\bigoplus_{k \geq 1} M_{k}$ is finitely generated over $\Gamma\left(X_{K}, \mathscr{O}_{C_{K}}\right)$ by Moret-Bailly's theorem. This is already an important finiteness statement on geometric modular forms, which is hard to prove by other methods. The geometric modular forms in our sense is identified with holomorphic modular forms with integral coefficients ( $q$-expansion principle). Summing up, we have the following statement.

Proposition 7.6 (cf. [16, Chap. V, §1] in the Siegel modular case). The following properties hold if bad primes are invertible in the coefficients:
(a) Koecher principle,
(b) $q$-expansion principle,
(c) the finiteness theorem for the space of geometric modular forms of given weight (including the vector valued case).

[^26]One can also show that $\overline{S h_{K}(G, X)}{ }_{\text {min }}=\operatorname{Proj} \bigoplus_{k \geq 1} M_{k}$ gives another compactification of $S h_{K}(G, X)$, which is in fact the arithmetic minimal (= Satake, Baily-Borel) compactification:

Proposition 7.7 (cf. [16, Chap. V, §2] in the Siegel modular case). The compactification $\overline{S h}_{K}(G, X)$ min of $S h_{K}(G, X)$ has the following property: for a Noetherian normal scheme $S$, an open dense subscheme $U$, and a morphism $f: U \rightarrow S h_{K}(G, X)$ such that the pull-back of the universal abelian scheme $f^{*}\left(\mathscr{A}^{\text {univ }}\right)$ admits a semi-stable reduction to $S$, $f$ has a unique extension $\bar{f}: S \rightarrow{\overline{S h_{K}(G, X)}}_{\min }$.

These integrality results have very important consequence in number theory. For example, one can use $q$-expansion principle to produce congruence between two modular forms. Deligne and Ribet [14] constructed $p$-adic $L$-functions for finite order characters over a totally real field by using Hilbert-Blumenthal varieties, and recently Urban and Skinner use similar method for unitary Shimura varieties in their study of Iwasawa main conjecture of elliptic curves over $\mathbb{Q}$.

## $\S 8$. Rigid spaces and Frobenius

The main subject to be dealt with in this section, as the second application, is an application of rigid geometry to theory of schemes. The category of rigid spaces is, as pointed out before, much broader than that of schemes. Hence, what we like to show is, so to speak, one of the "non-scheme-theoretic" methods for treating schemes. In fact, such methods that derail from scheme theory often reveal hidden and important features in scheme theory, which would be quite invisible only from the scheme-theoretic point of view.

In this section, we particularly focus on Frobenius. To do this, we first show the general technique to bridge between scheme theory and rigid geometry in the next subsection.

### 8.1. From schemes to rigid spaces; constant deformation technique

This is the general technique that is important for applying rigid geometry to geometry of schemes. The general picture is as follows (Figure 10).

First, start from a variety $X$ over a field $k$. From $X$ we are going to construct canonically a rigid space. Consider the ring of formal power series $k[[t]]$ endowed with the $t$-adic topology, and put $\widehat{X}_{k[[t]]}=X \times_{k}$ Spf $k[[t]]$ (constant deformation). Then one takes its associated rigid space $\left(\widehat{X}_{k[[t]]}\right)^{\text {rig }}$ over $(\operatorname{Spf} k[[t]])^{\text {rig }}$.

| Variety |  | Formal scheme |  | Rigid space |
| :---: | :---: | :---: | :---: | :---: |
| X <br> Spec $k$ | $\cdots$ | $\downarrow_{\operatorname{Spf} k[[t]]}^{\widehat{X}_{k[[t]]}}$ | $\rightsquigarrow$ | $\downarrow_{(\operatorname{Spf} k[[t]])^{\mathrm{rig}}}^{\left(\widehat{X}_{k[[t]]}\right)^{\mathrm{rig}}}$ |

Fig. 10. Constant deformation technique

In spite of its entirely trivial looking, this construction opens the way for several effective applications of rigid geometry to algebraic geometry.

### 8.2. Frobenius

Rigid geometry reveals a new feature of Frobenius morphisms in positive characteristic. This feature will be given in Claim 8.1.

Consider the following situation:

- $S$ : an $\mathbb{F}_{q}$-scheme,
- $\mathrm{Fr}_{q}: S \rightarrow S:$ Frobenius over $\mathbb{F}_{q}$, that is, the $q$-th power map,
- $\mathscr{C}_{S}$ : a category of geometric objects over $S$.

One of the properties of Frobenius morphisms that are already known to be very important in classical algebraic and arithmetic geometry is that, most of the time, the Frobenius induces a self-functor

$$
\operatorname{Fr}_{q}^{*}: \mathscr{C}_{S} \longrightarrow \mathscr{C}_{S}
$$

In other words, one has the "dynamical system" with the "phase space" $\mathscr{C}_{S}$ acted on by the "self-similarity map" $\mathrm{Fr}_{q}^{*}$. As usual in the theory of dynamical system, one is particularly interested in the " $\mathrm{Fr}_{q}^{*}$-fixed point", that is, the $\mathrm{Fr}_{q}$-structure

$$
\operatorname{Fr}_{q}^{*} A \cong A
$$

Once one has such a structure, one is interested in the following question.
Question: what happens near the " $\mathrm{Fr}_{q}^{*}$-fixed point"?
Constant deformation and Frobenius. In this context, the constant deformation technique proves to be useful. First observe that, in complex situation with $\mathbb{A}_{\mathbb{C}}^{1}=\operatorname{Spec} \mathbb{C}[X]$, the selfmap

$$
\mathbb{A}_{\mathbb{C}}^{1} \longrightarrow \mathbb{A}_{\mathbb{C}}^{1}, \quad X \mapsto X^{q}
$$

is a contracting map near the origin for the analytic topology. The rigid geometric counterpart of this is the following:

$$
\mathbb{D}^{1} \longrightarrow \mathbb{D}^{1}, \quad X \mapsto X^{q}
$$

is contracting near 0 in the sense that $\left(\operatorname{Fr}_{q}\left(\mathbb{D}^{1}(r)\right) \subset \mathbb{D}^{1}\left(r^{q}\right)\right.$.
More precisely, for an $\mathbb{F}_{q^{-}}$variety $X$, consider the rigid space

$$
\mathscr{X}=\left(\widehat{X}_{\mathbb{F}_{q}[[t]]}\right)^{\text {rig }}
$$

obtained by constant deformation. Let $Y \subseteq X$ be an $\operatorname{Fr}_{q}$-invariant subspace, and set

$$
\mathscr{Y}=\left(\widehat{Y}_{\mathbb{F}_{q}[[t]]}\right)^{\mathrm{rig}} .
$$

We think of $\mathscr{X}$ as the phase space equipped with the dynamical system

$$
\mathscr{X} \xrightarrow{\mathrm{Fr}_{q}} \mathscr{X} \xrightarrow{\mathrm{Fr}_{q}} \mathscr{X} \xrightarrow{\mathrm{Fr}_{q}} \cdots .
$$

This can be more concretely done by means of the associated ZariskiRiemann space $\langle\mathscr{X}\rangle$; by this, we have a topological space (in the usual sense) as the phase space.

Claim 8.1. "The Frobenius mapping is contracting near $Y$," i.e., $\mathrm{Fr}_{q}$ is contracting near $\langle\mathscr{Y}\rangle$ in $\langle\mathscr{X}\rangle$.

The claim can be shown by the reasoning similar to that in the case of the unit disk as above. The property of Frobenius that the claim shows is so essential in general that it actually simplifies arguments in many situations. The Lefschetz trace formula, which we are to discuss in the next subsection, is one of them.

### 8.3. Trace formula in characteristic $p$

The "dynamical system" approach to Frobenius as in $\S 8.2$ has already appeared and applied in the study of Lefschetz trace formula in characteristic $p$ by the first-named author (solution of Deligne's conjecture [19]). Let us briefly outline the argument therein.

Deligne's conjecture. Let $X$ be an algebraic variety over a field $k$. Consider a correspondence

$$
a: Y \longrightarrow X \times_{k} X
$$

such that $a_{1}=\operatorname{pr}_{1} \circ a$ is proper and that $a_{2}=\operatorname{pr}_{2} \circ a$ is quasi-finite. Let $K$ be a $\overline{\mathbb{Q}}_{\ell}$-complex (where $\frac{1}{\ell} \in k$ ) with a cohomological correspondence compatible with $a$. In this situation the Lefschetz number, an element of $\overline{\mathbb{Q}}_{\ell}$, is defined by

$$
\operatorname{Lef}\left(a, \operatorname{R} \Gamma_{\mathrm{c}}(X, K)\right)=\operatorname{Trace}\left(a^{*}, \operatorname{R} \Gamma_{\mathrm{c}}(X, K)\right)
$$

Theorem 8.2 (Deligne's conjecture; [19]). Let $k=\overline{\mathbb{F}}_{q}$. If the above data admit $\operatorname{Fr}_{q}$-structure, then there exists $N \in \mathbb{N}$ such that the following conditions are satisfied:
(1) $\operatorname{dim} \operatorname{Fix}\left(\operatorname{Fr}_{q}^{n} \circ a\right)=0$ for $q^{n}>N$;
(2) for $q^{n}>N$,

$$
\operatorname{Lef}\left(\operatorname{Fr}_{q}^{n} \circ a, \operatorname{R\Gamma }_{\mathrm{c}}(X, K)\right)=\sum_{D \in \operatorname{Fix}\left(\operatorname{Fr}_{q}^{n} \circ a\right)} \text { naive. } \operatorname{loc}_{D}\left(\operatorname{Fr}_{q}^{n} \circ a, K\right)
$$

Here naive. $\operatorname{loc}_{D}(a, K)$ vanishes if $\left.K\right|_{a_{2}(D)}=0$.
The proof is given by establishing the trace formula for certain rigid analytic correspondences; note that this argument is not completely scheme-theoretical.

Another way of proof was given by Shpiz and Pink in their work around 1990, in which they assume that $X$ is smooth and $K$ is a smooth sheaf, that there exists a good compactification, and that $K$ is tame. Recently, T. Saito and Y. Varshavsky [43] independently gave schemetheoretic proofs.

Remark 8.3. In [19] it was assumed that $X$ and $Y$ are schemes. But, by Equivalence Theorem (Theorem 6.12) and Nagata's Embedding Theorem for algebraic spaces, we may weaken the assumption to that $X$ and $Y$ are separated algebraic spaces of finite type over $k$. This generalization actually eliminates the use of an argument in [29] to show that the moduli space of Shtuka is a scheme. ${ }^{27}$

Applications of Deligne's conjecture. Finally, let us list some of the applications of Deligne's conjecture, which provides a very strong counting argument in arithmetic and many other areas in mathematics:

- Non-abelian class field theory (Shtuka moduli (L. Lafforgue [29]), Shimura varieties (Harris-Taylor...)),
- Representation theory of Chevalley groups (Digne-Rouquier, ...),
- Model theory (Hrushovski-Macintyre).


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# Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI, part II 

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#### Abstract

. In this paper, we show that the family of moduli spaces of $\boldsymbol{\alpha}^{\prime}$ stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic $\phi$-connections of rank 2 over $\mathbf{P}^{1}$ with 4 -regular singular points and the fixed determinant bundle of degree -1 is isomorphic to the family of Okamoto-Painlevé pairs introduced by Okamoto [O1] and [STT]. We also discuss about the generalization of our theory to the case where the rank of the connections and genus of the base curve are arbitrary. Defining isomonodromic flows on the family of moduli space of stable parabolic connections via the Riemann-Hilbert correspondences, we will show that a property of the Riemann-Hilbert correspondences implies the Painlevé property of isomonodromic flows.


## §1. Introduction

In part I [IIS1], we established a complete geometric background for Painlevé equations of type VI or more generally for Garnier systems from view points of moduli spaces of rank 2 stable parabolic connections, moduli spaces of $S L_{2}$-representations of $\pi_{1}\left(\mathbf{P}^{1} \backslash D(\mathbf{t})\right)$ and the RiemannHilbert correspondences between them.

In this formulation, Painlevé equations of type VI or Garnier systems are vector fields or systems of vector fields on each corresponding family of moduli spaces of stable parabolic connections arising from

[^28]isomonodromic deformations of linear connections. Most notably, we can give a complete geometric proof of the Painlevé property of Painlevé equations of type VI and Garnier systems by proving that the RiemannHilbert correspondences are bimeromorphic proper surjective holomorphic maps. Moreover, one can prove that the Riemann-Hilbert correspondences give analytic resolutions of singularities of moduli spaces of the $S L_{2}$-representations. Then on the inverse image of each singular point, which is a family of compact subvarieties in the family of moduli spaces of connections, the vector fields admit classical solutions such as Riccati solutions in Painlevé VI case. See [Iw1], [Iw2], [SU], [IIS0], [STe] and [IIS3], for further applications of our approach to explicit dynamics of the Painlevé VI equations such as the classification of Riccati solutions and rational solutions, nonlinear monodromy, and Bäklund transformations as well as the relation with the former results [Miwa], [Mal] on the Painlevé property.

In this paper, with the notation in $\S 3$, we study in detail the moduli space $\overline{M_{4}^{\boldsymbol{\alpha}^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$ of $\boldsymbol{\alpha}^{\prime}$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic $\phi$-connections of rank 2 over $\mathbf{P}^{1}$ with the fixed determinant bundle of degree -1 as well as the moduli space $M_{4}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda},-1)$ of corresponding $\boldsymbol{\alpha}$-stable $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections of rank 2 over $\mathbf{P}^{1}$. From a general result ([Theorem 1.1, [IIS1]] or [Theorem 5.1, §3]) which is also valid for $n \geq 5$, we can show that

- $\overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$ is a projective surface,
- $M_{4}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda},-1)$ is a smooth irreducible algebraic surface with a holomorphic symplectic structure and
- there exists a natural embedding $M_{4}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda},-1) \hookrightarrow \overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$.

In Theorem 4.1, which is the main theorem in this paper, we will show that the moduli space $\overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$ is isomorphic to a smooth projective rational surface $\overline{\mathcal{S}}_{\mathbf{t}, \boldsymbol{\lambda}}$. Moreover we can show that there exists a unique effective anti-canonical divisor $\mathcal{Y}_{\mathbf{t}, \boldsymbol{\lambda}} \in\left|-K_{\overline{\mathcal{S}}_{\mathbf{t}, \boldsymbol{\lambda}}}\right|$ of $\overline{\mathcal{S}}_{\mathbf{t}, \boldsymbol{\lambda}}$ such that $\overline{\mathcal{S}}_{\mathbf{t}, \boldsymbol{\lambda}} \backslash \mathcal{Y}_{\mathbf{t}, \boldsymbol{\lambda}, \text { red }} \simeq M_{4}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda},-1)$. Moreover $\left(\overline{\mathcal{S}}_{\mathbf{t}, \boldsymbol{\lambda}}, Y_{\mathbf{t}, \boldsymbol{\lambda}}\right)$ is a nonfibered rational Okamoto-Painlevé pairs of type $D_{4}^{(1)}$ which is defined in $[\mathrm{STT}]$ (cf. [Sakai]). Note that $\overline{\mathcal{S}}_{\mathbf{t}, \boldsymbol{\lambda}} \backslash \mathcal{Y}_{\mathbf{t}, \boldsymbol{\lambda}, \text { red }}$ is isomorphic to the space of initial conditions for Painlevé equations of type VI constructed by Okamoto [O1].

We should mention here that an algebraic moduli space of parabolic connections without stability conditions was essentially considered by D. Arinlin and S. Lysenco in [AL1], [AL2] and [A] and they constructed a nice moduli space for generic $\boldsymbol{\lambda}$. However for special $\boldsymbol{\lambda}$, we should consider certain stability condition to construct a nice moduli space.

There are also different approaches [ N$],[\mathrm{Ni}]$ for constructions of moduli spaces of logarithmic connections with or without parabolic structures.

The rough plan of this paper is as follows. In §2, we will explain about motivation of this paper and the theory of Okamoto-Painlevé pairs in [STa] and [STT]. In $\S 3$, we review results in part I [IIS1]. In $\S 4$, we will state Theorem 4.1 and the rest of the section will be devoted to show this theorem. In $\S 5$, we give a formulation of moduli theory of stable parabolic connection with regular singularities of any rank over any smooth curve. We also define the moduli space of representations of the fundamental group of $n$-punctured curve of genus $g$. Then we state the existence theorem of moduli space due to Inaba [Ina] without proof. In $\S 6$, we define the Riemann-Hilbert correspondence and state, also without proof, Theorem 6.1 which says that the Riemann-Hilbert correspondence is a proper surjective bimeromorphic analytic morphism. In $\S 7$, we will define isomonodromic flows on the family of the moduli spaces of $\boldsymbol{\alpha}$-stable parabolic connections. Assuming that Theorem 6.1 is true, we will show that isomonodromic flows satisfy the Painlevé property. (Note that, if rank $r=2$ and over $\mathbf{P}^{1}$, a proof of Theorem 6.1 is found in [IIS1]).

Throughout in this paper, we will work over the field $\mathbf{C}$ of complex numbers.

## §2. Motivation-Painlevé equations of type VI and OkamotoPainlevé pairs

Let us recall the theory of space of initial conditions of Painlevé equation of type VI. Fix $\boldsymbol{\lambda}=\left(\lambda_{1}, \cdots, \lambda_{4}\right) \in \Lambda_{4}=\mathbf{C}^{4}$ and consider the following ordinary differential equation of Painlevé VI type $P_{V I}(\boldsymbol{\lambda})$ parameterized by $\boldsymbol{\lambda}$ :
(1)

$$
\begin{aligned}
& P_{V I}(\boldsymbol{\lambda}): \\
& \frac{d^{2} x}{d t^{2}}= \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-t}\right)\left(\frac{d x}{d t}\right)^{2}- \\
&\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{x-t}\right)\left(\frac{d x}{d t}\right)+\frac{x(x-1)(x-t)}{t^{2}(t-1)^{2}} \times \\
& {\left[2\left(\lambda_{4}-\frac{1}{2}\right)^{2}-2 \lambda_{1}^{2} \frac{t}{x^{2}}+2 \lambda_{2}^{2} \frac{t-1}{(x-1)^{2}}+\left(\frac{1}{2}-2 \lambda_{3}^{2}\right) \frac{t(t-1)}{(x-t)^{2}}\right] . }
\end{aligned}
$$

It is known that this algebraic differential equation $P_{V I}(\boldsymbol{\lambda})$ is equivalent to the following nonautonomous Hamiltonian system:

$$
\left(H_{V I}(\boldsymbol{\lambda})\right):\left\{\begin{align*}
\frac{d x}{d t} & =\frac{\partial H_{V I}}{\partial y}  \tag{2}\\
\frac{d y}{d t} & =-\frac{\partial H_{V I}}{\partial x}
\end{align*}\right.
$$

where the Hamiltonian is given as follows.

$$
\begin{aligned}
H_{V I}(x, y, t)= & \frac{1}{t(t-1)}\left[x(x-1)(x-t) y^{2}-\left\{2 \lambda_{1}(x-1)(x-t)\right.\right. \\
& \left.\left.+2 \lambda_{2} x(x-t)+\left(2 \lambda_{3}-1\right) x(x-1)\right\} y+\lambda(x-t)\right] \\
(\lambda:= & \left.\left\{\left(\lambda_{1}+\lambda_{2}+\lambda_{3}-1 / 2\right)^{2}-\left(\lambda_{4}-\frac{1}{2}\right)^{2}\right\}\right)
\end{aligned}
$$

Let us set $T=\mathbf{C} \backslash\{0,1\}$ and consider the following algebraic vector fields on $\mathcal{S}^{(0)}=\mathbf{C}^{2} \times T \times \Lambda_{4} \ni(x, y, t, \boldsymbol{\lambda})$

$$
\begin{equation*}
v=\frac{\partial}{\partial t}+\frac{\partial H_{V I}}{\partial y} \frac{\partial}{\partial x}-\frac{\partial H_{V I}}{\partial x} \frac{\partial}{\partial y} \tag{3}
\end{equation*}
$$

Taking a relative compactification $\overline{\mathcal{S}}^{(0)}=\Sigma_{0} \times T \times \Lambda_{4}$ of $\mathcal{S}^{(0)}$ where $\Sigma_{0}=\mathbf{P}^{1} \times \mathbf{P}^{1}$ and setting $\mathcal{D}^{(0)}=\overline{\mathcal{S}}^{(0)} \backslash \mathcal{S}^{(0)}$, we obtain the commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{S}^{(0)} & \hookrightarrow & \overline{\mathcal{S}}^{(0)} & \hookleftarrow & \mathcal{D}^{(0)}  \tag{4}\\
& \searrow \pi & \downarrow \bar{\pi}^{(0)} & \swarrow & \\
& T \times \Lambda_{4} . & &
\end{array}
$$

We can extend the vector field $v$ in $(3)$ on $\mathcal{S}^{(0)}$ to a rational vector field

$$
\begin{equation*}
\tilde{v} \in H^{0}\left(\overline{\mathcal{S}}^{(0)}, \Theta_{\overline{\mathcal{S}}^{(0)}}\left(* \mathcal{D}^{(0)}\right)\right) \tag{5}
\end{equation*}
$$

In general, the rational vector field $\tilde{v}$ has accessible singularities at the boundary divisor $\mathcal{D}^{(0)}$. In [O1], Okamoto gave explicit resolutions of accessible singularities by successive blowings-up at points on the boundary divisor. Then finally, we obtain a smooth family of smooth projective rational surfaces

$$
\begin{array}{ccccc}
\mathcal{S} & \hookrightarrow & \overline{\mathcal{S}} & \hookleftarrow & \mathcal{D}  \tag{6}\\
& \searrow \pi & \downarrow \pi & \swarrow &
\end{array}
$$

such that $\mathcal{D}:=\overline{\mathcal{S}} \backslash \mathcal{S}$ is a reduced normal crossing divisor and $\mathcal{S}$ contains $\mathcal{S}^{(0)}$ as a Zariski open set. Moreover one can show that

$$
\begin{equation*}
\tilde{v} \in H^{0}\left(\overline{\mathcal{S}}, \Theta_{\overline{\mathcal{S}}}(-\log \mathcal{D})(\mathcal{D})\right) \tag{7}
\end{equation*}
$$

where $\Theta_{\overline{\mathcal{S}}}(-\log \mathcal{D})$ denotes the sheaf of germs of regular vector fields with logarithmic zero along $\mathcal{D}$ (cf. [STT]). The extended rational vector field $\tilde{v}$ on $\overline{\mathcal{S}}$ has poles of order 1 along $\mathcal{D}$ and is regular on $\mathcal{S}=\overline{\mathcal{S}} \backslash \mathcal{D}$.

For each fixed $(t, \boldsymbol{\lambda}) \in T \times \Lambda_{4}$, the fiber $\bar{\pi}^{-1}((t, \boldsymbol{\lambda}))=\overline{\mathcal{S}}_{t, \boldsymbol{\lambda}}$ has a unique effective anti-canonical divisor $\mathcal{Y}_{t, \boldsymbol{\lambda}} \in\left|-K_{\overline{\mathcal{S}}_{t, \boldsymbol{\lambda}}}\right|$ with the irreducible decomposition

$$
\mathcal{Y}_{t, \boldsymbol{\lambda}}=2 D_{0}+D_{1}+D_{2}+D_{3}+D_{4}
$$

such that $\mathcal{Y}_{t, \boldsymbol{\lambda}, \text { red }}=\sum_{i=0}^{4} D_{i}=\mathcal{D}_{t, \boldsymbol{\lambda}}$. Moreover it satisfies the following numerical conditions

$$
\begin{equation*}
\mathcal{Y}_{t, \boldsymbol{\lambda}} \cdot D_{i}=\operatorname{deg}\left(-K_{\overline{\mathcal{S}}_{t, \boldsymbol{\lambda} \mid D_{i}}}\right)=0 \text { for } i=0, \ldots, 4 \tag{8}
\end{equation*}
$$

In [STT], we give the following
Definition 2.1. (Cf. [STT], [STa], [Sakai]). A pair $(S, Y)$ of a smooth projective rational surface with an anti-canonical divisor $Y \in$ $\left|-K_{S}\right|$ with the irreducible decomposition $Y=\sum_{i} m_{i} Y_{i}$ is called a rational Okamoto-Painlevé pair if it satisfies the condition

$$
\begin{equation*}
Y \cdot Y_{i}=\operatorname{deg}\left(-K_{\overline{\mathcal{S}}_{t, \lambda} \mid Y_{i}}\right)=0 \text { for all } i \tag{9}
\end{equation*}
$$

A rational Okamoto-Painlevé pair $(S, Y)$ is called of fibered-type if there exists an elliptic fibration $f: S \longrightarrow \mathbf{P}^{1}$ such that $f^{*}(\infty)=n Y$ for some $n \geq 1$.

It is easy to see that for a rational Okamoto-Painlevé pair the configuration of $Y$ is in the list of degenerate fibers of elliptic surfaces due to Kodaira, which was classified by affine Dynkin diagrams. Therefore, we have a classification of rational Okamoto-Painlevé pairs $(S, Y)$ by the Dynkin diagram of $Y$. For the case of Painlevé VI, we can say that the pair $\left(\overline{\mathcal{S}}_{t, \boldsymbol{\lambda}}, \mathcal{Y}_{t, \boldsymbol{\lambda}}\right)$ appeared in a fiber of the family (6) is a rational Okamoto-Painlevé pair of type $D_{4}^{(1)}$. The family of the complement of the divisor $\mathcal{D}$ in (6) $\mathcal{S} \longrightarrow T \times \Lambda_{4}$, where the rational vector field $\tilde{v}$ is regular, should be the family of the space of initial conditions of Painleve equations of type VI or the phase space of the vector field $\tilde{v}$. Note that $\mathcal{S} \longrightarrow T \times \Lambda_{4}$ contains the original family $\mathcal{S}^{(0)} \longrightarrow T \times \Lambda_{4}$ as a proper Zariski open subset, that is, $\mathcal{S}^{(0)} \subsetneq \mathcal{S}$. Here we recall the following technical lemma proved in [Proposition 1.3, [STT]].

Lemma 2.1. Let $(S, Y)$ be a rational Okamoto-Painlevé pair. Then the following conditions are equivalent to each other.
(1) $(S, Y)$ is non-fibered type.
(2) $A$ regular algebraic functions on the complement $S \backslash Y_{\text {red }}$ must be a constant function.
In particular, for a non-fibered rational Okamoto-Painlevé pair $(S, Y)$, the complement $S \backslash Y_{\text {red }}$ is never an affine variety.

Since one can show that an Okamoto-Painlevé pair $\left(\overline{\mathcal{S}}_{t, \boldsymbol{\lambda}}, \mathcal{Y}_{t, \boldsymbol{\lambda}}\right)$ which appeared in a fiber of $\bar{\pi}$ in (6) is non-fibered type, we obtain the following

Corollary 2.1. As for the family (6) for Painlevé equations of type $V I$ constructed by Okamoto [O1], each fiber $\mathcal{S}_{t, \boldsymbol{\lambda}}=\overline{\mathcal{S}}_{t, \boldsymbol{\lambda}} \backslash \mathcal{D}_{t, \boldsymbol{\lambda}}$ is not an affine variety.

In Theorem 4.1, we will show that the family (6) $\overline{\mathcal{S}} \longrightarrow T \times \Lambda_{4}$ constructed by Okamoto in [O1] is isomorphic to the family of moduli spaces

$$
\overline{M_{4}^{\alpha^{\prime}}}(-1) \longrightarrow T_{4} \times \Lambda_{4}
$$

of $\boldsymbol{\alpha}^{\prime}$-stable parabolic $\phi$-connections of rank 2 over $\mathbf{P}^{1}$ with 4 regular singular points. (In order to identify, we need to normalize 4 points $\left(t_{1}, t_{2}, t_{3}, t_{4}\right)$ to $\left.(0,1, t, \infty)\right)$.

In [IIS1], for $\mathbf{a}=\left(a_{1}, \cdots, a_{4}\right) \in \mathcal{A}_{4} \simeq \mathbf{C}^{4}$, we can also consider the moduli space $\mathcal{R}\left(\mathcal{P}_{4, t}\right)_{\mathbf{a}}$ of $S L_{2}(\mathbf{C})$-representations $\rho$ of $\pi_{1}\left(\mathbf{P}^{1} \backslash D(\mathbf{t})\right)$ with the conditions $\operatorname{Tr}\left[\rho\left(\gamma_{i}\right)\right]=a_{i}$. Then we can define the Riemann-Hilbert correspondence

$$
\begin{equation*}
\mathbf{R H}_{t, \boldsymbol{\lambda}}: S_{t, \boldsymbol{\lambda}} \simeq M_{4}^{\boldsymbol{\alpha}}(t, \boldsymbol{\lambda},-1) \longrightarrow \mathcal{R}\left(\mathcal{P}_{4, t}\right)_{\mathbf{a}} \tag{10}
\end{equation*}
$$

where $a_{i}=2 \cos 2 \pi \lambda_{i}$.
Note that the Riemann-Hilbert correspondence is a highly transcendental analytic morphism, which is never an algebraic morphism. From results in [IIS1], we can show the following Theorem, which shows highly transcendental nature of the Riemann-Hilbert correspondence $\mathbf{R H}_{t, \boldsymbol{\lambda}}$.

Proposition 2.1. (Cf. [Theorem 1.4, Theorem 1.3, [IIS1]] )
(1) For all $(t, \boldsymbol{\lambda}) \in T \times \Lambda_{4}$, the Riemann-Hilbert correspondence $\mathbf{R H}_{t, \boldsymbol{\lambda}}$ is a bimeromorphic proper surjective analytic morphism. If $\boldsymbol{\lambda} \in \Lambda_{4}$ is generic, $\mathbf{R H}_{t, \boldsymbol{\lambda}}$ is an analytic isomorphism.
(2) For all $\mathbf{a} \in \mathcal{A}_{4}, \mathcal{R}\left(\mathcal{P}_{4, t}\right)_{\mathbf{a}}$ is an affine variety, while $S_{t, \boldsymbol{\lambda}} \simeq$ $M_{4}^{\alpha}(t, \boldsymbol{\lambda},-1)$ is not an affine variety. Hence if $\lambda \in \Lambda_{4}$ is generic, $\mathbf{R H}_{t, \boldsymbol{\lambda}}$ gives an analytic isomorphism between a nonaffine variety $S_{t, \boldsymbol{\lambda}} \simeq M_{4}^{\boldsymbol{\alpha}}(t, \boldsymbol{\lambda},-1)$ and an affine $\operatorname{variety} \mathcal{R}\left(\mathcal{P}_{4, t}\right)_{\mathbf{a}}$.
(3) For a generic $\boldsymbol{\lambda} \in \Lambda_{4}, S_{t, \boldsymbol{\lambda}} \simeq M_{4}^{\boldsymbol{\alpha}}(t, \boldsymbol{\lambda},-1)$ is a Stein manifold, but not an affine variety.
In $\S 4$, in order to obtain Okamoto-Painlevé pairs $\left(\overline{\mathcal{S}}_{\mathbf{t}, \boldsymbol{\lambda}}, \mathcal{Y}_{\mathbf{t}, \boldsymbol{\lambda}}\right)$, we use a process of blowings-up which is a little bit different from Okamoto's in [O1]. The process can be explained as follows. Take $\Sigma_{2}=\mathbf{P}\left(\mathcal{O}_{\mathbf{P}^{1}}(2) \oplus\right.$ $\left.\mathcal{O}_{\mathbf{P}^{1}}\right) \longrightarrow \mathbf{P}^{1}$, which is the Hirzebruch surface of degree 2. Let $D_{0}$ denote the unique infinite section with $D_{0}^{2}=-2$ and take the fibers $F_{i}$ over $t_{i}$ for $i=1, \ldots, 4$. From the data $\lambda_{i}$, we can determine two points $b_{i}^{+}$and $b_{i}^{-}$on $F_{i}$. (See $\S 4$ for precise definition of $b_{i}^{ \pm}$). By blowingup of $\Sigma_{2}$ at 8-points $\left\{b_{i}^{ \pm}\right\}_{i=1}^{4}$, we obtain the rational surface $\overline{\mathcal{S}}_{\mathbf{t}, \boldsymbol{\lambda}}$ and the unique effective anti-canonical divisor $\mathcal{Y}_{\mathbf{t}, \boldsymbol{\lambda}}$ can be given by $\mathcal{Y}_{\mathbf{t}, \boldsymbol{\lambda}}=$ $2 D_{0}+D_{1}+D_{2}+D_{3}+D_{4}$ where $D_{i}$ denotes the proper transform of $F_{i}$, (see Fig. 1).


Fig. 1. Okamoto-Painlevé pair of type $D_{4}^{(1)}$
§3. Moduli spaces of rank 2 stable parabolic connections on $P^{1}$ and their compactifications. A review of Part I.

In this section, we reproduce basic notation and definition in part I [IIS1] for reader's convenience.

### 3.1. Parabolic connections on $\mathbf{P}^{1}$.

Let $n \geq 3$ and set

$$
\begin{gather*}
T_{n}=\left\{\left(t_{1}, \ldots, t_{n}\right) \in\left(\mathbf{P}^{1}\right)^{n} \quad \mid \quad t_{i} \neq t_{j},(i \neq j)\right\},  \tag{11}\\
\Lambda_{n}=\left\{\boldsymbol{\lambda}=\left(\lambda_{1}, \ldots, \lambda_{n}\right) \in \mathbf{C}^{n}\right\}
\end{gather*}
$$

Fixing a data $(\mathbf{t}, \boldsymbol{\lambda})=\left(t_{1}, \ldots, t_{n}, \lambda_{1}, \ldots, \lambda_{n}\right) \in T_{n} \times \Lambda_{n}$, we define a reduced divisor on $\mathbf{P}^{1}$ as

$$
\begin{equation*}
D(\mathbf{t})=t_{1}+\cdots+t_{n} \tag{13}
\end{equation*}
$$

Moreover we fix a line bundle $L$ on $\mathbf{P}^{1}$ with a logarithmic connection $\nabla_{L}: L \longrightarrow L \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$.

Definition 3.1. A (rank 2) ( $\mathbf{t}, \boldsymbol{\lambda}$ )-parabolic connection on $\mathbf{P}^{1}$ with the determinant $\left(L, \nabla_{L}\right)$ is a quadruplet $\left(E, \nabla, \varphi,\left\{l_{i}\right\}_{1 \leq i \leq n}\right)$ which consists of
(1) a rank 2 vector bundle $E$ on $\mathbf{P}^{1}$,
(2) a logarithmic connection $\nabla: E \longrightarrow E \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$
(3) a bundle isomorphism $\varphi: \wedge^{2} E \xrightarrow{\simeq} L$
(4) one dimensional subspace $l_{i}$ of the fiber $E_{t_{i}}$ of $E$ at $t_{i}, l_{i} \subset E_{t_{i}}$, $i=1, \ldots, n$, such that
(a) for any local sections $s_{1}, s_{2}$ of $E$,

$$
\varphi \otimes i d\left(\nabla s_{1} \wedge s_{2}+s_{1} \wedge \nabla s_{2}\right)=\nabla_{L}\left(\varphi\left(s_{1} \wedge s_{2}\right)\right)
$$

(b) $\quad l_{i} \subset \operatorname{Ker}\left(\operatorname{res}_{t_{i}}(\nabla)-\lambda_{i}\right)$, that is, $\lambda_{i}$ is an eigenvalue of the residue $\operatorname{res}_{t_{i}}(\nabla)$ of $\nabla$ at $t_{i}$ and $l_{i}$ is a one-dimensional eigensubspace of $\operatorname{res}_{t_{i}}(\nabla)$.

Definition 3.2. Two ( $\mathbf{t}, \boldsymbol{\lambda}$ )-parabolic connections

$$
\left(E_{1}, \nabla_{1}, \varphi,\left\{l_{i}\right\}_{1 \leq i \leq n}\right), \quad\left(E_{2}, \nabla_{2}, \varphi^{\prime},\left\{l_{i}^{\prime}\right\}_{1 \leq i \leq n}\right)
$$

on $\mathbf{P}^{1}$ with the determinant $\left(L, \nabla_{L}\right)$ are isomorphic to each other if there is an isomorphism $\sigma: E_{1} \xrightarrow{\sim} E_{2}$ and $c \in \mathbf{C}^{\times}$such that the diagrams

$$
\begin{align*}
& E_{1} \xrightarrow{\nabla_{1}} E_{1} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \\
& \bigwedge^{2} E_{1} \xrightarrow[\cong]{\varphi} L \\
& \sigma \downarrow \cong \quad \cong \downarrow \sigma \otimes \mathrm{id}  \tag{14}\\
& E_{2} \xrightarrow{\nabla_{2}} E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \\
& \wedge^{2} \sigma \downarrow \cong \quad c \downarrow \cong \\
& \bigwedge^{2} E_{2} \xrightarrow[\cong]{\varphi^{\prime}} L
\end{align*}
$$

commute and $(\sigma)_{t_{i}}\left(l_{i}\right)=l_{i}^{\prime}$ for $i=1, \ldots, n$.

### 3.2. The set of local exponents $\boldsymbol{\lambda} \in \Lambda_{n}$

Note that a data $\boldsymbol{\lambda}=\left(\lambda_{1}, \ldots, \lambda_{n}\right) \in \Lambda_{n} \simeq \mathbf{C}^{n}$ specifies the set of eigenvalues of the residue matrix of a connection $\nabla$ at $\mathbf{t}=\left(t_{1}, \ldots, t_{n}\right)$, which will be called a set of local exponents of $\nabla$.

Definition 3.3. A set of local exponents $\boldsymbol{\lambda}=\left(\lambda_{1}, \ldots, \lambda_{n}\right) \in \Lambda_{n}$ is called special if
(1) $\boldsymbol{\lambda}$ is resonant, that is, for some $1 \leq i \leq n$,

$$
\begin{equation*}
2 \lambda_{i} \in \mathbf{Z} \tag{15}
\end{equation*}
$$

(2) or $\boldsymbol{\lambda}$ is reducible, that is, for some $\left(\epsilon_{1}, \ldots, \epsilon_{n}\right) \in\{ \pm 1\}^{n}$

$$
\begin{equation*}
\sum_{i=1}^{n} \epsilon_{i} \lambda_{i} \in \mathbf{Z} \tag{16}
\end{equation*}
$$

If $\boldsymbol{\lambda} \in \Lambda_{n}$ is not special, $\boldsymbol{\lambda}$ is said to be generic.

### 3.3. Parabolic degrees and $\alpha$-stability

Let us fix a series of positive rational numbers $\boldsymbol{\alpha}=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{2 n}\right)$, which is called a weight, such that

$$
\begin{equation*}
0 \leq \alpha_{1}<\alpha_{2}<\cdots<\alpha_{i}<\cdots<\alpha_{2 n}<\alpha_{2 n+1}=1 \tag{17}
\end{equation*}
$$

For a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection on $\mathbf{P}^{1}$ with the determinant $\left(L, \nabla_{L}\right)$, we can define the parabolic degree of $E=(E, \nabla, \varphi, l)$ with respect to the weight $\boldsymbol{\alpha}$ by

$$
\begin{align*}
\operatorname{pardeg}_{\boldsymbol{\alpha}} E & =\operatorname{deg} E+\sum_{i=1}^{n}\left(\alpha_{2 i-1} \operatorname{dim} E_{t_{i}} / l_{i}+\alpha_{2 i} \operatorname{dim} l_{i}\right)  \tag{18}\\
& =\operatorname{deg} L+\sum_{i=1}^{n}\left(\alpha_{2 i-1}+\alpha_{2 i}\right)
\end{align*}
$$

Let $F \subset E$ be a rank 1 subbundle of $E$ such that $\nabla F \subset F \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$. We define the parabolic degree of $\left(F, \nabla_{\mid F}\right)$ by

$$
\begin{equation*}
\operatorname{pardeg}_{\boldsymbol{\alpha}} F=\operatorname{deg} F+\sum_{i=1}^{n}\left(\alpha_{2 i-1} \operatorname{dim} F_{t_{i}} / l_{i} \cap F_{t_{i}}+\alpha_{2 i} \operatorname{dim} l_{i} \cap F_{t_{i}}\right) \tag{19}
\end{equation*}
$$

Definition 3.4. Fix a weight $\boldsymbol{\alpha}$. A ( $\mathbf{t}, \boldsymbol{\lambda}$ )-parabolic connection $(E, \nabla, \varphi, l)$ on $\mathbf{P}^{1}$ with the determinant $\left(L, \nabla_{L}\right)$ is said to be $\boldsymbol{\alpha}$-stable
(resp. $\boldsymbol{\alpha}$-semistable ) if for every rank-1 subbundle $F$ with $\nabla(F) \subset$ $F \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$
(20) $\quad \operatorname{pardeg}_{\alpha} F<\frac{\operatorname{pardeg}_{\alpha} E}{2}, \quad$ (resp. $\operatorname{pardeg}_{\alpha} F \leq \frac{\operatorname{pardeg}_{\alpha} E}{2}$ ).
(For simplicity," $\boldsymbol{\alpha}$-stable" will be abbreviated to "stable").

We define the coarse moduli space by

$$
M_{n}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)=\left\{\begin{array}{ll}
(E, \nabla, \varphi, l) ; & \text { an } \boldsymbol{\alpha} \text {-stable }(\mathbf{t}, \boldsymbol{\lambda}) \text {-parabolic }  \tag{21}\\
\text { the detion with } \\
\text { thermant }\left(L, \nabla_{L}\right)
\end{array}\right\} / \text { isom. }
$$

### 3.4. Stable parabolic $\phi$-connections

If $n \geq 4$, the moduli space $M_{n}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ never becomes projective nor complete. In order to obtain a compactification of the moduli space $M_{n}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$, we will introduce the notion of a stable parabolic $\phi$-connection, or equivalently, a stable parabolic $\Lambda$-triple. Again, let us fix $(\mathbf{t}, \boldsymbol{\lambda}) \in T_{n} \times \Lambda_{n}$ and a line bundle $L$ on $\mathbf{P}^{1}$ with a connection $\nabla_{L}: L \rightarrow L \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$.

Definition 3.5. The data $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}_{i=1}^{n}\right)$ is said to be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic $\phi$-connection of rank 2 with the determinant $\left(L, \nabla_{L}\right)$ if $E_{1}, E_{2}$ are rank 2 vector bundles on $\mathbf{P}^{1}$ with $\operatorname{deg} E_{1}=\operatorname{deg} L, \phi: E_{1} \rightarrow$ $E_{2}, \nabla: E_{1} \rightarrow E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ are morphisms of sheaves, $\varphi: \bigwedge^{2} E_{2} \xrightarrow{\sim} L$ is an isomorphism and $l_{i} \subset\left(E_{1}\right)_{t_{i}}$ are one dimensional subspaces for $i=1, \ldots, n$ such that
(1) $\quad \phi(f a)=f \phi(a)$ and $\nabla(f a)=\phi(a) \otimes d f+f \nabla(a)$ for $f \in \mathcal{O}_{\mathbf{P}^{1}}$, $a \in E_{1}$,
(2) $\quad(\varphi \otimes \mathrm{id})\left(\nabla\left(s_{1}\right) \wedge \phi\left(s_{2}\right)+\phi\left(s_{1}\right) \wedge \nabla\left(s_{2}\right)\right)=\nabla_{L}\left(\varphi\left(\phi\left(s_{1}\right) \wedge \phi\left(s_{2}\right)\right)\right)$ for $s_{1}, s_{2} \in E_{1}$ and
(3) $\left.\quad\left(\operatorname{res}_{t_{i}}(\nabla)-\lambda_{i} \phi_{t_{i}}\right)\right|_{l_{i}}=0$ for $i=1, \ldots, n$.

## Definition 3.6.

Two ( $\mathbf{t}, \boldsymbol{\lambda}$ ) parabolic $\phi$-connections

$$
\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right), \quad\left(E_{1}^{\prime}, E_{2}^{\prime}, \phi^{\prime}, \nabla^{\prime}, \varphi^{\prime},\left\{l_{i}^{\prime}\right\}\right)
$$

are said to be isomorphic to each other if there are isomorphisms $\sigma_{1}$ : $E_{1} \xrightarrow{\sim} E_{1}^{\prime}, \sigma_{2}: E_{2} \xrightarrow{\sim} E_{2}^{\prime}$ and $c \in \mathbf{C} \backslash\{0\}$ such that the diagrams

$$
\begin{aligned}
& E_{1} \xrightarrow{\phi} E_{2} \quad E_{1} \xrightarrow{\nabla} E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \\
& \sigma_{1} \downarrow \cong \quad \cong \downarrow \sigma_{2} \quad \sigma_{1} \downarrow \cong \downarrow \sigma_{2} \otimes \mathrm{id} \\
& E_{1}^{\prime} \xrightarrow{\phi^{\prime}} E_{2}^{\prime} \quad E_{1}^{\prime} \xrightarrow{\nabla^{\prime}} E_{2}^{\prime} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \\
& \bigwedge^{2} E_{2} \xrightarrow[\cong]{\varphi} L \\
& \wedge^{2} \sigma_{2} \downarrow \cong \quad c \downarrow \cong \\
& \bigwedge^{2} E_{2}^{\prime} \xrightarrow[\cong]{\varphi^{\prime}} L
\end{aligned}
$$

commute and $\left(\sigma_{1}\right)_{t_{i}}\left(l_{i}\right)=l_{i}^{\prime}$ for $i=1, \ldots, n$.
Remark 3.1. Assume that two vector bundles $E_{1}, E_{2}$ and morphisms $\phi: E_{1} \rightarrow E_{2}, \nabla: E_{1} \rightarrow E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ satisfying $\phi(f a)=$ $f \phi(a), \nabla(f a)=\phi(a) \otimes d f+f \nabla(a)$ for $f \in \mathcal{O}_{\mathbf{P}^{1}}, a \in E_{1}$ are given. If $\phi$ is an isomorphism, then $(\phi \otimes \mathrm{id})^{-1} \circ \nabla: E_{1} \rightarrow E_{1} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ becomes a connection on $E_{1}$.

Fix rational numbers $\alpha_{1}^{\prime}, \alpha_{2}^{\prime}, \ldots, \alpha_{2 n}^{\prime}, \alpha_{2 n+1}^{\prime}$ satisfying

$$
0 \leq \alpha_{1}^{\prime}<\alpha_{2}^{\prime}<\cdots<\alpha_{2 n}^{\prime}<\alpha_{2 n+1}^{\prime}=1
$$

and positive integers $\beta_{1}, \beta_{2}$. Setting $\boldsymbol{\alpha}^{\prime}=\left(\alpha_{1}^{\prime}, \ldots, \alpha_{2 n}^{\prime}\right), \boldsymbol{\beta}=\left(\beta_{1}, \beta_{2}\right)$, we obtain a weight $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$ for parabolic $\phi$-connections.

Definition 3.7. Fix a sufficiently large integer $\gamma$. Let

$$
\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}_{i=1}^{n}\right)
$$

be a parabolic $\phi$-connection. For any subbundles $F_{1} \subset E_{1}, F_{2} \subset E_{2}$ satisfying $\phi\left(F_{1}\right) \subset F_{2}, \nabla\left(F_{1}\right) \subset F_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$, we define

$$
\begin{aligned}
& \mu\left(\left(F_{1}, F_{2}\right)\right)_{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}=\frac{1}{\beta_{1} \operatorname{rank}\left(F_{1}\right)+\beta_{2} \operatorname{rank}\left(F_{2}\right)}\left(\beta_{1}\left(\operatorname{deg} F_{1}(-D(\mathbf{t}))\right)\right. \\
& \quad+\beta_{2}\left(\operatorname{deg} F_{2}-\gamma \operatorname{rank}\left(F_{2}\right)\right)+\sum_{i=1}^{n} \beta_{1}\left(\alpha_{2 i-1}^{\prime} d_{2 i-1}\left(F_{1}\right)+\alpha_{2 i}^{\prime} d_{2 i}\left(F_{1}\right)\right)
\end{aligned}
$$

where $d_{2 i-1}(F)=\operatorname{dim}\left(\left(F_{1}\right)_{t_{i}} / l_{i} \cap\left(F_{1}\right)_{t_{i}}\right), d_{2 i}\left(F_{1}\right)=\operatorname{dim}\left(\left(F_{1}\right)_{t_{i}} \cap l_{i}\right)$.
A parabolic $\phi$-connection $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}_{i=1}^{n}\right)$ is said to be $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$ stable (resp. $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$-semistable) if for any subbundles $F_{1} \subset E_{1}, F_{2} \subset$
$E_{2}$ satisfying $\phi\left(F_{1}\right) \subset F_{2}, \nabla\left(F_{1}\right) \subset F_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ and $\left(F_{1}, F_{2}\right) \neq$ $\left(E_{1}, E_{2}\right),(0,0)$, the inequality

$$
\begin{align*}
& \mu\left(\left(F_{1}, F_{2}\right)\right)_{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}<\mu\left(\left(E_{1}, E_{2}\right)\right)_{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}},  \tag{22}\\
(\text { resp. } & \mu\left(\left(F_{1}, F_{2}\right)\right)_{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}} \leq \mu\left(\left(E_{1}, E_{2}\right)\right)_{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta} .} .
\end{align*}
$$

We define the coarse moduli space of $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$-stable ( $\left.\mathbf{t}, \boldsymbol{\lambda}\right)$-parabolic $\phi$-connections with the determinant $\left(L, \nabla_{L}\right)$ by

$$
\begin{equation*}
\overline{M_{n}^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L):=\left\{\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)\right\} / \text { isom. } \tag{23}
\end{equation*}
$$

For a given weight $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$ and $1 \leq i \leq 2 n$, define a rational number $\alpha_{i}$ by

$$
\begin{equation*}
\alpha_{i}=\frac{\beta_{1}}{\beta_{1}+\beta_{2}} \alpha_{i}^{\prime} . \tag{24}
\end{equation*}
$$

Then $\boldsymbol{\alpha}=\left(\alpha_{i}\right)$ satisfies the condition

$$
\begin{equation*}
0 \leq \alpha_{1}<\alpha_{2}<\cdots<\alpha_{2 n}<\frac{\beta_{1}}{\left(\beta_{1}+\beta_{2}\right)}<1 \tag{25}
\end{equation*}
$$

hence $\boldsymbol{\alpha}$ defines a weight for parabolic connections. It is easy to see that if we take $\gamma$ sufficiently large $\left(E, \nabla, \varphi,\left\{l_{i}\right\}\right)$ is $\boldsymbol{\alpha}$-stable if and only if the associated parabolic $\phi$-connection $\left(E, E, \mathrm{id}_{E}, \nabla, \varphi,\left\{l_{i}\right\}\right)$ is stable with respect to $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$. Therefore we see that the natural map

$$
\begin{equation*}
\left(E, \nabla, \varphi,\left\{l_{i}\right\}\right) \mapsto\left(E, E, \operatorname{id}_{E}, \nabla, \varphi,\left\{l_{i}\right\}\right) \tag{26}
\end{equation*}
$$

induces an injection

$$
\begin{equation*}
M_{n}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L) \hookrightarrow \overline{M_{n}^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L) \tag{27}
\end{equation*}
$$

Conversely, assuming that $\boldsymbol{\beta}=\left(\beta_{1}, \beta_{2}\right)$ are given, for a weight $\boldsymbol{\alpha}=\left(\alpha_{i}\right)$ satisfying the condition (25), we can define $\alpha_{i}^{\prime}=\alpha_{i} \frac{\beta_{1}+\beta_{2}}{\beta_{1}}$ for $1 \leq i \leq 2 n$. Since $0 \leq \alpha_{1}^{\prime}<\alpha_{2}^{\prime}<\cdots<\alpha_{2 n}^{\prime}=\alpha_{2 n} \frac{\beta_{1}+\beta_{2}}{\beta_{1}}<1,\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$ give a weight for parabolic $\phi$-connections.

Moreover, considering the relative setting over $T_{n} \times \Lambda_{n}$, we can define two families of the moduli spaces

$$
\begin{equation*}
\bar{\pi}_{n}: \overline{M_{n}^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(L) \longrightarrow T_{n} \times \Lambda_{n}, \quad \pi_{n}: M_{n}^{\boldsymbol{\alpha}}(L) \longrightarrow T_{n} \times \Lambda_{n} \tag{28}
\end{equation*}
$$

such that the following diagram commutes;

$$
\begin{array}{ccc}
M_{n}^{\alpha}(L) & \stackrel{\iota}{\hookrightarrow} & \overline{M_{n}^{\alpha^{\prime} \beta}}(L) \\
\pi_{n} \downarrow & &  \tag{29}\\
T_{n} \times \Lambda_{n} & & \bar{\pi}_{n} \\
& T_{n} \times \Lambda_{n} .
\end{array}
$$

Here the fibers of $\pi_{n}$ and $\bar{\pi}_{n}$ over $(\mathbf{t}, \boldsymbol{\lambda}) \in T_{n} \times \Lambda_{n}$ are

$$
\begin{equation*}
\pi_{n}^{-1}(\mathbf{t}, \boldsymbol{\lambda})=M^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L), \quad \bar{\pi}_{n}^{-1}(\mathbf{t}, \boldsymbol{\lambda})=\overline{M^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L) . \tag{30}
\end{equation*}
$$

### 3.5. The existence of moduli spaces and their properties

The following theorem was proved in [IIS1].
Theorem 3.1. ( [Theorem 2.1, [IIS1]]).
(1) Fix a weight $\boldsymbol{\beta}=\left(\beta_{1}, \beta_{2}\right)$. For a generic weight $\boldsymbol{\alpha}^{\prime}$,

$$
\overline{\pi_{n}}: \overline{M_{n}^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(L) \longrightarrow T_{n} \times \Lambda_{n}
$$

is a projective morphism. In particular, the moduli space $\overline{M^{\alpha^{\prime} \boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is a projective algebraic scheme for all $(\mathbf{t}, \boldsymbol{\lambda}) \in$ $T_{n} \times \Lambda_{n}$.
(2) For a generic weight $\boldsymbol{\alpha}, \pi_{n}: M_{n}^{\boldsymbol{\alpha}}(L) \longrightarrow T_{n} \times \Lambda_{n}$ is a smooth morphism of relative dimension $2 n-6$ with irreducible closed fibers. Therefore, the moduli space $M_{n}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is a smooth, irreducible algebraic variety of dimension $2 n-6$ for all $(\mathbf{t}, \boldsymbol{\lambda}) \in$ $T_{n} \times \Lambda_{n}$.

Remark 3.2. (1) The structures of moduli spaces $M_{n}^{\alpha}(L)$ and $\overline{M_{n}^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(L)$ may depend on the weights $\boldsymbol{\alpha},\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$ and $\operatorname{deg} L$.
(2) The moduli spaces $M_{n}^{\alpha}(L)$ is a fine moduli space. In fact, we have the universal families over these moduli spaces.
(3) The moduli space $M_{n}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ admits a natural holomorphic symplectic structure. (See [Proposition 6.2, [IIS1]). This fact is a part of the reason why Painlevé VI and Garnier systems can be written in nonautonomous Hamiltonian systems.
(4) In case of $n=4$, we can show that $\overline{M_{4}^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth (cf. Proposition 4.3). However we do not know whether $\overline{M_{n}^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L)$ is smooth or not for $n \geq 5$.

When we describe the explicit algebraic or geometric structure of the moduli spaces $M_{n}^{\boldsymbol{\alpha}}(L)$ and $\overline{M_{n}^{\alpha^{\prime} \boldsymbol{\beta}}}(L)$, it is convenient to fix a determinant line bundle $\left(L, \nabla_{L}\right)$. As a typical example of the determinant bundle is

$$
\begin{equation*}
\left(L, \nabla_{L}\right)=\left(\mathcal{O}_{\mathbf{P}^{1}}\left(-t_{n}\right), d\right) \tag{31}
\end{equation*}
$$

where the connection is given by

$$
\begin{equation*}
\nabla_{L}\left(z-t_{n}\right)=d\left(z-t_{n}\right)=\left(z-t_{n}\right) \otimes \frac{d z}{z-t_{n}} \tag{32}
\end{equation*}
$$

Here $z$ is an inhomogeneous coordinate of $\mathbf{P}^{1}=\operatorname{Spec} \mathbf{C}[z] \cup\{\infty\}$. For this $\left(L, \nabla_{L}\right)=\left(\mathcal{O}_{\mathbf{P}^{1}}\left(-t_{n}\right), d\right)$, we set
$M_{n}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda},-1)=M_{n}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda}, L), \quad\left(\operatorname{resp} . \overline{M_{n}^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda},-1)=\overline{M_{n}^{\boldsymbol{\alpha}^{\prime} \boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L) \quad\right)$.

## §4. Explicit construction of moduli spaces for the case of $n=4$ (Painlevé VI case).

In this section, we will deal with the case of $n=4$ in detail. Let us fix a sufficiently large integer $\gamma$ and take a weight $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$ for parabolic $\phi$-connections where $\boldsymbol{\alpha}^{\prime}=\left(\alpha_{1}^{\prime}, \ldots, \alpha_{8}^{\prime}\right), \boldsymbol{\beta}=\left(\beta_{1}, \beta_{2}\right), \gamma$ and fix $(\mathbf{t}, \boldsymbol{\lambda})=$ $\left(t_{1}, \ldots, t_{4}, \lambda_{1}, \ldots, \lambda_{4}\right) \in T_{4} \times \Lambda_{4}$.

Then the corresponding weight $\boldsymbol{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{8}\right)$ for parabolic connections can be given by

$$
\alpha_{i}=\alpha_{i}^{\prime} \frac{\beta_{1}}{\beta_{1}+\beta_{2}} \quad 1 \leq i \leq 8
$$

For simplicity, we will assume that $\beta_{1}=\beta_{2}=1$, hence $\boldsymbol{\alpha}=\boldsymbol{\alpha}^{\prime} / 2$. We also assume $\left(L, \nabla_{l}\right)=\left(\mathcal{O}_{\mathbf{P}^{1}}\left(-t_{n}\right), d\right)$ and set

$$
\overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)=\overline{M_{4}^{\alpha^{\prime} \boldsymbol{\beta}}}(\mathbf{t}, \boldsymbol{\lambda}, L), \quad \overline{M_{4}^{\alpha^{\prime}}}(-1)=\overline{M_{4}^{\alpha^{\prime} \beta}}(L) .
$$

From Theorem 3.1, we can obtain the commutative diagram:

such that $\pi_{4}^{-1}((\mathbf{t}, \boldsymbol{\lambda})) \simeq M_{4}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda},-1)$ and $\bar{\pi}_{4}^{-1}(\mathbf{t}, \boldsymbol{\lambda}) \simeq \overline{M_{4}^{\boldsymbol{\alpha}^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$. (Note that $\boldsymbol{\alpha}=\boldsymbol{\alpha}^{\prime} / 2$ ). From Theorem 3.1, we see that for a generic weight $\boldsymbol{\alpha}^{\prime}, \bar{\pi}_{4}$ is a projective morphism and $\pi_{4}$ is a smooth morphism of relative dimension 2 .

### 4.1. Main Theorem (Explicit description for $n=4$ case).

Putting $\beta_{1}=\beta_{2}=1$, we further assume that $\left|\alpha_{j}^{\prime}\right| \ll 1$ for $i=$ $1, \ldots, 8$. Let $\tilde{t}_{1}, \ldots, \tilde{t}_{4} \subset \mathbf{P}^{1} \times \Lambda_{4} \times T_{4}$ be the pull-back of the universal sections on $\mathbf{P}^{1} \times T_{4}$ over $T_{4}$. Put $D(\tilde{\mathbf{t}}):=\tilde{t}_{1}+\cdots+\tilde{t}_{4}$ and consider the projective bundle

$$
\pi: \mathbf{P}\left(\Omega_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4} / T_{4} \times \Lambda_{4}}^{1}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4}}\right) \longrightarrow \mathbf{P}^{1} \times T_{4} \times \Lambda_{4}
$$

Note that since $\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \simeq \mathcal{O}_{\mathbf{P}^{1}}(2)$ the fiber of $p_{23} \circ \pi$ over $(\mathbf{t}, \boldsymbol{\lambda}) \in$ $T_{4} \times \Lambda_{4}$ is isomorphic to

$$
\mathbf{P}\left(\mathcal{O}_{\mathbf{P}^{1}}(2) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right) \simeq \Sigma_{2}
$$

where $\Sigma_{2}$ is the Hirzebruch surface of degree 2.
Let $\tilde{D}_{i} \subset \mathbf{P}\left(\Omega_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4} / T_{4} \times \Lambda_{4}}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4}}\right)$ be the inverse image of $\tilde{t}_{i}$. Since the residue map induces an isomorphism

$$
\left.\Omega_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4} / T_{4} \times \Lambda_{4}}^{1}(D(\tilde{\mathbf{t}}))\right|_{\tilde{t}_{i}} \xrightarrow{\sim} \mathcal{O}_{\tilde{t}_{i}},
$$

we have a canonical isomorphism $\tilde{D}_{i} \xrightarrow{\sim} \mathbf{P}^{1} \times T_{4} \times \Lambda_{4}$. Let $\tilde{b}_{i}^{+} \subset \tilde{D}_{i}$ (resp. $\tilde{b}_{i}^{-} \subset \tilde{D}_{i}$ ) be the inverse image of $\left[\lambda_{i}^{+}: 1\right] \subset \mathbf{P}^{1} \times T_{4} \times \Lambda_{4}$ (resp. $\left.\left[\lambda_{i}^{-}: 1\right] \subset \mathbf{P}^{1} \times T_{4} \times \Lambda_{4}\right)$. We denote by $B^{+}\left(\right.$resp. $\left.B^{-}\right)$the reduced induced structure on $\tilde{b}_{1}^{+} \cup \cdots \cup \tilde{b}_{4}^{+}\left(\right.$resp. $\left.\tilde{b}_{1}^{-} \cup \cdots \cup \tilde{b}_{4}^{-}\right)$and we consider the reduced induced structure on $B=B^{+} \cup B^{-}$. Let

$$
g: Z \rightarrow \mathbf{P}\left(\Omega_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4} / T_{4} \times \Lambda_{4}}^{1}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4}}\right)
$$

be the blow-up along $B^{+}$and $\overline{\mathcal{S}}$ be the blow-up of $Z$ along the closure of $g^{-1}\left(B^{-} \backslash\left(B^{+} \cap B^{-}\right)\right)$. (It is easy to see that $\overline{\mathcal{S}} \longrightarrow T_{4} \times \Lambda_{4}$ is isomorphic to the family constructed by Okamoto [O1]). Note that $Z$ is isomorphic to the blow-up of $Z$ along $g^{-1}(B)$.

The main purpose of this section is to prove the following theorem:
Theorem 4.1. Take $\boldsymbol{\alpha}^{\prime}=\left(\alpha_{i}^{\prime}\right)_{1 \leq i \leq 2 n}, \boldsymbol{\beta}=\left(\beta_{1}, \beta_{2}\right)$ and $\gamma$ such that $\beta_{1}=\beta_{2}=1, \gamma \gg 0,\left|\alpha_{i}^{\prime}\right| \ll 1$ for $1 \leq i \leq 2 n, \alpha_{2 i}^{\prime}-\alpha_{2 i-1}^{\prime}<$ $\sum_{j \neq i}\left(\alpha_{2 j}^{\prime}-\alpha_{2 j-1}^{\prime}\right)$ for $1 \leq i \leq n$ and that any $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$-semistable parabolic $\phi$-connection is $\left(\boldsymbol{\alpha}^{\prime}, \boldsymbol{\beta}\right)$-stable.
(1) There exists an isomorphism

$$
\begin{equation*}
\overline{M_{4}^{\alpha^{\prime}}}\left(\mathcal{O}_{\mathbf{P}^{1}}\left(-\tilde{t}_{4}\right)\right) \xrightarrow{\sim} \bar{S} \tag{34}
\end{equation*}
$$

over $T_{4} \times \Lambda_{4}$.
(2) Let $\mathcal{Y}$ be the closed subscheme of $\overline{M_{4}^{\alpha^{\prime}}}\left(\mathcal{O}_{\mathbf{P}^{1}}\left(-\tilde{t}_{4}\right)\right)$ defined by the condition $\wedge^{2} \phi=0$. Then

$$
\begin{equation*}
M_{4}^{\alpha^{\prime} / 2}\left(\mathcal{O}_{\mathbf{P}^{1}}\left(-\tilde{t}_{4}\right)\right)=\overline{M_{4}^{\alpha^{\prime}}}\left(\mathcal{O}_{\mathbf{P}^{1}}\left(-\tilde{t}_{4}\right)\right) \backslash \mathcal{Y} \tag{35}
\end{equation*}
$$

(3) For each $(\mathbf{t}, \boldsymbol{\lambda}) \in T_{4} \times \Lambda_{4}$, the fiber $\mathcal{Y}_{(\mathbf{t}, \boldsymbol{\lambda})}$ is the anti-canonical divisor of $\overline{M_{4}^{\alpha^{\prime}}}\left(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^{1}}\left(-\tilde{t}_{4}\right)\right)$ and the pair

$$
\left(\overline{M_{4}^{\alpha^{\prime}}}\left(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^{1}}\left(-\tilde{t}_{4}\right)\right), \mathcal{Y}_{(\mathbf{t}, \boldsymbol{\lambda})}\right)
$$

is an Okamoto-Painlevé pair of type $D_{4}^{(1)}$.
4.2. Construction of the morphism $\overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1) \rightarrow \Sigma_{2}$

We assume that $\left(\alpha_{i}\right)$ satisfies the condition of Lemma 4.2 below.
Take any point $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right) \in \overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$. There are unique trivial subbundles $L_{1}^{(0)} \subset E_{1}, L_{2}^{(0)} \subset E_{2}$, whose existence is confirmed by Proposition 4.1 bellow. Since the composite

$$
\mathcal{O}_{\mathbf{P}^{1}} \cong L_{1}^{(0)} \hookrightarrow E_{1} \xrightarrow{\phi} E_{2} \rightarrow E_{2} / L_{2}^{(0)} \cong \mathcal{O}_{\mathbf{P}^{1}(-1)}
$$

is zero, the composite

$$
\begin{equation*}
u: L_{1}^{(0)} \hookrightarrow E_{1} \xrightarrow{\nabla} E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \rightarrow E_{2} / L_{2}^{(0)} \otimes \Omega_{\mathbf{P}^{1}(D(\mathbf{t}))}^{1} \cong \mathcal{O}_{\mathbf{P}^{1}(1)} \tag{37}
\end{equation*}
$$

becomes a homomorphism. By Proposition 4.1 bellow, there is a unique point $q \in \mathbf{P}^{1}$ satisfying $u(q)=0$. Put $L_{1}^{(-1)}:=E_{1} / L_{1}^{(0)}, L_{2}^{(-1)}:=$ $E_{2} / L_{2}^{(0)}$ and let $p_{j}: E_{j} \rightarrow L_{j}^{(-1)}$ be the projection for $j=1,2$. We define a homomorphism $B: E_{1} \rightarrow L_{2}^{(-1)} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ by $B(a):=\left(p_{2} \otimes\right.$ id) $\nabla(a)-d\left(p_{2} \phi(a)\right)$ for $a \in E_{1}$, where $d$ is the canonical connection on $L_{2}^{(-1)} \cong \mathcal{O}_{\mathbf{P}^{1}}\left(-t_{4}\right)$. Since $u_{q}=0, B_{q}$ induces a homomorphism $h_{1}:\left(L_{1}^{(-1)}\right)_{q} \rightarrow\left(L_{2}^{(-1)} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right)_{q}$ which makes the diagram

$$
0 \rightarrow\left(L_{1}^{(0)}\right)_{q} \underset{u_{q}=0}{\longrightarrow} \begin{gathered}
\left(E_{1}\right)_{q} \\
B_{q} \downarrow \\
\left(L_{2}^{(-1)} \otimes \Omega_{\left.\mathbf{P}^{1}(D(\mathbf{t}))\right)_{q}}\right.
\end{gathered} \stackrel{\exists h_{1} \swarrow}{\longrightarrow} \begin{aligned}
& \left(L_{1}^{(-1)}\right)_{q} \rightarrow 0 \\
&
\end{aligned}
$$

commute. On the other hand, $\phi$ induces the following commutative diagram


We put $h_{2}:=\phi_{2}(q)$. Then $h_{1}, h_{2}$ determine a homomorphism (38)
$\iota:\left(L_{1}^{(-1)}\right)_{q} \longrightarrow\left(L_{2}^{(-1)} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus L_{2}^{(-1)}\right)_{q} ; \quad a \mapsto\left(-h_{1}(a), h_{2}(a)\right)$.
By Proposition 4.2, $\iota$ is injective and the inclusion

$$
\iota:\left(L_{1}^{(-1)}\right)_{q} \hookrightarrow\left(L_{2}^{(-1)}\right)_{q} \otimes\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)_{q}
$$

determines a point $p\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)$ of $\mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)$, where $\mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)$ means $\operatorname{Proj} S\left(\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)^{\vee}\right)$. So we can define a morphism

$$
\begin{array}{rlll}
p: \quad \overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1) & \longrightarrow & \mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right) ;  \tag{39}\\
\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right) & \mapsto & p\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right) .
\end{array}
$$

Proposition 4.1. For any member

$$
\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right) \in \overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)
$$

we have

$$
E_{1} \cong E_{2} \cong \mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1)
$$

Proof. Take decompositions

$$
\begin{array}{ll}
E_{1}=\mathcal{O}_{\mathbf{P}^{1}}\left(d_{1}\right) \oplus \mathcal{O}_{\mathbf{P}^{1}}\left(-d_{1}-1\right) & \left(d_{1} \geq 0\right) \\
E_{2}=\mathcal{O}_{\mathbf{P}^{1}}\left(d_{2}\right) \oplus \mathcal{O}_{\mathbf{P}^{1}}\left(-d_{2}-1\right) & \left(d_{2} \geq 0\right)
\end{array}
$$

Assume that $d_{1}+d_{2}>1$. Then we have $\phi\left(\mathcal{O}_{\mathbf{P}^{1}}\left(d_{1}\right)\right) \subset \mathcal{O}_{\mathbf{P}^{1}}\left(d_{2}\right)$. The composite

$$
\mathcal{O}_{\mathbf{P}^{1}}\left(d_{1}\right) \rightarrow E_{1} \xrightarrow{\nabla} E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \rightarrow \mathcal{O}_{\mathbf{P}^{1}}\left(-d_{2}-1\right) \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \cong \mathcal{O}_{\mathbf{P}^{1}}\left(1-d_{2}\right)
$$

becomes a homomorphism and must be zero since $H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}\left(1-\left(d_{1}+\right.\right.\right.$ $\left.\left.\left.d_{2}\right)\right)\right)=0$. So we have $\nabla\left(\mathcal{O}_{\mathbf{P}^{1}}\left(d_{1}\right)\right) \subset \mathcal{O}_{\mathbf{P}^{1}}\left(d_{2}\right) \otimes \Omega^{1}(D(\mathbf{t}))$. Then the subbundles $\left(\mathcal{O}_{\mathbf{P}^{1}}\left(d_{1}\right), \mathcal{O}_{\mathbf{P}^{1}}\left(d_{2}\right)\right)$ breaks the stability of $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)$.

If $d_{1}=1$ and $d_{2}=0$, then $\phi\left(\mathcal{O}_{\mathbf{P}^{1}}(1)\right)=0$ and the composite

$$
f: \mathcal{O}_{\mathbf{P}^{1}}(1) \hookrightarrow E_{1} \xrightarrow{\nabla} E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))
$$

becomes a homomorphism.
Put $L:=(\operatorname{Im} f) \otimes \Omega^{1}(D(\mathbf{t}))^{\vee}$. Then $L$ is a vector bundle and either $L=0$ or $L$ is a line bundle with $\operatorname{deg} L \geq-1$. Then the subsheaves $\left(\mathcal{O}_{\mathbf{P}^{1}}(1), L\right)$ breaks the stability of $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)$.

If $d_{1}=0$ and $d_{2}=1$, then the composite $E_{1} \xrightarrow{\phi} E_{2} \rightarrow \mathcal{O}_{\mathbf{P}^{1}(-2)}$ must be zero and the composite $f: E_{1} \xrightarrow{\nabla} E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \rightarrow \mathcal{O}_{\mathbf{P}^{1}}(-2) \otimes$ $\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ becomes a homomorphism. Put $L:=\operatorname{ker} f$. Then we have either $L=E_{1}$ or $L$ is a line bundle such that $\operatorname{deg} L \geq-1$. Then the subbundles $\left(L, \mathcal{O}_{\mathbf{P}^{1}}(1)\right)$ breaks the stability of $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)$.

Hence we have $d_{1}=d_{2}=0$ and $E_{1} \cong E_{2} \cong \mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1)$.
Q.E.D.

Lemma 4.1. For any $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right) \in \overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$, the homomorphism $u$ defined in (37) is injective.

Proof. Assume that $u=0$. Then the subbundles $\left(L_{1}^{(0)}, L_{2}^{(0)}\right)$ breaks the stability of $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)$. Thus $u \neq 0$ and $u$ is injective.
Q.E.D.

Lemma 4.2. Assume $\alpha_{2 i}^{\prime}-\alpha_{2 i-1}^{\prime}<\sum_{j \neq i}\left(\alpha_{2 j}^{\prime}-\alpha_{2 j-1}^{\prime}\right)$ for any $1 \leq i \leq n$. Then the homomorphism $\iota$ defined above is injective.

Proof. If $\phi$ is isomorphic, then $h_{2}:\left(L_{1}^{(-1)}\right)_{q} \rightarrow\left(L_{2}^{(-1)}\right)_{q}$ is isomorphic, and so $\iota$ is injective. So we assume that $\phi$ is not isomorphic, that is, $\wedge^{2} \phi=0$.

First consider the case $\operatorname{rank} \phi=1$. Take decompositions $E_{1}=$ $\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), E_{2}=\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1)$. Then the homomorphism $\phi$ can be represented by a matrix

$$
\left(\begin{array}{cc}
\phi_{1} & \phi_{3} \\
0 & \phi_{2}
\end{array}\right) \quad\left(\phi_{1}, \phi_{2} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}\right), \phi_{3} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(1)\right)\right)
$$

where the composite $E_{1} \xrightarrow{\phi} E_{2} \xrightarrow{p_{2}} \mathcal{O}_{\mathbf{P}^{1}}(-1)$ is represented by $\left(0, \phi_{2}\right)$ and $E_{1} \xrightarrow{\phi} E_{2} \rightarrow \mathcal{O}_{\mathbf{P}^{1}}$ by $\left(\phi_{1}, \phi_{3}\right)$.

Now assume that $p_{2} \circ \phi=0$. Then $\phi_{2}=0$. If moreover $\phi_{1}=$ 0 , then $\phi_{3} \neq 0$ since rank $\phi=1$. Take local bases $e_{1}$ of $\mathcal{O}_{\mathbf{P}^{1}} \subset E_{1}$ and $e_{2}$ of $\mathcal{O}_{\mathbf{P}^{1}}(-1) \subset E_{1}$. Then the condition $\nabla\left(e_{1}\right) \wedge \phi\left(e_{2}\right)+\phi\left(e_{1}\right) \wedge$ $\nabla\left(e_{2}\right)=0$ implies that $\nabla\left(e_{1}\right) \in \mathcal{O}_{\mathbf{P}^{1}} \otimes \Omega_{\mathbf{P}^{1}}(D(\mathbf{t}))$, which contradicts the result of Lemma 4.1. Thus we have $\phi_{1} \neq 0$. Then, by multiplying an automorphism of $E_{1}$ given by

$$
\left(\begin{array}{cc}
c_{1} & c_{3} \\
0 & c_{2}
\end{array}\right) \quad\left(c_{1}, c_{2} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}^{\times}\right), c_{3} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(1)\right)\right)
$$

the matrix representing $\phi$ changes into the form

$$
\left(\begin{array}{cc}
\phi_{1} & \phi_{3} \\
0 & 0
\end{array}\right)\left(\begin{array}{cc}
c_{1} & c_{3} \\
0 & c_{2}
\end{array}\right)=\left(\begin{array}{cc}
c_{1} \phi_{1} & c_{3} \phi_{1}+c_{2} \phi_{3} \\
0 & 0
\end{array}\right) .
$$

For a suitable choice of $c_{1}, c_{2}$ and $c_{3}$, we have $c_{1} \phi_{1}=1$ and $c_{3} \phi_{1}+c_{2} \phi_{3}=$ 0 . So we may assume without loss of generality that $\phi_{3}=0$ and $\phi_{1}=1$.

The homomorphism $B: E_{1} \rightarrow L_{2}^{(-1)} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))=\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(-1)$ defined by $B(a):=\left(p_{2} \otimes \mathrm{id}\right) \nabla(a)-d\left(p_{2} \phi(a)\right)$ for $a \in E_{1}$ can be represented by a matrix $\left(\omega_{3}, \omega_{4}\right)$ where $\omega_{3} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(-1)\right)$ and $\omega_{4} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right)$. Define a homomorphism $A: E_{1} \rightarrow \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ by $A(a):=\left(q_{2} \otimes \mathrm{id}\right) \nabla(a)-d\left(q_{2} \phi(a)\right)$ for $a \in E_{1}$, where $q_{2}: E_{2} \rightarrow \mathcal{O}_{\mathbf{P}^{1}}$
is the projection with respect to the given decomposition of $E_{2}$ and $d$ is the trivial connection on $\mathcal{O}_{\mathbf{P}^{1}}$. Then $A$ can be represented by a ma$\operatorname{trix}\left(\omega_{1}, \omega_{2}\right)$, where $\omega_{1} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right)$ and $\omega_{2} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(1)\right)$. Roughly speaking $\nabla$ is represented by the matrix

$$
\left(\begin{array}{ll}
\omega_{1} & \omega_{2} \\
\omega_{3} & \omega_{4}
\end{array}\right)
$$

Since $\phi\left(e_{2}\right)=0$ and $\phi\left(e_{1}\right) \in \mathcal{O}_{\mathbf{P}^{1}}$, the condition $\nabla\left(e_{1}\right) \wedge \phi\left(e_{2}\right)+\phi\left(e_{1}\right) \wedge$ $\nabla\left(e_{2}\right)=0$ implies that $\nabla\left(e_{2}\right) \in \mathcal{O}_{\mathbf{P}^{1}} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$. Thus we have $\omega_{4}=0$. Take a nonzero vector $v^{(i)} \in l_{i} \subset\left(E_{1}\right)_{t_{i}}$. Then we must have

$$
\begin{equation*}
\left(\operatorname{res}_{t_{i}} \nabla\right)\left(v^{(i)}\right)=\lambda_{i} \phi_{t_{i}}\left(v^{(i)}\right) \tag{40}
\end{equation*}
$$

Since $E_{1}=\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1)$, we can write $v^{(i)}=\binom{v_{1}^{(i)}}{v_{2}^{(i)}}$ with $v_{1}^{(i)} \in$ $\left(\mathcal{O}_{\mathbf{P}^{1}}\right)_{t_{i}}$ and $v_{2}^{(i)} \in\left(\mathcal{O}_{\mathbf{P}^{1}}(-1)\right)_{t_{i}}$. Then we have

$$
\begin{aligned}
\left(\operatorname{res}_{t_{i}} \nabla\right)\binom{v_{1}^{(i)}}{v_{2}^{(i)}} & =\binom{\operatorname{res}_{t_{i}}\left(\omega_{1}\right) v_{1}^{(i)}+\operatorname{res}_{t_{i}}\left(\omega_{2}\right) v_{2}^{(i)}}{\operatorname{res}_{t_{i}}\left(\omega_{3}\right) v_{1}^{(i)}} \\
\phi_{t_{i}}\binom{v_{1}^{(i)}}{v_{2}^{(i)}} & =\binom{v_{1}^{(i)}}{0}
\end{aligned}
$$

Thus the equality (40) is equivalent to the equalities

$$
\operatorname{res}_{t_{i}}\left(\omega_{1}\right) v_{1}^{(i)}+\operatorname{res}_{t_{i}}\left(\omega_{2}\right) v_{2}^{(i)}=\lambda_{i} v_{1}^{(i)}, \quad \operatorname{res}_{t_{i}}\left(\omega_{3}\right) v_{1}^{(i)}=0
$$

Since $u$ is injective by Lemma 4.1, $\omega_{3} \neq 0$. So there is at most one point $t_{i}$ which satisfies $\operatorname{res}_{t_{i}}\left(\omega_{3}\right)=0$, because $\omega_{3} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(-1)\right) \cong$ $H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(1)\right)$. Thus, for some $i$, we have $\operatorname{res}_{t_{j}}\left(\omega_{3}\right) \neq 0$ for $j \neq i$. Then we have $v_{1}^{(j)}=0$ for $j \neq i$. So we have $l_{j} \subset\left(\mathcal{O}_{\mathbf{P}^{1}}(-1)\right)_{t_{j}}$ for $j \neq i$. Recall that the image of $\left.\nabla\right|_{\mathcal{O}_{\mathbf{P}^{1}}(-1)}$ is contained in $\mathcal{O}_{\mathbf{P}^{1}} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ because $\omega_{4}=0$. Let $F_{*}\left(E_{1}\right)$ be the filtration of $E_{1}$ corresponding to $\left\{l_{j}\right\}$. Then $\left(\mathcal{O}_{\mathbf{P}^{1}}(-1), \mathcal{O}_{\mathbf{P}^{1}},\left.\Phi\right|_{\mathcal{O}_{\mathbf{P}^{1}}(-1)}, F_{*}\left(E_{1}\right) \cap \mathcal{O}_{\mathbf{P}^{1}}(-1)\right)$ is a parabolic $\phi$-subconnection of $\left(E_{1}, E_{2}, \Phi, F_{*}\left(E_{1}\right)\right)$. Since $2\left(\alpha_{2 i-1}^{\prime}+\sum_{j \neq i} \alpha_{2 j}^{\prime}\right)>$ $\sum_{j=1}^{8} \alpha_{j}^{\prime}$ by the assumption of the lemma, we have

$$
\begin{aligned}
& \mu\left(\left(\mathcal{O}_{\mathbf{P}^{1}}(-1), \mathcal{O}_{\mathbf{P}^{1}},\left.\Phi\right|_{\mathcal{O}_{\mathbf{P}^{1}}(-1)}, F_{*}\left(E_{1}\right) \cap \mathcal{O}_{\mathbf{P}^{1}}(-1)\right)\right) \\
& \quad \geq \frac{-1-4-1-\gamma+\alpha_{2 i-1}^{\prime}+\sum_{j \neq i} \alpha_{2 j}^{\prime}}{2} \\
& \quad>\frac{-2-8-2-2 \gamma+\sum_{j=1}^{4}\left(\alpha_{2 j-1}^{\prime}+\alpha_{2 j}^{\prime}\right)}{4}=\mu\left(\left(E_{1}, E_{2}, \Phi, F_{*}\left(E_{1}\right)\right)\right),
\end{aligned}
$$

which breaks the stability of $\left(E_{1}, E_{2}, \Phi, F_{*}\left(E_{1}\right)\right)$. Therefore $p_{2} \circ \phi \neq 0$ and the homomorphism $L_{1}^{(-1)} \rightarrow L_{2}^{(-1)}$ induced by $\phi$ is an isomorphism. Hence $h_{2}:\left(L_{1}^{(-1)}\right)_{q} \rightarrow\left(L_{2}^{(-1)}\right)_{q}$ is bijective and so $\iota$ is injective.

Next consider the case $\phi=0$. In this case, $\nabla: E_{1} \rightarrow E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ is a homomorphism. If we choose a decomposition $E_{1}=\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1)$, $E_{2}=\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \nabla$ is represented by a matrix

$$
\left(\begin{array}{ll}
\omega_{1} & \omega_{2} \\
\omega_{3} & \omega_{4}
\end{array}\right) \quad\left\{\begin{array}{l}
\omega_{1}, \omega_{4} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right) \\
\omega_{2} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(1)\right) \\
\omega_{3} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(-1)\right)
\end{array}\right.
$$

Notice that $\omega_{3}$ corresponds to the homomorphism $u: L_{1}^{(0)} \rightarrow E_{2} / L_{2}^{(0)} \otimes$ $\Omega_{\mathbf{P}^{1}}(D(\mathbf{t}))$ and so $\omega_{3} \neq 0$. Let $q$ be the point of $\mathbf{P}^{1}$ satisfying $\omega_{3}(q)=0$. Assume that $\omega_{4}(q)=0$. Multiplying an automorphism of $E_{1}$ given by

$$
\left(\begin{array}{cc}
c_{1} & c_{3} \\
0 & c_{2}
\end{array}\right) \quad\left(c_{1}, c_{2} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}^{\times}\right), c_{3} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(1)\right)\right)
$$

the matrix representing $\nabla$ changes into the form

$$
\left(\begin{array}{cc}
\omega_{1} & \omega_{2} \\
\omega_{3} & \omega_{4}
\end{array}\right)\left(\begin{array}{cc}
c_{1} & c_{3} \\
0 & c_{2}
\end{array}\right)=\left(\begin{array}{cc}
c_{1} \omega_{1} & c_{3} \omega_{1}+c_{2} \omega_{2} \\
c_{1} \omega_{3} & c_{3} \omega_{3}+c_{2} \omega_{4}
\end{array}\right)
$$

For a suitable choice of $c_{2}, c_{3}$, we have $c_{3} \omega_{3}+c_{2} \omega_{4}=0$. So we may assume without loss of generality that $\omega_{4}=0$. Take a nonzero element $v^{(i)}$ of $l_{i} \subset\left(E_{1}\right)_{t_{i}}$. We can write $v^{(i)}=\binom{v_{1}^{(i)}}{v_{2}^{(i)}}$ with $v_{1}^{(i)} \in\left(\mathcal{O}_{\mathbf{P}^{1}}\right)_{t_{i}}$ and $v_{2}^{(i)} \in\left(\mathcal{O}_{\mathbf{P}^{1}}(-1)\right)_{t_{i}}$. Then we have

$$
\begin{aligned}
\left(\operatorname{res}_{t_{i}} \nabla\right)\left(v^{(i)}\right) & =\left(\operatorname{res}_{t_{i}} \nabla\right)\binom{v_{1}^{(i)}}{v_{2}^{(i)}} \\
& =\binom{\operatorname{res}_{t_{i}}\left(\omega_{1}\right) v_{1}^{(i)}+\operatorname{res}_{t_{i}}\left(\omega_{2}\right) v_{2}^{(i)}}{\operatorname{res}_{t_{i}}\left(\omega_{3}\right) v_{1}^{(i)}}
\end{aligned}
$$

Since $\left(\operatorname{res}_{t_{i}} \nabla\right)\left(v^{(i)}\right)=\lambda_{i} \phi_{t_{i}}\left(v^{(i)}\right)=0$, we have $\operatorname{res}_{t_{i}}\left(\omega_{3}\right) v_{1}^{(i)}=0$ for $i=1, \ldots, 4$. There is at most one $i$ satisfying $\operatorname{res}_{t_{i}}\left(\omega_{3}\right)=0$ because $\omega_{3} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(-1)\right)$. So we may assume that for some $i, \omega_{3}\left(t_{j}\right) \neq 0$ for $j \neq i$. Then we have $v_{1}^{(j)}=0$ for $j \neq i$ and $l_{j} \subset \mathcal{O}_{\mathbf{P}^{1}}(-1)_{t_{j}}$ for $j \neq i$. Since $\omega_{4}=0, \nabla\left(\mathcal{O}_{\mathbf{P}^{1}}(-1)\right) \subset \mathcal{O}_{\mathbf{P}^{1}} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$. If $F_{*}\left(E_{1}\right)$ is the filtration of $E_{1}$ corresponding to $\left\{l_{j}\right\}$, then $\left(\mathcal{O}_{\mathbf{P}^{1}}(-1), \mathcal{O}_{\mathbf{P}^{1}},\left.\Phi\right|_{\mathcal{O}_{\mathbf{P}^{1}}(-1)}, F_{*}\left(E_{1}\right) \cap\right.$
$\left.\mathcal{O}_{\mathbf{P}^{1}}(-1)\right)$ is a parabolic $\phi$-subconnection of $\left(E_{1}, E_{2}, \Phi, F_{*}\left(E_{1}\right)\right)$ and

$$
\begin{aligned}
& \mu\left(\mathcal{O}_{\mathbf{P}^{1}}(-1), \mathcal{O}_{\mathbf{P}^{1}},\left.\Phi\right|_{\mathcal{O}_{\mathbf{P}^{1}}(-1)}, F_{*}\left(E_{1}\right) \cap \mathcal{O}_{\mathbf{P}^{1}}(-1)\right) \\
& \quad \geq \frac{-1-4-1-\gamma+\alpha_{2 i-1}^{\prime}+\sum_{j \neq i} \alpha_{2 j}^{\prime}}{2} \\
& \quad>\frac{-2-8-2-2 \gamma+\sum_{j=1}^{4}\left(\alpha_{2 j-1}^{\prime}+\alpha_{2 j}^{\prime}\right)}{4}=\mu\left(E_{1}, E_{2}, \Phi, F_{*}\left(E_{1}\right)\right)
\end{aligned}
$$

which contradicts the stability of $\left(E_{1}, E_{2}, \Phi, F_{*}\left(E_{1}\right)\right)$. Therefore we have $\omega_{4}(q) \neq 0$, which means that $h_{1}$ is bijective and so $\iota$ is injective. Q.E.D.

### 4.3. Smoothness of $\overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$

Let $\mathcal{Y}$ be the closed subscheme of $\overline{M_{4}^{\alpha^{\prime}}}(-1)$ defined by the condition $\wedge^{2} \phi=0$ and $Y(\mathbf{t}, \boldsymbol{\lambda})$ be the fiber of $\mathcal{Y}$ over $(\mathbf{t}, \boldsymbol{\lambda})$.

Proposition 4.2. Under the assumption of Lemma 4.2, the restriction $Y(\mathbf{t}, \boldsymbol{\lambda}) \xrightarrow{p} \mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)$ of the morphism $p$ defined above is injective.

Proof. Let $D_{0}$ be the section of $\mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)$ over $\mathbf{P}^{1}$ defined by the injection $\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \hookrightarrow \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}$. Take any point $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right) \in Y(\mathbf{t}, \boldsymbol{\lambda})$. From the proof of Lemma 4.2, we can see that $p\left(\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right) \in D_{0}\right.$ if and only if $\phi=0$.

First assume that $\operatorname{rank} \phi=1$. As in the proof of Lemma 4.2, We take decompositions $E_{1}=\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}(-1)}, E_{2}=\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}(-1)}$ and represent $\phi$ by a matrix

$$
\left(\begin{array}{cc}
\phi_{1} & \phi_{3} \\
0 & \phi_{2}
\end{array}\right) \quad\left(\phi_{1}, \phi_{2} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}\right), \phi_{3} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(1)\right)\right)
$$

By the proof of Lemma 4.2, $\phi_{2} \neq 0$. Multiplying a certain automorphism of $E_{2}$, we may assume that $\phi_{3}=0$ and $\phi_{2}=1$. Since $\operatorname{rank} \phi=1$, we have $\phi_{1}=0$. Consider the homomorphism $B: E_{1} \rightarrow$ $\mathcal{O}_{\mathbf{P}^{1}}(-1) \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ defined by $B(a)=p_{2} \nabla(a)-d\left(p_{2} \phi(a)\right)$. Let $\left(\omega_{3}, \omega_{4}\right)\left(\omega_{3} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(-1)\right), \omega_{4} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right)\right)$ be the matrix which represents $B$. Since $\phi_{1}=0, \phi_{3}=0$, the composite $E_{1} \xrightarrow{\nabla}$ $E_{2} \otimes \Omega_{\mathbf{P}^{1}}^{1}(\mathbf{t}) \xrightarrow{q_{2} \otimes 1} \mathcal{O}_{\mathbf{P}^{1}} \otimes \Omega_{\mathbf{P}^{1}}^{1}(\mathbf{t})$ becomes a homomorphism, which can be represented by a matrix $\left(\omega_{1}, \omega_{2}\right)$ with $\omega_{1} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(\mathbf{t})\right), \omega_{2} \in$ $H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(\mathbf{t})(1)\right)$. Roughly speaking, $\nabla$ is represented by the matrix

$$
\left(\begin{array}{ll}
\omega_{1} & \omega_{2} \\
\omega_{3} & \omega_{4}
\end{array}\right) .
$$

We use the same notation as in the proof of Lemma 4.2. Then we have $\nabla\left(e_{1}\right) \wedge \phi\left(e_{2}\right)+\phi\left(e_{1}\right) \wedge \nabla\left(e_{2}\right)=0$. Since $\phi\left(e_{1}\right)=0$ and $\phi\left(e_{2}\right) \in$
$\mathcal{O}_{\mathbf{P}^{1}}(-1)$, we have $\nabla\left(e_{1}\right) \in \mathcal{O}_{\mathbf{P}^{1}}(-1) \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ and so $\omega_{1}=0$. Take a nonzero element $v^{(i)}$ of $l_{i} \subset\left(E_{1}\right)_{t_{i}}$ and write $v^{(i)}=\binom{v_{1}^{(i)}}{v_{2}^{(i)}}$ where $v_{1}^{(i)} \in\left(\mathcal{O}_{\mathbf{P}^{1}}\right)_{t_{i}}$ and $v_{2}^{(i)} \in \mathcal{O}_{\mathbf{P}^{1}}(-1)_{t_{i}}$. Then we have

$$
\begin{aligned}
& \left(\operatorname{res}_{t_{i}} \nabla\right)\left(v^{(i)}\right)=\left(\operatorname{res}_{t_{i}} \nabla\right)\binom{v_{1}^{(i)}}{v_{2}^{(i)}} \\
& \quad=\binom{\operatorname{res}_{t_{i}}\left(\omega_{2}\right) v_{2}^{(i)}}{\operatorname{res}_{t_{i}}\left(\omega_{3}\right) v_{1}^{(i)}+\operatorname{res}_{t_{i}}\left(\omega_{4}\right) v_{2}^{(i)}+\operatorname{res}_{t_{i}}\left(\frac{d z}{z-t_{4}}\right) v_{2}^{(i)}} \\
& \phi_{t_{i}}\left(v^{(i)}\right)=\phi_{t_{i}}\binom{v_{1}^{(i)}}{v_{2}^{(i)}}=\binom{0}{v_{2}^{(i)}}
\end{aligned}
$$

Since $\left(\operatorname{res}_{t_{i}} \nabla\right)\left(v^{(i)}\right)=\lambda_{i} \phi_{t_{i}}\left(v^{(i)}\right)$, we have

$$
\begin{aligned}
& \operatorname{res}_{t_{i}}\left(\omega_{2}\right) v_{2}^{(i)}=0 \\
& \operatorname{res}_{t_{i}}\left(\omega_{3}\right) v_{1}^{(i)}+\operatorname{res}_{t_{i}}\left(\omega_{4}\right) v_{2}^{(i)}+\operatorname{res}_{t_{i}}\left(\frac{d z}{z-t_{4}}\right) v_{2}^{(i)}=\lambda_{i} v_{2}^{(i)}
\end{aligned}
$$

If $\omega_{2}\left(t_{i}\right)=0$ for any $i$, then $\omega_{2}=0$ because $\omega_{2} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(1)\right) \cong$ $H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(3)\right)$ and there is a decomposition

$$
\left(E_{1}, E_{2}, \phi, \nabla,\left\{l_{i}\right\}\right)=\left(E_{1}, \mathcal{O}_{\mathbf{P}^{1}}(-1), \phi, \nabla,\left\{l_{i}\right\}\right) \oplus\left(0, \mathcal{O}_{\mathbf{P}^{1}}, 0,0,\{0\}\right)
$$

which contradicts the stability of $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)$. On the other hand, if $\omega_{2}\left(t_{i}\right) \neq 0$, then $v_{2}^{(i)}=0, v_{1}^{(i)} \neq 0$ and $\omega_{3}\left(t_{i}\right)=0$. However, there is at most one $i$ which satisfies $\omega_{3}\left(t_{i}\right)=0$ because $\omega_{3} \in$ $H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(-1)\right) \cong H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(1)\right)$. Therefore there is only one $i$ which satisfies $\omega_{2}\left(t_{i}\right) \neq 0$. In this case, $\omega_{3}\left(t_{i}\right)=0$ and so $q=t_{i}$, which means that the image $p\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{j}\right\}\right)$ is contained in the fiber $D_{i}$ of $\mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)$ over $t_{i}$. Applying certain automorphisms of $E_{1}$ and $E_{2}$ represented by a matrix of the form

$$
\left(\begin{array}{ll}
c & 0 \\
0 & 1
\end{array}\right) \quad\left(c \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}^{\times}\right)\right)
$$

we may assume that

$$
\omega_{2}=\frac{\prod_{j \neq i}\left(z-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z, \quad \omega_{3}=\frac{z-t_{i}}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z
$$

where $z$ is a fixed inhomogeneous coordinate of $\mathbf{P}^{1}$. Then giving a value $\operatorname{res}_{t_{i}}\left(\omega_{4}\right)$ is equivalent to giving a point $p\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)$ in the
fiber $D_{i}$. Applying an automorphism of $E_{1}$ represented by a matrix of the form

$$
\left(\begin{array}{ll}
1 & c \\
0 & 1
\end{array}\right) \quad\left(c \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(1)\right)\right)
$$

we may assume that $\omega_{4}$ is of the form

$$
\omega_{4}=\frac{a d z}{\prod_{j=1}^{4}\left(z-t_{j}\right)}
$$

with $a \in$ C. $a$ is determined by the value $\operatorname{res}_{t_{i}}\left(\omega_{4}\right)$. Thus the matrices representing $\phi$ and $\nabla$ are determined uniquely, up to automorphisms of $E_{1}$ and $E_{2}$, by the point $p\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{j}\right\}\right)$. Recall that $v_{1}^{(i)} \neq 0$, $v_{2}^{(i)}=0$ and $\operatorname{res}_{t_{j}}\left(\omega_{3}\right) v_{1}^{(j)}+\operatorname{res}_{t_{j}}\left(\omega_{4}\right) v_{2}^{(j)}+\operatorname{res}_{t_{j}}\left(\frac{d z}{z-t_{4}}\right) v_{2}^{(j)}=\lambda_{j} v_{2}^{(j)}$ for $j \neq i$. Since $\operatorname{res}_{t_{j}}\left(\omega_{3}\right) \neq 0$ for $j \neq i$, every $v^{(j)}$ (including $v^{(i)}$ ) is uniquely determined up to a scalar multiplication. Thus the parabolic structure is determined by $\phi, \nabla$. Hence $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{j}\right\}\right)$ is uniquely determined by the point $p\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{j}\right\}\right)$.

Next we assume that $\phi=0$. Let

$$
\left(\begin{array}{ll}
\omega_{1} & \omega_{2} \\
\omega_{3} & \omega_{4}
\end{array}\right), \quad\left\{\begin{array}{l}
\omega_{1}, \omega_{4} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right) \\
\omega_{2} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(1)\right) \\
\omega_{3} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(-1)\right)
\end{array}\right.
$$

be a matrix representing $\nabla$. Let $q$ be the point of $\mathbf{P}^{1}$ satisfying $\omega_{3}(q)=0$. We may assume without loss of generality that $q \neq t_{i}$ for $i=1,2,3$. From the proof of Lemma 4.2, we have $\omega_{4}(q) \neq 0$. Applying an automorphism of $E_{1}$, we may assume

$$
\omega_{4}=\frac{\left(z-t_{1}\right)\left(z-t_{2}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z, \quad \omega_{3}=\frac{z-q}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z
$$

For a nonzero element $v^{(i)} \in l_{i}$, we have $\left(\operatorname{res}_{t_{i}} \nabla\right)\left(v^{(i)}\right)=\lambda_{i} \phi_{t_{i}}\left(v^{(i)}\right)=0$ for $i=1, \ldots, 4$. Thus $\operatorname{det}\left(\nabla_{t_{i}}\right)=\omega_{1}\left(t_{i}\right) \omega_{4}\left(t_{i}\right)-\omega_{2}\left(t_{i}\right) \omega_{3}\left(t_{i}\right)=0$ for $i=1, \ldots, 4$. Since $\omega_{3}\left(t_{i}\right) \neq 0$ for $i=1,2$, we have $\omega_{2}\left(t_{i}\right)=0$ for $i=1,2$. We write

$$
\omega_{2}=\frac{\left(z-t_{1}\right)\left(z-t_{2}\right) u}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z
$$

with $u$ a polynomial in $z$ of degree less than or equal to 1 . Applying a certain automorphism of $E_{2}$ of the form

$$
\left(\begin{array}{cc}
c_{1} & c_{2} \\
0 & 1
\end{array}\right) \quad\left(c_{1} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}^{\times}\right), c_{2} \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(1)\right)\right)
$$

we may assume that $u=z-t_{3}$. Note that $\nabla$ is of the form

$$
\frac{d z}{\prod_{j=1}^{4}\left(z-t_{j}\right)}\left(\begin{array}{cc}
\alpha & \left(z-t_{1}\right)\left(z-t_{2}\right)\left(z-t_{3}\right) \\
z-q & \left(z-t_{1}\right)\left(z-t_{2}\right)
\end{array}\right) \quad\left(\alpha \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}(2)\right)\right)
$$

Since $\operatorname{det}\left(\nabla_{t_{3}}\right)=0$, we have $\alpha\left(t_{3}\right)=0$. The condition $\operatorname{det}\left(\nabla_{t_{4}}\right)=0$ implies that $\alpha$ is of the form $\alpha=\left(z-t_{3}\right)\left(c\left(z-t_{4}\right)+t_{4}-q\right)$, where $c \in \mathbf{C}$. If $c=1$, we have $\nabla\left(E_{1}\right) \subset \mathcal{O}_{\mathbf{P}^{1}}(-1) \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ after applying a certain automorphism of $E_{2}$. Then there is a decomposition $\left(E_{1}, E_{2}, \phi, \nabla,\left\{l_{i}\right\}\right)=\left(E_{1}, \mathcal{O}_{\mathbf{P}^{1}}(-1), \phi, \nabla,\left\{l_{i}\right\}\right) \oplus\left(0, \mathcal{O}_{\mathbf{P}^{1}}, 0,0,\{0\}\right)$, which contradicts the stability of $\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)$. Thus we have $c \neq 1$. Applying a certain automorphism of $E_{2}$ of the form

$$
\left(\begin{array}{cc}
t & (1-t)\left(z-t_{3}\right) \\
0 & 1
\end{array}\right) \quad\left(t \in H^{0}\left(\mathcal{O}_{\mathbf{P}^{1}}^{\times}\right)\right)
$$

we may assume that $c=0$. Since $\nabla_{t_{i}} \neq 0, \operatorname{ker}\left(\nabla_{t_{i}}\right)=l_{i}$ for $i=$ $1, \ldots, 4$. Hence ( $E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}$ ) is uniquely determined by $q$ and it is determined by the point $p\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right)$.
Q.E.D.

Proposition 4.3. Under the assumption of Lemma 4.2, $\overline{M_{4}^{\alpha^{\prime}}}(-1)$ is smooth over $T_{4} \times \Lambda_{4}$.

Proof. Let $A$ be an artinian local ring over $T_{4} \times \Lambda_{4}$ with residue field $A / m=k$ and $I$ be an ideal of $A$ such that $m I=0$. It is sufficient to show that

$$
\overline{M_{4}^{\alpha^{\prime}}}(-1)(A) \longrightarrow \overline{M_{4}^{\alpha^{\prime}}}(-1)(A / I)
$$

is surjective. Take any member

$$
\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{i}\right\}\right) \in \overline{M_{4}^{\alpha^{\prime}}}(-1)(A / I)
$$

Note that $E_{1} \cong \mathcal{O}_{\mathbf{P}_{A / I}^{1}} \oplus \mathcal{O}_{\mathbf{P}_{A / I}^{1}}(-1)$ and $E_{2} \cong \mathcal{O}_{\mathbf{P}_{A / I}^{1}} \oplus \mathcal{O}_{\mathbf{P}_{A / I}^{1}}(-1)$. Then the homomorphism $\phi: E_{1} \rightarrow E_{2}$ can be represented by a matrix of the form

$$
\left(\begin{array}{cc}
\phi_{1} & \phi_{3} \\
0 & \phi_{2}
\end{array}\right) \quad\left(\phi_{1}, \phi_{2} \in A / I, \phi_{3} \in H^{0}\left(\mathcal{O}_{\mathbf{P}_{A / I}^{1}}(1)\right)\right)
$$

As in the proof of Proposition 4.2, we may assume that $\phi_{3} \in m \otimes$ $H^{0}\left(\mathcal{O}_{\mathbf{P}_{A / I}^{1}}(1)\right)$. Put
$A:=\left(q_{2} \otimes 1\right) \circ \nabla-d \circ q_{2} \circ \phi: E_{1} \longrightarrow \mathcal{O}_{\mathbf{P}_{A / I}^{1}} \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \cong \mathcal{O}_{\mathbf{P}_{A / I}^{1}}(2)$,
$B:=\left(p_{2} \otimes 1\right) \circ \nabla-d \circ p_{2} \circ \phi: E_{1} \longrightarrow \mathcal{O}_{\mathbf{P}_{A / I}^{1}}(-1) \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \cong \mathcal{O}_{\mathbf{P}_{A / I}^{1}}(1)$,
where $q_{2}: E_{2} \rightarrow \mathcal{O}_{\mathbf{P}_{A / I}^{1}}, p_{2}: E_{2} \rightarrow \mathcal{O}_{\mathbf{P}_{A / I}^{1}}(-1)$ are projections with respect to the decomposition of $E_{2}$. Let $\left(\omega_{1}, \omega_{2}\right)$ and $\left(\omega_{3}, \omega_{4}\right)$ be the matrices representing $A$ and $B$, respectively. We can see that the condition

$$
(\varphi \otimes 1)\left(\nabla\left(s_{1}\right) \wedge \phi\left(s_{2}\right)+\phi\left(s_{1}\right) \wedge \nabla\left(s_{2}\right)\right)=d\left(\varphi\left(\phi\left(s_{1}\right) \wedge \phi\left(s_{2}\right)\right)\right) \quad\left(s_{1}, s_{2} \in E_{1}\right)
$$

is equivalent to the equality

$$
\omega_{1} \phi_{2}-\omega_{3} \phi_{3}+\omega_{4} \phi_{1}=0
$$

Let $\left(t_{1}, \ldots, t_{4}\right) \in \mathbf{P}^{1}(A) \times \cdots \times \mathbf{P}^{1}(A),\left(\lambda_{1}, \ldots, \lambda_{4}\right) \in A \times \cdots \times A$ be the data corresponding to the structure morphism $\operatorname{Spec} A \rightarrow T_{4} \times \Lambda_{4}$. Let $v^{(i)}$ be a basis of $l_{i}$. Then we can write $v^{(i)}=\binom{v_{1}^{(i)}}{v_{2}^{(i)}}$ with $v_{1}^{(i)} \in \mathcal{O}_{\mathbf{P}_{A / I}^{1}} \mid t_{i}$ and $\left.v_{2}^{(i)} \in \mathcal{O}_{\mathbf{P}_{A / I}^{1}}(-1)\right|_{t_{i}}$ We must find lifts

$$
\tilde{\phi}_{1}, \tilde{\phi}_{2}, \tilde{\phi}_{3}, \tilde{\omega}_{1}, \tilde{\omega}_{2}, \tilde{\omega}_{3}, \tilde{\omega}_{4},\binom{v_{1}^{(i)}}{v_{2}^{(i)}}_{i=1, \ldots, 4}
$$

over $A$ of $\phi_{1}, \phi_{2}, \phi_{3}, \omega_{1}, \omega_{2}, \omega_{3}, \omega_{4},\binom{v_{1}^{(i)}}{v_{2}^{(i)}}_{i=1, \ldots, 4}$ satisfying the following conditions:

$$
\left\{\begin{array}{l}
\tilde{\omega}_{1} \tilde{\phi}_{2}-\tilde{\omega}_{3} \tilde{\phi}_{3}+\tilde{\omega}_{4} \tilde{\phi}_{1}=0 \\
\left(\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{1}\right)-\lambda_{i} \tilde{\phi}_{1}\right) \tilde{v}_{1}^{(i)}+\left(\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{2}\right)-\lambda_{i} \tilde{\phi}_{3}\left(t_{i}\right)\right) \tilde{v}_{2}^{(i)}=0 \\
\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{3}\right) \tilde{v}_{1}^{(i)}+\left(\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{4}\right)+\left(\operatorname{res}_{t_{i}}\left(\frac{d z}{z-t_{4}}\right)-\lambda_{i}\right) \tilde{\phi}_{2}\right) \tilde{v}_{2}^{(i)}=0 \\
\text { for } i=1, \ldots, 4
\end{array}\right.
$$

Since we have already proved the smoothness of $M_{4}^{\alpha / 2}(-1)$ over $T_{4} \times \Lambda_{4}$, we may assume that $\wedge^{2} \phi \in m A / I$.

Assume that $\phi_{1} \in m A / I$ and $\phi_{2} \in(A / I)^{\times}$. Still we may assume that $\phi_{3}=0$. In this case we can see from the proof of Proposition 4.2 that $\operatorname{res}_{t_{i}}\left(\omega_{3}\right) \in m A / I$ and $\operatorname{res}_{t_{i}}\left(\omega_{2}\right) \in(A / I)^{\times}$for some $i$. Take lifts $\tilde{\omega}_{2}^{(i)} \in \Omega_{\mathbf{P}_{A}^{1}}^{1}(D(\mathbf{t}))(1)_{t_{i}}, \tilde{\omega}_{4} \in H^{0}\left(\Omega_{\mathbf{P}_{A}^{1}}^{1}(D(\mathbf{t}))\right), \tilde{\phi}_{1} \in A$ and $\tilde{\phi}_{2} \in A$ of $\omega_{2}\left(t_{i}\right), \omega_{4}, \phi_{1}$ and $\phi_{2}$, respectively. Put $\tilde{\omega}_{1}:=-\tilde{\omega}_{4} \tilde{\phi}_{1} \tilde{\phi}_{2}^{-1}$. Then we can find a lift $\tilde{\omega}_{3} \in H^{0}\left(\Omega_{\mathbf{P}_{A}^{1}}^{1}(D(\mathbf{t}))(-1)\right)$ of $\omega_{3}$ satisfying

$$
\begin{aligned}
& \left(\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{1}\right)-\lambda_{i} \tilde{\phi}_{1}\right)\left(\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{4}\right)+\left(\operatorname{res}_{t_{i}}\left(\frac{d z}{z-t_{4}}\right)-\lambda_{i}\right) \tilde{\phi}_{2}\right) \\
& \quad-\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{2}^{(i)}\right) \operatorname{res}_{t_{i}}\left(\tilde{\omega}_{3}\right)=0 .
\end{aligned}
$$

Let $\tilde{\omega}_{2}$ be the element of $H^{0}\left(\Omega_{\mathbf{P}_{A}^{1}}^{1}(D(\mathbf{t}))(1)\right)$ satisfying

$$
\begin{aligned}
& \left(\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{1}\right)-\lambda_{j} \tilde{\phi}_{1}\right)\left(\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{4}\right)+\left(\operatorname{res}_{t_{j}}\left(\frac{d z}{z-t_{4}}\right)-\lambda_{j}\right) \tilde{\phi}_{2}\right) \\
& \quad-\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{2}\right) \operatorname{res}_{t_{j}}\left(\tilde{\omega}_{3}\right)=0
\end{aligned}
$$

for $j \neq i$ and $\tilde{\omega}_{2}\left(t_{i}\right)=\tilde{\omega}_{2}^{(i)}$. For $j=1, \ldots, 4$, we can take lifts $\tilde{v}_{1}^{(j)} \in$ $\left.\mathcal{O}_{\mathbf{P}_{A}^{1}}\right|_{t_{j}},\left.\tilde{v}_{2}^{(j)} \in \mathcal{O}_{\mathbf{P}_{A}^{1}}(-1)\right|_{t_{j}}$ of $v_{1}^{(j)}, v_{2}^{(j)}$ satisfying

$$
\left(\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{1}\right)-\lambda_{i} \tilde{\phi}_{1}\right) \tilde{v}_{1}^{(i)}+\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{2}\right) \tilde{v}_{2}^{(i)}=0
$$

and

$$
\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{3}\right) \tilde{v}_{1}^{(j)}+\left(\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{4}\right)+\left(\operatorname{res}_{t_{j}}\left(\frac{d z}{z-t_{4}}\right)-\lambda_{j}\right) \tilde{\phi}_{2}\right) \tilde{v}_{2}^{(j)}=0
$$

for $j \neq i$. Put $\tilde{\phi}_{3}:=0$. Then $\tilde{\phi}_{1}, \tilde{\phi}_{2}, \tilde{\phi}_{3}, \tilde{\omega}_{1}, \tilde{\omega}_{2}, \tilde{\omega}_{3}, \tilde{\omega}_{4},\left(\tilde{v}_{1}^{(j)}, \tilde{v}_{2}^{(j)}\right)_{j=1}^{4}$ are desired lifts.

Next assume that $\phi_{2} \in m / I$. In this case, we can see from the proof of Proposition 4.2 that $\phi_{1} \in m / I$ and $\phi_{2} \in m H^{0}\left(\mathcal{O}_{\mathbf{P}_{A / I}^{1}}(1)\right)$. Take a lift $\tilde{\omega}_{3} \in H^{0}\left(\Omega_{\mathbf{P}_{A}^{1}}^{1}(D(\mathbf{t}))(-1)\right)$ of $\omega_{3}$ and let $q \in \mathbf{P}^{1}(A)$ be the zero point of $\tilde{\omega}_{3}$. There exists $i \in\{1, \ldots, 4\}$ such that $\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{3}\right) \in A^{\times}$for $j \neq i$. Applying a certain auotomorphism of $E_{1}$, we may assume that $\operatorname{res}_{t_{i}}\left(\omega_{4}\right) \in$ $(A / I)^{\times}$. Take lifts $\tilde{\omega}_{4} \in H^{0}\left(\Omega_{\mathbf{P}_{A}^{1}}(D(\mathbf{t}))\right), \tilde{\omega}_{2}^{(i)} \in \Omega_{\mathbf{P}_{A}^{1}}(D(\mathbf{t})(1))_{t_{i}}$ and $\tilde{\phi}_{2} \in A$ of $\omega_{4}, \omega_{2}\left(t_{i}\right)$ and $\phi_{2}$, respectively. We can see from Lemma 4.2 that $\tilde{\omega}_{4}(q)$ is a basis of $\left.\Omega_{\mathbf{P}_{A}^{1}}^{1}(D(\mathbf{t}))\right|_{q}$. Then we can find an element $\tilde{\omega}_{1} \in H^{0}\left(\Omega_{\mathbf{P}_{A}^{1}}^{1}(D(\mathbf{t}))\right)$ such that

$$
\begin{aligned}
& \left(\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{1}\right) \tilde{\omega}_{4}(q)+\lambda_{i} \tilde{\omega}_{1}(q) \tilde{\phi}_{2}\right)\left(\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{4}\right)+\left(\operatorname{res}_{t_{i}}\left(\frac{d z}{z-t_{4}}\right)-\lambda_{i}\right) \tilde{\phi}_{2}\right) \\
& =\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{3}\right) \operatorname{res}_{t_{i}}\left(\tilde{\omega}_{2}^{(i)}\right) \tilde{\omega}_{4}(q)-\lambda_{i}\left(\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{1}\right) \tilde{\phi}_{2} \tilde{\omega}_{4}(q)-\operatorname{res}_{t_{i}}\left(\tilde{\omega}_{4}\right) \tilde{\omega}_{1}(q) \tilde{\phi}_{2}\right)
\end{aligned}
$$

We can take an element $\tilde{\phi}_{1}$ of $A$ such that $\tilde{\phi}_{2} \tilde{\omega}_{1}(q)+\tilde{\phi}_{1} \tilde{\omega}_{4}(q)=0$. Then there is an element $\tilde{\phi}_{3} \in H^{0}\left(\mathcal{O}_{\mathbf{P}_{A}^{1}}(1)\right)$ such that

$$
\tilde{\omega}_{1} \tilde{\phi}_{2}-\tilde{\omega}_{3} \tilde{\phi}_{3}+\tilde{\omega}_{4} \tilde{\phi}_{1}=0
$$

Let $\tilde{\omega}_{2}$ be the element of $H^{0}\left(\Omega_{\mathbf{P}_{A}^{1}}^{1}(D(\mathbf{t}))(1)\right)$ satisfying $\tilde{\omega}_{2}\left(t_{i}\right)=\tilde{\omega}_{2}^{(i)}$ and

$$
\begin{aligned}
& \left(\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{1}\right)-\lambda_{j} \tilde{\phi}_{1}\right)\left(\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{4}\right)+\left(\operatorname{res}_{t_{j}}\left(\frac{d z}{z-t_{4}}\right)-\lambda_{j}\right) \tilde{\phi}_{2}\right) \\
& =\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{3}\right)\left(\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{2}\right)-\lambda_{j} \tilde{\phi}_{3}\left(t_{j}\right)\right)
\end{aligned}
$$

for $j \neq i$. We can take lifts $\left.\tilde{v}_{1}^{(j)} \in \mathcal{O}_{\mathbf{P}_{A}^{1}}\right|_{t_{j}},\left.\tilde{v}_{2}^{(j)} \in \mathcal{O}_{\mathbf{P}_{A}^{1}}(-1)\right|_{t_{j}}$ of $v_{1}^{(j)}, v_{2}^{(j)}$ such that

$$
\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{3}\right) \tilde{v}_{1}^{(j)}+\left(\operatorname{res}_{t_{j}}\left(\tilde{\omega}_{4}\right)+\left(\operatorname{res}_{t_{j}}\left(\frac{d z}{z-t_{4}}\right)-\lambda_{j}\right) \tilde{\phi}_{2}\right) \tilde{v}_{2}^{(j)}=0
$$

for $j=1, \ldots, 4$. Then $\tilde{\phi}_{1}, \tilde{\phi}_{2}, \tilde{\phi}_{3}, \tilde{\omega}_{1}, \tilde{\omega}_{2}, \tilde{\omega}_{3}, \tilde{\omega}_{4},\left(\tilde{v}_{1}^{(j)}, \tilde{v}_{2}^{(j)}\right)_{j=1}^{4}$ are desired lifts.
Q.E.D.

### 4.4. Proof of Theorem 4.1

We put $\lambda_{i}^{+}:=\lambda_{i}$ for $i=1, \ldots, 4, \lambda_{i}^{-}:=-\lambda_{i}$ for $i=1, \ldots, 3$ and $\lambda_{4}^{-}:=1-\lambda_{4}$. Let $D_{i}$ be the fiber of $\mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)$ over $t_{i} \in \mathbf{P}^{1}$ and $b_{i}^{+}$(resp. $b_{i}^{-}$) be the point of $D_{i}$ corresponding to $\lambda_{i}^{+}$(resp. $\lambda_{i}^{-}$). Put $Z:=\left\{b_{1}^{+}, \ldots, b_{4}^{+}, b_{1}^{-}, \ldots, b_{4}^{-}\right\}$.

Proposition 4.4. Under the above notation,

$$
\begin{equation*}
\overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1) \backslash p^{-1}(Z) \xrightarrow{p} \mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right) \backslash Z \tag{41}
\end{equation*}
$$

is an isomorphism.
Proof. Let $D_{0}$ be the section of $\mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)$ over $\mathbf{P}^{1}$ defined by the injection $\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \hookrightarrow \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}$. First we will show that

$$
\begin{equation*}
\overline{M_{4}^{\alpha}}(\mathbf{t}, \boldsymbol{\lambda},-1) \backslash \bigcup_{i=0}^{4} p^{-1}\left(D_{i}\right) \longrightarrow \mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right) \backslash \bigcup_{i=0}^{4} D_{i} \tag{42}
\end{equation*}
$$

is an isomorphism. Fix a section

$$
\tau:\left(\pi_{2}\right)_{*}\left(\left.\pi_{1}^{*} \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right|_{\Delta}\right) \longrightarrow\left(\pi_{2}\right)_{*}\left(\pi_{1}^{*} \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right)
$$

of the canonical homomorphism

$$
\left(\pi_{2}\right)_{*}\left(\pi_{1}^{*} \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right) \longrightarrow\left(\pi_{2}\right)_{*}\left(\left.\pi_{1}^{*} \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right|_{\Delta}\right)
$$

where

$$
\pi_{1}: \mathbf{P}^{1} \times\left(\mathbf{P}^{1} \backslash D(\mathbf{t})\right) \rightarrow \mathbf{P}^{1}, \quad \pi_{2}: \mathbf{P}^{1} \times\left(\mathbf{P}^{1} \backslash D(\mathbf{t})\right) \rightarrow \mathbf{P}^{1} \backslash D(\mathbf{t})
$$

are projections and $\Delta \subset \mathbf{P}^{1} \times\left(\mathbf{P}^{1} \backslash D(\mathbf{t})\right)$ is the diagonal. Take a point $s$ of $\mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right) \backslash \bigcup_{i=0}^{4} D_{i}$, which is given by $q \in \mathbf{P}^{1}$ and an injection $\left(-h_{1}, h_{2}\right):\left.\left.\mathbf{C} \hookrightarrow \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right|_{q} \oplus \mathcal{O}_{\mathbf{P}^{1}}\right|_{q}$. We may assume that $h_{2}=1$. We put

$$
\begin{aligned}
\omega_{4} & :=\tau_{q}\left(h_{1}\right) \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))\right) \\
\omega_{3} & :=\frac{z-q}{\left(t_{4}-q\right) \prod_{j=1}^{4}\left(z-t_{j}\right)} d z \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(-1)\right)
\end{aligned}
$$

where $z$ is a fixed inhomogeneous coordinate of $\mathbf{P}^{1}$. Let $\omega_{2}$ be the element of $H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(1)\right)$ determined by

$$
\left(\operatorname{res}_{t_{i}}\left(\omega_{4}\right)+\lambda_{i}\right)\left(\operatorname{res}_{t_{i}}\left(\omega_{4}\right)+\operatorname{res}_{t_{i}}\left(\frac{d z}{z-t_{4}}\right)-\lambda_{i}\right)+\operatorname{res}_{t_{i}}\left(\omega_{2}\right) \operatorname{res}_{t_{i}}\left(\omega_{3}\right)=0
$$

for $i=1, \ldots, 4$. Define a rational connection $\nabla$ on $\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1)$ by

$$
\nabla\binom{f_{1}}{f_{2}}:=\binom{d f_{1}}{d f_{2}}+\binom{-f_{1} \omega_{4}+f_{2} \omega_{2}}{f_{1} \omega_{3}+f_{2} \omega_{4}}
$$

for $f_{1} \in \mathcal{O}_{\mathbf{P}^{1}}$ and $f_{2} \in \mathcal{O}_{\mathbf{P}^{1}}(-1)$. Then $s \mapsto\left(\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \nabla\right)$ determines a morphism

$$
\mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right) \backslash \bigcup_{i=0}^{4} D_{i} \longrightarrow \overline{M_{4}^{\boldsymbol{\alpha}^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1) \backslash \bigcup_{i=0}^{4} p^{-1}\left(D_{i}\right)
$$

which is just the inverse of the morphism (42). Then the morphism (41) is surjective, since it is proper and dominant. The morphism (41) is also injective by the above argument and Proposition 4.2. Thus, by Zariski's Main Theorem, the morphism (41) is an isomorphism.
Q.E.D.

Proposition 4.5. If $\lambda_{i}^{+} \neq \lambda_{i}^{-}$, then $p^{-1}\left(b_{i}^{+}\right) \cong \mathbf{P}^{1}, p^{-1}\left(b_{i}^{-}\right) \cong \mathbf{P}^{1}$ and these are $(-1)$-curves.

Proof. We can see that $p^{-1}\left(b_{i}^{+}\right)$is just the moduli space of $(\mathbf{t}, \boldsymbol{\lambda})$ parabolic $\phi$-connections $\left(\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \phi, \nabla, \varphi,\left\{l_{j}\right\}\right)$ satisfying

$$
\begin{aligned}
& \phi\binom{s_{1}}{s_{2}}=\binom{\phi_{1} s_{1}}{s_{2}} \\
& \nabla\binom{s_{1}}{s_{2}}=\binom{\phi_{1} s_{1}}{s_{2}}+\binom{s_{1} \phi_{1} \frac{\lambda_{i}^{+} \prod_{j \neq i}\left(t_{i}-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right.} d z+s_{2} \omega_{2}}{s_{1} \frac{\left(z-t_{i}\right) d z}{\prod_{j=1}^{4}\left(z-t_{j}\right)}-s_{2} \frac{\lambda_{i}^{+} \prod_{j \neq i}\left(t_{i}-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z}
\end{aligned}
$$

for $s_{1} \in \mathcal{O}_{\mathbf{P}^{1}}$ and $s_{2} \in \mathcal{O}_{\mathbf{P}^{1}}(-1)$, where $\phi_{1} \in \mathbf{C}, l_{j}=\operatorname{ker}\left(\operatorname{res}_{t_{j}}(\nabla)-\right.$ $\left.\left.\lambda_{j}^{+} \phi\right|_{t_{j}}\right)$ for $j=1, \ldots, 4$ and $\omega_{2} \in H^{0}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))(1)\right)$ satisfies the condition

$$
\begin{aligned}
& \phi_{1}\left(\operatorname{res}_{t_{k}}\left(\frac{\lambda_{i}^{+} \prod_{j \neq i}\left(t_{i}-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z\right)-\lambda_{k}^{+}\right)\left(\operatorname{res}_{t_{k}}\left(\frac{d z}{z-t_{4}}-\frac{\lambda_{i}^{+} \prod_{j \neq i}\left(t_{i}-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z\right)-\lambda_{k}^{+}\right) \\
& -\operatorname{res}_{t_{k}}\left(\frac{\left(z-t_{i}\right) d z}{\Pi_{j=1}^{4}\left(z-t_{j}\right)}\right) \operatorname{res}_{t_{k}}\left(\omega_{2}\right)=0 .
\end{aligned}
$$

for $k \neq i$. Then we can define a mapping

$$
\begin{array}{ccc}
p^{-1}\left(b_{i}^{+}\right) & \longrightarrow & \mathbf{P}^{1} \\
\left(\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \phi, \nabla, \varphi,\left\{l_{j}\right\}\right) & \mapsto & {\left[\phi_{1}: \operatorname{res}_{t_{i}}\left(\omega_{2}\right)\right]}
\end{array}
$$

which is an isomorphism.
Similarly we can see that $p^{-1}\left(b_{i}^{-}\right) \cong \mathbf{P}^{1}$. Since $\overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$ and $\mathbf{P}_{*}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)$ are smooth, $p^{-1}\left(b_{i}^{+}\right), p^{-1}\left(b_{i}^{-}\right)$must be $(-1)$-curves.
Q.E.D.

Proposition 4.6. Assume that $\lambda_{i}^{+}=\lambda_{i}^{-}$. Put

$$
\begin{aligned}
C_{1} & :=\left\{\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{j}\right\}\right) \in p^{-1}\left(b_{i}^{+}\right)\left|l_{i}=L_{1}^{(0)}\right| t_{i}\right\} \\
C_{2} & :=\left\{\left(E_{1}, E_{2}, \phi, \nabla, \varphi,\left\{l_{j}\right\}\right) \in p^{-1}\left(b_{i}^{+}\right) \mid \operatorname{res}_{t_{i}}(\nabla)=\lambda_{i} \phi_{t_{i}}\right\} .
\end{aligned}
$$

Then $C_{1} \cong \mathbf{P}^{1}, C_{2} \cong \mathbf{P}^{1}, C_{1} \cap C_{2}=\{$ one point $\}, C_{1} \cap Y(\mathbf{t}, \boldsymbol{\lambda})=$ $\{$ one point $\}, C_{2} \subset M_{4}^{\alpha}(\mathbf{t}, \boldsymbol{\lambda},-1),\left(C_{1}\right)^{2}=-1,\left(C_{2}\right)^{2}=-2$ and $p^{-1}\left(b_{i}^{+}\right)=$ $C_{1} \cup C_{2}$.

Proof. $p^{-1}\left(b_{i}^{+}\right)$is the moduli space of the objects

$$
\left(\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \phi, \nabla, \varphi,\left\{l_{j}\right\}\right)
$$

satisfying

$$
\begin{aligned}
& \phi\binom{s_{1}}{s_{2}}=\binom{\phi s_{1}}{s_{2}} \\
& \nabla\binom{s_{1}}{s_{2}}=\binom{\phi d s_{1}}{d s_{2}}+\binom{s_{1} \phi_{1} \frac{\lambda_{i}^{+} \prod_{j \neq i}\left(t_{i}-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z+s_{2} \omega_{2}}{s_{1} \frac{\left(z-t_{i}\right) d z}{\prod_{j=1}^{4}\left(z-t_{j}\right)}-s_{2} \frac{\lambda_{i}^{+} \prod_{j \neq i}\left(t_{i}-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z}
\end{aligned}
$$

for $s_{1} \in \mathcal{O}_{\mathbf{P}^{1}}$ and $s_{2} \in \mathcal{O}_{\mathbf{P}^{1}}(-1)$, where $\phi_{1} \in \mathbf{C}, l_{k}=\operatorname{ker}\left(\operatorname{res}_{t_{k}}(\nabla)-\right.$ $\left.\left.\lambda_{k} \phi\right|_{t_{k}}\right)$ for $k \neq i$ and $\omega_{2}$ satisfies the condition

$$
\begin{aligned}
& \phi_{1}\left(\operatorname{res}_{t_{k}}\left(\frac{\lambda_{i}^{+} \prod_{j \neq i}\left(t_{i}-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z\right)-\lambda_{k}^{+}\right)\left(\operatorname{res}_{t_{k}}\left(\frac{d z}{z-t_{4}}-\frac{\lambda_{i}^{+} \prod_{j \neq i}\left(t_{i}-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z\right)-\lambda_{k}^{+}\right) \\
& \quad-\operatorname{res}_{t_{k}}\left(\frac{\left(z-t_{i}\right) d z}{\prod_{j=1}^{4}\left(z-t_{j}\right)}\right) \operatorname{res}_{t_{k}}\left(\omega_{2}\right)=0 .
\end{aligned}
$$

for $k \neq i$. If $v^{(i)}=\binom{v_{1}^{(i)}}{v_{2}^{(i)}}$ is a basis of $l_{i}, \operatorname{res}_{t_{i}}\left(\omega_{2}\right) v_{2}^{(i)}=0$. Thus we have

$$
p^{-1}\left(b_{i}^{+}\right)=\left(\left\{v_{2}^{(i)}=0\right\} \cap p^{-1}\left(b_{i}^{+}\right)\right) \cup\left(\left\{\omega_{2}\left(t_{i}\right)=0\right\} \cap p^{-1}\left(b_{i}^{+}\right)\right) .
$$

We can see that $\left\{v_{2}^{(i)}=0\right\} \cap p^{-1}\left(b_{i}^{+}\right)=C_{1}$ and $\left\{\omega_{2}\left(t_{i}\right)=0\right\} \cap p^{-1}\left(b_{i}^{+}\right)=$ $C_{2}$. From the proof of Proposition 4.2, we can see that the objects of $C_{2}$ satisfies the condition $\phi_{1} \neq 0$. Thus we have $C_{2} \cap Y(\mathbf{t}, \boldsymbol{\lambda})=\emptyset$. We can also see that $C_{1} \cap C_{2}$ consists of one point corresponding to the object of $p^{-1}\left(b_{i}^{+}\right)$satisfying $\omega_{2}\left(t_{i}\right)=0, \phi_{1}=1$ and $l_{i}=\left.L_{1}^{(0)}\right|_{t_{i}} . C_{1} \cap Y(\mathbf{t}, \boldsymbol{\lambda})$ consists of one point corresponding to the object of $C_{1}$ satisfying $\phi_{1}=0$. We have $C_{1} \cong \mathbf{P}^{1}$ by the same proof as Proposition 4.5. $\phi, \nabla, \varphi$ and $l_{k}$ for $k \neq i$ are all constant on $C_{2}$. So $C_{2}$ is just the moduli of lines $\left.\left.l_{i} \subset \mathcal{O}_{\mathbf{P}^{1}}\right|_{t_{i}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1)\right|_{t_{i}}$, which is isomorphic to $\mathbf{P}^{1}$.

Let $N_{4}(\mathbf{t}, \boldsymbol{\lambda},-1)$ be the moduli space of rank 2 bundles $E$ with a connection $\nabla: E \rightarrow E \otimes \Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t}))$ and a horizontal isomorphism $\varphi: \Lambda^{2} E \xrightarrow{\sim} \mathcal{O}_{\mathbf{P}^{1}}\left(-x_{4}\right)$ satisfying
(1) $\operatorname{det}\left(\operatorname{res}_{t_{i}}(\nabla)-\lambda_{i} \operatorname{id}_{\left.E\right|_{t_{i}}}\right)=0$ for $i=1, \ldots, 4$ and
(2) $(E, \nabla)$ is stable in the sense of Simpson [Sim].

Then there is a canonical morphism

$$
M_{4}^{\boldsymbol{\alpha}}(\mathbf{t}, \boldsymbol{\lambda},-1) \longrightarrow N_{4}(\mathbf{t}, \boldsymbol{\lambda},-1)
$$

which is obtained by forgetting parabolic structure. We can see that the image of $C_{2}$ in $N_{4}(\mathbf{t}, \boldsymbol{\lambda},-1)$ is a singular point with $A_{1}$-singularity. Thus $C_{2}$ is a $(-2)$-curve and we can see that $C_{1}$ is a $(-1)$-curve. Q.E.D.

The morphism $p: \overline{M_{4}^{\alpha^{\prime}}}\left(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^{1}}\left(-t_{4}\right)\right) \rightarrow \mathbf{P}\left(\Omega_{\mathbf{P}^{1}}^{1}(D(\mathbf{t})) \oplus \mathcal{O}_{\mathbf{P}^{1}}\right)$ defined in (39) extends to the morphism
$p: \overline{M_{4}^{\alpha^{\prime}}}\left(\mathcal{O}_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4}}\left(-\tilde{t}_{4}\right)\right) \longrightarrow \mathbf{P}\left(\Omega_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4} / T_{4} \times \Lambda_{4}}^{1}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4}}\right)$.
We can check that the inverse image $p^{-1}\left(B^{+}\right)$is a Cartier divisor on $\overline{M_{4}^{\alpha^{\prime}}}\left(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^{1}}\left(-t_{4}\right)\right)$. Since $Z$ is a blow up of

$$
\mathbf{P}\left(\Omega_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4} / T_{4} \times \Lambda_{4}}^{1}(D(\tilde{\mathbf{t}})) \oplus \mathcal{O}_{\mathbf{P}^{1} \times T_{4} \times \Lambda_{4}}\right)
$$

along $B^{+}, p$ induces a morphism

$$
f: \overline{M_{4}^{\alpha^{\prime}}}\left(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^{1}}\left(-t_{4}\right)\right) \longrightarrow Z
$$

We can also check that $f^{-1}\left(g^{-1}(B)\right)=p^{-1}(B)$ is a Cartier divisor on $\overline{M_{4}^{\alpha^{\prime}}}\left(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^{1}}\left(-t_{4}\right)\right)$. Since $\bar{S}$ is a blow up of $Z$ along $g^{-1}(B), f$ induces a morphism

$$
f^{\prime}: \overline{M_{4}^{\alpha^{\prime}}}\left(\mathbf{t}, \boldsymbol{\lambda}, \mathcal{O}_{\mathbf{P}^{1}}\left(-t_{4}\right)\right) \longrightarrow \bar{S}
$$

We can see by Proposition 4.4, Proposition 4.5 and Proposition 4.6 that each fiber of $f^{\prime}$ over $T_{4} \times \Lambda_{4}$ is an isomorphism. Thus $f^{\prime}$ is an isomorphism and Theorem 4.1 (1) is proved.

Theorem 4.1 (2) is easy. It is well-known that $K_{\bar{S}_{(\mathbf{t}, \boldsymbol{\lambda})}} \equiv-\left(2 D_{0}+\right.$ $\left.D_{1}+D_{2}+D_{3}+D_{4}\right)$. So it is sufficient to prove the following proposition in order to prove Theorem 4.1 (3).

Proposition 4.7. $\mathcal{Y}$ is a Cartier divisor on $\overline{M_{4}^{\alpha^{\prime}}}(-1)$ flat over $T_{4} \times$ $\Lambda_{4}$ and the divisor $Y(\mathbf{t}, \boldsymbol{\lambda})$ on $\overline{M_{4}^{\boldsymbol{\alpha}^{\prime}}}\left(\mathbf{t}, \boldsymbol{\lambda},-\mathcal{O}_{\mathbf{P}^{1}}\left(-t_{4}\right)\right)$ has multiplicity 2 along $\left(\left.p\right|_{\mathcal{Y}(\mathbf{t}, \boldsymbol{\lambda})}\right)^{-1}\left(D_{0}\right)$ and 1 along $\left(\left.p\right|_{Y(\mathbf{t}, \boldsymbol{\lambda})}\right)^{-1}\left(D_{i}\right)$ for $i=1, \ldots, 4$.

Proof. Let $\left(\mathcal{E}_{1}, \mathcal{E}_{2}, \tilde{\phi}, \tilde{\nabla}, \tilde{\varphi},\left\{\tilde{l}_{i}\right\}\right)$ be a universal family on $\mathbf{P}^{1} \times \overline{M_{4}^{\alpha^{\prime}}}(-1)$. Then $\tilde{\phi}: \mathcal{E}_{1} \rightarrow \mathcal{E}_{2}$ determines a section $f$ of $\left(\pi_{M_{4}^{\alpha}}\right)_{*}\left(\operatorname{det}\left(\mathcal{E}_{1}\right)^{-1} \otimes \operatorname{det}\left(\mathcal{E}_{2}\right)\right)$, whose zero scheme is $\mathcal{Y}$. Since $\left(\pi_{M_{4}^{\alpha}}\right)_{*}\left(\operatorname{det}\left(\mathcal{E}_{1}\right)^{-1} \otimes \operatorname{det}\left(\mathcal{E}_{2}\right)\right)$ is a line bundle on $\overline{M_{4}^{\alpha^{\prime}}}(-1), \mathcal{Y}$ is a Cartier divisor on $\overline{M_{4}^{\alpha^{\prime}}}(-1) . Y(\mathbf{t}, \boldsymbol{\lambda})$ is also a Cartier divisor on $\overline{M_{4}^{\alpha^{\prime}}}(\mathbf{t}, \boldsymbol{\lambda},-1)$ and so $\mathcal{Y}$ is flat over $T_{4} \times \Lambda_{4}$.

Let $U_{i}$ be the open subscheme of $Y(\mathbf{t}, \boldsymbol{\lambda})$ whose underlying space is $\left(\left.p\right|_{Y(\mathbf{t}, \boldsymbol{\lambda})}\right)^{-1}\left(D_{i} \backslash\left(D_{0} \cap D_{i}\right)\right)$. Then $U_{i}$ is just the moduli space of the objects $\left(\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \phi, \nabla, \varphi,\left\{l_{j}\right\}\right)$ satisfying

$$
\begin{aligned}
& \phi\binom{f_{1}}{f_{2}}=\binom{0}{f_{2}} \\
& \nabla\binom{f_{1}}{f_{2}}=\binom{0}{d f_{2}}+\binom{f_{2} \frac{\prod_{j \neq i}\left(z-t_{j}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z}{f_{1} \frac{\left.z-t_{i}\right) d z}{\prod_{j=1}^{4}\left(z-t_{j}\right)}+f_{2} \frac{a d z}{\prod_{j=1}^{4}\left(z-t_{j}\right)}}
\end{aligned}
$$

for $f_{1} \in \mathcal{O}_{\mathbf{P}^{1}}$ and $f_{2} \in \mathcal{O}_{\mathbf{P}^{1}}(-1)$, where $a \in \mathbf{C}$ and $l_{j}=\operatorname{ker}\left(\operatorname{res}_{t_{j}}(\nabla)-\right.$ $\lambda_{j} \phi_{t_{j}}$ ) for $j=1, \ldots, 4$. Thus $U_{i} \cong \mathbf{A}^{1}$ and $U_{i}$ is reduced.

Let $U_{0}$ be the open subscheme of $Y(\mathbf{t}, \boldsymbol{\lambda})$ such that $p\left(U_{0}\right)=D_{0} \backslash$ $\bigcup_{j=1}^{4} D_{j}$ as sets. $U_{0}$ is the moduli space of the objects $\left(\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1)\right.$, $\left.\mathcal{O}_{\mathbf{P}^{1}} \oplus \mathcal{O}_{\mathbf{P}^{1}}(-1), \phi, \nabla, \varphi,\left\{l_{j}\right\}\right)$ satisfying

$$
\begin{aligned}
\phi\binom{f_{1}}{f_{2}} & =\binom{f_{1} \phi_{1}+f_{2} \phi_{3}}{f_{2} \phi_{2}} \\
\nabla\binom{f_{1}}{f_{2}} & =\binom{\phi_{1} d f_{1}+\phi_{3} d f_{2}}{\phi_{2} d f_{2}}+\binom{\omega_{1} f_{1}}{\omega_{3} f_{1}+\omega_{4} f_{2}}
\end{aligned}
$$

for $f_{1} \in \mathcal{O}_{\mathbf{P}^{1}}$ and $f_{2} \in \mathcal{O}_{\mathbf{P}^{1}(-1)}$ with the conditions $\phi_{1} \phi_{2}=0$ and $\omega_{1} \phi_{2}-\omega_{3} \phi_{3}+\omega_{4} \phi_{1}=0$, where $q \in \mathbf{P}^{1} \backslash\left\{t_{1}, \ldots, t_{4}\right\}, l_{j}=\operatorname{ker}\left(\operatorname{res}_{t_{j}}(\nabla)-\right.$ $\lambda_{j} \phi_{t_{j}}$ ) for $j=1, \ldots, 4$ and

$$
\begin{aligned}
& \omega_{1}=\frac{\prod_{k=3}^{4}\left(z-t_{k}+\left(t_{k}-t_{1}\right)\left(t_{k}-t_{2}\right) \lambda_{k} \phi_{1}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z, \omega_{3}=\frac{(z-q) d z}{\left(t_{4}-q\right) \prod_{j=1}^{4}\left(z-t_{j}\right)} \\
& \omega_{4}=\frac{\prod_{k=1}^{2}\left(z-t_{k}+\left(t_{k}-t_{3}\right)\left(t_{k}-t_{4}\right) \lambda_{k} \phi_{2}\right)}{\prod_{j=1}^{4}\left(z-t_{j}\right)} d z
\end{aligned}
$$

$\phi_{2}$ and $\phi_{3}$ are determined by $\phi_{1}$ and the conditions
$\omega_{1}(q) \phi_{2}+\omega_{4}(q) \phi_{1}=0, \quad \omega_{3}\left(t_{j}\right) \phi_{3}\left(t_{j}\right)=\omega_{1}\left(t_{j}\right) \phi_{2}+\omega_{4}\left(t_{j}\right) \phi_{1} \quad(j=1,2)$
and $\phi_{2}$ must satisfy the condition $\phi_{1}^{2}=0$. Thus $U_{0} \cong \mathbf{P}^{1} \backslash\left\{t_{1}, \ldots, t_{4}\right\} \times$ $\operatorname{Spec} \mathbf{C}\left[\phi_{1}\right] /\left(\phi_{1}^{2}\right)$ and $Y(\mathbf{t}, \boldsymbol{\lambda})$ has multiplicity 2 along $\left(\left.p\right|_{Y(\mathbf{t}, \boldsymbol{\lambda})}\right)^{-1}\left(D_{0}\right)$.
Q.E.D.

## §5. Moduli of stable parabolic connections in general case

In this section, we will formulate the general moduli theory of $\boldsymbol{\alpha}$ stable parabolic connections over a curve and state the existence theorem of the coarse moduli scheme due to Inaba [Ina]. We fix integers $g, d, r, n$ with $g \geq 0, r>0, n>0$ and let $(C, \mathbf{t})=\left(C, t_{1}, \ldots, t_{n}\right)$ be an $n$-pointed smooth projective curve of genus $g$, which consists of a smooth projective curve $C$ and a set of $n$-distinct points $\mathbf{t}=\left\{t_{i}\right\}_{1 \leq i \leq n}$ on $C$. We denote by $D(\mathbf{t})=t_{1}+\cdots+t_{n}$ the divisor associated to $\mathbf{t}$. Define the set of exponents as
$\Lambda_{r}^{n}(d):=\left\{\boldsymbol{\lambda}=\left(\lambda_{j}^{(i)}\right)_{\substack{1 \leq i \leq n \\ 0 \leq j \leq r-1}}^{\substack{\text { j }}} \mathbf{C}^{n r} \mid d+\sum_{1 \leq i \leq n, 0 \leq j \leq r-1} \lambda_{j}^{(i)}=0\right\}$.
Definition 5.1. A $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection of rank $r$ on $C$ is a collection of data $\left(E, \nabla,\left\{l_{*}^{(i)}\right\}_{1 \leq i \leq n}\right)$ consisting of:
(1) a vector bundle $E$ of rank $r$ on $C$,
(2) a logarithmic connection $\nabla: E \longrightarrow E \otimes \Omega_{C}^{1}(D(\mathbf{t}))$,
(3) and a filtration $l_{*}^{(i)}: E_{\mid t_{i}}=l_{0}^{(i)} \supset l_{1}^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_{r}^{(i)}=0$ for each $i, 1 \leq i \leq n$ such that $\operatorname{dim}\left(l_{j}^{(i)} / l_{j+1}^{(i)}\right)=1$ and $\left(\operatorname{res}_{t_{i}}(\nabla)-\right.$ $\left.\lambda_{j}^{(i)}\right)\left(l_{j}^{(i)}\right) \subset l_{j+1}^{(i)}$ for $j=0,1, \cdots, r-1$.
We set $\operatorname{deg} E=\operatorname{deg}\left(\wedge^{r} E\right)$ as usual.
Take a sequence of rational numbers $\boldsymbol{\alpha}=\left(\alpha_{j}^{(i)}\right)_{1 \leq j \leq r}^{1 \leq i \leq n}$ such that

$$
\begin{equation*}
0<\alpha_{1}^{(i)}<\alpha_{2}^{(i)}<\cdots<\alpha_{r}^{(i)}<1 \tag{44}
\end{equation*}
$$

for $i=1, \ldots, n$ and $\alpha_{j}^{(i)} \neq \alpha_{j^{\prime}}^{\left(i^{\prime}\right)}$ for $(i, j) \neq\left(i^{\prime}, j^{\prime}\right)$. We choose $\boldsymbol{\alpha}=$ $\left(\alpha_{j}^{(i)}\right)$ sufficiently generic. Let $\left(E, \nabla,\left\{l_{*}^{(i)}\right\}_{1 \leq i \leq n}\right)$ be a $(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connection, and $F \subset E$ a nonzero subbundle satisfying $\nabla(F) \subset F \otimes$ $\Omega_{C}^{1}(D(\mathbf{t}))$. We define integers len $(F)_{j}^{(i)}$ by

$$
\begin{equation*}
\operatorname{len}(F)_{j}^{(i)}=\operatorname{dim}\left(\left.F\right|_{t_{i}} \cap l_{j-1}^{(i)}\right) /\left(\left.F\right|_{t_{i}} \cap l_{j}^{(i)}\right) \tag{45}
\end{equation*}
$$

Note that $\operatorname{len}(E)_{j}^{(i)}=\operatorname{dim}\left(l_{j-1}^{(i)} / l_{j}^{(i)}\right)=1$ for $1 \leq j \leq r$.
Definition 5.2. A parabolic connection $\left(E, \nabla,\left\{l_{*}^{(i)}\right\}_{1 \leq i \leq n}\right)$ is $\boldsymbol{\alpha}$ stable if for any proper nonzero subbundle $F \varsubsetneqq E$ satisfying $\nabla(F) \subset$ $F \otimes \Omega_{C}^{1}(D(\mathbf{t}))$, the inequality

$$
\begin{equation*}
\frac{\operatorname{deg} F+\sum_{i=1}^{m} \sum_{j=1}^{r} \alpha_{j}^{(i)} \operatorname{len}(F)_{j}^{(i)}}{\operatorname{rank} F}<\frac{\operatorname{deg} E+\sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_{j}^{(i)} \operatorname{len}(E)_{j}^{(i)}}{\operatorname{rank} E} \tag{46}
\end{equation*}
$$

holds.
For a fixed $(C, \mathbf{t})$ and $\boldsymbol{\lambda}$, let us define the coarse moduli space by

$$
\begin{align*}
& \mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\boldsymbol{\alpha}}(r, n, d)=  \tag{47}\\
& \left\{\left(E, \nabla,\left\{l_{*}^{(i)}\right\}_{1 \leq i \leq n}\right) \mid\right. \\
& \left.\quad \begin{array}{l}
\text { an } \boldsymbol{\alpha} \text {-stable }(\mathbf{t}, \boldsymbol{\lambda}) \text {-parabolic connection } \\
\text { of rank } r \text { and degree } d \text { over } C
\end{array}\right\} / \simeq
\end{align*}
$$

Varying $(C, \mathbf{t})$ and $\boldsymbol{\lambda}$, we can also consider the moduli space in relative setting. Let $\mathcal{M}_{g, n}$ be the coarse moduli space of $n$-pointed curves of genus $g$. Here we assume that every point of $\mathcal{M}_{g, n}$ corresponds to an $n$-pointed smooth curve $(C, \mathbf{t})$ such that $\mathbf{t}=\left(t_{1}, \ldots, t_{n}\right)$ is a set of $n$-distinct points on $C$. We consider a finite covering $\mathcal{M}_{g, n}^{\prime} \rightarrow \mathcal{M}_{g, n}$ where $\mathcal{M}_{g, n}^{\prime}$ is the coarse moduli space of $n$-pointed curves of genus $g$ with a suitable level structure so that there exists the universal family $(\mathcal{C}, \tilde{\mathbf{t}})=\left(\mathcal{C}, \tilde{t}_{1}, \ldots, \tilde{t}_{n}\right)$ of $n$-pointed curves (with a level structure). From now on, for simplicity, we set

$$
\begin{equation*}
T=\mathcal{M}_{g, n}^{\prime} \tag{48}
\end{equation*}
$$

and let

$$
\begin{equation*}
(\mathcal{C}, \tilde{\mathbf{t}}) \longrightarrow T=\mathcal{M}_{g, n}^{\prime} \tag{49}
\end{equation*}
$$

be the universal family.
We can show the existence theorem of moduli space as a smooth quasi-projective algebraic scheme (cf. [IIS1], [Ina]).

Theorem 5.1. (Cf. [IIS1], [Ina]). Assume that $r, n, d$ are positive integers. There exists a relative moduli scheme

$$
\begin{equation*}
\varphi_{r, n, d}: \mathcal{M}_{(\mathcal{C}, \tilde{\mathbf{t}}) / T}^{\alpha}(r, n, d) \longrightarrow T \times \Lambda_{r}^{(n)}(d) \tag{50}
\end{equation*}
$$

of $\boldsymbol{\alpha}$-stable parabolic connections of rank $r$ and degree $d$, which is smooth and quasi-projective over $T \times \Lambda_{r}^{(n)}(d)$. Moreover the fiber $\mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\alpha}(r, n, d)$ of $\varphi_{r, n, d}$ over $((C, \mathbf{t}), \boldsymbol{\lambda}) \in T \times \Lambda_{r}^{(n)}(d)$ is the moduli space of $\boldsymbol{\alpha}$-stable
$(\mathbf{t}, \boldsymbol{\lambda})$-parabolic connections over $C$, which is a smooth algebraic scheme and

$$
\begin{equation*}
\operatorname{dim} \mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\boldsymbol{\alpha}}(r, n, d)=2 r^{2}(g-1)+n r(r-1)+2 \tag{51}
\end{equation*}
$$

## Remark 5.1.

(1) When $C=\mathbf{P}^{1}$ and $r=2$, Theorem 5.1 is proved in [IIS1].
(2) Inaba [Ina] showed that the moduli space $\mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\boldsymbol{\alpha}}(r, n, d)$ is irreducible in the following cases:
(a) $g \geq 2, n \geq 1$,
(b) $g=1, n \geq 2$,
(c) $g=0, r \geq 2, r n-2 r-2>0$

### 5.1. The moduli space of representations

For each $n$-pointed curve $(C, \mathbf{t})=\left(C, t_{1}, \cdots, t_{n}\right) \in T=\mathcal{M}_{g, n}^{\prime}(g \geq$ $0, n \geq 1)$, set $D(\mathbf{t})=t_{1}+\cdots+t_{n}$. By abuse of notation, we denote by $\pi_{1}(C \backslash D(\mathbf{t}) *)$ the fundamental group of $C \backslash\left\{t_{1}, \cdots, t_{n}\right\}$. The set

$$
\begin{equation*}
\operatorname{Hom}\left(\pi_{1}(C \backslash D(\mathbf{t}), *), G L_{r}(\mathbf{C})\right) \tag{52}
\end{equation*}
$$

of $G L_{r}(\mathbf{C})$-representations of $\pi_{1}(C \backslash D(\mathbf{t}), *)$ is an affine variety, and $G L_{r}(\mathbf{C})$ naturally acts on this space by the adjoint action.

We define the moduli space by

$$
\begin{equation*}
\mathcal{R} \mathcal{P}_{(C, \mathbf{t})}^{r}=\operatorname{Hom}\left(\pi_{1}(C \backslash D(\mathbf{t}), *), G L_{r}(\mathbf{C})\right) / / A d\left(G L_{r}(\mathbf{C})\right) \tag{53}
\end{equation*}
$$

Here the quotient // means the categorical quotient ([Mum]). More precisely, it is known that $\pi_{1}(C \backslash D(\mathbf{t}), *)$ is generated by $(2 g+n)$ elements $\alpha_{1}, \ldots, \alpha_{g}, \beta_{1}, \ldots, \beta_{g}, \gamma_{1}, \ldots, \gamma_{n}$ with one relation

$$
\prod_{i=1}^{g}\left[\alpha_{i}, \beta_{i}\right] \gamma_{1} \cdots \gamma_{n}=1
$$

Therefore if we denote by $R$ the ring of invariants of the simultaneous adjoint action of $G L_{r}(\mathbf{C})$ on the coordinate ring of $G L_{r}(\mathbf{C})^{2 g+n-1}$, then we have an isomorphism

$$
\begin{equation*}
\mathcal{R} \mathcal{P}_{(C, \mathbf{t})}^{r} \simeq \operatorname{Spec}(R) \tag{54}
\end{equation*}
$$

Hence the moduli space $\mathcal{R} \mathcal{P}_{(C, \mathbf{t})}^{r}$ becomes an affine algebraic scheme. Furthermore, each closed point of $\mathcal{R} \mathcal{P}_{(C, \mathbf{t})}^{r}$ corresponds to a Jordan equivalence class of a representation (cf. [Section 4, [IIS1]]).

Let us set

$$
\begin{equation*}
\mathcal{A}_{r}^{(n)}:=\left\{\mathbf{a}=\left(a_{j}^{(i)}\right)_{0 \leq j \leq r-1}^{1 \leq i \leq n} \in \mathbf{C}^{n r} \mid a_{0}^{(1)} a_{0}^{(2)} \cdots a_{0}^{(n)}=(-1)^{r n}\right\} \tag{55}
\end{equation*}
$$

For each $\mathbf{a}=\left(a_{j}^{(i)}\right) \in \mathcal{A}_{r}^{(n)}$ and $i, 1 \leq i \leq n$, we set $\mathbf{a}^{(i)}=\left(a_{0}^{(i)}, \cdots, a_{r-1}^{(i)}\right)$ and define

$$
\begin{equation*}
\chi_{\mathbf{a}^{(i)}}(s)=s^{r}+a_{r-1}^{(i)} s^{r-1}+\cdots+a_{0}^{(i)} \tag{56}
\end{equation*}
$$

Moreover we define a morphism

$$
\begin{equation*}
\phi_{(C, \mathbf{t})}^{r}: \mathcal{R} \mathcal{P}_{(C, \mathbf{t})}^{r} \longrightarrow \mathcal{A}_{r}^{(n)} \tag{57}
\end{equation*}
$$

by the relation

$$
\begin{equation*}
\operatorname{det}\left(s I_{r}-\rho\left(\gamma_{i}\right)\right)=\chi_{\mathbf{a}^{(i)}}(s) \tag{58}
\end{equation*}
$$

where $[\rho] \in \mathcal{R} \mathcal{P}_{(C, \mathbf{t})}^{r}$ and $\gamma_{i}$ is a counterclockwise loop around $t_{i}$.
For $\mathbf{a}=\left(a_{j}^{(i)}\right) \in \mathcal{A}_{r}^{(n)}$, we denote by $\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}$ the fiber of $\phi_{(C, \mathbf{t})}^{r}$ over a, that is,

$$
\begin{equation*}
\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}=\left\{[\rho] \in \mathcal{R} \mathcal{P}_{(C, \mathbf{t})}^{r} \mid \operatorname{det}\left(s I_{r}-\rho\left(\gamma_{i}\right)\right)=\chi_{\mathbf{a}^{(i)}}(s), 1 \leq i \leq n\right\} . \tag{59}
\end{equation*}
$$

For any covering $T^{\prime} \rightarrow T$, we can define a relative moduli space $\mathcal{R} \mathcal{P}_{n, T^{\prime}}^{r}=\coprod_{(C, \mathbf{t}) \in T^{\prime}} \mathcal{R} \mathcal{P}_{(C, \mathbf{t})}^{r}$ of representations with the natural morphism

$$
\begin{equation*}
\mathcal{R} \mathcal{P}_{n, T^{\prime}}^{r} \longrightarrow T^{\prime} \tag{60}
\end{equation*}
$$

As in Section 4, [IIS1], there exists a finite covering $T^{\prime} \longrightarrow T$ with the morphism

$$
\begin{equation*}
\phi_{n}^{r}: \mathcal{R} \mathcal{P}_{n, T^{\prime}}^{r} \longrightarrow T^{\prime} \times \mathcal{A}_{r}^{(n)} \tag{61}
\end{equation*}
$$

such that

$$
\left(\phi_{n}^{r}\right)^{-1}((C, \mathbf{t}), \mathbf{a})=\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}
$$

## §6. The Riemann-Hilbert correspondence

Next we define the Riemann-Hilbert correspondence from the moduli space of $\boldsymbol{\alpha}$-stable parabolic connections to the moduli space of the representations.

Let us fix positive integers $r, d, \boldsymbol{\alpha}=\left(\alpha_{j}^{(i)}\right)$ as in (44), and $(C, \mathbf{t}) \in$ $T^{\prime}=\mathcal{M}_{g, n}^{\prime}$. For simplicity, we set $\mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\boldsymbol{\alpha}}=\mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\boldsymbol{\alpha}}(r, n, d)(\mathrm{cf}$. (47)).

We define a morphism

$$
\begin{equation*}
r h: \Lambda_{r}^{(n)}(d) \longrightarrow \mathcal{A}_{r}^{(n)}, \quad r h(\boldsymbol{\lambda})=\mathbf{a} \tag{62}
\end{equation*}
$$

by the relation

$$
\begin{equation*}
\prod_{j=0}^{r-1}\left(s-\exp \left(-2 \pi \sqrt{-1} \lambda_{j}^{(i)}\right)\right)=s^{r}+a_{r-1}^{(i)} s^{r-1}+\cdots+a_{0}^{(i)} \tag{63}
\end{equation*}
$$

For each member $\left(E, \nabla,\left\{l_{j}^{(i)}\right\}\right) \in \mathcal{M}_{(C, \mathbf{t}), \boldsymbol{\lambda}}^{\boldsymbol{\alpha}}$, the solution subsheaf of $E^{a n}$

$$
\begin{equation*}
\operatorname{ker}\left(\left.\nabla^{a n}\right|_{C \backslash D(\mathbf{t})}\right) \subset E^{a n} \tag{64}
\end{equation*}
$$

becomes a local system on $C \backslash D(\mathbf{t})$ and corresponds to a representation

$$
\begin{equation*}
\rho: \pi_{1}(C \backslash\{\mathbf{t}\}, *) \longrightarrow G L_{r}(\mathbf{C}) \tag{65}
\end{equation*}
$$

Since the eigenvalues of the residue matrix of $\nabla^{a n}$ at $t_{i}$ are $\lambda_{j}^{(i)}, 0 \leq$ $j \leq r-1$, considering the local fundamental solutions of $\nabla^{a n}=0$ near $t_{i}$, the monodromy matrix of $\rho\left(\gamma_{i}\right)$ has eigenvalues $\exp \left(-2 \pi \sqrt{-1} \lambda_{j}^{(i)}\right)$, $0 \leq j \leq r-1$. Hence under the relation (63), or $\mathbf{a}=r h(\boldsymbol{\lambda})$, we can define a morphism

$$
\begin{equation*}
\mathbf{R H}_{(C, \mathbf{t}), \boldsymbol{\lambda}}: \mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\boldsymbol{\alpha}} \longrightarrow \mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r} \tag{66}
\end{equation*}
$$

Replacing $T=\mathcal{M}_{g, n}^{\prime}$ by a certain finite étale covering $u: T^{\prime} \longrightarrow T$ and varying $((C, \mathbf{t}), \boldsymbol{\lambda}) \in T^{\prime} \times \Lambda_{r}^{(n)}(d)$ we can define a morphism

$$
\begin{equation*}
\mathbf{R H}: \mathcal{M}_{(\mathcal{C}, \mathbf{t}) / T^{\prime}}^{\boldsymbol{\alpha}}(r, n, d) \longrightarrow \mathcal{R} \mathcal{P}_{n, T^{\prime}}^{r} \tag{67}
\end{equation*}
$$

which makes the diagram

$$
\begin{array}{ccc}
\mathcal{M}_{(\mathcal{C}, \tilde{\mathbf{t}}) / T^{\prime}}^{\boldsymbol{\alpha}}(r, n, d) & \xrightarrow{\mathbf{R H}} & \mathcal{R} \mathcal{P}_{n, T^{\prime}}^{r} \\
\varphi_{r, n, d} \downarrow & & \phi_{n}^{r}  \tag{68}\\
T^{\prime} \times \Lambda_{r}^{(n)}(d) & \xrightarrow{I d \times r h} T^{\prime} \times \mathcal{A}_{r}^{(n)}
\end{array}
$$

commute. The following result is proved in [Ina].
Theorem 6.1. ([Theorem 2.2, [Ina]] ). Assume that $\boldsymbol{\alpha}$ is so generic that $\boldsymbol{\alpha}$-stable $\Leftrightarrow \boldsymbol{\alpha}$-semistable. Moreover we assume that $r \geq 2, r n-2 r-$ $2>0$ if $g=0, n \geq 2$ if $g=1$ and $n \geq 1$ if $g \geq 2$. Then the morphism

$$
\begin{equation*}
\mathbf{R H}: \mathcal{M}_{(\mathcal{C}, \tilde{\mathbf{t}}) / T^{\prime}}^{\boldsymbol{\alpha}}(r, n, d) \longrightarrow \mathcal{R} \mathcal{P}_{n, T^{\prime}}^{r} \times_{\mathcal{A}_{r}^{(n)}} \Lambda_{r}^{(n)} \tag{69}
\end{equation*}
$$

induced by (67) is a proper surjective bimeromorphic analytic morphism. In particular, for each $((C, \mathbf{t}), \boldsymbol{\lambda}) \in T^{\prime} \times \Lambda_{r}^{(n)}(d)$, the restricted morphism

$$
\begin{equation*}
\mathbf{R H}_{((C, \mathbf{t}), \boldsymbol{\lambda})}: \mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\boldsymbol{\alpha}}(r, n, d) \longrightarrow \mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r} \tag{70}
\end{equation*}
$$

gives an analytic resolution of singularities of $\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}$ where $\mathbf{a}=$ $r h(\boldsymbol{\lambda})$.

Remark 6.1. Take $\boldsymbol{\lambda} \in \Lambda_{r}^{(n)}$ such that $r h(\boldsymbol{\lambda})=\mathbf{a}$. A representation $\rho$ such that $[\rho] \in \mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}$ is said to be resonant if

$$
\begin{equation*}
\operatorname{dim}\left(\operatorname{ker}\left(\rho\left(\gamma_{i}\right)-\exp \left(-2 \pi \sqrt{-1} \lambda_{j}^{(i)}\right)\right)\right) \geq 2 \text { for some } i, j \tag{71}
\end{equation*}
$$

The singular locus of $\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}$ is given by the set

$$
\left(\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}\right)^{\operatorname{sing}}:=\left\{[\rho] \in \mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r} \left\lvert\, \begin{array}{l}
\rho \text { is reducible or }  \tag{72}\\
\text { resonant }
\end{array}\right.\right\}
$$

Moreover we denote the smooth part of $\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}$ by

$$
\begin{equation*}
\left(\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}\right)^{\sharp}=\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r} \backslash\left(\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}\right)^{\operatorname{sing}} \tag{73}
\end{equation*}
$$

Theorem 6.1 implies that the restriction

$$
\begin{equation*}
\mathbf{R H}_{((C, \mathbf{t}), \boldsymbol{\lambda}) \mid\left(\mathcal{M}_{(C, \mathbf{t}), \boldsymbol{\lambda}}^{\alpha}\right)}:\left(\mathcal{M}_{(C, \mathbf{t}), \boldsymbol{\lambda}}^{\alpha}\right)^{\sharp} \xrightarrow{\simeq}\left(\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}\right)^{\sharp} \tag{74}
\end{equation*}
$$

is an analytic isomorphism, where

$$
\left(\mathcal{M}_{(C, \mathbf{t}), \boldsymbol{\lambda}}^{\boldsymbol{\alpha}}\right)^{\sharp}=\mathbf{R} \mathbf{H}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{-1}\left(\left(\mathcal{R} \mathcal{P}_{(C, \mathbf{t}), \mathbf{a}}^{r}\right)^{\sharp}\right) .
$$

## §7. Isomonodromic flows and Differential systems of Painlevé type

Consider the family of the moduli spaces of $\boldsymbol{\alpha}$-stable parabolic connections

$$
\begin{equation*}
\varphi_{r, n, d}: \mathcal{M}_{(\mathcal{C}, \mathbf{t}) / T}^{\boldsymbol{\alpha}}(r, d, n) \longrightarrow T \times \Lambda_{r}^{(n)}(d) \tag{75}
\end{equation*}
$$

where $T=\mathcal{M}_{g, n}^{\prime}$ as in (48).
Fix $\left(\left(C_{0}, \mathbf{t}_{0}\right), \boldsymbol{\lambda}_{0}\right) \in T \times \Lambda_{r}^{(n)}(d)$ and take an $\boldsymbol{\alpha}$-stable parabolic connection $\mathbf{x}=\left(E, \nabla,\left\{l_{*}^{(i)}\right\}_{1 \leq i \leq n}\right) \in \mathcal{M}_{\left(\left(\mathcal{C}_{0}, \mathbf{t}_{0}\right), \boldsymbol{\lambda}_{0}\right)}^{\alpha}(r, d, n)$. Let $\Delta=\{t \in$ $\mathbf{C}||t|<1\}$ be the unit disc and let $h: \Delta \longrightarrow T$ be a holomorphic
embedding such that $h(0)=\left(C_{0}, \mathbf{t}_{0}\right)$. Then pulling back the universal family, we obtain the family of $n$-pointed curves $f:(\mathcal{C}, \mathbf{t}) \longrightarrow \Delta$ with the central fiber $f^{-1}(0)=\left(C_{0}, \mathbf{t}_{0}\right)$. An $\boldsymbol{\alpha}$-stable parabolic connection $(\mathcal{E}, \nabla, l)$ on the family of $n$-pointed curves $(\mathcal{C}, \mathbf{t})$ over $\Delta$ is called a (1-parameter) deformation of $\left(E, \nabla,\left\{l_{*}^{(i)}\right\}_{1 \leq i \leq n}\right)$ if we have an isomor$\operatorname{phism}(\mathcal{E}, \nabla, l)_{\mid\left(C_{0}, \mathbf{t}_{0}\right)} \simeq\left(E, \nabla,\left\{l_{*}^{(i)}\right\}_{1 \leq i \leq n)}\right.$. Restricting the $\boldsymbol{\alpha}$-stable parabolic connection $(\mathcal{E}, \nabla, l)$ to each fiber $\left(\mathcal{C}_{t}, \mathbf{t}_{t}\right)$, we have a family of $\boldsymbol{\alpha}$-stable parabolic connections $\left(\mathcal{E}_{t}, \nabla_{t}, l_{t}\right)$ over $\left(\mathcal{C}_{t}, \mathbf{t}_{t}\right)$ which are automatically flat in the direction of each fiber. If the connection $\nabla$ on $\mathcal{E}$ is flat on the total space $\mathcal{C}$, which means that the curvature 2 -form of $\nabla$ vanishes over the total space $\mathcal{C}$, the associated representations $\rho_{t}: \pi_{1}\left(\mathcal{C}_{t} \backslash\left\{\mathbf{t}_{t}\right\}, *\right) \longrightarrow G L_{r}(\mathbf{C})$ is constant with respect to $t \in \Delta$. Moreover the converse is also true. Therefore such a deformation $(\mathcal{E}, \nabla, l)$ over $\mathcal{C} \longrightarrow \Delta$ is called an isomonodromic deformation of a $\boldsymbol{\alpha}$-stable parabolic connection. Under an isomonodromic deformation, local exponents $\boldsymbol{\lambda}_{t}$ of the connection $\left(\mathcal{E}_{t}, \nabla_{t}, l_{t}\right)$ are also constant, so we have $\boldsymbol{\lambda}_{t}=\boldsymbol{\lambda}_{0}$. Therefore an isomonodromic deformation determines a holomorphic map $\tilde{h}: \Delta \longrightarrow \mathcal{M}_{(\mathcal{C}, \mathbf{t}), \boldsymbol{\lambda}_{0} / T}^{\alpha}(r, d, n)$ which is a lift of $h: \Delta \longrightarrow T$ such that $\tilde{h}(0)=\mathbf{x} \in \mathcal{M}_{\left(\left(\mathcal{C}_{0}, \mathbf{t}_{0}\right), \boldsymbol{\lambda}_{0}\right)}^{\boldsymbol{\alpha}}(r, d, n)$.

$$
\begin{array}{llll} 
& & \mathcal{M}_{(\mathcal{C}, \mathbf{t}), \boldsymbol{\lambda}_{0} / T}^{\alpha}(r, n, d) \\
& \tilde{h} & \nearrow & \downarrow \varphi_{r, n, d, \boldsymbol{\lambda}_{0}} \\
\Delta & \xrightarrow{h} & & T \times\left\{\boldsymbol{\lambda}_{0}\right\}
\end{array}
$$

Next we will define a global foliation $\mathcal{I F}$ on the total space of $\mathcal{M}_{(\mathcal{C}, \mathbf{t}) / T}^{\alpha}(r, d, n)$ from isomonodromic deformations of the $\boldsymbol{\alpha}$-stable parabolic connections. We mean that a foliation $\mathcal{I F}$ is a subsheaf of the tangent sheaf $\Theta_{\mathcal{M}_{(\mathcal{C}, \mathrm{t}) / T}^{\alpha}(r, d, n)}$. We will show that the global foliation $\mathcal{I F}$ coming from isomonodromic deformations has the Painlevé property, whose precise meaning will be defined in Theorem 7.1.

Let us consider the universal covering map $u: \tilde{T} \rightarrow T=\mathcal{M}_{g, n}^{\prime}$. Note that $u$ factors thorough the morphism $u^{\prime}: \tilde{T} \rightarrow T^{\prime}$. Pulling back the fibration $\phi_{n}^{r}: \mathcal{R} \mathcal{P}_{n, T^{\prime}}^{r} \longrightarrow T^{\prime} \times \mathcal{A}_{r}^{(n)}$ in (61) by $u^{\prime}$, we obtain the fibration $\mathcal{R} \mathcal{P}_{n, T^{\prime}}^{r} \times_{T^{\prime}} \tilde{T} \longrightarrow \tilde{T}$, which becomes a trivial fibration as explained in Section 4 in [IIS1]. This means that if we fix a point $\left(C_{0}, \mathbf{t}_{0}\right) \in T$ there exists an isomorphism

$$
\begin{equation*}
\pi: \mathcal{R} \mathcal{P}_{n, T^{\prime}}^{r} \times T_{T^{\prime}} \tilde{T} \xrightarrow{\simeq} \mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right)}^{r} \times \tilde{T} \tag{76}
\end{equation*}
$$

which makes the following diagram commute.

\[

\]

Fixing $\mathbf{a} \in \mathcal{A}_{r}^{(n)}$, we set $\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r}=\left(\phi_{n}^{r}\right)^{-1}\left(T^{\prime} \times\{\mathbf{a}\}\right)$. From the morphisms (57) and (61), we also have the following commutative diagram:

$$
\begin{align*}
& \mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times{ }_{T^{\prime}} \tilde{T} \xrightarrow{\pi_{\mathbf{a}}} \mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r} \times \tilde{T} \\
& \widetilde{\phi_{n, \mathbf{a}}^{r}} \downarrow \quad \downarrow p_{2}  \tag{78}\\
& \tilde{T} \times\{\mathbf{a}\} \quad \simeq \quad \tilde{T} .
\end{align*}
$$

By using the isomorphism (78) we can define the smooth part of $\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times{ }_{T^{\prime}} \tilde{T}$ by

$$
\left(\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times{ }_{T^{\prime}} \tilde{T}\right)^{\sharp}=\pi_{\mathbf{a}}^{-1}\left(\left(\mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r}\right)^{\sharp} \times \tilde{T}\right)
$$

where $\left(\mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r}\right)^{\sharp}$ is the smooth locus of $\mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r}(\mathrm{cf} .(73))$. Note that for generic a the variety $\mathcal{R} \mathcal{P}^{r}{ }_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}$ is non-singular, but for special $\mathbf{a}, \mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r}$ does have singularities (cf. [(72), Remark 6.1]).

We also have the following commutative diagram

$$
\begin{array}{ccc}
\left(\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times T_{T^{\prime}} \tilde{T}\right)^{\sharp} & \stackrel{\pi_{\mathbf{a}}}{\simeq}\left(\mathcal{R P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r}\right)^{\sharp} \times \tilde{T} \\
\downarrow & & \downarrow_{2}  \tag{79}\\
\tilde{T} \times\{\mathbf{a}\} & \simeq & \tilde{T}
\end{array} .
$$

By using this isomorphism, for any fixed $\mathbf{a} \in \mathcal{A}_{r}^{(n)}$, we define the set of constant sections
$\operatorname{Isomd}\left(\tilde{T},\left(\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times{ }_{T^{\prime}} \tilde{T}\right)^{\sharp}\right)=\left\{\sigma: \tilde{T} \rightarrow\left(\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times{ }_{T^{\prime}} \tilde{T}\right)^{\sharp}\right.$, constant $\}$.
Note that by using the isomorphism (79), we have a natural isomorphism

$$
\begin{equation*}
\operatorname{Isomd}\left(\tilde{T},\left(\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times{ }_{T^{\prime}} \tilde{T}\right)^{\sharp}\right) \simeq\left(\mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r}\right)^{\sharp} \tag{81}
\end{equation*}
$$

A section $\sigma \in \operatorname{Isomd}\left(\tilde{T},\left(\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times T^{\prime} \tilde{T}\right)^{\sharp}\right)$ is called an isomonodromic section by trivial reason and its image $\sigma(\tilde{T})$ is called an isomonodromic flow.

Next, considering the pullback of $\varphi_{r, n, d}$ in (50) by $\tilde{T} \longrightarrow T$, we can obtain the family of moduli spaces of $\boldsymbol{\alpha}$-stable parabolic connections

$$
\begin{equation*}
\widetilde{\varphi_{r, n, d}}: \mathcal{M}_{(\mathcal{C}, \mathbf{t}) / \tilde{T}}^{\alpha} \longrightarrow \tilde{T} \times \Lambda_{r}^{(n)}(d) \tag{82}
\end{equation*}
$$

Fixing $\boldsymbol{\lambda} \in \Lambda$ such that $r h(\boldsymbol{\lambda})=\mathbf{a}$, we also obtain the restricted family over $\tilde{T} \times\{\boldsymbol{\lambda}\}$

$$
\begin{equation*}
\widetilde{\varphi_{r, n, d, \boldsymbol{\lambda}}}: \mathcal{M}_{((\mathcal{C}, \mathbf{t}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}} \longrightarrow \tilde{T} \times\{\boldsymbol{\lambda}\} \tag{83}
\end{equation*}
$$

Restricting the Riemann-Hilbert correspondence (68) to this space, we obtain the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\alpha}(r, n, d) & \xrightarrow{\mathbf{R H}_{\boldsymbol{\lambda}}} & \mathcal{R} \mathcal{P}_{n, T, \mathbf{a}}^{r} \times{ }_{T} \tilde{T}  \tag{84}\\
\widetilde{\varphi_{r, n, d, \boldsymbol{\lambda}}} \downarrow & \\
\tilde{T} \times\{\boldsymbol{\lambda}\} & \xrightarrow{I d \times r h} & \tilde{\phi_{n, \mathbf{a}}^{r}}
\end{array}
$$

Note that by Theorem 6.1 the morphism $\mathbf{R H}_{\boldsymbol{\lambda}}$ gives an analytic resolution of singularities. Set

$$
\begin{equation*}
\left.\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)\right)^{\sharp}=\mathbf{R H}_{\boldsymbol{\lambda}}^{-1}\left(\left(\mathcal{R} \mathcal{P}_{n, T, \mathbf{a}}^{r} \times_{T} \tilde{T}\right)^{\sharp}\right), \tag{85}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)\right)^{\operatorname{sing}}=\mathbf{R H}_{\boldsymbol{\lambda}}^{-1}\left(\left(\mathcal{R} \mathcal{P}_{n, T, \mathbf{a}}^{r} \times_{T} \tilde{T}\right)^{\operatorname{sing}}\right) \tag{86}
\end{equation*}
$$

(Cf. (72), (73)). Then we have an analytic isomorphism

$$
\left.\left(\mathbf{R H}_{\boldsymbol{\lambda}}\right)^{\sharp}:\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)\right)^{\sharp} \xrightarrow{\simeq}\left(\mathcal{R} \mathcal{P}_{n, T, \mathbf{a}}^{r} \times_{T} \tilde{T}\right)^{\sharp} .
$$

Now we define:
$\left.\operatorname{Isomd}\left(\tilde{T},\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)\right)^{\sharp}\right)=\mathbf{R H}_{\boldsymbol{\lambda}}^{-1}\left(\operatorname{Isomd}\left(\tilde{T},\left(\mathcal{R} \mathcal{P}_{n, T, \mathbf{a}}^{r} \times_{T} \tilde{T}\right)^{\sharp}\right)\right)$.
Each section $\sigma \in \operatorname{Isomd}\left(\tilde{T},\left(\mathcal{M}_{((\mathcal{C}, \tilde{t}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)^{\sharp}\right)$ is called an isomonodromic section on $\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)^{\sharp}$ and its image

$$
\sigma(\tilde{T}) \subset\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)^{\sharp}
$$

is called an isomonodromic flow. Note that since the Riemann-Hilbert correspondence $\left(\mathbf{R H}_{\boldsymbol{\lambda}}\right)^{\sharp}$ is a highly non-trivial analytic isomorphism, isomonodromic flows $\{\sigma(\tilde{T})\}$ are not constant any more and it is known that they define highly transcendental analytic functions.

From the morphism (83) restricted to $\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\alpha}(r, n, d)\right)^{\sharp}$, we obtain the natural sheaf homomorphism

$$
\Theta_{\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\alpha}(r, n, d)\right)^{\sharp}} \stackrel{\widetilde{\varphi_{r, n, d, \lambda}} \lambda^{*}}{ }{\widetilde{\varphi_{r, n, d, \lambda}}}^{*}\left(\Theta_{\tilde{T}}\right)_{\mid\left(\mathcal{M}_{(\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\alpha}(r, n, d)\right)^{\sharp} \longrightarrow 0 .} 0 .
$$

Then the set of all isomonodromic sections defines a sheaf homomorphism

$$
\begin{equation*}
\mathcal{V}_{\boldsymbol{\lambda}}:{\widetilde{\varphi_{r, n, d, \boldsymbol{\lambda}}}}^{*}\left(\Theta_{\tilde{T}}\right)_{\mid\left(\mathcal{M}_{((C, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\alpha}(r, n, d)\right)^{\sharp} \longrightarrow \Theta_{\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathrm{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\alpha}(r, n, d)\right)^{\sharp}} .} \tag{88}
\end{equation*}
$$

which gives a splitting of the homomorphism $\widehat{\varphi_{r, n, d, \boldsymbol{\lambda}}} *$. The splitting (88) is algebraic, because the condition of isomonodromic flows given by the vanishing of the curvature 2 -forms of the associated universal connections. Since the exceptional locus for $\mathbf{R H}=\cup_{\boldsymbol{\lambda}} \mathbf{R} \mathbf{H}_{\boldsymbol{\lambda}}$ has codimension at least 2 , by Hartogs' theorem, it is easy to see that this algebraic splitting (88) can be extend to the whole family of moduli spaces, and we obtain an extended homomorphism

$$
\begin{equation*}
\mathcal{V}_{\boldsymbol{\lambda}}: \widetilde{\varphi_{r, n, d, \boldsymbol{\lambda}}} *\left(\Theta_{\tilde{T}}\right) \longrightarrow \Theta_{\mathcal{M}_{((\mathcal{C}, \tilde{\mathfrak{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\alpha}(r, n, d)} \tag{89}
\end{equation*}
$$

Under the notation above, we have the following
Definition 7.1. (1) The foliation $\mathcal{I F}_{\boldsymbol{\lambda}}$ defined by the subsheaf

$$
\begin{equation*}
\mathcal{I} \mathcal{F}_{\boldsymbol{\lambda}}=\mathcal{V}_{\boldsymbol{\lambda}}\left({\widetilde{\varphi_{r, n, d, \boldsymbol{\lambda}}}}^{*}\left(\Theta_{\tilde{T}}\right)\right) \subset \Theta_{\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \bar{T}}^{\alpha}(r, n, d)} \tag{90}
\end{equation*}
$$

is called an isomonodromic foliation on $\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)$.
(2) Let $h: \Delta \longrightarrow \tilde{T}$ be a holomorphic embedding such that $h(t)=\left(C_{t}, \mathbf{t}_{t}\right)$ for $t \in \Delta$. A holomorphic map $\tilde{h}: \Delta \longrightarrow$ $\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)$ such that $\widetilde{\varphi_{r, n, d, \boldsymbol{\lambda}}} \circ \tilde{h}=h$ is called a $\mathcal{I F}_{\boldsymbol{\lambda}^{-}}$ lift of $h$ if $\tilde{h}$ is tangent to $\mathcal{I F}_{\boldsymbol{\lambda}}$, that is, $\tilde{h}_{*}\left(\Theta_{\Delta}\right) \subset \mathcal{I} \mathcal{F}_{\boldsymbol{\lambda}}$.
Lemma 7.1. Let $h: \Delta \longrightarrow \tilde{T}$ be a holomorphic embedding and $\tilde{h}$ : $\Delta \longrightarrow \mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)$ a $\mathcal{I F}_{\boldsymbol{\lambda}}$-lift of $h$. Then the image of $\mathbf{R H}_{\boldsymbol{\lambda}} \circ \tilde{h}$ lies in the image of a constant section $\sigma \in \operatorname{Isomd}\left(\tilde{T},\left(\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times{ }_{T^{\prime}} \tilde{T}\right)\right)$.

Proof. Note that a lift $\tilde{h}$ of $h$ corresponds to a 1-parameter deformation of $\boldsymbol{\alpha}$-stable parabolic connection under a deformation of $n$ pointed curves associated to $h: \Delta \longrightarrow \tilde{T}$. Since $\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)^{\sharp}$ is a Zariski dense open subset of $\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)$, we see that the curvature form vanishes on the $\mathcal{I F}$-foliation defined on the total space $\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)$. Therefore if $\tilde{h}$ is a $\mathcal{I} \mathcal{F}$-lift of $h$, we can conclude that the deformation of connections is isomonodromic. Hence the associated representations of the fundamental group of $\mathcal{C}_{t} \backslash\left\{\mathbf{t}_{t}\right\}$ are constant, which means that $\mathbf{R H}_{\boldsymbol{\lambda}}(\tilde{h}(\Delta))$ is contained in the image of a constant section of $\left(\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}}^{r} \times{ }_{T^{\prime}} \tilde{T}\right) \longrightarrow \tilde{T}$.
Q.E.D.

Now, we can show that the isomonodromic foliation is a differential system satisfying the Painlevé property (cf. [Mal], [Miwa] and [IIS3]).

Theorem 7.1. For any $\boldsymbol{\lambda} \in \Lambda_{r}^{(n)}(d)$, the isomonodromic foliation $\mathcal{I F}_{\boldsymbol{\lambda}}$ defined on $\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)$ has Painlevé property. That is, for any holomorphic embedding $h: \Delta \longrightarrow \tilde{T}$ of the unit disc $\Delta=$ $\left\{t \in \mathbf{C}||t|<1\}\right.$ such that $h(0)=(C, \mathbf{t})$ and $\mathbf{x}=\left(E, \nabla,\left\{l_{*}^{(i)}\right\}_{1 \leq i \leq n}\right) \in$ $\mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\boldsymbol{\alpha}}(r, n, d)$, there exists the unique $\mathcal{I F}_{\boldsymbol{\lambda}}$-lift

$$
\tilde{h}: \Delta \longrightarrow \mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)
$$

of $h$ such that $\tilde{h}(0)=\mathbf{x}$.
Proof. If $\mathbf{x} \in\left(\mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\alpha}(r, n, d)\right)^{\sharp}$, there is a unique isomonodromic section $\sigma: \tilde{T} \longrightarrow\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)^{\sharp}$ such that $\sigma((C, \mathbf{t}))=\mathbf{x}$. The holomorphic map $\tilde{h}=\sigma \circ h: \Delta \longrightarrow\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)\right)^{\sharp}$ is the unique $\mathcal{I} \mathcal{F}_{\boldsymbol{\lambda}}$-lift of $h$.

Let us consider the case when $\mathbf{x} \in\left(\mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\alpha}(r, n, d)\right)^{\text {sing }}$. Pulling back the commutative diagrams (84) and (78) via the embedding $h$ : $\Delta \longrightarrow \tilde{T}$, we obtain the commutative diagram


The restriction of the foliation $\mathcal{I} \mathcal{F}_{\boldsymbol{\lambda}}$ to $\mathcal{M}_{((\mathcal{C}, \tilde{\mathfrak{t}}), \boldsymbol{\lambda}) / \Delta}^{\boldsymbol{\alpha}}(r, n, d)$ determines a vector field $v_{\boldsymbol{\lambda}}$ on $\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \Delta}^{\boldsymbol{\alpha}}(r, n, d)$ such that $\widetilde{\varphi_{\Delta}}\left(v_{\boldsymbol{\lambda}}\right)=\frac{\partial}{\partial t}$ where $t$ is a coordinate of $\Delta$. We will show that there exist a unique section $\tilde{h}: \Delta \longrightarrow \mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \Delta}^{\boldsymbol{\alpha}}(r, n, d)$ such that $\tilde{h}(0)=\mathbf{x}$ and $\tilde{h}_{*}\left(\frac{\partial}{\partial t}\right)=v_{\boldsymbol{\lambda}}$, which gives a $\mathcal{I F}_{\boldsymbol{\lambda}}$-lift of $h$. Such a section $\tilde{h}$ can be locally given by an analytic solution of the Cauchy problem of an ordinary differential equation associated to the vector field $v_{\boldsymbol{\lambda}}$. Such an analytic solution can be locally given by holomorphic functions of $t$ on $\Delta_{\epsilon}=\{t \in \mathbf{C}| | t \mid<\epsilon\}$ for some $0<\epsilon<1$. This gives a section $\tilde{h}_{\epsilon}: \Delta_{\epsilon} \longrightarrow \mathcal{M}_{((\mathcal{C}, \tilde{\mathfrak{t}}), \boldsymbol{\lambda}) / \Delta_{\epsilon}}^{\alpha}(r, n, d)$ which is a $\mathcal{I} \mathcal{F}_{\boldsymbol{\lambda}}$-lift of $h_{\epsilon}=h_{\mid \Delta_{\epsilon}}$. Let $\epsilon_{1}$ be the supremum of $\epsilon$ such that a $\mathcal{I} \mathcal{F}_{\lambda}$ lift of $h_{\epsilon}$ exists. The above argument shows that $\epsilon_{1}>0$. Now $\tilde{\sim}_{\epsilon_{1}}$ we will show that $\epsilon_{1}=1$. Assume the contrary, that is, $\epsilon_{1}<1$, and let $\tilde{h}_{\epsilon_{1}}: \Delta_{\epsilon_{1}} \longrightarrow \mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \Delta_{\epsilon_{1}}}^{\alpha}(r, n, d)$ be the section over $\Delta_{\epsilon_{1}}$.

Let $p_{1}: \mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r} \times \Delta \longrightarrow \mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r}$ be the first projection and consider the morphism

$$
p_{1} \circ \pi_{a} \circ \mathbf{R H}_{\boldsymbol{\lambda}}: \mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \Delta}^{\boldsymbol{\alpha}}(r, n, d) \longrightarrow \mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r}
$$

By definition of $\left(\mathcal{M}_{((C, \mathbf{t}), \boldsymbol{\lambda})}^{\boldsymbol{\alpha}}(r, n, d)\right)^{\text {sing }}$, the point $\mathbf{y}=p_{1} \circ \pi_{a} \circ \mathbf{R} \mathbf{H}_{\boldsymbol{\lambda}}(\mathbf{x})$ is a singular point of $\mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r}$ and let

$$
\mathcal{K}_{\Delta, \mathbf{y}}=\left(\pi_{a} \circ \mathbf{R H}_{\boldsymbol{\lambda}}\right)^{-1}(\{\mathbf{y}\} \times \Delta) \subset\left(\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \Delta}^{\boldsymbol{\alpha}}(r, n, d)\right)^{\operatorname{sing}}
$$

denote the exceptional locus dominated over $\{\mathbf{y}\} \times \Delta$. Then restricting (91) to $\mathcal{K}_{\Delta, \mathbf{y}}$, we have the following commutative diagram:


From Theorem 6.1, we see that $\pi_{\mathbf{a}} \circ \mathbf{R} \mathbf{H}_{\boldsymbol{\lambda}}$ is a resolution of singularity of $\mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right), \mathbf{a}}^{r} \times \Delta$, hence each fiber of $\widetilde{\varphi_{\Delta, \mathbf{y}}}: \mathcal{K}_{\Delta, \mathbf{y}} \longrightarrow \Delta$ is compact. Now from Lemma 7.1 , we see that $\tilde{h}_{\epsilon_{1}}\left(\Delta_{\epsilon}\right) \subset \mathcal{K}_{\Delta_{\epsilon_{1}}, \mathbf{y}}$. Moreover since $\widetilde{\varphi_{\Delta, \mathbf{y}}}$ is proper, we see that $\tilde{h}_{\epsilon_{1}}\left(\overline{\Delta_{\epsilon_{1}}}\right) \subset \mathcal{K}_{\overline{\Delta_{\epsilon_{1}}}, \text {, }}$ where $\overline{\Delta_{\epsilon_{1}}}=\left\{t,|t| \leq \epsilon_{1}\right\}$. Take and fix $t=b$ such that $|b|=\epsilon_{1}$. Then

$$
\tilde{h}_{\epsilon_{1}}(b)=\mathbf{y}_{b} \in \mathcal{K}_{\overline{\Delta_{\epsilon_{1}}}, \mathbf{y}} \subset \mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \overline{\Delta_{\epsilon_{1}}}}^{\boldsymbol{\alpha}}(r, n, d)
$$

Starting from $t=b$ and $\mathbf{y}_{b}$, we can extend the section $\widetilde{h}_{\epsilon_{1}}$ over $\Delta\left(b, \epsilon_{b}\right)=$ $\left\{t \in \Delta\left||t-b|<\epsilon_{b}\right\}\right.$ with $0<\epsilon_{b} \leq 1-\epsilon_{1}$. Again, from the compactness
of the fiber of $\widetilde{\varphi_{\Delta, \mathbf{y}}}: \mathcal{K}_{\Delta, \mathbf{y}} \longrightarrow \Delta$, we can show that the minimum $\epsilon_{0}$ of $\epsilon_{b}$ for $|b|=\epsilon_{1}$ is positive, hence for $\epsilon=\epsilon_{1}+\epsilon_{0}$ the section $\tilde{h}_{\epsilon}$ exists and this contradicts to the fact that $\epsilon_{1}$ is the supremum and $\epsilon_{1}<\epsilon$. Q.E.D.

Remark 7.1. Let us remark that the isomonodromic foliation $\mathcal{I F}_{\boldsymbol{\lambda}}$ on $\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / \tilde{T}}^{\boldsymbol{\alpha}}(r, n, d)$ descends to a foliation on $\mathcal{M}_{((\mathcal{C}, \tilde{\mathbf{t}}), \boldsymbol{\lambda}) / T^{\prime}}^{\boldsymbol{\alpha}}(r, n, d)$ under the covering map $\tilde{T} \longrightarrow T^{\prime}$, which we also denote by $\mathcal{I} \mathcal{F}_{\boldsymbol{\lambda}}$. Recall that the isomonodromic section (81) is the constant section with respect to the isomorphism (76). Moreover, when the base point $* \in T^{\prime}$ corresponds to $\left(C_{0}, \mathbf{t}_{0}\right)$, the fundamental group $\pi_{1}\left(T^{\prime}, *\right)$ acts on the moduli space $\mathcal{R} \mathcal{P}_{\left(C_{0}, \mathbf{t}_{0}\right)}^{r}$ via the action to the generators of $\pi_{1}\left(C_{0} \backslash\right.$ $\left.D\left(\mathbf{t}_{0}\right), *^{\prime}\right)$. Therefore, we can define the local isomonodromic sections for $\mathcal{R} \mathcal{P}_{n, T^{\prime}, \mathbf{a}^{\prime}}^{r} \longrightarrow T^{\prime}$, which also defines a local isomonodromic sections for $\left(\mathcal{M}_{((\mathcal{C}, \mathbf{t}), \boldsymbol{\lambda}) / T^{\prime}}^{\boldsymbol{\alpha}}\right)^{\sharp} \longrightarrow T^{\prime}$. Now the set of local isomonodromic sections determines a splitting homomorphism $\mathcal{V}_{\boldsymbol{\lambda}}$ like (89), and it defines an isomonodromic foliation

$$
\mathcal{I} \mathcal{F}_{\boldsymbol{\lambda}}=\mathcal{V}_{\boldsymbol{\lambda}}\left(\Theta_{T^{\prime}}\right) \subset \Theta_{\mathcal{M}_{((\mathcal{C}, \mathbf{t}), \boldsymbol{\lambda}) / T^{\prime}}^{\alpha}}
$$

which is obviously the descent of the original isomonodromic foliation on $\mathcal{M}_{((\mathcal{C}, \mathbf{t}), \boldsymbol{\lambda}) / \tilde{T}^{\prime}}^{\boldsymbol{\alpha}}$

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[^1]:    ${ }^{1}$ In their preprint [6], Inaba, Iwasaki and Saito have independently given a moduli construction for parabolic connections when $X$ a curve, without any such restriction.

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[^4]:    ${ }^{1}$ The reason for the adjective "rigid" will be explained later (cf. §2.4).
    ${ }^{2}$ The non-trivial fact that $\mathbb{C}_{p}$ is algebraically closed is due to Krasner.

[^5]:    ${ }^{3}$ For generalities of valuations, we refer to [7, Chap. VI] and [46, Chap. VI].

[^6]:    ${ }^{4}$ In some literature, affinoid algebras are called Tate algebras. Here we follow the terminology of [5], where Tate algebra means affinoid algebra of a special kind; cf. Definition 2.3.
    ${ }^{5}$ That one has to use a Grothendieck topology is a fatal drawback of Tate's theory, which makes the theory look extremely difficult. It is one of our aims

[^7]:    ${ }^{7}$ Note that this set is an open subset of $K^{n}$ with respect to the metric topology.

[^8]:    ${ }^{8}$ Any $K$-algebra homomorphism between affinoid algebras is automatically continuous.

[^9]:    ${ }^{9}$ This is the reason for the name "rigid" geometry.

[^10]:    ${ }^{10}$ See [5, 9.1] for what "Grothendieck topology" means here.

[^11]:    ${ }^{11}$ Indeed, while the $a$-adic completion of $A\langle\langle X\rangle\rangle /(f X-1)$ obviously coincides with $A_{\{f\}}$, as we have mentioned in Remark $3.2, A\langle\langle X\rangle\rangle /(f X-1)$ is already complete, whence the equality.

[^12]:    ${ }^{12}$ Here we followed the commonly used notation as in $\left[E G A, \mathrm{I}_{\text {new }}\right.$, (10.10)].

[^13]:    ${ }^{13}$ Coherent schemes are the analogue of compact Hausdorff topological spaces in the category of schemes.

[^14]:    ${ }^{14}$ The scheme $S^{\prime \prime \prime}$ might be called the join of $S^{\prime}$ and $S^{\prime \prime}$.

[^15]:    ${ }^{15}$ See, for example, [46, Chap. VI] and [7, Chap. VI] for basics of valuation rings.

[^16]:    ${ }^{16}$ The adjective "long" indicates that it might be of large height.

[^17]:    ${ }^{17}$ It might be more precise to say deleted tubular neighborhood.

[^18]:    ${ }^{18} \mathrm{~A}$ sober topological space is said to be coherent if it is quasi-compact, quasi-separated (i.e., the intersection of finitely many quasi-compact open subsets is quasi-compact), and has an open basis consisting of quasi-compact open subsets. Notice that this condition is equivalent to that the associated topos is coherent in the sense of [SGA4-2, Exposé VI].

[^19]:    ${ }^{19}$ There is no reason why we should deal only with formal schemes, and a perhaps more reasonable formulation would be given by allowing formal spaces ( $=$ formal algebraic spaces) to enter in. See Theorem 6.12.

[^20]:    ${ }^{20}$ Let us list two reasons why it is necessary to work in the derived categorical language: (1) it is user-friendly for applications; (2) recently, the importance of derived categories has been more and more recognized in algebraic geometry and in mathematical physics. The last point is related to the cohomological mirror symmetries speculated on by Kontsevich-Soibelman and Fukaya et al., in which our theorems in terms of the derived categorical language, as well as our approach involving higher-height valuation rings, could be important.

[^21]:    ${ }^{21}$ Perhaps the reader might complain that there is too much use of "coherent." Do not mix up the coherence of sheaves and the coherence of spaces.

[^22]:    ${ }^{22}$ An essentially equivalent statement was proved by Huber [23]; a similar but different approach was taken by van der Put-Schneider [42].

[^23]:    ${ }^{23}$ It can be shown that any point of $\langle\mathscr{X}\rangle$ has a unique maximal generalization.

[^24]:    ${ }^{24} \mathrm{~A}$ rigid space $\mathscr{X}$ is said to be quasi-compact if the topological space $\langle\mathscr{X}\rangle$ is quasi-compact; this is equivalent to the small admissible topos $\mathscr{X}$ ad being quasi-compact.

[^25]:    ${ }^{25}$ The identification by $\delta_{n}$ can be different from the one which is already chosen

[^26]:    ${ }^{26}$ This method works over $\mathbb{Z}\left[\frac{1}{2}\right]$. One must exclude prime 2 since it is a bad prime for the theory of algebraic theta functions.

[^27]:    ${ }^{27}$ In fact, here, one need more general version of the trace formula, which has been proven [19].

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