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# Cobordism of fibered knots and related topics

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#### Abstract.

This is a survey article on the cobordism theory of non-spherical knots studied in [BM, B2, BS1, BMS, BS2, BS3]. Special emphasis is put on fibered knots.

We first recall the classical results concerning cobordisms of spherical knots. Then we give recent results on cobordisms of simple fibered (2n - 1)-knots for  $n \ge 2$  together with relevant examples. We discuss the Fox-Milnor type relation and show that the usual spherical knot cobordism group modulo the subgroup generated by the cobordism classes of fibered knots is infinitely generated for odd dimensions. The pull back relation on the set of knots is also discussed, which is closely related to the cobordism theory of knots via the codimension two surgery theory. We also present recent results on cobordisms of surface knots in  $S^4$  and 4-dimensional knots in  $S^6$ . Finally we give some open problems related to the subject.

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## §1. Introduction

#### 1.1. History

In the early fifties Rohlin [Rh1] and Thom [Th] studied the cobordism groups of manifolds. At the 1958 International Congress of Mathematicians in Edinburgh, René Thom received a Fields Medal for his development of cobordism theory. Then, Fox and Milnor [FM1, FM2] were the first to study cobordism of knots, i.e., cobordism of embeddings of the circle  $S^1$  into the 3-sphere  $S^3$ . Knot cobordism is slightly different from the general cobordism, since its definition is more restrictive. After Fox and Milnor, Kervaire [K1] and Levine [L2] studied embeddings of the *n*-sphere  $S^n$  (or homotopy *n*-spheres) into the (n+2)-sphere  $S^{n+2}$ , and gave classifications of such embeddings up to cobordism for  $n \ge 2$ . Moreover, Kervaire defined group structures on the set of concordance classes of *n*-spheres embedded in  $S^{n+2}$ . The structures of these groups for  $n \ge 2$  were clarified by Kervaire [K1], Levine [L2, L3] and Stoltzfus [Sf].

Note that embeddings of spheres were studied only in the codimension two case, since in the PL category Zeeman [Ze] proved that all such embeddings in codimension greater than or equal to three are unknotted, and Stallings [Sg] proved that it is also true in the topological category (here, one needs to assume the locally flatness condition), provided that the ambient sphere has dimension greater than or equal to five. In the smooth category Haefliger [Ha] proved that a cobordism of spherical knots in codimension greater than or equal to three implies isotopy.

Milnor [M3] showed that, in a neighborhood of an isolated singular point, a complex hypersurface is homeomorphic to the cone over the *algebraic knot* associated with the singularity. Hence, the embedded topology of a complex hypersurface around an isolated singular point is given by the algebraic knot, which is a special case of a *fibered knot*. After Milnor's work, the class of fibered knots has been recognized as an

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important class of knots to study. Usually algebraic knots are not homeomorphic to spheres, and this motivated the study of embeddings of general manifolds (not necessarily homeomorphic to spheres) into spheres in codimension two. Moreover, in the beginning of the seventies, Lê [Lê] proved that isotopy and cobordism are equivalent for 1-dimensional algebraic knots. Lê proved this for the case of *connected* (or spherical) algebraic 1-knots, and the generalization to arbitrary algebraic 1-knots follows easily (for details, see §4). About twenty years later, Du Bois and Michel [DM] gave the first examples of algebraic spherical knots that are cobordant but are not isotopic. These examples motivated the classification of fibered knots up to cobordism.

#### 1.2. Contents

This article is organized as follows. In  $\S2$  we give several definitions related to the cobordism theory of knots. Seifert forms associated with knots are also introduced. In  $\S3$  we review the classifications of (simple) spherical (2n-1)-knots with  $n \ge 2$  up to isotopy and up to cobordism. In  $\S4$  we review the properties of algebraic 1-knots and present the classification theorem of algebraic 1-knots up to cobordism due to Lê [Lê]. In §5 we present the classifications of simple fibered (2n-1)-knots with n > 3 up to isotopy and up to cobordism. The classification up to cobordism is based on the notion of the *algebraic cobordism*. In order to clarify the definition of algebraic cobordism, we give several explicit examples. We also explain why this relation might not be an equivalence relation on the set of bilinear forms defined on free Z-modules of finite rank. A classification of 3-dimensional simple fibered knots up to cobordism is given in  $\S6$ . In  $\S7$  we recall the Fox-Milnor type relation on the Alexander polynomials of cobordant knots. As an application, we show that the usual spherical knot cobordism group modulo the subgroup generated by the cobordism classes of fibered knots is infinitely generated for odd dimensions. In  $\S8$  we present several examples of knots with interesting properties in view of the cobordism theory of knots. In §9 we define the pull back relation for knots which naturally arises from the viewpoint of the codimension two surgery theory. We illustrate several results on pull back relations for fibered knots using some explicit examples. Some results for even dimensional knots are given in  $\S10$ , where we explain recent results about embedded surfaces in  $S^4$  and embedded 4-manifolds in  $S^6$ . Finally in §11, we give several open problems related to the cobordism theory of non-spherical knots.<sup>1</sup>

<sup>&</sup>lt;sup>1</sup>A "non-spherical manifold" in this article refers to a general manifold which may not necessarily be a homotopy sphere.

With all the results collected in this paper, we have classifications of knots up to cobordism in every dimension, except for the classical case of one dimensional knots and the case of three dimensional knots. In the latter two cases, a complete classification still remains open until now.

Throughout the article, we shall work in the smooth category unless otherwise specified. All the homology and cohomology groups are understood to be with integer coefficients. The symbol " $\cong$ " denotes an (orientation preserving) diffeomorphism between (oriented) manifolds, or an appropriated isomorphism between algebraic objects.

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#### $\S 2.$ Several definitions

Since our aim is to study cobordisms of codimension two embeddings of general manifolds, not necessarily homeomorphic to spheres, we define the following.

**Definition 2.1.** Let K be a closed n-dimensional manifold embedded in the (n + 2)-dimensional sphere  $S^{n+2}$ . We suppose that K is  $(\lfloor n/2 \rfloor - 1)$ -connected, where for  $a \in \mathbf{R}$ ,  $\lfloor a \rfloor$  denotes the greatest integer not exceeding a. (We adopt the convention that a space is (-1)-connected if it is not empty.) Equivalently, we suppose that K is

(k-2)-connected if n = 2k - 1 and  $k \ge 2$ , or

(k-1)-connected if n = 2k and  $k \ge 1$ .

When K is orientable, we further assume that it is oriented.<sup>2</sup> Then we call K or its (oriented) isotopy class an *n*-knot, or simply a knot.

An *n*-knot K is *spherical* if K is

- (1) diffeomorphic to the standard *n*-sphere  $S^n$  for  $n \leq 4$ , or
- (2) a homotopy *n*-sphere for  $n \ge 5$ .

Remark 2.2. We adopt the above definition of a spherical knot for  $n \leq 4$  in order to avoid the difficulty related to the smooth Poincaré conjecture in dimensions three and four.

Note that we impose the connectivity condition on the embedded submanifold in Definition 2.1. This is motivated by the following reasons. First, a knot associated with an isolated singularity of a complex

<sup>&</sup>lt;sup>2</sup>In this article, we always assume that *n*-knots are oriented if  $n \neq 2$ .

hypersurface satisfies the above connectivity condition as explained below. Second, if we assume that K is [n/2]-connected, then K is necessarily a homotopy sphere so that K is spherical at least for  $n \neq 3, 4$ . Third, the connectivity condition on K technically helps to perform certain embedded surgeries and this simplifies the arguments in various situations.

Remark 2.3. For the case of n = 1, i.e., for the classical knot case, a 1-knot in our sense is usually called a "link", and a connected (or spherical) 1-knot is usually called a "knot".

As mentioned in §1, Definition 2.1 is motivated by the study of the topology of isolated singularities of complex hypersurfaces. More precisely, let  $f: \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$  be a holomorphic function germ with an isolated singularity at the origin. If  $\varepsilon > 0$  is sufficiently small, then  $K_f = f^{-1}(0) \cap S_{\varepsilon}^{2n+1}$  is a (2n-1)-dimensional manifold which is naturally oriented, where  $S_{\varepsilon}^{2n+1}$  is the sphere in  $\mathbb{C}^{n+1}$  of radius  $\varepsilon$  centered at the origin. Furthermore, its (oriented) isotopy class in  $S_{\varepsilon}^{2n+1} = S^{2n+1}$  does not depend on the choice of  $\varepsilon$  (see [M3]). We call  $K_f$  the algebraic knot associated with f. Since the pair  $(D_{\varepsilon}^{2n+2}, f^{-1}(0) \cap D_{\varepsilon}^{2n+2})$  is homeomorphic to the cone over the pair  $(S_{\varepsilon}^{2n+1}, K_f)$ , the algebraic knot completely determines the local embedded topological type of  $f^{-1}(0)$  near the origin, where  $D_{\varepsilon}^{2n+2}$  is the disk in  $\mathbb{C}^{n+1}$  of radius  $\varepsilon$  centered at the origin.

In [M3], Milnor proved that algebraic knots associated with isolated singularities of holomorphic function germs  $f: \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$ are (2n-1)-dimensional closed, oriented and (n-2)-connected submanifolds of the sphere  $S^{2n+1}$ . This means that algebraic knots are in fact knots in the sense of Definition 2.1. Moreover, the complement of an algebraic knot  $K_f$  in the sphere  $S^{2n+1}$  admits a fibration over the circle  $S^1$ , and the closure of each fiber is a compact 2n-dimensional oriented (n-1)-connected submanifold of  $S^{2n+1}$  which has  $K_f$  as boundary. This motivates the following definition.

**Definition 2.4.** We say that an oriented *n*-knot *K* is fibered if there exists a smooth fibration  $\phi: S^{n+2} \smallsetminus K \to S^1$  and a trivialization  $\tau: N(K) \to K \times D^2$  of a closed tubular neighborhood N(K) of *K* in  $S^{n+2}$  such that  $\phi|_{N(K) \smallsetminus K}$  coincides with  $\pi \circ \tau|_{N(K) \smallsetminus K}$ , where  $\pi: K \times (D^2 \smallsetminus \{0\}) \to S^1$  is the composition of the projection to the second factor and the obvious projection  $D^2 \searrow \{0\} \to S^1$ . Note that then the closure of each fiber of  $\phi$  in  $S^{n+2}$  is a compact (n + 1)-dimensional oriented manifold whose boundary coincides with *K*. We shall often call the closure of each fiber simply a fiber. Furthermore, we say that a fibered *n*-knot K is simple if each fiber of  $\phi$  is [(n-1)/2]-connected.

Note that an algebraic knot is always a simple fibered knot.

Let us now recall the classical definition of Seifert forms of odd dimensional oriented knots, which were first introduced in [Se] and play an important role in the study of knots.

First of all, for every oriented *n*-knot K with  $n \ge 1$ , there exists a compact oriented (n+1)-dimensional submanifold V of  $S^{n+2}$  having K as boundary. Such a manifold V is called a *Seifert manifold* associated with K.

For the construction of Seifert manifolds (or Seifert surfaces) associated with 1-knots, see [R1], for example.

For general dimensions, the existence of a Seifert manifold associated with a knot K can be proved by using the obstruction theory as follows. It is known that the normal bundle of a closed orientable manifold embedded in a sphere in codimension two is always trivial (see [MS, Corollary 11.4], for example). Let  $N(K) \stackrel{\tau}{\cong} K \times D^2$  be a closed tubular neighborhood of K in  $S^{n+2}$ , and  $\Phi: \partial N(K) \stackrel{\cong}{\to} K \times S^1 \stackrel{pr_2}{\to} S^1$ the composite of the restriction of  $\tau$  to the boundary of N(K) and the projection  $pr_2$  to the second factor. Using the exact sequence

$$H^{1}(S^{n+2} \smallsetminus \operatorname{Int} N(K)) \to H^{1}(\partial N(K))$$
$$\to H^{2}(S^{n+2} \smallsetminus \operatorname{Int} N(K), \, \partial N(K)),$$

associated with the pair  $(S^{n+2} \setminus \operatorname{Int} N(K), \partial N(K))$ , we see that the obstruction to extending  $\Phi$  to  $\widetilde{\Phi} \colon S^{n+2} \setminus \operatorname{Int} N(K) \to S^1$  lies in the cohomology group

$$H^2(S^{n+2} \setminus \operatorname{Int} N(K), \partial N(K)) \cong H_n(S^{n+2} \setminus \operatorname{Int} N(K)).$$

By Alexander duality we have

$$H_n(S^{n+2} \setminus \operatorname{Int} N(K)) \cong H^1(K),$$

which vanishes if  $n \ge 4$ , since K is simply connected for  $n \ge 4$ . When  $n \le 3$ , we can show that by choosing the trivialization  $\tau$  appropriately, the obstruction in question vanishes. Therefore, a desired extension  $\widetilde{\Phi}$  always exists. Now, for a regular value y of  $\widetilde{\Phi}$ , the manifold  $\widetilde{\Phi}^{-1}(y)$  is a submanifold of  $S^{n+2}$  with boundary being identified with  $K \times \{y\}$  in  $K \times S^1$ . The desired Seifert manifold associated with K is obtained by gluing a small collar  $K \times [0, 1]$  to  $\widetilde{\Phi}^{-1}(y)$ .

When K is a fibered knot, the closure of a fiber is always a Seifert manifold associated with K.

**Definition 2.5.** We say that an *n*-knot is *simple* if it admits an [(n-1)/2]-connected Seifert manifold.

Now let us recall the definition of Seifert forms for odd dimensional knots.

**Definition 2.6.** Suppose that V is a compact oriented 2n-dimensional submanifold of  $S^{2n+1}$ , and let G be the quotient of  $H_n(V)$  by its **Z**-torsion. The *Seifert form* associated with V is the bilinear form  $A: G \times G \to \mathbf{Z}$  defined as follows. For  $(x, y) \in G \times G$ , we define A(x, y) to be the linking number in  $S^{2n+1}$  of  $\xi_+$  and  $\eta$ , where  $\xi$  and  $\eta$  are n-cycles in V representing x and y respectively, and  $\xi_+$  is the n-cycle  $\xi$  pushed off V into the positive normal direction to V in  $S^{2n+1}$ .

By definition a Seifert form associated with an oriented (2n-1)-knot K is the Seifert form associated with F, where F is a Seifert manifold associated with K. A matrix representative of a Seifert form with respect to a basis of G is called a Seifert matrix.

Remark 2.7. Some authors define A(x, y) to be the linking number of  $\xi$  and  $\eta_+$  instead of  $\xi_+$  and  $\eta$ , where  $\eta_+$  is the *n*-cycle  $\eta$  pushed off Vinto the positive normal direction to V in  $S^{2n+1}$ . There is no essential difference between such a definition and ours. However, one should be careful, since some formulas may take different forms.

*Remark 2.8.* For codimension two embeddings between general manifolds, similar invariants have been constructed by Cappell-Shaneson [CS1] and Matsumoto [Mt1, Mt2] (see also [St]). These invariants arose as obstructions for certain codimension two surgeries.

Let us illustrate the above definition in the case of the trefoil knot. Let us consider the Seifert manifold V associated with this knot as depicted in Fig. 1, where "+" indicates the positive normal direction. Note that rank  $H_1(V) = 2$ . We denote by  $\xi$  and  $\eta$  the 1-cycles which represent the generators of  $H_1(V)$ . Then, with the aid of Fig. 1, we see that the Seifert matrix for the trefoil knot is given by

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Note that a Seifert matrix is not symmetric in general. When A is a Seifert matrix associated with a Seifert manifold  $V \subset S^{2n+1}$  of a (2n-1)-knot  $K = \partial V$ , the matrix  $S = A + (-1)^n A^T$  is the matrix of the intersection form for V with respect to the same basis, where  $A^T$  denotes the transpose of A (for example, see [D]).

When a knot is fibered, its Seifert form associated with a fiber is always unimodular by virtue of Alexander duality (see [Kf]). In the



Fig. 1. Computing a Seifert matrix for the trefoil knot

following, for a fibered (2n-1)-knot, we use the Seifert form associated with a fiber unless otherwise specified.

Furthermore, when a (2n-1)-knot is simple, we consider an (n-1)connected Seifert manifold associated with this knot unless otherwise
specified.

Let us now focus on the cobordism classes of knots.

**Definition 2.9.** Two *n*-knots  $K_0$  and  $K_1$  in  $S^{n+2}$  are said to be *cobordant* if there exists a properly embedded (n+1)-dimensional manifold X of  $S^{n+2} \times [0, 1]$  such that

- (1) X is diffeomorphic to  $K_0 \times [0, 1]$ , and
- (2)  $\partial X = (K_0 \times \{0\}) \cup (K_1 \times \{1\})$

(see Fig. 2). The manifold X is called a *cobordism* between  $K_0$  and  $K_1$ . When the knots are oriented, we say that  $K_0$  and  $K_1$  are *oriented* cobordant (or simply cobordant) if there exists an oriented cobordism X between them such that  $\partial X = (-K_0 \times \{0\}) \cup (K_1 \times \{1\})$ , where  $-K_0$  is obtained from  $K_0$  by reversing the orientation.

In Fig. 2 the manifold  $X \cong K_0 \times [0, 1]$ , embedded in  $S^{n+2} \times [0, 1]$ , and its boundary  $(K_0 \times \{0\}) \cup (K_1 \times \{1\})$ , embedded in  $(S^{n+2} \times \{0\}) \cup (S^{n+2} \times \{1\})$ , are drawn by solid curves and black dots respectively, and the levels  $S^{n+2} \times \{t\}$ ,  $t \in (0, 1)$ , are drawn by dotted curves.

Recall that a manifold with boundary Y embedded in a manifold X with boundary is said to be *properly embedded* if  $\partial Y = \partial X \cap Y$  and Y is transverse to  $\partial X$ .

It is clear that isotopic knots are always cobordant. However, the converse is not true in general, since the manifold  $X \cong K_0 \times [0, 1]$  can be knotted in  $S^{n+2} \times [0, 1]$  as depicted in Fig. 3. For explicit examples, see §8.



Fig. 2. A cobordism between  $K_0$  and  $K_1$ 



Fig. 3. A cobordism which is not an isotopy

We also introduce the notion of *concordance* for embedding maps as follows.

**Definition 2.10.** Let K be a closed n-dimensional manifold. We say that two embeddings  $f_i: K \to S^{n+2}$ , i = 0, 1, are *concordant* if there exists a proper embedding  $\Phi: K \times [0, 1] \to S^{n+2} \times [0, 1]$  such that  $\Phi|_{K \times \{i\}} = f_i: K \times \{i\} \to S^{n+2} \times \{i\}, i = 0, 1.$ 

Note that an embedding map  $\varphi \colon Y \to X$  between manifolds with boundary is said to be *proper* if  $\partial Y = \varphi^{-1}(\partial X)$  and Y is transverse to  $\partial X$ .

Recall that for a simple (2n-1)-knot K with an (n-1)-connected Seifert manifold V, we have the following exact sequence

(2.1) 
$$0 \to H_n(K) \to H_n(V) \xrightarrow{S_*} H_n(V, K) \to H_{n-1}(K) \to 0$$

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where the homomorphism  $S_*$  is induced by the inclusion. Let

$$\widetilde{\mathfrak{P}} \colon H_n(V, K) \xrightarrow{\cong} \operatorname{Hom}_{\mathbf{Z}}(H_n(V), \mathbf{Z})$$

be the composite of the Poincaré-Lefschetz duality isomorphism and the universal coefficient isomorphism. Set  $S = A + (-1)^n A^T$  and let  $S^* \colon H_n(V) \to \operatorname{Hom}_{\mathbf{Z}}(H_n(V), \mathbf{Z})$  be the adjoint of S, where A is the Seifert form associated with V. Then we see easily that the homomorphisms  $S_*$  and  $S^*$  are related together by  $S^* = \widetilde{\mathfrak{P}} \circ S_*$ .

Cobordant knots are diffeomorphic. Hence, to have a cobordism between two given knots, we need to have topological informations about the knots as abstract manifolds. Since a simple fibered (2n - 1)-knot is the boundary of the closure of a fiber, which is an (n - 1)-connected Seifert manifold associated with the knot, by considering the above exact sequence (2.1) we can use the kernel and the cokernel of the homomorphism  $S^*$  to get topological data of the knot. Note that in the case of spherical knots, these considerations are not necessary, since  $S_*$  and  $S^*$ are isomorphisms.

#### $\S 3.$ Spherical knots

In this section, let us briefly review the case of spherical knots, which was studied mainly by Kervaire and Levine.

The Seifert form is the main tool to study cobordisms of odd dimensional spherical knots. In [L4] Levine described the possible modifications on Seifert forms of a spherical simple knot corresponding to alterations of Seifert manifolds as follows.

An enlargement A' of a square integral matrix A is defined as follows:

$$A' = \begin{pmatrix} A & \mathcal{O} & \mathcal{O} \\ \alpha & 0 & 0 \\ \mathcal{O}^T & 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} A & \beta & \mathcal{O} \\ \mathcal{O}^T & 0 & 1 \\ \mathcal{O}^T & 0 & 0 \end{pmatrix},$$

where  $\mathcal{O}$  is a column vector whose entries are all 0, and  $\alpha$  (resp.  $\beta$ ) is a row (resp. column) vector of integers. In this case, we also call A a *reduction* of A'.

Two square integral matrices are said to be S-equivalent if they are related each other by enlargement and reduction operations together with the congruence. We also say that two integral bilinear forms defined on free **Z**-modules of finite rank are S-equivalent if so are their matrix representatives.

Levine [L4] proved

**Theorem 3.1.** For  $n \ge 2$ , two spherical simple (2n-1)-knots are isotopic if and only if they have S-equivalent Seifert forms.

Remark 3.2. For spherical simple (2n-1)-knots, we have another algebraic invariant, called the *Blanchfield pairing*, which is closely related to the Seifert form (see [Ke1, T]). In fact, it is known that giving an *S*equivalence class of a Seifert form is equivalent to giving an isomorphism class of a Blanchfield pairing.

Kervaire showed that the set  $C_n$  of cobordism classes of spherical n-knots has a natural group structure. The group operation is given by the connected sum and the inverse of a knot K is given by its mirror image with reversed orientation  $-K^!$ . We say that an n-knot  $K \subset S^{n+2}$  is *null-cobordant* if it is cobordant to the trivial knot, i.e., if there exists an (n+1)-disk  $D^{n+1}$  properly embedded in the (n+3)-disk  $D^{n+3}$  such that  $\partial D^{n+1} = K \subset S^{n+2} = \partial D^{n+3}$ . Note that the neutral element of  $C_n$  is the class of null-cobordant n-knots.

In the case of spherical (2n-1)-knots Kervaire and Levine used the following notion for integral bilinear forms.

**Definition 3.3.** Let  $A: G \times G \to \mathbb{Z}$  be an integral bilinear form defined on a free Z-module G of finite rank. The form A is said to be *Witt associated to* 0 if the rank m of G is even and there exists a submodule M of rank m/2 in G such that M is a direct summand of G and A vanishes on M. Such a submodule M is called a *metabolizer* for A.

The following theorem was proved by Levine [L2] (see also [K2]).

**Theorem 3.4.** For  $n \ge 2$ , a spherical (2n-1)-knot is null-cobordant if and only if its Seifert form is Witt associated to 0.

*Remark 3.5.* For Blanchfield pairing (see Remark 3.2), there is also a notion of "null-cobordism", and we have a result similar to Theorem 3.4 (see [Ke2]).

For two spherical (2n-1)-knots  $K_0$  and  $K_1$  with Seifert forms  $A_0$  and  $A_1$  respectively, the oriented connected sum  $K = K_0 \sharp (-K_1^!)$  has  $A = A_0 \oplus (-A_1)$  as the Seifert form associated with the oriented connected sum along the boundaries of the Seifert manifolds associated with  $K_0$  and  $-K_1^!$ , where  $-K_1^!$  denotes the mirror image of  $K_1$  with reversed orientation. Hence, as a consequence of Theorem 3.4, we have that two spherical knots  $K_0$  and  $K_1$  are cobordant if and only if the form  $A = A_0 \oplus (-A_1)$  is Witt associated to 0. In this case we sometimes say that  $A_0$  and  $A_1$  are Witt equivalent.

For  $\varepsilon = \pm 1$ , let  $C^{\varepsilon}(\mathbf{Z})$  be the set of all Witt equivalence classes of integral bilinear forms A defined on free **Z**-modules of finite rank such that  $A + \varepsilon A^T$  is unimodular (for the notation, we follow [K2]). It can be shown that  $C^{\varepsilon}(\mathbf{Z})$  has a natural abelian group structure, where the addition is defined by the direct sum. Then we have the following.

**Theorem 3.6** (Levine [L2]). Let  $\Phi_n: C_{2n-1} \to C^{(-1)^n}(\mathbf{Z})$  be the (well-defined) homomorphism induced by the Seifert form. Then  $\Phi_n$  is an isomorphism for  $n \geq 3$ . For n = 2,  $\Phi_2$  is a monomorphism whose image  $C^{+1}(\mathbf{Z})^0$  is a specified subgroup of  $C^{+1}(\mathbf{Z})$  of index 2. For n = 1,  $\Phi_1: C_1 \to C^{-1}(\mathbf{Z})$  is merely an epimorphism.

Furthermore, Levine [L3] showed the following (see also Remark 7.4).

**Theorem 3.7.** For  $\varepsilon = \pm 1$ , we have

(3.1) 
$$C^{\varepsilon}(\mathbf{Z}) \cong \mathbf{Z}_{2}^{\infty} \oplus \mathbf{Z}_{4}^{\infty} \oplus \mathbf{Z}^{\infty},$$

where the right hand side is the direct sum of countably many (but infinite) copies of the cyclic groups  $\mathbf{Z}$ ,  $\mathbf{Z}_2$  and  $\mathbf{Z}_4$ .

Note that the right hand side of (3.1) is *not* an unrestricted direct sum, i.e., each element of the group is a linear combination of *finitely* many elements corresponding to the generators of the factors.

Remark 3.8. Michel [Mc] showed that for  $n \ge 1$ , spherical algebraic (2n - 1)-knots have infinite order in  $C_{2n-1}$ , provided that the associated holomorphic function germ has an isolated singularity at the origin and is not non-singular. Note, however, that they are not independent. See Remark 4.2.

For n = 1,  $\Phi_1: C_1 \to C^{-1}(\mathbf{Z})$  is far from being an isomorphism. The non-triviality of the kernel of this epimorphism was first shown by Casson-Gordon [CG]. The classification of spherical 1-knots up to cobordism is still an open problem. Moreover, for spherical 1-knots, we have also the important notion of a *ribbon knot* (see, for example, [R1]). Ribbon knots are null-cobordant. It is still an open problem whether the converse is true or not.

For even dimensions, we have the following vanishing theorem.

**Theorem 3.9** (Kervaire [K1]). For all  $n \ge 1$ ,  $C_{2n}$  vanishes.

Let  $\widetilde{C}_n$  be the group of concordance classes of embeddings into  $S^{n+2}$ 

- (1) the *n*-dimensional standard sphere  $S^n$  for  $n \leq 4$ , or
- (2) homotopy *n*-spheres for  $n \ge 5$ .

of

In [K1] Kervaire showed that the natural surjection  $i: \widetilde{C}_n \to C_n$  is a group homomorphism.

Let us denote by  $\Theta_n$  the group of *h*-cobordism classes of smooth oriented homotopy *n*-spheres, and by  $bP_{n+1}$  the subgroup of  $\Theta_n$  consisting of the *h*-cobordism classes represented by homotopy *n*-spheres which bound compact parallelizable manifolds [KM]. Then we have the following

**Theorem 3.10** (Kervaire [K1]). For  $n \leq 5$  we have  $\widetilde{C}_n \cong C_n$ , and for n > 6 we have the short exact sequence

$$0 \to \Theta_{n+1}/bP_{n+2} \to \widetilde{C}_n \xrightarrow{i} C_n \to 0.$$

Note that for  $n \ge 4$ ,  $\Theta_{n+1}/bP_{n+2}$  is a finite abelian group. For details, see [KM].

#### $\S4$ . Cobordism of algebraic 1-knots

As has been pointed out in the previous section, the classification of 1-knots up to cobordism is still unsolved. However, for algebraic 1-knots, a classification is known as follows.

Consider an algebraic 1-knot K associated with a holomorphic function germ  $f: \mathbb{C}^2, 0 \to \mathbb{C}, 0$  of two variables with an isolated critical point at the origin. Note that K is naturally oriented. Let us further assume that K is spherical. Then it is known that K is an iterated torus knot [Br]. An *iterated torus knot* is a knot obtained from a torus knot by an iteration of the cabling operation (for example, see [Rl]). Furthermore, the relevant operations are always "positive" cablings, which is peculiar to algebraic knots.

For a knot, the fundamental group of its complement in the ambient sphere is called the *knot group*. In [Z1] Zariski explicitly gave generators and relations of the knot group of a spherical algebraic 1-knot. When two spherical algebraic 1-knots are isotopic, they have isomorphic knot groups. Although the converse is not true for general spherical (not necessarily algebraic) 1-knots, it was proved that two spherical algebraic 1-knots with isomorphic knot groups are isotopic (see [Bu1, Z1, Re, Lê]). Furthermore, Burau [Bu1] proved that two spherical algebraic 1knots with the same Alexander polynomial are isotopic. For a definition of the Alexander polynomial, see §7. It is known that the Alexander polynomial of a spherical 1-knot is determined by its knot group (see, for example, [CF]).

For general algebraic 1-knots which are not necessarily spherical, the following is known. Let  $K = K_1 \cup K_2 \cup \cdots \cup K_s$  and  $L = L_1 \cup L_2 \cup \cdots \cup L_t$ 

be algebraic 1-knots, where  $K_i$ ,  $1 \leq i \leq s$ , and  $L_j$ ,  $1 \leq j \leq t$ , are components of K and L respectively. Then K and L are isotopic if and only if s = t,  $K_i$  is isotopic to  $L_i$ ,  $1 \leq i \leq s$ , and the linking number of  $K_i$  and  $K_j$  coincides with that of  $L_i$  and  $L_j$  for  $i \neq j$ , after renumbering the indices if necessary (for example, see [Re]). It is also known that the multi-variable Alexander polynomial classifies algebraic 1-knots [Bu2, Re, Y].

As to the classification of algebraic 1-knots up to cobordism, we have the following result due to Lê [Lê]. Let K and L be two cobordant spherical algebraic 1-knots. Let us denote their Alexander polynomials by  $\Delta_K(t)$  and  $\Delta_L(t)$  respectively, where we normalize them so that their degree 0 terms are positive. In [FM2], Fox and Milnor proved that then there exists a polynomial  $f(t) \in \mathbf{Z}[t]$  such that  $\Delta_K(t)\Delta_L(t) =$  $t^d f(t)f(1/t)$ , where d is the degree of f(t) (for details, see §7 of the present survey). Using this, one can conclude that the product of the Alexander polynomials of two cobordant spherical algebraic 1-knots is a square in  $\mathbf{Z}[t]$ . In fact, Lê [Lê] proved that two cobordant spherical algebraic 1-knots have the same Alexander polynomial, and hence the following holds.

**Theorem 4.1** ([L $\hat{e}$ ]). Two cobordant spherical algebraic 1-knots are isotopic.

For general (not necessarily spherical) algebraic 1-knots, since the linking numbers between the components are cobordism invariants, we see that the same conclusion as in Theorem 4.1 holds also for the general case of not necessarily spherical algebraic 1-knots.

Remark 4.2. It has been shown that the images of the cobordism classes of spherical algebraic 1-knots by  $\Phi_1: C_1 \to C^{-1}(\mathbf{Z})$  are not independent. An explicit example is given in [LM].

# §5. Cobordism of simple fibered (2n-1)-knots

In this section, we will give a classification of simple fibered (2n-1)-knots up to cobordism for  $n \geq 3$ .

Let us first recall that Durfee [D] and Kato [Kt] independently proved an analogue of Theorem 3.1 for (not necessarily spherical) simple fibered knots as follows. Recall that Seifert forms associated with simple fibered knots are unimodular.

**Theorem 5.1.** For  $n \geq 3$ , there is a one-to-one correspondence between the isotopy classes of simple fibered (2n-1)-knots in  $S^{2n+1}$  and the isomorphism classes of integral unimodular bilinear forms, where the correspondence is given by the Seifert form. Note that isomorphism classes of integral bilinear forms correspond to congruence classes of integral square matrices.

The study of cobordism of (not necessarily spherical) odd dimensional simple fibered knots cannot be done by a direct generalization of the results proved by Kervaire and Levine for spherical (2n - 1)-knots with  $n \ge 2$ , since we have to consider the topological data contained in the kernel and the cokernel of the intersection form of the fiber (see the exact sequence (2.1)).

For  $n \geq 3$ , Du Bois and Michel [DM] constructed the first examples of spherical algebraic (2n - 1)-knots which are cobordant but are not isotopic. Hence, algebraic knots of dimension greater than or equal to five do not have the nice behavior of algebraic 1-knots, since the notion of cobordism and isotopy are distinct.

Moreover, there exist plenty of examples of knots, not necessarily spherical nor algebraic, which are cobordant but are not isotopic for any dimension. For example, in the case of dimension one, the square knot, which is the connected sum of the right hand and the left hand trefoil knots, is cobordant but is not isotopic to the trivial knot. (For more explicit examples, see  $\S8$ .)

Using Seifert forms, we have a complete characterization of cobordism classes of simple fibered knots as follows (see [BM, B1, B3]).

**Theorem 5.2** ([BM]). For  $n \ge 3$ , two simple fibered (2n - 1)-knots are cobordant if and only if their Seifert forms are algebraically cobordant.

The definition of algebraically cobordant forms will be given later in this section.

Remark 5.3. Related results had been obtained by Vogt [V1, V2], who proved that if two simple (not necessarily fibered) (2n - 1)-knots,  $n \ge 3$ , are cobordant, then their Seifert forms are Witt equivalent and satisfy certain properties which are weaker than the algebraic cobordism. Conversely, if two simple (2n - 1)-knots,  $n \ge 3$ , with torsion free homologies have algebraically cobordant Seifert forms, then they are cobordant.

In Theorem 5.2 the condition on the integer n is only used to prove the sufficiency, and we have the following theorem which is valid for all odd dimensions.

**Theorem 5.4** ([BM]). For  $n \ge 1$ , two cobordant simple fibered (2n-1)-knots have algebraically cobordant Seifert forms.

Furthermore, the following holds for (not necessarily fibered) simple knots.

**Theorem 5.5** ([BM]). For  $n \ge 3$ , two simple (2n - 1)-knots are cobordant if their Seifert forms associated with (n - 1)-connected Seifert manifolds are algebraically cobordant.

To define the algebraic cobordism, we first need to fix some notations and definitions. Let  $\mathcal{A}$  be the set of all bilinear forms defined on free  $\mathbb{Z}$ -modules of finite rank. Set  $\varepsilon = (-1)^n$ . Let  $A: G \times G \to \mathbb{Z}$  be a bilinear form in  $\mathcal{A}$ . We denote by  $A^T$  the transpose of A, by S the  $\varepsilon$ symmetric form  $A + \varepsilon A^T$  associated with A, by  $S^*: G \to G^*$  the adjoint of S with  $G^*$  being the dual  $\operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$  of G, and by  $\overline{S}: \overline{G} \times \overline{G} \to \mathbb{Z}$ the  $\varepsilon$ -symmetric non-degenerate form induced by S on  $\overline{G} = G/\operatorname{Ker} S^*$ . For a submodule M of G, we denote by  $\overline{M}$  the image of M in  $\overline{G}$  by the natural projection map. A submodule M of a free  $\mathbb{Z}$ -module G of finite rank is said to be *pure* if G/M is torsion free, or equivalently if M is a direct summand of G. For a submodule M of a free  $\mathbb{Z}$ -module G of finite rank, we denote by  $M^{\wedge}$  the smallest pure submodule of G which contains M.

**Definition 5.6** ([BM]). Let  $A_i: G_i \times G_i \to \mathbb{Z}$ , i = 0, 1, be two bilinear forms in  $\mathcal{A}$ . Set  $G = G_0 \oplus G_1$ ,  $A = A_0 \oplus (-A_1)$ ,  $S_i = A_i + \varepsilon A_i^T$ and  $S = A + \varepsilon A^T$ . We say that  $A_0$  is algebraically cobordant to  $A_1$  if there exist a metabolizer M for A in the sense of Definition 3.3 with  $\overline{M}$ pure in  $\overline{G}$ , an isomorphism  $\psi$ : Ker  $S_0^* \to \text{Ker } S_1^*$ , and an isomorphism  $\theta$ : Tors(Coker  $S_0^*$ )  $\to$  Tors(Coker  $S_1^*$ ) which satisfy the following two conditions:

(c1) 
$$M \cap \operatorname{Ker} S^* = \{(x, \psi(x)) \colon x \in \operatorname{Ker} S^*_0\} \subset \operatorname{Ker} S^*_0 \oplus \operatorname{Ker} S^*_1$$
  
=  $\operatorname{Ker} S^*$ ,

(c2) 
$$d(S^*(M)^{\wedge}) = \{(y, \theta(y)) : y \in \operatorname{Tors}(\operatorname{Coker} S_0^*)\} \\ \subset \operatorname{Tors}(\operatorname{Coker} S_0^*) \oplus \operatorname{Tors}(\operatorname{Coker} S_1^*) \\ = \operatorname{Tors}(\operatorname{Coker} S^*),$$

where d is the quotient map  $G^* \to \operatorname{Coker} S^*$  and "Tors" means the torsion subgroup.

In the above situation, we also say that  $A_0$  and  $A_1$  are algebraically cobordant with respect to  $\psi$  and  $\theta$ .

Recall that the knot cobordism is an equivalence relation. Furthermore, any unimodular matrix can be realized as a Seifert matrix associated with a simple fibered (2n-1)-knot,  $n \ge 3$ . Therefore, Theorem 5.2 implies the following

**Theorem 5.7.** Algebraic cobordism is an equivalence relation on the set of unimodular forms. *Example 5.8.* In [BM, Theorem 1], it is claimed that the algebraic cobordism is an equivalence relation on the whole set of integral bilinear forms  $\mathcal{A}$ . However, this may be not true as explained below.

Let us consider the three matrices

$$A_0 = \begin{pmatrix} 0 & 4 & -2 & -3 \\ -4 & 0 & -2 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & -1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 4 & 1 & 2 \\ -4 & 0 & 1 & -2 \\ -1 & -1 & 0 & 0 \\ -2 & 2 & -1 & 0 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} 0 & 4 & -6 & 1 \\ -4 & 0 & -2 & -1 \\ 6 & 2 & 0 & 1 \\ -1 & 1 & 0 & 0 \end{pmatrix},$$

which are given in [V2, p. 45]. We identify  $A_i$  with the corresponding bilinear form  $A_i: G_i \times G_i \to \mathbf{Z}$  with  $G_i \cong \mathbf{Z}^4$ , i = 0, 1, 2. Set

$$\begin{split} m_1 &= (0, 0, 1, 0, 0, 0, -2, 0) \in G_0 \oplus G_1, \\ m_2 &= (0, 1, 0, 2, 0, 0, 0, -1) \in G_0 \oplus G_1, \\ m_3 &= (1, 0, 0, 0, 1, 0, 0, 0) \in G_0 \oplus G_1, \\ m_4 &= (0, 1, 0, 0, 0, 1, 0, 0) \in G_0 \oplus G_1, \\ n_1 &= (0, 0, 2, 0, 0, -1, -1, 0) \in G_1 \oplus G_2, \\ n_2 &= (0, 0, 0, 1, 0, 0, 0, 2) \in G_1 \oplus G_2, \\ n_3 &= (1, 0, 0, 0, 1, 0, 0, 0) \in G_1 \oplus G_2, \\ n_4 &= (0, 1, 0, 0, 0, 1, 0, 0) \in G_1 \oplus G_2. \end{split}$$

Then we see that the subgroup generated by  $m_1$ ,  $m_2$ ,  $m_3$ ,  $m_4$  of  $G_0 \oplus G_1$ gives a metabolizer for  $A_0 \oplus (-A_1)$ , and that the subgroup generated by  $n_1$ ,  $n_2$ ,  $n_3$ ,  $n_4$  of  $G_1 \oplus G_2$  gives a metabolizer for  $A_1 \oplus (-A_2)$ . Furthermore, it is easy to check that  $A_i$  and  $A_{i+1}$  are algebraically cobordant for  $\varepsilon = +1$  with respect to the "identity"

$$\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{0} \oplus \mathbf{0} = \operatorname{Ker} S_i^* \to \operatorname{Ker} S_{i+1}^* = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{0} \oplus \mathbf{0},$$

i = 0, 1, where  $S_i = A_i + A_i^T$ , i = 0, 1, 2.

However, in [V2] it is shown that  $A_0$  and  $A_2$  are not algebraically cobordant with respect to the "identity".

In the proof given in [BM, pp. 38–39], it is shown that if  $A_i$  and  $A_{i+1}$  are algebraically cobordant with respect to  $\psi_i$ , i = 0, 1 (see Definition 5.6

(c1)), then  $A_0$  and  $A_2$  are algebraically cobordant with respect to  $\psi_1 \circ \psi_0$ . So, this contradicts Vogt's result mentioned above.

In fact, in general we may not have the direct sum decomposition  $G_i = \text{Ker } S_i^* \oplus T_i, \ i = 0, 1, 2$ , mentioned in the proof given in [BM, p. 39].

Presumably, the above example would show that the algebraic cobordism is not an equivalence relation on the set of general (not necessarily unimodular) integral bilinear forms defined on free Z-modules of finite rank. Since the relation introduced by Vogt [V2] and that of Definition 5.6 are slightly different, we do not know at present if the relation of algebraic cobordism is an equivalence relation or not.

Remark 5.9. For general forms which are not necessarily unimodular, we can consider the equivalence relation generated by the algebraic cobordism, called the *weak algebraic cobordism*. Then by using Theorem 5.5,<sup>3</sup> we can show that if two simple (2n - 1)-knots,  $n \ge 3$ , have weakly algebraically cobordant Seifert forms with respect to (n - 1)connected Seifert manifolds, then they are cobordant.

Furthermore, we can prove the following. A simple (2n - 1)-knot is said to be *C*-algebraically fibered if its Seifert form is algebraically cobordant to a unimodular form (see [BS1]). Then, two simple *C*-algebraically fibered (2n - 1)-knots,  $n \ge 3$ , are cobordant if and only if their Seifert forms are weakly algebraically cobordant. We do not know if this is true for all simple (2n - 1)-knots,  $n \ge 3$ .

Let  $A_i$  be Seifert forms associated with (n-1)-connected Seifert manifolds  $V_i$  of simple (2n-1)-knots  $K_i$ , i = 0, 1, and  $S_i^*$  the adjoint of the intersection form of  $V_i$ . Since we have the exact sequence

$$0 = H_{n+1}(V_i, K_i) \to H_n(K_i) \to H_n(V_i) \xrightarrow{S_i^*} H_n(V_i, K_i)$$
$$\to H_{n-1}(K_i) \to H_{n-1}(V_i) = 0$$

associated with the pair  $(V_i, K_i)$ , where we identify  $H_n(V_i, K_i)$  with the dual of  $H_n(V_i)$  (see (2.1)), Ker  $S_i^*$  and Coker  $S_i^*$  are naturally identified with  $H_n(K_i)$  and  $H_{n-1}(K_i)$  respectively.

As remarked before, in the case of a spherical knot K we have  $H_n(K) = H_{n-1}(K) = 0$ , and the intersection form is an isomorphism. Hence the algebraic cobordism for Seifert forms associated with *spherical* simple knots is reduced to the Witt equivalence, and Theorem 5.2

<sup>&</sup>lt;sup>3</sup>Here, we also need the fact that every form in  $\mathcal{A}$  can be realized as the Seifert form of a simple (2n-1)-knot.

follows from the classical result of Kervaire and Levine (see Theorem 3.4 and the paragraph just after Remark 3.5).

In order to clarify the relation of algebraic cobordism, we present here several examples.

*Example 5.10.* (1) Let us consider any integral bilinear form A in  $\mathcal{A}$  such that  $A + \varepsilon A^T$  is unimodular. Then,  $A \oplus (-A)$  is always algebraically cobordant to the zero form.

(2) Let us consider the integral bilinear forms  $A_0$  and  $A_1$  represented by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & -1 \\ -2 & 4 \end{pmatrix}$$

respectively, which are given in [K2, p. 93]. Then it is easy to check that the subgroup of  $\mathbf{Z}^4$  generated by  $(3, 1, 3, 0)^T$  and  $(0, 1, 2, 1)^T$  is a metabolizer for  $A_0 \oplus (-A_1)$ . Since  $A_i - A_i^T$  are unimodular, i = 0, 1, we see that  $A_0$  and  $A_1$  are algebraically cobordant for  $\varepsilon = -1$ . Note that  $A_0$  and  $A_1$  are not congruent to each other.

(3) The following example is a generalization of those given in [BMS]. Let us consider the two matrices

$$A_0 = \begin{pmatrix} p^2 & 1 \\ -1 & 0 \end{pmatrix}$$
 and  $A_1 = \begin{pmatrix} q^2 & 1 \\ -1 & 0 \end{pmatrix}$ ,

which are identified with the corresponding integral bilinear forms, where p and q are odd integers with  $1 \le p < q$ . Note that they are both unimodular and

$$S_0 = A_0 + \varepsilon A_0^T = S_1 = A_1 + \varepsilon A_1^T = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

where  $\varepsilon = -1$ . Let us show that  $A_0$  and  $A_1$  are algebraically cobordant in the sense of Definition 5.6 for  $\varepsilon = -1$ .

Let r be the greatest common divisor of p and q and set p = rp' and q = rq'. Furthermore, set  $m = (q', 0, p', 0)^T$  and  $m' = (0, p', 0, q')^T$ . Then it is easy to see that the submodule M of  $\mathbb{Z}^4$  generated by m and m' constitutes a metabolizer for  $A = A_0 \oplus (-A_1)$ . Since  $S_0 = S_1$  are non-degenerate, we have only to verify condition (c2) of Definition 5.6.

Set  $S = S_0 \oplus (-S_1) = A - A^T$ . Let  $S^* \colon \mathbf{Z}^4 \to \mathbf{Z}^4$ ,  $S_0^* \colon \mathbf{Z}^2 \to \mathbf{Z}^2$ and  $S_1^* \colon \mathbf{Z}^2 \to \mathbf{Z}^2$  be the adjoints of S,  $S_0$  and  $S_1$  respectively. It is easy to see that Coker  $S_0^* = \operatorname{Coker} S_1^*$  is naturally identified with  $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ . Furthermore, we have

$$S^*(m) = m^T S = (0, 2q', 0, -2p')$$
 and  
 $S^*(m') = (m')^T S = (-2p', 0, 2q', 0).$ 

Therefore,  $S^*(M)^{\wedge}$ , the smallest direct summand of  $\mathbf{Z}^4$  containing  $S^*(M)$ , is the submodule of  $\mathbf{Z}^4$  generated by (0, q', 0, -p') and (-p', 0, q', 0). Hence, for the natural quotient map  $d: \mathbf{Z}^4 \to \operatorname{Coker} S^* = (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \oplus (\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ , we have

$$d(S^*(M)^{\wedge}) = \{(x, x) \colon x \in \operatorname{Coker} S_0^* = \mathbf{Z}_2 \oplus \mathbf{Z}_2\},\$$

since Im  $S_i^*$  is generated by (2, 0) and (0, 2), i = 0, 1, and Im  $S^*$  is generated by (2, 0, 0, 0), (0, 2, 0, 0), (0, 0, 2, 0) and (0, 0, 0, 2). Therefore, we conclude that the unimodular matrices  $A_0$  and  $A_1$  are algebraically cobordant.

Note that  $A_0$  and  $A_1$  are not congruent, since there exists an element  $x \in \mathbb{Z}^2$  such that  $x^T A_0 x = p^2$ , while such an element does not exist for  $A_1$ .

Let us give a sketch of the proof of Theorem 5.2. Let  $K_0 = \partial F_0$  and  $K_1 = \partial F_1$  be two simple fibered (2n-1)-knots with  $n \ge 3$  with fibers  $F_0$  and  $F_1$  respectively. Denote by  $A_0$  and  $A_1$  the Seifert forms associated with  $F_0$  and  $F_1$  respectively.

To prove the necessity in Theorem 5.2, we first suppose that  $K_0 \subset S^{2n+1} \times \{0\}$  and  $K_1 \subset S^{2n+1} \times \{1\}$  are cobordant. Then we see that the union of the cobordism and the fibers bound a compact oriented (2n+1)-dimensional manifold W embedded in  $S^{2n+1} \times [0, 1]$  by using the obstruction theory as in §2. Using the kernel of the homomorphism induced by the inclusion  $F_0 \cup F_1 \to W$ , we can construct a metabolizer for  $A_0 \oplus (-A_1)$  which fulfills all the conditions in the definition of algebraic cobordism. (For this we need to have that  $A_0$  and  $A_1$  are unimodular, which is guaranteed since  $K_0$  and  $K_1$  are fibered.) We refer to [BM] for details.

For sufficiency we suppose that  $A_0$  and  $A_1$  are algebraically cobordant with respect to a metabolizer M. We consider  $F_i$  to be embedded in  $S^{2n+1} \times \{i\}$ , i = 0, 1, and denote by F the connected sum  $F = F_0 \sharp F_1$  embedded in  $S^{2n+1} \times [0, 1]$ . Note that we naturally have  $H_n(F) = H_n(F_0) \oplus H_n(F_1)$ . Then, since  $n \ge 3$ , we can show that one can perform embedded surgeries on the connected sum of Seifert manifolds in  $S^{2n+1} \times [0, 1]$  so that we obtain a simply connected submanifold X of  $S^{2n+1} \times [0, 1]$  with  $\partial X = (K_0 \times \{0\}) \coprod (K_1 \times \{0\})$  and  $H_*(X, K_i) = 0$ for i = 0, 1. According to Smale's *h*-cobordism theorem [Sm2, M2] we have  $X \cong K_0 \times [0, 1]$ , and thus X gives a cobordism between  $K_0$  and  $K_1$ . This is where we need to have (2n - 1)-dimensional knots with  $n \ge 3$ , since the *h*-cobordism theorem is valid only for dim  $X \ge 6$ .

The crucial point in the proof is to see that the technical conditions imposed on the metabolizer in Definition 5.6 give a strategy to perform the right embedded surgeries. For details, see [BM, B3].

#### §6. 3-Dimensional knots

In this section, we deal with 3-dimensional knots.<sup>4</sup> This case is much more difficult than that of higher dimensional knots, since the dimension of the Seifert manifolds associated with 3-knots is equal to four. The topology of 4-dimensional manifolds is exceptional, and the usual technics like the Whitney trick [W2] used in the case of higher dimensional manifolds are not available any more.

The algebraic cobordism of Seifert forms is a necessary condition for the existence of a cobordism between two simple fibered (2n - 1)-knots for all  $n \ge 1$  (see Theorem 5.4). Furthermore, two isotopic simple fibered (2n - 1)-knots have isomorphic Seifert forms for all  $n \ge 1$  (for example, see [D, Kt, S1]). However, it is known that there exist 3-dimensional simple fibered knots which are abstractly diffeomorphic and have isomorphic Seifert forms but which are not isotopic (see Example 6.1 below). This shows that the one-to-one correspondence between the isotopy classes of knots and the isomorphism classes of Seifert forms stated in Theorem 5.1 does not hold for n = 2. In fact, these fibered 3-knots are even not cobordant (see Remark 6.7). Hence, for 3-dimensional knots, isotopy classes and cobordism classes must be characterized by new equivalence relations. Isotopy classes of 3-knots were studied in [S1, S2, S4] (see also [Hi]). For cobordism classes we will define a new equivalence relation. For this we need to use Spin structures on manifolds.

Recall that a Spin *structure* on a manifold X means the homotopy class of a trivialization of  $TX \oplus \varepsilon^N$  over the 2-skeleton  $X^{(2)}$  of X, where TX denotes the tangent bundle and  $\varepsilon^N$  is a trivial vector bundle of dimension N sufficiently large. Note that X admits a Spin structure if and only if its second Stiefel-Whitney class  $w_2(X) \in H^2(X; \mathbb{Z}_2)$  vanishes and that if it admits, then the set of all Spin structures on X is in oneto-one correspondence with  $H^1(X; \mathbb{Z}_2)$ .

Let K be an oriented 3-knot, with a Seifert manifold V, embedded in  $S^5$ . Then K has a natural normal 2-framing  $\nu = (\nu_1, \nu_2)$  in  $S^5$  such that the first normal vector field  $\nu_1$  is obtained as the inward normal vector field of  $K = \partial V$  in V. The homotopy class of this 2-framing does

<sup>&</sup>lt;sup>4</sup>In the following, all 3-knots will be oriented.

not depend on the choice of the Seifert manifold V. Then K carries a tangent 3-framing on its 2-skeleton  $K^{(2)}$  such that the juxtaposition with the above 2-framing gives the standard framing of  $S^5$  restricted to  $K^{(2)}$  up to homotopy. This means that K carries a natural Spin structure, which is determined uniquely up to homotopy. Furthermore, this Spin structure coincides with that induced from the Seifert manifold V, which is endowed with the natural Spin structure induced from  $S^5$ .

In the case of 3-knots, Spin structures must be considered as the following example shows.

Example 6.1. Let  $K_0$  and  $K_1$  be the simple fibered 3-knots which are abstractly diffeomorphic to  $S^1 \times \Sigma_g$ , constructed in [S4, Proposition 3.8], where  $\Sigma_g$  is the closed connected orientable surface of genus  $g \ge 2$ . They have the property that their Seifert forms are isomorphic, but that there exists no diffeomorphism between  $K_0$  and  $K_1$  which preserves their Spin structures. Consequently they are not isotopic.

In order to study cobordisms of 3-knots, we will use some results valid only for 3-dimensional manifolds without torsion on the first homology group. Hence, we define

**Definition 6.2** ([BS1]). We say that a 3-knot K is *free* if  $H_1(K)$  is torsion free over **Z**.

Moreover, for free knots we do not need to consider condition (c2) in the definition of the algebraic cobordism (see Definition 5.6), which simplifies the argument.

**Definition 6.3** ([BS1]). Consider two simple 3-knots  $K_0$  and  $K_1$ . Let  $A_0$  and  $A_1$  be the Seifert forms of  $K_0$  and  $K_1$  respectively with respect to 1-connected Seifert manifolds. We say that the pairs  $(K_0, A_0)$ and  $(K_1, A_1)$  are Spin *cobordant* (for simplicity, we also say that the Seifert forms  $A_0$  and  $A_1$  are Spin *cobordant*) if there exists an orientation preserving diffeomorphism  $h: K_0 \to K_1$  such that

- (1) h preserves their Spin structures,
- (2)  $A_0$  and  $A_1$  are algebraically cobordant with respect to  $h_*: H_2(K_0) \to H_2(K_1)$  and  $h_*|_{\text{Tors } H_1(K_0)}: \text{Tors } H_1(K_0) \to \text{Tors } H_1(K_1)$ , where we identify  $H_2(K_i)$  and  $H_1(K_i)$  with Ker  $S_i^*$  and Coker  $S_i^*$ respectively (see the exact sequence (2.1)) and  $S_i = A_i + A_i^T$ , i = 0, 1.

Note that if  $K_0$  and  $K_1$  are free 3-knots, then we do not need to consider condition (c2) of Definition 5.6 and hence the isomorphism  $h_*|_{\text{Tors } H_1(K_0)}$ in the above definition.

In [BS1] we proved the following.

**Theorem 6.4.** Two simple fibered free 3-knots are cobordant if and only if their Seifert forms with respect to 1-connected fibers are Spin cobordant.

Remark 6.5. Note that in the case of homology 3-spheres embedded in  $S^5$ , the corresponding result had been obtained in [S3].

Since the cobordism for knots is an equivalence relation, the Spin cobordism is an equivalence relation on the set of Seifert forms of simple fibered free 3-knots with respect to 1-connected Seifert manifolds.

Let us show that the Spin cobordism is a necessary condition for the existence of a knot cobordism between given two simple fibered 3knots. Let  $K_0$  and  $K_1$  be two cobordant simple fibered 3-knots with fibers  $F_0$  and  $F_1$  respectively. Denote by  $X \cong K_0 \times [0, 1]$  a submanifold of  $S^5 \times [0, 1]$  which gives a cobordism between  $K_0$  and  $K_1$ , and set  $N = F_0 \cup X \cup (-F_1)$ . By classical obstruction theory as described in §2, we see that the closed oriented 4-manifold  $N \subset S^5 \times [0, 1]$  is the boundary of a compact oriented 5-dimensional submanifold W of  $S^5 \times [0, 1]$ . Using a normal 2-framing of X in  $S^5 \times [0, 1]$  induced from the inward normal vector field along  $N = \partial W$  in W, we see that the diffeomorphism hbetween  $K_0$  and  $K_1$  induced by X preserves their Spin structures.

Moreover, in [BM], it has been shown that the two forms  $A_0$  and  $A_1$ , associated with the fibers, are algebraically cobordant with respect to  $h_*: H_2(K_0) \to H_2(K_1)$  and  $h_*|_{\operatorname{Tors} H_1(K_0)}: \operatorname{Tors} H_1(K_0) \to \operatorname{Tors} H_1(K_1)$ .

Finally we get the following result, in which the knots may not necessarily be free.

**Proposition 6.6** ([BS1]). If two simple fibered 3-knots are cobordant, then their Seifert forms with respect to 1-connected fibers are Spin cobordant.

Remark 6.7. In Example 6.1 above, the Seifert forms of  $K_0$  and  $K_1$  are algebraically cobordant, but are not Spin cobordant. Hence they cannot be cobordant by Proposition 6.6 (or Theorem 6.4). Example 6.1 shows that Spin structures are essential in the theory of cobordism of 3-knots as well.

We have another example as follows.

Example 6.8. Let P be a non-trivial orientable  $S^1$ -bundle over the closed connected orientable surface of genus  $g \ge 2$ . Note that  $H_1(P)$  is not torsion free in general. For every positive integer n, let  $K_1, K_2, \ldots, K_n$  be the simple fibered 3-knots constructed in [S4, Theorem 3.1] which are all abstractly diffeomorphic to P. They have the property that their fibers are all diffeomorphic and their Seifert forms are isomorphic to each other, but any such isomorphism restricted to  $H_2(K_i)$  cannot be realized by a diffeomorphism. Thus, the Seifert forms of  $K_i$ , i = 1, 2, ..., n, are algebraically cobordant to each other, but are not Spin cobordant. Hence they are not cobordant by Proposition 6.6, which is valid also for non-free simple fibered 3-knots.

Using the 5-dimensional stable h-cobordism theorem due to Lawson [La] and Quinn [Q] together with Boyer's work [Bo], we also have the following theorem, in which the 3-knots are simple and free, but may not be fibered.

**Theorem 6.9** ([BS1]). Consider two simple free 3-knots in  $S^5$ . If their Seifert forms with respect to 1-connected Seifert manifolds are Spin cobordant, then they are cobordant.

The proof of the above theorem is very technical and complicated, and we refer to [BS1] for details. Finally Proposition 6.6 and Theorem 6.9 imply Theorem 6.4.

*Remark 6.10.* Some of the results in [BS1] depend on the possibly erroneous hypothesis that the algebraic cobordism is an equivalence relation on the whole set of integral bilinear forms. However, all the results are valid if we replace the algebraic cobordism with the weak algebraic cobordism as introduced in Remark 5.9 and the Spin cobordism with the equivalence relation generated by the Spin cobordism.

## $\S7$ . Fox-Milnor type relation

In [FM2] Fox and Milnor showed that the Alexander polynomials of two cobordant 1-knots should satisfy a certain property. In this section, we explain this property for odd dimensional knots and present an application to the cobordism classes of spherical fibered knots.

In the following, for a polynomial  $f(t) \in \mathbf{Z}[t]$ , we set

$$f^*(t) = t^d f(t^{-1}),$$

where d is the degree of f(t). We say that a polynomial  $f(t) \in \mathbf{Z}[t]$  is symmetric if  $f^*(t) = \pm t^a f(t)$  for some  $a \in \mathbf{Z}$ .

Let K be either a spherical (2n-1)-knot or a simple (2n-1)-knot with Seifert matrix A. As mentioned before, we still assume that A is associated with an (n-1)-connected Seifert manifold when K is simple. Then the polynomial

$$\Delta_K(t) = \det(tA + (-1)^n A^T)$$

is called the Alexander polynomial of K (see [Al, L1]). It is known to be an isotopy invariant of K up to a multiple of  $\pm t^a$ ,  $a \in \mathbb{Z}$ . For fibered knots, we use (unimodular) Seifert matrices with respect to fibers so that the Alexander polynomial is well-defined up to a multiple of  $\pm 1$ and has leading coefficient  $\pm 1$ . Note that the Alexander polynomial of a knot is always symmetric.

The following relation is called the *Fox-Milnor type relation* (for proofs, see [L2, BM], for example).

**Proposition 7.1.** Let  $K_0$  and  $K_1$  be two (2n-1)-knots which are both spherical or both simple. If they are cobordant, then there exists a polynomial  $f(t) \in \mathbf{Z}[t]$  such that

(7.1) 
$$\Delta_{K_0}(t)\Delta_{K_1}(t) = \pm t^a f(t) f^*(t)$$

for some  $a \in \mathbf{Z}$ .

For example, in [DM], Du Bois and Michel showed that the algebraic knots constructed in [Sz] are in fact not cobordant by exploiting the Fox-Milnor type relation.

Let us show that the above relation, although very simple, gives us a lot of information on the cobordism of knots.

Let us recall that  $C_n$  denotes the cobordism group of spherical *n*-knots. Let us denote by  $F_n$  the subgroup of  $C_n$  generated by the cobordism classes of fibered knots. Note that  $F_n$  coincides with the set of all cobordism classes which contain a fibered knot.

Then we can prove the following proposition by using the Fox-Milnor type relation. Although it might be implicit in the works of Levine [L2, L3], Kervaire [K2] and Stoltzfus [Sf], here we give a detailed proof in order to clarify how to apply the Fox-Milnor type relation.

**Proposition 7.2.** The group  $C_n/F_n$  is infinitely generated if n is odd.

*Proof.* Set n = 2k - 1. We have only to prove that  $(C_n/F_n) \otimes \mathbb{Z}_2$  contains  $\mathbb{Z}_2^{\infty}$ .

First we consider the case where k is odd. For each positive integer p, set  $\Delta_p(t) = pt^2 + (1-2p)t + p$ . Note that  $\Delta_p(t)$  is irreducible over  $\mathbf{Z}$ . According to Levine (see [L2]), there exists a simple spherical (2k-1)-knot  $K_p$  in  $S^{2k+1}$  whose Alexander polynomial  $\Delta_{K_p}(t)$  is equal to  $\Delta_p(t)$ . Let  $[K_p]$  denote the class in  $(C_n/F_n) \otimes \mathbf{Z}_2 = (C_n/F_n)/2(C_n/F_n) = C_n/(F_n + 2C_n)$  represented by  $K_p$ . In order to show that  $(C_n/F_n) \otimes \mathbf{Z}_2$  contains  $\mathbf{Z}_2^{\infty}$ , we have only to show that  $\{[K_p]\}_{p\geq 2}$  are linearly independent over  $\mathbf{Z}_2$ . Suppose that  $K_{p_1} # K_{p_2} # \cdots # K_{p_\ell}$  is cobordant to L # L # L', where  $p_1, p_2, \ldots, p_\ell$  are distinct positive integers with  $p_i \ge 2, L$  is a spherical (2k-1)-knot, and L' is a spherical fibered (2k-1)-knot. Then by Proposition 7.1 we have

$$\Delta_{K_{p_1}}(t)\Delta_{K_{p_2}}(t)\cdots\Delta_{K_{p_\ell}}(t)\Delta_L(t)^2\Delta_{L'}(t) = \pm t^a f(t)f^*(t)$$

for some  $a \in \mathbf{Z}$  and  $f(t) \in \mathbf{Z}[t]$ .

Since  $\Delta_{K_{p_i}}(t)$  are irreducible and symmetric, each  $\Delta_{K_{p_i}}(t)$  should appear an even number of times in the irreducible decomposition of  $f(t)f^*(t)$ . Therefore,  $\Delta_{K_{p_i}}(t)$  should divide  $\Delta_{L'}(t)$ , since  $\Delta_{K_{p_1}}(t)$ ,  $\Delta_{K_{p_2}}(t), \ldots, \Delta_{K_{p_\ell}}(t)$  are pairwise relatively prime.

On the other hand, since L' is fibered, its Seifert matrix is unimodular and hence  $\Delta_{L'}(t)$  has leading coefficient  $\pm 1$ . This is a contradiction, since the leading coefficient of  $\Delta_{K_{p_i}}(t)$  is equal to  $p_i \geq 2$ .

Therefore,  $\{[K_p]\}_{p\geq 2} \subset (C_n/F_n)\otimes \mathbb{Z}_2$  are linearly independent over  $\mathbb{Z}_2$ .

When k is even, by considering the polynomial  $\widetilde{\Delta}_p(t) = pt^4 - (2p - 1)t^2 + p, p \ge 2$ , instead of  $\Delta_p(t)$  in the above argument, we get the desired conclusion. This completes the proof. Q.E.D.

Remark 7.3. The above polynomials  $\Delta_p(t)$  and  $\Delta_p(t)$  were used by Kervaire in [K1, Théorème III.12] for showing that  $C_{2k-1}$  is infinitely generated.

Remark 7.4. When k is even, every degree two symmetric polynomial which arises as the Alexander polynomial of a (2k - 1)-knot is reducible. In fact, in [L2], it is mentioned that such a polynomial should be of the form

$$a(a+1)t^{2} - (2a(a+1)+1)t + a(a+1) = (at - (a+1))((a+1)t - a).$$

The degree two symmetric polynomial constructed in [L3, p. 109] for  $\varepsilon = 1$  is also reducible, and it seems that the proof of Theorem 3.7 (or [L3, Theorem, p. 108]) given there should appropriately be modified.

#### $\S$ 8. Examples

In this section, we review some examples constructed in [B2, BMS, BS1].

First we construct non-spherical 3-knots which are cobordant but are not isotopic.

*Example 8.1* ([BS1]). A *stabilizer* is a simple fibered spherical 3knot whose fiber F is diffeomorphic to  $(S^2 \times S^2) \not | (S^2 \times S^2) \setminus \text{Int } D^4$ . Such a stabilizer does exist (see [S2, §4]). Moreover, we denote by  $K_S$  a stabilizer with Seifert matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

with respect to a basis of  $H_2(F)$  denoted by  $a_1, a_2, a_3, a_4$  (see [S1, p. 600] or [S4, §10]).

Since A is not congruent to the zero form,  $K_S$  is a non-trivial 3-knot.

Moreover, the submodule generated by  $a_1$  and  $a_3$  is a metabolizer for A, and one can perform embedded surgeries on the two cycles  $a_1$ and  $a_3$ , represented by two embedded 2-spheres in F. The result of this embedded surgery in  $D^6$  is a 4-dimensional disk properly embedded in  $D^6$  with  $K_S$  as boundary. Thus  $K_S$  is null-cobordant, i.e., it is cobordant to the trivial spherical 3-knot.

Then consider any simple fibered 3-knot K which is not spherical. The two simple fibered 3-knots  $K \sharp K_S$  and K are not isotopic, since the ranks of the second homology groups of their fibers are distinct. However, these knots are cobordant.

In the following example, we construct non-spherical simple fibered (2n-1)-knots with  $n \geq 3$  which are cobordant but are not isotopic. These knots are constructed using algebraic knots.

Example 8.2 ([B2]). Let  $K_i$ , with i = 0, 1, be the spherical algebraic (2n - 1)-knots,  $n \ge 3$ , associated with the isolated singularity at 0 of the polynomial functions  $h_i: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$  defined by

$$h_i(x_0, x_1, \dots, x_n) = g_i(x_0, x_1) + x_2^p + x_3^q + \sum_{k=4}^n x_k^2$$

with

$$g_0(x_0, x_1) = (x_0 - x_1) \left( (x_1^2 - x_0^3)^2 - x_0^{s+6} - 4x_1 x_0^{(s+9)/2} \right) \left( (x_0^2 - x_1^5)^2 - x_1^{r+10} - 4x_0 x_1^{(r+15)/2} \right),$$

and

$$g_1(x_0, x_1) = (x_0 - x_1) \left( (x_1^2 - x_0^3)^2 - x_0^{r+14} - 4x_1 x_0^{(r+17)/2} \right) \left( (x_0^2 - x_1^5)^2 - x_1^{s+2} - 4x_0 x_1^{(s+7)/2} \right),$$

where  $s \ge 11$ ,  $s \ne r + 8$ , s and r are odd, and p and q are distinct prime numbers which do not divide the product 330(30+r)(22+s) (see [DM, p. 166]). Note that the algebraic knots  $K_i$  associated with  $h_i$  are spherical for i = 0, 1. It has been shown in [DM] that the algebraic knots  $K_0$  and  $K_1$  are cobordant but are not isotopic.

Now let L be the algebraic (2n-1)-knot associated with the isolated singularity at 0 of the polynomial function  $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ defined by

$$f(x_0, x_1, \dots, x_n) = \sum_{k=0}^n x_k^2.$$

Note that L is not spherical.

Let us consider the connected sums  $L_i = K_i \sharp L$ , i = 0, 1, which are simple fibered (2n - 1)-knots. Then in [B2] it has been shown that  $L_0$ and  $L_1$  are cobordant but are not isotopic.

Note that according to [A, Theorem 4, p. 117], the knots  $L_0$  and  $L_1$ , which are connected sums of two algebraic knots, are not algebraic.

Let K be a knot. A stabilization of K is the operation of taking the connected sum  $K \sharp K_S$  for some null-cobordant spherical knot  $K_S$ . As the above examples show, stabilization is a natural way to construct knots that are cobordant but are not isotopic. We have other types of constructions as follows.

Example 8.3. The matrices given in Example 5.10 (2) give two spherical simple (2n-1)-knots with  $n \ge 3$  odd which are cobordant but are not isotopic. Similarly, the matrices given in Example 5.10 (3) give two simple fibered non-spherical (2n-1)-knots with  $n \ge 3$  odd which are cobordant but are not isotopic.

#### $\S$ 9. Pull back relation for knots

For cobordisms of non-spherical knots, Yukio Matsumoto asked the following question.

 $(\mathcal{Q})$  If two non-spherical knots (of sufficiently high dimension) are simple homotopy equivalent as abstract manifolds, then are they cobordant after taking connected sums with some spherical knots? In other words, consider the action of the spherical knot cobordism group on the set of cobordism classes of codimension two embeddings of manifolds of a fixed simple homotopy type into a sphere. Then, is the action transitive?

According to the codimension two surgery theory [Mt2], the answer to the above question is affirmative provided that the two non-spherical knots satisfy some connectivity conditions and that one of them is obtained as the inverse image of the other one by a certain degree one map between the ambient spheres. This motivates the following definition.

**Definition 9.1** ([BMS]). Let  $K_0$  and  $K_1$  be oriented *m*-knots in  $S^{m+2}$ . We say that  $K_0$  is a *pull back* of  $K_1$  if there exists a degree one smooth map  $g: S^{m+2} \to S^{m+2}$  with the following properties:

- (1) g is transverse to  $K_1$ ,
- (2)  $g^{-1}(K_1) = K_0,$
- (3)  $g|_{K_0} \colon K_0 \to K_1$  is an orientation preserving simple homotopy equivalence.

In this case, we write  $K_0 \succ K_1$ . We say that two *m*-knots are *pull back* equivalent if they are equivalent with respect to the equivalence relation generated by the pull back relation.

The following properties are direct consequences of the previous definition.

- (1)  $K \succ K$  for any *m*-knot *K*.
- (2)  $K_0 \succ K_1$  and  $K_1 \succ K_2$  imply  $K_0 \succ K_2$  for any *m*-knots  $K_0$ ,  $K_1$  and  $K_2$ .
- (3)  $K_0 \succ K_1$  and  $K'_0 \succ K'_1$  imply  $K_0 \sharp K'_0 \succ K_1 \sharp K'_1$  for any *m*-knots  $K_0, K'_0, K_1$  and  $K'_1$ .

Furthermore, if we restrict ourselves to spherical *m*-knots, then it is not difficult to see that the *trivial m-knot*  $K_U$  is the minimal element, i.e.,  $K \succ K_U$  for every spherical *m*-knot K, where  $K_U$  is defined to be the isotopy class of the boundary of an (m + 1)-dimensional disk embedded in  $S^{m+2}$ .

Here are some basic results on the pull back relation for simple fibered (2n-1)-knots,  $n \geq 3$ .

**Theorem 9.2** ([BMS]). Let  $K_0$  and  $K_1$  be simple fibered (2n-1)knots in  $S^{2n+1}$  with  $n \ge 3$ . If  $K_0 \succ K_1$  and  $K_1 \succ K_0$ , then  $K_0$  is isotopic to  $K_1$ . In other words, the relation " $\succ$ " defines a partial order for simple fibered (2n-1)-knots in  $S^{2n+1}$  for  $n \ge 3$ .

**Theorem 9.3** ([BMS]). Let  $K_0$  and  $K_1$  be simple fibered (2n-1)knots in  $S^{2n+1}$  with  $n \ge 3$ . Then  $K_0 \succ K_1$  if and only if there exists a spherical simple fibered (2n-1)-knot  $\Sigma$  in  $S^{2n+1}$  such that  $K_0$  is isotopic to the connected sum  $K_1 \sharp \Sigma$ . Remark 9.4. For n = 1, Theorem 9.3 does not hold<sup>5</sup>. Let  $K_1$  be a non-trivial spherical prime fibered 1-knot in  $S^3$  and  $K_0$  a spherical prime satellite fibered 1-knot with companion  $K_1$ , where their fibering structures are compatible. Then we can show that  $K_0 > K_1$ . However,  $K_0$  is not isotopic to the connected sum  $K_1 \not\models \Sigma$  for any non-trivial 1-knot  $\Sigma$ . Note that such a construction does not give a counter example to Theorem 9.3 for  $n \geq 3$ , since such a satellite knot in higher dimensions is always a connected sum by virtue of Theorem 5.1.

Let  $K_0$  and  $K_1$  be two simple fibered (2n-1)-knots with  $n \ge 3$ . By Theorem 9.3 if  $K_0$  is pull back equivalent to  $K_1$ , then they are cobordant after taking connected sums with some spherical knots. In the following proposition, we show that the converse is not true in general.

**Proposition 9.5** ([BMS]). For every odd integer  $n \ge 3$ , there exists a pair  $(K_0, K_1)$  of simple fibered (2n - 1)-knots with the following properties:

- (1) the knots  $K_0$  and  $K_1$  are cobordant, but
- (2) the knots  $K_0$  and  $K_1$  are not pull back equivalent.

*Proof.* Let us consider the two matrices  $A_0$  and  $A_1$  given in Example 5.10 (3).

By Theorem 5.1, there exists a simple fibered (2n - 1)-knot  $K_i$ which realizes  $A_i$  as its Seifert form with respect to the fiber, i = 0, 1. By Theorem 5.5,  $K_0$  and  $K_1$  are cobordant.

Let us now show that  $K_0$  and  $K_1$  are not pull back equivalent. By Theorem 9.3, we have only to show that for any spherical simple fibered (2n-1)-knots  $\Sigma_0$  and  $\Sigma_1$  in  $S^{2n+1}$ ,  $K_0 \sharp \Sigma_0$  is never isotopic to  $K_1 \sharp \Sigma_1$ .

Since  $K_i \sharp \Sigma_i$  is a fibered knot, we can consider the monodromy on the *n*-th homology group of the fiber, i = 0, 1. Let us denote by  $H_i$  the monodromy matrix of  $K_i \sharp \Sigma_i$  and by  $\widetilde{A}_i$  its Seifert matrix with respect to the same basis. Here, we choose a basis which is the union of a basis of the homology of the fiber for  $K_i$  and that for  $\Sigma_i$ . It is known that  $H_i = (-1)^{n+1} \widetilde{A}_i^{-1} \widetilde{A}_i^T$  (for example, see [D]). Therefore, we have

$$H_0 = \begin{pmatrix} -1 & 0\\ 2p^2 & -1 \end{pmatrix} \oplus H'_0$$
 and  $H_1 = \begin{pmatrix} -1 & 0\\ 2q^2 & -1 \end{pmatrix} \oplus H'_1$ ,

where  $H'_i$  is the monodromy matrix of  $\Sigma_i$ , i = 0, 1.

Let us consider Ker  $((I + H_i)^2)$ , where I is the unit matrix, i = 0, 1. Since  $\Sigma_i$  are spherical knots, the monodromy matrices  $H'_i$  cannot have

<sup>&</sup>lt;sup>5</sup>The authors are indebted to Shicheng Wang for the construction in this remark.

eigenvalue -1. Therefore, Ker  $((I + H_i)^2)$  corresponds exactly to the homology of the fiber of  $K_i$ .

Suppose that  $K_0 \sharp \Sigma_0$  is isotopic to  $K_1 \sharp \Sigma_1$ . Then the Seifert form of  $K_0 \sharp \Sigma_0$  restricted to Ker  $((I + H_0)^2)$  should be isomorphic to that of  $K_1 \sharp \Sigma_1$  restricted to Ker  $((I + H_1)^2)$ . This means that  $A_0$  should be congruent to  $A_1$ . However, as we saw in Example 5.10 (3), this is a contradiction. Thus, we conclude that  $K_0$  and  $K_1$  are not pull back equivalent. Q.E.D.

Let us now give some examples of pairs of knots which are diffeomorphic but not cobordant even after taking connected sums with (not necessarily simple or fibered) spherical knots. For this, we use the following proposition (see [BMS, V2]).

**Proposition 9.6.** Let  $K_0$  and  $K_1$  be simple fibered (2n-1)-knots with fibers  $F_0$  and  $F_1$  respectively,  $n \ge 3$ . For i = 0, 1, we denote by  $I(K_i)$  the image of the homomorphism  $H_n(K_i) \to H_n(F_i)$  induced by the inclusion. If  $K_0 \sharp \Sigma_0$  and  $K_1 \sharp \Sigma_1$  are cobordant for some spherical knots  $\Sigma_0$  and  $\Sigma_1$ , then the Seifert forms of  $K_0$  and  $K_1$  restricted to  $I(K_0)$  and  $I(K_1)$ , respectively, are isomorphic to each other.

In the following example we give a pair of diffeomorphic knots for which their connected sums with any spherical knots are never cobordant. This answers question (Q) mentioned at the beginning of this section negatively.

*Example 9.7* ([BMS]). Let us consider the following unimodular matrices:

$$A_0 = \begin{pmatrix} 0 & 1\\ (-1)^{n+1} & 0 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ (-1)^{n+1} & 0 & 0 & 1\\ 0 & (-1)^{n+1} & 0 & 0 \end{pmatrix}$$

Then, for every integer  $n \geq 3$ , there exist simple fibered (2n-1)-knots  $K_i$  in  $S^{2n+1}$  whose Seifert matrices are given by  $A_i$ , i = 0, 1. Note that if we denote their fibers by  $F_i$ , i = 0, 1, then  $F_1$  is orientation preservingly diffeomorphic to  $F_0 \sharp (S^n \times S^n)$ . In particular,  $K_0$  and  $K_1$  are orientation preservingly diffeomorphic to each other.

It is easy to verify that the Seifert form restricted to  $I(K_1)$  is the zero form, while it is not zero for  $K_0$ . Hence, by Proposition 9.6,  $K_0 \sharp \Sigma_0$  is never cobordant to  $K_1 \sharp \Sigma_1$  for any spherical (not necessarily simple or fibered) knots  $\Sigma_0$ ,  $\Sigma_1$ .

Note that for this example, we have  $H_{n-1}(K_i) \cong \mathbb{Z} \oplus \mathbb{Z}$ , i = 0, 1.

Let us give another kind of an example together with an argument using the Alexander polynomial.

Example 9.8 ([BMS]). Let us consider the unimodular matrices

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and their associated simple fibered (2n - 1)-knots  $K_i$ , i = 0, 1, with  $n \ge 4$  even. As in Example 9.7 we see that  $K_0$  and  $K_1$  are orientation preservingly diffeomorphic to each other.

Now, suppose that for some spherical (2n-1)-knots  $\Sigma_i$ , i = 0, 1,  $K_0 \sharp \Sigma_0$  is cobordant to  $K_1 \sharp \Sigma_1$ . We may assume that  $\Sigma_0$  and  $\Sigma_1$  are simple. The Alexander polynomials of  $K_0$  and  $K_1$  are given by

$$\Delta_{K_0}(t) = \det(tA_0 + A_0^T) = t^2 + t + 1$$

and

$$\Delta_{K_1}(t) = \det(tA_1 + A_1^T) = -(t^4 + t^3 - t^2 + t + 1)$$

respectively. Both of these polynomials are irreducible over **Z**. If  $K_0 \sharp \Sigma_0$  is cobordant to  $K_1 \sharp \Sigma_1$ , then by Proposition 7.1, we must have a Fox-Milnor type relation

(9.1) 
$$\Delta_{K_0}(t)\Delta_{\Sigma_0}(t)\Delta_{K_1}(t)\Delta_{\Sigma_1}(t) = \pm t^a f(t)f^*(t)$$

for some  $a \in \mathbb{Z}$  and  $f(t) \in \mathbb{Z}[t]$ , where  $\Delta_{\Sigma_i}(t)$  denotes the Alexander polynomial of  $\Sigma_i$ , i = 0, 1.

Note that we have  $|\Delta_{K_0}(1)| = |\Delta_{K_1}(1)| = 3$  and  $|\Delta_{\Sigma_0}(1)| = |\Delta_{\Sigma_1}(1)| = 1$ . Since  $\Delta_{K_0}(t)$  is irreducible of degree 2, and  $\Delta_{K_1}(t)$  is irreducible of degree 4, the relation (9.1) leads to a contradiction.

Hence,  $K_0 \sharp \Sigma_0$  is not cobordant to  $K_1 \sharp \Sigma_1$  for any spherical (not necessarily simple or fibered) (2n-1)-knots  $\Sigma_0$ ,  $\Sigma_1$ . In this example we have  $H_{n-1}(K_i) \cong \mathbb{Z}_3$ , for i = 0, 1.

#### $\S$ **10.** Even dimensional knots

In this section, we study cobordism classes of non-spherical 2n-knots for n = 1, 2.

Recall that in [K1] Kervaire showed that  $C_{2n}$ , the cobordism group of spherical 2*n*-knots in  $S^{2n+2}$ , is trivial for all  $n \ge 1$ . In particular, any two such knots are cobordant. For  $n \ge 3$ , Vogt [V1, V2] showed that two 2*n*-knots in  $S^{2n+2}$  are cobordant if and only if they have the same *n*-th Betti number. Note that the technics used by Vogt are only available for  $2n \ge 6$ , since it is difficult to perform embedded surgeries in low dimensions, and the *h*-cobordism theorem is not available for low dimensions.

# 10.1. Cobordism of surfaces in $S^4$

In [K1] Kervaire proved that a 2*n*-sphere embedded in  $S^{2n+2} = \partial(D^{2n+3})$  is the boundary of a (2n+1)-disk properly embedded in  $D^{2n+3}$ . This implies that  $C_{2n}$  is trivial.

Although there is no group structure on the set of cobordism classes of non-spherical 2-knots, we have a similar result. In fact we show that any connected, closed and orientable surface embedded in  $S^4$  is the boundary of an orientable handlebody properly embedded in the disk  $D^5$ . When the surface is non-orientable, it is the boundary of a non-orientable handlebody properly embedded in  $D^5$  if and only if the Euler number of the normal bundle vanishes.

Recall that the normal Euler number of an orientable surface embedded in  $S^4$  always vanishes (see [MS]). Let us recall the definition of the normal Euler number of a closed non-orientable surface M embedded in  $S^4$ , where  $S^4$  is considered to be oriented. (Throughout this section, we use the letter "M" for 2n-knots rather than "K", since the letter "K" will be used for another purpose.) The tubular neighborhood N of Mmay be regarded as a normal disk bundle over M. Let  $p: \widetilde{M} \to M$  be the orientation double cover of M. Consider the induced bundle  $\widetilde{N}$  over  $\widetilde{M}$  so that we have the commutative diagram

$$\begin{array}{cccc} \widetilde{N} & \stackrel{\widetilde{p}}{\longrightarrow} & N \\ & \downarrow & & \downarrow \\ \widetilde{M} & \stackrel{p}{\longrightarrow} & M. \end{array}$$

We orient  $\widetilde{N}$  so that the induced map  $\widetilde{p} \colon \widetilde{N} \to N$  preserves the orientations. The normal Euler number e(M) of the surface M is then defined by  $e(M) = (\widetilde{M} \cdot \widetilde{M})/2$ , where  $\widetilde{M} \cdot \widetilde{M}$  denotes the self-intersection number of  $\widetilde{M}$  in  $\widetilde{N}$ , which is always even.

Let us denote by  $N_g$  the closed connected non-orientable surface of non-orientable genus g. For a closed connected non-orientable surface  $M \cong N_g$  embedded in  $S^4$ , it is known that  $e(M) \in \{-2g, 4 - 2g, 8 - 2g, \ldots, 2g\}$ . Furthermore, all the values in the set can be realized as the normal Euler number of some  $N_g$  embedded in  $S^4$  (see [W1, Ms, Km]). In [BS2] we characterized those closed connected surfaces embedded in  $S^4$  which are the boundary of a handlebody properly embedded in  $D^5$ . For this purpose, we need to use Pin<sup>-</sup> structures on manifolds.

A Pin<sup>-</sup> structure on a manifold X is the homotopy class of a trivialization of  $TX \oplus \det TX \oplus \varepsilon^N$  over the 2-skeleton  $X^{(2)}$  of X, where TXdenotes the tangent bundle, det TX denotes the orientation line bundle, and  $\varepsilon^N$  is a trivial vector bundle of dimension N sufficiently large. A Pin<sup>-</sup> structure is equivalent to a Spin structure when X is orientable.

When M is a closed surface embedded in  $S^4$ , there is a canonical Pin<sup>-</sup> structure defined on M. More precisely, since M is characteristic, i.e., as a submanifold of  $S^4$  it represents the  $\mathbb{Z}_2$  homology class dual to the second Stiefel-Whitney class of  $S^4$ , there exists a unique Spin structure on  $S^4 \smallsetminus M$  which cannot be extended to any normal 2-disk of M. This Spin structure on  $S^4 \smallsetminus M$  induces a unique Pin<sup>-</sup> structure on M (see [KT1]).

We denote by  $H_g$  the orientable handlebody of dimension three which is obtained by gluing g orientable 1-handles to a 0-handle. The boundary of  $H_g$  is the closed connected orientable surface of genus g, denoted by  $\Sigma_g$ . Furthermore, we denote by  $I_g$  the non-orientable handlebody of dimension three which is obtained by gluing g non-orientable 1-handles to a 0-handle. Then the boundary of  $I_g$  is identified with  $N_{2g}$ . In the following we will denote by  $K_g$  the handlebody  $H_g$  or  $I_g$ .

**Definition 10.1** ([BS2]). Let M be a closed connected surface embedded in  $S^4$ . Suppose that M has genus g if M is orientable and 2gif M is non-orientable. Let  $\psi: \partial K_g \to M$  be a diffeomorphism. We say that  $\psi$  is Pin<sup>-</sup> compatible if the Pin<sup>-</sup> structure on  $\partial K_g$  induced by  $\psi$ extends through  $K_g$ .

When M is oriented, there always exists a compact oriented 3dimensional submanifold V of  $S^4$  such that  $\partial V = M$  as oriented manifolds (see, for example, [E]). Such a manifold V is again called a *Seifert* manifold associated with M (see the definition of Seifert manifolds associated with odd dimensional knots in §2). When M is non-orientable, a compact 3-dimensional submanifold V of  $S^4$  with  $\partial V = M$  is also called a Seifert manifold. Such a (non-orientable) Seifert manifold exists for M if and only if e(M) = 0 (see [GL, Km]). When a surface M admits a Seifert manifold V, the unique Spin structure on  $S^4$  induces a Pin<sup>-</sup> structure on V and this induces a Pin<sup>-</sup> structure on M, which coincides with the Pin<sup>-</sup> structure described above (see [Fi]).

In [BS2] we proved the following theorem.

**Theorem 10.2.** Let M be a closed connected surface embedded in  $S^4 = \partial D^5$ , and  $\psi \colon \partial K_q \to M$  a diffeomorphism, where  $K_q$  denotes the 3-dimensional handlebody with g 1-handles. Then, there exists an embedding  $\tilde{\psi}: K_g \to D^5$  with  $\tilde{\psi}|_{\partial K_g} = \psi$  if and only if e(M) = 0 and  $\psi$  is Pin<sup>-</sup> compatible.

Remark 10.3. Since every closed connected 3-dimensional manifold admits a Heegaard splitting of genus  $g \ge 0$ , as a consequence of Theorem 10.2 we have a new proof of Rohlin's theorem [Rh2] on the existence of an embedding of an arbitrary closed 3-dimensional manifold into  $\mathbf{R}^5$  (see also [Wl, WZ] and [GM, p. 90]). For details, see [BS2].

Let us give a sketch of a proof of Theorem 10.2. First, it is easy to see that the vanishing of e(M) and the Pin<sup>-</sup> compatibility of  $\psi$  are necessary conditions. The proof of the sufficiency is based on embedded surgeries inside the disk  $D^5$  on a Seifert manifold V of M. To do that we start with the abstract closed 3-manifold  $V' = V \cup_{\psi} K_g$  obtained by attaching V and  $K_g$  along their boundaries by using  $\psi$ . Since the 3-dimensional cobordism group  $\Omega_3^{\text{Spin}}$  (resp.  $\Omega_3^{\text{Pin}^-}$ ) of Spin (resp. Pin<sup>-</sup>) manifolds is trivial (see [M1], [K1, Lemme III.7, p. 265], [GM, p. 91], [MK] or [Ki] for  $\Omega_3^{\text{Spin}}$ , and [ABP, KT1, KT2] for  $\Omega_3^{\text{Pin}^-}$ ), there exists a compact (oriented if so is M) Pin<sup>-</sup> 4-manifold W such that  $\partial W = V'$ as (oriented) Pin<sup>-</sup> manifolds. Let f be a Morse function  $f: W \to [0, 1]$ which extends the projection to the second factor  $\partial W = (V \times \{0\}) \cup_{\psi}$  $(\partial K_g \times [0, 1]) \cup (K_g \times \{1\}) \to [0, 1]$ . Note that f can be chosen so that all its critical values lie in the interval  $(\varepsilon, 1 - \varepsilon)$  for  $\varepsilon > 0$  small enough. Moreover, we may assume that the critical points have index 1, 2 or 3.

Consider the handlebody decomposition of W associated with this Morse function. We can remove handles of index 1 and 3 using modifications described by Wallace in [Wc], respecting the Pin<sup>-</sup> structure. Then we get a new (oriented) Pin<sup>-</sup> manifold W' such that  $\partial W = \partial W'$ . Since the handlebody decomposition of the manifold W' has only handles of index 2, we can attach the handles to  $V \times [0, 1]$  inside  $D^5$  to get an embedding of W' into  $D^5$ . Finally we have a proper embedding of  $K_g \cong (\partial K_g \times [0, 1]) \cup (K_g \times \{1\}) \subset \partial W'$  into the disk  $D^5$  such that  $\partial K_g = M$ .

As a corollary to Theorem 10.2 we have

**Corollary 10.4** ([BS2]). Let M be a closed connected surface embedded in  $S^4 = \partial D^5$ . Then there exists a 3-dimensional handlebody embedded in  $D^5$  such that its boundary coincides with M if and only if e(M) = 0.

Using Theorem 10.2, we can characterize cobordism classes of closed connected surfaces embedded in  $S^4$  as follows.

**Theorem 10.5** ([BS2]). Let  $M_0$  and  $M_1$  be two closed connected surfaces embedded in  $S^4$ . Then they are cobordant if and only if they are diffeomorphic as abstract manifolds and have the same normal Euler number.

*Remark 10.6.* The above theorem in the orientable case is proved by Ogasa [O], although his proof is slightly different from ours explained below.

When two closed connected surfaces embedded in  $S^4$  are cobordant, it is clear that they are diffeomorphic as abstract manifolds and have the same normal Euler number (for details, see [BS2]). Thus we have the necessity in Theorem 10.5.

For the sufficiency, start with two closed connected surfaces  $M_0$ and  $M_1$  in  $S^4$  which are diffeomorphic as abstract manifolds and have the same normal Euler number. In the following, we consider the case where  $M_0$  and  $M_1$  are non-orientable of non-orientable genus g. (For the orientable case, the proof is similar. For details, see [BS2].)

By changing  $M_0$  and  $M_1$  by isotopies, we may assume that for a 4-disk  $D^4$  in  $S^4$ , we have  $M_0 \cap D^4 = M_1 \cap D^4 = D^2$  and  $(D^4, D^2)$  is the standard disk pair. Set  $\Delta = (S^4 \setminus \text{Int } D^4) \times [0, 1] \cong D^5$  and

$$\widetilde{M} = (M_0 \setminus \operatorname{Int} D^2) \cup (\partial D^2 \times [0, 1]) \cup (M_1 \setminus \operatorname{Int} D^2) = M_0^! \sharp M_1 \subset \partial \Delta,$$

where  $M_0^!$  denotes the mirror image of  $M_0$ . Since  $e(M_0) = e(M_1)$ , we have  $e(\widetilde{M}) = 0$ . Furthermore, one can prove that there exists a Pin<sup>-</sup> compatible diffeomorphism between  $\partial((N_g \setminus \text{Int } D^2) \times [0, 1]) \cong \partial I_g$  and  $\widetilde{M}$  which sends  $(N_g \setminus \text{Int } D^2) \times \{i\}$  diffeomorphically onto  $M_i \setminus \text{Int } D^2$ .

According to Theorem 10.2 we can embed  $I_g$  in  $\Delta$  so that  $M_0^! \sharp M_1 = \partial I_g$ . The cobordism between  $M_0$  and  $M_1$  is then obtained by gluing back  $D^4 \times [0, 1]$  to  $\Delta$  and by replacing  $I_g \cong (N_g \setminus \text{Int } D^2) \times [0, 1]$  by  $N_q \times [0, 1]$ .

As a consequence of Theorem 10.5 we have that two closed connected orientable surfaces embedded in  $S^4$  are cobordant if and only if they have the same genus. Hence, the monoid of cobordism classes of closed connected orientable surfaces embedded in  $S^4$  is isomorphic to the monoid of non-negative integers  $\mathbf{Z}_{>0}$ .

Let us consider non-orientable surfaces. First note that by adding the cobordism class of an embedding of  $S^2$  into  $S^4$  to the associative groupoid (or the associative magma or the semigroup) of cobordism classes of closed connected non-orientable surfaces embedded in  $S^4$ , we get a monoid denoted by  $\mathfrak{N}$ . We can also describe the monoid structure of  $\mathfrak{N}$  as follows. Let  $\mathbb{R}P^2_+$  (resp.  $\mathbb{R}P^2_-$ ) be the projective plane standardly
embedded in  $S^4$  with normal Euler number being equal to +2 (resp. -2) (see [HK]). For a pair of non-negative integers (k, l) such that  $k + l \geq 1$ , let  $M_{k,l}$  be the non-orientable surface embedded in  $S^4$  obtained by taking the connected sum of k copies of  $\mathbb{R}P^2_+$  and l copies of  $\mathbb{R}P^2_-$ . Then we have  $e(M_{k,l}) = 2(k-l)$  and the genus of  $M_{k,l}$  is equal to k+l. Hence, the set of non-orientable surfaces  $\{M_{k,l}: k, l \in \mathbb{Z}, k, l \geq 0, k+l \geq 1\}$ constitutes a complete set of representatives of the cobordism classes of closed connected non-orientable surfaces embedded in  $S^4$ . Therefore,  $\mathfrak{N}$ is isomorphic to the monoid of pairs of non-negative integers  $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ . If we denote by [M] the cobordism class of a closed connected nonorientable surface M embedded in  $S^4$ , and by g(M) the genus of M, then the isomorphism  $\mathfrak{N} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$  is given by mapping [M] to

$$\left(\frac{2g(M) + e(M)}{4}, \frac{2g(M) - e(M)}{4}\right).$$

#### 10.2. Concordance of embeddings of a surface

In this subsection, we consider the concordance classification of embeddings of closed connected surfaces into  $S^4$ . For the definition of the concordance, see Definition 2.10.

Examining the proof of Theorem 10.5 carefully, we see that the following characterization of concordant embeddings of surfaces into  $S^4$  holds.

**Theorem 10.7** ([BS2]). Let  $\Sigma$  be a closed connected surface. Two embeddings of  $\Sigma$  into  $S^4$  are concordant if and only if the Pin<sup>-</sup> structures induced by these embeddings coincide and the normal Euler numbers of these embeddings coincide.

When the knots are spherical of dimension two, the notions of cobordism and concordance coincide with each other, since every diffeomorphism of  $S^2$  which preserves the orientation is isotopic to the identity [Sm1]. However, when  $g \ge 1$ , for an arbitrary embedding  $f: \Sigma_g \to S^4$ there exists an orientation preserving diffeomorphism  $h: \Sigma_g \to \Sigma_g$  which does not preserve the Pin<sup>-</sup> structure induced by f. Therefore, the embeddings  $f \circ h$  and f are not concordant. This means that contrary to the spherical case, the notions of cobordism and concordance differ for orientable surfaces of genus  $g \ge 1$ .

The group of orientation preserving diffeomorphisms of a closed connected oriented surface acts transitively on the set of Pin<sup>-</sup> structures with trivial Brown invariant (see, for example, [BS2]). This set is naturally identified with the set of Spin structures with trivial Arf invariant, since the surface is assumed to be orientable. This implies that

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$\beta$	g:  odd	g: even
0	0	$2^{(g-2)/2}(2^{(g-2)/2}+1)$
1	$2^{(g-3)/2}(2^{(g-1)/2}+1)$	0
2	0	$2^{g-2}$
3	$2^{(g-3)/2}(2^{(g-1)/2}-1)$	0
4	0	$2^{(g-2)/2}(2^{(g-2)/2}-1)$
5	$2^{(g-3)/2}(2^{(g-1)/2}-1)$	0
6	0	$2^{g-2}$
7	$2^{(g-3)/2}(2^{(g-1)/2}+1)$	0

Table 1. Number of Pin<sup>-</sup>structures on the non-orientable surface  $N_g$  with Brown invariant  $\beta \in \mathbb{Z}_8$ 

the number of concordance classes of embeddings of a closed connected oriented surface is equal to the number of Spin structures with trivial Arf invariant on this surface. According to [J] this number is equal to  $2^{g-1}(2^g + 1)$ , where g is the genus of the surface. If we denote by  $\omega_g$ the number of concordance classes of embeddings of  $\Sigma_g$ , then we have  $\omega_g = 2^{g-1}(2^g + 1)$ .

Let us denote by  $\nu_g$  the number of concordance classes of embeddings of the closed connected non-orientable surface  $N_g$  of non-orientable genus g. According to [Ms, Km], the set of possible normal Euler numbers for such embeddings coincides with  $\{-2g, 4 - 2g, 8 - 2g, \ldots, 2g\}$ . Hence, we have

$$\nu_g = \sum_{i=0}^g \nu_{g,-2g+4i},$$

where  $\nu_{g,-2g+4i}$  denotes the number of concordance classes of embeddings of  $N_g$  into  $S^4$  with normal Euler number equal to -2g+4i. Moreover, according to [KT1, Theorem 6.3],  $\nu_{g,-2g+4i}$  is equal to the number of Pin<sup>-</sup> structures with Brown invariant equal to -g + 2i modulo 8. Such numbers can be calculated as in Table 1 (see [DP]).

Using the values given in Table 1, we get

$$\nu_g = \begin{cases} 2^{g-2}(g+1) & \text{if } g \text{ is odd,} \\ 2^{g-2}(g+1) + 2^{(g-2)/2} & \text{if } g \text{ is even.} \end{cases}$$

#### 10.3. Cobordism of 4-knots

In the study of cobordism of embeddings of even dimensional manifolds, the only case which remains to be studied is the case of 4dimensional manifolds embedded in  $S^6$ . In [BS3] we proved the following

**Theorem 10.8.** Let M be a closed simply connected 4-dimensional manifold. Then all the embeddings of M into  $S^6$  are concordant.

In particular, two 4-knots in  $S^6$ , i.e., two closed simply connected 4-dimensional manifolds embedded in  $S^6$ , are (oriented) cobordant if and only if they are abstractly (orientation preservingly) diffeomorphic to each other.

One can prove Theorem 10.8 by imitating the proofs of Theorems 10.2 and 10.5, and the proof is based essentially on Kervaire's original idea [K1].

Remark 10.9. It is known that a closed connected orientable 4dimensional manifold M can be embedded in  $S^6$  if and only if it is Spin and its signature vanishes (see [CS2]). If in addition M is simply connected, then it can be embedded in  $S^6$  if and only if it is homeomorphic to a connected sum of some copies of  $S^2 \times S^2$  by the homeomorphism classification of closed simply connected 4-dimensional manifolds due to Freedman [Fr].

Remark 10.10. By Park [P], for any sufficiently large odd integer m, there exist infinitely many smooth manifolds which are all homeomorphic to the connected sum of m copies of  $S^2 \times S^2$  but which are not diffeomorphic to each other. Let us denote by  $\mathfrak{O}_4$  the monoid of (oriented) cobordism classes of closed simply connected 4-manifolds embedded in  $S^6$ , and by  $\mathbf{Z}_{\geq 0}$  the monoid of non-negative integers. Then the homomorphism  $\mathfrak{O}_4 \to \mathbf{Z}_{\geq 0}$  which associates to a 4-knot one half of its second Betti number is an epimorphism. The above result of Park shows that this homomorphism is far from being an isomorphism. Compare this with the result of Vogt [V1, V2]: the corresponding homomorphism  $\mathfrak{O}_{2n} \to \mathbf{Z}_{\geq 0}$  for  $n \geq 3$  is an isomorphism, where  $\mathfrak{O}_{2n}$  denotes the monoid of (oriented) cobordism classes of 2n-knots in  $S^{2n+2}$ .

Remark 10.11. When  $n \neq 2$ , for an arbitrary 2n-knot M, its orientation reversal -M is oriented cobordant to M. For n = 2, there exists a closed 4-dimensional manifold N homeomorphic to a connected sum of some copies of  $S^2 \times S^2$  such that N is not oriented diffeomorphic to -N. In fact, by Kotschick [Ko2], every simply connected compact complex surface of general type which is Spin and has vanishing signature gives such an example. Such a complex surface has been constructed by Moishezon and Teicher [MT1, MT2, Ko1]. Hence, there exists a closed simply connected oriented 4-dimensional manifold embedded in  $S^6$  which is not oriented cobordant to its orientation reversal.

### §11. Open problems

To conclude this survey article, we would like to list some open problems.

*Problem 11.1.* In Definition 2.1, if we remove the connectivity condition on the embedded manifolds, then is it still possible to characterize their isotopy and cobordism classes?

*Problem 11.2.* Construct efficient invariants of algebraic cobordism.

*Problem 11.3.* Is the algebraic cobordism an equivalence relation on the whole set of integral bilinear forms?

See Theorem 5.7, Example 5.8, Remarks 5.9 and 6.10 for the above problem.

Problem 11.4. Is it true that two simple (2n - 1)-knots,  $n \ge 3$ , are cobordant if and only if their Seifert forms associated with (n - 1)-connected Seifert manifolds are weakly algebraically cobordant? In particular, is there a pair of two simple (2n - 1)-knots,  $n \ge 3$ , which are cobordant, but whose Seifert forms are not (weakly) algebraically cobordant?

Note that for C-algebraically fibered simple knots, the above equivalence is true (see Remark 5.9).

*Problem 11.5.* Is the Spin cobordism of Seifert forms associated with non-free 3-knots a sufficient condition of cobordism?

Problem 11.6. Does Theorem 9.3 (a characterization of the pull back relation for simple fibered (2n-1)-knots) hold for n = 2?

As noted in Remark 9.4, the above characterization does not hold for n = 1.

*Problem 11.7.* Let us fix an oriented simple homotopy type (or an oriented diffeomorphism type) of manifolds, and consider the set of all embeddings of such manifolds into a sphere in codimension two. Then, does there exist a minimal element with respect to the pull back relation?

As mentioned in §9, for spheres, the trivial knot is such a minimal element.

Problem 11.8. Is  $C_n/F_n$  isomorphic to  $\mathbf{Z}_2^{\infty} \oplus \mathbf{Z}_4^{\infty} \oplus \mathbf{Z}^{\infty}$  for odd n? Determine the group structure of  $F_n$  for odd n. Is  $F_n$  a direct summand of  $C_n$ ?

*Problem 11.9.* Is the multiplicity of a complex holomorphic function germ at an isolated singular point a cobordism invariant of the associated algebraic knot?

This is known to be true for the case of algebraic 1-knots. See also [Z2].

*Problem 11.10.* Let us consider Brieskorn type polynomials of the form

$$z_1^{a_1} + z_2^{a_2} + \dots + z_{n+1}^{a_{n+1}}.$$

If two algebraic knots associated with Brieskorn type polynomials are cobordant, then do their exponents coincide?

A related result is obtained in [S3]. Note that the associated Seifert matrix has been explicitly determined (for example, see [Sk]). It is also known that two algebraic (2n-1)-knots associated with Brieskorn polynomials with the same Alexander polynomial have the same exponents [YS].

Problem 11.11. Two fibered *n*-knots in  $S^{n+2}$  are said to be fibered cobordant if there exists a cobordism  $X \subset S^{n+2} \times [0, 1]$  between them whose complement  $S^{n+2} \smallsetminus X$  fibers over the circle in a sense similar to Definition 2.4. Is there a pair of two fibered knots which are cobordant but are not fibered cobordant?

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# Proportionality of indices of 1-forms on singular varieties

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## §0. Introduction

M.-H. Schwartz in [20, 21] introduced the technique of radial extension of stratified vector fields and frames on singular varieties, and used this to construct cocycles representing classes in the cohomology  $H^*(M, M \setminus V)$ , where V is a singular variety embedded in a complex manifold M; these are now called the Schwartz classes of V. A basic property of radial extension is that the index of the vector fields (or frames) constructed in this way is the same when measured in the strata or in the ambient space; this is called the Schwartz index of the vector field (or frame). MacPherson in [15] introduced the notion of local Euler obstruction, an invariant defined at each point of a singular variety using an index of an appropriate radial 1-form, and used this (among other things) to construct the homology Chern classes of singular varieties. Brasselet and Schwartz in [3] proved that the Alexander isomorphism  $H^*(M, M \setminus V) \cong H_*(V)$  carries the Schwartz classes into the MacPherson classes; a key ingredient for this proof is their *proportionality theorem* relating the Schwartz index and the local Euler obstruction.

These were the first indices of vector fields and 1-forms on singular spaces, in the literature. Later in [8] was introduced another index for vector fields on isolated hypersurface singularities, and this definition was extended in [23] to vector fields on complete intersection germs. This is known as the GSV-index and one of its main properties is that it is invariant under perturbations of both, the vector field and the functions that define the singular variety. The definition of this index was recently extended in [4] for vector fields with isolated singularities on

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hypersurface germs with non-isolated singularities, and it was proved that this index satisfies a proportionality property analogous to the one proved in [3] for the Schwartz index and the local Euler obstruction, the proportionality factor being now the Euler-Poincaré characteristic of a local Milnor fiber.

In [5] Ebeling and Gusein-Zade observed that when dealing with singular varieties, 1-forms have certain advantages over vector fields, as for instance the fact that for a vector field on the ambient space the condition of being tangent to a (stratified) singular variety is very stringent, while every 1-form on the ambient space defines, by restriction, one on the singular variety. They adapted the definition of the GSV-index to 1-forms on complete intersection germs with isolated singularities, and proved a very nice formula for it in the case when the form is holomorphic, generalizing the well-known formula of Lê-Greuel for the Milnor number of a function.

This article is about 1-forms on complex analytic varieties and it is particularly relevant when the variety has non-isolated singularities. We show in Section 2 how the radial extension technique of M.-H. Schwartz can be adapted to 1-forms, allowing us to define the Schwartz index of 1-forms with isolated singularities on singular varieties. Then we see (Section 3) how MacPherson's local Euler obstruction, adapted to 1forms in general, relates to the Schwartz index, thus obtaining a proportionality theorem for these indices analogous to the one in [3] for vector fields. We then extend (in Section 4) the definition of the GSV-index to 1-forms with isolated singularities on (local) complete intersections with non-isolated singularities that satisfy the Thom  $a_f$ -condition (which is always satisfied if the variety is a hypersurface), and we prove the corresponding proportionality theorem for this index. When the form is the differential of a holomorphic function h, this index measures the number of critical points of a generic perturbation of h on a local Milnor fiber, so it is analogous to invariants studied by a number of authors (see for instance [9, 11, 22]). Section 1 is a review of well-known facts about real and complex valued 1-forms.

The radial extension of 1-forms can be made global on compact varieties, and it can also be made for frames of differential 1-forms. One gets in this way the dual Schwartz classes of singular varieties, which equal the usual ones up to sign. One also has the dual Chern-Mather classes of V, already envisaged in [17], and the proportionality formula 3.3 can be used as in [3] to express the dual Chern-Mather classes as "weighted" dual Schwartz classes, the weights been given by the local Euler obstruction. Similarly, in analogy with Theorem 1.1 in [4], the corresponding GSV-index and the proportionality Theorem 4.4 extend to frames and can be used to express the dual Fulton-Johnson classes of singular hypersurfaces embedded with trivial normal bundle in compact complex manifolds, as "weighted" dual Schwartz classes, the weights been now given by the Euler-Poincaré characteristic of the local Milnor fiber.

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#### $\S1$ . Some basic facts about 1-forms

In this section we study some basic facts about the geometry of 1-forms and the relation between real and complex valued 1-forms on (almost) complex manifolds, which plays an important role in the sequel. The material here is all contained in the literature; we include it for completeness and to set up our notation with no possible ambiguities. We give precise references when appropriate.

Let M be an almost complex manifold of real dimension 2m > 0. Let TM be its complex tangent bundle. We denote by  $T^*M$  the cotangent bundle of M, dual of TM; each fiber  $(T^*M)_x$  consists of the  $\mathbb{C}$ -linear maps  $TM_x \to \mathbb{C}$ . Similarly, we denote by  $T_{\mathbb{R}}M$  the underlying real tangent bundle of M; it is a real vector bundle of fiber dimension 2m, endowed with a canonical orientation. Its dual  $T^*_{\mathbb{R}}M$  has as fiber the  $\mathbb{R}$ -linear maps  $(T_{\mathbb{R}}M)_x \to \mathbb{R}$ .

**1.1 Definition.** Let A be a subset of M. By a real (valued) 1form  $\eta$  on A we mean the restriction to A of a continuous section of the bundle  $T^*_{\mathbb{R}}M$ , i.e., for each  $x \in A$ ,  $\eta_x$  is an  $\mathbb{R}$ -linear map  $(T_{\mathbb{R}}M)_x \to \mathbb{R}$ . We usually drop the word "valued" and say simply real 1-forms on A. Similarly, a complex 1-form  $\omega$  on A means the restriction to A of a continuous section of the bundle  $T^*M$ , i.e., for each  $x \in A$ ,  $\omega_x$  is a  $\mathbb{C}$ -linear map  $(TM)_x \to \mathbb{C}$ .

Notice that the kernel of a real form  $\eta$  at a point x is either the whole fiber  $(T_{\mathbb{R}}M)_x$  or a real hyperplane in it. In the first case we say that x is a singular point (or zero) of  $\eta$ . In the second case the kernel

ker  $\eta_x$  splits  $(T_{\mathbb{R}}M)_x$  in two half spaces  $(T_{\mathbb{R}}M^{\pm})_x$ ; in one of these the form takes positive values and negative in the other.

We recall that a vector field v in  $\mathbb{R}^N$  is radial at a point  $x_o$  if it is transverse to every sufficiently small sphere around  $x_o$  in  $\mathbb{R}^N$ . The duality between real 1-forms and vector fields assigns to each tangent vector  $\partial/\partial x_i$  the form  $dx_i$  (extending it by linearity to all tangent vectors). This refines the classical duality that assigns to each hyperplane in  $\mathbb{R}^N$  the line orthogonal to it and motivates the following definition (c.f. [5, 6]):

**1.2 Definition.** A real 1-form  $\eta$  on M is **radial** (outwards-pointing) at a point  $x_o \in M$  if, locally, it is dual over  $\mathbb{R}$  to a radial outwardspointing vector field at  $x_o$ . Inwards-pointing radial vector fields are defined similarly.

In other words,  $\eta$  is **radial** at a point  $x_o$  if it is everywhere positive when evaluated in some radial vector field at  $x_o$ .

Thus, for instance, if for a fixed  $x_o \in M$  we let  $\rho_{x_o}(x)$  be the function  $||x - x_o||^2$  (for some Riemmanian metric), then its differential is a radial form.

**1.3 Remark.** The concept of radial forms was introduced in [5]. In [6] radial forms are defined using more relaxed conditions than we do here. However this is a concept inspired by the corresponding notion of radial vector fields, so we use Definition 1.2.

A complex 1-form  $\omega$  on  $A \subset M$  can be written in terms of its real and imaginary parts:

$$\omega = \operatorname{Re}(\omega) + i \operatorname{Im}(\omega).$$

Both  $\operatorname{Re}(\omega)$  and  $\operatorname{Im}(\omega)$  are real 1-forms, and the linearity of  $\omega$  implies that for each tangent vector one has:

$$\operatorname{Im}(\omega)(v) = -\operatorname{Re}(\omega)(iv),$$

thus

$$\omega(v) = \operatorname{Re}(\omega)(v) - i\operatorname{Re}(\omega)(iv).$$

In other words the form  $\omega$  is determined by its real part and one has a 1to-1 correspondence between real and complex forms, assigning to each complex form its real part, and conversely, to a real 1-form  $\eta$  corresponds the complex form  $\omega$  defined by:

$$\omega(v) = \eta(v) - i\eta(iv).$$

This statement (noted in [6]) refines the obvious fact that a complex hyperplane P in  $\mathbb{C}^m$ , say defined by a linear form H, is the intersection of the real hyperplanes  $\hat{H} := \{\operatorname{Re} H = 0\}$  and  $i\hat{H}$ . This justifies the following definition:

**1.4 Definition.** A complex 1-form  $\omega$  is **radial** at a point  $x \in M$  if its real part is radial at x.

Recall that the Euler class of an oriented vector bundle is the primary obstruction to constructing a non-zero section [24]. In the case of the bundle  $T_{\mathbb{R}}^*M$ , this class equals the Euler class Eu(M) of the underlying real tangent bundle  $T_{\mathbb{R}}M$ , since they are isomorphic. Thus, if M is compact then its Euler class evaluated on the orientation cycle of M gives the Euler-Poincaré characteristic  $\chi(M)$ . We can say this in different words: let  $\eta$  be a real 1-form on M with isolated (hence finitely many) singularities  $x_1, \ldots, x_r$ . At each  $x_i$  this 1-form defines a map,  $\mathbb{S}_{\varepsilon} \xrightarrow{\eta/||\eta||} \mathbb{S}^{2m-1}$ , from a small sphere in M around  $x_i$  into the unit sphere in the fiber  $(T_{\mathbb{R}}^*M)_x$ . The degree of this map is the **Poincaré-Hopf** local index of  $\eta$  at  $x_i$ , that we may denote by  $\mathrm{Ind}_{\mathrm{PH}}(\eta, x_i)$ . Then the total index of  $\eta$  in M is by definition the sum of its local indices at the  $x_i$  and it equals  $\chi(M)$ . Its Poincaré dual class in  $H^{2m}(M)$  is the Euler class of  $T_{\mathbb{R}}^*M \cong T_{\mathbb{R}}M$ .

More generally, if M is a compact,  $C^{\infty}$  manifold of real dimension 2m with non-empty boundary  $\partial M$  and a complex structure in its tangent bundle, one can speak of real and complex valued 1-forms as above. Elementary obstruction theory (see [24]) implies that one can always find real and complex 1-forms on M with isolated singularities, all contained in the interior of M. In fact, if a real 1-form  $\eta$  is defined in a neighborhood of  $\partial M$  in M and it is non-singular there, then we can always extend it to the interior of M with finitely many singularities, and its total index in M does not depend on the choice of the extension.

**1.5 Definition.** Let M be an almost complex manifold with boundary  $\partial M$  and let  $\omega$  be a (real or complex) 1-form on M, nonsingular on a neighborhood of  $\partial M$ ; let  $\operatorname{Re} \omega$  be its real part if  $\omega$  is a complex form, otherwise  $\operatorname{Re} \omega = \omega$  for real forms. The form  $\omega$  is **radial** at the boundary if for each vector  $v(x) \in TM$ ,  $x \in \partial M$ , which is normal to the boundary (for some metric), pointing outwards of M, one has  $\operatorname{Re} \omega(v(x)) > 0$ .

By the theorem of Poincaré-Hopf for manifolds with boundary, if a real 1-form  $\eta$  is radial at the boundary and M is compact, then the total index of  $\eta$  is  $\chi(M)$ .

We now make similar considerations for complex 1-forms. We let M be a compact,  $C^{\infty}$  manifold of real dimension 2m (with or without boundary  $\partial M$ ), with a complex structure in its tangent bundle TM. Let  $T^*M$  be as before, the cotangent bundle of M, i.e., the bundle of complex valued continuous 1-forms. The top Chern class  $c^m(T^*M)$  is the primary obstruction to constructing a section of this bundle, i.e., if M has empty boundary, then  $c^m(T^*M)$  is the number of points, counted with their local indices, of the zeroes of a section  $\omega$  of  $T^*M$  (i.e., a complex 1-form) with isolated singularities (i.e., points where it vanishes). It is well known (see for instance [16]) that one has:

$$c^m(T^*M) = (-1)^m c^m(TM).$$

This corresponds to the fact that at each isolated singularity  $x_i$  of  $\omega$  one has two local indices: one of them is the index of its real part defined as above,  $\operatorname{Ind}_{PH}(\operatorname{Re} \omega, x_i)$ ; the other is the degree of the map  $\mathbb{S}_{\varepsilon} \xrightarrow{\omega/||\omega||} \mathbb{S}^{2m-1}$ , that we denote by  $\operatorname{Ind}_{PH}(\omega, x_i)$ . These two indices are related by the equality:

$$\operatorname{Ind}_{\operatorname{PH}}(\omega, x_i) = (-1)^m \operatorname{Ind}_{\operatorname{PH}}(\operatorname{Re} \omega, x_i),$$

and the index on the right corresponds to the local Poincaré-Hopf index of the vector field defined by duality near  $x_i$ . For instance, the form  $\omega = \sum z_i dz_i$  in  $\mathbb{C}^m$  has index 1 at 0, while its real part  $\sum (x_i dx_i - y_i dy_i)$ has index  $(-1)^m$ .

If we take M as above, compact and with possibly non-empty boundary, and  $\omega$  is a complex 1-form with isolated singularities in the interior of M and radial on the boundary, then (by the previous considerations) the total index of  $\omega$  in M is  $(-1)^m \chi(M)$ . We summarize some of the previous discussion in the following theorem (c.f. [5, 6]):

**1.6 Theorem.** Let M be a compact,  $C^{\infty}$  manifold of real dimension 2m (with or without boundary  $\partial M$ ), with a complex structure in its tangent bundle TM. Let  $T^*_{\mathbb{R}}M$  and  $T^*M$  be as before, the bundles of real and complex valued continuous 1-forms on M, respectively. Then: i) Every real 1-form  $\eta$  on M determines a complex 1-form  $\omega$  by the formula

$$\omega(v) = \eta(v) - i\eta(iv);$$

so the real part of  $\omega$  is  $\eta$ .

**ii)** The local Poincaré-Hopf indices at an isolated singularity of a complex 1-form and its real part are related by:

$$\operatorname{Ind}_{\operatorname{PH}}(\omega, x_i) = (-1)^m \operatorname{Ind}_{\operatorname{PH}}(\operatorname{Re} \omega, x_i).$$

iii) If a real 1-form on M is radial at the boundary  $\partial M$ , then its total Poincaré-Hopf index in M is  $\chi(M)$ . In particular, a radial real 1-form has local index 1.

**vi)** If a complex 1-form on M is radial at the boundary  $\partial M$ , then its total Poincaré-Hopf index in M is  $(-1)^m \chi(M)$ .

1.7 Remark. One may consider frames of complex 1-forms on M instead of a single 1-form. This means considering sets of k complex 1-forms, whose singularities are the points where these forms become linearly dependent over  $\mathbb{C}$ . By definition (see [24]) the primary obstruction to constructing such a frame is the Chern class  $c^{m-k+1}(T^*M)$ , so these classes also have an expression similar to 1.6 but using indices of frames of 1-forms. One always has  $c^i(T^*M) = (-1)^i c^i(TM)$ . Thus the Chern classes, and all the Chern numbers of M, can be computed using indices of either vector fields or 1-forms.

#### $\S$ 2. Radial extension and the Schwartz index

In the sequel we will be interested in considering forms defined on singular varieties in a complex manifold, so we introduce some standard notation. Let V be a reduced, equidimensional complex analytic space of dimension n in a complex manifold M of dimension m, endowed with a Whitney stratification  $\{V_{\alpha}\}$  adapted to V, i.e., V is a union of strata.

The following definition is an immediate extension for 1-forms of the corresponding (standard) definition for functions on stratified spaces in terms of its differential (c.f. [6, 7, 12]).

**2.1 Definition.** Let  $\omega$  be a (real or complex) 1-form on V, i.e., a continuous section of either  $T^*_{\mathbb{R}}M|_V$  or  $T^*M|_V$ . A singularity of  $\omega$  with respect to the Whitney stratification  $\{V_\alpha\}$  means a point x where the kernel of  $\omega$  contains the tangent space of the corresponding stratum.

This means that the pull back of the form to  $V_{\alpha}$  vanishes at x.

In Section 1 we introduced the notion of radial forms, which is dual to the "radiality" for vector fields. We now extend this notion relaxing the condition of radiality in the directions tangent to the strata. From now on, unless it is otherwise stated explicitly, by a singularity of a 1-form on V we mean a singularity in the stratified sense, i.e., in the sense of 2.1.

**2.2 Definition.** Let  $\omega$  be a (real or complex) 1-form on V. The form is **normally radial** at a point  $x_o \in V_\alpha \subset V$  if it is radial when restricted to vectors which are not tangent to the stratum  $V_\alpha$  that contains  $x_o$ . In other words, for every vector v(x) tangent to M at a point  $x \notin V_\alpha$ , x sufficiently close to  $x_o$  and v(x) pointing outwards a tubular

neighborhood of the stratum  $V_{\alpha}$ , one has  $\operatorname{Re} \omega(v) > 0$  (or  $\operatorname{Re} \omega(v) < 0$  for all such vectors; if  $\omega$  is real then it equals  $\operatorname{Re} \omega$ ).

Obviously a radial 1-form is also normally radial, since it is radial in all directions.

For each point x in a stratum  $V_{\alpha}$ , one has a neighborhood  $U_x$  of x in M which is diffeomorphic to the product  $U_{\alpha} \times \mathbb{D}_{\alpha}$ , where  $U_{\alpha} = U_x \cap V_{\alpha}$  and  $\mathbb{D}_{\alpha}$  is a small disc in M transverse to  $V_{\alpha}$ . Let  $\pi$  be the projection  $\pi: U_x \to U_{\alpha}$  and p the projection  $p: U_x \to \mathbb{D}_{\alpha}$ . One has an isomorphism:

$$T^*U_x \cong \pi^*T^*U_\alpha \oplus p^*T^*\mathbb{D}_\alpha.$$

That a (real or complex) 1-form  $\omega$  be normally radial at x means that up to a local change of coordinates in M,  $\omega$  is the direct sum of the pull back of a (real or complex) form on  $U_{\alpha}$ , i.e., a section of the (real or complex) cotangent bundle  $T^*U_{\alpha}$ , and a section of the (real or complex) cotangent bundle  $T^*\mathbb{D}_{\alpha}$  which is a (real or complex) radial form in the disc.

It is possible to make for 1-forms the classical construction of **radial** extension introduced by M.-H. Schwartz in [20, 21] for stratified vector fields and frames. Locally, the construction can be described as follows. We consider first real 1-forms. Let  $\eta$  be a 1-form on  $U_{\alpha}$ , denote by  $\hat{\eta}$ its pull back to a section of  $\pi^*T^*_{\mathbb{R}}U_{\alpha}$ . This corresponds to the **parallel** extension of stratified vector fields done by Schwartz. Now look at the function  $\rho$  given by the square of the distance to the origin in  $\mathbb{D}_{\alpha}$ . The form  $p^*d\rho$  on  $U_x$  vanishes on  $U_{\alpha}$  and away from  $U_{\alpha}$  its kernel is transverse to the strata of V by Whitney conditions. The sum  $\eta' = \hat{\eta} + p^*d\rho$  defines a normally radial 1-form on  $U_x$  which coincides with  $\eta$  on  $U_{\alpha}$ ; away from  $U_{\alpha}$  its kernel is transverse to the strata of V. Thus, if  $\eta$  is non-singular at x, then  $\eta'$  is non-singular everywhere on  $U_x$ . If  $\eta$  has an isolated singularity at  $x \in V_{\alpha}$ , then  $\eta'$  also has an isolated singularity there. In particular, if the dimension of the stratum  $V_{\alpha}$  is zero then  $\eta'$  is a radial form in the sense of Section 1.

Following the terminology of [20, 21] we say that the form  $\eta'$  is obtained from  $\eta$  by **radial extension**. Since the index in M of a normally radial form is its index in the stratum times the index of a radial form in the disc  $\mathbb{D}_{\alpha}$ , we obtain the following important property of forms constructed by radial extension.

**2.3 Proposition.** Let  $\eta$  be a real 1-form on the stratum  $V_{\alpha}$  with an isolated singularity at a point x with local Poincaré-Hopf index  $\operatorname{Ind}_{PH}(\eta, V_{\alpha}; x)$ . Let  $\eta'$  the 1-form on a neighborhood of x in M obtained by radial extension. Then the index of  $\eta$  in the stratum equals the index of  $\eta'$  in M:

$$\operatorname{Ind}_{\operatorname{PH}}(\eta, V_{\alpha}; x) = \operatorname{Ind}_{\operatorname{PH}}(\eta', M; x).$$

**2.4 Definition.** The Schwartz index of the continuous real 1form  $\eta$  at an isolated singularity  $x \in V_{\alpha} \subset V$ , denoted  $\operatorname{Ind}_{\operatorname{Sch}}(\eta, V; x)$ , is the Poincaré-Hopf index of the 1-form  $\eta'$  obtained from  $\eta$  by radial extension; or equivalently, if the stratum of x has dimension more than 0,  $\operatorname{Ind}_{\operatorname{Sch}}(\eta, V; x)$  is the Poincaré-Hopf index at x of  $\eta$  in the stratum  $V_{\alpha}$ .

If x is an isolated singularity of V then every 1-form on V must be singular at x since its kernel contains the "tangent space" of the stratum. In this case the index of the form in the stratum is defined to be 1, and this is consistent with the previous definition since in this case the radial extension of  $\eta$  is actually radial at x, so it has index 1 in the ambient space.

The previous process is easily adapted to give radial extension for complex 1-forms. Let  $\omega$  be such a form on  $V_{\alpha}$ ; let  $\eta$  be its real part. We extend  $\eta$  as above, by radial extension, to obtain a real 1-form  $\eta'$ which is normally radial at x. Then we use statement i) in Theorem 1.6 above to obtain a complex 1-form  $\omega'$  on  $U_x$  that extends  $\omega$  and is also normally radial at x. If we prefer, we can make this process in a different but equivalent way: first make a parallel extension of  $\omega$  to  $U_x$ as above, using the projection  $\pi$ ; denote by  $\hat{\omega}$  this complex 1-form. Now use 1.6.i) to define a complex 1-form  $\hat{d\rho}$  on  $U_x$  whose real part is  $d\rho$ , and take the direct sum of  $\hat{\omega}$  and  $\hat{d\rho}$  at each point to obtain the extension  $\omega'$ . We say that  $\omega'$  is obtained from  $\omega$  by **radial extension**.

We have the equivalent of Proposition 2.3 for complex forms, modified with the appropriate signs:

$$(-1)^{s} \operatorname{Ind}_{\operatorname{PH}}(\omega, V_{\alpha}; x) = (-1)^{m} \operatorname{Ind}_{\operatorname{PH}}(\omega', M; x),$$

where 2s is the real dimension of  $V_{\alpha}$  and 2m that of M.

**2.5 Definition.** The Schwartz index of the continuous complex 1-form  $\omega$  at an isolated singularity  $x \in V_{\alpha} \subset V$ , denoted  $\operatorname{Ind}_{\operatorname{Sch}}(\omega, V; x)$ , is  $(-1)^n$ -times the index of its real part:

$$\operatorname{Ind}_{\operatorname{Sch}}(\omega, V; x) = (-1)^n \operatorname{Ind}_{\operatorname{Sch}}(\operatorname{Re} \omega, V; x).$$

#### $\S$ 3. Local Euler obstruction and the Proportionality Theorem

We are now concerned only with a local situation, so we take V to be embedded in an open ball  $\mathbb{B} \subset \mathbb{C}^m$  centered at the origin 0. On the regular part of V one has the map  $\sigma: V_{\text{reg}} \to G_{n,m}$  into the Grassmannian of  $n(= \dim V)$ -planes in  $\mathbb{C}^m$ , that assigns to each point the corresponding tangent space of  $V_{\text{reg}}$ . The Nash blow up  $\widetilde{V} \xrightarrow{\nu} V$  of V is by definition the closure in  $\mathbb{B} \times G_{n,m}$  of the graph of the map  $\sigma$ . One also has the Nash bundle  $\widetilde{T} \xrightarrow{p} \widetilde{V}$ , restriction to  $\widetilde{V}$  of the tautological bundle over  $\mathbb{B} \times G_{n,m}$ .

The corresponding dual bundles of complex and real 1-forms are denoted by  $\widetilde{T}^* \xrightarrow{p} \widetilde{V}$  and  $\widetilde{T}^*_{\mathbb{R}} \xrightarrow{p} \widetilde{V}$ , respectively. Observe that a point in  $\widetilde{T}^*$  is a triple  $(x, P, \omega)$  where x is in V, P is an n-plane in the tangent space  $T_x \mathbb{B}$  which is limit of a sequence  $\{(TV_{\text{reg}})_{x_i}\}$ , where the  $x_i$  are points in the regular part of V converging to x, and  $\omega$  is a  $\mathbb{C}$ -linear map  $P \to \mathbb{C}$ . (Similarly for  $\widetilde{T}^*_{\mathbb{R}}$ .)

Let us denote by  $\rho$  the function given by the square of the distance to 0. We recall that MacPherson in [15] observed that the Whitney condition (a) implies that the pull-back of the differential  $d\rho$  defines a never-zero section  $\widetilde{d\rho}$  of  $\widetilde{T}^*_{\mathbb{R}}$  over  $\nu^{-1}(\mathbb{S}_{\varepsilon} \cap V) \subset \widetilde{V}$ , where  $\mathbb{S}_{\varepsilon}$  is the boundary of a small ball  $\mathbb{B}_{\varepsilon}$  in  $\mathbb{B}$  centered at 0. The obstruction to extending  $\widetilde{d\rho}$  as a never-zero section of  $\widetilde{T}^*_{\mathbb{R}}$  over  $\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V) \subset \widetilde{V}$  is a cohomology class in  $H^{2n}(\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V), \nu^{-1}(\mathbb{S}_{\varepsilon} \cap V);\mathbb{Z})$ , and MacPherson defined **the local Euler obstruction**  $\operatorname{Eu}_V(0)$  of V at 0 to be the integer obtained by evaluating this class on the orientation cycle  $[\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V), \nu^{-1}(\mathbb{S}_{\varepsilon} \cap V)]$ .

More generally, given a section  $\eta$  of  $T^*_{\mathbb{R}}\mathbb{B}|_A$ ,  $A \subset V$ , there is a canonical way of constructing a section  $\tilde{\eta}$  of  $\tilde{T}^*_{\mathbb{R}}|_{\tilde{A}}$ ,  $\tilde{A} = \nu^{-1}A$ , which is described in the following. The same construction works for complex forms. First, taking the pull-back  $\nu^*\eta$ , we get a section of  $\nu^*T^*_{\mathbb{R}}\mathbb{B}|_V$ . Then  $\tilde{\eta}$  is obtained by projecting  $\nu^*\eta$  to a section of  $\tilde{T}^*_{\mathbb{R}}$  by the canonical bundle homomorphism

$$\nu^* T^*_{\mathbb{R}} \mathbb{B}|_V \longrightarrow \tilde{T}^*_{\mathbb{R}}.$$

Thus the value of  $\tilde{\eta}$  at a point (x, P) is simply the restriction of the linear map  $\eta(x): (T_{\mathbb{R}}\mathbb{B})_x \to \mathbb{R}$  to P. We call  $\tilde{\eta}$  the **canonical lifting** of  $\eta$ .

By the Whitney condition (a), if  $a \in V_{\alpha}$  is the limit point of the sequence  $\{x_i\} \in V_{\text{reg}}$  such that  $P = \lim(TV_{\text{reg}})_{x_i}$  and if the kernel of  $\eta$ is transverse to  $V_{\alpha}$ , then the linear form  $\tilde{\eta}$  will be non-vanishing on P. Thus, if  $\eta$  has an isolated singularity at the point  $0 \in V$  (in the stratified sense), then we have a never-zero section  $\tilde{\eta}$  of the dual Nash bundle  $\tilde{T}^*_{\mathbb{R}}$ over  $\nu^{-1}(\mathbb{S}_{\varepsilon} \cap V) \subset \tilde{V}$ . Let  $o(\eta) \in H^{2n}(\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V), \nu^{-1}(\mathbb{S}_{\varepsilon} \cap V); \mathbb{Z})$  be the cohomology class of the obstruction cycle to extend this to a section of  $\tilde{T}^*_{\mathbb{R}}$  over  $\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V)$ . Then define (c.f. [2, 6]): **3.1 Definition.** The local Euler obstruction of the real differential form  $\eta$  at an isolated singularity is the integer  $\operatorname{Eu}_V(\eta, 0)$  obtained by evaluating the obstruction cohomology class  $o(\eta)$  on the orientation cycle  $[\nu^{-1}(\mathbb{B}_{\varepsilon} \cap V), \nu^{-1}(\mathbb{S}_{\varepsilon} \cap V)]$ .

The local Euler obstruction  $\operatorname{Eu}_V(0)$  of MacPherson corresponds to taking the differential of the squared function distance to 0. In the complex case, one can perform the same construction, using the corresponding complex bundles. If  $\omega$  is a complex differential form, section of  $T^*\mathbb{B}|_A$ with an isolated singularity, one can define the local Euler obstruction  $\operatorname{Eu}_V(\omega, 0)$ . Notice that it is equal to that of its real part up to sign:

(3.2) 
$$\operatorname{Eu}_V(\omega, 0) = (-1)^n \operatorname{Eu}_V(\operatorname{Re}\omega, 0).$$

This is an immediate consequence of the relation between the Chern classes of a complex vector bundle and those of its dual (see for instance [16]).

We note that the idea to consider the (complex) dual Nash bundle was already present in [17], where Sabbah introduces a local Euler obstruction  $\mathrm{E}\check{u}_V(0)$  that satisfies  $\mathrm{E}\check{u}_V(0) = (-1)^n \mathrm{E}\mathrm{u}_V(0)$ . See also Schürmann [18], sec. 5.2.

Just as for vector fields (see [3]), one has in this situation the following:

**3.3 Theorem.** Let  $V_{\alpha} \subset V$  be the stratum containing 0,  $\operatorname{Eu}_{V}(0)$  the local Euler obstruction of V at 0 and  $\omega$  a (real or complex) 1-form on  $V_{\alpha}$  with an isolated singularity at 0. Then the local Euler obstruction of the radial extension  $\omega'$  of  $\omega$  and the Schwartz index of  $\omega$  at 0 are related by the following proportionality formula:

$$\operatorname{Eu}_V(\omega', 0) = \operatorname{Eu}_V(0) \cdot \operatorname{Ind}_{\operatorname{Sch}}(\omega, V; 0).$$

*Proof.* By 3.2 and Theorem 1.6 above, it is enough to prove 3.3 for either real or complex 1-forms, each case implying the other. We prove it for real forms.

Let  $\omega$  and  $\omega'$  be as above. Also, let  $\eta_{\rm rad}$  denote a real radial form at 0.

By construction and definition, we have

(3.4) 
$$\operatorname{Ind}_{\operatorname{PH}}(\omega, V_{\alpha}; 0) = \operatorname{Ind}_{\operatorname{PH}}(\omega', \mathbb{B}; 0) = \operatorname{Ind}_{\operatorname{Sch}}(\omega, V; 0).$$

By definition of  $\operatorname{Ind}_{PH}(\omega', \mathbb{B}; 0)$ , there is a homotopy

$$\Psi\colon [0,\,1]\times\mathbb{S}_{\varepsilon}\longrightarrow T^*_{\mathbb{R}}\mathbb{B}|_{\mathbb{S}_{\varepsilon}}$$

such that its image satisfies:

(3.5) 
$$\partial \operatorname{Im} \Psi = \omega'(\mathbb{S}_{\varepsilon}) - \operatorname{Ind}_{\operatorname{PH}}(\omega', \mathbb{B}; 0) \cdot \eta_{\operatorname{rad}}(\mathbb{S}_{\varepsilon})$$

as chains in  $T^*_{\mathbb{R}}\mathbb{B}|_{\mathbb{S}_{\varepsilon}}$ . The restriction of  $\Psi$  gives a homotopy

$$\psi \colon [0, 1] \times (\mathbb{S}_{\varepsilon} \cap V) \longrightarrow T^*_{\mathbb{R}} \mathbb{B}|_{\mathbb{S}_{\varepsilon} \cap V}$$

such that (c.f. (3.4))

$$\partial \operatorname{Im} \psi = \omega'(\mathbb{S}_{\varepsilon} \cap V) - \operatorname{Ind}_{\operatorname{Sch}}(\omega, V; 0) \cdot \eta_{\operatorname{rad}}(\mathbb{S}_{\varepsilon} \cap V).$$

Now we can lift  $\psi$ ,  $\omega'$  and  $\eta_{\rm rad}$  to sections  $\nu^*\psi$ ,  $\nu^*\omega'$  and  $\nu^*\eta_{\rm rad}$  of  $\nu^*T^*_{\mathbb{R}}\mathbb{B}$  to get a homotopy

$$\nu^*\psi\colon [0,\,1]\times\nu^{-1}(\mathbb{S}_{\varepsilon}\cap V)\longrightarrow\nu^*T^*_{\mathbb{R}}\mathbb{B}|_{\nu^{-1}(\mathbb{S}_{\varepsilon}\cap V)}$$

and, since  $\nu$  is an isomorphism away from the singularity of V, we still have

$$\partial \operatorname{Im} \nu^* \psi = \nu^* \omega' (\nu^{-1}(\mathbb{S}_{\varepsilon} \cap V)) - \operatorname{Ind}_{\operatorname{Sch}}(\omega, V; 0) \cdot \nu^* \eta_{\operatorname{rad}}(\nu^{-1}(\mathbb{S}_{\varepsilon} \cap V))$$

as chains in  $\nu^* T^*_{\mathbb{R}} \mathbb{B}|_{\nu^{-1}(\mathbb{S}_{\varepsilon} \cap V)}$ . Recall that we get the canonical liftings  $\tilde{\psi}, \tilde{\omega}'$  and  $\tilde{\eta}_{rad}$  of  $\psi, \omega'$  and  $\eta_{rad}$  by taking the images of  $\nu^* \psi, \nu^* \omega'$  and  $\nu^* \eta_{rad}$  by the canonical bundle homomorphism  $\nu^* T^*_{\mathbb{R}} \mathbb{B} \longrightarrow \tilde{T}^*_{\mathbb{R}}$ . Thus we have

$$\partial \operatorname{Im} \tilde{\psi} = \tilde{\omega}'(\nu^{-1}(\mathbb{S}_{\varepsilon} \cap V)) - \operatorname{Ind}_{\operatorname{Sch}}(\omega, \, V; 0) \cdot \tilde{\eta}_{\operatorname{rad}}(\nu^{-1}(\mathbb{S}_{\varepsilon} \cap V))$$

as chains in  $\tilde{T}^*_{\mathbb{R}}|_{\nu^{-1}(\mathbb{S}_{\varepsilon}\cap V)}$ . The sections  $\tilde{\omega}'$  and  $\tilde{\eta}_{rad}$  are non-vanishing on  $\nu^{-1}(\mathbb{S}_{\varepsilon}\cap V)$ , by the Whitney condition, and by definition of the Euler obstructions, we have the theorem by the Whitney condition, and by definition of the Euler obstructions, we have the theorem. Q.E.D.

#### $\S4.$ The GSV-index

We recall ([8, 23]) that the GSV-index of a vector field v on an isolated complete intersection germ V can be defined to be the Poincaré-Hopf index of an extension of v to a Milnor fiber F. Similarly, the GSVindex of a 1-form  $\omega$  on V can be defined to be the Poincaré-Hopf index of the form on F, i.e., the number of singularities of  $\omega$  in F counted with multiplicities [5]. When V has non-isolated singularities one may not have a Milnor fibration in general, but one does if V has a Whitney stratification with Thom's  $a_f$ -condition,  $f = (f_1, \ldots, f_k)$  being the functions that define V (c.f. [13, 14, 4]). Let (V, 0) be a complete intersection of complex dimension n defined in a ball  $\mathbb{B}$  in  $\mathbb{C}^{n+k}$  by functions  $f = (f_1, \ldots, f_k)$ , and assume 0 is a singular point of V (not necessarily an isolated singularity). As before, we endow  $\mathbb{B}$  with a Whitney stratification adapted to V, and we assume that we can choose  $\{V_{\alpha}\}$  so that it satisfies the  $a_f$ -condition of Thom (see for instance [14]). In particular one always has such a stratification if k = 1, by [10].

Let  $\omega$  be as before, a (real or complex) 1-form on  $\mathbb{B}$ , and assume its restriction to V has an isolated singularity at 0. This means that the kernel of  $\omega(0)$  contains the tangent space of the stratum  $V_{\alpha}$  containing 0, but everywhere else it is transverse to each stratum  $V_{\alpha} \subset V$ . Now let  $F = F_t$  be a Milnor fiber of V, i.e.,  $F = f^{-1}(t) \cap \mathbb{B}_{\varepsilon}$ , where  $\mathbb{B}_{\varepsilon}$  is a sufficiently small ball in  $\mathbb{B}$  around 0 and  $t \in \mathbb{C}^k$  is a regular value of f with ||t|| sufficiently small with respect to  $\varepsilon$ . Notice that the  $a_f$ -condition implies that for every sequence  $t_n$  of regular values converging to 0, and for every sequence  $\{x_n\}$  of points in the corresponding Milnor fibers converging to a point  $x_o \in V$  so that the sequence of tangent spaces  $\{(TF)_{x_n}\}$  has a limit T, one has that T contains the space  $(TV_{\alpha})_{x_n}$ , tangent to the stratum that contains  $x_o$ . By transversality this implies that choosing the regular value t sufficiently close to 0 we can assure that the kernel of  $\omega$  is transverse to the Milnor fiber at every point in its boundary  $\partial F$ . Thus its pull-back to F is a 1-form on this smooth manifold, and it is never-zero on its boundary, thus  $\omega$  has a well defined Poincaré-Hopf index in F as in Section 1. This index is well-defined and depends only on the restriction of  $\omega$  to V and the topology of the Milnor fiber F, which is well-defined once we fix the defining function f (which is assumed to satisfy the  $a_f$ -condition for some Whitney stratification).

**4.1 Definition.** The GSV-index of  $\omega$  at  $0 \in V$  relative to f,  $\operatorname{Ind}_{GSV}(\omega, 0)$ , is the Poincaré-Hopf index of  $\omega$  in F.

In other words this index measures the number of points (counted with signs) in which a generic perturbation of  $\omega$  is tangent to F. In fact the inclusion  $F \xrightarrow{i} M$  pulls the form  $\omega$  to a section of the (real or complex, as the case may be) cotangent bundle of F, which is never-zero near the boundary because  $\omega$  has an isolated singularity at 0 and, by hypothesis, the map f satisfies the  $a_f$ -condition of Thom. If the form  $\omega$ is real then

(4.2) 
$$\operatorname{Ind}_{\mathrm{GSV}}(\omega, 0) = \operatorname{Eu}(F; \omega)[F],$$

where  $\operatorname{Eu}(F; \omega) \in H^{2n}(F, \partial F)$  is the Euler class of the real cotangent bundle  $T^*_{\mathbb{R}}F$  relative to the section defined by  $\omega$  on the boundary, and [F] is the orientation cycle of the pair  $(F, \partial F)$ . If  $\omega$  is a complex form, then one has:

(4.3.i) 
$$\operatorname{Ind}_{\mathrm{GSV}}(\omega, 0) = c^n (T^*F; \omega)[F],$$

where  $c^n(T^*F;\omega)$  is the top Chern class of the cotangent bundle of F relative to the form  $\omega$  on its boundary. In this case one can, alternatively, express this index as the relative Chern class:

(4.3.ii) 
$$\operatorname{Ind}_{\mathrm{GSV}}(\omega, 0) = c^n (T^* M|_F; \Omega)[F],$$

where  $\Omega$  is the frame of k + 1 complex 1-forms on the boundary of F given by

$$\Omega = (\omega, df_1, df_2, \ldots, df_k),$$

since the forms  $(df_1, \ldots, df_k)$  are linearly independent everywhere on F. Notice that if the form  $\omega$  is holomorphic, then this index is necessarily non-negative because it can be regarded as an intersection number of complex submanifolds. For every complex 1-form one has:

$$\operatorname{Ind}_{\operatorname{GSV}}(\omega, 0) = (-1)^n \operatorname{Ind}_{\operatorname{GSV}}(\operatorname{Re}\omega, 0).$$

We remark that if V has an isolated singularity at 0, this is the index envisaged in [5], i.e., the degree of the map from the link K of V into the Stiefel manifold of complex (k + 1)-frames in the dual  $\mathbb{C}^{n+k}$  given by the map  $(\omega, df_1, \ldots, df_k)$ . Also notice that this index is somehow dual to the index defined in [4] for vector fields, which is related to the top Fulton-Johnson class of singular hypersurfaces.

So, given the (non-isolated) complete intersection singularity (V, 0)and a (real or complex) 1-form  $\omega$  on V with an isolated singularity at 0, one has three different indices: the Euler obstruction (Section 2), the GSV-index just defined and the index of its pull back to a 1-form on the stratum of 0. One also has the index of the form in the ambient manifold M. For forms obtained by radial extension, the index in the stratum equals its index in M, and this is by definition the Schwartz index. The following proportionality theorem is analogous to the one in [4] for vector fields.

**4.4 Theorem.** Let  $\omega$  be a (real or complex) 1-form on the stratum  $V_{\alpha}$  of 0 with an isolated singularity at 0. Then the GSV index of its radial extension  $\omega'$  is proportional to the Schwartz index, the proportionality factor being the Euler-Poincaré characteristic of the Milnor fiber F:

$$\operatorname{Ind}_{\operatorname{GSV}}(\omega', 0) = \chi(F) \cdot \operatorname{Ind}_{\operatorname{Sch}}(\omega, V; 0).$$

*Proof.* It is enough to prove 4.4 either for complex forms or for real forms, each one implying the other. The proof is similar to that of 3.3. Let  $\omega'$  and  $\eta_{\rm rad}$  be as in the proof of Theorem 3.3. Then 4.4 is proved by taking the retriction to F of each section in (3.5) as a differential form, noting that  $\operatorname{Ind}_{\rm GSV}(\eta_{\rm rad}, 0) = \chi(F)$ . Q.E.D.

4.5 Remark. We notice that 4.4 and 3.3 can also be proved using the stability of the index under perturbations; this works for vector fields too. More precisely, one can easily show that the Euler obstruction  $\operatorname{Eu}_V(\omega, x)$  and the GSV-index are stable when we perturb the 1-form (or the vector field) in the stratum and then extend it radially; then the sum of the indices at the singularities of the new 1-form (vector field) give the corresponding index for the original singularity. This implies the proportionality of the indices.

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# Motivic sheaves and intersection cohomology

## Masaki Hanamura

We propose a motivic refinement of a result in [BBFGK]. The formulation involves the notion of intersection Chow group, introduced by A. Corti and the author.

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## $\S1$ . Intersection Chow groups and lifting theorems

We consider quasi-projective varieties over  $k = \mathbb{C}$ . For a quasiprojective variety Z,  $\operatorname{CH}_s(Z)$  denotes the Chow group of s-cycles on Z tensored with  $\mathbb{Q}$ ; if Z is smooth,  $\operatorname{CH}^r(Z) := \operatorname{CH}_{\dim Z-r}(Z)$ . We consider only constructible sheaves of  $\mathbb{Q}$ -vector spaces. The singular (co-)homology, Borel-Moore homology, and intersection cohomology are those with  $\mathbb{Q}$ -coefficients.

Relative canonical filtration.

The study of filtration on the Chow group of a smooth projective variety was started by Bloch and continued by several people; of most relevance to us now are the works of Beilinson, Murre and Shuji Saito. Beilinson explained the filtration in terms of the conjectural framework of mixed motives. Murre proposed a set of conjectures, Murre's conjectures, on a decomposition of the diagonal class in the Chow ring of self-correspondences; he relates the decomposition to the filtration of Chow groups.

For X a smooth projective variety, its Chow group of codimension r cycles  $\operatorname{CH}^r(X)$  should have a filtration  $F^{\bullet}$  such that  $\operatorname{CH}^r(X) = F^0 \operatorname{CH}^r(X)$ ,  $F^1 \operatorname{CH}^r(X)$  is the homologically trivial part,  $F^2 \operatorname{CH}^r(X)$ is perhaps the kernel of Abel-Jacobi map, and so on. The subquotient  $\operatorname{Gr}_F^{\nu} \operatorname{CH}^r(X)$  should in some way be determined by cohomology  $H^{2r-\nu}(X, \mathbb{Q})$ .

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A candidate for the filtration was proposed by S. Saito, see [Sa 1] [Sa 2]. We extend his definition as follows. If  $S = \operatorname{Spec} k$ , it coincides with Saito's filtration.

Let S be a quasi-projective variety, and X a smooth variety with a projective map  $p: X \to S$ . For another smooth variety W with a projective map  $q: W \to S$ , an element  $\Gamma \in \operatorname{CH}_{\dim X-s}(W \times_S X)$  induces a map  $\Gamma_*$ :  $\operatorname{CH}^{r-s}(W) \to \operatorname{CH}^r(X)$ , see [CH]. The cycle class of  $\Gamma$  in Borel-Moore homology gives a map  $\Gamma_* \colon Rq_*\mathbb{Q}_W[-2s] \to Rp_*\mathbb{Q}_X$ ; passing to perverse cohomology one has a map (for each  $\nu$ )

$${}^{p}\mathcal{H}^{2r-\nu}\Gamma_{*}: {}^{p}\mathcal{H}^{2r-2s-\nu}Rq_{*}\mathbb{Q}_{W} \to {}^{p}\mathcal{H}^{2r-\nu}Rp_{*}\mathbb{Q}_{X}.$$

(Here  ${}^{p}\mathcal{H}^{*}$  stands for perverse cohomology.)

We define a filtration  $F_S^{\bullet}$  on  $\operatorname{CH}^r(X)$  as follows. Let  $\operatorname{CH}^r(X) =$  $F_S^{-\dim S} \operatorname{CH}^r(X)$ . Assume  $F_S^{\tilde{\nu}}$  has been defined. Define

$$F_S^{\nu+1} \operatorname{CH}^r(X) := \sum \operatorname{Image}[\Gamma_* \colon F_S^{\nu} \operatorname{CH}^{r-s}(W) \to \operatorname{CH}^r(X)]$$

where the sum is over  $(q: W \to S, \Gamma \in \operatorname{CH}_{\dim X-s}(W \times_S X))$  satisfying the following condition: the map  ${}^{p}\mathcal{H}^{2r-\nu}\Gamma_{*}: {}^{p}\mathcal{H}^{2r-2s-\nu}Rq_{*}\mathbb{Q}_{W} \to$  ${}^{p}\mathcal{H}^{2r-\nu}Rp_{*}\mathbb{Q}_{X}$  is zero. One can show:

**Proposition 1.1.** The filtration  $F_S^{\bullet}$  on  $CH^r(X)$  has the following properties.

(1)  $\operatorname{CH}^{r}(X) = F_{S}^{-\dim S} \operatorname{CH}^{r}(X).$  For any  $\Gamma \in \operatorname{CH}_{\dim X-s}(W \times_{S} X)$ , the induced map  $\Gamma_{*} \colon \operatorname{CH}^{r-s}(W) \to \operatorname{CH}^{r}(X)$  respects  $F_{S}^{\bullet}$ . (2) If  ${}^{p}\mathcal{H}^{2r-\nu}\Gamma_{*} \colon {}^{p}\mathcal{H}^{2r-2s-\nu}Rq_{*}\mathbb{Q}_{W} \to {}^{p}\mathcal{H}^{2r-\nu}Rp_{*}\mathbb{Q}_{X}$  is zero,

then  $\Gamma_*$  sends  $F_S^{\nu} \operatorname{CH}^{r-s}(W)$  to  $F_S^{\nu+1} \operatorname{CH}^r(X)$ .

The filtration is the smallest one with properties (1) and (2). (3)

#### Intersection Chow group.

We refer to a forthcoming paper with A. Corti for details on intersection Chow groups.

Let S be a quasi-projective variety, X a smooth variety, and  $p: X \rightarrow X$ S a projective map. There is an algebraic Whitney stratification

$$S = S_0 \supset S_1 \supset \cdots \supset S_\alpha \supset \cdots \supset S_{\dim S}$$

of S, so that  $S_{\alpha} - S_{\alpha+1}$  is smooth of codimension  $\alpha$ , satisfying the following condition.

(i) p is smooth projective over  $S^0 := S - S_1$ , and

(ii) there is an algebraic stratification of X such that p is a stratified fiber bundle over each stratum  $S^0_{\alpha} := S_{\alpha} - S_{\alpha+1}$ .

We then say  $p: X \to S$  is a stratified map with respect to  $\{S_{\alpha}\}$ . The stratification can be chosen to satisfy a stronger condition as follows.

Let  $X_{\alpha} = p^{-1}S_{\alpha}$ . There exist resolutions  $\tilde{X}_{\alpha} \to X_{\alpha}$  (with  $\tilde{X}_0 = X$ ) such that

(i) the induced map  $\tilde{p}_{\alpha} \colon \tilde{X}_{\alpha} \to S_{\alpha}$  is smooth over  $S_{\alpha}^{0}$ , and

(ii) there is a stratification on  $\tilde{X}_{\alpha}$  such that  $\tilde{p}_{\alpha}$  is a stratified fiber bundle over  $S^0_{\beta}$  for  $\beta \geq \alpha$ . (In other words,  $\tilde{p}_{\alpha}$  is a stratified map with respect to  $\{S_{\beta}\}_{\beta \geq \alpha}$ .)

In this case we say the data  $(p: X \to S, \{\tilde{X}_{\alpha} \to X_{\alpha}\})$  is stratified with respect to  $\{S_{\alpha}\}$ .



Let  $\iota_{\alpha} \colon \tilde{X}_{\alpha} \to X$  be the induced map.

We now restrict ourselves to the birational case: let S be a quasiprojective variety and  $p: X \to S$  a resolution of singularities. One has maps  $(d = \dim S)$ 

$$\operatorname{CH}_{d-r}(\tilde{X}_{\alpha}) \xrightarrow{\iota_{\alpha*}} \operatorname{CH}^{r}(X) \xrightarrow{\iota_{\alpha}^{*}} \operatorname{CH}^{r}(\tilde{X}_{\alpha})$$

Each group has filtration  $F_S^{\bullet}$ .

Define the *intersection Chow group* as a subquotient of the Chow group of X given by:

$$\operatorname{ICH}^{r}(S) := \frac{\bigcap_{\alpha \ge 1} (\iota_{\alpha}^{*})^{-1} F_{S}^{2r-d+1} \operatorname{CH}^{r}(\tilde{X}_{\alpha})}{\sum_{\alpha \ge 1} \iota_{\alpha *} F_{S}^{2r-d+1} \operatorname{CH}_{d-r}(\tilde{X}_{\alpha})}$$

**Theorem 1.2.** ICH<sup>r</sup>(S) is well-defined (independent of the choice of stratification and resolution).

Denote by  $IH^{i}(S)$  the intersection cohomology with middle perversity and with  $\mathbb{Q}$ -coefficients.

**Proposition 1.3.** There is a natural map

$$\operatorname{ICH}^{r}(S) \to IH^{2r}(S).$$

The Conjectures.

We recall three well-known conjectures concerning cohomology, Chow group, and higher Chow group of a smooth projective variety over a field. In this paper we refer to them as Conjectures. The addition of the third conjecture is needed to prove the existence of a t-structure on the triangulated category of mixed motives. See [Ha].

1. Grothendieck's Standard conjecture.

This concerns the functorial behavior of cycle classes in (singular or étale) cohomology. It has two components, the Lefschetz type conjecture and the Hodge type conjecture. For  $k = \mathbb{C}$ , the latter holds true (Hodge index theorem). The Lefschetz type conjecture itself consists of three statements, Conjecture (A), (B) and (C). Conjecture (C) says: the Künneth components of the diagonal class of a smooth projective variety are algebraic.

The standard conjecture implies the semi-simplicity of the category of pure homological motives (Grothendieck).

2. Murre's conjecture (Bloch-Beilinson-Murre conjecture)

One of the formulation of the conjectural filtration on Chow group is due to Murre, and stated as the existence of a orthogonal decomposition to projectors of the diagonal class  $\Delta_X$  in  $CH(X \times X)$ . To be precise, the conjecture states:

(A) Let X be a smooth projective variety. There exists a decomposition  $\Delta_X = \sum \Pi^i$  to orthogonal projectors in the Chow ring such that the cohomology class of  $\Pi^i$  is the Künneth component  $\Delta(2 \dim X - i, i)$ . The decomposition is called the *Chow-Künneth decomposition*.

(B)  $\Pi^i$  with  $i = 0, \ldots, r-1$  or  $i = 2d, \ldots, 2r+1$  acts as zero on  $\operatorname{CH}^r(X)$ .

(C) Put  $F^0 = \operatorname{CH}^r(X)$ ,  $F^1 = \operatorname{Ker} \Pi^{2r}$ ,  $F^2 = \operatorname{Ker}(\Pi^{2r-1}|F^1)$ , ...,  $F^r = \operatorname{Ker}(\Pi^{r+1}|F^{r-1})$ ,  $F^{r+1} = 0$ . This is independent of the choice of the decomposition in (A).

(D)  $F^1 = \operatorname{CH}^r(X)_{\text{hom}}$ , the homologically trivial part.

Note a Chow-Künneth decomposition gives a decomposition in the category of Chow motives over k  $h(X) = \bigoplus h^i(X)$ , where  $h^i(X)$  carries cohomology in degree i only. For the category of Chow motives, see §2.

3. Variant of Beilinson-Soulé vanishing conjecture: Let (X, 0, P) be an object of the category of Chow motives CHM(k) whose realization is of cohomological degree  $\geq 2r - n$  if n > 0 and > 2r if n = 0. Then  $P_* \operatorname{CH}^r(X, n) = 0$ .

When we give results that hold under the three <u>Conjectures</u>, we will always say so; some of them require only the first two. For example,

**Proposition 1.4** (Under <u>Conjectures</u>).  $F_S^{\nu} \operatorname{CH}^r(X) = 0$  for  $\nu$  large enough.

We have:

**Theorem 1.5** (Under Conjectures). The map

$$p_* \colon \operatorname{CH}^r(X) \to \operatorname{CH}_{d-r}(S)$$

induces a surjective map  $\operatorname{ICH}^r(S) \to \operatorname{CH}_{d-r}(S)$ .

Under <u>Conjectures</u>, one has (1.5), which immediately implies the following Theorem (1.6) in [BBFGK]. One has the cycle class map  $cl: \operatorname{CH}_{d-r}(S) \to H^{BM}_{2(d-r)}(S)$  (the latter is the Borel-Moore homology). There is a natural map  $IH^{2r}(S) \to H^{BM}_{2(d-r)}(S)$ .

**Theorem 1.6.** For any  $z \in CH_{d-r}(S)$ , its class  $cl(z) \in H^{BM}_{2(d-r)}(S)$ can be (non-canonically) lifted to an element of intersection cohomology.

Indeed, we can show (1.6) without assuming <u>Conjectures</u>, but still using the same ideas as for the proof of (1.5).

### $\S 2.$ Motivic categories and decompositions of motives

Theory of Chow motives.

Let S be a quasi-projective variety over  $k = \mathbb{C}$ . Let  $CH\mathcal{M}(S)$  be the pseudo-abelian category of Chow motives over S. It has the following properties (for details see [CH]).

• An object of  $CH\mathcal{M}(S)$  is of the form

$$(X, r, P) = (X/S, r, P)$$

where X is a smooth variety over k with a projective (not necessarily smooth) map  $p: X \to S, r \in \mathbb{Z}$ , and if X has connected components  $X_i$ ,

$$P \in \bigoplus_i \operatorname{CH}_{\dim X_i}(X \times_S X_i)$$

such that  $P \circ P = P$ . Here  $\circ$  denotes composition of relative correspondences defined in [CH], which makes  $\bigoplus_i \operatorname{CH}_{\dim X_i}(X \times_S X_i)$  a ring with the diagonal  $\Delta_X$  as the identity element. If (Y, s, Q) is another object,  $Y_j$  the components of Y, then

$$\operatorname{Hom}((X, r, P), (Y, s, Q)) = Q \circ \left(\bigoplus_{j} \operatorname{CH}_{\dim Y_{j}-s+r}(X \times_{S} Y_{j})\right) \circ P.$$

Composition of morphisms is induced from the composition of relative correspondences.

• Let h(X/S) = (X, 0, ip) and h(X/S)(r) = (X, r, ip). More generally, Tate twist is defined to be the functor  $(t \in \mathbb{Z})$ 

$$K = (X, r, P) \mapsto K(t) = (X, r+t, P)$$

on objects.

• One has a functor

$$\operatorname{CH}^t \colon CH\mathfrak{M}(S) \to Vect_{\mathbb{Q}}, \quad \operatorname{CH}^t((X, r, P)) = P_* \operatorname{CH}^{r+t}(X).$$

Note  $\operatorname{CH}^t(K) = \operatorname{CH}^0(K(t))$  and  $\operatorname{CH}^r(h(X/S)) = \operatorname{CH}^0(h(X/S)(r)) = \operatorname{CH}^r(X)$ .

• If X and Y are smooth varieties with projective maps to S and  $f: X \to Y$  a map over S, there corresponds a morphism

$$f^* \colon h(Y/S) \to h(X/S).$$

If X, Y are equidimensional, there corresponds

$$f_* \colon h(X/S) \to h(Y/S)(\dim Y - \dim X).$$

It is of use to define the homological motive of X/S: if X has components  $X_i$ ,

$$h'(X/S) := \bigoplus h(X_i/S)(\dim X_i).$$

Then a map  $f: X \to Y$  induces a morphism  $f_*: h'(X/S) \to h'(Y/S)$ .

• Let  $D_c^b(S) = D_c^b(S, \mathbb{Q})$  be the derived category of sheaves of  $\mathbb{Q}$ -vector spaces on S with constructible cohomology. There is the realization functor

$$\rho\colon CH\mathcal{M}(S) \to D^b_c(S)$$

such that on objects

$$(X, r, P) \mapsto P_* R p_* \mathbb{Q}_X[2r],$$

 $(P_* \in \operatorname{End}_{D_c^b(S)}(Rp_*\mathbb{Q}_X))$  is a projector, and  $P_*Rp_*\mathbb{Q}_X$  is its image, which exists since  $D_c^b(S)$  is pseudo-abelian.) Note  $\rho(h(X/S)(r)) = Rp_*\mathbb{Q}_X[2r]$  and

$$\rho(h'(X/S)(r)) = Rp_*D_X[2r],$$

where  $D_X$  is the dualizing complex of X. Recall  $D_X = \mathbb{Q}_X[2 \dim X]$  if X is smooth.

Theory of Grothendieck motives.

We also have the pseudo-abelian category of Grothendieck motives over S. The main properties are the following.

Denote by  $\operatorname{Perv}(S)$  be the abelian category of perverse sheaves of  $\mathbb{Q}$ -vector spaces on S. There is a canonical full functor cano:  $CH\mathcal{M}(S) \rightarrow$ 

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$\mathcal{M}(S)$  and a faithful realization functor  $\rho \colon \mathcal{M}(S) \to \operatorname{Perv}(S)$ . The following diagram commutes.



Here  ${}^{p}\mathcal{H}^{*} = \bigoplus_{i} {}^{p}\mathcal{H}^{i}$  is the total perverse cohomology functor.

Relative decomposition of motives.

The following is in [CH] (for this, we only need the first two of the three Conjectures). This is a motivic analogue of the decomposition theorem for the total direct image in [BBD].

**Theorem 2.1** (Under <u>Conjectures</u>). Let  $p: X \to S$  be as before. Let  $\{S_{\alpha}\}$  be a Whitney stratification of S, and  $\tilde{X}_{\alpha} \to X_{\alpha}$  resolutions such that  $(p: X \to S, \{\tilde{X}_{\alpha} \to X_{\alpha}\})$  is stratified with respect to  $\{S_{\alpha}\}$ . Then:

(1) There are local systems  $\mathcal{V}^{j}_{\alpha}$  on  $S_{\alpha} - S_{\alpha+1}$ , non-canonical direct sum decomposition in  $CH\mathcal{M}(S)$ 

$$h(X/S) = \bigoplus_{j, \alpha} h^j_{\alpha}(X/S)$$

and isomorphisms

$$\rho(h_{\alpha}^{j}(X/S)) \cong IC_{S_{\alpha}}(\mathcal{V}_{\alpha}^{j})[-j + \dim S_{\alpha}]$$

in  $D^b_c(S)$ .

(2) For each *i*, the sum  $\bigoplus_{j \le i, \alpha} h^j_{\alpha}(X/S)$  is a well-defined subobject of h(X/S) (independent of the decomposition).

(3) The category  $\mathcal{M}(S)$  is semi-simple abelian, and the functor  $\rho: \mathcal{M}(S) \to \operatorname{Perv}(S)$  is exact and faithful.

Relative canonical filtration and motives.

For a projective map  $p: X \to S$  with X smooth, the filtration on  $\operatorname{CH}^r(X)$  can be interpreted in terms of motives as follows. Keeping the notation in the above theorem, define subobjects of h(X/S) by

$${}^{p}\tau_{\leq i}h(X/S) := \bigoplus_{j \leq i, \alpha} h^{j}_{\alpha}(X/S)$$

(the sum over  $(j, \alpha)$  with  $j \leq i$ ) and subquotients

$${}^{p}\mathcal{H}^{i}h(X/S) := \bigoplus_{\alpha} h^{i}_{\alpha}(X/S).$$

More generally for  $r \in \mathbb{Z}$ , subobjects of h(X/S)(r)

$${}^{p}\tau_{\leq i}\big(h(X/S)(r)\big) := \bigoplus_{j \leq i+2r, \alpha} h^{j}_{\alpha}(X/S)(r)$$

and subquotients

$${}^{p}\mathcal{H}^{i}(h(X/S)(r)) := \bigoplus_{\alpha} h_{\alpha}^{i+2r}(X/S)(r)$$

are defined. Then we have

$$CH^{r}(X) = CH^{0}(h(X/S)(r))$$
  
=  $CH^{0}(\bigoplus_{\alpha,\nu} h_{\alpha}^{2r-\nu}(X/S)(r)),$   
 $F_{S}^{\nu} CH^{r}(X) = CH^{0}({}^{p}\tau_{\leq -\nu}(h(X/S)(r)))$   
=  $CH^{0}(\bigoplus_{\mu \leq 2r-\nu, \alpha} h_{\alpha}^{\mu}(X/S)(r)),$ 

and

$$\operatorname{Gr}_{F_S}^{\nu} \operatorname{CH}^r(X) = \operatorname{CH}^0({}^p \mathcal{H}^{-\nu}(h(X/S)(r)))$$
$$= \operatorname{CH}^0(\bigoplus h_{\alpha}^{2r-\nu}(X/S)(r)).$$

### §3. Outline of the proof of (1.5)

We start with a result on perverse cohomology. Let X be smooth,  $p: X \to S$  a projective map, and assume  $(p: X \to S, \{\tilde{X}_{\alpha} \to X_{\alpha}\})$  is stratified with respect to  $\{S_{\alpha}\}$ . There are local systems  $\mathcal{V}^{i}_{\alpha}$  on  $S^{0}_{\alpha}$  such that  $Rp_*\mathbb{Q}_X \cong \bigoplus IC_{S_{\alpha}}(\mathcal{V}^{j}_{\alpha})[-j + \dim S_{\alpha}]$ . Let  $d = \dim X$ .

**Proposition 3.1.** (1) Let  $\iota_{\alpha}^* \colon Rp_*\mathbb{Q} \to i_{\alpha*}R\tilde{p}_{\alpha*}\mathbb{Q}_{\tilde{X}_{\alpha}}$  be the natural map  $\iota_{\alpha}$  induces, and

$${}^{p}\mathcal{H}^{i}(\iota_{\alpha}^{*}) \colon {}^{p}R^{i}p_{*}\mathbb{Q} \to i_{\alpha*}{}^{p}R^{i}\tilde{p}_{\alpha*}\mathbb{Q}_{\tilde{X}_{\alpha}}$$

the induced map on perverse cohomology of degree *i*. The restriction to the direct summand  $IC_{S_{\alpha}}(\mathcal{V}_{\alpha}^{i})[\dim S_{\alpha}]$ 

$${}^{p}\mathfrak{H}^{i}(\iota_{\alpha}^{*})\colon IC_{S_{\alpha}}(\mathcal{V}_{\alpha}^{i})[\dim S_{\alpha}]\to i_{\alpha*}{}^{p}R^{i}\tilde{p}_{\alpha*}\mathbb{Q}_{\tilde{X}_{\alpha}}$$

is a split injection.

(2) Let  $\iota_{\alpha*} : \iota_{\alpha*} R\tilde{p}_{\alpha*} D_{\tilde{X}_{\alpha}}(-d)[-2d] \to Rp_* \mathbb{Q}$  be the natural map, and

$${}^{p}\mathcal{H}^{i}\iota_{\alpha*} \colon \iota_{\alpha*}{}^{p}\mathcal{H}^{i}R\tilde{p}_{\alpha*}D_{\tilde{X}_{\alpha}}(-d)[-2d] \to {}^{p}R^{i}p_{*}\mathbb{Q}$$

the induced map on perverse cohomology; here  $D_{\tilde{X}_{\alpha}}$  is the dualizing complex. This map factors through a split surjection

$${}^{p}\mathcal{H}^{i}\iota_{\alpha*} \colon \iota_{\alpha*}{}^{p}\mathcal{H}^{i}R\tilde{p}_{\alpha*}D_{\tilde{X}_{\alpha}}(-d)[-2d] \to IC_{S_{\alpha}}(\mathcal{V}^{i}_{\alpha})[\dim S_{\alpha}]$$

to the direct summand of the target.

We can extend the definition of the filtration  $F_S^{\bullet}$  as follows. For any quasi-projective (possibly singular) variety Z with a quasi-projective map to S, one can define a filtration  $F_S^{\bullet}$  on the Chow group  $\operatorname{CH}_s(Z)$ . This was done in [CH, §5] in the case  $S = \operatorname{Spec} k$ , and the general case is similar. For a projective map of varieties over  $S, f: X \to Y$ , the induced map  $f_*: \operatorname{CH}_s(X) \to \operatorname{CH}_s(Y)$  respects the filtrations  $F_S^{\bullet}$ . If  $S \to S'$  is a closed immersion, and  $Z \to S$ , then the filtrations  $F_S^{\bullet}$  and  $F_{S'}^{\bullet}$  on  $\operatorname{CH}_s(Z)$  coincide.

For a quasi-projective variety T, viewing it as a variety over T, one has filtration  $F_T^{\bullet}$  on  $\operatorname{CH}_s(T)$ . For this filtration, one has the following result. The proof uses the triangulated category of mixed motives over a base, the perverse *t*-structure on it, and the interpretation of the filtration  $F_S^{\bullet}$  on  $\operatorname{CH}_s(Z)$  in terms of the perverse truncation (similar to the interpretation in §2). See [Ha] for the case where the base is Spec k.

**Lemma 3.2** (Under Conjectures). For an irreducible quasi-projective variety T,  $F_T^{-2s+\dim T+1} \operatorname{CH}_s(T) = 0$ .

From now on we assume the Conjectures throughout.

Let  $p: X \to S$  be a desingularization. We have a decomposition  $h(X/S) = \bigoplus h^j_{\alpha}(X/S)$  as in (2.1). In this case  $h^{\nu}_0 = 0$  for  $\nu \neq d$ , and it can be shown  $CH^r(h^d_0) = \operatorname{ICH}^r(S)$  as a subquotient of  $\operatorname{CH}^r(X)$ .

Lemma (3.2) implies that  $p_* \colon \operatorname{CH}^r(X) \to \operatorname{CH}_{d-r}(S)$  passes to a map  $\operatorname{ICH}^r(S) \to \operatorname{CH}_{d-r}(S)$ .

For the surjectivity we must show: For any  $a \in CH_{d-r}(S)$ , there is an element  $b \in CH^{r}(X)$  such that

(i)  $p_*(b) = a$ , and

(ii)  $\iota_{\alpha}^{*}(b) \in F_{S}^{2r-d+1} \operatorname{CH}^{r}(\tilde{X}_{\alpha})$  for each  $\alpha \geq 1$ .

Let  $a \in CH_{d-r}(S)$  and  $\nu \leq 2r - d + 1$ . Consider the following Claim  $(I)_{\nu}$ .

Claim  $(I)_{\nu}$ .

(1) (Case  $\nu \leq 2r - d$ ) there is an element  $b^{\nu} \in CH^{r}(X)$  with (i)  $p_{*}(b^{\nu}) = a$ , and (ii)  $b^{\nu} \in F_{S}^{\nu} CH^{r}(X)$ .

(2) (Case  $\nu = 2r - d + 1$ ) there is an element  $b^{2r-d+1} \in \operatorname{CH}^r(X)$ satisfying the following (let  $b = b^{2r-d+1}$  for short): (i)  $p_*(b) = a$ , and (ii)  $b \in F_S^{2r-d} \operatorname{CH}^r(X)$  (not  $F_S^{2r-d+1} \operatorname{CH}^r(X)$ !), and  $b \mod F_S^{2r-d+1} \in$  $\operatorname{Gr}_{F_S}^{2r-d} \operatorname{CH}^r(X) = \bigoplus_{\alpha \ge 0} \operatorname{CH}^r(h^d_{\alpha}(X/S))$  is contained in the first summand  $ICH^r(S) = \operatorname{CH}^r(h^d_0(X/S))$ .

For  $\nu$  small enough  $(I)_{\nu}$  obviously holds: one can take any element satisfying (i). The larger  $\nu$  is, the stronger  $(I)_{\nu}$  is. What we must show is  $(I)_{2r-d+1}$ .

**Proposition 3.3.** Let  $\nu \leq 2r - d$ . We have  $(I)_{\nu} \Rightarrow (I)_{\nu+1}$ .

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The proof of Proposition (3.3) is achieved by an argument that uses Proposition (3.1), the motivic interpretation of the filtration in §2, the two Lemmas (3.2) and (3.4), and semi-simplicity of the category  $\mathcal{M}(S)$ .

**Lemma 3.4.** If  $\nu < 2r - 2 \dim \tilde{X}_{\alpha} + \dim S_{\alpha}$ , then  $h_{\alpha}^{2r-\nu}(X/S)$  is zero.

Indeed using (3.1) one shows the realization of  $h_{\alpha}^{2r-\nu}(X/S)$  is zero. Since  $\rho \colon \mathcal{M}(S) \to \operatorname{Perv}(S)$  is exact and faithful, it follows  $h_{\alpha}^{2r-\nu}(X/S)$  itself is zero.

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# On hyperbolic perturbations of algebraic links and small Mahler measure

# Eriko Hironaka

### Abstract.

This paper surveys some results surrounding Lehmer's problem in the context of fibered links and Hopf plumbing. Topics addressed here are Mahler measures of fibered links, the relation between iterated Hopf plumbings and Salem-Boyd polynomials, and the question of when monotone growth occurs under iterated plumbing. Explicit calculations for certain "perturbations" of links associated to the ADE singularities are computed.

### $\S1.$ Introduction

The *Mahler measure* of a monic integer polynomial is the absolute value of the product of roots with norm greater than one. Lehmer's problem [Leh] asks whether the Mahler measure of a monic integer polynomial can be made arbitrarily close to but greater than one. So far, there is no known monic integer polynomial with Mahler measure greater than one but less than Lehmer's number  $\alpha_L = 1.17628...$ , which is the Mahler measure of the polynomial

$$f_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

To solve Lehmer's problem it is enough to answer the question for Alexander polynomials of fibered links. A polynomial f(t) is reciprocal if  $f(t) = t^d f(1/t)$ , where  $d = \deg(f)$ . Smyth [Smy] showed that the Mahler measures of irreducible non-reciprocal polynomials not vanishing at zero are bounded below by  $\theta_0 = 1.32472...$ , a number greater than Lehmer's number. Thus, it remains to search among reciprocal polynomials. Any monic reciprocal polynomial occurs as the Alexander polynomial of a fibered link  $K \subset S^3$  up to cyclotomic factors [Kan]. Lehmer's number

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 $\alpha_L$  appears in this context as the Mahler measure of the Alexander polynomial of the (-2, 3, 7)-pretzel knot.

The Mahler measure of a fibered link  $(K, \Sigma)$  can be considered to be a weak measure of "hyperbolicity" of the link in the following sense. Let  $K \subset S^3$  be a fibered link with monodromy  $h: \Sigma \to \Sigma$ . Define the Mahler measure  $M(K, \Sigma)$  to be the Mahler measure of  $\Delta_{(K, \Sigma)}$ , where  $\Delta_{(K, \Sigma)}$  is the characteristic polynomial of the automorphism on the first singular homology group of  $\Sigma$ 

$$h_* \colon \mathrm{H}_1(\Sigma; \mathbb{R}) \to \mathrm{H}_1(\Sigma; \mathbb{R})$$

induced by h. The Mahler measure  $M(K, \Sigma)$  is bounded from below by the leading eigenvalue  $\lambda(K, \Sigma)$  of  $h_*$ , known as the homological dilatation of the monodromy h. If h is isotopic to a pseudo-Anosov map, then  $\lambda(K, \Sigma)$  is also a lower bound for the (geometric) dilatation of h. In particular, if  $\lambda(K, \Sigma) > 1$ , and h is irreducible, then h is isotopic to a pseudo-Anosov homeomorphism [Thu] (see also [FLP], [CB]).

As a first guess, it seems natural to expect small Mahler measures to be attained by "small perturbations" of non-hyperbolic links, for example, algebraic links. Here, we will take *small perturbations* to mean Hopf or trefoil plumbing along a suitable path on the fibering surface. For example, the smallest Mahler measures of degrees 2,4,6,8,10 (listed by Lehmer in [Leh]) all arise from Hopf or trefoil plumbings of torus links (see Section 4).

Two problems arise in this approach. The first is that the Alexander polynomial is only a weak indicator of geometric properties of the fibered link; a hyperbolic fibered link  $(K, \Sigma)$  may have  $M(K, \Sigma) = \lambda(K, \Sigma) =$ 1. The second is that Mahler measure and homological dilatation are not always monotone increasing or decreasing under iterations of Hopf plumbing. Useful connections between Mahler measure and geometry do hold, however, when we restrict our attention to certain subfamilies of fibered links.

We begin by defining and stating properties of Hopf plumbings in Section 2. In particular, we give a formula for the Alexander polynomials of fibered links obtained by iterated Hopf plumbing. These have the form of Salem-Boyd polynomials introduced in [Sal], developed further in [Boyd], and applied to Hopf plumbings in [Hir2].

In Section 3 we present two families of fibered links with the monotonicity property. The first example is the family of Coxeter links studied in [Hir1]. For Coxeter links, the homological dilatations grow or decrease monotonically with iterations of Hopf plumbing. If the underlying Coxeter graph is a star graph, then the homological dilatation equals the Mahler measure for any associated Coxeter link. Furthermore, Leininger [Lei] showed that for pseudo-Anosov Coxeter links associated to a bi-colored graph, the monodromy is orientable. It follows that the homological and geometric dilatations are equal for these examples [Ryk]. The second example is the family of Salem links. These are fibered links whose homological dilatation is equal to the Mahler measure of the Alexander polynomial. The Coxeter links associated to star graphs are either cyclotomic or Salem links. We give a criterion for a sequence of fibered links obtained by iterated Hopf plumbing to be eventually Salem, and show that for such Salem sequences, the dilatations grow or decrease monotonically.

Section 4 contains examples and speculations.

# $\S$ **2.** Iterated Hopf plumbings

In this section, we review some basic definitions and properties of fibered links, and their monodromy. Any fibered link can be converted to any other by a finite sequence of Hopf plumbings and deplumbings [Gir]. We recall the definition of Hopf plumbing, and give a formula for the Alexander polynomial of the fibered link for sequences of links obtained by iterated Hopf plumbing.

A link  $K \subset S^3$  is *fibered*, with fiber  $\Sigma$ , if for a regular neighborhood U(K) of K in  $S^3$ , there is a fibration

$$S^3 \setminus U(K) \to S^1$$

of the complement U(K) in  $S^3$ , where  $\Sigma$  is a general fiber, and the boundary of  $\Sigma$  equals K. Let  $(K, \Sigma)$  denote the fibered link. There is a homeomorphism  $h: \Sigma \to \Sigma$ , so that  $S^3 \setminus U(K)$  can be identified with the mapping torus for  $\Sigma$  with respect to h. The map h is called the (geometric) monodromy of the fibered link  $(K, \Sigma)$ .

Let  $h_*$  be the restriction of h to the first homology group  $H_1(\Sigma; \mathbb{R})$ . The transformation  $h_*$  is the homological monodromy of  $(K, \Sigma)$ , and its characteristic polynomial is the Alexander polynomial  $\Delta_{(K,\Sigma)}(t)$  of  $(K, \Sigma)$ . This definition of Alexander polynomial is associated to the pair  $(K, \Sigma)$  and not to the link itself; if K has more than one component, the fibering structure is not in general unique, and each fibering structure gives rise to a different Alexander polynomial. The homological dilatation of  $(K, \Sigma)$  is the maximum among absolute values of roots of  $\Delta_{(K,\Sigma)}(t)$ , or eigenvalues of  $h_*$ .

Let  $\tau$  be a properly embedded path on  $\Sigma$ . The *positive (negative)* Hopf plumbing on  $(K, \Sigma)$  along  $\tau$  is obtained by gluing a positive (negative) Hopf band onto  $\Sigma$  along a thickening of  $\tau$ . Fig. 1 shows the result



Fig. 1. Positive Hopf plumbing.

of a positive Hopf plumbing. The *n*-th iterated Hopf plumbing on  $(K, \Sigma)$ based at  $\tau$  is shown in Fig. 2. We will write  $(K_n^{\pm}, \Sigma_n^{\pm})$  for the result of the *n*-th iterated Hopf plumbing. By this convention,  $(K, \Sigma) = (K_1^{\pm}, \Sigma_1^{\pm})$ . If  $(K, \Sigma)$  is a fibered link, so is the result of any Hopf plumbing [Sta]. Thus,  $(K_n^{\pm}, \Sigma_n^{\pm})$  is fibered for all *n*.



Fig. 2. Fourth iterated Hopf plumbing.

As we will show, the Alexander polynomials of links resulting via iterated Hopf plumbings from a fixed  $(K, \Sigma)$  based at a path  $\tau$  satisfy a simple formula. Before stating the result, we give some definitions.

Given two integer polynomials f and g, we write  $f \doteq g$  if there exists cyclotomic polynomials  $c_1, \ldots, c_k, d_1, \ldots, d_\ell$ , and an integer r such that

$$f(t)c_1(t)\cdots c_k(t) = \pm t^r g(t)d_1(t)\cdots d_\ell(t).$$

If f(t) is a polynomial of degree d, define its reciprocal

$$f^*(t) = t^d f(1/t).$$

A polynomial f(t) is said to be a reciprocal polynomial if  $f = f^*$ , and anti-reciprocal if  $f = -f^*$ . If f(t) is anti-reciprocal, then  $f \doteq g$ , where g is reciprocal. This is because, if f(t) is anti-reciprocal, then (t-1)divides f(t) and f(t)/(t-1) is reciprocal.

The following theorem is proved in [Hir2].

**Theorem 1.** Let  $(K, \Sigma)$  be a fibered link, and  $\tau$  a properly embedded path on  $\Sigma$ . Then there is a polynomial  $P = P_{(\Sigma,\tau)}^{\pm}$  depending on  $\Sigma$ ,  $\tau$  and the orientation of the plumbings, such that the Alexander polynomials  $\Delta_n(t) = \Delta_{(K_n,\Sigma_n)}(t)$  satisfy

(1) 
$$\Delta_n(t) \doteq t^n P(t) + (-1)^{n+r} P^*(t),$$

where r is the number of components of K.

Polynomials of the form given in Equation (1) were studied by Salem [Sal], and Boyd [Boyd] in their investigations of Salem and P-V numbers. We will call Equation (1), the *Salem-Boyd form* of the polynomial  $\Delta_n$ . Given a polynomial f, let N(f) be the number of roots outside the unit circle,  $\lambda(f)$  (called the *radius of* f) the maximum among absolute values of roots of f, and M(f) the Mahler measure of f. The following is proved in [Boyd] (see also, [Hir2]).

**Theorem 2.** Let P(t) be a monic integer polynomial and

$$Q_n(t) = t^n P(t) \pm P^*(t).$$

Then  $Q_n$  is reciprocal or anti-reciprocal for all n, and furthermore

- (1)  $N(Q_n) \leq N(P)$  for all n;
- (2)  $\lim_{n\to\infty} \lambda(Q_n) = \lambda(P); and$
- (3)  $\lim_{n \to \infty} M(Q_n) = M(P).$

Analogously define, for a fibered link  $(K, \Sigma)$ ,  $N(K, \Sigma)$  (respectively,  $\lambda(K, \Sigma)$ , and  $M(K, \Sigma)$ ), to be  $N(\Delta_{(K,\Sigma)})$  (respectively,  $\lambda(\Delta_{(K,\Sigma)})$ ,  $M(\Delta_{(K,\Sigma)})$ ). Then Theorem 3 below follows immediately from Theorem 2.

**Theorem 3.** Let  $(K_n, \Sigma_n)$  be fibered links obtained from  $(K, \Sigma)$ by iterated Hopf plumbing. Then  $N(K_n, \Sigma_n)$  is eventually constant, and  $\lambda(K_n, \Sigma_n)$  and  $M(K_n, \Sigma_n)$  are convergent sequences. E. Hironaka

We give two explicit formulae for  $P_{(\Sigma, \tau)}$ . Before doing this, recall that for any link K and Seifert surface  $\Sigma$ , there is an associated Seifert matrix S with respect to some choice of basis for  $H_1(\Sigma; \mathbb{R})$  (see, for example, [Rolf] for terminology). Then the Alexander polynomial of Kwith respect to  $\Sigma$  is given by  $\Delta_{(K, \Sigma)}(t) = |tS - S^{tr}|$  up to multiplies of  $\pm t$ , where |A| denotes the determinant of A and  $A^{tr}$  the transpose of A. This definition specializes to our previous definition of Alexander polynomials for fibered links. For an invertible matrix A, let s(A) be the sign of the determinant of A. For example, if K is a fibered knot with fiber  $\Sigma$ , and Sis any invertible Seifert matrix for K, then  $s(S) = \Delta_{(K, \Sigma)}(1)$ . Since s(S)doesn't depend on the choice of basis, we will define  $s(K, \Sigma) = s(S)$ . If  $(K, \Sigma)$  is fibered and S is a Seifert matrix with respect to some choice of basis for  $H_1(\Sigma; \mathbb{R})$ , then  $S^{-1}S^t$  represents the homological monodromy  $h_*$  with respect to this basis.

Let  $(K, \Sigma)$  be a fibered link, and let  $\tau$  be a properly embedded path in  $\Sigma$ . Let  $\Sigma_{\tau}$  be the surface in  $S^3$  obtained by taking  $\Sigma$  and removing a regular neighborhood of  $\tau$ . Let  $K_{\tau}$  be the boundary of  $\Sigma_{\tau}$ . The first formula is reminiscent of the skein relations, where one keeps track of the associated Seifert surfaces.

(2) 
$$P_{(\Sigma,\tau)}(t) = \Delta_{(K,\Sigma)}(t) \pm \mathbf{s}(\Sigma)\mathbf{s}(\Sigma_{\tau})\Delta_{(K_{\tau},\Sigma_{\tau})}(t).$$

The second formula is given as a determinant:

(3) 
$$P_{(\Sigma,\tau)}(t) = |tS - (S^{\mathrm{tr}} \mp vv^{\mathrm{tr}})|,$$

where  $v \in H_1(\Sigma; \mathbb{R})$  is the dual vector to  $\tau$  considered as an element of  $H_1(\Sigma, \partial \Sigma; \mathbb{R})$ .

**Remark.** Silver and Williams proved the following related result [SW].

**Theorem 4.** Let K be any link, and let  $\ell$  be an unknot disjoint from K, whose linking number with K is nonzero. Let  $K^{(n)}$  be obtained by 1/n surgery on a tubular neighborhood of  $\ell$ , and let  $\widetilde{\Delta}_{K^{(n)}}$  be the multi-variable Alexander polynomial of  $K^{(n)}$ . Then the multi-variable Mahler measures of  $\widetilde{\Delta}_{K^{(n)}}$  converge to the multi-variable Mahler measure of  $\widetilde{\Delta}_{K\cup\ell}$ .

If K is a knot, then  $K^{(n)}$  is a knot for all n, and we have  $\widetilde{\Delta}_{K^{(n)}} = \Delta_{K^{(n)}}$ . If  $(K, \Sigma)$  is a fibered knot, and  $(K_n^{\pm}, \Sigma_n^{\pm})$  is obtained from  $(K, \Sigma)$  by iterated Hopf plumbing, then  $K^{(n)} = K_{2n}^{\pm}$  is a sequence satisfying the conditions of Theorem 4.

### $\S$ **3.** Monotone sequences

In general, the sequences described in Theorem 2 are not monotone. This section contains two large families of examples where monotonicity does hold.

# 3.1. Coxeter links

Let  $(K, \Sigma)$  be the fibered link obtained by positive Hopf plumbing along an ordered system of chords  $\ell_1, \ldots, \ell_k$  on an oriented disk in  $S^3$ . Let  $\Gamma$  be the dual graph. We say that  $(K, \Sigma)$  is a *Coxeter link* for  $\Gamma$ , if

- (1) all plumbings are positive; and
- (2) whenever i < j, the intersection of  $\ell_i$  with  $\ell_j$  on the disk is negative with respect to the skew-symmetric intersection form on the disk.

Recall that for any ordered finite graph  $\Gamma$  with no self-or doubleedges, there is an associated *simply-laced Coxeter system* (see for example, [Hum]). Let  $c(\Gamma)$  be the associated Coxeter element.

The Coxeter element gives important information about the Coxeter link. For example, an irreducible Coxeter system is spherical or affine if and only if  $\lambda(c(\Gamma)) = 1$ , where  $\lambda(c(\Gamma))$  is the leading eigenvalue of  $c(\Gamma)$ [Hum], [A'C]. It follows that the Coxeter links whose Mahler measure equals one are those associated to disjoint unions of spherical and affine Coxeter diagrams. In the irreducible case, these are just  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$ , and  $E_8$ , and their affine extensions. For the irreducible spherical cases, the graphs are trees, and it follows that the Coxeter links are uniquely determined (see [Hir1]), and are the algebraic links associated to the A-D-E plane curve singularities.

For a graph  $\Gamma$ , let  $\mu(\Gamma)$  be the leading eigenvalue of the adjacency matrix for  $\Gamma$ , known as the *radius of the graph*  $\Gamma$ . Let  $\lambda(\Gamma)$  be the leading eigenvalue of  $c(\Gamma)$ . Let  $\mu = \mu(\Gamma)$ , and consider the equation

$$\lambda + \lambda^{-1} = \mu^2 - 2$$

The solutions  $\lambda$  are roots of unity if and only if  $\mu \leq 2$ , and we set  $\lambda(\Gamma) = 1$ . Otherwise the solutions are real and positive, and we set  $\lambda(\Gamma)$  to be the larger (real) solution.

An ordered bi-colored graph is a graph with ordered vertices  $\nu_1, \ldots, \nu_k$  such that for some s with  $1 \leq s \leq k, \nu_i$  and  $\nu_j$  are not connected by an edge whenever  $i, j \leq s$  or i, j > s. In the following theorem, McMullen shows that  $\lambda(c(\Gamma))$  is bounded from below by  $\lambda(\Gamma)$  ([Mc] Theorem 1.3).

**Theorem 5.** Let  $\Gamma$  be any Coxeter graph. Then

$$\lambda(\Gamma) \le \lambda(c(\Gamma)),$$

and equality holds if  $\Gamma$  is bi-colored.

Since  $\mu(\Gamma) \mapsto \lambda(\Gamma)$  is order preserving, one can get information about the smallest possible values of  $\lambda(c(\Gamma))$  using properties of graph radii.

An arm of  $\Gamma$ , is a chain of edges  $\xi_1, \ldots, \xi_n$  and vertices  $\nu_0, \ldots, \nu_n$ , so that

- (1)  $\deg(\nu_0) = 1;$
- (2)  $\deg(\nu_i) = 2$  for i = 1, ..., n 1; and
- (3) The end vertices of  $\xi_i$  are  $\nu_{i-1}$  and  $\nu_i$  for each  $i = 1, \ldots, n$ .

Choose an edge  $\xi$  on  $\Gamma$  connecting vertices  $\gamma_1$  and  $\gamma_2$ . A graph  $\Gamma_{\xi,n}$  is obtained from  $\Gamma$  by extending the edge  $\xi$  if  $\Gamma_{\xi,n}$  is obtained by replacing  $\xi$  on  $\Gamma$  with n edges  $\xi_1, \ldots, \xi_n$  and vertices  $\nu_1, \ldots, \nu_{n-1}$  so that

- (1)  $\xi_1$  connects  $\gamma_1$  and  $\nu_1$ ;
- (2)  $\xi_i$  connects  $\nu_i$  and  $\nu_{i+1}$  for  $i = 2, \ldots, n-1$ ; and
- (3)  $\xi_{n-1}$  connects  $\nu_{n-1}$  with  $\gamma_2$ .

Fig. 3 gives an illustration.



Fig. 3. Extending an edge of a graph.

Hoffman proves the following theorem about monotonic of  $\mu(\Gamma)$ and hence of  $\lambda(\Gamma)$  with respect to extending edges [Hof].

**Theorem 6.** Let  $\xi$  be an edge of a graph  $\Gamma$ , and let  $\Gamma_{\xi,n}$  be obtained by extending  $\Gamma$  along  $\xi$ . There exists N such that

$$\mu(\Gamma_{\xi,n}) \le 2$$

if and only if n < N. For  $n \ge N$ ,  $\mu(\Gamma_{\xi,n})$  is monotone increasing if  $\xi$  lies on a free arm of  $\Gamma$ , and  $\mu(\Gamma_{\xi,n})$  is monotone decreasing otherwise.

The following property is proved in [Hir1].

**Theorem 7.** If  $(K, \Sigma)$  is a Coxeter link associated to  $\Gamma$ , then after a natural identification of underlying vector spaces,

$$h_* = -c(\Gamma).$$

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It follows that in this case  $\lambda(K, \Sigma) = \lambda(c(\Gamma))$ .

Let  $(K, \Sigma)$  be a Coxeter link associated to a graph  $\Gamma$ . Then extending an edge of  $\Gamma$  corresponds to performing an iterated Hopf plumbing on  $(K, \Sigma)$ . Thus, Hoffman's theorem implies the following.

**Theorem 8.** Let  $\Gamma$  be a Coxeter graph that is not the union of spherical and affine Coxeter graphs. Let  $(K, \Sigma)$  be an associated Coxeter link, and let  $(K_n, \Sigma_n)$  be obtained by an iterated Hopf plumbing on  $(K, \Sigma)$  associated to extending an edge  $\Gamma$ . Then, for some N, the sequence  $\lambda(K_n, \Sigma_n)$ , n > N, is monotone.

Lehmer's number  $\alpha_L$  occurs as the Mahler measure of the  $E_{10}$  Coxeter graph, which is also known as the (2, 3, 7) star-like graph (cf. [MRS]). The following theorem was proved in greater generality for all Coxeter systems in [Mc], but we give a simpler version here that applies to Coxeter links.

**Theorem 9.** If  $\Gamma$  is any connected Coxeter graph, then either  $\Gamma$  is spherical or affine, or

$$M(E_{10}) = \lambda(E_{10}) \le \lambda(\Gamma) \le M(\Gamma).$$

The (-2, 3, 7)-pretzel knot  $K_{2,3,7}$  is a Coxeter link associated to  $E_{10}$  (see [Hir1]). Thus, we have the following corollary to Theorem 9.

**Theorem 10.** If  $(K, \Sigma)$  is a Coxeter link, then either  $M(K, \Sigma) = 1$ , or

$$M(K, \Sigma) \ge M(\Delta_{K_{2,3,7}}).$$

If  $\Gamma$  is bi-colored, the monodromy of the Coxeter link is pseudo-Anosov if and only if  $\Gamma$  is connected and the simply-laced Coxeter system associated to  $\Gamma$  is not spherical or affine [Lei]. Furthermore, the invariant stable and unstable foliations are orientable, and hence the homological and geometric dilatations are equal. By Rykken's result [Ryk], we have the following.

**Theorem 11.** If  $(K, \Sigma)$  is a Coxeter link associated to a connected bi-colored graph which is not spherical or affine, then the homological and geometric dilatations of  $(K, \Sigma)$  are equal.

**Theorem 12.** Let  $(K, \Sigma)$  be a Coxeter link associated to a nonspherical or affine connected Coxeter graph  $\Gamma$ . Let  $(K_n, \Sigma_n)$  be obtained by iterated Hopf plumbing on  $(K, \Sigma)$  associated to extending an edge of  $\Gamma$ . Then for some N > 0, the sequence of geometric dilatations of  $(K_n, \Sigma_n)$  is monotone.

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### **3.2.** Salem sequences

A Salem number is a real algebraic integer  $\alpha > 1$  such that all other algebraic conjugates lie on or within the unit circle C with at least one on C. The minimal polynomial of a Salem number is always reciprocal. For convenience, we will also include among Salem numbers real quadratic integers  $\alpha > 1$  whose other algebraic conjugate equals  $\alpha^{-1}$ . With this addition,  $\alpha$  is a Salem number if and only if it is the Mahler measure of a reciprocal monic integer polynomial f and satisfies N(f) = 1 (see notation in Section 2). Lehmer's problem is still open for Salem numbers, for example, it is not known if there is a Salem number smaller than Lehmer's number. Furthermore, it is not known whether the minimization problem for Salem numbers is equivalent to the minimization problem for Mahler measures greater than one.

Closely related to Salem numbers are *P-V numbers*, or *Pisot* -*Vijayaraghavan* numbers. These are algebraic integers  $\theta > 1$  all of whose other algebraic conjugates lie strictly within the unit circle. For our purposes we will redefine P-V numbers to be the Mahler measure of a monic integer polynomial f such that  $f \neq f^*$ ,  $f \neq -f^*$ , and N(f) = 1. The set of P-V numbers is closed [Sal] and its smallest element is  $\theta_0 = 1.32472...$  [Sie].

If  $(K, \Sigma)$  is a fibered link whose homological dilatation is a Salem number, we say that  $(K, \Sigma)$  is a *Salem (fibered) link*. If  $(K_n, \Sigma_n)$  is a sequence obtained from  $(K, \Sigma)$  by an iteration of Hopf plumbings, and if  $(K_n, \Sigma_n)$  is a Salem link for large enough n, we call  $(K_n, \Sigma_n)$  a *Salem* sequence.

The minimal polynomial of a P-V number will be called a P-V polynomial, and the minimal polynomial of a Salem number will be called a Salem polynomial. Theorem 2 has a stronger form when restricting to the case when P(t) is a P-V polynomial (see [Sal], [Boyd]).

**Theorem 13.** If P(t) is a P-V polynomial, then there exist constants  $N_{\pm}$  such that  $M(Q_n^{\pm}) = 1$  for  $n < N_{\pm}$ , and  $N(Q_n^{\pm}) = 1$  for  $n \ge N_{\pm}$ . Furthermore,  $M(Q_n^{\pm})$  converges monotonically to M(P) from below (respectively, above) if and only if  $\pm P(0) > 0$  (respectively < 0).

From Theorem 13 it follows that to each Salem sequence  $(K_n, \Sigma_n)$  there corresponds a P-V number  $\theta_{(\Sigma, \tau)} \geq \theta_0$  to which the Salem numbers converge. Furthermore, one has an effective way to find the smallest Salem number occurring in the sequence, as seen in the following corollary.

**Corollary 14.** If  $(K_n, \Sigma_n)$  is a Salem sequence associated to a *P*-*V* polynomial *P*, then the values greater than one attained by  $M(K_n, \Sigma_n)$  On hyperbolic perturbations of algebraic links and small Mahler measure87

are bounded from below by the minimum of  $\theta_0$ , and the first nontrivial terms in the sequences  $M(K_{2n}, \Sigma_{2n})$  and  $M(K_{2n+1}, \Sigma_{2n+1})$ .

**Remark.** The role of Salem links in studying Mahler measures of fibered links is still mysterious. For Salem links, the homological dilatation and the Mahler measure of  $\Delta_{(K,\Sigma)}$  are equal. While both geometric and homological dilatation can be made arbitrarily close to one, a lower bound greater than one for Salem numbers would imply a lower bound greater than one for dilatations for Salem links. This leads to the following problem, which we leave for further research.

**Problem 15.** Give a geometric interpretation for the algebraic conjugates of the dilatation of a fibered link, and characterize the Salem links.

# §4. Small perturbations of A-D-E singularities

We make use of the Salem-Boyd equations given in Section 3 to find the minimal Mahler measures greater than one occuring in certain families.

The fibered links in this section are obtained by positive or negative Hopf plumbings along an ordered system of chords arranged on a disk in  $S^3$ . Let  $\Gamma$  be the dual graph of the chord arrangement. The polynomials  $P_{\Sigma,\tau}^{\pm}$  of Theorem 1 are easy to compute from the combinatorics of  $\Gamma$ using Equation 2, especially in the case when  $\Gamma$  is a tree, and the locus of plumbing is one of its nodes. A filled (unfilled) vertex  $\nu$  corresponds to positive (negative) Hopf plumbing, as shown in Fig. 4. We will refer to  $\Gamma$  as the plumbing graph for the associated link.



Fig. 4. Graphs and plumbing.

If  $\Gamma$  is a tree, then the fibered link associated to any realization is an arborescent link with underlying graph  $\Gamma$ . If  $\Gamma$  is a tree and has no vertices of degree greater than 3, then the link is determined by  $\Gamma$ . E. Hironaka

It is not hard to see that for fixed degree, there is a positive gap between 1 and the next smallest Mahler measure. In [Leh], Lehmer lists polynomials with the smallest Mahler measures for non-cyclotomic polynomials in all even degrees up to 10. For degree 2 the minimal Mahler measure is attained by the Fig. 8 knot, which can also be thought of a (2, 3, 1)-pretzel link. This appears in the sequence described in Section 4.3. For degrees 4,6,8 and 10, the minimal Mahler measures can be obtained by Coxeter links of star graphs (see Section 4.1).

We end by giving an application of Theorem 1, Theorem 2, and Theorem 13, by computing the minimum Salem number occurring for certain positive (Section 4.2) and negative (Section 4.3) perturbations of the algebraic links associated to  $A_n$ .

### 4.1. Coxeter links and pretzel links from star graphs

The (-2, m, n)-pretzel links  $K_{-2, m, n}$  and more generally the  $(p_1, \ldots, p_k, -1, \ldots, -1)$ -pretzel links, where the number of-1's is k-2, are Coxeter links associated to  $(p_1, \ldots, p_k)$ -star graphs [Hir1].

The star graphs are defined as follows. Let  $A_p$  be the graph consisting of p nodes  $\nu_0, \ldots, \nu_p$  and edges between  $\nu_i$  and  $\nu_{i+1}$  for  $i = 1, \ldots, p-1$ . The vertex  $\nu_0$  will be called the *base* of the  $A_p$ . A  $(p_1, \ldots, p_k)$ -star graph is a connected tree  $\Gamma$  that is the union of subgraphs isomorphic to  $A_{p_1}, \ldots, A_{p_k}$  with their bases identified as in Fig. 5.



Fig. 5. The (2, 3, 4)-star graph.

For star graphs with less than or equal to 3 branches, the Coxeter link is an arborescent link completely determined by the graph. If the star graph is one of  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$  or  $E_8$ , or their affine extensions, then the links are iterated torus links, and the geometric and homological monodromy equal 1. In all other cases, the fibered links have pseudo-Anosov monodromy with orientable stable and unstable invariant foliations [Lei], and hence the homological and geometric dilatations are also equal [Ryk]. Furthermore, the dilatations are Salem numbers and hence are equal to the Mahler measures of the Alexander polynomials [MRS].

The minimal hyperbolic extensions of  $D_4$ ,  $E_6$ ,  $E_7$  and  $E_8$  are respectively the (2, 2, 2, 3), (3, 3, 4), (2, 4, 5), and (2, 3, 7) star links. The Mahler measures for the characteristic polynomials of these links are the minimal ones greater than one in degrees 4,6, 8 and 10 (cf. [Mc], Proposition 7.3 and page 175).

# 4.2. Positive perturbations of $A_n$

For the calculations in this Section, and the next, we will make use of the following Lemma. Let C denote the unit circle |z| = 1. Let  $\theta_G$ , known as the *golden mean*, be the sole root of  $t^2 - t - 1$  that is greater than one.

**Lemma 16.** Consider the polynomials

$$f_m^{\pm}(t) = t^m (t^2 - t - 1) \pm 1$$

Then  $f_m^-$  has exactly one root  $\theta_m^-$  outside C for all  $m \ge 1$ , and the sequences  $\theta_m^-$  converge to  $\theta_G$  monotonically from above. The roots of  $f_m^+$  are roots of unity for m = 1, 2, and for  $m \ge 3$ , they have exactly one root  $\theta_m^+$  outside C. The sequences  $\theta_m^+$  converge to  $\theta_G$  monotonically from below.

*Proof.* To show that  $f_m^-$  has at most one root outside C, we will use an argument similar to that of Boyd in [Boyd]. Consider the polynomials

$$F_m^{\pm}(t, s) = t^m (t^2 - t - 1) \pm s$$

where s is a variable ranging in the interval [0, 1]. Let  $\alpha(s)$  be any branch of  $F_m^{\pm}(t, s) = 0$  considered as curve lying over [0, 1]. Then  $\alpha(s)$ can never lie on C as long as  $0 \le s < 1$ , since, on C,  $|t^2 - t - 1|$  is bounded from below by 1. If such an  $\alpha = \alpha(s)$  existed, we would have

$$|\alpha^2 - \alpha - 1| > s = |\alpha^2 - \alpha - 1|$$

yielding a contradiction. It follows that the number of roots of  $f_m^{\pm}(t)$  outside C is bounded from above by  $N(t^2 - t - 1) = 1$ .

The cases for small m can be checked by hand. Monotonicity follows from the fact that as soon as  $f_m^{\pm}(t)$  has a root  $\alpha$  outside C, then  $f_{m+1}^{\pm}(t)$ is forced to have a root strictly between  $\alpha$  and  $\theta_G$ . Q.E.D.

The Coxeter link  $K_{A_n}$  associated to  $A_n$  is the torus link T(2, n+1), and the Alexander polynomial is

(4) 
$$\Delta_{A_n} = \frac{t^{n+1} + (-1)^n}{t+1} = t^n - t^{n-1} + \dots + (-1)^n.$$

The (-2, m, n)-pretzel links  $K_{-2, m, n}$  are obtained by positive iterated Hopf plumbing on  $K_{A_{m+1}}$  along  $\tau$ , where  $[\tau]^{\text{dual}} = [0, 1, 0, \dots, 0]$ . The link  $K_{-2, m, 1}$  has one component if m is odd and two components if mis even. Thus, the Alexander polynomial for  $K_{-2, m, n}$  is given by

$$\Delta_{K_{-2,m,n}}(t) = t^n P_m(t) + (-1)^{m+n} (P_m)^*(t),$$

where

$$P_m(t) = \Delta_{A_{m+1}}(t) + \Delta_{A_1}(t)\Delta_{A_{m-1}}(t)$$
  
=  $(t^{m+1} - t^m + \dots + (-1)^{m+1})$   
+  $(t-1)(t^{m-1} - t^{m-2} + \dots + (-1)^{m-1})$   
=  $t^{m+1} - t^{m-1} + t^{m-2} - \dots + (-1)^m t.$ 

The polynomials  $P_m(t)$  satisfy

$$P_m(t) + P_{m+1}(t) = t^{m+2} + t^{m+1} - t^m = t^m(t^2 + t - 1).$$

Thus

$$P_m(t) + (-1)^m P_1(t) = \sum_{i=1}^{m-1} (-1)^{m-i-1} (P_i(t) + P_{i+1}(t))$$
$$= (t^{m-1} - t^{m-2} + \dots + (-1)^m t)(t^2 + t - 1)$$

and

$$P_m(t) = \frac{(t^{m-1} + (-1)^m)t(t^2 + t - 1) + (-1)^{m+1}t^2(t+1)}{t+1}$$
$$= \frac{t^m(t^2 + t - 1) + (-1)^{m+1}t}{t+1}$$

Let  $(\overline{P}_m)(t) = P_m(-t)$ . Then

$$\overline{P}_m(t) = \frac{(-1)^m \left[ t^m (t^2 - t - 1) + t \right]}{t + 1},$$

and

$$(\overline{P}_m)^*(t) = \frac{-(P_m)^*(-t)}{t+1}$$

By Lemma 16,  $P_m(-t)$  is cyclotomic for m = 1, 2, and is a P-V polynomial for  $\theta_m$ , for  $m \ge 3$  where  $\theta_m$  converges monotonically to  $\theta_G$  from below.

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Replacing t by -t in the formula for  $\Delta_{K-2,m,n}$ , we have

$$\Delta_{K_{-2,m,n}}(-t) \doteq t^n \overline{P}_m(t) + (\overline{P}_m)^*(t)$$
  
=  $t^n P_m(-t) - (P_m)^*(-t).$ 

By Theorem 13, all the Salem sequences arising from (2, m, n)-stars are monotone increasing. The minimal elements in this family are listed below.

pretzel type	Salem number
(-2, 3, 7)	$\approx 1.17628$
(-2, 4, 5)	$\approx 1.36$

Thus, the (-2, 3, 7)-pretzel is minimal in this family.

For the particular case when m = 3, we have

$$P_3(t) = t^4 - t^2 + t = t(t^3 - t + 1) = tg(-t),$$

where g is the minimal polynomial for the smallest P-V number  $\theta_0$ . Lehmer's polynomial  $f_L(t)$  can thus be written as

$$f_L(t) = t^8(g(t)) - g^*(t) = \Delta_{K_{-2,3,7}}(-t).$$

# 4.3. Negative perturbations of $A_n$

We now consider the positive (2, m, n)-pretzel links. These are not Coxeter links, since they have a negative twist in their plumbing graph as in Fig. 6. Just as in the previous example, these links are arborescent links, and the Alexander polynomials are independent of the choice of directions on the plumbing graphs.



Fig. 6. Plumbing graph for the (2, 3, 4)-pretzel.

We begin with the (2, m, 1)-pretzel links. These have plumbing graph as in Fig. 7.

Let  $K_m$  be the (2, m, 1)-pretzel link. When m = 1, 3, 5, 7 these links are, respectively, denoted by  $4_2, 6_2, 8_2$ , and  $10_2$  in Rolfsen's knot table ([Rolf] p. 391–429). The knot  $4_2$  is more commonly known as the E. Hironaka

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Fig. 7. Plumbing graph for the (2, 3, 1)-pretzel.

figure eight knot. By Theorem 1, the Alexander polynomials of  $K_n$  are given by

$$\Delta_{K_m}(t) = \frac{t^{m+1}P(t) + (-1)^{m+1}P^*(t)}{t+1},$$

where

$$P(t) = \Delta_{K_1} + \Delta_{K_0}$$
  
=  $(t^2 - 3t + 1) + (t - 1)$   
=  $t^2 - 2t = t(t - 2).$ 

It follows that

$$\Delta_{K_m}(t) = t^{m+1} - 3t^m + 3t^{m-1} - \dots (-1)^m (3t - 1).$$

Since P(t) has one root outside C, the  $K_m$  are eventually Salem links. Looking at the even and odd subsequences, we see that the only cyclotomic link that occurs is  $K_2$ . Thus, the minimal elements in this family are the figure eight knot  $K_1$ , and  $K_4$ . The sequences are decreasing for nodd and increasing for n even. Thus, the smallest Salem number arising in this sequence is  $1.8832 \cdots = \alpha(K_4)$ .

Let  $K_{m,n}$  be the (2, m, n)-pretzel link. Then this is an iterated Hopf sequence using the index n, and starting with the (2, m, 1)-pretzel. We find  $P_m(t)$  as follows.

$$P_m(t) = \Delta_{K_m}(t) + \Delta_{A_1}(t)\Delta_{A_{m-1}}(t)$$
  
=  $\Delta_{K_m}(t) + (t-1)(t^{m-1} - t^{m-2} + \dots + (-1)^{m-1})$   
=  $t^{m+1} - 2t^m + t^{m-1} - t^{m-2} + \dots + (-1)^m t$ 

Adding consecutive functions, yields the formula

$$P_m(t) + P_{m+1}(t) = t^m(t^2 - t - 1).$$

Thus,

$$P_m(t) + (-1)^{m-1} P_1(t) = \sum_{i=1}^{m-1} (-1)^{m-i-1} (P_i(t) + P_{i+1}(t))$$
$$= \sum_{i=1}^{m-1} (-1)^{m-i-1} t^i (t^2 - t - 1).$$

Isolating  $P_m(t)$ , we get

$$P_m(t) = \frac{(t^{m-1} + (-1)^m)t(t^2 - t - 1) + (-1)^{m-1}t(t - 2)(t + 1)}{t + 1}$$
$$= \frac{t^m(t^2 - t - 1) + (-1)^m t}{t + 1}.$$

By Lemma 16,  $P_m(t)$  has exactly one root  $\theta_m$  outside C for m = 1 and  $m \ge 3$ , and  $\theta_m$  tends to the root  $\theta_G$  of  $P_G(t) = t^2 - t - 1$  from above (for odd m) and below (for even m).

The number r of components of  $K_m$  is 1 if m is odd and 2 if m is even. We thus have,

$$\Delta_{K_{m,n}}(t) \doteq P_m(t) + (-1)^{m+n} (P_m)^*(t).$$

and the leading coefficient of  $(-1)^{m+n}P_m(t)$  is  $(-1)^n$ . It follows from an argument similar to that in the proof of Lemma 16 that  $M(K_{m,2n+1})$  is monotone decreasing and  $M(K_{m,2n})$  is monotone increasing.

Since the (2, 4, 4)-and all (2, 2, n)-pretzel links are cyclotomic, the minimal elements of (2, m, n)-pretzel knots with respect to trefoil plumbing are those listed below.

pretzel type	Salem number
(2,1,1)	$\approx 2.61803$
(2,1,4)	$\approx 1.8832$
(2, 4, 6)	$\approx 1.36$

Of these only the (2, 4, 6)-pretzel gives Salem number smaller than  $\theta_G$ . Thus,  $M(K_{4,6}) \approx 1.36$  is the minimal Mahler measure greater than one among the (2, m, n)-pretzel links.

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# Stably hyperbolic polynomials

# Vladimir Petrov Kostov

### Abstract.

A real polynomial in one real variable is *hyperbolic* if all its roots are real. Denote the set of monic hyperbolic polynomials of degree n by  $\Pi_n$ . Suppose that for a real polynomial P(x) of degree n there exists  $k \in \mathbb{N}$  and a polynomial Q(x) of degree  $\leq k-1$  such that  $x^k P + Q \in \Pi_{n+k}$ . Denote the set of such polynomials P by  $\Pi_n(k)$ . Call the set  $\Pi_n(\infty) = \overline{\bigcup_{k=0}^{\infty} \Pi_n(k)}$  the domain of *stably hyperbolic polynomials* of degree n. In the present paper we explore the geometric properties of the set  $\Pi_4(\infty)$ .

### $\S1.$ Introduction

Consider the family of polynomials  $P(x, a) = x^n + a_1 x^{n-1} + \dots + a_n$ ,  $a_i, x \in \mathbf{R}$ .

**Definition 1.** Call a polynomial from the family P hyperbolic (resp. strictly hyperbolic) if it has only real (resp. real and distinct) roots. Denote by  $\Pi_n$  the hyperbolicity domain of the family P, i.e. the subset of  $\mathbf{R}^n$  consisting of the values of the *n*-tuple of coefficients  $(a_1, \ldots, a_n)$ for which P is hyperbolic. Geometric properties of the hyperbolicity domain are given in papers [Ko1], [Ko2], [Me1] and [Me2]. In the proofs in the first two of them the results of the papers [Ar] and [Gi] are used.

Notice that  $\Pi_n \cap \{a_1 = 0, a_2 > 0\} = \emptyset$  and  $\Pi_n \cap \{a_1 = 0, a_2 = 0\} = 0 \in \mathbb{R}^n$ . Indeed, if a polynomial is hyperbolic, then such are its nonconstant derivatives as well. For  $a_1 = 0$  one has  $P^{(n-2)} = (n!/2)x^2 + (n-2)!a_2$  which is hyperbolic only if  $a_2 \leq 0$ . If one has  $a_1 = a_2 = 0$ , then one has  $P^{(n-3)} = (n!/6)x^3 + (n-3)!a_3$  which is hyperbolic only if  $a_3 = 0$ , and in a similar way one must have  $a_4 = \cdots = a_n = 0$ . Therefore

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in what follows we set once for all  $a_1 = 0$  (this can be achieved by the shift  $x \mapsto x - a_1/n$ ) and  $a_2 = -1$  (recall that  $\Pi_n$  is invariant for the one-parameter group of stretchings  $a_j \mapsto e^{jt}a_j$ ).

**Notation 2.** Set  $\Pi_n(0) = \Pi_n$ . Denote for  $k \in \mathbb{N}$  by  $\Pi_n(k)$  the set of polynomials P for which there exist polynomials Q of degree  $\leq k-1$ such that  $R(x) := x^k P + Q \in \Pi_{n+k}$ . Hence, one has  $\Pi_n(k+1) \supset \Pi_n(k)$ because if  $P \in \Pi_{n+k}$ , then  $xP \in \Pi_{n+k+1}$ . Set  $\Pi_n(\infty) = \bigcup_{k=0}^{\infty} \Pi_n(k)$ . Notice that for a polynomial from  $\partial \Pi_n(\infty)$ , the boundary of  $\Pi_n(\infty)$ , one cannot find k and Q as above.

**Definition 3.** We call the set  $\Pi_n(\infty)$  the domain of stably hyperbolic polynomials of degree n.

**Proposition 4.** For any  $n \in \mathbf{N}$ ,  $n \geq 2$ , the set  $\Pi_n(\infty)$  (with  $a_1 = 0, a_2 = -1$ ) is bounded.

*Proof.* Denote by  $x_1 \ge \cdots \ge x_{n+k}$  the roots of the polynomial R, see the above notation. One has  $x_1 + \cdots + x_{n+k} = 0$ ,  $\sum_{1 \le i < j \le n+k} x_i x_j = -1$ , hence,  $\sum_{i=1}^{n+k} x_i^2 = 2$ . This means that one can have  $|x_i| \ge 1$  only for one value of i, say, for i = n + k.

Hence, for each  $n \in \mathbf{N}^*$ ,  $n \ge 2$ , and for  $k \ge 0$  one has  $|\sum_{i=1}^{n+k} x_i^m| \le 2^{m/2} + 2$ . Indeed, one has  $|x_{n+k}| \le \sqrt{2}$  and  $|x_{n+k}^m| \le 2^{m/2}$ . For  $i \ne n+k$  one has  $|x_i^m| \le |x_i^2| = x_i^2$ , hence,  $|\sum_{i=1}^{n+k-1} x_i^m| \le \sum_{i=1}^{n+k-1} x_i^2 \le 2$ . The Vieta symmetric functions  $\sigma_l$  of  $x_1, \ldots, x_{n+k}$  (where  $\sigma_l = 1$ ).

The Vieta symmetric functions  $\sigma_l$  of  $x_1, \ldots, x_{n+k}$  (where  $\sigma_l = \sum_{1 \le i_1 < \cdots < i_l \le n+k} x_{i_1} \cdots x_{i_l}$ ) can be expressed as polynomials of the Newton symmetric functions  $\varphi_l = \sum_{i=1}^{n+k} x_i^l$ . Recall that there exist polynomials  $M_{\nu}, M_{\nu}^*$  such that

(1) 
$$\varphi_l = (-1)^{l-1} l \sigma_l + M_l(\sigma_1, \dots, \sigma_{l-1}),$$
$$(-1)^{l-1} l \sigma_l = \varphi_l + M_l^*(\varphi_1, \dots, \varphi_{l-1})$$

i.e. the passage from the Newton to the Vieta functions and its inverse are described by "triangular" formulas.

Hence, the first n Vieta functions, i.e. the first n coefficients  $a_m$  up to a sign of the polynomial R, are bounded by constants not depending on k (but only on n). Q.E.D.

**Notation 5.** In what follows we set  $a_3 = a$ ,  $a_4 = b$ , and we denote by  $\Pi'_n$  the projections of the sets  $\Pi_n$  on the space of the variables (a, b). Notice that one has  $\Pi'_n = \Pi_4(n-4) \cap \{a_1 = 0, a_2 = -1\}$ .

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**Remarks 6.** 1) Proposition 4 and Theorem 14 can be given shorter proofs if one uses the results of papers [Ko3] and [Ko4] concerning the so-called *very hyperbolic*<sup>1</sup> polynomials. We prefer to make the present text self-contained, therefore we do not use these results and we give direct proofs instead. Moreover, the proofs contain an explicit parametrization of the set  $\partial \Pi'_n$ , the boundary of  $\Pi'_n$ .

2) It is shown in [Ko3] that the mapping

$$\tau: a_j \mapsto \beta_j a_j$$
 where  $\beta_j = (n(n-1))^{j/2}/n(n-1)\cdots(n-j+1)$ 

defines a diffeomorphism between the set  $\Pi_n(\infty)$  and the set  $V\Pi_n$  of very hyperbolic polynomials. Set  $\beta_j = ((n(n-1))^{n/2}/n!)((n-j)!/(n(n-1)))^{(n-j)/2}$ . This allows one to view the mapping  $\tau$  as a superposition of the mappings  $\Phi: a_j \mapsto ((n(n-1))^{n/2}/n!)a_j$  (multiplication with a non-zero constant),  $\Psi: a_j \mapsto a_j/(n(n-1))^{(n-j)/2}$  (change of the scale of the x-axis) and  $\Xi: a_j \mapsto (n-j)!a_j$ .

The latter mapping is related to the *Laplace transform* which transforms the monomial  $x^k$  into  $\int_0^\infty t^k e^{-\xi t} dt = k!/\xi^{k+1}$  (the formula is meaningful for  $\operatorname{Re} \xi > 0$ ). Therefore the mapping  $\Xi$  is the Laplace transform followed by  $\xi \mapsto 1/x$  and by a division by x.

The mapping  $\Xi^{-1}$  results from the *Borel transform* which maps the formal power series  $\sum a_k x^k$  into the series  $\sum a_k x^k/k!$  (this accelerates the convergence). We call its inverse the *anti-Borel* transform. Thus the Borel (the anti-Borel) transform maps stably hyperbolic (very hyperbolic) polynomials into very hyperbolic (into stably hyperbolic) ones.

The following lines were communicated to the au-Comments 7. thor by B.Z. Shapiro and J. Borcea. Stably hyperbolic polynomials are interesting to study for the following reasons. Consider a linear operator T acting on the space of polynomials of degree < n which does not increase the degree of the polynomials. More exactly, suppose that it is "triangular":  $T(x^k) = x^k + R_k$  where  $R_k$  is a polynomial of degree  $\leq k-1, k=0, 1, \ldots, n$ . A theorem of Carnicer, Peña and Pinkus (see [CaPePi] states that if the operator T preserves hyperbolicity, then it is a differential one, i.e. of the form  $1 + c_1 D + \cdots + c_n D^n$  (\*),  $c_i \in \mathbf{C}$ , D := d/dx. This result has been recently generalized in [BoSh]. It is shown in [Bo] (see also [BoSh]) that an operator of the form (\*) (with  $c_i \in \mathbf{R}$ ) preserves hyperbolicity if and only if the polynomial  $T(x^n)$  is hyperbolic. In this case a partially proved conjecture due to J. Borcea and B.Z. Shapiro claims that the polynomial  $1 + c_1 x + \cdots + c_n x^n$  is stably hyperbolic.

<sup>&</sup>lt;sup>1</sup>i.e. hyperbolic and having hyperbolic primitives of all orders.

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### $\S 2$ . Properties of the set of stably hyperbolic polynomials

**Definition 8.** We stratify the sets  $\Pi_n$  and  $\Pi'_n$  the strata being defined by the *multiplicity vectors* (*MVs*) of the polynomials. A MV is a vector whose components are the multiplicities of the distinct roots of the polynomial given in decreasing order. Example: if n = 4 and if one has  $x_1 = x_2 > x_3 > x_4$ , then the MV of the polynomial is (2, 1, 1). We identify the strata with their MVs.

**Comments 9.** Recall that (see [Ko2]) the sets  $\Pi'_n$  look as shown on Fig. 1. The picture is symmetric w.r.t. Ob, the tangent lines and their limits at the strata of the form (k, n - k) are nowhere vertical.



Fig. 1.

Hence, the sets  $\Pi'_n$  and  $\Pi'_{n-1}$  together look as shown on Fig. 1. First of all, it is clear that  $\Pi'_n \supset \Pi'_{n-1}$  because if  $P \in \Pi_{n-1}$ , then  $xP \in \Pi_n$ . The set  $\partial \Pi'_n$  consists of the closures of all strata with MVs of the form (l, 1, n - l - 1) and (1, n - 2, 1). No point X of a stratum  $S = (l, 1, n - l - 2) \subset \Pi'_{n-1}$  lies on the boundary  $\partial \Pi'_n$  of  $\Pi'_n$ . Indeed, if the middle root (which is a simple one) of a polynomial  $P \in S$  is not 0, then the MV of the polynomial xP would be of the form (l, 1, 1, n - l - 2) (the left or the right root of P is not 0 because one has  $a_1 = 0$ ). This is not the MV of a stratum of  $\partial \Pi'_n$ . If the middle root of P is 0, then the MV of xP must be (l, 2, n - l - 2) which is not the MV of a stratum of  $\partial \Pi'_n$  is not the MV of a stratum of  $\partial \Pi'_n$ .

On the other hand, there exists a single point from the stratum  $(s, 1, n - s - 1) \subset \partial \Pi'_n$  or  $(1, n - 2, 1) \subset \partial \Pi'_n$  for which the middle root equals 0 (we leave the proof for the reader). Hence, this point is

the stratum  $(s, n - s - 1) \subset \partial \Pi'_{n-1}$  (resp. a point from the stratum  $(1, n - 3, 1) \subset \partial \Pi'_{n-1}$ ; clearly, this must be the point  $(0, 0) \in Oab$ ).

Using the above comments one can draw the sets  $\Pi'_n$  for  $n = 4, 5, \ldots$  together, see Fig. 2.



Fig. 2.

**Proposition 10.** The limits of the strata (n - s - 1, 1, s) and (s, 1, n - s - 1) of  $\partial \Pi'_n$  exist (for s fixed and  $n \to \infty$ ) as well as the limit for  $n \to \infty$  of the stratum (1, n - 2, 1). These limits are algebraic arcs (denoted by  $A_s$ ,  $B_s$  and C, see Fig. 2).

*Proof.* The closure of the stratum (n - s - 1, 1, s) can be parametrized by the three roots  $\xi \ge \eta \ge \zeta$  for which one has

(2) 
$$(n-s-1)\xi + \eta + s\zeta = 0, \quad (n-s-1)\xi^2 + \eta^2 + s\zeta^2 = 2$$

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These two equations define an ellipse in  $\mathbb{R}^3$ . Adding the inequalities  $\xi \ge \eta \ge \zeta$  means cutting off an arc of the ellipse. Hence, Vieta's formulas imply

$$\begin{aligned} -a &= C_{n-s-1}^3 \xi^3 + C_{n-s-1}^2 \xi^2 \eta + C_{n-s-1}^2 s \xi^2 \zeta + (n-s-1) s \xi \eta \zeta \\ &+ (n-s-1) C_s^2 \xi \zeta^2 + C_s^2 \eta \zeta^2 + C_s^3 \zeta^3 \\ b &= C_{n-s-1}^4 \xi^4 + \cdots \end{aligned}$$

Set  $\xi = \varphi/n$ . Hence, for  $n \to \infty$  equations (2) look like this:

(3) 
$$\varphi + \eta + s\zeta = 0, \quad \eta^2 + s\zeta^2 = 2$$

Indeed, the second of equations (2) implies that the quantities  $\eta$  and  $\zeta$  are uniformly bounded in  $n \in \mathbf{N}$ . The first of these equations implies that then  $\varphi$  is uniformly bounded as well. Hence, the term  $(n-s-1)\xi^2 = (n-s-1)\varphi^2/n^2$  in the second of equations (2) tends to 0 when  $n \to \infty$ .

Equations (3) are again a couple of equations defining an ellipse in  $\mathbf{R}^3$ . If  $\eta > 0$ , then for  $n \to \infty$  the inequality  $\varphi \ge n\eta$  implies that  $\varphi$  cannot be chosen such that  $(\varphi, \eta, \zeta)$  belong to the ellipse. Hence, one must have  $0 \ge \eta \ge \zeta$  (and there is no restriction upon  $\varphi$  other than the first of equations (3)). For  $n \to \infty$  one has

$$-a = \frac{\varphi^3}{6} + \frac{\varphi^2 \eta}{2} + \frac{s\varphi^2 \zeta}{2} + s\varphi \eta \zeta + C_s^2 \varphi \zeta^2 + C_s^2 \eta \zeta^2 + C_s^3 \zeta^3 + O\left(\frac{1}{n}\right),$$

i.e. for  $n \to \infty$  the limit of the quantity *a* is a homogeneous polynomial of degree 3 in  $\varphi$ ,  $\eta$  and  $\zeta$  which satisfy conditions (3) and the inequalities  $0 \ge \eta \ge \zeta$ . In the same way one shows that the limit of *b* is such a polynomial of degree 4. This proves the proposition for the arcs  $A_s$ , for the arcs  $B_s$  and *C* the proof is analogous. The reader can find the parametrization of the arc *C* in 7<sup>0</sup> of the proof of Theorem 14. Q.E.D.

**Remark 11.** One checks directly that neither of the arcs  $A_s$ ,  $B_s$ and C is a line segment. As each stratum (n - s - 1, 1, s), (1, n - 2, 1)and (s, 1, n - s - 1) of  $\partial \Pi'_n$  has a curvature of constant sign (see [Me1] or [Ko2]) such that the concavity is towards the interior of  $\Pi'_n$ , this is also the case of the arcs  $A_s$ ,  $B_s$  and C w.r.t.  $\Pi_4(\infty)$ .

**Notation 12.** Denote by D the point from  $\Pi_4(\infty)$  lying on the *b*-axis and with greatest *b*-coordinate.

**Remark 13.** The point D is the common limit of the right endpoints of the arcs  $A_s$  or of the left endpoints of the arcs  $B_s$  when  $s \to \infty$ . It can be computed also as the limit of the strata  $(k, k) \subset \Pi'_{2k}$  for  $k \to \infty$ . The computation gives D = (0, 1/2).

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**Theorem 14.** 1) The tangent lines to the arcs  $A_s$ ,  $B_s$  and C are never vertical. Their limits at the endpoints of these arcs exist and are not vertical either.

2) The slopes of these tangent lines (together with their limits at the endpoints) are uniformly bounded. These slopes (and their limits at the endpoints) are positive for the arcs  $A_s$  and negative for the arcs  $B_s$ .

3) At the common endpoint of two arcs  $A_s$ ,  $A_{s+1}$  or  $B_s$ ,  $B_{s+1}$  the slopes of the two limits of tangent lines (from left and right) are different.

4) At the common endpoints of the arcs  $A_1$  and C and of  $B_1$  and C the two limits of tangent lines are the same.

5) The limit of the slope of the tangent lines exists when the point from  $\partial \Pi_4(\infty)$  tends to D; this limit equals 0.

**Remarks 15.** 1) The boundary of the set  $\Pi_4(\infty)$  consists of countably many arcs whose endpoints accumulate towards the point D. These points are singular points for  $\Pi_4(\infty)$ , see 3) of the theorem. Hence, the set  $\Pi_4(\infty)$  is not semi-algebraic.

2) It is decidable whether a point  $U = (a^0, b^0) \in Oab$  represents a polynomial from  $\Pi_4(\infty)$  (in particular, from  $\partial \Pi_4(\infty)$ ) or not. This follows from the fact that one knows explicit parametrizations of the arcs  $A_s$ ,  $B_s$  and C and the coordinates of the point D.

Indeed, denote by  $(\alpha_s, \beta_s)$  (resp. by  $(\alpha_s^*, \beta_s^*)$ ) the left (resp. the right) endpoint of the arc  $A_s$  (resp.  $B_s$ ). By  $2^0$  of the proof of Theorem 14, see below, one has  $(\alpha_s, \beta_s) = ((-2/3)\sqrt{2/s}, 1/2-1/s)$ . One has first to check whether  $a^0 \in [\alpha_1, \alpha_1^*]$  or not. If not, then  $U \notin \Pi_4(\infty)$ . If yes, then one has to check whether  $a^0 = 0$  or not. If yes, then  $U \in \Pi_4(\infty)$  if and only if  $b^0 \in [0, 1/2]$ . If  $a^0 \neq 0$ , then one checks for which s one has  $a^0 \in [\alpha_s, \alpha_{s+1})$  or  $a^0 \in (\alpha_{s+1}^*, \alpha_s^*]$  (and which of these two conditions holds). After this one has to compare  $b^0$  with the b-coordinate of the points of the arcs  $A_s$ , C or  $B_s$ , C whose a-coordinates equal  $a^0$ .

Proof of Theorem 14.

1<sup>0</sup>. We use the notation from the proof of Proposition 10. Our first aim is to give explicit parametrization of the arc  $A_s$ . The one of the arc  $B_s$  is given by analogy and the one of the arc C is given in 7<sup>0</sup>. Consider first the stratum  $(n - s - 1, 1, s) \subset \partial \Pi'_n$ . Instead of operating with Vieta's functions  $a_j$  (in the variables  $\xi \geq \eta \geq \zeta$ , of multiplicities n-s-1, 1 and s), we use the sums  $b_j$  of the j-th powers of these variables (taking their multiplicities into account). Recall that (see formulas (1))

$$b_3 = 3a_3 + \alpha a_1 a_2 + \beta a_1^3, \quad b_4 = -4a_4 + \gamma a_1^4 + \delta a_1^2 a_2 + \varepsilon a_2^2 + \theta a_1 a_3$$

for some  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ ,  $\varepsilon$ ,  $\theta \in \mathbf{R}$ . As  $a_1 = 0$ ,  $a_2 = -1$ , we have  $b_3 = 3a_3$ ,  $b_4 = -4a_4 + \varepsilon$ . By computing the values of the symmetric functions for

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the quadruple  $1/\sqrt{2}$ ,  $1/\sqrt{2}$ ,  $-1/\sqrt{2}$ ,  $-1/\sqrt{2}$  one finds that  $\varepsilon = 2$ . Thus the stratum (n - s - 1, 1, s) is parametrized (in the variables  $\varphi$ ,  $\eta$ ,  $\zeta$ ) in the following form:

$$\varphi + \eta + s\zeta + O(1/n) = 0$$
  

$$\eta^2 + s\zeta^2 + O(1/n) = 2$$
  

$$a = a_3 = (1/3)(\eta^3 + s\zeta^3 + O(1/n))$$
  

$$b = a_4 = (-1/4)(\eta^4 + s\zeta^4) + 1/2 + O(1/n)$$

(see (2)) and after deleting the terms O(1/n) one obtains a parametrization of the arc  $A_s$ .

2<sup>0</sup>. Set  $\eta = \sqrt{2} \cos t$ ,  $\zeta = \sqrt{2/s} \sin t$ . Recall that  $0 \ge \eta \ge \zeta$  (see the proof of Proposition 10). The endpoints of the arc  $A_s$  are such that either  $(\eta, \zeta) = (0, -\sqrt{2/s})$  (and one has  $(a_3, a_4) = ((-2/3)\sqrt{2/s}, 1/2 - 1/s)$ , this is the left endpoint of  $A_s$ ) or  $\eta = \zeta = -\sqrt{2/(s+1)}$  (and one has  $(a_3, a_4) = ((-2/3)\sqrt{2/(s+1)}, 1/2 - 1/(s+1))$ , this is the right endpoint of  $A_s$ ).

In the new parametrization of the arc  $A_s$  one has

$$a = a_3 = (2/3)\sqrt{2}\cos^3 t + (2/3)\sqrt{2/s}\sin^3 t,$$
  
$$b = a_4 = 1/2 - \cos^4 t + (-1/s)\sin^4 t.$$

One has

(4) 
$$db/da = (db/dt)/(da/dt) = -(\sqrt{2}\cos t + \sqrt{2/s}\sin t) = -\eta - \zeta$$

This expression depends continuously on t and is uniformly bounded (both in s and t). In the case of arcs  $A_s$  we have  $0 \ge \eta \ge \zeta$  (and one cannot have both equalities at the same time), hence, db/da > 0. This proves parts 1) and 2) of the theorem for the arcs  $A_s$  (for the arcs  $B_s$ the proof is analogous).

3<sup>0</sup>. Recall that one has  $0 \ge \eta \ge -\sqrt{2/(s+1)}, -\sqrt{2/(s+1)} \ge \zeta \ge -\sqrt{2/s}$ . Hence, for  $s \to \infty$  the sum  $-\eta - \zeta$  (see (4)) tends to 0 uniformly in t. This proves part 5) of the theorem for the arcs  $A_s$  (in the same way one proves it for the arcs  $B_s$ ).

 $4^{0}$ . To prove part 3) of the theorem it suffices to compute the two values of db/da obtained for  $\eta$ ,  $\zeta$  corresponding to the right endpoint of  $A_{s}$  and to the left endpoint of  $A_{s+1}$ , see  $2^{0}$ . These values are  $2/\sqrt{s+1}$  and  $2/\sqrt{s+2}$ . Hence, they are different. For the arcs  $B_{s}$  the proof is analogous.

 $5^0$ . Part 4) of the theorem can be proved either by direct computation or by observing that the common endpoints in question are the

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limits of the strata (n-1, 1) and (1, n-1) of the sets  $\Pi'_n$  where the limits of the tangent lines to the strata (n-2, 1, 1), (1, n-2, 1) and (1, 1, n-2), (1, n-2, 1) coincide, see [Ko2]. We leave the details for the reader.

 $6^0$ . To extend the proof of parts 1) and 2) of the theorem to the arc *C* it suffices to observe that the slope of the tangent line to this arc is comprised between its limit values at the common endpoints with  $A_1$  and  $B_1$  due to the constant sign of the curvature, see Remark 11.

7<sup>0</sup>. Give the parametrization of the arc *C*. For a point of the closure of the stratum  $(1, n-2, 1) \subset \partial \Pi'_n$  defined by the roots  $\xi \ge \eta \ge \zeta$ , of multiplicities 1, n-2, 1, one has

$$\begin{split} \xi + (n-2)\eta + \zeta &= 0\\ \xi^2 + (n-2)\eta^2 + \zeta^2 &= 2\\ - a &= (n-2)\xi\eta\zeta + C_{n-2}^2(\xi\eta^2 + \zeta\eta^2) + C_{n-2}^3\eta^3\\ b &= C_{n-2}^2\xi\eta^2\zeta + C_{n-2}^3\eta^3(\xi + \zeta) + C_{n-2}^4\eta^4 \end{split}$$

Set  $\eta = \psi/n$ . Hence, when  $n \to \infty$  (and the given point tends to a point from C) one has  $\xi \ge 0 \ge \zeta$  and

$$\xi + \psi + \zeta = 0$$
  

$$\xi^{2} + \zeta^{2} = 2$$
  

$$-a = \xi \psi \zeta + (\xi \psi^{2} + \zeta \psi^{2})/2 + \psi^{3}/6$$
  

$$b = \xi \psi^{2} \zeta/2 + (\xi + \zeta) \psi^{3}/6 + \psi^{4}/24$$

These formulas provide the parametrization of the arc C. Q.E.D.

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# On weighted-degrees for algebraic local cohomologies associated with semiquasihomogeneous singularities

# Yayoi Nakamura and Shinichi Tajima

### Abstract.

In this paper, a notion of a weighted-degree is introduced to algebraic local cohomology classes associated with a semiquasihomogeneous function. Utilizing weighted-degrees, computations of a dual basis of Milnor algebra and membership problems are considered as applications.

# $\S1.$ Introduction

Let X be a neighbourhood of the origin O of n-dimensional affine space  $\mathbb{C}^n$ . Let f be a holomorphic function on X and S the hypersurface defined by the function f, i.e.,  $S = \{x \in X \mid f(x) = 0\}$ . We assume that the function f has an isolated singularity at the origin, i.e.,

$$\{x \in X \mid f_{x_1}(x) = \dots = f_{x_n}(x) = 0\} \cap S = \{O\},\$$

where  $f_{x_j} = \partial f / \partial x_j$  and  $x = (x_1, \ldots, x_n)$ . Let  $\mathcal{J}$  be Jacobi ideal in  $\mathcal{O}_{X,O}$  of the function f, and  $\mathcal{H}_f$  the set of algebraic local cohomology classes annihilated by  $\mathcal{J}$ , i.e.,

$$\mathcal{J} = \mathcal{O}_{X,O} \langle f_{x_1}, \dots, f_{x_n} \rangle \subset O_{X,O},$$
$$\mathcal{H}_f = \{ \eta \in \mathcal{H}^n_{[O]}(\mathcal{O}_X) \mid g\eta = 0, \ \forall g \in \mathcal{J} \}$$

where  $\mathcal{O}_{X,O}$  is the stalk at O of the sheaf  $\mathcal{O}_X$  of germs of holomorphic functions and  $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$  is the sheaf of *n*-th algebraic local cohomology groups, supported at the origin. Then,  $\mathcal{H}_f$  and  $\mathcal{O}_{X,O}/\mathcal{J}$  are finite dimensional vector spaces of the same dimension, i.e., Milnor number.  $\mathcal{H}_f$ 

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is isomorphic to  $\mathcal{E}xt^n_{\mathcal{O}_{X,O}}(\mathcal{O}_{X,O}/\mathcal{J},\mathcal{O}_{X,O})$  and thus there exists nondegenerate pairing,

(1.1) 
$$\operatorname{res}_O(\,\cdot\,,\,\cdot\,)\colon O_{X,\,O}/\mathcal{J}\times\mathcal{H}_f\to\mathbb{C}$$

between them defined by Grothendieck local residues ([3], [4]).

As  $\mathcal{H}_f$  is the dual space of  $\mathcal{O}_{X,O}/\mathcal{J}$ , the space  $\mathcal{H}_f$  reflects properties of a given singularity. Furthermore, these algebraic local cohomology classes in  $\mathcal{H}_f$  and the associated holonomic systems exhibit some characteristic features of the singularity ([5], [6]). This indicates therefore that further studies of  $\mathcal{H}_f$  in the context of D-modules would be of interest. In this paper, we study basic properties of  $\mathcal{H}_f$  for semiquasihomogeneous function and give a method for constructing a basis of  $\mathcal{H}_f$ . As applications, we consider a membership problem and a computation of standard basis. We show that Grothendieck local duality provides us with an effective method for these problem. Note that the approach and the results presented in this paper have applications to the study of holonomic systems attached to semiquasihomogeneous isolated singularities.

In Section 2, we define a notion of weighted-degrees of algebraic local cohomologies and study their properties. For semiquasihomogeneous functions f, we clarify relations of these properties for  $\mathcal{H}_f$  to Poincaré polynomial. In Section 3, by combining Grothendieck duality (1.1) and the notion of the weighted-degree with respect to the weight vector of the function f, we derive a method for constructing a basis of the space  $\mathcal{H}_f$  that gives rise to a dual basis of  $\mathcal{O}_{X,O}/\mathcal{J}$ . In Section 4, as applications, we study a membership problem for the ideal  $\mathcal{J}$  and a computation of a standard basis of  $\mathcal{J}$ . By making the most of the dual basis, we give an effective method for computing a standard basis of  $\mathcal{J}$  with examples.

### $\S$ **2.** Algebraic local cohomologies

We introduce a notion of weighted-degrees for algebraic local cohomology classes and study the dual space of  $\mathcal{O}_{X,O}/\mathcal{J}$  associated with semiquasihomogeneous singularities.

### 2.1. Definition of weighted-degrees

Let us fix a weight vector  $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n_+$  for a fixed coordinate system  $x = (x_1, \ldots, x_n) \in X$ . Put  $|\mathbf{w}| = w_1 + \cdots + w_n$  and  $\langle \mathbf{w}, \lambda \rangle = \lambda_1 w_1 + \cdots + \lambda_n w_n$  for  $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$ . Any algebraic local cohomology class  $\eta$  in  $\mathcal{H}^n_{[O]}(\mathcal{O}_X)$  can be expressed in terms of a relative Čech cohomology.

$$\eta = \left[\sum_{\lambda \in \Lambda_{\eta}} c_{\lambda} \frac{1}{x^{\lambda}}\right]$$

where  $c_{\lambda} \in \mathbb{C}$ ,  $x^{\lambda} = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$  with  $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_{\eta}$ ,  $\Lambda_{\eta}$  is a finite subset of  $\mathbb{N}^n_+$ . Then, we define the weighted-degree of an algebraic local cohomology class  $[1/x^{\lambda}]$  to be  $-\langle \mathbf{w}, \lambda \rangle$ . We call algebraic local cohomology classes, represented by a single term, monomials. An algebraic local cohomology class  $\eta \in \mathcal{H}^n_{[O]}(\mathcal{O}_X)$  can be written in the form

$$\eta = \left[ \sum_{\lambda \in \Lambda_{\eta}} c_{\lambda} \frac{1}{x^{\lambda}} \right] \quad \text{where} \quad c_{\lambda} \in \mathbb{C}, \ x^{\lambda} = x_{1}^{\lambda_{1}} \cdots x_{n}^{\lambda_{n}}$$
$$\text{with} \quad \lambda = (\lambda_{1}, \dots, \lambda_{n}) \in \Lambda_{\eta},$$

 $\Lambda_{\eta}$  is a finite subset of  $\mathbb{N}^{n}_{+}$  and  $[\cdot]$  stands for a relative Čech cohomology.

**Definition.** We define the weighted-degree  $\deg_{\mathbf{w}}(\eta)$  of a cohomology class

(2.1) 
$$\eta = \left[\sum_{\lambda \in \Lambda_{\eta}} c_{\lambda} \frac{1}{x^{\lambda}}\right]$$

by the smallest degree of monomials  $[1/x^{\lambda}]$  for  $\lambda \in \Lambda_{\eta}$ :

 $\deg_{\mathbf{w}}(\eta) = \min\{-\langle \mathbf{w}, \ \lambda \rangle \mid \lambda \in \Lambda_{\eta}\},\$ 

where  $\Lambda_{\eta}$  is a set of all exponents  $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n_+$  with non-zero coefficients  $c_{\lambda}$  in the above expression (2.1) of the cohomology class  $\eta$ .

### 2.2. Basic properties

Let  $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n_+$  be a weight vector. A polynomial f(x) is said to be weighted homogeneous of degree d with weight  $\mathbf{w}$  if f(x) is a sum of weighted homogeneous monomials of weighted degree d, i.e.,

$$f(x) = \sum_{\langle \mathbf{w}, \kappa \rangle = d} c_{\kappa} x^{\kappa}$$

where  $c_{\kappa} \in \mathbb{C}$ ,  $x^{\kappa} = x_1^{k_1} \cdots x_n^{k_n}$  and  $\langle \mathbf{w}, \kappa \rangle = w_1 k_1 + \cdots + w_n k_n$  for  $\kappa = (k_1, \ldots, k_n) \in \mathbb{N}^n$ . We define a weighted degree of a holomorphic

function h(x) to be the smallest degree of monomials  $x^{\kappa}$  constituting h(x);

$$\deg_{\mathbf{w}}(h) = \min\{\langle \mathbf{w}, \kappa \rangle \mid h(x) = \sum_{c_{\kappa} \neq 0} c_{\kappa} x^{\kappa}, \ c_{\kappa} \neq 0\}.$$

**Definition.** A polynomial f is said to be semiquasihomogeneous of weighted degree d if f is of the form  $f = f_0 + g$  where  $f_0$  is a weighted homogeneous polynomial of weighted degree d defining an isolated singularity at the origin and g is a function of weighted degree strictly greater than d.

Let f be a semiquasihomogeneous function with respect to a weight vector  $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{N}^n_+$ . Let  $\mathcal{J}_0$  denote Jacobi ideal of the weighted homogeneous part of the function f and  $E_0$  the canonical monomial basis of  $\mathcal{O}_{X,O}/\mathcal{J}_0$ . It is known ([1]) that  $E_0$  also gives a monomial basis of  $\mathcal{O}_{X,O}/\mathcal{J}$ . We use the notation E when we regard  $E_0$ as a monomial basis of  $\mathcal{O}_{X,O}/\mathcal{J}$ .

Let us recall the following result (see [1]):

**Lemma 2.1.** There exists exactly one basis monomial in E of degree  $n \cdot \deg_{\mathbf{w}}(f) - 2|\mathbf{w}|$ . Monomials  $x^{\kappa}$  belong to the ideal  $\mathcal{J}$  if  $\langle \mathbf{w}, \kappa \rangle > n \cdot \deg_{\mathbf{w}}(f) - 2|\mathbf{w}|$ .

Based on these results, we have the following:

**Proposition 2.1.** Any cohomology class  $\eta \in \mathcal{H}_f \setminus \{0\}$  satisfies the following inequality:

$$-n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}| \le \deg_{\mathbf{w}}(\eta) \le -|\mathbf{w}|.$$

And there exist cohomology classes in  $\mathcal{H}_f$  of degree  $-|\mathbf{w}|$  and  $-n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$ .

*Proof.* Since the set  $\Lambda_{\eta}$  of exponents for the cohomology class  $\eta$  is a subset of  $\mathbb{N}^n_+$ , we have  $\deg_{\mathbf{w}}(\eta) \leq -\langle \mathbf{w}, \mathbf{1} \rangle = -|\mathbf{w}|$  where  $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^n_+$ . The equality holds if and only if  $\Lambda_{\eta} = \{\mathbf{1}\}$  which corresponds to Dirac's delta function  $\delta = [1/(x_1 \cdots x_n)]$ . It is evident that  $\delta$  is in  $\mathcal{H}_f$ .

Assume, for the moment, that there exists a cohomology class  $\eta \in \mathcal{H}_f$  satisfying  $\deg_{\mathbf{w}}(\eta) < -n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$ . Put  $\deg_{\mathbf{w}}(\eta) = -n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}| - r$  for some positive integer  $r \in \mathbb{N}_+$ . Then there exists  $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_\eta$  such that  $\langle \mathbf{w}, \lambda \rangle = n \cdot \deg_{\mathbf{w}}(f) - |\mathbf{w}| + r$ . Then, for an exponent  $\kappa = \lambda - \mathbf{1} = (\lambda_1 - 1, \ldots, \lambda_n - 1) \in \mathbb{N}^n$ , we have  $x^{\kappa} \eta = c\delta$ 

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where c is a non-zero constant. On the other hand, since

$$\begin{aligned} \deg_{\mathbf{w}}(x^{\kappa}) &= \langle \mathbf{w}, \kappa \rangle \\ &= \langle \mathbf{w}, \lambda \rangle - \langle \mathbf{w}, \mathbf{1} \rangle \\ &= n \cdot \deg_{\mathbf{w}}(f) - |\mathbf{w}| + r - |\mathbf{w}| \\ &= n \cdot \deg_{\mathbf{w}}(f) - 2|\mathbf{w}| + r > n \cdot \deg_{\mathbf{w}}(f) - 2|\mathbf{w}|, \end{aligned}$$

we have  $x^{\kappa} \in \mathcal{J}$ , i.e.,  $x^{\kappa} \eta = 0$ , which is a contradiction.

Let  $e \in E$  be a monomial with the weighted-degree  $n \cdot \deg_{\mathbf{w}}(f) - 2|\mathbf{w}|$ . There exists a cohomology class  $\tau \in \mathcal{H}_f$  such that  $e\tau \in \mathcal{H}_f \setminus \{0\}$ . Then we have

$$deg_{\mathbf{w}}(e) + deg_{\mathbf{w}}(\tau) = n \cdot deg_{\mathbf{w}}(f) - 2|\mathbf{w}| + deg_{\mathbf{w}}(\tau)$$
  
$$\leq deg_{\mathbf{w}}(e\tau)$$
  
$$\leq -|\mathbf{w}|.$$

Thus  $\deg_{\mathbf{w}}(\tau) \leq -n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$ . On the other hand, since  $\tau \in \mathcal{H}_f$ ,  $\deg_{\mathbf{w}}(\tau) \geq -n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$ . Therefore we have  $\deg_{\mathbf{w}}(\tau) = -n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$ . Q.E.D.

**Proposition 2.2.** Let  $\eta$  be an algebraic local cohomology class belonging to  $\mathcal{H}_f$ . Then the following conditions are equivalent:

- (1)  $\deg_{\mathbf{w}}(\eta) = -n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|.$
- (2)  $\eta$  generates  $\mathcal{H}_f$  over  $\mathcal{O}_{X,O}$ .

Proof. It is obvious that a generator of  $\mathcal{H}_f$  over  $\mathcal{O}_{X,O}$  has a degree  $-n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$  since a degree of any holomorphic function in  $\mathcal{O}_{X,O}$  is greater than or equal to 0 and the smallest degree of classes in  $\mathcal{H}_f$  is  $-n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$ . Let  $\sigma$  be a generator of  $\mathcal{H}_f$ . There exists a function  $h = h(z) \in \mathcal{O}_{X,O}$  satisfying  $\eta = h\sigma$ . Since both  $\eta$  and  $\sigma$  are of degree  $-n \cdot \deg_{\mathbf{w}}(f) + |\mathbf{w}|$ , we have  $\deg_{\mathbf{w}}(h) = 0$  equivalently  $h(0) \neq 0$ . Thus, the function 1/h is in  $\mathcal{O}_{X,O}$  and  $\sigma$  can be represented by  $\sigma = (1/h)\eta$ . This completes the proof. Q.E.D.

**Corollary 2.1.** For any basis monomial  $e \in E$ , there exists a cohomology class  $\eta \in \mathcal{H}_f$  satisfying the following condition:

- (i)  $e\eta = c\delta$ , where  $\delta$  is the delta function with support at the origin and c is a non zero constant.
- (ii)  $\deg_{\mathbf{w}}(\eta) = -|\mathbf{w}| \deg_{\mathbf{w}}(e).$

*Proof.* Put  $d = n \cdot \deg_{\mathbf{w}}(f) - 2|\mathbf{w}|$ . Let  $e \in E$  be a monomial with the weighted-degree w. It is known that the number of the basis monomial of weighted-degree w is the same with that of d - w. Let

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 $e' \in E$  be a monomial with weighted-degree d - w. Then a monomial ee' is in  $\mathcal{O}_{X,0}/\mathcal{J}$ . For a generator  $\sigma$  of  $\mathcal{H}_f$  over  $\mathcal{O}_{X,O}$ , we have

$$deg_{\mathbf{w}}(ee'\sigma) = d + deg_{\mathbf{w}}(\sigma)$$
  
=  $(n \cdot deg_{\mathbf{w}}(f) - 2|\mathbf{w}|) + (-n \cdot deg_{\mathbf{w}}(f) + |\mathbf{w}|)$   
=  $-|\mathbf{w}|.$ 

Since the only element in  $\mathcal{H}_f$  with the weighted-degree  $-|\mathbf{w}|$  is the delta function  $\delta$ , we have  $ee'\sigma = c\delta$  with some non-zero constant c. Thus the algebraic local cohomology class  $\eta = e'\sigma$  satisfies the conditions (i) and (ii). Q.E.D.

The next corollary is obvious from Corollary 2.1.

**Corollary 2.2.** Let  $\chi_{\mathcal{J}}(t) = \sum_{j=1}^{p} \mu_{d_j} t^{d_j}$  be Poincaré polynomial of  $\mathcal{J}_0$  where  $\mu_{d_j} \in \mathbb{N}, j = 1, ..., \ell$ . There is a basis of  $\mathcal{H}_f$  consisting of  $\mu_{d_j}$  classes of the degree  $-d_j - |\mathbf{w}|$ .

For instance, for any generator  $\sigma$  over  $\mathcal{O}_{X,O}$  of  $\mathcal{H}_f$ , the set  $\{e_1\sigma, \ldots, e_\mu\sigma\}$  with  $e_j \in E$  gives a basis of  $\mathcal{H}_f$  enjoying Corollary 2.2.

In general, weighted-degrees of basis monomials of  $\mathcal{O}_{X,O}/\mathcal{J}$  depend on representatives and thus some monomial bases do not meet the condition of Poincaré polynomial. In contrast, the semiquasihomogeneity of f always warrants the existence of a basis of  $\mathcal{H}_f$  as claimed in Corollary 2.2.

### $\S$ **3.** Computation of the dual basis

In this section, we give a method for constructing relative Čech cohomologies that constitute the dual basis of E with respect to Grothendieck local residues.

Let  $f_0$  be the quasihomogeneous part of the semiquasihomogeneous function  $f \in \mathcal{O}_{X,O}$ . Let  $K_0$  be the set of exponents  $\kappa$  of basis monomials in  $E_0$ ;

$$K_0 = \{ \kappa \in \mathbb{N}^n \mid x^\kappa \in E_0 \}.$$

For an exponent  $\kappa \in \mathbb{N}^n$ , let  $\Gamma_{\kappa}$  be a set of multi indices  $\lambda \in \mathbb{N}^n_+$  satisfying  $\lambda - \mathbf{1} \notin K_0$  and  $\langle \mathbf{w}, \lambda \rangle = \langle \mathbf{w}, \kappa \rangle + |\mathbf{w}|;$ 

$$\Gamma_{\kappa} = \{ \lambda \in \mathbb{N}^n_+ \mid \lambda - \mathbf{1} \notin K_0, \ \langle \mathbf{w}, \lambda \rangle = \langle \mathbf{w}, \kappa \rangle + |\mathbf{w}| \}.$$

We have the following:

#### Proposition 3.1.

(1) For every exponent  $\kappa \in K_0$ , there exists an algebraic local cohomology class  $\eta_{0,\kappa}$  in  $\mathcal{H}_{f_0}$  of the form

(3.1) 
$$\eta_{0,\kappa} = \left[\frac{1}{x^{\kappa+1}} + \sum_{\lambda \in \Gamma_{\kappa}} c_{\lambda} \frac{1}{x^{\lambda}}\right].$$

(2) For  $K_0 = {\kappa_1, \ldots, \kappa_\mu}$ , algebraic local cohomology classes  $\eta_{0,\kappa_1}$ , ...,  $\eta_{0,\kappa_\mu}$  in (3.1) constitute the dual basis of  $E_0$  with respect to Grothendieck local duality between  $\mathcal{O}_{X,O}/\mathcal{J}_0$  and  $\mathcal{H}_{f_0}$ .

Let  $H_0 = \{\eta_{0,\kappa_1}, \ldots, \eta_{0,\kappa_{\mu}}\}$  be the dual basis given in Proposition 3.1 of  $E_0$ . Since the basis  $E_0$  for  $\mathcal{O}_{X,O}/\mathcal{J}_0$  also gives a basis E of  $\mathcal{O}_{X,O}/\mathcal{J}$ , we have the following:

**Proposition 3.2.** For each  $\eta_{0,\kappa} \in H_0$ , there exists algebraic local cohomology class  $\tau_{\kappa}$  satisfying  $\Lambda_{\tau_{\kappa}} \cap \Gamma_{\kappa} = \emptyset$  and  $\deg_{\mathbf{w}}(\tau_{\kappa}) > \deg_{\mathbf{w}}(\eta_{0,\kappa})$  or  $\tau_{\kappa} = 0$  such that the algebraic local cohomology class

(3.2) 
$$\eta_{\kappa} = \eta_{0,\kappa} + \tau_{\kappa}$$

belongs to  $\mathcal{H}_f$ .

Let us discuss a method for constructing algebraic local cohomology classes  $\eta_{\kappa}$  based on the above proposition.

There are monomials  $\eta_{\kappa}$  in  $\mathcal{H}_f$  that are determined by the conditions  $f_{x_j}\eta_{\kappa} = 0$  for all  $j = 1, \ldots, n$ . Note that such monomials also belong to  $H_0$ . Denote the set of these monomials  $\eta_{\kappa}$  in  $H_0$  by  $H_M$ :

$$H_M = \left\{ \left[ \frac{1}{x^{\lambda}} \right] \in H_0 \ \middle| \ f_{x_j} \left[ \frac{1}{x^{\lambda}} \right] = 0, \ \forall j = 1, \dots, n \right\}.$$

Let  $\Lambda_0$  be the set of multi indices defined by  $\{\lambda \in \mathbb{N}^n_+ \mid \lambda - \mathbf{1} \in K_0\}$ . Let  $L_\eta = \{\lambda \in \mathbb{N}^n_+ \mid \lambda \notin \Lambda_0, \langle \mathbf{w}, \lambda \rangle \leq -\deg_{\mathbf{w}}(\eta)\}$  for an algebraic local cohomology class  $\eta \in H_0$ . Then, in order for  $\eta_{\kappa}$  given in (3.2) to constitute the dual basis of E, we may take  $\tau_{\kappa}$  for  $\eta_{0,\kappa} \in H_0 \setminus H_M$  by a linear combination of monomials  $[1/x^{\lambda}]$  with  $\lambda \in L_{\eta_{0,\kappa}}$ .

We give a procedure for constructing the dual basis of E with respect to Grothendieck duality among  $\mathcal{O}_{X,O}/\mathcal{J}$  and  $\mathcal{H}_f$ . Put  $\Lambda_{\eta,x_j} = \Lambda_{f_{x_j}\eta}$ . Let  $R_\eta$  denote a set of multi indices defined by

$$R_{\eta} = \{\nu \in N \mid \exists j \in \{1, \ldots, n\}, \text{ s.t., } \Lambda_{\eta, x_j} \cap \Lambda_{[1/x^{\nu}], x_j} \neq \emptyset\}$$

where N is a given set of multi indices.

**Procedure 1.** Put  $H = H_M$ . For each  $\eta_0 = \eta_{0,k} \in H_0 \setminus H_M$ ,

- (1) Put  $\eta = \eta_0$  and  $N = L_{\eta_0}$ .
- (2) Until  $R_{\eta} = \emptyset$ , compute  $R_{\eta}$ , put  $\eta := \eta + \sum_{\nu \in R_{\eta}} c_{\nu} [1/x^{\nu}]$  with undetermined coefficients  $c_{\nu}$  and put  $N := N \setminus R_{\eta}$ .
- (3) Determine coefficients  $c_{\nu}$  in  $\eta$  by the condition

(4) 
$$f_{x_j}\eta = 0 \text{ for all } j = 1, \dots, n.$$
$$H = H \cup \{\eta\}.$$

**Theorem 3.1.** The set H of algebraic local cohomology classes constructed by Procedure 1 gives rise to the dual basis of E.

*Proof.* It is obvious that each  $\eta_{\kappa_j}$  constructed by the above procedure satisfies the condition

$$\operatorname{res}_0(\eta_{\kappa_j}, \, x^{\kappa_i}) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

It completes the proof.

**Example 1.** Let  $f = x^3y + y^6 + axy^5$  with a parameter *a*. This is a normal form of  $Z_{13}$  type semiquasihomogeneous function with the weighted-degree  $\deg_{\mathbf{w}}(f) = 18$  with respect to the weight vector  $\mathbf{w} = (5, 3) \in \mathbb{N}^2_+$ . The quasihomogeneous part of the function f is  $f_0 = x^3y + y^6$  and thus

$$E = \{1, y, x, y^2, xy, y^3, x^2, xy^2, xy^3, y^5, xy^4, xy^5\}.$$

We have

$$\begin{bmatrix} \frac{1}{xy} \end{bmatrix}, \begin{bmatrix} \frac{1}{xy^2} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y} \end{bmatrix}, \begin{bmatrix} \frac{1}{xy^3} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y^2} \end{bmatrix}, \begin{bmatrix} \frac{1}{xy^4} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^3y} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y^3} \end{bmatrix}, \begin{bmatrix} \frac{1}{xy^5} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y^4} \end{bmatrix}, \begin{bmatrix} \frac{1}{xy^6} - 6\frac{1}{x^4y} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y^5} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y^6} - 6\frac{1}{x^5y} \end{bmatrix} \in \mathcal{H}_{f_0}.$$

The partial derivatives of the function f are  $f_x = 3x^2y + ay^5$  and  $f_y = x^3 + 6y^5 + 5axy^4$ . Then, the set  $H_M$  is given by the following ten monomials:

$$\begin{bmatrix} \frac{1}{xy} \end{bmatrix}, \begin{bmatrix} \frac{1}{xy^2} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y} \end{bmatrix}, \begin{bmatrix} \frac{1}{xy^3} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y^2} \end{bmatrix}, \begin{bmatrix} \frac{1}{xy^4} \end{bmatrix}, \\ \begin{bmatrix} \frac{1}{x^3y} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y^3} \end{bmatrix}, \begin{bmatrix} \frac{1}{xy^5} \end{bmatrix}, \begin{bmatrix} \frac{1}{x^2y^4} \end{bmatrix}.$$

Q.E.D.

Thus, in order to construct the dual basis of E, it suffices to compute other three cohomology classes in  $\mathcal{H}_f$  with quasihomogeneous parts

$$\left[\frac{1}{xy^6} - 6\frac{1}{x^4y}\right], \left[\frac{1}{x^2y^5}\right], \left[\frac{1}{x^2y^6} - 6\frac{1}{x^5y}\right].$$

- (1) Let  $\eta_0 = [(1/xy^6) 6(1/x^4y)]$ . Then  $L_{\eta_0} = \{(3, 2), (4, 1)\}$ . Put  $\eta = \eta_0$ . Then  $\Lambda_{\eta, x} = \Lambda_{\eta, y} = \{(1, 1)\}$ . Put  $N = \{(3, 2), (4, 1)\}$ . Since  $R_\eta = \{(3, 2)\}$ , put  $\eta = \eta_0 + c[1/x^3y^2]$ . By the condition  $f_x\eta = f_y\eta = 0$ , we have c = -(1/3)a.
- (2) Let  $\eta_0 = [1/x^2y^5]$ . Then  $L_{\eta_0} = \{(3, 2), (4, 1), (3, 3)\}$ . Put  $\eta = \eta_0$ . Then  $\Lambda_{\eta, x} = \emptyset$  and  $\Lambda_{\eta, y} = \{(1, 1)\}$ . Put  $N = \{(3, 2), (4, 1), (3, 3)\}$ . Since  $R_\eta = \{(4, 1)\}$ , put  $\eta = \eta_0 + c[1/x^4y]$ . By the condition  $f_x\eta = f_y\eta = 0$ , we have c = -5a.

(3) Let 
$$\eta_0 = [(1/x^2y^6) - 6(1/x^5y)]$$
. Then

$$L_{\eta_0} = \{(3, 2), (4, 1), (3, 3), (1, 7), (4, 2), (3, 4), (5, 1)\}.$$

Put  $\eta = \eta_0$ . Then  $\Lambda_{\eta, x} = \{(2, 1)\}, \Lambda_{\eta, y} = \{(2, 1), (1, 2)\}.$ Put  $N = L_{\eta_0}$ . Since  $R_\eta = \{(1, 7), (4, 2), (3, 3)\}$ , put

$$\eta = \eta_0 + s[1/xy^7] + t[1/x^4y^2] + u[1/x^3y^3].$$

Then  $\Lambda_{\eta, x} = \{(2, 1), (1, 2)\}$  and  $\Lambda_{\eta, y} = \{(2, 1), (1, 2)\}.$ Put  $N = \{(3, 2), (4, 1), (3, 4), (5, 1)\}.$ Then,  $R_{\eta} = \emptyset.$ Now,  $\eta = [(1/x^2y^6) - 6(1/x^5y) + s(1/xy^7) + t(1/x^4y^2) + u(1/x^3y^3)].$ By the condition  $f_x \eta = f_y \eta = 0$ , we have s = -(7/9)a, t = -(1/3)a and  $u = (7/27)a^2.$ 

Thus, the dual basis of E with respect to Grothendieck pairing between  $\mathcal{O}_{X,O}/\mathcal{J}$  and  $\mathcal{H}_f$  is given by

$$\left\{ \left[\frac{1}{xy}\right], \left[\frac{1}{xy^2}\right], \left[\frac{1}{x^2y}\right], \left[\frac{1}{xy^3}\right], \left[\frac{1}{x^2y^2}\right], \left[\frac{1}{xy^4}\right], \left[\frac{1}{x^3y}\right], \left[\frac{1}{x^2y^3}\right], \left[\frac{1}{x^2y^3}\right], \left[\frac{1}{xy^6} - 6\frac{1}{x^4y} - \frac{1}{3}a\frac{1}{x^3y^2}\right], \left[\frac{1}{x^2y^5} - 5a\frac{1}{x^4y}\right], \left[\frac{1}{x^2y^6} - 6\frac{1}{x^5y} - \frac{7}{9}a\frac{1}{xy^7} - \frac{1}{3}a\frac{1}{x^4y^2} + \frac{7}{27}a^2\frac{1}{x^3y^3}\right] \right\}.$$

## $\S4.$ Applications

We give two applications of results in Section 3; one is a method for solving a membership problem for Jacobi ideal  $\mathcal{J}$ , the other is a method for computing a standard basis of  $\mathcal{J}$ .

### 4.1. A membership problem

Let us recall the following result which immediately follows from Grothendieck local duality (1.1):

**Proposition 4.1.** Let  $p(x) \in \mathcal{O}_{X,O}$ . Then,  $\operatorname{res}_O(p(x), \eta) = 0$  for all  $\eta \in \mathcal{H}_f$  is a necessary and sufficient condition for p(x) to be in the ideal  $\mathcal{J}$ .

By using the dual basis H of E constructed by Procedure 1, we can find whether a given p(x) is in  $\mathcal{J}$  based on Proposition 4.1. For the dual basis H of E, let

$$K = \bigcup_{\eta \in H} \{ \kappa \in \mathbb{N}^n \mid \kappa + \mathbf{1} \in \Lambda_\eta \}$$

and  $K_M = \{ \kappa \in K \mid [1/x^{\kappa+1}] \in H_M \}$ . Then,

(1) if there are monomials  $x^{\kappa}$  in p(x) with  $\kappa \in K_M$ , p(x) does not belong to the ideal  $\mathcal{J}$ .

On the other hand, Proposition 4.1 assures that

(2) linear combinations of monomials  $x^{\kappa}$  with exponents  $\kappa$  satisfying  $\kappa \notin K$  belong to  $\mathcal{J}$ .

Let  $K(p) = \{\kappa \in \mathbb{N}^n \mid p(x) = \sum a_{\kappa} x^{\kappa}, a_{\kappa} \neq 0\}$  for a function p(x)and  $K' = K \setminus K_M$ . Then, after testing the above two conditions (1) and (2), it suffices to find if the part q(x) of a given function satisfying  $K(q) \subset K'$  belongs to  $\mathcal{J}$  or not. By following the procedure below, one can solve the membership problem for the ideal  $\mathcal{J}$ .

**Procedure 2.** For a given function p(x),

If  $K(p) \cap K_M \neq \emptyset$ , then  $p(x) \notin \mathcal{J}$ . Else, let q(x) be the part of p(x) given by the linear combination of monomials  $x^{\kappa}$  with  $\kappa \in K'$ , i.e.,  $p(x) = q(x) + \sum_{\kappa \notin K'} c_{\kappa} x^{\kappa}$ . if q(x) satisfies  $\operatorname{res}_O(q(x), \eta) = 0$  for all  $\eta \in H \setminus H_M$ , then  $p(x) \in \mathcal{J}$ . else,  $p(x) \notin \mathcal{J}$ .

# 4.2. A standard basis

Making use of the dual basis of E constructed by the above procedure, we can compute a standard basis of the ideal  $\mathcal{J}$ . Note that the method described below is also applicable to the case where the given function f contains parameters. Let us illustrate a procedure for computing a standard basis of  $\mathcal{J}$  by using examples. Following notations will be used in examples,

$$K_{\eta} = \{ \kappa \in \mathbb{N}^n \mid \kappa + \mathbf{1} \in \Lambda_{\eta} \}, \quad \Delta = \sum_{\eta \in H \setminus H_M} K_{\eta}.$$

Let  $\succ$  be the lexicographical ordering, and  $\succ_{\mathbf{w}}$  defined by

$$x^{\alpha} \succ_{\mathbf{w}} x^{\beta} \Leftrightarrow (\langle \mathbf{w}, \alpha \rangle < \langle \mathbf{w}, \beta \rangle) \text{ or } (\langle \mathbf{w}, \alpha \rangle = \langle \mathbf{w}, \beta \rangle \text{ and } x^{\alpha} \succ x^{\beta}).$$

**Example 2.** Let us compute a standard basis of  $\mathcal{J}$  for the same function f with Example 1. As seen in Example 1, three algebraic local cohomology classes

$$\eta_1 = \left[\frac{1}{xy^6} - 6\frac{1}{x^4y} - \frac{1}{3}a\frac{1}{x^3y^2}\right], \ \eta_2 = \left[\frac{1}{x^2y^5} - 5a\frac{1}{x^4y}\right]$$

and

$$\eta_3 = \left[\frac{1}{x^2y^6} - 6\frac{1}{x^5y} - \frac{7}{9}a\frac{1}{xy^7} - \frac{1}{3}a\frac{1}{x^4y^2} + \frac{7}{27}a^2\frac{1}{x^3y^3}\right]$$

together with  $H_M$  constitute the dual basis of E. Then,

$$K_{\eta_1} = \{ (2, 1), (3, 0), (0, 5) \}, K_{\eta_2} = \{ (3, 0), (1, 4) \}, K_{\eta_3} = \{ (2, 2), (3, 1), (4, 0), (0, 6), (1, 5) \}$$

and

$$\Delta = \{ (2, 1), (3, 0), (2, 2), (3, 1), (4, 0), (0, 5), (1, 4), (0, 6), (1, 5) \}.$$

Put  $G = \Delta$ .

(1) The exponent (2, 1) is the smallest one in  $\Delta$  with respect to  $\succ_{\mathbf{w}}$ . Since (2, 1) is only in  $K_{\eta_1}$ , take the biggest one (0, 5) from  $(K_{\eta_1} \cap G) \setminus \{(2, 1)\}$ . Since (2, 1)  $\succ_{\mathbf{w}}$  (0, 5), put  $p(x, y) = x^2y + sy^5$ . By the conditions  $\operatorname{res}_0(p(x, y), \eta_1) = 0$ , we have  $p(x, y) = x^2y + (1/3)ay^5 \in \mathcal{J}$ . Put  $G = G \setminus (G \cap \{(i, j) \mid i \geq 2, j \geq 1\})$  $= \{(3, 0), (4, 0), (0, 5), (1, 4), (0, 6), (1, 5)\}.$ 

(2) The exponent (3, 0) appears both in  $K_{\eta_1}$  and  $K_{\eta_2}$ . Take the biggest ones (1, 4) from  $(K_{\eta_1} \cap G) \setminus \{(3, 0)\}$  and (0, 5) from  $(K_{\eta_2} \cap G) \setminus \{(3, 0)\}$  respectively. Since  $x^3 \succ_{\mathbf{w}} y^5 \succ_{\mathbf{w}} xy^4$ , put  $q(x, y) = x^3 + sy^5 + txy^4$ . By the conditions  $\operatorname{res}_0(q(x, y), \eta_1) = \operatorname{res}_0(q(x, y), \eta_2) = 0$ , we have  $q(x, y) = x^3 + 6y^5 + 5axy^4 \in \mathcal{J}$ . Put  $G = G \setminus (G \cap \{(i, j) \mid i \ge 3, j \ge 0\})$  $= \{(0, 5), (1, 4), (0, 6), (1, 5)\}.$ 

- (3) While the exponent  $(0, 5) \in K_{\eta_1}$  is the smallest one in  $G, y^5$  can not become the leading term of any generator of  $\mathcal{J}$  because  $(K_{\eta_1} \cap G) \setminus \{(0, 5)\} = \emptyset$ . Put  $G = G \setminus \{(0, 5)\} = \{(1, 4), (0, 6), (1, 5)\}.$
- (4) While the exponent  $(1, 4) \in K_{\eta_2}$  is the smallest one in G,  $xy^4$  can not become the leading term of any generator of  $\mathcal{J}$  because  $(K_{\eta_2} \cap G) \setminus \{(1, 4)\} = \emptyset$ . Put  $G = G \setminus \{(1, 4)\} = \{(0, 6), (1, 5)\}.$
- (5) The exponent  $(0, 6) \in K_{\eta_3}$  is the smallest one in  $\Delta$  and the other exponent (1, 5) in G is also belong to  $K_{\eta_3}$ . Since  $(0, 6) \succ_{\mathbf{w}} (1, 5)$ , put  $r(x, y) = y^6 + sxy^5$ . By the condition  $\operatorname{res}_0(r(x, y), \eta_3) = 0$ , we have  $r(x, y) = y^6 + (7/9)ax^5$ .

By the condition of the weighted-degrees, we have  $y^7 \in \mathcal{J}$ . Then, we have constructed a standard basis

$$\{y^7,\,y^6+(7/9)axy^5,\,x^3+6y^5+5axy^4,\,x^2y+(1/3)ay^5\}$$

of the ideal  $\mathcal{J}$  with respect to  $\succ_{\mathbf{w}}$ .

**Example 3.** Let us consider a plane curve defined by  $x = t^5$  and  $y = t^{16} + t^{54}$ . The defining equation of this curve is

$$f(x, y) = x^{16} - y^5 + 5x^{14}y^3 - 5x^{28}y + x^{54}.$$

This is a semiquasihomogeneous function with a weighted homogeneous part  $f_0(x, y) = x^{16} - y^5$  of the weighted-degree 80 with respect to the weight vector  $\mathbf{w} = (5, 16)$ . Then, the dual basis  $H_{f_0}$ of  $E_0 = \{x^i y^j \mid 0 \le i \le 14, 0 \le j \le 3\}$  is given by monomials  $\{[1/x^k y^l] \mid 1 \le k \le 15, 1 \le l \le 4\}.$ 

Since,  $H_M = H_{f_0} \setminus \{[1/x^{15}y^3], [1/x^{14}y^4], [1/x^{15}y^4]\}$ , in order to construct the dual basis of  $E(=E_0)$ , it suffices to find cohomology classes with terms  $[1/x^{15}y^3], [1/x^{14}y^4], [1/x^{15}y^4]$  respectively. By direct computations, we obtain algebraic local cohomology classes  $[(1/x^{15}y^3) + 3(1/xy^5)]$  and  $[(1/x^{14}y^4) - (35/8)(1/x^{16}y)]$  that belong to  $H_f$ . It is easy to verify that  $[(1/x^{15}y^4) - (35/8)(1/x^{17}y) + 3(1/xy^6)]$  belongs to  $H_f$ .

Then, cohomology classes

$$\eta_1 = \left[\frac{1}{x^{15}y^3} + 3\frac{1}{xy^5}\right], \ \eta_2 = \left[\frac{1}{x^{14}y^4} - \frac{35}{8}\frac{1}{x^{16}y}\right],$$
$$\eta_3 = \left[\frac{1}{x^{15}y^4} - \frac{35}{8}\frac{1}{x^{17}y} + 3\frac{1}{xy^6}\right]$$

together with  $H_M$  constitute the dual basis of E.

In order to construct a standard basis of  $\mathcal{J}$ , it suffices to use  $\eta_1$  and  $\eta_2$ . It is easy to see that  $35x^{13}y^3 + 8x^{15}$  and  $3x^{14}y^2 - y^4$  constitute a standard basis of  $\mathcal{J}$  with respect to the total lexicographic ordering.

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# The geometry of continued fractions and the topology of surface singularities

# Patrick Popescu-Pampu

## Abstract.

We survey the use of continued fraction expansions in the algebraical and topological study of complex analytic singularities. We also prove new results, firstly concerning a geometric duality with respect to a lattice between plane supplementary cones and secondly concerning the existence of a canonical plumbing structure on the *abstract boundaries* (also called *links*) of normal surface singularities. The duality between supplementary cones gives in particular a geometric interpretation of a duality discovered by Hirzebruch between the continued fraction expansions of two numbers  $\lambda > 1$  and  $\lambda/(\lambda - 1)$ .

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# $\S1$ . Introduction

Continued fraction expansions appear naturally when one resolves germs of plane curves by sequences of plane blowing-ups, or Hirzebruch-Jung (that is, cyclic quotient) surface singularities by toric modifications.

They also appear when one passes from the natural plumbing decomposition of the abstract boundary of a normal surface singularity to its minimal JSJ decomposition. In this case it is very important to keep track of natural orientations. In general, as was shown by Neumann [57], if one changes the orientation of the boundary, the resulting 3-manifold is no more orientation-preserving diffeomorphic to the boundary of an isolated surface singularity. The only exceptions are *Hirzebruch-Jung* singularities and cusp-singularities. This last class of singularities got

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its name from its appearance in Hirzebruch's work [37] as germs at the compactified cusps of Hilbert modular surfaces. For both classes of singularities, one gets an involution on the set of analytical isomorphism types of the singularities in the class, by changing the orientation of the boundary. From the viewpoint of computations, Hirzebruch saw that both types of singularities have structures which can be encoded in continued fraction expansions of positive integers, and that the previous involution manifests itself in a duality between such expansions.

In the computations with continued fractions alluded to before, there appear in fact two kinds of continued fraction expansions. Some are constructed using only additions - we call them in the sequel *Euclidean* continued fractions - and the others using only subtractions - we call them *Hirzebruch-Jung continued fractions*. There is a simple formula, also attributed to Hirzebruch, which allows to pass from one type of continued fraction expansion of a number to the other one. Both types of expansions have geometric interpretations in terms of polygonal lines  $P(\sigma)$ . If  $(L, \sigma)$  is a pair consisting of a 2-dimensional lattice L and a strictly convex cone  $\sigma$  in the associated real vector space,  $P(\sigma)$  denotes the boundary of the convex hull of the set of lattice points situated inside  $\sigma$  and different from the origin.

For Euclidean continued fractions this interpretation is attributed to Klein [45], while for the Hirzebruch-Jung ones it is attributed to Cohn [12].

It is natural to try to understand how both geometric interpretations fit together. By superimposing the corresponding drawings, we were led to consider *two supplementary cones* in a real plane, in the presence of a lattice. By *supplementary cones* we mean two closed strictly convex cones which have a common edge and whose union is a half-plane.

Playing with examples made us understand that the algebraic duality between continued fractions alluded to before has as geometric counterpart a duality between two supplementary cones in the plane with respect to a lattice. This duality is easiest to express in the case where the edges of the cones are irrational:

Suppose that the edges of the supplementary cones  $\sigma$  and  $\sigma'$  are irrational. Then the edges of each polygonal line  $P(\sigma)$  and  $P(\sigma')$  correspond bijectively in a natural way to the vertices of the other one.

When at least one of the edges is rational, the correspondence is slightly more complicated (there is a defect of bijectivity near the intersection points of the polygonal lines with the edges of the cones), as explained in Proposition 5.3. In this duality, points correspond to lines and conversely, as in the classical polarity relation between points and lines with respect to a conic. But the duality relation described in this paper is more elementary, in the sense that it uses only parallel transport in the plane. For this reason it can be explained very simply by drawing on a piece of cross-ruled paper.

The duality between supplementary cones gives a simple way to think about the relation between the pair  $(L, \sigma)$  and its dual pair  $(\check{L}, \check{\sigma})$ and in particular about the relations between various invariants of toric surfaces (see Section 6). Indeed (see Proposition 5.11):

The supplementary cone of  $\sigma$  is canonically isomorphic over the integers with the dual cone  $\check{\sigma}$ , once an orientation of L is fixed.

As stated at the beginning of the introduction, computations with continued fractions appear also when one passes from the canonical plumbing structure on the boundary of a normal surface singularity to its minimal JSJ structure. Using this, Neumann [57] showed that the topological type of the minimal good resolution of the germ is determined by the topological type of the link. In fact all continued fractions appearing in Neumann's work are the algebraic counterpart of pairs  $(L, \sigma)$  canonically determined by the topology of the boundary. Using this remark, we prove the stronger statement (see Theorem 9.7):

The plumbing structure on the boundary of a normal surface singularity associated to the minimal normal crossings resolution is determined up to isotopy by the oriented ambient manifold. In particular, it is invariant up to isotopy under the group of orientation-preserving self-diffeomorphisms of the boundary.

In order to prove this theorem we have to treat separately the boundaries of Hirzebruch-Jung and cusp singularities. In both cases, we show that the oriented boundary determines naturally a pair  $(L, \sigma)$  as before. If one changes the orientation of the boundary, one gets a supplementary cone. In this way, the involution defined before on both sets of singularities is a manifestation of the geometric duality between supplementary cones (see Propositions 9.3 and 9.6).

For us, the moral of the story we tell in this paper is the following one:

If one meets computations with either Euclidean or Hirzebruch-Jung continued fractions in a geometrical problem, it means that somewhere behind is present a natural 2-dimensional lattice L and a couple of lines in the associated real vector space. One has first to choose one of the two pairs of opposite cones determined by the four lines and secondly an ordering of the edges of those cones. These choices may be dictated by choices of orientations of the manifolds which led to the construction of the lattice and the cones. So, in order to think geometrically at the computations with continued fractions, recognize the lattice, the lines and the orientation choices.

Let us outline now the content of the paper.

Someone who is interested only in the algebraic relations between the Euclidean and the Hirzebruch-Jung continued fraction expansions of a number can consult only Section 2. If one is also interested in their geometric interpretation, one can read Sections 3 and 4.

In Section 5 we prove geometrically the relations between the two kinds of continued fractions using the duality between supplementary cones described before. We introduce also a new kind of graphical representation which we call the *zigzag diagram*, allowing to visualize at the same time the algebra and the geometry of the continued fraction expansions of a number.

In Section 6 we give applications of zigzag diagrams to the algebraic description of special curve and surface singularities, defined using toric geometry.

Sections 8 and 9 are dedicated to the study of topological aspects of the links of normal surface singularities, after having recalled in Section 7 general facts about Seifert, graph, plumbing and JSJ-structures on 3manifolds.

We think that the new results of the paper are Proposition 5.3, Theorem 9.1 and Theorem 9.7, as well as the very easy Proposition 5.11, which is nevertheless essential in order to understand the relation between dual cones in terms of parallelism, using Proposition 5.3.

We wrote this paper having in mind as a potential reader a graduate student who wants to be initiated either to the algebra of surface singularities or to their topology. That is why we tried to communicate basic intuitions, often referring to the references for complete proofs.

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# §2. Algebraic comparison of Euclidean and Hirzebruch-Jung continued fractions

**Definition 2.1.** If  $x_1, \ldots, x_n$  are variables, we consider two kinds of continued fractions associated to them:

$$[x_1, \dots, x_n]^+ := x_1 + \frac{1}{x_2 + \frac{1}{\dots + \frac{1}{x_n}}}$$
$$[x_1, \dots, x_n]^- := x_1 - \frac{1}{x_2 - \frac{1}{\dots - \frac{1}{x_n}}}$$

We call  $[x_1, \ldots, x_n]^+$  a Euclidean continued fraction (abbreviated E-continued fraction) and  $[x_1, \ldots, x_n]^-$  a Hirzebruch-Jung continued fraction (abbreviated HJ-continued fraction).

The first name is motivated by the fact that E-continued fractions are tightly related to the Euclidean algorithm: if one applies this algorithm to a couple of positive integers (a, b) and the successive quotients are  $q_1, \ldots, q_n$ , then  $a/b = [q_1, \ldots, q_n]^+$ . See Hardy & Wright [32], Davenport [15] for an introduction to their arithmetics and Fowler [22] for the relation with the Greek theories of proportions. An extended bibliography on their applications can be found in Brezinski [7] and Shallit [72].

The second name is motivated by the fact that HJ-continued fractions appear naturally in the Hirzebruch-Jung method of resolution of singularities, originating in Jung [42] and Hirzebruch [35], as explained after Definition 6.4 below.

Define two sequences  $(Z^{\pm}(x_1, \ldots, x_n))_{n \ge 1}$  of polynomials with integer coefficients, by the initial data

$$Z^{\pm}(\emptyset) = 1, \ Z^{\pm}(x) = x$$

and the recurrence relations:

(1) 
$$Z^{\pm}(x_1, \ldots, x_n)$$
  
=  $x_1 Z^{\pm}(x_2, \ldots, x_n) \pm Z^{\pm}(x_3, \ldots, x_n), \quad \forall n \ge 2.$ 

Then one proves immediately by induction on n the following equality of rational fractions:

(2) 
$$[x_1, \ldots, x_n]^{\pm} = \frac{Z^{\pm}(x_1, \ldots, x_n)}{Z^{\pm}(x_2, \ldots, x_n)}, \quad \forall n \ge 1.$$

Also by induction on n, one proves the following twin of relation (1):

(3) 
$$Z^{\pm}(x_1, \ldots, x_n)$$
  
=  $Z^{\pm}(x_1, \ldots, x_{n-1})x_n \pm Z^{\pm}(x_1, \ldots, x_{n-2}), \quad \forall n \ge 2.$ 

which, combined with (1), proves the following symmetry property:

(4) 
$$Z^{\pm}(x_1, \ldots, x_n) = Z^{\pm}(x_n, \ldots, x_1), \quad \forall n \ge 1.$$

If  $(y_1, \ldots, y_k)$  is a finite sequence of numbers or variables and  $m \in \mathbb{N} \cup \{+\infty\}$ , we denote by

$$(y_1,\ldots,y_k)^m$$

the sequence obtained by repeating m times the sequence  $(y_1, \ldots, y_k)$ . By convention, when m = 0, the result is the empty sequence.

Each number  $\lambda \in \mathbf{R}$  can be expanded as (possibly infinite) Euclidean and Hirzebruch-Jung continued fractions:

$$\lambda = [a_1, a_2, \dots]^+ = [\alpha_1, \alpha_2, \dots]^-$$

with the conditions:

(5) 
$$a_1 \in \mathbf{Z}, \ a_n \in \mathbf{N} - \{0\}, \quad \forall n \ge 1$$

(6) 
$$\alpha_1 \in \mathbf{Z}, \ \alpha_n \in \mathbf{N} - \{0, 1\}, \quad \forall n \ge 1$$

Of course, we consider only indices n effectively present. For an infinite number of terms, these conditions ensure the existence of the limits  $[a_1, a_2, \ldots]^+ := \lim_{n \to +\infty} [a_1, \ldots, a_n]^+$  and  $[\alpha_1, \alpha_2, \ldots]^- := \lim_{n \to +\infty} [\alpha_1, \ldots, \alpha_n]^-$ .

Any sequence  $(a_n)_{n\geq 1}$  which verifies the restrictions (5) can appear and the only ambiguity in the expansion of a number as a E-continued fraction comes from the identity:

(7) 
$$[a_1, \ldots, a_n, 1]^+ = [a_1, \ldots, a_{n-1}, a_n + 1]^+$$

We deduce that any real number  $\lambda \neq 1$  admits a unique expansion as a E-continued fraction such that condition (5) is satisfied and in the case that the sequence  $(a_n)_n$  is finite, its last term is different from 1. When we speak in the sequel about the E-continued fraction expansion of a number  $\lambda \neq 1$ , it will be about this one. By analogy with the vocabulary of the Euclidean algorithm, we say that the numbers  $(a_n)_{n\geq 1}$ are the E-partial quotients of  $\lambda$ .

Similarly, any sequence  $(\alpha_n)_{n\geq 1}$  which verifies the restrictions (6) can appear and the only ambiguity in the expansion of a number as a HJ-continued fraction comes from the identity:

(8) 
$$[\alpha_1, \ldots, \alpha_n, (2)^{\infty}]^- = [\alpha_1, \ldots, \alpha_{n-1}, \alpha_n - 1]^-$$

We see that any real number  $\lambda$  admits a unique expansion as a HJcontinued fraction such that condition (6) is satisfied and the sequence  $(\alpha_n)_n$  is not infinite and ultimately constant equal to 2. When we speak in the sequel about the HJ-continued fraction expansion of a number  $\lambda$ , it will be about this one. We call the numbers  $(\alpha_n)_{n\geq 1}$  the HJ-partial quotients of  $\lambda$ .

The following lemma (see Hirzebruch [37, page 257]) can be easily proved by induction on the integer  $b \ge 1$ .

**Lemma 2.2.** If  $a \in \mathbb{Z}$ ,  $b \in \mathbb{N} - \{0\}$  and x is a variable, then:

$$[a, b, x]^+ = [a+1, (2)^{b-1}, x+1]^-$$

Using this lemma one sees how to pass from the E-continued fraction expansion of a real number  $\lambda$  to its HJ-continued fraction expansion:

**Proposition 2.3.** If  $(a_n)_{n\geq 1}$  is a (finite or infinite) sequence of positive integers, then:

$$[a_1, \ldots, a_{2n}]^+ = [a_1 + 1, (2)^{a_2 - 1}, a_3 + 2, (2)^{a_4 - 1}, \ldots, (2)^{a_{2n} - 1}]^- [a_1, \ldots, a_{2n+1}]^+ = [a_1 + 1, (2)^{a_2 - 1}, a_3 + 2, (2)^{a_4 - 1}, \ldots, (2)^{a_{2n} - 1}, a_{2n+1} + 1]^- [a_1, a_2, a_3, a_4, \ldots]^+ = [a_1 + 1, (2)^{a_2 - 1}, a_3 + 2, (2)^{a_4 - 1}, a_5 + 2, (2)^{a_6 - 1}, \ldots]^-$$

(recall that, by convention,  $(2)^0$  denotes the empty sequence).

**Example 2.4.**  $11/7 = [(1)^3, 3]^+ = [2, 3, (2)^2]^-.$ 

Notice that this procedure can be inverted. In particular, an immediate consequence of the previous proposition is that a number has bounded E-partial quotients if and only if it has bounded HJ-partial quotients. Similarly, it has ultimately periodic E-continued fraction (which happens if and only if it is a quadratic number, see Davenport [15]) if and only if it has ultimately periodic HJ-continued fraction. In this case, Proposition 2.3 explains how to pass from its E-period to its HJ-period. The continued fraction expansions of two numbers which differ by an integer are related in an evident and simple way. For this reason, from now on we restrict our attention to real numbers  $\lambda > 1$ . The map

(9) 
$$\lambda \longrightarrow \frac{\lambda}{\lambda - 1}$$

is an involution of the interval  $(1, +\infty)$  on itself. The E-continued fraction expansions of the numbers in the same orbit of this involution are related in a very simple way:

**Lemma 2.5.** If  $\lambda \in (1, +\infty)$  and  $\lambda = [a_1, a_2, \ldots]^+$  is its expansion as a (finite or infinite) continued fraction, then:

$$\frac{\lambda}{\lambda - 1} = \begin{cases} [1 + a_2, a_3, a_4, \dots]^+, & \text{if } a_1 = 1, \\ [1, a_1 - 1, a_2, a_3, \dots]^+, & \text{if } a_1 \ge 2 \end{cases}$$

The proof is immediate, once one notices that  $\lambda/(\lambda-1) = [1, \lambda-1]^+$ . Notice also that the involutivity of the map (9) shows that the first equality in the previous lemma is equivalent to the second one.

**Example 2.6.** If  $\lambda = 11/7 = [(1)^3, 3]^+$ , then  $11/4 = \lambda/(\lambda - 1) = [2, 1, 3]^+$ .

By combining Proposition 2.3 and Lemma 2.5, we get the following relation between the HJ-continued fraction expansions of the numbers in the same orbit of the involution (9):

**Proposition 2.7.** If  $\lambda \in \mathbf{R}$  is greater than 1 and

$$\lambda = [(2)^{m_1}, n_1 + 3, (2)^{m_2}, n_2 + 3, \dots]^{-1}$$

is its expression as a (finite or infinite) continued fraction, with  $m_i, n_i \in \mathbf{N}, \forall i \geq 1$ , then:

$$\frac{\lambda}{\lambda-1} = [m_1+2, (2)^{n_1}, m_2+3, (2)^{n_2}, m_3+3, \dots]^{-1}$$

For  $\lambda$  rational, this was proved in a different way by Neumann [57, Lemma 7.2]. It reads then:

$$\lambda = [(2)^{m_1}, n_1 + 3, (2)^{m_2}, \dots, n_s + 3, (2)^{m_{s+1}}]^- \Longrightarrow \frac{\lambda}{\lambda - 1} = [m_1 + 2, (2)^{n_1}, m_2 + 3, \dots, (2)^{n_s}, m_{s+1} + 2]^-$$

The important point here is that even a value  $m_{s+1} = 0$  contributes to the number of partial quotients in the HJ-continued fraction expansion of  $\lambda/(\lambda - 1)$ .

The next proposition is equivalent to the previous one, as an easy inspection shows. Its advantage is that it gives a graphical way to pass from the HJ-continued fraction expansion of a number  $\lambda > 1$  to the analogous expansion of  $\lambda/(\lambda - 1) > 1$ .

**Proposition 2.8.** Consider a number  $\lambda \in \mathbf{R}$  greater than 1 and let

$$\lambda = [\alpha_1, \alpha_2, \dots]^-, \quad \frac{\lambda}{\lambda - 1} = [\beta_1, \beta_2, \dots]^-$$

be the expressions of  $\lambda$  and  $\lambda/(\lambda-1)$  as (finite or infinite) HJ-continued fractions. Construct a diagram made of points organized in lines and columns in the following way:

• its lines are numbered by the positive integers;

• the line numbered  $k \geq 1$  contains  $\alpha_k - 1$  points;

• the first point in the line numbered k + 1 is placed under the last point of the line numbered k.

Then the k-th column contains  $\beta_k - 1$  points.

This graphical construction seems to have been first noticed by Riemenschneider in [66] when  $\lambda \in \mathbf{Q}_+$ . Nowadays one usually speaks about *Riemenschneider's point diagram* or *staircase diagram*.

**Example 2.9.** If  $\lambda = 11/7 = [2, 3, (2)^2]^-$ , the associated point diagram is:



One deduces from it that  $\lambda/(\lambda - 1) = [3, 4]^-$ .

# §3. Klein's geometric interpretation of Euclidean continued fractions

We let Klein [46] himself speak about his interpretation, in order to emphasize his poetical style:

Let us now enliven these considerations with geometric pictures. Confining our attention to positive numbers, let us *mark all those points* in the positive quadrant of the xy plane *which have integral coordinates*, forming thus a so-called *point lattice*. Let us examine this lattice, I am tempted to say this "firmament" of points, with our point of view at the origin. [...]

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Looking from 0, then, one sees points of the lattice in all rational directions and only in such directions. The field of view is everywhere "densely" but not completely and continuously filled with "stars". One might be inclined to compare this view with that of the milky way. With the exception of 0 itself there is not a single integral point lying upon an irrational ray  $x/y = \omega$ , where  $\omega$  is irrational, which is very remarkable. If we recall Dedekind's definition of irrational number, it becomes obvious that such a ray makes a *cut in the field* of integral points by separating the points into two point sets, one lying to the right of the ray and one to the left. If we inquire how these point sets converge toward our ray  $x/y = \omega$ , we shall find a very simple relation to the continued fraction for  $\omega$ . By marking each point  $(x = p_{\nu}, y = q_{\nu})$ , corresponding to the convergent  $p_{\nu}/q_{\nu}$ , we see that the rays to these points approximate to the ray  $x/y = \omega$  better and better, alternately from the left and from the right, just as the numbers  $p_{\nu}/q_{\nu}$ approximate to the number  $\omega$ . Moreover, if one makes use of the known number-theoretic properties of  $p_{\nu}$ ,  $q_{\nu}$ , one finds the following theorem: Imagine pegs or needles affixed at all the integral points, and wrap a tightly drawn string about the sets of pegs to the right and to the left of the  $\omega$ -ray, then the vertices of the two convex string-polygons which bound our two point sets will be precisely the points  $(p_{\nu}, q_{\nu})$  whose coordinates are the numerators and denominators of the successive convergents to  $\omega$ , the left polygon having the even convergents, the right one the odd. This gives a new and, one may well say, an extremely graphical definition of a continued fraction.

In the original article [45], one finds moreover the following interpretation of the E-partial quotients:

Each edge of the polygons  $[\ldots]$  may contain integral points. The number of parts in which the edge is decomposed by such points is exactly equal to a partial quotient.

Before Klein, Smith expressed a related idea in [73]:

If with a pair of rectangular axes in a plane we construct a system of unit points (*i.e.* a system of points of which the coordinates are integral numbers), and draw the line  $y = \theta x$ , we learn from that theorem that if (x, y) be a unit point lying nearer to that line than any other unit point having a less abscissa (or, which comes to the same thing, lying at a less distance from the origin), y/x is a convergent to  $\theta$ ; and, vice

versa, if y/x is a convergent, (x, y) is one of the 'nearest points'. Thus the 'nearest points' lie alternately on opposite sides of the line, and the double area of the triangle, formed by the origin and any two consecutive 'nearest points', is unity.

Proofs of the preceding properties can be found in Stark [75]. Here we only sketch the reason of Klein's interpretation. For explanations about our vocabulary, read next section.

Let  $\lambda > 1$  be a real number. In the first quadrant  $\sigma_0$ , consider the half-line  $L_{\lambda}$  of slope  $\lambda$  (see Fig. 1). It is defined by the equation  $y = \lambda x$ , which shows that  $\lambda = \omega^{-1} = \theta$ , where  $\omega$  is Klein's notation and  $\theta$  is Smith's. It subdivides the quadrant  $\sigma_0$  into two closed cones with vertex the origin,  $\sigma_x(\lambda)$  adjacent to the axis of the variable x and  $\sigma_y(\lambda)$ adjacent to the axis of the variable y.

**Lemma 3.1.** The segment which joins the lattice points of coordinates (1, 0) and  $(1, a_1)$  is a compact edge of the convex hull of the set of lattice points different from the origin contained in the cone  $\sigma_x(\lambda)$ , where  $\lambda = [a_1, a_2, \ldots]^+$  is the E-continued fraction expansion of  $\lambda$ .

*Proof.* Indeed, the half-line starting from (1, 0) and directed towards  $(1, a_1)$  cuts the half-line  $L_{\lambda}$  inside the segment  $[(1, [\lambda]), (1, [\lambda] + 1))$ , where  $[\lambda]$  is the integral part of  $\lambda$ . But  $[\lambda] = a_1$ , which finishes the proof. Q.E.D.



Fig. 1. Figure illustrating the proof of Lemma 3.1

Replace now the initial basis of the lattice by (0, 1),  $(1, a_1)$ . With respect to this new basis, the slope of the half-line  $L_{\lambda}$  is  $(\lambda - a_1)^{-1} =$   $[a_2, a_3, \ldots]^+$ . This allows one to prove Klein's interpretation by induction.

If one considers all lattice points on the compact edges of the boundaries of the two previous convex hulls instead of only the vertices, and then one looks at the slopes of the lines which join them to the origin, one obtains the so-called *slow approximating sequence* of  $\lambda$ . This kind of sequence appears naturally when one desingularizes germs of complex analytic plane curves by successively blowing up points (see Enriques & Chisini [19], Michel & Weber [53] and Lê, Michel & Weber [51]). We leave as an exercise for the interested reader to interpret this geometrically (first, read Section 6.3).

As explained by Klein himself in [45], his interpretation suggests to generalize the notion of continued fraction to higher dimensions by taking the boundaries of convex hulls of lattice points situated inside convex cones. For references about recent research in this area, see Arnold [1] and Moussafir [54].

# $\S4.$ Cohn's geometric interpretation of Hirzebruch-Jung continued fractions

A geometric interpretation of HJ-continued fractions analogous to Klein's interpretation of Euclidean ones was given by Cohn [12] (see the comment on his work in Hirzebruch [37, 2.3]). It seems to have soon become folklore among people doing toric geometry (see Section 6). Before describing this interpretation, let us introduce some vocabulary in order to speak with more precision about convex hulls of lattice points in the plane.

Let L be a lattice of rank 2, that is, a free abelian group of rank 2. It embeds canonically into the associated real vector space  $L_{\mathbf{R}} = L \otimes_{\mathbf{Z}} \mathbf{R}$ . When we picture the elements of L as points in the affine plane  $L_{\mathbf{R}}$ , we call them the *integral points* of the plane. When A and B are points of the affine plane  $L_{\mathbf{R}}$ , we denote by AB the element of the vector space  $L_{\mathbf{R}}$  which translates A into B, by [AB] the closed segment in  $L_{\mathbf{R}}$  of extremities A, B and by [AB the closed half-line having A as an extremity and directed towards B.

If (u, v) is an ordered basis of  $L_{\mathbf{R}}$  and l is a line of  $L_{\mathbf{R}}$ , its *slope* is the quotient  $\beta/\alpha \in \mathbf{R} \cup \{\infty\}$ , where  $\alpha u + \beta v$  generates l.

**Definition 4.1.** A (closed convex) triangle ABC in  $L_{\mathbf{R}}$  is called **elementary** if its vertices are integral and they are the only intersections of the triangle with the lattice L.

If the triangle ABC is elementary, then each pair of vectors (AB, AC), (BC, BA), (CA, CB) is a basis of the lattice L. Conversely, if one of these pairs is a basis of the lattice, then the triangle is elementary.

We call a line or a half-line in  $L_{\mathbf{R}}$  rational if it contains at least two integral points. If so, then it contains an infinity of them. If A and B are two integral points, the integral length  $l_{\mathbf{Z}}[AB]$  of the segment [AB] is the number of subsegments in which it is divided by the integral points it contains. A vector OA of L is called primitive if  $l_{\mathbf{Z}}[OA] = 1$ .

Let  $\sigma$  be a closed strictly convex 2-dimensional cone in the plane  $L_{\mathbf{R}}$ , that is, the convex "angle" (in the language of plane elementary geometry) delimited by two non-opposing half-lines originating from 0. These half-lines are called the *edges* of  $\sigma$ . The cone  $\sigma$  is called *rational* if its edges are rational. A cone is called *regular* if its edges contain points A, B such that the triangle OAB is elementary. The name is motivated by the fact that the associated toric surface  $\mathcal{Z}(L, \sigma)$  is smooth (that is, all its local rings are *regular*) if and only if  $\sigma$  is regular (see Section 6.1).

Let  $K(\sigma)$  be the convex hull of the set of lattice points situated inside  $\sigma$ , with the exception of the origin, that is:

$$K(\sigma) := \operatorname{Conv}(\sigma \cap (L - \{0\})).$$

The closed convex set  $K(\sigma)$  is unbounded. Denote by  $P(\sigma)$  its boundary: it is a connected polygonal line. It has two ends (in the topological sense), each one being asymptotic to (or contained inside) an edge of  $\sigma$ (see Fig. 2). An edge of  $\sigma$  intersects  $P(\sigma)$  if and only if it is rational.

Denote by  $\mathcal{V}(\sigma)$  the set of vertices of  $P(\sigma)$  and by  $\mathcal{E}(\sigma)$  the set of its (closed) edges. For example, in Fig. 3 the vertices are the points  $A_0, A_2, A_5$  and the edges are the segments  $[A_0A_2], [A_2A_5]$  and two halflines contained in  $l_-, l_+$ , starting from  $A_0$ , respectively  $A_5$ .

Now order arbitrarily the edges of  $\sigma$ . Denote by  $l_{-}$  the first one and by  $l_{+}$  the second one. This orients the plane  $L_{\mathbf{R}}$ , by deciding to turn from  $l_{-}$  towards  $l_{+}$  inside  $\sigma$ . If we orient  $P(\sigma)$  from the end which is asymptotic to  $l_{-}$  towards the end which is asymptotic to  $l_{+}$ , we get induced orientations of its edges.

Suppose now that the edge  $l_{-}$  of  $\sigma$  is rational. Denote then by  $A_{-} \neq 0$  the integral point of the half-line  $l_{-}$  which lies nearest to 0, and by  $V_{-} \neq A_{-}$  the vertex of  $P(\sigma)$  which lies nearest to  $A_{-}$ . Define in the same way  $A_{+}$  and  $V_{+}$  whenever  $l_{+}$  is rational. Denote by  $(A_{n})_{n\geq 0}$  the sequence of integral points on  $P(\sigma)$ , enumerated as they appear when one travels on this polygonal line in the positive direction, starting from  $A_{0} = A_{-}$ . If  $l_{+}$  is a rational half-line, then we stop this sequence when we arrive at the point  $A_{+}$ . If  $l_{+}$  is irrational, then this sequence is



Fig. 2. The polygonal line associated to a convex cone

infinite. Define  $r \ge 0$  such that  $A_{r+1} = A_+$ . So,  $r = +\infty$  if and only if  $l_+$  is irrational.

**Example 4.2.** We consider the lattice  $\mathbf{Z}^2 \subset \mathbf{R}^2$  and the cone  $\sigma$  with rational edges, generated by the vectors (1, 0) and (4, 11) (see Fig. 3). The small dots represent integral points in the plane and the bigger ones represent integral points on the polygonal lines  $P(\sigma)$ . In this example we have  $V_+ = V_- = A_2$ .



Fig. 3. An illustration of Example 4.2

Each triangle  $OA_nA_{n+1}$  is elementary, by the construction of the convex hull  $K(\sigma)$ , which implies that all the couples  $(OA_n, OA_{n+1})$  are bases of L. This shows that for any  $n \in \{1, \ldots, r\}$ , one has a relation of the type:

(10) 
$$OA_{n+1} + OA_{n-1} = \alpha_n \cdot OA_n$$

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with  $\alpha_n \in \mathbf{Z}$ , and the convexity of  $K(\sigma)$  shows that:

(11) 
$$\alpha_n \ge 2, \quad \forall n \in \{1, \ldots, r\}$$

Conversely:

**Proposition 4.3.** Suppose that  $(OA_n)_{n\geq 0}$  is a (finite or infinite) sequence of primitive vectors of L, related by relations of the form (10). Then we have

$$OA_n = Z^-(\alpha_1, \dots, \alpha_{n-1})OA_1 - Z^-(\alpha_2, \dots, \alpha_{n-1})OA_0, \quad \forall n \ge 1$$

and the slope of the half-line  $l_+ = \lim_{n \to \infty} [OA_n)$  in the base  $(-OA_0, OA_1)$  is equal to  $[\alpha_1, \alpha_2, \ldots]^-$ .

*Proof.* Recall that the polynomials  $Z^-$  were defined by the recursion formula (1). The first assertion can be easily proved by induction, using the relations (10). The second one is a consequence of formula (2), which shows that the slope of the half-line  $[OA_n]$  in the base  $(-OA_0, OA_1)$  is equal to  $[\alpha_1, \ldots, \alpha_{n-1}]^-$ . Q.E.D.

**Proposition 4.4.** Let  $\sigma$  be the closure of the convex hull of the union of the half-lines  $([OA_n)_{n\geq 0}$ . Then  $\sigma$  is strictly convex and the points  $\{A_n\}_{n\geq 1}$  are precisely the integral points on the compact edges of the polygonal line  $P(\sigma)$  if and only if the conditions (11) are satisfied and the sequence  $(\alpha_n)_{n\geq 1}$  is not infinite and ultimately constant equal to 2.

**Proof.** • What remains to be proved about the *necessity* is that if the sequence  $(\alpha_n)_{n\geq 1}$  is infinite, then it cannot be ultimately constant equal to 2. If this was the case, by relation (8) we would deduce that  $[\alpha_1, \alpha_2, \ldots]^-$  is rational, and Proposition 4.3 would imply that  $l_+$  is rational. Then  $P(\sigma)$  would contain a finite number of integral points on its compact edges, which would contradict the infinity of the sequence  $(\alpha_n)_{n\geq 1}$ .

• Let us prove now the sufficiency. As  $\alpha_n \geq 2, \forall n \in \{1, \ldots, r\}$ , we see that the triangles  $(OA_nA_{n+1})_{n\geq 0}$  turn in the same sense. Moreover, Proposition 4.3 shows that  $\sigma$  is a strictly convex cone. The vertices of the polygonal line  $P = A_0A_1A_2\ldots$  are precisely those points  $A_n$  for which  $\alpha_n \geq 3$ . As all the triangles  $OA_nA_{n+1}$  are elementary, we see that the origin O is the only integral point of the connected component of  $\sigma - P$  which contains it. Moreover, conditions (11) show that the other component is convex. So,  $P \subset P(\sigma)$ .

The proposition is proved.

Q.E.D.

# $\S 5.$ Geometric comparison of Euclidean and HJ-continued fractions

In Section 5.1 we relate the two preceding interpretations, by describing a duality between two supplementary cones in the plane, an underlying lattice being fixed (see Proposition 5.3). In Section 5.2 we introduce a so-called *zigzag diagram* based on this duality, which makes it very easy to visualize the various relations between continued fractions proved algebraically in Section 2. In Section 5.3 we give a proof of the isomorphism between the supplementary cone  $(L, \sigma')$  and the dual cone  $(\check{L}, \check{\sigma})$  of a given cone  $(L, \sigma)$ .

## 5.1. A geometric duality between supplementary cones

Suppose again that  $\sigma$  is any strictly convex cone in  $L_{\mathbf{R}}$ , whose edge  $l_{-}$  is not necessarily rational. Let  $l'_{-}$  be the half-line opposite to  $l_{-}$  and  $\sigma'$  be the closed convex cone bounded by  $l_{+}$  and  $l'_{-}$ . So,  $\sigma$  and  $\sigma'$  are supplementary cones:

**Definition 5.1.** Two strictly convex cones in a real plane are called **supplementary** if they have a common edge and if their union is a half-plane.

By analogy with what we did in the previous section for  $\sigma$ , orient the polygonal line  $P(\sigma')$  from  $l'_{-}$  towards  $l_{+}$ . If  $l_{-}$  is rational, define the point  $A'_{-}$  and the sequence  $(A'_{n})_{n\geq 0}$ , with  $A'_{0} = A'_{-}$ . They are the analogs for  $\sigma'$  of the points  $A_{-}$  and  $(A_{n})_{n\geq 0}$  for  $\sigma$ . In particular,  $OA_{-} + OA'_{-} = 0$ .

**Example 5.2.** Consider the same cone as in Example 4.2. Then the polygonal lines  $P(\sigma)$  and  $P(\sigma')$  are represented in Fig. 4 using heavy segments.

The basis for our geometric comparison of Euclidean and Hirzebruch-Jung continued fractions is the observation that the polygonal line  $P(\sigma')$ can be constructed in a very simple way once one knows  $P(\sigma)$ . Namely, starting from the origin, one draws the half-lines parallel to the oriented edges of  $P(\sigma)$ . On each half-line, one considers the integer point which is nearest to the origin. Then the polygonal line which joins those points is the union of the compact edges of  $P(\sigma')$ .

Now we describe this with more precision. If  $e \in \mathcal{E}(\sigma)$  is an edge of  $P(\sigma)$ , denote by  $\mathcal{I}(e) \in L$  the integral point such that  $O\mathcal{I}(e)$  is a primitive vector of L positively parallel to e (where e is oriented according to the chosen orientation of  $P(\sigma)$ ). Then it is an easy exercise to see that  $\mathcal{I}(e) \in \sigma'$  (use the fact that the line containing e intersects  $l_-$  and  $l_+$  in



Fig. 4. An illustration of Example 5.2

interior points). We can define a map:

(12) 
$$\begin{aligned} \mathcal{I}: \quad \mathcal{E}(\sigma) & \longrightarrow \quad \sigma' \cap L \\ e & \longrightarrow \quad \mathcal{I}(e) \end{aligned}$$

As the edges of  $P(\sigma)$  always turn in the same direction, one sees that the map  $\mathcal{I}$  is injective.

**Proposition 5.3.** The map  $\mathcal{I}$  respects the orientations and the image of  $\mathcal{I}$  verifies the double inclusion

$$\mathcal{V}(\sigma') \subset \operatorname{Im}(\mathcal{I}) \subset P(\sigma') \cap L.$$

The difference  $\operatorname{Im}(\mathcal{I}) - \mathcal{V}(\sigma')$  contains at most the points  $\mathcal{I}[A_-V_-]$ ,  $\mathcal{I}[V_+A_+]$ . Such a point is a vertex of  $P(\sigma')$  if and only if the integral length of the corresponding edge of  $P(\sigma)$  is  $\geq 2$ . In particular, one has the equality  $\mathcal{V}(\sigma') = \operatorname{Im}(\mathcal{I})$  if and only if  $l_{\mathbf{Z}}[A_-V_-] \geq 2$  and  $l_{\mathbf{Z}}[V_+A_+] \geq 2$ , whenever these segments exist.

*Proof.* Denote by  $(V_j)_{j \in J}$  the vertices of  $P(\sigma)$ , enumerated in the positive direction. The indices form a set of consecutive integers, well-defined only up to translations.

For any  $j \in J$ , denote by  $V_j^-$  and  $V_j^+$  respectively the integral points of  $P(\sigma)$  which precede and follow  $V_j$ . If  $V_j$  is an interior point of  $\sigma$ , denote by  $W_j \in L$  the point such that  $OW_j = OV_j^- + OV_j^+$ , and by  $W_j^-$  its nearest integral point in the interior of the segment  $[OW_j]$  (see Fig. 5).

As  $OV_j^-V_j$  and  $OV_jV_j^+$  are elementary triangles, it implies that both  $(OV_j^-, OV_j)$  and  $(OV_j, OV_j^+)$  are bases of L. So, there exists an integer



Fig. 5. The first illustration for the proof of Proposition 5.3

 $n_j$  such that

(13) 
$$OV_j^- + OV_j^+ = (n_j + 3)OV_j.$$

As  $V_j$  is a vertex of  $P(\sigma)$ , we see that  $n_j \ge 0$ . We deduce that the points  $O, V_j, W_j^-, W_j$  are aligned in this order, that  $V_j V_j^- + V_j V_j^+ = V_j W_j^-$  and that  $l_{\mathbf{Z}}[V_j W_j^-] = n_j + 1$ .

Let us join each one of the  $n_j$  interior points of  $[V_j W_j^-]$  to  $V_j^-$ . This gives a decomposition of the triangle  $V_j^- V_j W_j^-$  into  $(n_j + 1)$  triangles. These are necessarily elementary, because the triangle  $OV_i^- V_j$  is. Denote

$$V'_j = \mathcal{I}[V_{j-1}V_j]$$
 and  $V'_{j+1} = \mathcal{I}[V_jV_{j+1}].$ 

By the definition of the map  $\mathcal{I}$ , we see that  $OV'_j = V_j^- V_j$  and  $OV'_{j+1} = V_j V_j^+ = V_j^- W_j^-$ . This implies that the triangle  $OV'_j V'_{j+1}$  is the translated image by the vector  $V_j^- O$  of the triangle  $V_j^- V_j W_j^-$ . The preceding arguments show that its only integral points are its vertices and  $n_j$  other points in the interior of the segment  $[V'_j V'_{j+1}]$ . Indeed:

(14) 
$$V'_{j}V'_{j+1} = V_{j}W^{-}_{j} = (n_{j}+1)OV_{j}$$

Moreover, the triangle  $OV'_jV'_{j+1}$  is included in the cone  $\sigma'$  and the couple of vectors  $(OV'_j, OV'_{j+1})$  has the same orientation as  $(l'_-, l_+)$ .

This shows that the triangles  $(OV'_jV'_{j+1})_{j\in J}$  are pairwise disjoint and that their union does not contain integral points in its interior.

• If both edges of  $\sigma$  are irrational, then the closure of the union of the cones  $\mathbf{R}_+ OV'_j + \mathbf{R}_+ OV'_{j+1}$  is the cone  $\sigma'$ , as the edges  $l_-$  and  $l_+$  are asymptotic to  $P(\sigma)$ . We deduce from relation (14) that the sequence  $(\lambda_j)_{j\in J}$  of slopes of the vectors  $(V'_j V'_{j+1})_{j\in J}$ , expressed in a base  $(u_-, u_+)$  of  $L_{\mathbf{R}}$  which verifies  $l_{\pm} = \mathbf{R}_+ u_{\pm}$  is strictly increasing, and that  $\lim_{j\to-\infty} \lambda_j = 0$ ,  $\lim_{j\to+\infty} \lambda_j = +\infty$ . This shows that the closure of the connected component of  $\sigma' - \bigcup_{j\in J} [V'_j V'_{j+1}]$  which does not contain the origin is convex. As a consequence,

$$\bigcup_{j \in J} [V'_j V'_{j+1}] = P(\sigma').$$

Moreover, as  $n_j \geq 0$ , the strict monotonicity of the sequence  $(\lambda_j)_{j \in J}$ implies that the points  $(V'_j)_{j \in J}$  are precisely the vertices of  $P(\sigma')$ . The proposition is proved in this case.

• Suppose now that  $l_{-}$  is rational. Then choose the index set J such that  $V_0 = A_{-}$  and  $V_1 = V_{-}$ . By the construction of the map  $\mathcal{I}$ , the triangle  $OV'_0V'_1$  is the translated image of  $V_0OV^+_0$  by the vector  $V_0O$  (see Fig. 6).



Fig. 6. The second illustration for the proof of Proposition 5.3

In particular,  $V'_0V'_1 = OV_0^+$ . But  $V'_1V'_2 = (n_1 + 1)OV_1$  by relation (14), which shows that the vectors  $V'_0V'_1$  and  $V'_1V'_2$  are proportional if and only if  $V_0^+ = V_1$ , which is equivalent to  $l_{\mathbf{Z}}[A_-V_-] = 1$ . Moreover, the property of monotonicity for the slopes of the vectors  $(V'_jV'_{j+1})_{j\in J}$  is true as before, if one starts from j = 0.

• An analogous reasoning is valid for  $l_+$  if this edge of  $\sigma$  is rational. By combining all this, the proposition is proved. Q.E.D.

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The previous proposition explains a geometric *duality* between the supplementary cones  $\sigma$ ,  $\sigma'$  with respect to the lattice L. We see that, with possible exceptions for the compact edges which intersect the edges of  $\sigma$  and  $\sigma'$ , the compact edges of  $P(\sigma)$  correspond to the vertices of  $P(\sigma')$  interior to  $\sigma'$  and conversely (by permuting the roles of  $\sigma$  and  $\sigma'$ ), which is a kind of point-line polarity relation.

The next corollary shows that the involution (9) studied algebraically in Section 2 is closely related to the previous duality.

**Corollary 5.4.** Suppose that  $l_{-}$  is rational and that  $\sigma$  is not regular. If  $(OA'_0, U)$  is a basis of L with respect to which the slope of  $l_+$ is greater than 1, then  $U = OA_1$ . If  $\lambda > 1$  denotes the slope of the half-line  $l_+$  in the base  $(OA'_0, OA_1)$ , then  $\lambda/(\lambda - 1)$  is its slope in the base  $(OA_0, OA'_1)$ .

*Proof.* We leave the first affirmation to the reader (look at Fig. 6). As the triangles  $OA_0A_1$  and  $OA'_0A'_1$  are elementary, we see that  $(OA_0, OA_1)$  and  $(OA'_0, OA'_1)$  are indeed two bases of the lattice L. Proposition 5.3 shows that  $OA'_0 = A_0A_1$ , which allows us to relate the two bases:

(15) 
$$\begin{cases} OA'_{0} = -OA_{0} \\ OA'_{1} = OA_{1} - OA_{0} \end{cases}$$

Let  $v \in L_{\mathbf{R}}$  be a vector which generates the half-line  $l_+$ . We want to express it in these two bases. As  $l_+$  lies between the half-lines  $[OA'_0]$ and  $[OA_1]$ , we see that:

(16) 
$$v = -qOA_0 + pOA_1, \text{ with } p, q \in \mathbf{R}^*_+$$

The equations (15) imply then that:

(17) 
$$v = -(p-q)OA'_0 + pOA'_1$$

which shows that p - q > 0, as  $l_+$  lies between the half-lines  $[OA'_1]$  and  $[OA_0]$ . This implies that  $\lambda := p/q > 1$ . We then deduce the corollary from equation (17). Q.E.D.

The previous corollary shows that the number  $\lambda > 1$  can be canonically attached to the pair  $(L, \sigma)$ , once a rational edge of  $\sigma$  is chosen as the first edge  $l_{-}$ . This motivates the following definition:

**Definition 5.5.** Suppose that  $l_{-}$  is rational and that the cone  $\sigma$  is not regular. We say that the pair  $(L, \sigma)$  with the chosen ordering of sides is of type  $\lambda > 1$  if  $\lambda$  is the slope of the half-line  $l_{+}$  in the base  $(OA'_{0}, OA_{1})$ .

Proposition 4.3 shows that, if  $(L, \sigma)$  is of type  $\lambda > 1$ , then  $\lambda = [\alpha_1, \alpha_2, \ldots]^-$ , where the sequence  $(\alpha_n)_{n\geq 1}$  was defined using relation (10).

Suppose now that both edges of  $\sigma$  are rational. Then one can choose  $p, q \in \mathbb{N}^*$  with gcd(p, q) = 1 in relation (16), condition which determines them uniquely. So,  $\lambda = p/q$ . The following proposition describes the type of  $(L, \sigma)$  after changing the ordering of the sides.

**Proposition 5.6.** If  $(L, \sigma)$  is of type p/q with respect to the ordering  $l_-, l_+$ , then it is of type  $p/\overline{q}$  with respect to the ordering  $l_+, l_-$ , where  $q\overline{q} \equiv 1 \pmod{p}$ .

*Proof.* By relation (16), we have  $OA_+ = -qOA_- + pOA_1$ . Multiply both sides by  $\overline{q}$ . By the definition of  $\overline{q}$ , there exists  $k \in \mathbf{N}$  such that  $q\overline{q} = 1 + kp$ . We deduce that  $OA_- = -\overline{q}OA_+ + p(\overline{q}OA_1 - kOA_-)$ . So,  $(-OA_+, \overline{q}OA_1 - kOA_-)$  is a base of L in which the slope of  $l_-$  is  $p/\overline{q} > 1$ . By the first affirmation of Corollary 5.4, the proposition is proved. Q.E.D.

By combining the previous proposition with Proposition 4.3, we deduce the following classical fact (see [4, section III.5]):

**Corollary 5.7.** If  $p/q = [\alpha_1, \alpha_2, \dots, \alpha_r]^-$ , then  $p/\overline{q} = [\alpha_r, \alpha_{r-1}, \dots, \alpha_1]^-$ .

Another immediate consequence of Corollary 5.4 is:

**Proposition 5.8.** If  $(L, \sigma)$  is of type p/q with respect to the ordering  $l_-, l_+$ , then  $(L, \sigma')$  is of type p/(p-q) with respect to the ordering  $l'_-, l_+$ .

The previous proposition describes the relation between the types of two supplementary cones. In Section 5.2, we describe more precisely the relation between numerical invariants attached to the edges and the vertices of  $P(\sigma)$  and  $P(\sigma')$ .

# 5.2. A diagram relating Euclidean and HJ-continued fractions

We introduce now a diagram which allows one to "see" the duality between  $P(\sigma)$  and  $P(\sigma')$ , as well as the relations between the various numerical invariants attached to these polygonal lines.

• Suppose first that both  $l_{-}$  and  $l_{+}$  are irrational. Consider two consecutive vertices  $V_j$ ,  $V_{j+1}$  of  $P(\sigma)$ . Let us attach the weight  $n_j + 3$  to the vertex  $V_j$ , where  $n_j \ge 0$  was defined by relation (13). Introduce also the integer  $m_{j+1} \ge 0$  such that  $l_{\mathbf{Z}}[V_jV_{j+1}] = m_{j+1} + 1$ . The relation (14) shows that  $l_{\mathbf{Z}}[V'_jV'_{j+1}] = n_j + 1$ . By reversing the roles of the polygonal lines  $P(\sigma')$  and  $P(\sigma)$ , we deduce that the weight of the vertex  $V'_{j+1}$  of  $P(\sigma')$  is  $m_{j+1} + 3$ .

We can visualize the relations between the vertices  $V_j, V_{j+1}, V'_j, V'_{j+1}$ as well as the numbers associated to them and to the segments  $[V_jV_{j+1}]$ ,  $[V'_jV'_{j+1}]$  by using a diagram, in which the heavy lines represent the polygonal lines  $P(\sigma)$ ,  $P(\sigma')$ , and each vertex  $V_j$  is joined to  $V'_j$  and  $V'_{j+1}$ (see Fig. 7). In this way, the region contained between the two curves representing  $P(\sigma)$  and  $P(\sigma')$  is subdivided into triangles. Each edge Eof  $P(\sigma)$ ,  $P(\sigma')$  is contained in only one of those triangles. Look at its opposite vertex. We say that E is the opposite edge of that vertex in the zigzag diagram. We see that the weight of a vertex is equal to the length of the opposite edge augmented by 2.



Fig. 7. Local aspect of the zigzag diagram

As an edge and its opposite vertex are dual through the morphism  $\mathcal{I}$  (see Proposition 5.3) and its analog  $\mathcal{I}'$  attached to the cone  $\sigma'$ , the triangles appearing in the zigzag diagram are a convenient graphical representation of the duality explained in Section 5.1.

• When  $l_{-}$  is rational and  $l_{+}$  is irrational, we draw a little differently the diagram (see Fig. 8). The curves representing  $P(\sigma)$  and  $P(\sigma')$ start from points  $V_0$  and  $V'_0$  of a horizontal line representing the line which contains  $l_{-}$ . We represent the integral point  $V'_1$  differently from the points  $V'_2$ ,  $V'_3$ , ..., because it may not be a vertex of  $P(\sigma')$ , as explained in Proposition 5.3. The length of  $[V'_0V'_1]$  is always 1. The relation between the length of an edge and the weight of the opposite vertex is the same as before, with the exception of the triangle  $V'_1V_0V_1$ , where the weight of  $V'_1$  is equal to  $l_{\mathbf{Z}}[V_0V_1] + 1$ .



Fig. 8. The zigzag diagram when  $l_{-}$  is rational

• When both  $l_{-}$  and  $l_{+}$  are rational and there is at least one vertex on  $P(\sigma)$  lying strictly between  $A_{-}$  and  $A_{+}$  (that is,  $s \geq 1$ ), the curves representing  $P(\sigma)$  and  $P(\sigma')$  start again from a horizontal line, but now they join in a point  $A_{+}$  (see Fig. 9).



Fig. 9. The zigzag diagram when both  $l_{-}$  and  $l_{+}$  are rational

• When both  $l_{-}$  and  $l_{+}$  are rational and  $[A_{-}A_{+}]$  is an edge of  $P(\sigma)$  (that is, s = 0), the diagram is represented in Fig. 10.



Fig. 10. The zigzag diagram when  $P(\sigma)$  has only one compact edge

To summarize, we have the following procedure for constructing and decorating the diagram when  $l_{-}$  is rational:

**Procedure.** Suppose that  $l_{-}$  is rational. Then draw a horizontal line with three marked points  $V'_0 = A'_-$ ,  $O, V_0 = A_-$  in this order,  $V'_0$  on the left and  $V_0$  on the right. Starting from  $V'_0$  and  $V_0$ , draw in the upper half-plane two curves  $P(\sigma')$ , respectively  $P(\sigma)$ , concave towards 0 and coming closer and closer from one another. If  $l_+$  is rational, join them in a point  $A_+$ . Draw a zigzag line starting from  $V_0$  and going alternatively from  $P(\sigma)$  to  $P(\sigma')$ . Denote its successive vertices by  $V'_1, V_1, V'_2, \ldots$ and stop at the point  $V'_{s+1}$ . Decorate the edges  $V'_0V'_1$  and  $V'_{s+1}A_+$  by 1. The other edges and vertices will be decorated using the initial data (discussed in the sequel), by respecting the following rule:

**Rule.** The weight of a vertex is equal to the length of the opposite edge augmented by the number of its vertices distinct from the points  $A_{-}, A'_{-}, A_{+}$ .

**Initial data.** If  $\sigma$  is of type  $\lambda$ , write the HJ-continued fraction expansion of  $\lambda$  in the form:

(18) 
$$\lambda = [(2)^{m_1}, n_1 + 3, (2)^{m_2}, n_2 + 3, \ldots]^{-1}$$

Then decorate the edges of  $P(\sigma)$  with the numbers  $m_1 + 1, m_2 + 1, \ldots$ and the vertices with the numbers  $n_1 + 3, n_2 + 3, \ldots$ 

**Definition 5.9.** We call the previous diagram the **zigzag diagram** associated to the pair  $(L, \sigma)$  and to the chosen ordering of the edges of  $\sigma$ , or to the number  $\lambda > 1$ , where  $(L, \sigma)$  is of type  $\lambda$  with respect to this ordering. We denote it by  $ZZ(\lambda)$ .

The zigzag diagrams allow one to visualize the relations between Euclidean and Hirzebruch-Jung continued fractions, proved algebraically in Section 2. Indeed, one can read the HJ-continued fraction expansion of  $\lambda > 1$  on the right-hand curved line of  $ZZ(\lambda)$ . By Corollary 5.4, we can read the HJ-continued fraction expansion of  $\lambda/(\lambda - 1)$  on the lefthand curved line  $P(\sigma)$  of  $ZZ(\lambda)$ . So, by looking at Fig. 9, which can be easily constructed from the initial data by respecting the rule, we get:

(19) 
$$\frac{\lambda}{\lambda - 1} = [m_1 + 2, (2)^{n_1}, m_2 + 3, (2)^{n_2}, m_3 + 3, \dots]^{-1}$$

which gives a geometric proof of Proposition 2.7.

Now, by Klein's geometric interpretation of E-continued fractions (see Section 3), we see that the E-continued fraction expansion of  $\lambda/(\lambda - 1)$  can be obtained by writing alternatively the integral lengths of the edges of the polygonal lines  $P(\sigma)$  and  $P(\sigma') - [V'_0V'_1]$  (indeed,  $\lambda/(\lambda - 1)$ is the slope of  $l_+$  in the base  $(OV_0, OV'_1)$ ):

(20) 
$$\frac{\lambda}{\lambda-1} = [m_1+1, n_1+1, m_2+1, n_2+1, m_3+1, \dots]^+.$$

This proves geometrically Proposition 2.3.

In order to read the E-continued fraction expansion of  $\lambda$  on the diagram, one has to look at  $ZZ(\lambda)$  from left to right instead of from right to left and draw a new zigzag line starting from  $V'_0$ . The important point here is that one has to discuss according to the alternative  $m_1 = 0$  or  $m_1 > 0$ . In the first case, the zigzag line joins  $V'_0$  to  $V_1$  and  $V_1$  to  $V'_2$ . In the second case, it joins  $V'_0$  to a new point representing  $A_1$  and  $A_1$  to  $V'_1$ . Compare this with Lemma 2.5.

**Example 5.10.** Take  $\lambda = 11/7$ . After computing  $\lambda = [2, 3, 2, 2]^-$ , we can construct the associated zigzag diagram ZZ(11/7). We see that the extreme points  $V'_1$ ,  $V'_2$  are vertices of  $P(\sigma')$ . One can read on it the results of the Examples 2.4, 2.6, 2.9.

If one had starts instead from  $\lambda = 11/4 = [3, 4]^-$ , the corresponding diagram would be ZZ(11/4). In this case the extreme points are not vertices of  $P(\sigma')$ , because their weights are equal to 2.


Fig. 11. The first illustration for Example 5.10: ZZ(11/7)



Fig. 12. The second illustration for Example 5.10: ZZ(11/4)

## 5.3. Relation with the dual cone

Denote by  $\check{L} := \text{Hom}(L, \mathbb{Z})$  the *dual lattice* of L. Inside the associated vector space  $\check{L}_{\mathbf{R}}$  lives the *dual cone*  $\check{\sigma}$  of  $\sigma$ , defined by:

$$\check{\sigma} := \{\check{u} \in \check{L}_{\mathbf{R}} \mid \check{u}.u \ge 0, \ \forall u \in \sigma\}.$$

Let  $\omega$  be the volume form on  $L_{\mathbf{R}}$  which verifies  $\omega(u_1, u_2) = 1$  for any basis  $(u_1, u_2)$  of L defining the *opposite* orientation to  $(l_-, l_+)$ . It is a symplectic form, that is, a non-degenerate alternating bilinear form on  $L_{\mathbf{R}}$ . But we prefer to look at it as a morphism (obtained by making interior products with the elements of L):

$$\omega\colon L\longrightarrow \check{L}.$$

**Proposition 5.11.** The mapping  $\omega$  realizes an isomorphism between the pairs  $(L, \sigma')$  and  $(\check{L}, \check{\sigma})$ .

*Proof.* Indeed we have:

$$\omega^{-1}(\check{\sigma}) = \{ u \in L \mid \omega(u) \in \check{\sigma} \} = \{ u \in L \mid \omega(u, v) \ge 0, \forall v \in L \} = \sigma'.$$

While writing the last equality, we used our convention on the orientation of  $\omega$ . Notice that the dual cone  $\check{\sigma}$  can be defined without the help of any orientation, in contrast with the morphism  $\omega$ . Q.E.D.

The previous proposition shows that the construction of the polygonal line  $P(\sigma')$  explained in Proposition 5.3 describes also the polygonal line  $P(\check{\sigma})$ . This observation is crucial when one wants to use zigzag diagrams for understanding computations with invariants of toric surfaces (see next section).

It also helps to understand geometrically the duality between the convex polygons  $K(\sigma)$  and  $K(\check{\sigma})$  explained in Gonzalez-Sprinberg [30] and in Oda [60, pages 27-29]. As Dimitrios Dais kindly informed us after seeing a version of this paper on ArXiv, a better algebraic understanding of that duality is explained in Dais, Haus & Henk [14, Section 3]. In particular, modulo Proposition 5.11, the Theorem 3.16 in the previous reference leads easily to an algebraic proof of our Proposition 5.3.

(Added in proof) Emmanuel Giroux has informed us that he had realized the existence of a duality between supplementary cones (see [25, section 1.G]).

## $\S 6.$ Relations with toric geometry

First we introduce elementary notions of toric geometry (see Section 6.1). In Section 6.2 we explain how to get combinatorially various invariants of a normal affine toric surface and of the corresponding Hirzebruch-Jung analytic surface singularities. In Section 6.3 we explain how to read the combinatorics of the minimal embedded resolution of a plane monomial curve on an associated zigzag diagram.

The basics about resolutions of surface singularities needed in order to understand this section are recalled in Section 8.1.

## 6.1. Elementary notions of toric geometry

For details about toric geometry, general references are the books of Oda [60] and Fulton [23], as well as the first survey of it by Kempf, Knudson, Mumford & St. Donat [44].

In the previous section, our fundamental object of study was a pair  $(L, \sigma)$ , where L is a lattice of rank 2 and  $\sigma$  is a strictly convex cone in the 2-dimensional vector space  $L_{\mathbf{R}}$ .

Suppose now that the lattice L has arbitrary finite rank  $d \ge 1$  and that  $\sigma$  is a strictly convex *rational* cone in  $L_{\mathbf{R}}$ . The pair  $(L, \sigma)$  gives rise canonically to an affine algebraic variety:

$$\mathcal{Z}(L,\,\sigma) := \operatorname{Spec} \mathbf{C}[\check{\sigma} \cap \check{L}].$$

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This means that the algebra of regular functions on  $\mathcal{Z}(L, \sigma)$  is generated by the monomials whose exponents are elements of the semigroup  $\check{\sigma} \cap \check{L}$  of integral points in the dual cone of  $\sigma$ . If  $v \in \check{\sigma} \cap \check{L}$ , we formally write such a monomial as  $X^v$ . One can show that the variety  $\mathcal{Z}(L, \sigma)$ is *normal* (see the definition at the beginning of Section 8.1).

The closed points of  $\mathcal{Z}(L, \sigma)$  are the morphisms of semigroups  $(\check{\sigma} \cap \check{L}, +) \to (\mathbf{C}, \cdot)$ . Among them, those whose image is contained in  $\mathbf{C}^*$  form a *d*-dimensional algebraic torus  $\mathcal{T}_L = \operatorname{Spec} \mathbf{C}[\check{L}]$ , that is, a complex algebraic group isomorphic to  $(\mathbf{C}^*)^d$ . The elements of L correspond to the 1-parameter subgroups of  $\mathcal{T}_L$ , that is, the group morphisms  $(\mathbf{C}^*, \cdot) \to (\mathcal{T}_L, \cdot)$ . The action of  $\mathcal{T}_L$  on itself by multiplication extends canonically to an algebraic action on  $\mathcal{Z}(L, \sigma)$ , such that  $\mathcal{T}_L$  is the unique open orbit. If  $(\overline{L}, \overline{\sigma})$  is a second pair and  $\phi \colon \overline{L} \to L$  is a morphism such that  $\phi(\overline{\sigma}) \subset \sigma$ , one gets an associated toric morphism:

$$\phi_* \colon \mathcal{Z}(\overline{L},\,\overline{\sigma}) \to \mathcal{Z}(L,\,\sigma)$$

It is birational if and only if  $\phi$  realizes an isomorphism between  $\overline{L}$  and L. In this case  $\phi_*$  identifies the tori contained inside  $\mathcal{Z}(\overline{L}, \overline{\sigma})$  and  $\mathcal{Z}(L, \sigma)$ .

In general:

**Definition 6.1.** Given an algebraic torus  $\mathcal{T}$ , a **toric variety**  $\mathcal{Z}$  is an algebraic variety containing  $\mathcal{T}$  as a dense Zariski open set and endowed with an action  $\mathcal{T} \times \mathcal{Z} \to \mathcal{Z}$  which extends the group multiplication of  $\mathcal{T}$ .

Oda [60] and Fulton [23] study mainly the *normal* toric varieties. For an introduction to the study of non-necessarily normal toric varieties, one can consult Sturmfels [76] and González Pérez & Teissier [29].

A normal toric variety can be described combinatorially using *fans*, that is finite families of rational strictly convex cones, closed under the operations of taking faces or intersections. If L is a lattice and  $\mathcal{F}$  is a fan in  $L_{\mathbf{R}}$ , we denote by  $\mathcal{Z}(L, \mathcal{F})$  the associated normal toric variety. It is obtained by glueing the various affine toric varieties  $\mathcal{Z}(L, \sigma)$  when  $\sigma$  varies among the cones of the fan  $\mathcal{F}$ . As glueing maps, one uses the toric birational maps  $\mathcal{Z}(\overline{L}, \overline{\sigma}) \to \mathcal{Z}(L, \sigma)$  induced by the inclusion morphisms  $(L, \overline{\sigma}) \to (L, \sigma)$ , for each pair  $\overline{\sigma} \subset \sigma$  of cones of  $\mathcal{F}$ .

The variety  $\mathcal{Z}(L, \mathcal{F})$  is smooth if and only if each cone of the fan  $\mathcal{F}$  is *regular*, that is, generated by a subset of a basis of the lattice L.

## 6.2. Toric surfaces

We restrict now to the case of surfaces. Consider a 2-dimensional normal toric surface  $\mathcal{Z}(L, \sigma)$ , where  $\sigma$  is a strictly convex cone with non-empty interior. There is a unique 0-dimensional orbit O, whose maximal ideal is generated by the monomials with exponents in the semigroup  $\check{\sigma} \cap \check{L} - O$ . The surface is smooth outside O, and O is a smooth point of it if and only if  $\sigma$  is a regular cone. Supposing that  $\sigma$  is not regular, we explain how to describe combinatorially the minimal resolution morphism of  $\mathcal{Z}(L, \sigma)$  and the effect of blowing-up the point O. We also give a formula for the embedding dimension of the germ  $(\mathcal{Z}(L, \sigma), O)$ , which is a so-called *Hirzebruch-Jung singularity*.

With the notations of Section 4, let us subdivide  $\sigma$  by drawing the half-lines starting from O and passing through the points  $A_k$ ,  $\forall k \in \{1, \ldots, r\}$ . In this way we decompose  $\sigma$  in a finite number of regular subcones. They form the minimal regular subdivision of  $\sigma$ , in the sense that any subdivision of  $\sigma$  by regular cones is necessarily a refinement of the preceding one.

The family consisting of the 2-dimensional cones in the subdivision, of their edges and of the origin form a fan  $\mathcal{F}(\sigma)$ . For each such subcone  $\sigma'$  of  $\sigma$ , there is a canonical birational morphism  $\mathcal{Z}(L, \sigma') \to \mathcal{Z}(L, \sigma)$ , which realizes an isomorphism of the tori. Using these morphisms, one can glue canonically the tori contained in the surfaces  $\mathcal{Z}(L, \sigma')$  when  $\sigma'$  varies, and obtain a new toric surface  $\mathcal{Z}(L, \mathcal{F}(\sigma))$ , endowed with a morphism:

$$\mathcal{Z}(L, \mathcal{F}(\sigma)) \xrightarrow{p_{\sigma}} \mathcal{Z}(L, \sigma)$$

**Proposition 6.2.** The morphism  $p_{\sigma}$  is the minimal resolution of singularities of the surface  $\mathcal{Z}(L, \sigma)$ . Moreover, its exceptional locus  $E_{\sigma}$  is a normal crossings divisor and the dual graph of  $E_{\sigma}$  is topologically a segment.

*Proof.* For details, see [23]. Here we outline only the main steps. The morphism  $p_{\sigma}$  is proper, birational and realizes an isomorphism over  $\mathcal{Z}(L, \sigma) - O$ . As  $\mathcal{Z}(L, \mathcal{F}(\sigma))$  is smooth,  $p_{\sigma}$  is a resolution of singularities of  $\mathcal{Z}(L, \sigma)$  (see Definition 8.2). There is a canonical bijection between the irreducible components  $E_k$  of the exceptional divisor  $E_{\sigma} = p_{\sigma}^{-1}(0)$  and the half-lines  $[OA_k, \text{ for } k \in \{1, \ldots, r\}$ . Moreover,  $E_k$  is a smooth compact rational curve and

(21) 
$$E_k^2 = -\alpha_k, \quad \forall k \in \{1, \dots, r\}$$

where the numbers  $\alpha_k$  were introduced in relation (10).

Using the inequality (11), we deduce that no component of  $E_{\sigma}$  is exceptional of the first kind (see the comments which follow Definition 8.2). This implies that  $p_{\sigma}$  is the minimal resolution of singularities of  $\mathcal{Z}(L, \sigma)$ . The proposition is proved. Q.E.D. Notice that relation (21) gives an intersection-theoretical interpretation of the weights attached through relation (10) to the integral points situated on  $P(\sigma)$  which are interior to  $\sigma$ .

Conversely (see [4] and [64]):

**Proposition 6.3.** Suppose that a smooth surface  $\mathcal{R}$  contains a compact normal crossings divisor E whose components are smooth rational curves of self-intersection  $\leq -2$  and whose dual graph is topologically a segment. Denote by  $\alpha_1, \ldots, \alpha_r$  the self-intersection numbers read orderly along the segment. Then E can be contracted by a map  $p: (\mathcal{R}, E) \to (\mathcal{S}, 0)$  to a normal surface  $\mathcal{S}$  and the germ  $(\mathcal{S}, 0)$  is analytically isomorphic to a germ of the form  $(\mathcal{Z}(L, \sigma), O)$ , where  $\sigma$  is of type  $\lambda := [\alpha_1, \ldots, \alpha_r]^-$ .

This motivates:

**Definition 6.4.** A normal surface singularity  $(\mathcal{S}, 0)$  isomorphic to a germ of the form  $(\mathcal{Z}(L, \sigma), O)$  is called a **Hirzebruch-Jung singularity**.

Hirzebruch-Jung singularities can also be defined as *cyclic quotient* singularities (see [4] and [64]). They appear naturally in the so-called *Hirzebruch-Jung method* of studying an arbitrary surface singularity. Namely, one projects the given singularity by a finite morphism on a smooth surface, then one makes an embedded resolution of the discriminant curve and takes the pull-back of the initial surface by this morphism. In this case, the normalization of the new surface has only Hirzebruch-Jung singularities (see Laufer [47], Lipman [52], Brieskorn [8] for details and Popescu-Pampu [64] for a generalization to higher dimensions).

The proof of Proposition 6.2 shows that the germs  $(\mathcal{Z}(L, \sigma), O)$  and  $(\mathcal{Z}(\overline{L}, \overline{\sigma}), O)$  are analytically isomorphic if and only if there exists an isomorphism of the lattices L and  $\overline{L}$  sending  $\sigma$  onto  $\overline{\sigma}$ . The same is true for strictly convex cones in arbitrary dimensions, as proved by González Pérez & Gonzalez-Sprinberg [28]. Previously we had proved this for simplicial cones in [64].

A Hirzebruch-Jung singularity isomorphic to  $(\mathcal{Z}(L, \sigma), O)$  is said to be of type  $\mathcal{A}_{p,q}$ , with  $1 \leq q < p$  and gcd(p, q) = 1 if (using Definition 5.5) the pair  $(L, \sigma)$  is of type p/q with respect to one of the orderings of the sides of  $\sigma$ . Then, by Proposition 4.3, we have  $p/q = [\alpha_1, \ldots, \alpha_r]^-$ . By Proposition 5.6, one has  $\mathcal{A}_{p,q} \simeq \mathcal{A}_{p',q'}$  if and only if p = p' and  $q' \in \{q, \overline{q}\}$ , where  $q\overline{q} \equiv 1 \pmod{p}$ .

The singularities of type  $\mathcal{A}_{n+1,n}$  are also called *of type*  $\mathbf{A}_n$ . They are those for which the polygonal line  $P(\sigma)$  has only one compact edge,

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as  $(n + 1)/n = [(2)^n]^-$  (a case emphasized in Section 5.2), and also the only Hirzebruch-Jung singularities of embedding dimension 3 (more precisely, they can be defined by the equation  $z^{n+1} = xy$ ). Indeed:

**Proposition 6.5.** If  $p/q = [\alpha_1, \ldots, \alpha_r]^- = [(2)^{m_1}, n_1 + 3, \ldots, n_s + 3, (2)^{m_{s+1}}]^-$ , then:

embdim
$$(\mathcal{A}_{p,q}) = 3 + \sum_{i=1}^{r} (\alpha_i - 2) = 3 + s + \sum_{k=1}^{s} n_k.$$

**Proof.** If S is a generating system of the semigroup  $\dot{L} \cap \check{\sigma} - O$ , then the monomials  $(X^v)_{v \in S}$  form a generating system of the Zariski cotangent space  $\mathcal{M}/\mathcal{M}^2$  of the germ at the singular point, where  $\mathcal{M}$  is the maximal ideal of the local algebra of the singularity  $\mathcal{A}_{p,q}$ . By taking a minimal generating system, one gets a basis of this cotangent space. But such a minimal generating system is unique, and consists precisely of the integral points of  $P(\check{\sigma})$  interior to  $\check{\sigma}$ . By Propositions 5.11 and 2.7, we see that this number is as given in the Proposition. Q.E.D.

Hirzebruch-Jung singularities are particular cases of rational singularities, introduced by M. Artin [2], [3] in the 60's (see also [4]). In [79], Tjurina proved that the blow-up of a rational surface singularity is a normal surface which has again only rational singularities (see also the comments of Lê [50, 4.1]). As any surface can be desingularized by a sequence of blow-ups of its singular points followed by normalizations (Zariski [87], see also Cossart [13] and the references therein), this shows that a rational singularity can be desingularized by a sequence of blowups of closed points. In particular this is true for a Hirzebruch-Jung singularity. As the operation of blow-up is analytically invariant, we can describe the blow-up of O in the model surface  $\mathcal{Z}(L, \sigma)$ . We use notations introduced at the beginning of the proof of Proposition 5.3.

**Proposition 6.6.** Suppose that the cone  $\sigma$  is not regular. Subdivide it by drawing the half-lines starting from O and passing through the points  $A_1, V_1, V_2, \ldots, V_s, A_r$ . Denote by  $\mathcal{F}_0(\sigma)$  the fan obtained in this way. Then the natural toric morphism  $\mathcal{Z}(L, \mathcal{F}_0(\sigma)) \xrightarrow{p_0} \mathcal{Z}(L, \sigma)$  is the blow-up of O in  $\mathcal{Z}(L, \sigma)$ .

*Proof.* A proof is sketched by Lipman in [52]. Here we give more details.

Let  $(\mathcal{S}, 0)$  be any germ of normal surface. Consider its minimal resolution  $p_{\min}$ :  $(\mathcal{R}_{\min}, E_{\min}) \to (\mathcal{S}, 0)$  and its exceptional divisor  $E_{\min} = \sum_{k=1}^{r} E_k$ . The divisors  $Z \in \sum_{k=1}^{r} \mathbf{Z} E_k$  which satisfy  $Z \cdot E_k \leq 0, \forall k \in$ 

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 $\{1, \ldots, r\}$  form an additive semigroup with a unique minimal element  $Z_{top}$ , called the *fundamental cycle* of the singularity. It verifies

(22) 
$$Z_{\text{top}} \ge \sum_{k=1}^{r} E_k$$

for the componentwise order on the set of cycles with integral coefficients. In the case of a rational singularity, Tjurina [79] showed that the divisors  $E_k$  which appear in the blow-up of 0 on S can be characterized using the fundamental cycle: they are precisely those for which  $Z_{top} \cdot E_k < 0$ .

In our case, where  $(\mathcal{S}, 0) = (\mathcal{Z}(L, \sigma), O)$ , Proposition 6.2 shows that  $p_{\min} = p_{\sigma}$ . Using the relations (21) and (22), we see that  $Z_{\text{top}} = \sum_{k=1}^{r} E_k$ . Again using relation (21), we get:

$$Z_{\text{top}} \cdot E_k < 0 \iff \text{ either } k \in \{1, r\} \text{ or } \alpha_k \ge 3.$$

This shows that the components of  $E_{\sigma}$  which appear when one blowsup the origin, are precisely those which correspond to the half-lines  $[OA_1, [OV_1, [OV_2, \ldots, [OV_s, [OA_r]] But the surface obtained by blow$ ing-up the origin is again normal, by Tjurina's theorem, which shows $that it coincides with <math>\mathcal{Z}(L, \mathcal{F}_0(\sigma))$ . Q.E.D.

One sees that after the first blow-up, the new surface has only singularities of type  $\mathbf{A}_n$ , where *n* varies in a finite set of positive numbers. The singular points are contained in the set of 0-dimensional orbits of the toric surface  $\mathcal{Z}(L, \mathcal{F}_0(\sigma))$ , which in turn correspond bijectively to the 2-dimensional cones of the fan  $\mathcal{F}_0(\sigma)$ . The germs of the surface at those points are Hirzebruch-Jung singularities of types  $\mathbf{A}_{n_0}, \ldots, \mathbf{A}_{n_s}$ , where  $n_0 = l_{\mathbf{Z}}[A_1V_1], n_1 = l_{\mathbf{Z}}[V_1V_2], \ldots, n_s = l_{\mathbf{Z}}[V_sA_r]$ .

We have spoken until now of algebraic aspects of Hirzebruch-Jung singularities. We discuss their topology in Section 8.3.

## 6.3. Monomial plane curves

Suppose that  $(\mathcal{S}, 0)$  is a germ of *smooth* surface and that  $(\mathcal{C}, 0) \subset (\mathcal{S}, 0)$  is a germ of reduced curve. A proper birational morphism  $p: \mathcal{R} \to \mathcal{S}$  is called an *embedded resolution* of the germ  $(\mathcal{C}, 0)$  if  $\mathcal{R}$  is smooth, p is an isomorphism above  $\mathcal{S} - 0$  and the *total transform*  $p^{-1}(\mathcal{C})$  of  $\mathcal{C}$  is a divisor with normal crossings on  $\mathcal{R}$  in a neighborhood of the *exceptional divisor*  $E := p^{-1}(0)$ . The closure in  $\mathcal{R}$  of the difference  $p^{-1}(\mathcal{C}) - p^{-1}(0)$  is called the *strict transform* of  $\mathcal{C}$  by the morphism p.

It is known since the XIX-th century that any germ of plane curve can be resolved in an embedded way by a sequence of blow-ups of points (see Enriques & Chisini [19], Laufer [47], Brieskorn & Knörrer [9]). The combinatorics of the exceptional divisor of the resolution can be determined starting from the Newton-Puiseux exponents of the irreducible components of the curve and from their intersection numbers using Econtinued fraction expansions. We explain here how to read the sequence of self-intersection numbers of the components of the exceptional divisor of the minimal embedded resolution of a monomial plane curve by using a zigzag diagram, instead of just doing blindly computations with continued fractions.

If  $p, q \in \mathbf{N}^*$ ,  $1 \leq q < p$  and gcd(p, q) = 1, consider the plane curve  $C_{p/q}$  defined by the equation:

$$(23) x^p - y^q = 0$$

It can be parametrized by:

(24) 
$$\begin{cases} x = t^q \\ y = t^p \end{cases}$$

As p and q are relatively prime, one sees that (24) describes the normalization morphism for  $C_{p/q}$  (see its definition at the beginning of Section 8.1). As  $t^p$  and  $t^q$  are monomials, one says that  $C_{p/q}$  is a monomial curve. There is a natural generalization to higher dimensions (see Teissier [77]).

If one identifies the plane  $\mathbf{C}^2$  of coordinates (x, y) with the toric surface  $\mathcal{Z}(L_0, \sigma_0)$ , where  $L_0 = \mathbf{Z}^2$  and  $\sigma_0$  is the first quadrant, then it is easy to see (look at equation (24)) that  $C_{p/q}$  is the closure in  $\mathbf{C}^2$  of the image of the 1-parameter subgroup of the complex torus  $\mathcal{T}_{L_0} = (\mathbf{C}^*)^2$ corresponding to the point (q, p).

Consider again the notations introduced before Lemma 3.1. Let  $l_{-} := [O(1, 0) \text{ and } l_{+} := [O(q, p) \text{ be the edges of the cone } \sigma_{x}(p/q).$  We leave to the reader the proof of the following lemma, which is very similar to the proof of Lemma 3.1. Recall that the *type* of a cone was introduced in Definition 5.5.

**Lemma 6.7.** With respect to the chosen ordering of its edges, the cone  $\sigma_x(p/q)$  is of type p/(p-q). Moreover, with the notations of Section 5,  $A_1 = (1, 1)$ ,  $A'_1 = (0, 1)$  and  $A_+ = (q, p)$ .

Even if the proof is very easy, it is important to be conscious of this result, as it allows to apply the study done in Section 5 to our context.

Given the pair (p, q), we want to describe the process of embedded resolution of the curve  $C_{p/q}$  by blow-ups, as well as the final exceptional divisor, the self-intersections of its components and their orders of appearance during the process. **Lemma 6.8.** The blow-up  $\pi_0: \mathcal{R}_0 \to \mathbf{C}^2$  of 0 in  $\mathbf{C}^2$  is a toric morphism corresponding to the subdivision of  $\sigma_0$  obtained by joining O to  $A_1 = (1, 1)$ . The strict transform of  $C_{p/q}$  passes through the 0dimensional orbit of  $\mathcal{R}_0$  associated to the cone  $\mathbf{R}_+OA_1 + \mathbf{R}_+OA'_1$ .

*Proof.* With the notations of Section 3, we consider the fan  $\mathcal{F}_0$  subdividing  $\sigma_0$  which consists of the cones  $\sigma_x(1)$ ,  $\sigma_y(1)$ , their edges and the origin. Let  $\pi_{\mathcal{F}_0} \colon \mathcal{Z}(L, \mathcal{F}_0) \to \mathcal{Z}(L, \sigma_0)$  be the associated toric morphism. It is obtained by gluing the maps  $\pi_x \colon \mathcal{Z}(L, \sigma_x(1)) \to \mathcal{Z}(L, \sigma_0)$  and  $\pi_y \colon \mathcal{Z}(L, \sigma_y(1)) \to \mathcal{Z}(L, \sigma_0)$  over  $(\mathbf{C}^*)^2$ . With respect to the coordinates given by the monomials associated to the primitive vectors of L situated on the edges of the cones  $\sigma_0$ ,  $\sigma_x(1)$ ,  $\sigma_y(1)$ , the maps  $\pi_x$  and  $\pi_y$  are respectively described by:

 $\begin{cases} x = x_1 y_1 \\ y = y_1 \end{cases} \quad \text{and} \quad \begin{cases} x = x_2 \\ y = x_2 y_2 \end{cases}$ 

One recognizes the blow-up of 0 in  $\mathbb{C}^2$ . Now, in order to compute the strict transform of  $C_{p/q}$ , one has to make the previous changes of variables in equation (19). The lemma follows immediately. Q.E.D.

Starting from Lemma 6.7 and using the previous lemma as an induction step, we get:

**Proposition 6.9.** The following procedure constructs the dual graph of the total transform of  $C_{p/q}$  by the minimal embedded resolution morphism, starting from the zigzag diagram ZZ(p/(p-q)):

• On each edge of integral length  $l \ge 1$ , add (l-1) vertices of weight 2. Then erase the weights of the edges (that is, their integral length).

• Attach the weight 1 to the vertex  $A_+$ . Then change the signs of all the weights of the vertices.

• Label the vertices by the symbols  $E_1, E_2, E_3, \ldots$  starting from  $A_1$ on  $P(\sigma)$  till arriving at  $V_1$ , continuing from the first vertex which follows  $V'_1$  on  $P(\sigma')$  till arriving at  $V'_2$ , coming then back to  $P(\sigma)$  at the first vertex which follows  $V_1$  and so on, till labelling the vertex  $A_+$ .

• Erase the horizontal line, the zigzag line and the curved segment between  $V'_0$  and the first vertex which follows  $V'_1$ .

• Add an arrow to the vertex  $A_+$  and keep only the weights of the vertices and their labels  $E_n$ .

The arrowhead vertex represents the strict transform of the curve  $C_{p/q}$  and the indices of the components  $E_i$  correspond to the orders of appearance during the process of blow-ups.

It is essential to remark that in the previous construction one starts from ZZ(p/(p-q)) and not from ZZ(p/q) (look again at Lemma 6.7).

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**Example 6.10.** Consider the curve  $x^{11} - y^4 = 0$ . Then  $\lambda = 11/(11 - 4) = 11/7$ . Its zigzag diagram ZZ(11/7) was constructed in Example 5.10. So, the dual graph of the total transform of  $C_{11/4}$  by the minimal embedded resolution morphism has 6 vertices, of easy computable weights (see Fig. 13).



Fig. 13. The dual graph of the total transform of  $C_{11,4}$ 

Proposition 6.9 endows us with an easy way of remembering the following classical description of the minimal embedded resolution of a monomial plane curve (see Jurkiewicz [43], who attributes it to Hirzebruch; Spivakovsky [74] extends it to the case of monomial-type valuations on function-fields of surfaces):

**Proposition 6.11.** If  $p/q = [m_1 + 1, n_1 + 1, m_2 + 1, ..., n_s + 1, m_{s+1}+1]^+$ , then the dual graph of the total transform of the monomial curve  $C_{p/q}$  is the one which appears in Fig. 14.

*Proof.* Combine formulae (20) and (18) with Fig. 9 and Proposition 6.9. Q.E.D.

In Fig. 14 we have indicated only the orders of appearance of the components of the exceptional divisor corresponding to the extremities of the graph. We leave as an exercise for the reader to complete the diagram with the sequence  $(E_k)_{k>1}$ .

Notice that in the E-continued fraction expansion of p/q used in the previous proposition, there is the possibility that  $m_{s+1} = 0$ . In this case, the canonical expansion is obtained using relation (7). But in order to express in a unified form the result of the application of the algorithm, it was important for us to use an expansion of  $\frac{p}{q}$  with an *odd* number of partial quotients (which is always possible, precisely according to formula (7)).

One can use the combinatorics of the embedded resolution of monomial plane curves as building blocks for the description of the combinatorics of the resolution of any germ of plane curve. A detailed description of the passage between *the Eggers tree*, which encodes the Newton-Puiseux exponents of the components of the curve, and the dual graph



Fig. 14. The dual graph of the total transform of  $C_{p/q}$ 

of the total transform of the curve by its embedded resolution morphism can be found in García Barroso [24] (see also Brieskorn & Knörrer [9, section 8.4] and Wall [85]). A topological interpretation of the trees appearing in these two encodings was given in Popescu-Pampu [62, chapter 4].

In higher dimensions, González Pérez [27] used toric geometry in order to describe embedded resolutions of quasi-ordinary hypersurface singularities. Again, the building blocks are monomial varieties. A prototype for his study is the method of resolution of an irreducible germ of plane curve by only one toric morphism, developed by Goldin & Teissier [26].

In the classical treatise of Enriques & Chisini [19], resolutions of curves by blow-ups of points are not studied using combinatorics of divisors, but instead using the *infinitely near points* through which the strict transforms of the curve pass during the process of blowing ups. Those combinatorics were also encoded in a diagram, called nowadays *Enriques diagram* (see Casas-Alvero [10]). Enriques diagrams are very easily constructed using the knowledge of the orders of appearance of

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the divisors during the process of blowing ups. For this reason, zigzag diagrams combined with Proposition 6.9 give an easy way to draw them for a monomial plane curve. We leave the details to the interested reader. Then one uses this again as building blocks for the analysis of general plane curve singularities (see [10]).

## $\S7$ . Graph structures and plumbing structures on 3-manifolds

This section contains preparatory material for the topological study of the 3-manifolds appearing as abstract boundaries of normal surface singularities, done in sections 8 and 9.

We recall general facts about Seifert, graph and plumbing structures on 3-manifolds, as well as about JSJ theory. We also define particular classes of plumbing structures on thick tori and solid tori, starting from naturally arising pairs  $(L, \sigma)$ , where L is a 2-dimensional lattice and  $\sigma$ is a rational strictly convex cone in  $L_{\mathbf{R}}$ . Namely, given a pair of essential curves on the boundary of a thick torus M, their classes generate two lines in the lattice  $L := H_1(M, \mathbf{Z})$ . A choice of orientations of these lines distinguishes one of the four cones in which the lines divide the plane...

## 7.1. Generalities on manifolds and their splittings

We denote by **I** the interval [0, 1], by **D** the closed disc of dimension 2 and by  $\mathbf{S}^n$  the sphere of dimension *n*. An *annulus* is a surface diffeomorphic to  $I \times \mathbf{S}^1$ .

A simple closed curve on a 2-dimensional torus is called *essential* if it is non-contractible. It is classical that an oriented essential curve on a torus T is determined up to isotopy by its image in  $H_1(T, \mathbf{Z})$  (see [21, Section 2.3]). Moreover, the vectors of  $H_1(T, \mathbf{Z})$  which are homology classes of essential curves are precisely the primitive ones.

We say that a manifold is *closed* if it is compact and without boundary. If M is a manifold with boundary, we denote by  $\stackrel{\circ}{M}$  its interior and by  $\partial M$  its boundary. If moreover M is oriented, we orient  $\partial M$  in such a way that at a point of  $\partial M$ , an outward pointing tangent vector to M, followed by a basis of the tangent space to  $\partial M$ , gives a basis of the tangent space to M (this is the convention which makes Stokes' theorem  $\int_M d\omega = \int_{\partial M} \omega$  true). We say then that  $\partial M$  is oriented *compatibly with* M.

If M is an oriented manifold, we denote by -M the same manifold with reversed orientation. If M is a closed oriented surface, then -M is orientation-preserving diffeomorphic to M. This fact is no longer true in dimension 3, that is why it is important to describe carefully the choice of orientation. In this sense, see Theorem 8.11, as well as Propositions 9.3 and 9.6.

We denote by Diff(M) the group of self-diffeomorphisms of M, by  $\text{Diff}^{\circ}(M)$  the subgroup of self-diffeomorphisms which are isotopic to the identity and by  $\text{Diff}^+(M)$  the subgroup of diffeomorphisms which preserve the orientation of M (when M is orientable).

**Definition 7.1.** Let M be a 3-manifold with boundary. We say that M is a **thick torus** if it is diffeomorphic to  $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{I}$ . We say that M is a **solid torus** if it is diffeomorphic to  $\mathbf{D} \times \mathbf{S}^1$ . We say that M is a **thick Klein bottle** if it is diffeomorphic to a unit tangent circle bundle to the Möbius band.

In the definition of a thick Klein bottle M we use an arbitrary riemannian metric on a Möbius band. The manifold obtained like this is independent of the choices up to diffeomorphism. Moreover, it is orientable, because any tangent bundle is orientable and the manifold we define appears as the boundary of a unit tangent disc bundle. The preimage of a central circle of the Möbius band by the fibration map is a Klein bottle, and the manifold M appears then as a tubular neighborhood of it, which explains the name. For details, see [82, Section 3] and [21, Section 10.11].

On the boundary of a solid torus M there exists an essential curve which is contractible in M. Such a curve, which is unique up to isotopy (see [21]), is called a *meridian* of M. A 3-manifold M is called *irreducible* if any embedded sphere bounds a ball. A surface embedded in M is called *incompressible* if its  $\pi_1$  injects in  $\pi_1(M)$ . Two tori embedded in M are called *parallel* if they are disjoint and they cobound a thick torus embedded in M. The manifold M is called *atoroidal* if any embedded incompressible torus is parallel to a component of  $\partial M$ .

**Definition 7.2.** Let M be an orientable manifold and S be an orientable closed (not necessarily connected) hypersurface of M. A manifold with boundary  $M_S$  endowed with a map  $M_S \xrightarrow{r_{M,S}} M$  is called **a** splitting of M along S if:

•  $r_{M,S}$  is a local embedding;

•  $\partial M_S = (r_{M,S})^{-1}(S)$  and the restriction  $r_{M,S}|_{\partial M_S}$  is a trivial double covering of S;

• the restriction  $(r_{M,S})|_{\overset{\circ}{M}_S} : \overset{\circ}{M}_S \longrightarrow M - S$  is a diffeomorphism.

If this is the case, the map  $r_{M,S}$  is called the **reconstruction map** associated to the splitting. We say that S splits M into  $M_S$  and that the connected components of  $M_S$  are the **pieces** of the splitting. If N

is a piece of  $M_S$  and  $P \subset M$  is a set, we say that P contains N if  $r_{M,S}(N) \subset P$ .

It can be shown easily that splittings of M along S exist and are unique up to unique isomorphism. The idea is very intuitive, one simply thinks at M being split open along each connected component of S. A way to realize this is to take the complement of an open tubular neighborhood of S in M and to deform the inclusion mapping in an arbitrarily small neighborhood of the boundary in order to push it towards S (see Waldhausen [83] and Jaco [39]).

If  $\phi \in \text{Diff}^+(M)$ , one can also canonically split  $\phi$  and get a diffeomorphism  $\phi_S$  of manifolds with boundary (we leave the axiomatic definition of  $\phi_S$  to the reader):

$$\phi_S \colon M_S \longrightarrow M_{\phi(S)}$$

Among closed 3-manifolds, two particular classes will be especially important for us, the lens spaces and the torus fibrations. The reason why we treat them simultaneously will appear clearly in Section 8.3.

**Definition 7.3.** Let M be an orientable 3-manifold. We say that M is a **lens space** if it contains an embedded torus T such that  $M_T$  is the disjoint union of two solid tori whose meridians have non-isotopic images on T. We say that M is a **torus fibration** if it contains an embedded torus T such that  $M_T$  is a thick torus.

Lens spaces can also be defined as quotients of  $\mathbf{S}^3$  by linear free cyclic actions or - and this explains the name - as manifolds obtained by gluing in a special way the faces of a lens-shaped polyhedron (see [71] or [21, Section 4.3]). We impose the condition on the meridians in order to avoid the manifold  $\mathbf{S}^1 \times \mathbf{S}^2$ , which can also be split into two solid tori, but whose universal cover is not the 3-dimensional sphere, a difference which makes it to be excluded from the set of lens spaces by most authors. There exists a classical encoding of oriented lens spaces by positive integers. We recall it at the end of Section 9.1 (see Proposition 9.4).

If M is a torus fibration and  $T \subset M$  splits it into a thick torus, then a trivial foliation of  $M_T$  by tori parallel to the boundary components is projected by  $r_{M,T}$  onto a foliation by pairwise parallel tori. The space of leaves is topologically a circle and the projection  $\pi: M \to \mathbf{S}^1$  is a locally trivial fibre bundle whose fibres are tori, which explains the name.

**Definition 7.4.** Let  $\pi: M \to \mathbf{S}^1$  be a locally trivial fibre bundle whose fibres are tori. Fix a fibre of  $\pi$  (for example the initial torus T) and also an orientation of the base space  $\mathbf{S}^1$ . The **algebraic monodromy** 

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**operator** m is by definition the first return map of the natural parallel transport on the first homology fibration over  $S^1$ , when one travels in the positive direction.

The map m is a well-defined linear automorphism  $m \in SL(H_1(T, \mathbf{Z}))$ , once an orientation of  $\mathbf{S}^1$  was chosen. Its conjugacy class in  $SL(2, \mathbf{Z})$ is independent of the choice of the fibre. If one changes the orientation of  $\mathbf{S}^1$ , then m is replaced by  $m^{-1}$ . This shows that the trace of m is independent of the choice of T and of the orientation of  $\mathbf{S}^1$ . Remark that no choice of orientation of M is needed in order to define it.

For more information about torus fibrations, see Neumann [57] and Hatcher [33]. We come back to them in Section 9.2, with special emphasis on subtleties related to their orientations.

## 7.2. Seifert structures

Seifert manifolds are special 3-manifolds whose study can be reduced in some way to the study of lower-dimensional spaces.

**Definition 7.5.** A Seifert structure on a 3-manifold M is a foliation by circles such that any leaf has a compact orientable saturated neighborhood. A leaf with trivial holonomy is called a **regular fibre**. A leaf which is not regular is called an **exceptional fibre**. The space of leaves is called the **base** of the Seifert structure. We say that a Seifert structure is **orientable** if there is a continuous orientation of all the leaves of the foliation. If such an orientation is fixed, one says that the Seifert structure is **oriented**. If there exists a Seifert structure on M, we say that M is a **Seifert manifold**.

The condition on the leaves to have compact saturated neighborhoods is superfluous if the ambient manifold M is compact, it is enough then to ask that any leaf be orientation-preserving, as was shown by Epstein [20]. This is no longer true on non-compact manifolds, as was shown by Vogt [81].

The initial definition of Seifert [70] was slightly different:

a) He did not speak of "foliation", but of "fibration".

b) He gave models for the possible neighborhoods of the leaves.

In what concerns point a), Seifert's definition is one of the historical sources of the concept of fibration and fibre bundle. For him a fibration is a decomposition of a manifold into "fibres"; only in a second phase can one try to construct the associated "orbit space", or the "base" with our vocabulary. This shows that his definition is closer to the present notion of foliation; in fact his "fibration" is a foliation, but this can be seen only by using the required condition on model neighborhoods. We prefer to speak about "Seifert structure" and not "Seifert fibration" precisely because what is important to us is to see the structure as living *inside* the manifold, which makes possible to speak about isotopies. For details about the historical development of different notions of fibrations, see Zisman [88].

In what concerns point b), the possible orientable saturated neighborhoods of foliations by circles coincide up to a leaf-preserving diffeomorphism with Seifert's model neighborhoods. If one drops the orientability condition, appears a new model which was not considered by Seifert, but which is very useful in the classification of *non-orientable* 3-manifolds (see Scott [69], Bonahon [6]). Some general references about Seifert manifolds are Orlik [61], Neumann & Raymond [58] (where the base was defined as an orbifold), Scott [69], Fomenko & Matveev [21] and Bonahon [6].

In the sequel, we are interested in Seifert structures only up to isotopy.

**Definition 7.6.** Two Seifert structures  $\mathcal{F}_1$  and  $\mathcal{F}_2$  on M are called **isotopic** if there exists  $\phi \in \text{Diff}^{\circ}(M)$  such that  $\phi(\mathcal{F}_1) = \mathcal{F}_2$ .

The following proposition is proved in Jaco [39] and Fomenko & Matveev [21].

**Proposition 7.7.** The only orientable compact connected 3-manifolds with non-empty boundary which admit more than one Seifert structure up to isotopy are the thick torus, the solid torus and the thick Klein bottle.

a) If M is a thick torus, any essential curve on one of its boundary components is the fibre of a Seifert structure on M, unique up to isotopy, and devoid of exceptional fibres. Moreover, M appears like this as the total space of a trivial circle bundle over an annulus.

b) If M is a solid torus and  $\gamma$  is a meridian of it, an essential curve c on its boundary is a fibre of a Seifert structure on M if and only if their homological intersection number  $[c] \cdot [\gamma]$  (once they are arbitrarily oriented) is non-zero. In this case, the associated structure is unique up to isotopy and has at most one exceptional fibre. All fibres are regular if and only if  $[c] \cdot [\gamma] = \pm 1$ . In this last case, M appears as the total space of a trivial circle bundle over a disc.

c) If M is a thick Klein bottle, it admits up to isotopy two Seifert structures. One of them is devoid of exceptional fibres and its space of orbits is a Möbius band. The other one has two exceptional fibres with holonomy of order 2 and its space of orbits is topologically a disc.

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The closed orientable 3-manifolds which admit more than one Seifert structure up to isotopy are also classified (see Bonahon [6] and the references therein). In this paper we need only the following less general result, which can be deduced by combining [6] with [57] (see Definition 8.1):

**Proposition 7.8.** The only 3-manifolds which are diffeomorphic to abstract boundaries of normal surface singularities and which admit non-isotopic Seifert structures are the lens spaces.

## 7.3. Graph structures and JSJ decomposition theory

If one glues various Seifert manifolds along components of their boundaries, one obtains so-called *graph-manifolds*:

**Definition 7.9.** A graph structure on a 3-manifold M is a pair  $(\mathcal{T}, \mathcal{F})$ , where  $\mathcal{T}$  is an embedded surface in M whose connected components are tori and where  $\mathcal{F}$  is a Seifert structure on  $M_{\mathcal{T}}$  (see Definition 7.2). We say that a graph structure is **orientable** if  $\mathcal{F}$  is an orientable Seifert structure on  $M_{\mathcal{T}}$ . If there exists a graph structure on M, we say that M is a graph manifold.

Notice that no particular graph structure is specified when one speaks about a graph manifold. One only supposes that there exists one. In the sequel we are interested in graph structures on a given manifold only *up to isotopy*:

**Definition 7.10.** Two graph structures  $(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2)$  on M are called **isotopic** if there exists  $\phi \in \text{Diff}^{\circ}(M)$  such that  $\phi(\mathcal{T}_1) = \mathcal{T}_2$  and  $\phi_{\mathcal{T}_1}(\mathcal{F}_1)$  is isotopic to  $\mathcal{F}_2$ .

Graph manifolds were introduced by Waldhausen [82], generalizing von Randow's *tree manifolds* (see their definition in the next paragraph) studied in [65]. Following Mumford [55] who proved Poincaré conjecture for the abstract boundaries of normal surface singularities (see Definition 8.1), von Randow proved it for tree manifolds; his proof contained a gap which was later filled by Scharf [68].

Waldhausen's definition was different from Definition 7.9. On one side he did not allow exceptional fibres in the Seifert structure on  $M_{\mathcal{T}}$ and on another side he did not fix (up to isotopy) a precise fibration by circles, but only supposed that such a fibration existed. He represented a graph structure by a finite graph with decorated vertices and edges (corresponding respectively to the pieces of  $M_{\mathcal{T}}$  and to the components of  $\mathcal{T}$ ), which explains the name. Tree manifolds are then the graph manifolds which admit a graph structure ( $\mathcal{T}, \mathcal{F}$ ) such that the corresponding graph is a tree and the base of the Seifert structure on  $\mathcal{F}$  has genus 0. With our definition, graph structures can also be encoded by graphs. One has only to add more decorations to the vertices, in order to keep in memory the exceptional fibres of the corresponding Seifert fibred pieces.

With his definition, Waldhausen solved the homeomorphism problem for graph-manifolds, by giving normal forms for the graph structures on a given manifold and by showing that with exceptions in a finite explicit list, any irreducible graph-manifold has a graph-structure in normal form which is unique *up to isotopy*.

Later, Jaco & Shalen [40] and Johannson [41] showed that there remains no exception in the classification up to isotopy if one modifies the notion of graph-structure by allowing exceptional fibres, that is, when one works with Definition 7.9. More generally, they proved:

**Theorem 7.11.** Let M be a compact, connected, orientable and irreducible 3-manifold (with possible non-empty boundary). Then Mcontains an embedded surface  $\mathcal{T}$  whose connected components are incompressible tori and such that any piece of  $M_{\mathcal{T}}$  is either a Seifert manifold or is atoroidal. Moreover, if  $\mathcal{T}$  is minimal for the inclusion among surfaces with this property, then it is well-defined up to isotopy.

We say that a minimal family  $\mathcal{T}$  as in the previous theorem is a JSJ family of tori in M.

A variant of the previous theorem considers also embedded annuli. These various theorems of canonical decomposition are called nowadays *Jaco-Shalen-Johannson (JSJ) decomposition theory*, and were the starting point of Thurston's geometrization program, as well as of the theory of JSJ decompositions for groups. For details about JSJ decompositions, in addition to the previously quoted books one can consult Jaco [39], Neumann & Swarup [59], Hatcher [33] and Bonahon [6]. In [62] and [63], we showed that also knot theory inside an irreducible 3-manifold reflects the ambient JSJ decomposition.

We define now a notion of *minimality* for graph structures on a given manifold.

**Definition 7.12.** Suppose that  $(\mathcal{T}, \mathcal{F})$  is a graph structure on M. We say that it is **minimal** if the following conditions are verified:

• No piece of  $M_{\mathcal{T}}$  is a thick torus or a solid torus.

• One cannot find a Seifert structure  $\mathcal{F}'$  on  $M_{\mathcal{T}}$  such that the images of its leaves by the reconstruction mapping  $r_{M,\mathcal{T}}$  coincide on a component of  $\mathcal{T}$ .

As a corollary of Theorem 7.11, if  $(\mathcal{T}, \mathcal{F})$  is a minimal graph structure on M, then  $\mathcal{T}$  is the minimal JSJ system of tori in M. But one can prove more:

**Theorem 7.13.** Each closed orientable irreducible graph manifold which is not a torus fibration with  $|\operatorname{tr} m| \geq 3$  admits a minimal graph structure. Moreover, the family T of tori associated to a minimal graph structure coincides with the JSJ family of tori. In particular, it is unique up to an isotopy.

Suppose that  $(\mathcal{T}, \mathcal{F})$  is a given graph structure without thick tori and solid tori among its pieces. In view of Proposition 7.7, its only pieces which can have non-isotopic Seifert structures are the thick Klein bottles. This shows that, in order to check whether  $(\mathcal{T}, \mathcal{F})$  is minimal or not, one has only to consider the possible choices of Seifert structures on them up to isotopy (that is  $2^n$  possibilities, where *n* is the number of such pieces).

Suppose that M is a graph manifold which is neither a torus fibration with  $|\operatorname{tr} m| \geq 3$ , nor a Seifert manifold which admits non-isotopic Seifert structures. Then, if  $\mathcal{T}$  is a family of tori associated to a minimal graph structure, there is a unique Seifert structure on  $M_{\mathcal{T}}$  up to isotopy, such that each piece which is a thick Klein bottle has an orientable base.

**Definition 7.14.** Suppose that M is an orientable graph manifold which is neither a torus fibration with  $|\operatorname{tr} m| \geq 3$  nor a Seifert manifold which admits non-isotopic Seifert structures. We say that a minimal graph structure is the **canonical graph structure** on M if each piece which is a thick Klein bottle has an orientable base.

## 7.4. Plumbing structures

Plumbing structures are special types of graph structures:

**Definition 7.15.** A plumbing structure on a 3-manifold M is a graph structure without exceptional fibres  $(\mathcal{T}, \mathcal{F})$  on M, such that for any component T of  $\mathcal{T}$ , the homological intersection number on T of two fibres of  $\mathcal{F}$  coming from opposite sides is equal to  $\pm 1$ .

Plumbing structures are the ancestors of graph structures. They were introduced by Mumford [55] in the study of singularities of complex analytic surfaces (see Hirzebruch [36], Hirzebruch, Neumann & Koh [38], as well as our explanations in Section 8.2). In fact Mumford does not speak about "plumbing structure". Instead, he describes a way to construct the abstract boundary of a normal surface singularity (see Definition 8.1) by gluing total spaces of circle-bundles over real surfaces using "plumbing fixtures". Later on, "plumbing" was more used as a verb than as a noun. That is, one concentrated more on the operations needed to construct a new object from elementary pieces, than on the structure obtained on the manifold resulting from the construction. The fact that we are interested precisely in this structure up to isotopy and not on the graph which encodes it, is a difference with Neumann [57] for example.

In [57], Neumann describes an algorithm for deciding if two manifolds obtained by plumbing are diffeomorphic. He uses as an essential ingredient Waldhausen's classification theorem of graph manifolds (according to the definition which does not allow exceptional fibres, see the comments made in Section 7.3). In fact, by using the uniqueness up to isotopy of the JSJ-tori, we can deduce the uniqueness up to isotopy for special plumbing structures on singularity boundaries. This is the subject of Section 9.

Even if Definition 7.15 seems to suggest the opposite, the class of graph manifolds is the same as the class of manifolds which admit a plumbing structure. A way to see this is to use the construction of plumbing structures on thick tori and solid tori described in Section 7.5. For a detailed comparison of graph structures and plumbing structures, as well as for a study of the elementary operations on them, one can consult Popescu-Pampu [62, Chapter 4].

# 7.5. Hirzebruch-Jung plumbing structures on thick tori and solid tori

In this section we define special classes of plumbing structures on thick tori and solid tori, which will be used in Section 9. The starting point is in both cases a pair  $(L, \sigma)$  of a 2-dimensional lattice and a rational strictly convex cone  $\sigma \subset L_{\mathbf{R}}$ , naturally attached to essential curves on the boundary of the 3-manifold.

• Suppose first that M is an oriented thick torus.

On each component of its boundary, we consider an essential curve. Denote by  $\gamma$ ,  $\delta$  these curves. We suppose that their homology classes (once they are arbitrarily oriented) in  $H_1(M, \mathbf{R}) \simeq \mathbf{R}^2$  are non-proportional. So, we are in presence of a 2-dimensional lattice  $L = H_1(M, \mathbf{Z})$ and of two distinct rational lines in it, generated by the homology classes  $[\gamma], [\delta].$ 

Orient  $\partial M$  compatibly with M. Then *order* in an arbitrary way the components of  $\partial M$ : call the first one  $T_-$  and the second one  $T_+$ . Denote by  $\gamma_-$  the simple closed curve drawn on  $T_-$  and by  $\gamma_+$  the one drawn on  $T_+$ . Then *orient*  $\gamma_-$  and  $\gamma_+$ . By hypothesis, their homology classes  $[\gamma_-], [\gamma_+]$  are non-proportional primitive vectors in the 2-dimensional lattice  $L = H_1(M, \mathbf{Z})$ . This shows that  $([\gamma_-], [\gamma_+])$  is a basis of  $L_{\mathbf{R}} =$ 

 $H_1(M, \mathbf{R})$  which induces an orientation of this vector space. As  $T_+$  is a deformation retract of M, one has canonically  $H_1(T_+, \mathbf{Z}) = L$ , and so the ordered pair  $(\gamma_-, \gamma_+)$  induces an orientation of  $T_+$ .

**Definition 7.16.** We say that  $\gamma_{-}$  and  $\gamma_{+}$  are oriented **compati**bly with the orientation of M if, when taken in the order  $(\gamma_{-}, \gamma_{+})$ , they induce on  $T_{+}$  an orientation which coincides with its orientation as a component of  $\partial M$ .

Of course, a priori there is no reason for choosing this notion of compatibility rather than the opposite one. Our choice was done in order to get a more pleasant formulation for Lemma 8.5.

Let  $\sigma$  be the cone generated by  $[\gamma_{-}]$  and  $[\gamma_{+}]$  in  $L_{\mathbf{R}}$ . As these homology classes were supposed non-proportional, the cone  $\sigma$  is strictly convex and has non-empty interior. Denote by  $l_{\pm}$  the edge of  $\sigma$  which contains the integral point  $[\gamma_{\pm}]$ . Then, with the notations of Section 4,  $A_{\pm} = [\gamma_{\pm}]$ . Indeed, as  $\gamma_{\pm}$  is an essential curve of  $T_{\pm}$ , its homology class is a primitive vector of L.

Let  $(A_n)_{0 \le n \le r+1}$  be the integral points on the compact edges of  $P(\sigma)$ , defined in Section 4. So,  $OA_0 = [\gamma_-]$  and  $OA_{r+1} = [\gamma_+]$ . Let  $(T_n)_{0 \le n \le r+2}$  be a sequence of pairwise parallel tori in M, such that  $T_0 = T_-$  and  $T_{r+2} = T_+$ . Moreover, we number them in the order in which they appear between  $T_-$  and  $T_+$ . Denote  $\mathcal{T} := \bigsqcup_{n=1}^{r+1} T_n$ . If  $M_n$  denotes the piece of  $M_{\mathcal{T}}$  whose boundary components are  $T_n$  and  $T_{n+1}$ , where  $n \in \{0, \ldots, r+1\}$ , we consider on it a Seifert structure such that the homology class of its fibres in L is  $OA_n$ .



Fig. 15. Hirzebruch-Jung plumbing structures on thick tori

We get like this a plumbing structure on M, well-defined up to isotopy, and depending only on the triple  $(M, \gamma_{-}, \gamma_{+})$ . We see that the simultaneous change of the orientations of  $\gamma_{-}$  and  $\gamma_{+}$  or the change of their ordering (in order to respect the compatibility condition of Definition 7.16) leads to the same (unoriented) plumbing structure.

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**Definition 7.17.** We say that the previous unoriented plumbing structure on the oriented thick torus M is **the Hirzebruch-Jung plumbing structure** associated to  $(\gamma, \delta)$  and we denote it by  $\mathcal{P}(M, \gamma, \delta)$ .

• Suppose now that M is an oriented solid torus.



Fig. 16. Hirzebruch-Jung plumbing structures on solid tori

We consider an essential curve  $\gamma$  on  $\partial M$  which is not a meridian. Take a torus T embedded in  $\stackrel{\circ}{M}$  and parallel to  $\partial M$ . Denote by N the thick torus contained between  $\partial M$  and T. Put  $T_- = \partial M$ ,  $T_+ = T$ ,  $\gamma_- = \gamma$  and let  $\gamma_+$  be an essential curve on  $T_+$  which is a meridian of the solid torus  $M - \stackrel{\circ}{N}$  (see Fig. 16). Consider the Hirzebruch-Jung plumbing structure  $\mathcal{P}(N, \gamma_-, \gamma_+)$ . With the notations of the construction done for thick tori, denote  $\mathcal{T}(M, \gamma) := \bigsqcup_{n=1}^{r} T_n$ . Then the pieces of  $M_{\mathcal{T}(M, \gamma)}$  are the thick tori  $M_0, M_1, \ldots, M_{r-1}$  and a solid torus which is the "union" of  $M_r, M_{r+1}$  and  $M - \stackrel{\circ}{N}$ . On  $M_0, \ldots, M_{r-1}$  we keep the Seifert structure of  $\mathcal{P}(N, \gamma_-, \gamma_+)$ . On the solid torus we extend the Seifert structure of  $M_r$ . By Proposition 7.7 b), we see that this Seifert structure has no exceptional fibres. This shows that we have constructed a plumbing structure on M. It is obviously well-defined up to isotopy, once the isotopy class of  $\gamma$  is fixed.

**Definition 7.18.** We say that the previous unoriented plumbing structure on the oriented solid torus M is **the Hirzebruch-Jung plumbing structure** associated to  $\gamma$  and we denote it by  $\mathcal{P}(M, \gamma)$ .

## $\S$ 8. Generalities on the topology of surface singularities

In this section we look at the boundaries  $M(\mathcal{S})$  of normal surface singularities  $(\mathcal{S}, 0)$ . We explain how to associate to any normal crossings resolution p of  $(\mathcal{S}, 0)$  a plumbing structure on  $M(\mathcal{S})$ . Then we explain how to pass from the plumbing structure associated to the minimal normal crossings resolution of  $(\mathcal{S}, 0)$  to the canonical graph structure on  $M(\mathcal{S})$  (see Definition 7.14).

We recommend the survey articles of Némethi [56] and Wall [84] for an introduction to the classification of normal surface singularities.

## 8.1. Resolutions of normal surface singularities and their dual graphs

First we recall basic facts about normal analytic spaces. Let  $\mathcal{V}$  be a reduced analytic space. It is called *normal* if for any point  $P \in \mathcal{V}$ , the germ  $(\mathcal{V}, P)$  is irreducible and its local algebra is integrally closed in its field of fractions. If  $\mathcal{V}$  is not normal, then there exists a finite map  $\nu \colon \tilde{\mathcal{V}} \to \mathcal{V}$  which is an isomorphism over a dense open set of  $\mathcal{V}$ and such that  $\tilde{\mathcal{V}}$  is normal. Such a map, which is unique up to unique isomorphism, is called a *normalization* map of  $\mathcal{V}$ .

A reduced analytic curve is normal if and only if it is smooth. If a germ  $(\mathcal{S}, 0)$  of reduced surface is normal, then there exists a representative of it, which we keep calling  $\mathcal{S}$ , such that  $\mathcal{S} - 0$  is smooth. The converse is not true.

Let  $(\mathcal{S}, 0)$  be a germ of normal complex analytic surface. We say also that  $(\mathcal{S}, 0)$  is a normal surface singularity (even if the point 0 is regular on  $\mathcal{S}$ ). In the sequel, we use the same notation  $(\mathcal{S}, 0)$  for the germ and for a sufficiently small representative of it. If  $e: (\mathcal{S}, 0) \to (\mathbb{C}^N, 0)$  is any local embedding, denote by  $\mathcal{S}_{e,r}$  the intersection of  $\mathcal{S}$  with a euclidean ball of  $\mathbb{C}^N$  of radius  $r \ll 1$  and by  $M_{e,r}(\mathcal{S})$  the boundary of  $\mathcal{S}_{e,r}$ .

By general transversality theorems due to Whitney, when r > 0is small enough,  $M_{e,r}(S)$  is a smooth manifold, *naturally oriented* as the boundary of the complex manifold  $S_{e,r}$ . It does not depend on the choices of embedding e and radius  $r \ll 1$  made to define it (see Durfee [16]).

**Definition 8.1.** An oriented 3-manifold  $M(\mathcal{S})$  orientation-preserving diffeomorphic with the manifolds  $M_{e,r}(\mathcal{S})$ , where r > 0 is small enough, is called the (**abstract**) **boundary** or the **link** of the singularity  $(\mathcal{S}, 0)$ .

It is important to keep in mind that in the sequel M(S) is supposed naturally oriented as explained before. In order to understand better this remark, look at Theorem 8.11.

The easiest way to describe the topological type of the manifold M(S) is (as first done by Mumford [55]) by retracting it to the exceptional divisor of a resolution of (S, 0). Let us first define this last notion.

**Definition 8.2.** An analytic map  $p: (\mathcal{R}, E) \to (\mathcal{S}, 0)$  is called a **resolution** of the singularity  $(\mathcal{S}, 0)$  with **exceptional divisor**  $E = p^{-1}(0)$  if the following conditions are simultaneously satisfied:

•  $\mathcal{R}$  is a smooth surface;

• *p* is a proper morphism;

• the restriction of p to  $\mathcal{R} - E = \mathcal{R} - f^{-1}(0)$  is an isomorphism onto  $\mathcal{S} - 0$ .

We say that  $p: (\mathcal{R}, E) \to (\mathcal{S}, 0)$  is a normal crossings resolution if one has moreover:

• E is a divisor with normal crossings.

Recall that, by definition, a divisor on a smooth complex surface has *normal crossings* if in the neighborhood of any of its points, its support is either smooth, or the union of transverse smooth curves.

Normal crossings resolutions always exist (see Laufer [47] and Lipman [52] for a careful presentation of the Hirzebruch-Jung method of resolution, as well as Cossart [13] for Zariski's method of resolution by normalized blow-ups).

There is a unique *minimal* resolution, which we denote

 $p_{\min}: (\mathcal{R}_{\min}, E_{\min}) \to (\mathcal{S}, 0)$ . The minimality property means that any other resolution  $p: (\mathcal{R}, E) \to (\mathcal{S}, 0)$  can be factorized as  $p = p_{\min} \circ q$ , where  $q: \mathcal{R} \to \mathcal{R}_{\min}$  is a proper bimeromorphic map. The minimal resolution  $p_{\min}$  is characterized by the fact that  $E_{\min}$  contains no component  $E_i$  which is smooth, rational and of self-intersection -1 (classically called an *exceptional curve of the first kind*).

Analogously, there is a unique resolution which is minimal among normal crossings ones. We denote it:

$$p_{\rm mnc}$$
:  $(\mathcal{R}_{\rm mnc}, E_{\rm mnc}) \to (\mathcal{S}, 0)$ 

It is characterized by the fact that  $E_{\rm mnc}$  has normal crossings and each component  $E_i$  of  $E_{\rm mnc}$  which is an exceptional curve of the first kind contains at least 3 points which are singular on  $E_{\rm mnc}$ .

If a normal crossings resolution has moreover only smooth components, one says usually that the resolution is *good*; there exists also a unique minimal good resolution, but in this paper we don't consider it.

The following criterion allows one to recognize the divisors which are exceptional with respect to some resolution of a normal surface singularity.

**Theorem 8.3.** Let E be a reduced compact connected divisor in a smooth surface  $\mathcal{R}$ . Denote by  $(E_i)_{1 \leq i \leq n}$  its components. Then E is the exceptional divisor of a resolution of a normal surface singularity if and only if the intersection matrix  $(E_i \cdot E_j)_{i,j}$  is negative definite. The necessity is classical (see [38, Section 9], where is presented Mumford's proof of [55] and where the oldest reference is to Du Val [80]). The sufficiency was proved by Grauert [31] (see also Laufer [47]). If E verifies the conditions which are stated to be equivalent in the theorem, one also says that E can be contracted on  $\mathcal{R}$ .

From now on we suppose that  $p: (\mathcal{R}, E) \to (\mathcal{S}, 0)$  is a normal crossings resolution of  $(\mathcal{S}, 0)$ .

Denote by  $\Gamma(p)$  its weighted dual graph. Its set of vertices  $\mathcal{V}(p)$  is in bijection with the irreducible components of E. Depending on the context, we think about  $E_i$  as a curve on  $\mathcal{R}$  or a vertex of  $\Gamma(p)$ . The vertices which represent the components  $E_i$  and  $E_j$  are joined by as many edges as  $E_i$  and  $E_j$  have intersection points on  $\mathcal{R}$ . In particular, there are as many *loops* based at the vertex  $E_i$  as singular points (that is, self-intersections) on the curve  $E_i$  (see Fig. 17). Each vertex  $E_i$  is decorated by two weights, the geometric genus  $g_i$  of the curve  $E_i$  (that is, the genus of its normalization) and its self-intersection number  $e_i \leq -1$ in  $\mathcal{R}$ . Denote also by  $\delta_i$  the valency of the vertex  $E_i$ , that is, the number of edges starting from it (where each loop counts for 2). For example, in Fig. 17 one has  $\delta_1 = 9$ ,  $\delta_2 = 5$ , etc.



Fig. 17. A normal crossings divisor and its dual graph

## 8.2. The plumbing structure associated to a normal crossings resolution

By Definition 8.1,  $M(\mathcal{S})$  is diffeomorphic to  $M_{e,r}(\mathcal{S})$ , where  $e: (\mathcal{S}, 0) \to (\mathbb{C}^N, 0)$  is an embedding and  $r \ll 1$ . But  $M_{e,r}(\mathcal{S})$  is the level-set at level r of the function  $\rho_e: (\mathcal{S}, 0) \to (\mathbb{R}_+, 0)$ , the restriction to  $e(\mathcal{S})$  of the distance-function to the origin in  $\mathbb{C}^N$ .

As the resolution p realizes by definition an isomorphism between  $\mathcal{R} - E$  and  $\mathcal{S} - 0$ , it means that  $M_{e,r}(\mathcal{S}) = \rho_e^{-1}(r)$  is diffeomorphic to  $\psi_e^{-1}(r)$ , where  $\psi_e := \rho_e \circ p$ . The advantage of this changed viewpoint on  $M(\mathcal{S})$  is that it appears now orientation-preserving diffeomorphic to the boundary of a "tubular neighborhood" of the curve E in the smooth manifold  $\mathcal{R}$ . As in general E has singularities, one has to discuss the precise meaning of the notion of tubular neighborhood. We quote Mumford [55, pages 230–231]:

Now the general problem, given a complex  $K \subset E^n$ , Euclidean *n*-space, to define a tubular neighborhood, has been attacked by topologists in several ways although it does not appear to have been treated definitively as yet. J.H.C. Whitehead [86], when K is a subcomplex in a triangulation of  $E^n$ , has defined it as the boundary of the star of K in the second barycentric subdivision of the given triangulation. I am informed that Thom [78] has considered it more from our point of view: for a suitably restricted class of positive  $C^{\infty}$  fcns. f such that f(P) = 0 if and only if  $P \in K$ , define the tubular neighborhood of K to be the level manifolds  $f = \epsilon$ , small  $\epsilon$ . The catch is how to suitably restrict f; here the archtype for  $f^{-1}$  may be thought of as the potential distribution due to a uniform charge on K.

In [34] M. W. Hirsch has constructed a theory of tubular neighborhoods well-adapted to complexes K appearing in analytic singularity theory.

Let us come back to the normal crossings divisor E in the smooth surface  $\mathcal{R}$ .

If E is smooth, then one can construct a diffeomorphism between a tubular neighborhood U(E) of E in  $\mathcal{R}$  and of E in the total space  $N_{\mathcal{R}}E$  of its normal bundle in  $\mathcal{R}$ . As  $N_{\mathcal{R}}E$  is naturally fibred by discs, this is also true for U(E). The fibration of U(E) can be chosen in such a way that the levels  $\psi_e^{-1}(r)$  are transversal to the fibres for  $r \ll 1$ . In this way one gets a Seifert structure without singular fibres on  $\psi_e^{-1}(r) \simeq M(\mathcal{S})$ .

Suppose now that E is not smooth, but that its irreducible components are so. One can also define in this situation a notion of tubular neighborhood U(E) of E in  $\mathcal{R}$ . One way to do it is to take the union of conveniently chosen tubular neighborhoods  $U(E_i)$  of E's components  $E_i$ . Abstractly, one has to glue the 4-manifolds with boundary  $U(E_i)$  by identifying well-chosen neighborhoods of the points which get identified on E. This procedure is what is called the "plumbing" of disc-bundles over surfaces (see Hirzebruch [36], Hirzebruch & Neumann & Koh [38], Brieskorn [8]). Its effect on the boundaries  $\partial U(E_i)$  is to take out saturated filled tori and to identify their boundaries, by a diffeomorphism which permutes fibres and meridians in an orientation-preserving way. This is the 3-dimensional "plumbing" operation introduced by Mumford [55], alluded to in Section 7.4.

In order to understand what happens near a singular point of E, it is convenient to choose local coordinates (x, y) on E in the neighborhood of the singular point, such that E is defined by the equation xy = 0. So, y = 0 defines locally an irreducible component  $E_i$  of E and similarly x = 0 defines  $E_j$ . It is possible that  $E_i = E_j$ , a situation excluded in the previous paragraph for pedagogical reasons. If this equality is true, then the same plumbing procedure can be applied, this time by identifying well-chosen neighborhoods of points of the same 4-manifold with boundary  $U(E_i)$ .

At this point appears a subtlety: the 4-manifold  $U(E_i)$  to be considered is no longer a tubular neighborhood of  $E_i$  in  $\mathcal{R}$ , but instead of the normalization  $\tilde{E}_i$  of  $E_i$  inside the modified normal bundle  $\nu_i^* T \mathcal{R} / T \tilde{E}_i$ .

Here  $\nu_i \colon \tilde{E}_i \to \mathcal{R}$  denotes the normalization map of  $E_i$  and  $T\mathcal{R}$ , respectively  $T\tilde{E}_i$  denote the holomorphic tangent bundles to the smooth complex manifolds  $\mathcal{R}$  and  $\tilde{E}_i$ . As a real differentiable bundle of rank 2, this vector bundle over  $\tilde{E}_i$  is characterized by its Euler number  $\tilde{e}_i$ , which is equal to the self-intersection number of  $\tilde{E}_i$  inside the total space of the bundle. This number is related to the self-intersection of  $E_i$  inside  $\mathcal{R}$  in the following way (see Neumann [57, page 333]):

**Lemma 8.4.** If  $\tilde{e}_i$  is the Euler number of the real bundle  $\nu_i^* T\mathcal{R}/T\tilde{E}_i$  over  $\tilde{E}_i$ , where  $\nu_i \colon \tilde{E}_i \to \mathcal{R}$  is the normalization map of  $E_i$ , then  $\tilde{e}_i = e_i - \delta_i$ .

*Proof.* In order to understand this formula, just think at the effect of a small isotopy of  $E_i$  inside  $\mathcal{R}$ . Near each self-crossing point of  $E_i$ , the intersection point of one branch of  $E_i$  with the image of the other branch after the isotopy is not counted when one computes  $\tilde{e}_i$ . Q.E.D.

Notice that Theorem 8.3 is *true* if one takes as diagonal entries of the matrix the numbers  $e_i = E_i^2$ , but is *false* if one takes instead the numbers  $\tilde{e}_i$ . The easiest example is given by an irreducible divisor  $E = E_1$ , with  $e_1 = 1 > 0$  and  $\delta_1 = 2$  which, by Lemma 8.4 implies that  $\tilde{e}_1 = -1 < 0$ .

In Fig. 18 we represent in two ways the local situation near the chosen singular point of E. On the left we simply draw the union of the two neighborhoods  $U(E_i)$  and  $U(E_j)$ . On the right, "the corners are smoothed". This is precisely what happens when we look at the

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levels of the function  $\psi_e$ . Moreover, we represent by interrupted lines the real analytic set defined by the equation |x| = |y|. Its intersection with  $\psi_e^{-1}(r) \simeq \partial U(E) \simeq M(S)$  is a two-dimensional torus T. This is the way in which such tori appear naturally as structural elements of the 3-manifolds M(S). One also sees how the complement of T in  $\partial U(E)$  is fibred by boundaries of discs transversal to  $E_i$  or  $E_j$ .

By considering model neighborhoods of the singular points of E structured as in the right-hand side of Fig. 18 and conveniently extending them to a tubular neighborhood of all of E, one gets a retraction

$$\Phi \colon U(E) \to E$$

which restricts to a locally trivial disc-fibration over the smooth locus of E and whose fibre over each singular point of E is a cone over a 2dimensional torus. By considering the restriction  $\Phi|_{\partial U(E)}$ , we see that the fibres over the singular points of E are embedded tori, and that their complement gets fibred by circles.

As  $\partial U(E)$  is orientation-preserving diffeomorphic to M(S), we see that M(S) gets endowed with a graph structure  $(\mathcal{T}(p), \mathcal{F}(p))$  welldefined up to isotopy. It is a good test of the understanding of the complexifications of Fig. 18 to show that  $(\mathcal{T}(p), \mathcal{F}(p))$  is in fact a plumbing structure (see Definition 7.15).



Fig. 18. The local configuration which leads to plumbing

The pieces of  $M(\mathcal{S})_{\mathcal{T}(p)}$  correspond to the irreducible components of E, that is to the vertices of  $\Gamma(p)$ . Denote by  $M(E_i)$  the piece which corresponds to  $E_i$ . The fibres of  $M(E_i)$  are obtained up to isotopy by cutting the boundary of the chosen sufficiently small tubular neighborhood of E with smooth holomorphic curves transversal to E at smooth points of  $E_i$ . So, the plumbing structure  $(\mathcal{T}(p), \mathcal{F}(p))$  is naturally oriented.

**Lemma 8.5.** With their natural orientations, the fibres on both sides of any component of  $\mathcal{T}(p)$  are oriented compatibly with the orientation of  $M(\mathcal{S})$ .

*Proof.* The notion of compatibility we speak about is the one of Definition 7.16. We mean that, if we take an arbitrary component T of  $\mathcal{T}(p)$ , and a tubular neighborhood N(T) such that its preimage in  $M(\mathcal{S})_{\mathcal{T}(p)}$  is saturated by the leaves of the foliation  $\mathcal{F}(p)$ , then two fibres, one in each boundary component of N(T), are oriented compatibly with the orientation of N(T). Now, this is an instructive exercise on the geometrical understanding of the relations between the orientations of various objects in the neighborhood of a normal crossing on a smooth surface. Just think of the complexification of Fig. 18.

**Corollary 8.6.** The orientation of the fibres of  $(\mathcal{T}(p), \mathcal{F}(p))$  is determined by the associated unoriented plumbing structure up to a simultaneous change of orientation of all the fibres.

**Proof.** Consider the unoriented plumbing structure. Start from an arbitrary piece  $M(E_i)$ , and choose one of the two continuous orientations of its fibres. Then propagate this orientations farther and farther through the components of  $\mathcal{T}(p)$ , by respecting the compatibility condition on the neighboring orientations. As M(S) is connected, we know that after a finite number of steps one has oriented the fibres of all the pieces. As one orientation exists which is compatible in the neighborhood of all the tori, we see that our process cannot arrive at a contradiction (that is, a non-trivial monodromy around a loop of  $\Gamma(p)$  in the choice of orientations). Q.E.D.

The following lemma is a particular case of the study done in Mumford [55, page 11] and Hirzebruch [36, page 250-03].

**Lemma 8.7.** Suppose that  $E_i$  is a component of E which is smooth, rational and whose valency in the graph  $\Gamma(p)$  is 2. In the thick torus  $M(E_i)$  which corresponds to it in the plumbing structure  $(\mathcal{T}(p), \mathcal{F}(p))$ , consider an oriented fibre f of  $M(E_i)$ , as well as oriented fibres f', f'' of the two (possibly coinciding) adjacent pieces. Then one has the following relation in the homology group  $H_1(M(E_i), \mathbb{Z})$ :

$$[f'] + [f''] = |e_i| \cdot [f].$$

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## 8.3. The topological characterization of HJ and cusp singularities

We want now to understand how to pass from the plumbing structure  $(\mathcal{T}(p), \mathcal{F}(p))$  on  $M(\mathcal{S})$  to the canonical graph structure on it (see Definition 7.14). We see that the pieces of  $M(\mathcal{S})_{\mathcal{T}(p)}$  which are thick tori correspond to components  $E_i$  which are smooth and rational with  $\delta_i = 2$ , and those which are solid tori correspond to components  $E_i$  which are smooth and rational with  $\delta_i = 1$ . It is then natural to introduce the following:

**Definition 8.8.** We say that a vertex  $E_i$  of  $\Gamma(p)$  is a **chain vertex** if  $E_i$  is smooth,  $g_i = 0$  and  $\delta_i \leq 2$ . If moreover  $\delta_i = 2$ , we call it an **interior chain vertex**, otherwise we call it a **terminal chain vertex**. We say that a vertex of  $\Gamma(p)$  is a **node** if it is not a chain vertex.

In [51], Lê, Michel & Weber used the name "rupture vertex" for a node in the dual graph associated to the minimal embedded resolution of a plane curve singularity. In their situation, where all the vertices represent smooth rational curves, nodes are simply those of valency  $\geq 3$ . In our case this is no longer true, as one can have also vertices of valency  $\leq 2$ , if they correspond to curves  $E_i$  which are either not smooth or of genus  $g_i \geq 1$ .

Denote by  $\mathcal{N}(p)$  the set of nodes of  $\Gamma(p)$ . It is an empty set if and only if  $\Gamma(p)$  is topologically a segment or a circle and all the components  $E_i$  are smooth rational curves. The first situation occurs precisely for the Hirzebruch-Jung singularities, defined in Section 6.2 (see Proposition 6.2), and the second one for cusp singularities, introduced by Hirzebruch [37] in the number-theoretical context of the study of Hilbert modular surfaces.

**Definition 8.9.** A germ  $(\mathcal{S}, 0)$  of normal surface singularity is called a **cusp singularity** if it has a resolution p such that  $\Gamma(p)$  is topologically a circle and  $\mathcal{N}(p) = \emptyset$ .

For other definitions and details about them, see Hirzebruch [37], Laufer [49] (where they appear as special cases of *minimally elliptic singularities*), Ebeling & Wall [17] (where they appear as special cases of *Kodaira singularities*), Oda [60], Wall [84] and Némethi [56]. They were generalized to higher dimensions by Tsuchihashi (see Oda [60, Chapter 4]).

In the previous definition it is not possible to replace the resolution p by the minimal normal crossings one. Indeed:

**Lemma 8.10.** If (S, 0) is a cusp singularity, then  $\Gamma(p_{mnc})$  is topologically a circle and either  $\mathcal{N}(p_{mnc}) = \emptyset$ , or  $E_{mnc}$  is irreducible, rational, with one singular point where it has normal crossings.

*Proof.* One passes from p to  $p_{mnc}$  by successively contracting components F which are smooth, rational and verify  $F^2 = -1$  (that is, exceptional curves of the first kind, by a remark which follows Definition 8.2). The new exceptional divisor verifies the same hypothesis as the one of p, except when one passes from a divisor with 2 components to a divisor with one component. In this last situation, this second irreducible divisor is rational, as its strict transform F is so. Moreover, it has one singular point with normal crossing branches passing through it, as by hypothesis F cuts transversely the other component of the first divisor in exactly two points. Q.E.D.

We would like to emphasize the following theorem due to Neumann [57, Theorem 3], which characterizes Hirzebruch-Jung and cusp singularities among normal surface singularities.

**Theorem 8.11.** Let (S, 0) be a normal surface singularity. The manifold -M(S) is orientation-preserving diffeomorphic to the abstract boundary of a normal surface singularity if and only if (S, 0) is either a Hirzebruch-Jung singularity or a cusp-singularity.

Recall that  $-M(\mathcal{S})$  denotes the manifold  $M(\mathcal{S})$  with reversed orientation.

We will bring more light on this theorem with Propositions 9.3 and 9.6, which show that for both Hirzebruch-Jung and cusp singularities, the involutions  $M(S) \rightsquigarrow -M(S)$  are manifestations of the duality described in Section 5.

As Hirzebruch-Jung singularities, cusp singularities can also be defined using toric geometry (see Oda [60, Chapter 4]). In the same spirit, as a particular case of Laufer's [48] classification of *taut* singularities, we have:

**Theorem 8.12.** *Hirzebruch-Jung and cusp singularities are taut, that is, their analytical type is determined by their topological type.* 

For this reason, it is natural to ask which 3-manifolds are obtained as abstract boundaries of Hirzebruch-Jung singularities and cusp singularities. This question is answered by:

**Proposition 8.13.** 1) (S, 0) is a Hirzebruch-Jung singularity if and only if M(S) is a lens space. Moreover, each oriented lens space appears like this. 2)  $(\mathcal{S}, 0)$  is a cusp singularity if and only if  $M(\mathcal{S})$  is a torus fibration with algebraic monodromy of trace  $\geq 3$ . Moreover, each oriented torus fibration of this type appears like this.

*Proof.* This proposition is a particular case of Neumann [57, Corollary 8.3]. Here we sketch the proofs of the necessities, in order to develop tools for sections 9.1 and 9.2.

Let  $p: (\mathcal{R}, E) \to (\mathcal{S}, 0)$  be the minimal normal crossings resolution of  $(\mathcal{S}, 0)$  (for notational convenience, we drop the index "mnc"). Denote by U(E) a (closed) tubular neighborhood of E in  $\mathcal{R}$  and by  $\Phi: U(E) \to E$  a preferred retraction, as defined in Section 8.2. Denote also by

$$\Psi \colon \partial U(E) \to E$$

the restriction of  $\Phi$  to  $\partial U(E) \simeq M(\mathcal{S})$ .

1) Suppose that  $(\mathcal{S}, 0)$  is a Hirzebruch-Jung singularity.

Orient the segment  $\Gamma(p)$ . Denote then by  $E_1, \ldots, E_r$  the components of E in the order in which they appear along  $\Gamma(p)$  in the positive direction. For each  $i \in \{1, \ldots, r-1\}$ , denote by  $A_{i,i+1}$  the intersection point of  $E_i$  and  $E_{i+1}$ . Consider also two other points  $A_{0,1} \in E_1$ ,  $A_{r,r+1} \in E_r$  which are smooth points of E. Then consider on each component  $E_i$  a Morse function

$$\Pi_i \colon E_i \to \left[\frac{i-1}{r}, \frac{i}{r}\right]$$

having as its only critical points  $A_{i-1,i}$  (where  $\Pi_i$  attains its minimum) and  $A_{i,i+1}$  (where  $\Pi_i$  attains its maximum). As  $\Pi_i(A_{i,i+1}) = \Pi_{i+1}(A_{i,i+1})$  for all  $i \in \{1, \ldots, r-1\}$ , we see that the maps  $\Pi_i$  can be glued together in a continuous map

$$\Pi \colon E \to [0, 1].$$

Consider the composed continuous map  $\Pi \circ \Psi \colon M(\mathcal{S}) \to [0, 1]$  (see Fig. 19).

Our construction shows that its fibres over 0 and 1 are circles and that those over interior points of [0, 1] are tori. Moreover, each such torus splits M into two solid tori. By Definition 7.3, we see that M is a lens space.

It remains now to prove that each oriented lens space appears like this.

Denote  $L := H_1(M(\mathcal{S}) - (\Pi \circ \Psi)^{-1}\{0, 1\}, \mathbb{Z})$ . As  $M(\mathcal{S}) - (\Pi \circ \Psi)^{-1}\{0, 1\}$  is the interior of a thick torus foliated by the tori  $(\Pi \circ \Psi)^{-1}(c)$ , where  $c \in (0, 1)$ , we see that L is a 2-dimensional lattice. With the



Fig. 19. The maps  $\Pi$  and  $\Psi$  for a Hirzebruch-Jung singularity

notations of Section 8.2, let  $f_i$  be an oriented fibre in the piece  $M(E_i)$  of the plumbing structure  $(\mathcal{T}(p), \mathcal{F}(p))$  which corresponds to  $E_i$ . Consider also  $f_0$  and  $f_{r+1}$ , canonically oriented meridians on the boundaries of tubular neighborhoods of  $(\Pi \circ \Psi)^{-1}(0)$ , respectively  $(\Pi \circ \Psi)^{-1}(1)$ .

For each  $i \in \{0, \ldots, r+1\}$ , denote by  $v_i := [f_i] \in L$  the homology class of  $f_i$ . Recall that  $e_i := E_i^2$ . By Lemma 8.7, we see that

(25) 
$$v_{i+1} = |e_i| \cdot v_i - v_{i-1}, \quad \forall i \in \{0, \dots, r\}.$$

By Proposition 6.2, p is also the minimal resolution of  $(\mathcal{S}, 0)$ , which shows that  $|e_i| \geq 2$ ,  $\forall i \in \{1, \ldots, r\}$ . Now apply Proposition 4.4. We deduce that the numbers  $e_i$  are determined by the oriented topological type of the lens space  $M(\mathcal{S})$ , once the isotopy class of the tori  $(\Pi \circ \Psi)^{-1}(c)$ is fixed.

This shows that, starting from any oriented lens space M and torus  $T \subset M$  which splits M into two solid tori, one can construct a Hirzebruch-Jung singularity  $(\mathcal{S}, 0)$  such that  $M(\mathcal{S}) \simeq M$  only by looking at the classes of the meridians of the two solid tori in the lattice  $L = H_1(T, \mathbb{Z})$ . One has only to be careful to orient them compatibly with the orientation of M (as explained at the beginning of the proof of Lemma 8.5).

2) Suppose that  $(\mathcal{S}, 0)$  is a cusp singularity.

• Consider first the case where  $r \ge 2$ . Orient the circle  $\Gamma(p)$  and choose one of its vertices. Denote then by  $E_1, \ldots, E_r$  the components

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of E in the order in which they appear along  $\Gamma(p)$  in the positive direction, starting from  $E_1$ . For each  $i \in \{1, \ldots, r\}$ , denote by  $A_{i, i+1}$ the intersection point of  $E_i$  and  $E_{i+1}$ , where  $E_{r+1} = E_1$ . Consider then functions  $\Pi_i : E_i \to [(i-1)/r, i/r]$  with the same properties as in the case of Hirzebruch-Jung singularities. By passing to the quotient  $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$ , we can glue the previous maps into a continuous map:

$$\Pi \colon E \to \mathbf{R}/\mathbf{Z}.$$

Consider then the map  $\Pi \circ \Psi \colon M(\mathcal{S}) \to \mathbf{R}/\mathbf{Z}$  (see Fig. 20).



Fig. 20. The maps  $\Pi$  and  $\Psi$  for a cusp singularity

Our construction shows that  $\Pi$  realizes  $M(\mathcal{S})$  as the total space of a torus fibration over  $\mathbf{R}/\mathbf{Z}$ .

Denote by  $T_{i,i+1} := \Psi^{-1}(A_{i,i+1})$  the torus of  $\mathcal{T}(p)$  which corresponds to the intersection point of  $E_i$  and  $E_{i+1}$ . Denote  $T := T_{r,1}$  and let N(T) be a (closed) tubular neighborhood of T, which does not intersect any other torus  $T_{i,i+1}$ , for  $i \in \{1, \ldots, r-1\}$  (see Fig. 20).

Denote  $L := H_1(M(\mathcal{S}) - N(T), \mathbf{Z})$ . As  $M(\mathcal{S}) - N(T)$  is the interior of a thick torus, we see that L is a 2-dimensional lattice. With the notations of Section 8.2, let  $f_i$  be an oriented fibre in the piece  $M(E_i)$ . We suppose moreover that  $f_1$  and  $f_r$  are situated on the boundary of N(T). Consider two other circles  $f_0$  and  $f_{r+1}$  on  $\partial N(T)$ , such that  $f_0$ ,  $f_r$ are isotopic *inside* N(T) and situated on distinct boundary components and such that the same is true for the pair  $f_1$ ,  $f_{r+1}$ .

For each  $i \in \{0, \ldots, r+1\}$ , denote by  $v_i := [f_i] \in L$  the homology class of  $f_i$ . By Lemma 8.7, we see that:

(26) 
$$v_{i+1} = |e_i| \cdot v_i - v_{i-1} = -e_i \cdot v_i - v_{i-1}, \quad \forall i \in \{0, \dots, r\}$$

where  $E_0 := E_r$ .

Denote by  $n \in GL(L)$  the automorphism which sends the basis  $(v_0, v_1)$  of L into the basis  $(v_r, v_{r+1})$ . The relations (26) show that its matrix in the basis  $(v_0, v_1)$  is:

$$\begin{pmatrix} 0 & -1 \\ 1 & e_1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & e_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & e_r \end{pmatrix}$$

A little thinking shows that n is the inverse of the algebraic monodromy  $m \in GL(L)$  in the positive direction along  $\mathbf{R}/\mathbf{Z}$ . So, the matrix of m in the basis  $(v_0, v_1)$  is:

$$\begin{pmatrix} e_r & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e_{r-1} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} e_1 & 1 \\ -1 & 0 \end{pmatrix}$$

We have reproved like this Theorem 6.1 IV in Neumann [57]. We deduce by induction the following expression for its trace, where the polynomials  $Z^-$  were defined by formula (1):

(27) 
$$\operatorname{tr} m = Z^{-}(|e_1|, \ldots, |e_r|) - Z^{-}(|e_2|, \ldots, |e_{r-1}|).$$

The negative definiteness of the intersection matrix of E (see Theorem 8.3) shows that there exists  $i \in \{1, \ldots, r\}$  such that  $|e_i| \geq 3$ . As p is supposed to be the minimal resolution of  $(\mathcal{S}, 0)$ , we have also  $e_j \geq 2, \forall j \in \{1, \ldots, r\}$ . Using equation (27), we deduce then easily by induction on r that tr  $m \geq 3$ .

• Consider now the case r = 1. Then, by Lemma 8.10, E is a rational curve with one singular point P, where E has normal crossings. Let  $p': (\mathcal{R}', E') \to (\mathcal{S}, 0)$  be the resolution of  $(\mathcal{S}, 0)$  obtained by blowing up  $P \in \mathcal{R}$ . Then E' is a normal crossings resolution with smooth components  $E_1, E_2$ , where  $E_1^2 = -1$  and  $E_2$  is the strict transform of E. As  $(p')^*E = 2E_1 + E_2$  and  $((p')^*E)^2 = E^2$ , we deduce that  $E_2^2 = E^2 - 4 \leq -5$ . Now we apply the same argument as in the case  $r \geq 2$ , but for the resolution p'.

An alternative proof could use Lemma 8.4.

The fact that each oriented torus fibration with tr  $m \geq 3$  appears like this is a consequence of the study done in Section 9.2. Indeed, there we show how to extract the numbers  $(e_1, \ldots, e_r)$  from the oriented topological type of  $M(\mathcal{S})$ . Q.E.D.

By Neumann [57], there exist also abstract boundaries M(S) which are torus fibrations with algebraic monodromy of trace 2. But in that case the exceptional divisor of the minimal resolution is an elliptic curve (then, following Saito [67], one speaks about *simple elliptic singularities*, which are other particular cases of minimally elliptic ones).

## 8.4. Construction of the canonical graph structure

Consider again an arbitrary normal surface singularity  $(\mathcal{S}, 0)$  and a normal crossings resolution p of it.

**Definition 8.14.** Suppose that the set of nodes  $\mathcal{N}(p)$  is nonempty. Conceive the graph  $\Gamma(p)$  as a 1-dimensional CW-complex and take the complement  $\Gamma(p) - \mathcal{N}(p)$ . This complement is the disjoint union of segments, which we call **chains**. If a chain is open at both extremities we call it an **interior chain**. If it is half-open we call it a **terminal chain**.

In Fig. 20 we represent the chains of Fig. 17, with the hypothesis that  $E_4, E_5, E_7 \notin \mathcal{N}(p)$  and  $E_6 \in \mathcal{N}(p)$ . That is, we suppose that  $E_4, E_5, E_6, E_7$  are smooth and that  $g(E_4) = g(E_5) = g(E_7) = 0$ ,  $g(E_6) \geq 1$ . There is only one terminal chain, which contains the terminal chain vertex  $E_7$ .

Denote by  $\mathcal{C}(p)$  the set of chains. This set can be written as a disjoint union

$$\mathcal{C}(p) = \mathcal{C}_i(p) \sqcup \mathcal{C}_t(p)$$

where  $C_i(p)$  denotes the set of interior chains and  $C_t(p)$  the set of terminal chains. The edges of  $\Gamma(p)$  contained in a chain  $C \in C(p)$  correspond to a set of parallel tori in M(S). Choose one torus  $T_C$  among them and define:

$$\mathcal{T}'(p) := \bigsqcup_{C \in \mathcal{C}_i(p)} T_C.$$



Fig. 21. The chains of Fig. 17 when  $E_6$  is a node
By construction, each piece of  $M(\mathcal{S})_{\mathcal{T}'(p)}$  contains a unique piece  $M(E_i)$  of  $M(\mathcal{S})_{\mathcal{T}(p)}$  such that  $E_i$  is a node of  $\Gamma(p)$ . If  $E_i \in \mathcal{N}(p)$ , denote by  $M'(E_i)$  the piece of  $M(\mathcal{S})_{\mathcal{T}'(p)}$  which contains  $M(E_i)$ . One can extend in a unique way up to isotopy the natural Seifert structure without exceptional fibres on  $M(E_i)$  to a Seifert structure on  $M'(E_i)$ . One obtains like this a graph structure  $(\mathcal{T}'(p), \mathcal{F}'(p))$  on  $M(\mathcal{S})$ .

Till now we have worked with any normal crossings resolution p. We consider now a special one, the minimal normal crossings resolution  $p_{\rm mnc}$ .

**Proposition 8.15.** Suppose that (S, 0) is neither a Hirzebruch-Jung singularity, nor a cusp singularity. Then the graph structure  $(\mathcal{T}'(p_{mnc}), \mathcal{F}'(p_{mnc}))$  is the canonical graph structure on M(S).

*Proof.* If  $\mathcal{T}'(p_{mnc})$  is empty, as  $(\mathcal{S}, 0)$  is not a cusp singularity we deduce that  $(\mathcal{T}'(p_{mnc}), \mathcal{F}'(p_{mnc}))$  is a Seifert structure. By Proposition 7.8, we see that it is the canonical graph structure on  $M(\mathcal{S})$ .

Suppose now that  $\mathcal{T}'(p_{mnc})$  is non-empty. One has to verify two facts (see Definition 7.14):

• first, that all the fibrations induced by  $\mathcal{F}'(p_{\rm mnc})$  on the pieces which are thick Klein bottles have orientable basis;

• second, that by taking the various choices of Seifert structures on the pieces of  $M(\mathcal{S})_{\mathcal{T}'(p_{mnc})}$ , one does not obtain isotopic fibres coming from different sides on one of the tori of  $\mathcal{T}'(p_{mnc})$ .

The first fact is immediate, as one starts from Seifert structures with orientable basis on the pieces of  $M(\mathcal{S})_{\mathcal{T}(p_{mnc})}$  before eliminating tori of  $\mathcal{T}(p_{mnc})$  in order to remain with  $\mathcal{T}'(p_{mnc})$ .

In what concerns the second fact, the idea is to look at the fibres corresponding to the chain vertices of any interior chain C. The union of the pieces of  $M(\mathcal{S})_{\mathcal{T}(p_{mnc})}$  which are associated to those vertices is a thick torus  $N_R$ . Take a fibre in each piece (remember that they are naturally oriented as boundaries of holomorphic discs) and look at their images in  $L = H_1(N_R, \mathbb{Z})$ . One gets like this a sequence of vectors  $v_1, \ldots, v_s \in L$ . Consider also the images  $v_0$  and  $v_{s+1}$  of the fibres coming from the nodes of  $\Gamma(p_{mnc})$  to which C is adjacent, the order of the indices respecting the order of the vertices along the chain.

By Lemma 8.7,  $v_{k+1} = \alpha_k v_k - v_{k-1}$  for any  $k \in \{1, \ldots, s\}$ , where  $\alpha_k$  is the absolute value of the self-intersection of the component  $E_i$  of  $\Gamma(p_{\rm mnc})$  which gave rise to the vector  $v_k$ . Here plays the hypothesis that  $p_{\rm mnc}$  is *minimal*: this implies that  $\alpha_k \geq 2$ . Then one can conclude by using Proposition 7.7.

The analysis of thick Klein bottles is similar. It is based on the fact that a thick Klein bottle can appear only from a portion of the

graph  $\Gamma(p)$  as in Fig. 21, where  $E_1, E_2, E_3$  are smooth rational curves of self-intersections -2, -2, respectively -n (see Neumann [57, pages 305, 334]). The important point is that  $n \ge 2$ . Otherwise the complete sub-graph of  $\Gamma(p)$  with vertices  $E_1, E_2, E_3$  would have a non-definite intersection matrix, which contradicts Theorem 8.3. Q.E.D.



Fig. 22. The appearance of a thick Klein bottle

The plumbing structure  $(\mathcal{T}(p_{mnc}), \mathcal{F}(p_{mnc}))$  on  $N(\mathcal{S})$  is associated to the resolution  $p_{mnc}$  of  $(\mathcal{S}, 0)$ . One can wonder if the canonical graph structure  $(\mathcal{T}'(p_{mnc}), \mathcal{F}'(p_{mnc}))$  is also associated to some analytic morphism with target  $(\mathcal{S}, 0)$ .

This is indeed the case. In order to see it, start from  $p_{mnc}$  and its exceptional divisor E. Then contract all the components of E which correspond to chain vertices. One gets like this a normal surface with only Hirzebruch-Jung singularities. The image of E on it is a divisor Fwith again only normal crossings when seen as an abstract curve. Take then as a representative of M(S) the boundary of a tubular neighborhood of F in the new surface and split it into pieces which project into the various components of F. The splitting is done using tori which are associated bijectively to the singular points of F. Namely, in a system of (toric) local coordinates (x, y) such that F is defined by xy = 0, one proceeds as for the definition of the plumbing structure associated to a normal crossings resolution (see Section 8.2). Then this system of tori is isotopic to  $\mathcal{T}'(p_{mnc})$ .

# $\S$ 9. Invariance of the canonical plumbing structure on the boundary of a normal surface singularity

In this section we describe how to reconstruct the plumbing structure  $(\mathcal{T}(p_{mnc}), \mathcal{F}(p_{mnc}))$  on  $M(\mathcal{S})$  associated to the minimal normal crossings resolution of  $(\mathcal{S}, 0)$ , only from the abstract oriented manifold  $M(\mathcal{S})$ . Namely, using the classes of plumbing structures on thick tori defined in Section 7.5, we define a plumbing structure  $\mathcal{P}(M(\mathcal{S}))$  on  $M(\mathcal{S})$ and we prove: **Theorem 9.1.** 1) When considered as an unoriented structure, the plumbing structure  $\mathcal{P}(M(\mathcal{S}))$  depends up to isotopy only on the natural orientation of  $M(\mathcal{S})$ . We call it the canonical plumbing structure on  $M(\mathcal{S})$ .

2) The plumbing structure  $(\mathcal{T}(p_{mnc}), \mathcal{F}(p_{mnc}))$  associated to the minimal normal crossings resolution of  $(\mathcal{S}, 0)$  is isotopic to the canonical plumbing structure  $\mathcal{P}(M(\mathcal{S}))$ .

As a corollary we get the theorem of invariance of the plumbing structure ( $\mathcal{T}(p_{mnc}), \mathcal{F}(p_{mnc})$ ) announced in the introduction (see Theorem 9.7). We also explain how the orientation reversal on the boundary of a Hirzebruch-Jung or cusp singularity reflects the duality between supplementary cones explained in Section 5.1 (see Propositions 9.4 and 9.6).

In order to prove Theorem 9.1, we consider three cases, according to the nature of M(S). In the first one it is supposed to be a lens space, in the second one a torus fibration with algebraic monodromy of trace  $\geq 3$  and in the last one none of the two (so, by Proposition 8.13, this corresponds to the trichotomy: (S, 0) is a Hirzebruch-Jung singularity/ a cusp singularity/ none of the two).

The idea is to start from some structure on  $M(\mathcal{S})$  which is welldefined up to isotopy, and to enrich it by canonical constructions of Hirzebruch-Jung plumbing structures (defined in Section 7.5). When  $M(\mathcal{S})$  is neither a lens space nor a torus fibration with algebraic monodromy of trace  $\geq 3$ , this starting structure will be the canonical graph structure (see Definition 7.14). Otherwise we need some special theorems of structure (Theorems 9.2 and 9.5).

#### 9.1. The case of lens spaces

Notice that by Proposition 8.13 1),  $M(\mathcal{S})$  is a lens space if and only if  $(\mathcal{S}, 0)$  is a Hirzebruch-Jung singularity.

The following theorem was proved by Bonahon [5]:

**Theorem 9.2.** Up to isotopy, a lens space contains a unique torus which splits it into two solid tori.

We say that a torus embedded in a lens space and splitting it into two solid tori is a *central torus*. By the previous theorem, a central torus is well-defined up to isotopy.

Let M be an oriented lens space and T a central torus in M. Consider a tubular neighborhood N(T) of T in M, whose boundary components we denote by  $T_{-}$  and  $T_{+}$ , ordered in an arbitrary way. Then  $M_{T_{-}\sqcup T_{+}}$  has three pieces, one being sent diffeomorphically by the reconstruction map  $r_{M, T_{-}\sqcup T_{+}}$  on N(T) - by a slight abuse of notations, we

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keep calling it N(T) - and the others,  $M_{-}$  and  $M_{+}$ , having boundaries sent by  $r_{M, T_{-} \sqcup T_{+}}$  on  $T_{-}$ , respectively  $T_{+}$  (see Fig. 33). The manifolds  $M_{-}$  and  $M_{+}$  are solid tori, as T was supposed to be a central torus. Let  $\gamma_{-}$  and  $\gamma_{+}$  be meridians of  $M_{-}$ , respectively  $M_{+}$ , oriented compatibly with the orientation of N(T) (see Definition 7.16). Consider the Hirzebruch-Jung plumbing structure  $\mathcal{P}(N(T), \gamma_{-}, \gamma_{+})$  on N(T), whose tori are denoted by  $T_{0} = T_{-}, T_{1}, \ldots, T_{r+2} = T_{+}$ , as explained in Section 7.5.



Fig. 23. Construction of the canonical plumbing structure on a lens space

Denote  $\mathcal{T}_M := \mathcal{T}_2 \sqcup \cdots \sqcup \mathcal{T}_r$ . Then  $M_{\mathcal{T}_M}$  contains four pieces less than the manifold  $M_{\mathcal{T}_- \sqcup \mathcal{T}_M \sqcup \mathcal{T}_+}$ . Denote by  $M'_-$  and  $M'_+$  the piece which "contains"  $M_-$ , respectively  $M_+$ . On  $M'_-$  we consider the Seifert structure which extends the Seifert structure of  $M_1$  and on  $M'_+$  the one which extends the Seifert structure of  $M_r$ . By applying the intersection theoretical criterion of Proposition 7.7 b), we see that those Seifert structures have no exceptional fibres (we used a similar argument to construct in Section 7.5 the Hirzebruch-Jung plumbing structure on solid tori). On the other pieces of  $M_{\mathcal{T}_M}$  we consider the Seifert structure coming from the plumbing structure  $\mathcal{P}(M, \gamma_-, \gamma_+)$ . Denote by  $\mathcal{P}(M)$  the plumbing structure constructed like this on the oriented manifold M.

Proof of Theorem 9.1.

1) This is obvious by construction (we use Theorem 9.2).

2) In the construction of  $\mathcal{P}(M(\mathcal{S}))$ , one can take as central torus Tany torus  $(\Pi \circ \Psi)^{-1}(c)$ , with  $c \in (0, 1)$ , in the notations of the proof of Proposition 8.13, 1). Then one sees that  $[\gamma_{-}] = [f_0]$  and  $[\gamma_{+}] = [f_{r+1}]$ 

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in the lattice  $L = H_1(M(S) - (\Pi \circ \Psi)^{-1}\{0, 1\}, \mathbf{Z}) = H_1(T, \mathbf{Z})$ . Using the relations (25) and the definition of a Hirzebruch-Jung plumbing structure on a thick torus (see Section 7.5), we deduce that the images of the fibres  $f_i$  in L are equal to the images of the fibres of  $\mathcal{P}(M(S))$  (see also Proposition 4.4). The proposition follows by the fact that on a 2-torus, any oriented essential curve is well-defined up to isotopy by its homology class. Q.E.D.

Let  $\sigma$  be the strictly convex cone of  $L_{\mathbf{R}}$  whose edges are generated by  $[\gamma_{-}]$  and  $[\gamma_{+}]$ . If one changes the ordering of the components of  $\partial N(T)$ , then one gets the same cone  $\sigma$ , and if one changes simultaneously the orientations of  $\gamma_{-}$  and  $\gamma_{+}$ , then one gets the opposite cone. But if one changes the orientation of M, then the cone  $\sigma$  is replaced by a supplementary cone. So, in view of Section 5.3, the two cones are in duality. In this sense, the canonical plumbing structure  $\mathcal{P}(-M(\mathcal{S}))$  is dual to  $\mathcal{P}(M(\mathcal{S}))$ . We get:

**Proposition 9.3.** Let (S, 0) be a Hirzebruch-Jung singularity. Then the canonical plumbing structures with respect to the two possible orientations of M(S) are dual to each other. More precisely, if  $(S, 0) \simeq (\mathcal{Z}(L, \sigma), 0) \simeq \mathcal{A}_{p,q}$ , then -M(S) is orientation preserving diffeomorphic to  $M(\check{S})$ , where, with the notations of Section 4,  $(\check{S}, 0) \simeq (\mathcal{Z}(\check{L}, \check{\sigma}), 0) \simeq \mathcal{A}_{p,p-q}$ .

Let  $\lambda := p/q$  be the type of the cone  $(L, \sigma)$  in the sense of Definition 5.5, where 0 < q < p and gcd(p, q) = 1. The oriented lens space  $M(\mathcal{S})$ , where  $(\mathcal{S}, 0) \simeq (\mathcal{Z}(L, \sigma), 0) \simeq \mathcal{A}_{p,q}$ , is said classically to be of type L(p, q). By Propositions 5.6 and 5.8, combined with Theorem 9.2, we get the following classical fact:

**Proposition 9.4.** 1) The lens spaces L(p, q) and L(p, q') are orientation-preserving diffeomorphic if and only if p = p' and  $q' \in \{q, \overline{q}\}$ , where  $0 < \overline{q} < p, \ q\overline{q} \equiv 1 \pmod{p}$ .

2) The lens spaces L(p, q) and L(p, q') are orientation-reversing diffeomorphic if and only if p = p' and  $q' \in \{p - q, p - \overline{q}\}$ .

#### 9.2. The case of torus fibrations with tr $m \geq 3$

Notice that by Proposition 8.13 2), M(S) is a torus fibration whose algebraic monodromy verifies tr  $m \geq 3$  if and only if (S, 0) is a cusp singularity. First we study with a little more detail torus fibrations.

Let M be an orientable torus fibration. Take a fibre torus T. Then consider the lattice  $L = H_1(T, \mathbb{Z})$  and the algebraic monodromy operator  $m \in SL(L)$  (see Definition 7.4) associated with one of the two possible orientations of the base. The following theorem is a consequence of Waldhausen [83, section 3] (see also Hatcher [33, section 5]):

**Theorem 9.5.** Up to isotopy, an orientable torus fibration M such that tr  $m \ge 3$  contains a unique torus which splits it into a thick torus (see Definition 7.2).

We say that a torus embedded in an orientable torus fibration whose algebraic monodromy m verifies tr  $m \ge 3$  and which splits it into a thick torus is a *fibre torus*. By the previous theorem, a fibre torus is welldefined up to isotopy.

From now on, we suppose that indeed tr  $m \geq 3$  (see Proposition 8.13, 2)). As M is orientable, m preserves the orientation of L, which shows that det m = 1. This implies that the characteristic polynomial of m is  $X^2 - (\operatorname{tr} m)X + 1$ . We deduce that m has two strictly positive eigenvalues with product 1, and so the eigenspaces are two distinct real lines in  $L_{\mathbf{R}}$ .

But the most important point is that these lines are *irrational*. Indeed, the eigenvalues are  $\nu_{\pm} := (1/2)(\operatorname{tr} m \pm \sqrt{(\operatorname{tr} m)^2 - 4})$  and  $(\operatorname{tr} m)^2 - 4$  is never a square if  $\operatorname{tr} m \geq 3$ .

Denote by  $d_{-}$  and  $d_{+}$  the eigenspaces corresponding to  $\nu_{-}$ , respectively  $\nu_{+}$ . Then m is strictly contracting when restricted to  $d_{-}$  and strictly expanding when restricted to  $d_{+}$ . Choose arbitrarily one of the two half-lines in which 0 divides the line  $d_{-}$ , and call it  $l_{-}$ .

At this point we have not used any orientation of M. Suppose now that M is oriented. Then the chosen orientation on the basis of the torus fibration induces an orientation of the fibre torus T, by deciding that this orientation, followed by the transversal orientation which projects on the orientation of the base induces the ambient orientation on M.

Denote by  $l_+$  the half-line bounded by 0 on  $d_+$  into which  $l_-$  arrives first when turned in the *negative* direction. Let  $\sigma$  be the strictly convex cone bounded by these two half-lines (see Fig. 24).

We arrive like this at a pair  $(L, \sigma)$  where both edges of  $\sigma$  are irrational. As m preserves L and  $\sigma$ , it preserves also the polygonal line  $P(\sigma)$ .

Let  $P_1$  be an arbitrary integral point of  $P(\sigma)$ . Consider the sequence  $(P_n)_{n\geq 1}$  of integral points of  $P(\sigma)$  read in the positive direction along  $P(\sigma)$ , starting from  $P_1$ . There exists an index  $t \geq 1$  such that  $P_{t+1} = m(P_1)$ . It is the *period* of the action of m on the linearly ordered set of integral points of  $P(\sigma)$ .

Consider t parallel tori  $T_1, \ldots, T_t$  inside M, where  $T_1 = T$  and the indices form an increasing function of the orders of appearance of the tori when one turns in the positive direction. Denote  $\mathcal{T} := \bigsqcup_{1 \le k \le t} T_k$ 



Fig. 24. The case of torus bundles with tr  $m \geq 3$ 

and  $T_{t+1} := T_1$ . For each  $k \in \{1, \ldots, t\}$ , denote by  $M_k$  the piece of  $M_T$  whose boundary components project by  $r_{M,T}$  on  $T_k$  and  $T_{k+1}$  (see Fig. 25). Then look at the thick torus  $M_T$ . Let  $T_-$  be its boundary component through which one "enters inside"  $M_T$  when one turns in the positive direction, and  $T_+$  be the one by which one "leaves"  $M_T$ . Identify then  $H_1(M_T, \mathbf{Z})$  with  $H_1(T_-, \mathbf{Z})$  through the inclusion  $T_- \subset M_{T_1}$  and  $H_1(T_-, \mathbf{Z})$  with  $H_1(T, \mathbf{Z}) = L$  through the reconstruction mapping  $r_{M,T}|_{T_-}: T_- \to T$ .



Fig. 25. Construction of the canonical plumbing structure on a torus fibration with tr $m\geq 3$ 

Consider now on each piece  $M_k$  an oriented Seifert fibration  $\mathcal{F}_k$  such that the class of a fibre in L (after projection in  $M_T$  and identification of  $H_1(M_T, \mathbf{Z})$  with L, as explained before) is equal to  $OP_k$ . Denote

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by  $\mathcal{F}$  the Seifert structure on  $M_{\mathcal{T}}$  obtained by taking the union of the structures  $\mathcal{F}_k$ . We get like this a plumbing structure on M. Denote it by  $\mathcal{P}(M)$ .

This plumbing structure does not depend, up to isotopy, on the choice of the initial integral point on  $P(\sigma)$ . Indeed, by lifting to M a vector field of the form  $\partial/\partial\theta$  on the base of the torus fibration and by considering its flow, one sees that one gets isotopic torus fibrations by starting from any integral point of  $P(\sigma)$ .

Notice that it does neither depend on the choice of the half-line  $l_{-}$ . An opposite choice would lead to the choice of an opposite cone, that is to the same unoriented plumbing structure.

Proof of Theorem 9.1.

1) This is obvious by construction (we use Theorem 9.5).

2) In the construction of  $\mathcal{P}(M(\mathcal{S}))$ , one can take as fibre torus T the torus  $T_{r,1}$ , with the notations of the proof of Proposition 8.13, 2). Using the relations (26) and Proposition 4.4, we get the Proposition. Q.E.D.

By Theorem 8.12, cusp singularities are determined up to analytic isomorphism by the topological type of the oriented manifold M(S). By Theorem 9.5, this manifold can be encoded by a pair  $(T, \mu)$ , where T is an oriented fibre and  $\mu$  is a geometric monodromy diffeomorphism of Tobtained by turning in the positive direction determined by the chosen orientation of T (recall that this is precisely the point were we use the given orientation of M(S)). But it is known that  $\mu$  can be reconstructed up to isotopy by its action on  $L = H_1(T, \mathbb{Z})$ , that is, by the algebraic monodromy operator  $m \in SL(L)$ . Moreover, to fix an orientation of T is the same as to fix an orientation of L. As explained in Section 5.3, such an orientation can be encoded in a symplectic isomorphism  $\omega: L \to \check{L}$ .

Denote by  $\mathcal{C}(L, \omega, m)$  the cusp singularity associated to an oriented lattice  $(L, \omega)$  and an algebraic monodromy operator  $m \in SL(L)$  with tr  $m \geq 3$ . If one changes the orientation of the base of the torus fibration, one gets the triple  $(L, -\omega, m^{-1})$ . This shows that:

$$\mathcal{C}(L, \,\omega, \,m) \simeq \mathcal{C}(L, \,-\omega, \,m^{-1}).$$

When one changes the orientation of M(S), we see that the cone  $(L, \sigma)$  is replaced by a supplementary one. In view of Section 5.3, we deduce that the two cones are dual to each other. In this sense, we get the following analog of Proposition 9.3:

**Proposition 9.6.** Let (S, 0) be a cusp singularity. Then the canonical plumbing structures with respect to the two possible orientations of M(S) are dual to each other. More precisely, if  $(S, 0) \simeq$ 

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 $(\mathcal{C}(L, \omega, m), 0)$ , then  $-M(\mathcal{S})$  is orientation preserving diffeomorphic to  $M(\check{\mathcal{S}})$ , where  $(\check{\mathcal{S}}, 0) \simeq (\mathcal{C}(L, -\omega, m), 0)$ .

#### 9.3. The other singularity boundaries

As in the two previous cases, we first define the plumbing structure  $\mathcal{P}(M(\mathcal{S}))$ .

Consider the canonical graph structure  $(\mathcal{T}_{can}, \mathcal{F}_{can})$  on  $M(\mathcal{S})$ . We do our construction starting from the neighborhoods of the JSJ tori (the elements of  $\mathcal{T}_{can}$ ) and the exceptional fibres in  $\mathcal{F}_{can}$ .

• For each component T of  $\mathcal{T}_{can}$ , consider a saturated tubular neighborhood N(T). We choose them pairwise disjoint. So, each manifold N(T) is a thick torus. We consider on each one of its boundary components a fibre of  $\mathcal{F}_{can}$ . Denote these fibres by  $\gamma(T)$ ,  $\delta(T)$ . We consider on N(T) the restriction of the orientation of  $M(\mathcal{S})$ . Consider the associated Hirzebruch-Jung plumbing structure  $\mathcal{P}(N(T), \gamma(T), \delta(T))$  (see Definition 7.17). Replace the Seifert structure on N(T) induced from  $\mathcal{F}_{can}$  with this plumbing structure. Then eliminate the boundary components of N(T) from the tori present in  $M(\mathcal{S})$  (by construction, the fibrations coming from both sides agree on them up to isotopy).

• For each exceptional fibre F, consider a solid torus N(F), which is a saturated tubular neighborhood of F. Choose those neighborhoods pairwise disjoint. On the boundary of N(F), take a fiber  $\gamma(F)$ of  $\mathcal{F}_{\text{can}}$ . Consider the associated Hirzebruch-Jung plumbing structure  $\mathcal{P}(N(F), \gamma(F))$  (see Definition 7.18). Replace the Seifert structure on N(F) induced from  $\mathcal{F}_{\text{can}}$  with this plumbing structure. Then eliminate the boundary component of N(F) from the tori present inside M(S)(by construction, the fibrations coming from both sides agree on it up to isotopy). Denote by  $\mathcal{P}(M(S))$  the plumbing structure constructed like this on M(S).

Proof of Theorem 9.1. The proof is very similar to the ones explained in the two previous cases, but starting this time from the canonical graph structure on M(S). The main point is Proposition 8.15. We leave the details to the reader. Q.E.D.

#### 9.4. The invariance theorem

Let  $(\mathcal{S}, 0)$  be a normal surface singularity. In [57], Neumann proved that the weighted dual graph  $\Gamma(p_{\rm mnc})$  of the exceptional divisor of its minimal normal crossings resolution  $p_{\rm mnc}$  is determined by the oriented manifold  $M(\mathcal{S})$ . But he says nothing about the action of the group Diff<sup>+</sup> $(M(\mathcal{S}))$  on  $(\mathcal{T}(p_{\rm mnc}), \mathcal{F}(p_{\rm mnc}))$ . As a corollary of Theorem 9.1 we get: **Theorem 9.7.** The plumbing structure  $(\mathcal{T}(p_{mnc}), \mathcal{F}(p_{mnc}))$  is invariant up to isotopy by the group  $\text{Diff}^+(M(\mathcal{S}))$ .

*Proof.* Suppose first that M(S) is not a lens space or a torus fibration. As the canonical graph structure on it is invariant by the group  $\text{Diff}^+(M(S))$  up to isotopy, we deduce that the canonical plumbing structure is also invariant up to isotopy by this group. This conclusion is also true when M(S) is a lens space or a torus fibration, as one starts in the construction of  $\mathcal{P}(M(S))$  from tori which are invariant up to isotopy. Then we apply Theorem 9.1. Q.E.D.

An easy study of the fibres of  $\mathcal{F}(p_{\rm mnc})$  in the neighborhoods of the tori of  $\mathcal{T}(p_{\rm mnc})$  which correspond to self-intersection points of components of  $E_{\rm mnc}$  show that the analogous statement about the minimal good normal crossings resolution of  $\mathcal{S}$  is also true.

We arrived at the conclusion that the affirmation of Theorem 9.7 was true while we were thinking about the natural contact structure on M(S)(see Caubel, Némethi & Popescu-Pampu [11]). Indeed, in that paper we prove that for normal surface singularities, the natural contact structure depends only on the topology of M(S) up to contactomorphisms. It was then natural to look at the subgroup of Diff<sup>+</sup>(M(S)) which leaves it invariant up to isotopy. Presently, we do not know how to characterize it. But we realized that the homotopy type of the underlying unoriented plane field was invariant by the full group Diff<sup>+</sup>(M(S)), provided that Theorem 9.7 was true (see [11, section 5]).

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# Exemples de fonctions de Artin de germes d'espaces analytiques

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#### Abstract.

We define here the Artin functions of a germ of analytic space. Artin functions are analytic invariants of the germ and a measure of its singularity. In general these functions are very difficult to compute. We give a few properties of these functions and we present some examples.

#### §1. Préliminaires

Soit k un corps valué (par exemple  $k = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \ldots$ ). Pour tout entier  $N \geq 1$ , notons  $\mathcal{O}_N$  l'anneau local  $k\{T_1, \ldots, T_N\}$ ,  $\mathfrak{m}$  son idéal maximal et  $\widehat{\mathcal{O}}_N := k[[T_1, \ldots, T_N]]$  son complété pour la topologie  $\mathfrak{m}$ adique. Nous noterons ord la valuation  $\mathfrak{m}$ -adique sur  $\mathcal{O}_N$ :

$$\operatorname{ord}(x) := \max\{n \in \mathbb{N} | x \in \mathfrak{m}^n\}.$$

Soit I un idéal de  $\Bbbk \{X_1, \ldots, X_n\}$  définissant un germe d'espace analytique (X, 0) plongé dans  $(\Bbbk^n, 0)$ . Nous allons définir ici la suite des fonctions de Artin de (X, 0) qui sont des fonctions numériques de  $\mathbb{N}$  dans  $\mathbb{N}$ . Ces fonctions sont des invariants analytiques du germe mais, malheureusement, sont très difficiles à calculer en général. Nous présentons ici les résultats connus à propos de ces fonctions puis nous présentons quelques exemples.

Pour tout entier  $p \geq 1$ , notons  $X_p^N$  l'ensemble des morphismes de k-algèbres locales  $\mathcal{O}_{X,0} \longrightarrow \widehat{\mathcal{O}}_N/\mathfrak{m}^{p+1}$ , et notons  $X_\infty^N$  l'ensemble des morphismes de k-algèbres locales  $\mathcal{O}_{X,0} \longrightarrow \widehat{\mathcal{O}}_N$ . Les projections canoniques

$$\widehat{\mathcal{O}}_N \longrightarrow \widehat{\mathcal{O}}_N/\mathfrak{m}^{p+1} \longrightarrow \widehat{\mathcal{O}}_N/\mathfrak{m}^{q+1}, \ \forall p \ge q$$

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définissent par composition des morphismes

$$X_{\infty}^{N} \xrightarrow{\pi_{p}} X_{p}^{N} \xrightarrow{\pi_{p,q}} X_{q}^{N} , \ \forall p \ge q.$$

Les fonctions de Artin de (X, 0) sont des fonctions qui donnent des conditions nécessaires et suffisantes pour qu'un élément de  $X_p^N$  puisse se relever en un élément de  $X_{\infty}^N$ .

Tout d'abord, nous avons le théorème suivant (qui est un cas légèrement plus particulier que le théorème énoncé originallement):

**Théorème 1.1** ([Wa]). Soit k un corps valué, complet de caractéristique nulle. Soient  $f_1, \ldots, f_r \in k[[T, X]]$ , où  $T = (T_1, \ldots, T_N)$ et  $X = (X_1, \ldots, X_n)$ . Alors il existe  $\beta \colon \mathbb{N} \longrightarrow \mathbb{N}$  telle que: Pour tout  $p \in \mathbb{N}$  et pour tout  $x(T) \in \widehat{\mathcal{O}}_N^n$  tel que

$$\begin{aligned} x(0) &= 0, \\ et \ f_i(x(T)) \in \mathfrak{m}^{\beta(p)+1}, \ 1 \le i \le r, \end{aligned}$$

il existe  $\overline{x}(T) \in \widehat{\mathcal{O}}_N^n$  tel que

$$f_i(\overline{x}(T)) = 0, \ 1 \le i \le r, \ et \ x(T) - \overline{x}(T) \in \mathfrak{m}^{p+1}.$$

Dans la suite, k sera toujours un corps valué, complet de caractéristique nulle, sauf mention du contraire.

Nous appellerons fonction de Artin des  $f_i$  la plus petite fonction qui vérifie le théorème précédent. Il n'est pas difficile à vérifier que cette fonction de Artin ne dépend que du morphisme  $\mathbb{k}[[T]] \longrightarrow \mathbb{k}[[T, X]]/(f_1, \ldots, f_r)$ . Nous avons alors la définition suivante

**Définition 1.2.** Soit  $I = (f_1, \ldots, f_r)$  un idéal de  $\Bbbk \{X_1, \ldots, X_n\}$ définissant un germe d'espace analytique (X, 0) plongé dans  $(\Bbbk^n, 0)$ . Soit  $N \ge 1$  un entier. Notons  $I_N$  l'idéal de  $\Bbbk[[T_1, \ldots, T_N, X]]$  engendré par I. Nous appellerons N-ième fonction de Artin de (X, 0) la plus petite fonction  $\beta_N$  qui vérifie le théorème de Artin précédent pour l'idéal  $I_N$ .

Dans le cas N = 1, un théorème de Greenberg nous permet d'affirmer que  $\beta_1$  est bornée par une fonction affine [Gr]. Pour  $N \ge 2$ , ceci a été conjecturé pendant longtemps, mais c'est en général faux [Ro2]. Nous verrons plus loin certains exemples où ce n'est pas le cas. Nous pouvons réénoncer l'existence de la fonction  $\beta_N$  en écrivant

(1) 
$$\pi_p(\mathcal{X}_{\infty}^N) = \pi_{\beta_N(p),p}(\mathcal{X}_{\beta_N(p)}^N), \ \forall p \ge 1.$$

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C'est-à-dire qu'un élément de  $X_p^N$  se relève en un élément de  $\mathcal{X}_{\infty}^N$  si l'on peut le relever en un élément de  $\mathcal{X}_{\beta_N(p)}^N$ .

Nous avons alors le résultat suivant qui découle d'un théorème de Chevalley, énonçant que l'image d'un ensemble algébrique par un morphisme est un sous-ensemble constructible de ensemble d'arrivée:

**Proposition 1.3.** Soient (X, 0) un germe d'espace analytique sur un corps valué algébriquement clos, complet de caractéristique nulle et  $N \in \mathbb{N}$  fixés. Alors pour tout p entier,  $\pi_p(\mathcal{X}_{\infty}^N)$  est un sous-ensemble constructible de  $\mathcal{X}_p^N$ .

La suite des fonctions de Artin d'un germe de variété analytique est un invariant analytique de celui-ci; d'après ce qui précède, nous voyons que  $\beta_N$  ne dépend que du morphisme  $\mathbb{k}[[T]] \longrightarrow \mathbb{k}[[T, X]]/(f_1, \ldots, f_r)$ . Par ailleurs, cette suite est, en quelque sorte, une mesure de la singularité du germe. En effet nous avons la proposition suivante:

**Proposition 1.4** ([H1]). Soit (X, 0) un germe d'espace analytique sur un corps  $\Bbbk$  et  $N \ge 1$  un entier. Alors la N-ième fonction de Artin de (X, 0) est égale à l'identité si et seulement si le germe est non-singulier.

Le but de ce travail est d'étudier la suite des fonctions de Artin d'un germe d'espace analytique. Dans cette optique nous allons tout d'abord énoncer les premiers résultats connus à propos de ces fonctions (essentiellement sur  $\beta_1$ ). Ensuite nous allons introduire deux outils utiles dans l'étude de fonctions de Artin: le théorème d'Izumi et un théorème d'approximation diophantienne dû à l'auteur. Enfin nous allons donner une liste des exemples connus de fonctions de Artin de germes d'espaces analytiques.

# §2. Propriétés des fonctions de Artin d'un germe d'espace analytique

Nous pouvons énoncer les propriétés suivantes de ces fonctions.

**Proposition 2.1.** Nous avons le propriétés suivantes:

i) Soient (X, 0) un germe d'espace analytique et  $(\beta_N)_N$  sa suite de fonctions de Artin. Nous avons les inégalités

$$\forall N \ge 1, \forall p \in \mathbb{N}, \ \beta_N(p) \le \beta_{N+1}(p)$$

ii) Soient (X, 0) et (Y, 0) deux germes d'espaces analytiques, définis respectivement par les idéaux I et J, et  $(\beta_N)_N$  et  $(\beta'_N)_N$  leurs suites de fonctions de Artin respectives. Notons  $(\gamma_N)_N$  la suite

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de fonctions de Artin du germe  $(X \cup Y, 0)$  défini par l'idéal  $I \cap J$ . Nous avons alors les inégalités

$$\forall N \ge 1, \, \forall p \in \mathbb{N}, \, \gamma_N(p) \le \beta_N(p) + \beta'_N(p).$$

Preuve. Montrons i). Soit  $f_1, \ldots, f_r$  un système de générateurs de l'idéal I définissant le germe (X, 0). Soit  $x(T) \in \widehat{\mathcal{O}}_N^n$  tel que  $f_i(x(T)) \in \mathfrak{m}^{\beta_{N+1}(p)+1}, 1 \leq i \leq r$ . Nous avons donc l'existence d'un  $\overline{x}(T) \in \widehat{\mathcal{O}}_{N+1}^n$  tel que  $f(\overline{x}(T)) = 0$  et  $x(T) - \overline{x}(T) \in \mathfrak{m}^{p+1}$ . En annulant  $T_{N+1}$  dans l'écriture de  $\overline{x}(T)$ , nous trouvons  $\overline{x}(T) \in \widehat{\mathcal{O}}_N^n$  tel que  $f(\overline{x}(T)) = 0$  et  $x(T) - \overline{x}(T) \in \widehat{\mathcal{O}}_N^n$  tel que  $f(\overline{x}(T)) = 0$  et  $x(T) - \overline{x}(T) \in \widehat{\mathcal{O}}_N^n$  tel que  $f(\overline{x}(T)) = 0$  et  $x(T) - \overline{x}(T) \in \widehat{\mathcal{O}}_N^n$  tel que  $f(\overline{x}(T)) = 0$  et  $x(T) - \overline{x}(T) \in \widehat{\mathcal{O}}_N^n$  tel que  $f(\overline{x}(T)) = 0$  et  $x(T) - \overline{x}(T) \in \mathfrak{M}^{p+1}$ .

Montrons maintenant ii). Soit  $f_1, \ldots, f_r$  (resp.  $g_1, \ldots, g_s$ ) un système de générateurs de l'idéal I définissant le germe (X, 0) (resp. (Y, 0)). Soit  $x \in \widehat{\mathcal{O}}_N^n$  tel que  $h(x) \in \mathfrak{m}^{\beta_N(p)+\beta'_N(p)+1}$  pour tout  $h \in I \cap J$ . En particulier  $f_j(x)g_k(x) \in \mathfrak{m}^{\beta_N(p)+\beta'_N(p)+1}$  pour tout j et k. Alors nous avons soit  $f_j(x) \in \mathfrak{m}^{\beta_N(p)+1}$  pour tout j, soit  $g_k(x) \in \mathfrak{m}^{\beta_N(p)+1}$  pour tout k. D'où l'existence de  $\overline{x}$  tel que  $x - \overline{x} \in \mathfrak{m}^{p+1}$  et  $f(\overline{x})g(\overline{x}) = 0$ , et donc tel que h(x) = 0 pour tout  $h \in I \cap J$ . Q.E.D.

Nous verrons plus tard que la première de ces inégalités peut être stricte.

Dans le cas des hypersurfaces, plusieurs auteurs ont étudié  $\beta_1$ , appelée parfois fonction de Artin-Greenberg du germe [LJ], [H1]. Le calcul explicite de  $\beta_1$  pour les courbes planes a même été effectué [H2], et montre que celle-ci, avec la donnée de la multiplicité, est un invariant topologique complet pour les courbes planes. M. Hickel a aussi montré que la constante égale à  $\theta := \overline{\lim_p} \beta_1(p)/p$  est une contrainte sur le nombre d'éclatements nécessaires à désingulariser un germe d'hypersurface:

**Proposition 2.2** ([H2]). Soit (X, 0) un germe d'hypersurface complexe singulier défini par une équation. Soit une suite d'éclatements  $\varphi_j$  de centres lisses  $Z_j$  où  $X_j$  est équimultiple le long de  $Z_j$ ,  $X_{j+1}$  est la transformée stricte de  $X_j$  et où  $X_n$  est lisse:



Alors nous avons

(2) 
$$n \ge \frac{\theta - 1}{m_0}$$

où  $\theta := \overline{\lim}_p \beta_1(p)/p$ ,  $\beta_1$  est la première fonction de Artin de (X, 0) et  $m_0$  est la multiplicité de X en l'origine.

La première fonction de Artin de (X, 0) nous donne donc une condition nécessaire quant à la désingularisation d'un germe d'espace analytique.

Dans le cas des hypersurfaces, une première majoration effective de  $\beta_1$  a été obtenue [LJ], améliorée ensuite par M. Hickel:

**Théorème 2.3** ([H1]). Soient f un germe de fonction holomorphe à l'origine de  $\mathbb{C}^n$  et  $J_f$  l'idéal jacobien de f (i.e. l'idéal de  $\Bbbk\{X_1, \ldots, X_n\}$ engendré par les dérivées partielles de f). Notons  $(X, 0) \subset (\mathbb{C}^n, 0)$  (resp.  $(X_J, 0) \subset (\mathbb{C}^n, 0)$ ) le germe de variété associé à f (resp. à  $J_f$ ) et  $\beta_1$ (resp.  $\beta'_1$ ) sa première fonction de Artin. Alors nous avons

(3) 
$$\beta_1(p) \le \beta'_1(p) + p, \ \forall i \in \mathbb{N}$$

Ce théorème relie la fonction de Artin de (f) avec celle de son jacobien, et cette inégalité est bien meilleure que celle qui apparaît dans la preuve du théorème de Greenberg, et qui est de la forme  $\beta_1 \leq 2\beta'_1$ (cf. [Gr]). En particulier, dans le cas d'un germe à singularité isolée, ce théorème permet d'obtenir le résultat suivant:

**Théorème 2.4** ([H1]). Soit f un germe de fonction holomorphe à l'origine de  $\mathbb{C}^n$  définissant un germe de variété analytique (X, 0). Supposons que (X, 0) est à singularité isolée et notons  $\nu$  son exposant de Lojasiewicz ([LJ-T] ou [H1]). Alors

(4) 
$$\beta_1(i) \le \lfloor \nu i \rfloor + i, \ \forall i \in \mathbb{N}.$$

Ces deux derniers résultats sont cependant faux si  $N \ge 2$ , comme nous le verrons dans la dernière partie.

# §3. théorème d'Izumi et théorème d'approximation diophantienne

Nous présentons ici deux résultats (l'un d'algèbre commutative, l'autre d'arithmétique) qui sont deux cas particuliers de majoration affine d'une fonction de Artin. Ces deux résultats seront utilisés par la suite pour estimer certaines fonctions de Artin.

#### 3.1. Théorème d'Izumi

Ce théorème donne une caractérisation des algèbres analytiques intègres:

Théorème 3.1 ([Iz][Re]). Soient

 $R := \mathbb{k}[[T_1, \ldots, T_N]]/(f_1, \ldots, f_p),$ 

avec  $\Bbbk$  un corps, et  $\mathfrak{m}$  son idéal maximal. Alors R est intègre si et seulement si il existe deux constantes a et b telles que:

$$\forall x, y \in R, \ \nu(xy) \le a(\nu(x) + \nu(y)) + b$$

où  $\nu$  est l'ordre  $\mathfrak{m}$ -adique sur R.

Pour tout anneau local R, nous avons toujours  $\nu(xy) \ge \nu(x) + \nu(y)$ , quelques soient x et y dans R. Ce théorème peut se réénoncer sous la forme suivante:

**Théorème 3.2** ([Ro3]). Si  $(f_1, \ldots, f_p)$  est un idéal premier de  $\widehat{\mathcal{O}}_N$ , alors la fonction de Artin de  $XY - \sum_i f_i Z_i \in \widehat{\mathcal{O}}_N[[X, Y, Z_i]]$  est bornée par une fonction affine de la forme  $p \mapsto 2ap + c$  où a est la constante du théorème précédent.

**Exemple 3.3.** Soit  $f \in \widehat{\mathcal{O}}_N$  une série irréductible. Alors, XY - f n'admet pas de zéro (x(T), y(T)) tel que x(0) = y(0) = 0. Sa fonction de Artin est donc constante. Notons c(f) cette constante. D'autre part, la fonction de Artin de XY - fZ est bornée par une fonction affine d'après le Théorème 3.2.

Si  $c(f) = \operatorname{ord}(f)$ , c'est-à-dire si le cône tangent à la variété formelle  $\{f = 0\}$  est irréductible, alors  $\nu$  est une valuation, et les constantes a et b du Théorème 3.1 peuvent être choisies respectivement égales à 1 et 0. On peut alors voir que la fonction  $p \mapsto 2p + c$  (où c est une constante bien choisie) majore la fonction de Artin de XY - fZ.

La question naturelle qui se pose est de savoir, en général, comment relier le coefficient de linéarité d'une fonction affine majorant la fonction de Artin de XY - fZ, et les constantes c(f) et ord(f).

# 3.2. Théorème d'approximation diophantienne

Nous allons noter ici  $V_N := \{ (x/y) / x, y \in \widehat{\mathcal{O}}_N \text{ et } \operatorname{ord}(x) \ge \operatorname{ord}(y) \},$ l'anneau de valuation discrète qui domine  $\widehat{\mathcal{O}}_N$  pour ord et

$$\hat{V}_N := \Bbbk(T_1/T_N, \dots, T_{N-1}/T_N)[[T_N]]$$

le complété pour la topologie **m**-adique de  $V_N$ . Les corps  $\mathbb{K}_N$  et  $\widehat{\mathbb{K}}_N$  sont respectivement les corps de fractions de  $\widehat{\mathcal{O}}_N$  et de  $\widehat{V}_N$ . La valuation ord définit une norme  $| \ |$  sur  $\widehat{\mathcal{O}}_N$  en posant  $|x| = e^{-\operatorname{ord}(x)}$  et cette norme induit une topologie appelée topologie **m**-adique. Cette norme s'étend naturellement à  $\mathbb{K}_N$  et  $\widehat{\mathbb{K}}_N$ . On peut remarquer que  $\widehat{\mathbb{K}}_N$  est le complété de  $\mathbb{K}_N$  pour la norme | |. Nous avons alors le théorème suivant: **Théorème 3.4** ([Ro4]). Soit  $z \in \widehat{\mathbb{K}}_N$  algébrique sur  $\mathbb{K}_N$  tel que  $z \notin \mathbb{K}_N$ . Alors il existe  $a \ge 1$  et  $K \ge 0$  tels que

(5) 
$$\left|z - \frac{x}{y}\right| \ge K|y|^a, \forall x, y \in \widehat{\mathcal{O}}_N.$$

Ce théorème est un cas particulier de majoration affine de fonctions de Artin. En effet, celui-ci est équivalent au théorème suivant:

**Théorème 3.5** ([Ro4]). Soit P(X, Y) un polynôme homogène en X et Y à coefficients dans  $\widehat{\mathcal{O}}_N$ . Alors P admet une fonction de Artin bornée par une fonction affine de la forme  $p \mapsto (d+a)p+c$ , où d est le degré de P, a est la constante du Théorème 3.4 précédent et c est une constante.

**Exemple 3.6.** Nous pouvons faire le parallèle avec le théorème d'Izumi. Soit Q(Z) un polynôme en une variable à coefficients dans  $\widehat{\mathcal{O}}_N$ . Supposons que Q n'ait pas de zéro dans  $\widehat{\mathcal{O}}_N$ . Alors la fonction de Artin de Q est constante. L'exemple le plus caractéristique est le cas où  $Q(Z) = Z^d - u$  et où u n'est pas une puissance d-ième dans  $\widehat{\mathcal{O}}_N$ . Dans ce cas notons c(u) la valeur constante de la fonction de Artin de Q. Notons P(X, Y) l'homogénéisé de Q (i.e.  $P(X, Y) = Y^d Q(X/Y)$ ). D'après le Théorème 3.5, P admet une fonction de Artin majorée par une fonction affine.

Si  $c(u) = \operatorname{ord}(u)$ , c'est-à-dire si le terme initial de u n'est pas une puissance p-ième dans  $\widehat{\mathcal{O}}_N$ , alors u n'est pas une puissance p-ième dans  $\widehat{V}_N$ et donc dans  $\widehat{\mathbb{K}}_N$ . Le Théorème 3.4 est donc vide dans ce cas, et l'on peut montrer que la fonction de Artin de P est bornée par une fonction de la forme  $p \longmapsto dp + c$ , avec c bien choisie.

Là encore, la question naturelle qui se pose est de savoir, en général, comment relier le coefficient de linéarité d'une fonction affine majorant la fonction de Artin de P(X, Y) (où, de manière équivalente, une constante a intervenant dans le Théorème 3.4 pour  $z \in \widehat{\mathbb{K}}_N$  tel que  $z^d = u$ ), et les constantes c(u) et  $\operatorname{ord}(u)$ .

#### $\S4.$ Exemples

Nous allons donner ici, pour plusieurs germes d'espaces analytiques, le comportement des différentes fonctions de Artin de ceux-ci. Pour  $N \geq 2$ , un tel comportement est en général très difficile à déterminer. Dans chaque cas,  $\beta_N$  est la N-ième fonction de Artin du germe considéré. Nous considérerons parfois le cas où  $\Bbbk$  est de caratéristique positive, si les  $\beta_N$  existent alors.

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#### 4.1. Germe de variété défini par un monôme

Ce premier cas, très simple, est celui d'un germe d'espace  $(X, 0) \subset (\mathbb{k}^n, 0)$  défini par une équation de la forme  $X_1^{n_1} \dots X_n^{n_n} = 0$ . Nous avons alors la proposition:

#### **Proposition 4.1.** *Pour tout* $N \ge 1$ *, nous avons*

$$\beta_N(p) = (n_1 + \dots + n_n)p, \, \forall p \in \mathbb{N}.$$

Preuve. Il est clair que si  $x_1^{n_1} \cdots x_n^{n_n} \in \mathfrak{m}^{(n_1+\cdots+n_n)p+1}$ , alors il existe *i* tel que  $\operatorname{ord}(x_i) \ge p+1$ . D'autre part, si pour tout *i* nous posons  $x_i = T^p$ , alors  $x_1^{n_1} \cdots x_n^{n_n} = T^{(n_1+\cdots+n_n)p} \in \mathfrak{m}^{(n_1+\cdots+n_n)p}$ . Cependant  $\operatorname{ord}(x_i) < p+1$ . D'où l'égalité. Q.E.D.

# 4.2. Germe de variété réduite à composantes irréductibles lisses

Les fonctions de Artin d'un germe d'espace analytique réduit dont chaque composante irréductible est lisse sont toutes bornées par une fonction linéaire  $p \mapsto m p$  où m est le nombre de composantes irréductibles du germe en 0. Ceci découle du fait que les fonctions de Artin de  $(X \cup$ Y, 0) sont bornées par celles de (X, 0) plus celles de (Y, 0), et que les fonctions de Artin d'un germe lisse sont toutes égales à l'identité. Plus précisément nous avons la proposition:

**Proposition 4.2.** Soient (X, 0) un germe d'espace analytique réduit dont toutes les composantes irréductibles sont lisses. Supposons que le corps de base n'est pas de cardinal fini. Alors

$$\beta_N(p) = m \, p, \, \forall p \in \mathbb{N}, \, \forall N \in \mathbb{N}^*$$

où m est le nombre de composantes irréductibles du germe en 0.

Preuve. D'après la remarque précédente, il suffit de montrer que  $\beta_N(p) \ge m p$  pour tout p. Supposons que k n'est pas de cardinal fini. Nous allons donner la preuve de ce résultat dans le cas de 2 composantes irréductibles (i.e. m = 2), le cas général étant identique. Soient I et J les idéaux de  $\Bbbk\{X_1, \ldots, X_n\}$  définissant respectivement (X, 0) et (Y, 0) deux germes lisses distincts. Nous avons  $\Bbbk\{X_1, \ldots, X_n\}/I \cap J \equiv \Bbbk\{Y_1, \ldots, Y_r\}/K$  où K est inclu dans  $\{X_1, \ldots, X_n\}^2$  car  $X \cup Y$  n'est pas lisse. Soient  $I/I \cap K$  et  $J/J \cap K$  les idéaux de  $\Bbbk\{Y_1, \ldots, Y_r\}/K$  engendrés par I et J, et I' et J' les idéaux de  $\Bbbk\{Y_1, \ldots, Y_r\}$  engendrés par les images reciproques de  $I/I \cap K$  et  $J/J \cap K$  respectivement. Comme X et Y sont lisses, ces idéaux sont inclus dans  $(Y_1, \ldots, Y_r)$  mais pas dans  $(Y_1, \ldots, Y_r)^2$ . Soient f et g deux éléments de I' et J' respectivement

qui ne sont pas dans  $(Y_1, \ldots, Y_r)^2$ . Soit  $(y_1, \ldots, y_r)$  un élément de  $\Bbbk^r$ n'appartenant ni aux zéros de la forme initiale de f, ni aux zéros de celle de g. Un tel  $(y_1, \ldots, y_r)$  existe car  $\Bbbk$  n'est pas un corps fini. Pour tout  $h \in I \cap J$ ,  $h(y_1T_1^p, \ldots, y_rT_1^p) \in \mathfrak{m}^{2p}$  car  $K \subset (Y_1, \ldots, Y_r)^2$ . Si nous avions  $\beta_N(p) < 2p$ , alors il existerait  $\overline{y}_1(T), \ldots, \overline{y}_r(T)$  tels que  $h(\overline{y}(T)) = 0$  pour tout  $h \in I \cap J$ , et  $\overline{y}_i(T) - y_iT_1^p \in \mathfrak{m}^{p+1}$  pour tout i. En particulier,  $f(\overline{y})g(\overline{y}) = 0$ . Or  $(y_1, \ldots, y_r)$  n'appartenant ni aux zéros de la forme initiale de f, ni aux zéros de celle de g, ceci est impossible et  $\beta_N(p) \ge 2p$ . Q.E.D.

# 4.3. Cusp

Soit  $(X, 0) \subset (\mathbb{k}^2, 0)$  le germe d'espace analytique défini par  $X^2 - Y^3 = 0$ . Nous avons alors la proposition:

**Proposition 4.3.** Nous avons les cas suivants: i) Si N = 1 et k = C nous avons

i) Si N = 1 et  $\Bbbk = \mathbb{C}$ , nous avons

$$\beta_1(p) = 3p, \ \forall p \notin (2)$$
  
$$\beta_1(2) = 4 \ et \ \beta_1(p) = 3(p-1), \ \forall p \in (2) \setminus \{2\}.$$

ii) Si N = 1 et k est un corps qui contient un élément  $\lambda$  qui n'est pas un carré (ex:  $k = \mathbb{R}$ ), nous avons

$$\beta_1(p) = 3p, \ \forall p \in \mathbb{N}.$$

iii) Si  $N \ge 2$  et si k est un corps de caractéristique différente de 2 ou 3, nous avons

$$\lim_{p \longrightarrow +\infty} \beta_N(p)/p \ge 4.$$

*Preuve.* i) La première assertion découle du calcul de la première fonction de Artin d'une branche plane (cf. Théorème 2.1 [H2]).

ii) Supposons que N = 1 et k est un corps qui contient un élément  $\lambda$  qui n'est pas un carré. En particulier  $\lambda^3$  n'est pas un carré (si  $\lambda^3 = \mu^2$  alors  $(\mu/\lambda)^2 = \lambda$  ce qui est impossible).

Posons alors x = 0 et  $y = \lambda T^p$ . Nous avons  $x^2 - y^3 = -\lambda^3 T^{3p}$ . Si  $\beta_1(p) < 3p$ , alors il existe  $\overline{x}$  et  $\overline{y}$  avec  $\overline{x}^2 - \overline{y}^3 = 0$  et  $\overline{x} - x$  et  $\overline{y} - y$  sont dans  $\mathfrak{m}^{p+1}$ . Dans ce cas, néssairement le terme initial de  $\overline{x}^2$  doit être égal au terme initial de  $\overline{y}^3$  qui est égal à  $\lambda^3 T^{3(p-1)}$ . Or  $\lambda^3$  n'est pas un carré, donc ceci est impossible et  $\beta_1(p) \ge 3p$ .

D'autre part, si  $x^2 - y^3 \in \mathfrak{m}^{3p+1}$ , alors deux cas peuvent se produire: si  $\operatorname{ord}(x) \ge p + 1$  et  $\operatorname{ord}(y) \ge p + 1$ , alors nous posons  $\overline{x} = \overline{y} = 0$ , et nous avons  $\overline{x}^2 - \overline{y}^3 = 0$  et  $\overline{x} - x$ ,  $\overline{y} - y \in \mathfrak{m}^{p+1}$ . Si  $\operatorname{ord}(x) ou$  $<math>\operatorname{ord}(y) , alors les termes initiaux de <math>x^2$  et de  $y^3$  sont égaux, car  $x^2 - y^3 \in \mathfrak{m}^{3p+1}$ . Donc  $y^3$  est un carré dans  $\Bbbk[[T]]$ , et nous avons alors  $(x - y^{3/2})(x + y^{3/2}) \in \mathfrak{m}^{3p+1}$ . Donc  $\operatorname{ord}(x - y^{3/2}) \ge p + 1$  par exemple, et nous posons  $\overline{x} = y^{3/2}$  et  $\overline{y} = y$ . Dans tous les cas nous avons  $\overline{x}^2 - \overline{y}^3 = 0$  et  $\overline{x} - x$ ,  $\overline{y} - y \in \mathfrak{m}^{p+1}$ . Donc  $\beta_1(p) \le 3p$ . La troisième assertion découle de [Ro1]. Q.E.D.

## 4.4. Parapluie de Whitney

Soit  $(X, 0) \subset (\mathbb{k}^3, 0)$  le germe de variété défini par  $X^2 - ZY^2 = 0$ où  $\mathbb{k}$  est un corps muni d'une norme. Nous avons alors la proposition:

**Proposition 4.4.** Nous avons les cas suivants: i) Si N = 1, nous avons

$$\beta_1(p) \le 3p, \ \forall p \ge 1.$$

ii) Si  $N \ge 2$  et car k = 2, nous avons

$$\beta_1(p) \le 3p, \ \forall p \ge 1.$$

iii) Si  $N \ge 2$  et car  $\mathbb{k} \ne 2$ , nous avons

$$\beta_N(p) \ge \frac{p^2}{4} + p - 4, \ \forall p \ge 1.$$

Preuve. Notons  $P(X, Y, Z) = X^2 - ZY^2$ .

Le cas i) découle du Théorème 2.3 si  $\mathbb{k} = \mathbb{C}$ . Dans le cas général, supposons que nous ayons x, y et z dans  $\mathcal{O}_1$  tels que  $x^2 - zy^2 \in \mathfrak{m}^{3p+1}$ . Alors trois cas se présentent:

- Supposons que  $\operatorname{ord}(y) < p+1$  et  $\operatorname{ord}(x^2) = \operatorname{ord}(zy^2)$ . Donc  $x^2/y^2 - z \in \mathfrak{m}^{p+1}$  et nous posons  $\overline{x} = x, \, \overline{y} = y$  et  $\overline{z} = x^2/y^2$ .

- Supposons que  $\operatorname{ord}(x^2) \neq \operatorname{ord}(zy^2)$ . Alors  $\operatorname{ord}(x) \geq p+1$ , et soit  $\operatorname{ord}(z) \geq p+1$  soit  $\operatorname{ord}(y) \geq p+1$ . Nous posons alors  $\overline{x} = 0$ , et  $\overline{z} = 0$  et  $\overline{y} = y$  dans le premier cas, et  $\overline{z} = z$  et  $\overline{y} = 0$  dans le second cas.

- Supposons enfin que  $\operatorname{ord}(y) \ge p+1$  et  $\operatorname{ord}(x^2) = \operatorname{ord}(zy^2)$ . Alors  $\operatorname{ord}(x) \ge p+1$ . Nous posons alors  $\overline{x} = 0$ ,  $\overline{y} = 0$  et  $\overline{z} = z$ . Dans tous les cas  $P(\overline{x}, \overline{y}, \overline{z}) = 0$  et

$$\overline{x} - x, \, \overline{y} - y, \, \overline{z} - z \in \mathfrak{m}^{p+1}$$

ii) Soient x, y et z dans  $\mathcal{O}_N$ , avec  $N \ge 2$ , tels que  $x^2 - zy^2 \in \mathfrak{m}^{3p+1}$ . Notons

$$\alpha = \operatorname{ord}(x), \ \beta = \operatorname{ord}(y), \ \gamma = \operatorname{ord}(z).$$

Si  $\beta > p$ , nous posons  $\overline{x} = 0$ ,  $\overline{y} = 0$  et  $\overline{z} = z$ . Nous avons alors

$$(x - \overline{x})^2 = x^2 \in \mathfrak{m}^{\min\{3p+1, 2\beta + \gamma\}} \subset \mathfrak{m}^{2p+2}$$

et donc  $\overline{x} - x \in \mathfrak{m}^{p+1}$ . Clairement  $\overline{y} - y$  et  $\overline{z} - z$  sont dans  $\mathfrak{m}^{p+1}$  et  $\overline{x}^2 - \overline{zy}^2 = 0$ .

Supposons maintenant que  $\beta \leq p$ . Nous pouvons écrire  $z = z_{\gamma} + z_{\gamma+1} + z_{\gamma+2} + \cdots$  où  $z_d$  est homogène de degré d. Soit  $\gamma_1$  le plus petit entier pour lequel  $z_{\gamma_1}$  n'est pas un carré. Comme car  $\Bbbk = 2$ , les monômes apparaissant dans l'écriture de  $x^2$  et de  $y^2$  sont tous des carrés et donc  $\gamma_1 + 2\beta \geq 3p + 1$ . Notons  $\overline{z} = z_{\gamma} + z_{\gamma+1} + \cdots + z_{\gamma_1-1}$ ; en particulier  $\overline{z} - z \in \mathfrak{m}^{3p+1-2\beta} \subset \mathfrak{m}^{p+1}$ . Cet élément est un carré, disons  $\overline{z} = u^2$ , car car  $\Bbbk = 2$ . Nous avons alors  $x^2 - (uy)^2 = (x - uy)(x + uy) \in \mathfrak{m}^{3p+1}$ . Donc, par exemple,  $x - uy \in \mathfrak{m}^{\lfloor (3p+1)/2 \rfloor} \subset \mathfrak{m}^{p+1}$ . Posons alors  $\overline{x} = uy$  et  $\overline{y} = y$ . Nous avons alors  $x - \overline{x} \in \mathfrak{m}^{p+1}$ ,  $y - \overline{y} \in \mathfrak{m}^{p+1}$  et  $z - \overline{z} \in \mathfrak{m}^{p+1}$ , et de plus  $\overline{x}^2 - \overline{zy}^2 = 0$ .

iii) Nous allons donner ici une idée de la preuve de ce résultat. Pour plus de détails, on pourra se reporter à [Ro2]. Considérons le polynôme  $P_k(X, Y) = X^2 - u_k Y^2$  avec  $u_k = T_1^2 + T_2^k$  et k > 2. L'idée est de voir que toute majoration affine de la fonction de Artin de  $P_k$  a un coefficient de linéraité supérieur à k, et ensuite de considérer ce polynôme comme une "spécialisation" du polynôme P. L'élément  $u_k$  n'est pas un carré dans  $\mathcal{O}_N$ , et donc  $P_k$  admet une fonction de Artin majorée par une fonction affine de la forme  $p \mapsto a(k)p + b(k)$  d'après le théorème 3.5. L'idée est de voir que le plus petit a(k) possible est minoré par k/2 + 1. Pour cela, il suffit de voir que  $u_k$  est un carré dans  $\widehat{V}_N$ .

Notons

$$z_k := T_1 \left( 1 + \frac{1}{2} \frac{T_2^p}{T_1^2} - \frac{1}{8} \frac{T_2^{2p}}{T_1^4} + \dots + \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!n!} \frac{T_2^{np}}{T_1^{2n}} + \dots \right).$$

Nous avons  $z_k^2 = u_k$ . Notons  $z_{k,p}$  la troncation de  $z_k$  à l'ordre p et soient  $x_{k,p}$  et  $y_{k,p}$  deux éléments de  $\mathcal{O}_N$  tels que  $x_{k,p}/y_{k,p} = z_{k,p}$ , et tels que  $x_{k,p}$  et  $y_{k,p}$  sont premiers entre eux. Alors  $|z_k - (x_{k,p}/y_{k,p})| = K_k |y_{k,p}|^{(k/2)-1}$ . En utilisant le théorème 3.5, nous voyons donc que le plus petit a(k) possible est minoré par k/2 + 1.

Maintenant,  $P(x_{k,p}, y_{k,p}, u_k) \in \mathfrak{m}^{(k+2)p-4}$ . Or les zéros (x, y, z) de P sont de deux formes: soit x = y = 0, soit z est un carré. Donc

$$\sup_{\substack{P(x,y,z)=0}} \left( \min\{\operatorname{ord}(x_{k,p}-x), \operatorname{ord}(y_{k,p}-y), \operatorname{ord}(u_k-z)\} \right)$$
  
$$\leq \max(2k-3, p).$$

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Donc en posant p = k+2, nous voyons que  $P(x_{k, k+2}, y_{k, k+2}, u_k)\mathfrak{m}^{k^2+4k}$ mais

$$\sup_{\substack{P(x, y, z)=0}} \left( \min\{ \operatorname{ord}(x_{k, k+2} - x), \operatorname{ord}(y_{k, k+2} - y), \operatorname{ord}(u_k - z) \} \right) \\ \leq 2k - 3.$$

Nous avons donc une solution approchée de P à l'ordre  $k^2 + 4k$  mais la différence entre cette solution approchée et une vraie solution est d'ordre inférieur à 2k - 3. Q.E.D.

## 4.5. Le germe de variété défini par $X_1X_2 - X_3X_4 = 0$

Considérons  $(X, 0) \subset (\mathbb{k}^4, 0)$  le germe de variété défini par  $X_1X_2 - X_3X_4 = 0$  où  $\mathbb{k}$  est un corps muni d'une norme. Nous avons alors la

# **Proposition 4.5.** Nous avons les cas suivants:

i) Si N = 1, nous avons

$$\beta_1(p) = 2p, \ \forall p \ge 1.$$

ii) Si  $N \ge 3$ , nous avons

$$\beta_1(p) \ge p^2 - 1, \ \forall p \ge 1.$$

Preuve. Notons  $P(X_1, X_2, X_3, X_4) = X_1X_2 - X_3X_4$ . Montrons i). Soient  $x_1, x_2, x_3, x_4$  tels que  $x_1x_2 - x_3x_4 \in \mathfrak{m}^{2p+1}$ . Si  $\operatorname{ord}(x_1) \ge p+1$  ou  $\operatorname{ord}(x_2) \ge p+1$ , alors  $\operatorname{ord}(x_3) \ge p+1$  ou  $\operatorname{ord}(x_4) \ge p+1$ . Par exemple  $\operatorname{ord}(x_1) \ge p+1$  et  $\operatorname{ord}(x_3) \ge p+1$ . Dans ce cas nous posons  $\overline{x}_1 = 0, \overline{x}_2 = x_2, \overline{x}_3 = 0$  et  $\overline{x}_4 = x_4$ .

Si, pour tout *i*,  $\operatorname{ord}(x_i) \leq p$ , alors les termes initiaux de  $x_1x_2$  et de  $x_3x_4$  sont égaux. Donc, par exemple  $x_1$  divise  $x_3$ . D'où

$$\varepsilon = x_2 - \frac{x_3}{x_1} x_4 \in \mathfrak{m}^{2p - \operatorname{ord}(x_1) + 1} \subset \mathfrak{m}^{p+1}.$$

Nous posons alors  $\overline{x}_1 = x_1$ ,  $\overline{x}_2 = x_2 - \varepsilon$ ,  $\overline{x}_3 = x_3$  et  $\overline{x}_4 = x_4$ . Dans tous les cas nous avons  $P(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4) = 0$  et

$$\overline{x}_1 - x_1, \overline{x}_2 - x_2, \overline{x}_3 - x_3, \overline{x}_4 - x_4 \in \mathfrak{m}^{p+1}.$$

Cela prouve que  $\beta_1(p) \leq 2p$  pour tout p.

Enfin, supposons qu'il n'y ait pas égalité, c'est-à-dire qu'il existe p tel que  $\beta_1(p) \leq 2p - 1$ . Posons alors  $x_1 = x_2 = T^p$  et  $x_3 = x_4 = 0$ . Nous avons alors  $x_1x_2 - x_3x_4 \in \mathfrak{m}^{2p}$ . Il existe donc des  $\overline{x}_i$  pour  $1 \leq i \leq 4$  tels

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que  $P(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4) = 0$  et  $\overline{x}_i - x_i \in \mathfrak{m}^p$ . Nous pouvons alors écrire  $\overline{x}_i = x_i + \varepsilon_i$  où  $\varepsilon_i \in \mathfrak{m}^p$ . Nous avons alors

(6) 
$$T^{2p} + T^p(\varepsilon_1 + \varepsilon_2) + \varepsilon_1 \varepsilon_2 - \varepsilon_3 \varepsilon_4 = 0$$

Or  $\operatorname{ord}(T^p(\varepsilon_1 + \varepsilon_2))$ ,  $\operatorname{ord}(\varepsilon_1 \varepsilon_2)$  et  $\operatorname{ord}(\varepsilon_3 \varepsilon_4)$  sont tous plus grand que 2p + 1, donc nous ne pouvons avoir l'égalité 6.

ii) Là encore nous n'allons donner ici qu'une idée de la preuve. Pour plus de détails, on pourra se reporter au théorème 5.1 de [Ro3].

Notons  $P_k(X_1, X_2, X_3) := X_1X_2 - (T_1T_2 - T_3^k)X_3$ . Comme  $T_1T_2 - T_3^k$ est irréductible dans  $\mathcal{O}_N$ , alors  $P_k$  admet une fonction de Artin majorée par une fonction affine de la forme  $p \mapsto a(k)p + b(k)$ . On peut montrer que le plus petit a(k) possible est au moins égal à k (cf. [Iz]). Soient  $x_{1,p} = T_1^p, x_{2,p} = T_2^p$ . Alors  $x_{1,p}x_{2,p} \in (T_1T_2 - T_3^k) + \mathfrak{m}^{pk}$ . Donc il existe  $x_{3,p,k}$  tel que  $x_{1,p}x_{2,p} - (T_1T_2 - T_3^k)x_{3,p,k} \in \mathfrak{m}^{pk}$ .

Posons k = p. Notons  $x_{4,p} = T_1T_2 - T_3^p$ . Soit  $(x_1, x_2, x_3, x_4)$  une vraie solution de P.

Si  $x_4 - x_{4,p} \in \mathfrak{m}^{p+1}$ , alors on peut montrer que  $x_4$  est irréductible. Dans ce cas  $x_1$  ou  $x_2$  est divisible par  $x_4$ . Or les termes initiaux de  $x_{1,p}$  et de  $x_{2,p}$  ne sont pas divisibles par le terme initial de  $x_{4,p}$  qui est le terme initial de  $x_4$ . Donc soit  $\operatorname{ord}(x_1 - x_{1,p}) \leq p$ , soit  $\operatorname{ord}(x_2 - x_{2,p}) \leq p$ . Le second cas est  $x_4 - x_{4,p} \notin \mathfrak{m}^{p+1}$ .

Dans les deux cas nous avons

$$\sup_{P(\underline{x})=0} \left( \min\{ \operatorname{ord}(x_{1,p} - x_1), \operatorname{ord}(x_{2,p} - x_2), \\ \operatorname{ord}(x_{3,p,p} - x_3), \operatorname{ord}(x_{4,p} - x_4) \} \right) \le p.$$

Nous avons donc une solution approchée de P à l'ordre  $p^2$  mais la différence entre cette solution approchée et une vraie solution de P est d'ordre inférieur à p. Q.E.D.

Nous pouvons remarquer ici que la première fonction de Artin ne différencie pas ce germe d'un germe défini par un monôme de degré 2. Cependant les *N*-ièmes fonctions de Artin de ces deux germes, pour  $N \ge 3$ , sont différentes. Une question naturelle qui se pose est la suivante: Soit (X, 0) un germe d'espace analytique. Si il existe *a* tel que  $\beta_N(p) =$ *ap* pour tous *p* et *N* entiers, alors l'idéal des fonctions de (X, 0) est-il un produit d'idéaux définissants des germes lisses ?

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# Perverse sheaves and Milnor fibers over singular varieties

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#### Abstract.

We review some recent applications of perverse sheaves (intersection cohomologies) in singularity theory. Milnor fibers over general complete intersection varieties will be treated. We also give a proof of a result announced in [31].

# §1. Introduction

The aim of this note is to introduce some recent applications of perverse sheaves (intersection cohomologies) to the study of complex hypersurface singularities. In the last two decades the theory of Milnor fibrations (see for example, Milnor [29], Dimca [7] etc.) was extended to the Milnor fibers over singular varieties. In particular, for any holomorphic function f with (stratified) isolated singularity on any complete intersection variety, Lê [21], Siersma [34] and Tibar [35] proved that the Milnor fiber of f admits a bouquet decomposition. This result is of course a vast generalization of Milnor's result, but the fact that the cohomological type of the Milnor fiber of f is the same as that of a bouquet of spheres can be easily deduced from the theory of perverse sheaves (see Theorem 2.2 below). It seems therefore that the above mentioned authors studied the topological or homotopy types of Milnor fibers motivated by this cohomological result obtained by perverse sheaves. This example shows that a general result in the theory of perverse sheaves sometimes can become a good guide principle in the study of singularity theory. In this short note, we explain some new topological constraints of general hypersurface singularities obtained by perverse sheaves. We hope that these results will help our understanding of non-isolated hypersurface singularities.

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#### $\S$ 2. Milnor fibers over complete intersection varieties

In this section we review some recent results on the topology of Milnor fibers over singular varieties. Let X be an irreducible analytic subset (or an algebraic subvariety) of  $\mathbb{C}^N$  of dimension n + 1 containing the origin  $0 \in \mathbb{C}^N$ . Throughout this note, unless otherwise stated, we assume moreover that X is locally a complete intersection (we write it CI for short) in the ambient affine space  $\mathbb{C}^N$ . This weak assumption is necessary because we use the fact that the shifted constant sheaf  $\mathbb{C}_X[n+1]$  on a CI variety X is a perverse sheaf (see Section 4). Now let  $f: X \to \mathbb{C}$  be a (non-constant and reduced) holomorphic function on X satisfying the condition  $0 \in Y = \{z \in X \mid f(z) = 0\}$ . Then we have a topological fibration over a sufficiently small punctured disk  $D_n^* = \{t \in \mathbb{C} \mid 0 < |t| < \eta\} \subset \mathbb{C}$ :

$$f\colon f^{-1}(D_{\eta}^*)\cap B_{\varepsilon}\longrightarrow D_{\eta}^*,$$

where  $B_{\varepsilon} = \{z \in \mathbb{C}^N \cap X \mid ||z|| < \varepsilon\}$  is a small open neighborhood of  $0 \in X$  in X and  $0 < \eta << \varepsilon$ . The general fiber  $F_0 = f^{-1}(t) \cap B_{\varepsilon}$  $(0 < |t| < \eta)$  is called the Milnor fiber of  $f: X \to \mathbb{C}$  at 0. Note that when X is not smooth the Milnor fiber may have singularities. Nevertheless we have now a nice bouquet decomposition theorem for the Milnor fibers of functions  $f: X \to \mathbb{C}$  which have stratified isolated singularities at  $0 \in X$ in the following sense. First take a Whitney stratification  $X = \bigsqcup_{\alpha \in A} X_{\alpha}$ of X and denote it by S. Then the stratified singular locus  $\operatorname{sing}^{\mathcal{S}}(f)$  of  $f: X \to \mathbb{C}$  w.r.t.  $\mathcal{S}$  is defined by  $\operatorname{sing}^{\mathcal{S}}(f) = \bigsqcup_{\alpha \in A} \operatorname{sing}(f \mid_{X_{\alpha}})$ . Using the Whitney conditons of  $\mathcal{S}$  we can easily check that  $\operatorname{sing}^{\mathcal{S}}(f)$  is a closed analytic subset of X. It is also easy to see (essentially by the curve selection lemma) that  $\operatorname{sing}^{\mathcal{S}}(f)$  is contained in the complex hypersurface  $Y = \{z \in X \mid f(z) = 0\} \subset X$  in an open neighborhood of Y in X (see for example Proposition 1.3 of Massey [24]). Now we say that a holomorphic function  $f: X \to \mathbb{C}$  has a stratified isolated singular point at  $0 \in X$  w.r.t. S if the dimension of sing<sup>S</sup>(f) at  $0 \in X$  is zero. Then Milnor's bouquet decomposition theorem over non-singular varieties can be generalized as follows.

**Theorem 2.1** (Lê [21], Siersma [34], Tibar [35]). Let  $f: X \to \mathbb{C}$ be a holomorphic function having a stratified isolated singular point at  $0 \in X$  w.r.t. a Whitney stratification S of X. Then the Milnor fiber  $F_0$  of f at 0 has the homotopy type of a bouquet of n-dimensional spheres:

$$F_0 \sim_h S^n \lor S^n \lor \cdots \lor S^n.$$

This theorem was obtained by developing the so-called polar curve method, which dates back to the work of Lê-Perron in [22]. By the same method we can also explicitly construct the handle decomposition of the Milnor fiber  $F_0$  when X is smooth. Namely for smooth X we can completely determine the topological type of  $F_0$ , though it might be still difficult to compute the Betti numbers of  $F_0$  if Y has non-isolated singularities at 0. For these important results we recommend the reader to see a series of papers by Lê or the recent book [25] by Massey etc. Note also that Massey's paper [24] gives also a method to compute the number of spheres in the above bouquet decomposition (i.e. the generalized Milnor number of f at 0). Now let us consider the general case where f does not necessarily have a stratified isolated singular point at 0. Then we have the following cohomological result.

**Theorem 2.2** (the generalized Kato-Matsumoto's theorem). Assume that the dimension of the stratified singular locus  $\operatorname{sing}^{\mathcal{S}}(f)$  of f at  $0 \in X$  is  $s \geq 0$ . Then for the reduced cohomology groups  $\widetilde{H}^{j}(F_{0}; \mathbb{C})$  of  $F_{0}$  we have

$$H^{j}(F_{0};\mathbb{C}) = 0 \quad for \quad \forall j \notin [n-s, n].$$

In Section 5 we will show that this theorem can be easily deduced from some well-known properties of perverse sheaves. To end this section we define the complex link CL(X;0) of X at 0, which is an important example of Milnor fibers over singular varieties. Recall that X is embedded in a smooth affine space  $\mathbb{C}^N$ . We take a linear form  $l: \mathbb{C}^N \to \mathbb{C}$  (l(0) = 0)on  $\mathbb{C}^N$  and consider its restriction  $l \mid_X$  to  $X \subset \mathbb{C}^N$ . Then we can show that for a sufficiently generic linear form l the dimension of the stratified singular locus  $\operatorname{sing}^{\mathcal{S}}(l \mid_X)$  of  $l \mid_X$  at  $0 \in X$  is zero. Therefore if we define the complex link CL(X;0) of X at 0 to be the Milnor fiber of such a function  $l \mid_X : X \to \mathbb{C}$  at 0, then we obtain a bouquet decomposition

$$CL(X;0) \sim_h S^n \lor S^n \lor \cdots \lor S^n$$

by Theorem 2.1. Note that the topological type of the complex link does not depend on the choice of linear forms  $l: \mathbb{C}^N \to \mathbb{C}$  nor embeddings  $X \hookrightarrow \mathbb{C}^N$ . This notion plays an important role also in stratified Morse theory (see Goresky-MacPherson [13]).

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#### $\S$ **3.** Some results and their generalizations

In this section we introduce some results obtained in Nang-T [30], [31] and Dimca [8]. Recall that X is a CI variety (or a CI analytic set) of dimension n + 1. Then for a non-constant holomorphic function  $f: X \to \mathbb{C}$  satisfying f(0) = 0 ( $0 \in X$ ) the dimensions of  $Y = \{z \in$  $X \mid f(z) = 0\} \subset X$  and the Milnor fiber  $F_0$  are n. We thus have the monodromy operators

$$T(j)_0: H^j(F_0; \mathbb{C}) \longrightarrow H^j(F_0; \mathbb{C})$$

for j = 0, n-s, n-s+1, ..., n-1, n at  $0 \in Y \subset X$   $(s = \dim_0 \operatorname{sing}^{\mathcal{S}}(f))$ . Since the lower dimensional monodromy operators

$$T(j)_0 \colon H^j(F_0; \mathbb{C}) \xrightarrow{\sim} H^j(F_0; \mathbb{C}) \quad (j = 0, n - s, \dots, n - 1)$$

are relatively simple as we shall see in Section 5 (in particular

$$T(0)_0 \colon H^0(F_0; \mathbb{C}) \longrightarrow H^0(F_0; \mathbb{C})$$

is the identity map of  $\mathbb{C}$ ), here we focus our attention on the top dimensional monodromy operator  $T(n)_0: H^n(F_0; \mathbb{C}) \xrightarrow{\sim} H^n(F_0; \mathbb{C})$ .

**Definition 3.1.** For a complex number  $a \in \mathbb{C}$ , we denote by  $N_a$  the number of Jordan blocks with the eigenvalue a in the monodromy operator  $T(n)_0 \colon H^n(F_0; \mathbb{C}) \xrightarrow{\sim} H^n(F_0; \mathbb{C})$ .

Then we have the following result which gives an upper bound for the multiplicities of eigenvalues of the monodromy

$$T(n)_0 \colon H^n(F_0; \mathbb{C}) \longrightarrow H^n(F_0; \mathbb{C}).$$

Note that as an upper bound for the sizes of Jordan blocks in the monodromy operators we have the famous monodromy theorem (see the references cited in the paper [10]). For a topological space W we denote by  $b_j(W)$  the *j*-th Betti number of W.

**Theorem 3.2** (Nang-T [30], [31] and Dimca [8]). For any nonzero complex number  $a \in \mathbb{C}$  we have

$$N_a \le b_{n-1}(CL(Y;0)) + b_n(CL(X;0)).$$

In particular if X is smooth (e.g.  $X = \mathbb{C}^{n+1}$ ) then  $N_a \leq b_{n-1}(CL(Y;0))$ . Namely  $N_a$ 's are bounded by the number of (n-1)-dimensional spheres in the bouquet decomposition  $CL(Y;0) \sim_h S^{n-1} \lor \cdots \lor S^{n-1}$  of the complex link of Y. This theorem was first obtained by Nang-T [30] for  $X = \mathbb{C}^{n+1}$  and the complex numbers  $a \neq 0$  satisfying a technical condition. Then Proposition 6.4.17 of Dimca [8] generalized it to the case of Milnor fibers over singular varieties assuming the same condition on  $a \neq 0$ . Finally Nang-T [31] removed this technical assumption. The proof of Theorem 3.2 will be given in Section 4. Note that if X is  $\mathbb{C}^{n+1}$  and f is a quasi-homogeneous polynomial then the monodromy operators are periodic ( $\Longrightarrow$  semisimple) and hence  $N_a$  is nothing but the multiplicity of the eigenvalue a in the map  $T(n)_0: H^n(F_0; \mathbb{C}) \xrightarrow{\sim} H^n(F_0; \mathbb{C})$ . Even in such simplest cases Theorem 3.2 seems to be new, because for  $Y = \{z \in X \mid f(z) = 0\}$  with a non-isolated singular point at 0 it is in general very difficult to compute the monodromy operators. For general hypersurface singularities we can compute only the monodromy zeta function

$$Z_f(\lambda) = \prod_{j=0}^n \det(\mathrm{Id} - \lambda T(j)_0)^{(-1)^j}$$

by constructing an embedded resolution of singularities (see Bierstone-Milman [3] for an algorithm to construct embedded resolutions) of each given complex hypersurface Y in  $X = \mathbb{C}^{n+1}$  (A'Campo [1]). If the hypersurface  $Y \subset X = \mathbb{C}^{n+1}$  has an isolated singular points at 0, Varchenko's formula ([36]) for the characteristic polynomial of  $T(n)_0$  obtained by this monodromy zeta function (and a result of Kouchnirenko [20]) is very useful. However to use his formula, the defining function f of Ymust satisfy the so-called Newton non-degeneracy condition. Hence it would be difficult to prove Theorem 3.2 along this line even for all complex hypersurfaces  $Y \subset X = \mathbb{C}^{n+1}$  having isolated singular points at 0. For such hypersurfaces we have the following corollary. Let L(Y; 0) be the real link of Y at  $0 \in Y$ . Namely we set

$$L(Y;0) = Y \cap S_{\varepsilon} \quad (0 < \varepsilon << 1),$$

where  $S_{\varepsilon} \subset \mathbb{C}^N$  is a small sphere centered at 0 with radius  $\varepsilon$ .

**Corollary 3.3.** Assume that  $X = \mathbb{C}^{n+1}$  and the complex hypersurface  $Y = \{z \in X \mid f(z) = 0\} \subset X$  has an isolated singular point at  $0 \in Y$ . Then we have

$$b_{n-1}(L(Y;0)) \le b_{n-1}(CL(Y;0)).$$

Under the assumptions of this corollary we can easily prove

$$N_1 = b_{n-1}(L(Y;0))$$

by Alexander duality. Therefore Corollary 3.3 immediately follows from Theorem 3.2. In order to understand the topological meaning of Corollary 3.3, recall that if Y has an isolated singular point at  $0 \in Y$  then the real link L(Y;0) is a smooth compact orientable (2n-1)-manifold whose non-zero Betti numbers are  $b_0$ ,  $b_{n-1}$ ,  $b_n$ ,  $b_{2n-1}$ . Since  $b_0 = b_{2n-1} = 1$ and  $b_{n-1} = b_n$  by Poincaré duality, the only interesting number among them is  $b_{n-1}(L(Y;0))$ . On the other hand, as we saw in Section 2 the complex link CL(Y;0) has only two non-zero Betti numbers  $b_0 =$  $1, b_{n-1}$ . So the inequality  $b_{n-1}(L(Y;0)) \leq b_{n-1}(CL(Y;0))$  means that the most interesting invariant of the topology of the real link and that of the complex link are related each other. Finally we remark that this result was generalized in Proposition 6.1.22 and Corollary 6.1.24 of [8] to the case where Y is higher-codimensional in  $X = \mathbb{C}^{n+1}$ .

### $\S4.$ Proof of Theorem 3.2

In this section we quickly review the theory of perverse sheaves and give a proof of Theorem 3.2. For the detail of the theory of perverse sheaves and constructible sheaves, we refer to Beilinson-Bernstein-Deligne [2], Dimca [8], Hotta-T-Tanisaki [14], Kashiwara-Schapira [18] and Schürmann [33] etc. Now let X be a complex analytic set or an algebraic variety (endowed with the classical topology). As usual we denote by  $\mathbf{D}^b(X)$  the derived category of bounded complexes of sheaves of  $\mathbb{C}_X$ -modules on X. The category of perverse sheaves is a full abelian subcategory of  $\mathbf{D}^b(X)$  which correponds to that of regular holonomic  $\mathcal{D}_X$ -modules (when X is smooth) through the Riemann-Hilbert correspondence (see Kashiwara [17] and Hotta-T-Tanisaki [14] etc.). In order to recall the definition of perverse sheaves by Beilinson-Bernstein-Deligne [2], denote by  $\mathbf{D}_c^b(X)$  the full subcategory of  $\mathbf{D}^b(X)$  consisting of complexes of sheaves with  $\mathbb{C}$ -constructible cohomology sheaves.

**Definition 4.1** ([2]). Let  $\mathcal{F} \in \mathbf{D}_c^b(X)$ . Then  $\mathcal{F}$  is a perverse sheaf on X if the following two conditions are satisfied.

(i) For any  $i \in \mathbb{Z}$ , we have dim[supp  $H^i \mathcal{F}$ ]  $\leq -i$ .

(ii) The Verdier dual  $D(\mathcal{F})$  of  $\mathcal{F}$  satisfies the condition

$$\dim[\operatorname{supp} H^i D(\mathcal{F})] \leq -i \quad \text{for any} \quad i \in \mathbb{Z}.$$

We denote by  $\operatorname{Perv}(\mathbb{C}_X)$  the full subcategory of  $\mathbf{D}_c^b(X)$  consisting of perverse objects.

As a special class of perverse sheaves we have the following.
**Theorem 4.2.** Assume that X is pure-dimensional and locally a CI. Then the shifted constant sheaf  $\mathbb{C}_X[\dim X] \in \mathbf{D}_c^b(X)$  is a perverse sheaf on X. Moreover for any local system (i.e. a locally constant sheaf of finite rank over  $\mathbb{C}_X$ )  $\mathcal{L}$ , we have  $\mathcal{L}[\dim X] \in \text{Perv}(\mathbb{C}_X)$ .

For the proof, see for example Sorite 1.8 (page 15) of Brylinski [5] etc. The proof of [5] uses  $\mathcal{D}$ -modules. A purely topological proof can be found in Theorem 5.1.20 of [8].

Now let us prove Theorem 3.2. For the sake of simplicity let X be an (n+1)-dimensional CI variety in  $\mathbb{C}^N$  containing the origin 0. Recall that  $f: X \to \mathbb{C}$  is a holomorphic function s.t.  $0 \in Y = \{z \in X \mid f(z) = 0\}$ . We prove the theorem by constructing a special perverse sheaf  $\mathcal{G}^{\cdot}$  on  $\mathbb{C}^N$  which contains the information of the top dimensional monodromy operator  $T(n)_0: H^n(F_0; \mathbb{C}) \xrightarrow{\sim} H^n(F_0; \mathbb{C})$ . To begin with, for the given complex number  $a \neq 0$  we define a local system  $\mathcal{L}_a$  on  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  by the representation

$$\pi_1(\mathbb{C}^*) \simeq \mathbb{Z} \longrightarrow GL(1, \mathbb{C}) = \mathbb{C}^*$$
$$n \longmapsto a^n.$$

Next consider a local system  $\widetilde{\mathcal{L}_a}$  on  $X \setminus Y$  obtained by taking the inverse image of  $\mathcal{L}_a$  by  $f: X \setminus Y \to \mathbb{C}^*$ . Then by Theorem 4.2 the complex of sheaves  $\mathcal{L}_a[n+1]$  is a perverse sheaf on  $X \setminus Y$ . Now let us set  $j: X \setminus Y \hookrightarrow$  $X, j_0: X \setminus Y \hookrightarrow X \setminus \{0\}$  and  $j_1: X \setminus \{0\} \hookrightarrow X$   $(j = j_1 \circ j_0)$ . We will extend the perverse sheaf  $\widetilde{\mathcal{L}_a}[n+1]$  to the whole  $\mathbb{C}^N$  in three steps. First, since  $j_0$  is a Stein map, the direct image  $Rj_{0*}\widetilde{\mathcal{L}_a}[n+1]$  is a perverse sheaf on  $X \setminus \{0\}$  by M. Artin's theorem (see for example Corollary 5.2.17 of [8]). Next we define a perverse sheaf  $\mathcal{F}^{\cdot}$  on X by  $\mathcal{F}^{\cdot} = j_{1!}(\mathrm{R}j_{0*}\widetilde{\mathcal{L}_{a}}[n+1])$ . Here  $j_{1!}(*)$  stands for the so-called Deligne-Goresky-MacPherson extension functor. By using the truncation functor  $\tau^{\leq -1}(*)$  we can rewrite it as  $j_{1!}(*) \simeq (\tau^{\leq -1} \circ \mathbf{R} j_{1*})(*)$ . Finally we define a perverse sheaf  $\mathcal{G}^{\cdot}$  on  $\mathbb{C}^N$  by  $\mathcal{G}^{\cdot} = \iota_* \mathcal{F}^{\cdot}$ , where we set  $\iota \colon X \hookrightarrow \mathbb{C}^N$ . By the Riemann-Hilbert correspondence there exists a unique regular holonomic  $\mathcal{D}_{\mathbb{C}^N}$ -module  $\mathcal{M}$ which corresponds to the perverse sheaf  $\mathcal{G}$ . Then, just as the proof of Theorem 5.4 of Nang-T [30] (or Proposition 6.4.17 of Dimca [8]), Theorem 3.2 can be proved by calculating the multiplicity  $m \in \mathbb{Z}_{>0}$  of  $\mathcal{M}$  along the conormal bundle  $T^*_{\{0\}}\mathbb{C}^N \subset T^*X$ . Namely we obtain the equality

$$m = -N_a + \{b_{n-1}(CL(Y;0)) + b_n(CL(X;0))\}$$

and the theorem follows from the non-negativity of the multiplicity m. Q.E.D.

#### $\S5.$ Some other consequences of perversity

In this section, using the notations of previous sections, we introduce some other important consequences of perversity in the topology of complex hypersurface singularities. In particular we show that the lower dimensional monodromy operators  $T(j)_0: H^j(F_0; \mathbb{C}) \xrightarrow{\sim} H^j(F_0; \mathbb{C})$  for  $j = 0, n - s, \dots, n - 1$  ( $s = \dim_0 \operatorname{sing}^{\mathcal{S}}(f)$ ) are usually much simpler than the top dimensional one. First of all, for a given holomorphic function  $f: X \to \mathbb{C}$  we associate to it the shifted vanishing cycle functor

$${}^{p}\phi_{f}(*)\colon \mathbf{D}^{b}(X)\longrightarrow \mathbf{D}^{b}(Y)$$

satisfying the condition

$$H^{j}(^{p}\phi_{f}(\mathbb{C}_{X}))_{x} \simeq \widetilde{H}^{j-1}(F_{x};\mathbb{C})$$

for any  $x \in Y = \{z \in X \mid f(z) = 0\}$  and  $j \in \mathbb{Z}$ . Here  $F_x$  is the Milnor fiber of f at  $x \in Y$ . Then it is well-known that this functor preserves the perversity. For the proof, see for example, Corollary 10.3.13 of Kashiwara-Schapira [18] and Theorem 6.0.2 of Schürmann [33] etc. This important result was first obtained by [2] in the algebraic case. The proof for the analytic case was given by Kashiwara [16] in his study of vanishing cycle functors for  $\mathcal{D}$ -modules (see also Goresky-MacPherson [12] for a topological approach to this problem). Now we can easily deduce Theorem 2.2 (the generalized Kato-Matsumoto's theorem) from this very general result. Indeed, applying it to the perverse sheaf  $\mathbb{C}_X[n+1]$  (we assume that X is locally a CI) we see that the vanishing cycle  $\mathcal{G}^{\cdot} = {}^p \phi_f(\mathbb{C}_X[n+1])$  is a perverse sheaf whose support is contained in the stratified singular locus  $\operatorname{sing}^{\mathcal{S}}(f)$  of f. Then it remains to apply the following very elemetary property of perserse sheaves to  $\mathcal{G}^{\cdot}$ .

**Lemma 5.1.** Let  $\mathcal{G}$  be a perverse sheaf on an analytic set X whose support is contained in an s-dimensional analytic subset S of X. Then we have  $H^j(\mathcal{G})_x \simeq 0$  for any  $x \in X$  and  $j \notin [-s, 0]$ .

Namely we obtain  $\widetilde{H}^{n+j}(F_0;\mathbb{C}) \simeq H^{n+j+1}({}^p\phi_f(\mathbb{C}_X))_0 \simeq H^j(\mathcal{G})_0 \simeq 0$  for  $j \notin [-s, 0]$   $(s = \dim_0 \operatorname{sing}^{\mathcal{S}}(f))$ . This completes the proof of Theorem 2.2. By refining this proof, we can obtain also the following interesting results on the propagation of monodromy eigenvalues up to the center  $0 \in Y$ . Let us consider the monodromy operators  $T(j)_x \colon H^j(F_x;\mathbb{C}) \xrightarrow{\sim} H^j(F_x;\mathbb{C})$  at points  $x \in Y$  outside the origin. Then we have

**Theorem 5.2.** Let  $a \in \mathbb{C}$  be a complex number.

- (i) (Corollary 6.1.7 of Dimca [8]) Assume that a is an eigenvalue of a lower dimensional monodromy  $T(j)_0: H^j(F_0; \mathbb{C}) \xrightarrow{\sim} H^j(F_0; \mathbb{C})$  $(j \le n-1)$  at 0. Then for any open neighborhood U of 0 in Y there exists a point  $x \ne 0$  in  $U \setminus \{0\}$  such that a is an eigenvalue of  $T(k)_x: H^k(F_x; \mathbb{C}) \xrightarrow{\sim} H^k(F_x; \mathbb{C})$  for some k.
- (ii) (Theorem 0.4 of Dimca-Saito [9]) Assume that a lower dimensional monodromy  $T(j)_0: H^j(F_0; \mathbb{C}) \xrightarrow{\sim} H^j(F_0; \mathbb{C})$   $(j \leq n-1)$ at 0 has a Jordan block with the eigenvalue a of size m. Then there exist points  $x_k \neq 0$  sufficiently close to 0 for  $k \leq j$  such that the monodromy  $T(k)_{x_k}: H^k(F_{x_k}; \mathbb{C}) \xrightarrow{\sim} H^k(F_{x_k}; \mathbb{C})$  at  $x_k$  has a Jordan block with the eigenvalue a of size  $m_k$  and  $\sum_{k \leq j} m_k \geq m$ .

To prove (i) of this theorem we use the direct sum decomposition

$${}^{p}\phi_{f}(\mathbb{C}_{X}[n+1]) \simeq \bigoplus_{a \in \mathbb{C}} [{}^{p}\phi_{f}(\mathbb{C}_{X}[n+1])]_{a}$$

in the category  $\operatorname{Perv}(\mathbb{C}_Y)$ . Here  $[{}^p\phi_f(\mathbb{C}_X[n+1])]_a$  denotes the generalized eigenspace for the eigenvalue a of the monodromy map

$$T: {}^{p}\phi_{f}(\mathbb{C}_{X}[n+1]) \to {}^{p}\phi_{f}(\mathbb{C}_{X}[n+1])$$

in Perv( $\mathbb{C}_Y$ ). Namely  $[{}^p\phi_f(\mathbb{C}_X[n+1])]_a$  is the kernel of  $(a \operatorname{Id} - T)^k$  for k >> 0. Then we can easily prove (i) by considering the supports of perverse sheaves  $[{}^p\phi_f(\mathbb{C}_X[n+1])]_a$  as in the proof of Theorem 2.2. To prove (ii) we use the restriction of  ${}^p\phi_f(\mathbb{C}_X[n+1])$  to the real link L(Y;0) of Y and a spectral sequence. See [9] for the precise proof. Q.E.D.

Roughly speaking, Theorem 5.2 asserts that some important parts of the lower dimensional monodromy operators  $T(j)_0$   $(j \leq n-1)$  at 0 are determined by the monodromy operators at points  $x \in Y, x \neq 0$ . We can observe a similar phenomenon also in Randell's theorem for two-dimensional complex hypersurfaces in  $\mathbb{C}^3$  obtained by deprojectivizing plane curves (see Oka [32] for a survey of this subject and related results). Theorem 5.2 (i) in particular implies that if the singularity of Y is normal crossing outside the origin then all the eigenvalues of the lower dimensional monodromy operators  $T(j)_0$   $(j \leq n-1)$  at the origin are 1 (see Example 6.1.8 of [8]). In this case, if an embedded resolution of  $Y \subset X = \mathbb{C}^{n+1}$  is given, using the monodromy zeta function obtained by the methods of [1] we can determine the multiplicities of the eigenvalues  $a \neq 1$  in the top dimensional monodromy  $T(n)_0: H^n(F_0; \mathbb{C}) \xrightarrow{\sim} H^n(F_0; \mathbb{C}).$ 

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Finally let us list up some related subjects that we could not explain precisely in this short note. As applications of perverse sheaves (intersection cohomologies) in singularity theory we have also the following results.

- (i) We can generalize vanishing theorems (obtained by Esnault-Schechtman-Viehweg, Kohno and Schechtman-Terao-Varchenko etc.) for twisted cohomology groups of the complements to hyperplane arrangements. See Cohen-Dimca-Orlik [6] etc.
- (ii) Recently using the theory of perverse sheaves Maxim [28] found a new construction of Alexander modules of hypersurface complements (see [23] and [32] for the definition) and generalized the results of Libgober [23] to the case where the hypersurface has non-isolated singularities.
- (iii) The classical theory of projective duality (i.e. the study of dual varieties in projective geometry) was reformuled in terms of constructible sheaves. After the fundamental work by Brylinski [5], Ernström proved that the topological Radon transform of the Euler obstruction of a projective variety V is that of the dual variety  $V^*$  modulo constant functions (see also [26] and [27] etc. for its generalizations).

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## On Horn-Kapranov uniformisation of the discriminantal loci

## Susumu Tanabé

#### Abstract.

In this note we give a rational uniformisation equation of the discriminant loci associated to a non-degenerate affine complete intersection variety. To show this formula we establish a relation of the fibre-integral with the hypergeometric function of Horn and that of Gel'fand-Kapranov-Zelevinski.

#### $\S 0.$ Introduction

In this note we give a concrete rational uniformisation equation for the discriminantal loci of non-degenerate affine complete intersection depending on deformation parameters.

First of all, let us fix the situation. For the complex varieties  $X = \mathbf{C}^{\times^N}$  and  $S = \mathbf{C}^k$ , we consider the mapping,

$$(0.1) f: X \to S$$

such that  $X_s := \{(x_1, \ldots, x_N) \in X; f_1(x) + s_1 = 0, \ldots, f_k(x) + s_k = 0\}$ . Let  $f_1(x), \ldots, f_k(x)$  be polynomials that define a non-degenerate complete intersection (CI) in the sense of Danilov-Khovanski [3] with the following specific form:

(0.2) 
$$f_{\ell}(x) = x^{\vec{\alpha}_{1,\ell}} + \dots + x^{\vec{\alpha}_{\tau_{\ell},\ell}}, \ 1 \le \ell \le k,$$

where  $\vec{\alpha}_{i,\ell} \in (\mathbf{Z}_{\geq 0})^N$ . Let *n* be the dimension of the variety  $X_0$ , dim  $X_0 = n \geq 0$ .  $W_s := \{(x_1, \ldots, x_N, y_1, \ldots, y_k) \in X \times (\mathbf{C})^k; y_1(f_1(x))\}$ 

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 $+s_1$ )  $+\cdots + y_k(f_k(x) + s_k) = 0$ }. Then it is known that the discriminantal loci of  $X_s$  coincides with that of  $W_s$ . That is to say, the study of the discriminantal loci of a CI can be reduced to that of an hypersurface associated with the original CI in a special manner. This fact has been discovered by Arthur Cayley [5] and thus the method to reduce the geometric study of a CI to that of a hypersurface is named "Cayley trick" in general, even in contexts apart from the study of discriminantal loci (e.g. the description of the mixed Hodge structure of the former by means of the latter given by T. Terasoma, A. Mavlyutov [9] and others). Here we return to the initial spirit of Cayley who treated the question of the discriminantal loci.

The main idea is based on that of the paper [6] which states that the singular loci of the linear differential operators annihilating the fibre integrals of  $X_s$  coincide with the discriminantal loci of  $X_s$ . In the modern terminology of the A-hypergeometric functions (HGF), it is equivalent to say that A-discriminantal loci are singular loci for generalized A-HGF. This fact has been proven in [7] and we give a more precise description of the discriminantal loci by means of combinatorial data of the polynomial mapping f and the toric geometry of  $W_s$  (see Theorem 2.6).

Let us review the contents of the note in short. In §1 we recall some basic facts on the Cayley trick and Néron-Severi torus. In §2, we calculate the Mellin transform of the fibre integral in an explicit manner. Using a representation of the Mellin transform we show that fibre integral satisfies the Horn type system of differential equations (Theorem 2.4). From this expression of the Horn type system, we get the discriminantal loci as the boundary of a convergence domain of solutions to the system. In §3, we show that the fibre integral calculated in §2 is nothing but the quotient of the Gel'fand-Kapranov-Zelevinski generalized hypergeometric function (HGF) by the torus action. In §4 we give two computational examples: discriminantal loci for the  $D_4$  type singularity and the simplest non-quasihomogeneous complete intersection.

Finally we remark that this note is an abridged version of some parts from [13] where one can find more details.

### §1. Cayley trick and Néron-Severi torus

Throughout this section we keep the notation of §0. Further we introduce the following notations. Let  $\mathbf{T}^m = (\mathbf{C} \setminus \{0\})^m = (\mathbf{C}^{\times})^m$  be the complex algebraic torus of dimension m. We denote by  $x^{\mathbf{i}}$  the monomial  $x^{\mathbf{i}} := x_1^{i_1} \cdots x_N^{i_N}$  with multi-index  $\mathbf{i} = (i_1, \ldots, i_N) \in \mathbf{Z}^N$ , and by dx the N-volume form  $dx := dx_1 \wedge \cdots \wedge dx_N$ . We shall also use the notations  $x^{\mathbf{i}} := x_1 \cdots x_N, y^{\zeta} = y_1^{\zeta_1} \cdots y_k^{\zeta_k}, s^{\mathbf{z}} = s_1^{z_1} \cdots s_k^{z_k}$  and  $ds = ds_1 \wedge \cdots \wedge ds_k$ 

and their analogies for each variable. In this section we consider an extension of the mapping f to that defined from  $\mathbf{P}_{\tilde{\Sigma}}$  to  $\mathbf{C}^k$ . We follow the construction by [2] and [9]. Let us define M as the dimension of a minimal ambient space so that we can quasihomogenize simultaneously the polynomials  $(f_1(x), \ldots, f_k(x))$  by multiplying certain terms by new variables:

$$x^{\mathbf{i}} \longmapsto x'_{j} x^{\mathbf{i}}, \quad j = 1, 2, \dots$$

Let us denote by  $(f_1(x, x'), \ldots, f_k(x, x'))$  the new polynomials obtained in such a way. These polynomials are quasi-homogeneous with respect to certain weight system i.e. there exists a set of positive integers  $(w_1, \ldots, w_N, w'_1, \ldots, w'_{M-N})$  such that their G.C.D. equals 1 and the following relation holds:

$$E(x, x')(f_{\ell}(x, x')) = p_{\ell}f_{\ell}(x, x') \text{ for } \ell = 1, \dots, k,$$

where  $p_{\ell}$  is some positive integer and

(1.1) 
$$E(x, x') = \sum_{i=1}^{N} w_i x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{M-N} w'_j x'_j \frac{\partial}{\partial x'_j}$$

E an Euler vector field.

**Example.** We modify the polynomial  $f(x) = x_1^a + x_1x_2 + x_2^b$ , with a, b > 2, GCD(a, b) = 1, in adding a new variable  $x'_1$  so that the new polynomial  $f(x, x') = x_1^a + x'_1x_1x_2 + x_2^b$ , becomes quasihomogeneous with respect to the weight system (b, a, ab - a - b).

In general there are of course many choices of terms that we modify to realize the quasihomogeneiety.

From now on we will use the notation  $X := (X_1, \ldots, X_M) := (x_1, \ldots, x_N, x'_1, \ldots, x'_{M-N})$  and that of the polynomial  $f_{\ell}(X) := f_{\ell}(x, x')$ . If we introduce the Euler vector field,

$$E(X') = \sum_{i=1}^{N} w_i x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{M-N} w'_j x'_j \frac{\partial}{\partial x'_j} + X_{M+1} \frac{\partial}{\partial X_{M+1}},$$

we have the following relation:

$$E(X')(f_{\ell}(X) + X_{M+1}^{p_{\ell}}s_{\ell}) = p_{\ell}(f_{\ell}(X) + X_{M+1}^{p_{\ell}}s_{\ell}) \quad \text{for} \quad \ell = 1, \dots, k.$$

From now on we denote  $X' := (X, X_{M+1})$ . Let  $\mathbf{M}_{\mathbf{Z}}$  be an integer lattice of rang N and  $\mathbf{N}_{\mathbf{Z}}$  be its dual,  $\mathbf{N}_{\mathbf{Z}} = \operatorname{Hom}(\mathbf{M}_{\mathbf{Z}}, \mathbf{Z})$ . We denote by  $\mathbf{M}_{\mathbf{R}}$ (resp.  $\mathbf{N}_{\mathbf{R}}$ ) the natural extension of  $\mathbf{M}_{\mathbf{Z}}$  (resp.  $\mathbf{N}_{\mathbf{Z}}$ ) to its real space. Let

us take  $\vec{e}_1, \ldots, \vec{e}_{M+1}$  a set of generators of one dimensional cones such that  $\sum_{\ell=1}^{M+1} \mathbf{R} \vec{e}_{\ell} = \mathbf{N}_{\mathbf{R}}$ . We can define a simplicial fan  $\Sigma$  in  $\mathbf{N}_{\mathbf{R}}$  as a set of simplicial cones spanned by the above  $\vec{e}_1, \ldots, \vec{e}_{M+1}$ . Our construction of the Euler vector field E(X') correspond to the superstructure  $\mathbf{N}_{\mathbf{R}} \times \mathbf{N}'_{\mathbf{R}}$  with a basis of generators  $\vec{e}_{N+1}, \ldots, \vec{e}_{M+1}$  such that

$$\sum_{i=1}^{N} w_i \vec{\tilde{e}}_i + \sum_{j=1}^{M-N} w'_j \vec{\tilde{e}}_j + \vec{\tilde{e}}_{M+1} = 0$$

Here we have  $p_{\mathbf{N}}(\vec{\tilde{e}}_j) = \vec{e}_j$  for the projection  $p_{\mathbf{N}} \colon \mathbf{N}_{\mathbf{R}} \times \mathbf{N}'_{\mathbf{R}} \to \mathbf{N}_{\mathbf{R}}$ . While the dimension of the vector space  $\mathbf{N}_{\mathbf{R}} \times \mathbf{N}'_{\mathbf{R}}$  must be minimal i.e.  $\dim(\mathbf{N}_{\mathbf{R}} \times \mathbf{N}'_{\mathbf{R}}) = M$ .

We introduce a polynomial,

(1.2) 
$$H(x, y) := y_1 f_1(x) + \dots + y_k f_k(x) \in \mathbf{Z}[x_1, \dots, x_N, y_1, \dots, y_k],$$

in adding new variables  $y_1, \ldots, y_k$ . Let  $\vec{n}_1, \ldots, \vec{n}_{M+k}$  be the elements of the set  $\operatorname{supp}(H(x, y)) \subset \mathbb{Z}^{N+k}$ . We define a simplicial rational fan  $\tilde{\Sigma}$ in  $\mathbb{R}^{N+k}$  as a set of simplicial cones generated by  $\vec{n}_1, \ldots, \vec{n}_{M+k}$ . We consider the injective homomorphism

$$\varphi \colon \mathbf{\tilde{M}}_{\mathbf{Z}} \to \mathbf{Z}^{M+k},$$

for  $\mathbf{\tilde{M}}_{\mathbf{Z}} = \mathbf{M}_{\mathbf{Z}} \times \mathbf{Z}^k$ , defined by

$$\varphi(\vec{\tilde{m}}) = (\langle \vec{\tilde{m}}, \vec{n}_1 \rangle, \dots, \langle \vec{\tilde{m}}, \vec{n}_{M+k} \rangle).$$

The cokernel of this mapping is a free abelian group,

$$Cl(\tilde{\Sigma}) = \mathbf{Z}^{M+k} / \varphi(\tilde{\mathbf{M}}_{\mathbf{Z}})$$

for which the following group can be defined

(1.3) 
$$\mathbf{D}(\tilde{\Sigma}) := \operatorname{Spec} \mathbf{C}[Cl(\tilde{\Sigma})].$$

As a matter of fact this group  $\mathbf{D}(\tilde{\Sigma})$  is isomorphic to an algebraic torus  $\mathbf{T}^{M-N}$ . One can define the toric variety  $\mathbf{P}_{\tilde{\Sigma}}$  associated to the affine space,

$$\mathbf{A}^{M+k} = \operatorname{Spec} \mathbf{C}[X_1, \ldots, X_M, y_1, \ldots, y_k].$$

To this end we proceed following way after the method initiated by M. Audin. Let  $\hat{X}_{\sigma} := \prod_{1 \leq i \leq M, \ \vec{n}_i \notin \sigma} X_i \prod_{1 \leq j \leq k, \ \vec{n}_{M+j} \notin \sigma} y_j$ , be a monomial defining a coordinate plane and the ideal

$$B(\tilde{\Sigma}) = \langle \hat{X}_{\sigma}; \sigma \in \tilde{\Sigma} \rangle \subset \mathbf{C}[X_1, \ldots, X_M, y_1, \ldots, y_k].$$

Let  $Z(\tilde{\Sigma}) := \mathbf{V}(B(\tilde{\Sigma})) \subset \mathbf{A}^{M+k}$  be the variety defined by the ideal  $B(\tilde{\Sigma})$ . We construct the toric variety  $\mathbf{P}_{\tilde{\Sigma}}$  as the quotient of  $U(\tilde{\Sigma}) := \mathbf{A}^{M+k} \setminus Z(\tilde{\Sigma})$  by the group action  $\mathbf{D}(\tilde{\Sigma})$ :

$$\mathbf{P}_{\tilde{\Sigma}} = U(\tilde{\Sigma}) / \mathbf{D}(\tilde{\Sigma}),$$

with dim  $\mathbf{D}(\tilde{\Sigma}) = M - N$ , dim  $U(\tilde{\Sigma}) = M + k$ .

**Definition 1.** This group  $\mathbf{D}(\tilde{\Sigma}) \cong \mathbf{T}^{M-N}$  is called the Néron-Severi torus associated to the fan  $\tilde{\Sigma}$ .

We introduce the following polynomial (named phase function below),

(1.4) 
$$F(X, s, y) := y_1(f_1(X) + s_1) + \dots + y_k(f_k(X) + s_k),$$

that will play essential rôle in our further studies. In  $\S3$ , we treat the following affine variety defined for (1.4):

(1.5) 
$$Z_{F(x, \mathbf{1}, \mathbf{1}, y)+1} = \{(x, y) \in \mathbf{T}^{N+k}; F(x, \mathbf{1}, \mathbf{1}, y) + 1 = 0\}.$$

Further on we shall prepare several lemmata on combinatorics which are useful for the derivation of the discriminant loci equation. We denote by L the number of monomials in (X, s, y) that take part in the phase function (1.4) for (0.2). That is to say  $L = \sum_{q=1}^{k} (\tau_q + 1)$  Here we introduce new variables  $(T_1, \ldots, T_L) \in \mathbf{T}^L$  that satisfy the following relations,

(1.6) 
$$T_1 = y_1 x^{\vec{\alpha}_{1,1}}, T_2 = y_1 x^{\vec{\alpha}_{2,1}}, \dots, T_L = y_k s_k.$$

Each  $T_q$  represents the q-th monomial present in  $F(x, \mathbf{1}, s, y)$  (see (2.3) below). We will use the following matrix M(A) whose column is a vertex of the Newton polyhedron  $\Delta(F(x, \mathbf{1}, \mathbf{1}, y))$ ,

$$(1.7)$$
 M(A)

:=	$     \begin{bmatrix}       1 \\       0 \\       0     \end{bmatrix} $	$egin{array}{c} 1 \\ 0 \\ 0 \end{array}$	· · · · · · ·	$\begin{array}{c} 1 \\ 0 \\ 0 \end{array}$	$\begin{array}{c} 0 \\ 1 \\ 0 \end{array}$	$egin{array}{c} 0 \ 1 \ 0 \end{array}$	· · · · · · ·	$egin{array}{c} 0 \ 1 \ 0 \end{array}$	$\begin{array}{c} 0 \\ 0 \\ 1 \end{array}$	· · · · · · ·	0 0 0	0 0 0	· · · · · · ·	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
	: 0	: 0	:	: 0	: 0	: 0	:	: 0	: 0	:	: 1	: 1	:	: 1
	0	$\alpha_{111}$		$\alpha_{\tau_1 1 1}$	0	$\alpha_{121}$		$\alpha_{\tau_2 21}$	0	• • •	0	$\alpha_{1k1}$	• • •	$\alpha_{\tau_k k 1}$
	÷	:	÷	÷	÷	:	÷	÷	÷	÷	÷	÷	÷	:
	0	$\alpha_{11N}$	•••	$\alpha_{\tau_1 1N}$	0	$\alpha_{12N}$	•••	$\alpha_{\tau_2 2N}$	0	• • •	0	$\alpha_{1kN}$	•••	$\alpha_{\tau_k kN}$

Further we assume that  $\operatorname{rank}(\mathsf{M}(A)) = k + N$ . We always assume the inequality  $N + 2k \leq L$  for (0.2).

In this situation we can define a non-negative integer m as the minimal number of variables

(1.8) 
$$x'' = (x'_1, \dots, x'_m)$$

to make the number of variables present in the expression (1.4) equal to L. That is to say L = N + m + 2k. For example, the relation (1.6) may be modified into the following form:

(1.6)' 
$$T_1 = y_1 x_1' x^{\vec{\alpha}_{1,1}}, T_2 = y_1 x_2' x^{\vec{\alpha}_{2,1}}, \cdots,$$
  
 $T_{L-1} = y_k x_m' x^{\vec{\alpha}_{\tau_k,k}}, T_L = y_k s_k.$ 

In other words, proper addition of new variables  $x'' = (x'_1, \ldots, x'_m)$  to  $f_1(x), \ldots, f_k(x)$  makes the polynomial F(X, 0, y) quasihomogeneous. In this way we have

$$(1.9) M = N + m.$$

Further we shall consider a simple parametrisation of the variety

(1.10) 
$$Z_{F(X, s, y)} = \{(X, y) \in \mathbf{T}^{M+k}; F(X, s, y) = 0\}.$$

Namely we denote,

(1.11) 
$$\Xi := {}^{t}(x_{1}, \ldots, x_{N}, x'_{1}, \ldots, x'_{m}, s_{1}, \ldots, s_{k}, y_{1}, \ldots, y_{k}),$$
  
(1.12) 
$$\log T := {}^{t}(\log T_{1}, \ldots, \log T_{L})$$
  
(1.13) 
$$\log \Xi := {}^{t}(\log x_{1}, \ldots, \log x_{N}, \log x'_{1}, \ldots, \log x'_{m}, \log s_{1}, \ldots, \log s_{k}, \log y_{1}, \cdots, \log y_{k})$$

Then we have, for example, a linear equation equivalent to (1.6)' that can be written down as follows,

(1.14)  

$$\log T_{1} = \log y_{1} + \log x'_{1} + \langle \vec{\alpha}_{1,1}, \log x \rangle,$$

$$\log T_{2} = \log y_{1} + \log x'_{2} + \langle \vec{\alpha}_{2,1} \log x \rangle,$$

$$\vdots$$

$$\log T_{L-1} = \log y_{k} + \log x'_{m} + \langle \vec{\alpha}_{\tau_{k},k}, \log x \rangle,$$

$$\log T_{L} = \log y_{k} + \log s_{k}.$$

Let us write down the relation between (1.12) and (1.13) by means of a matrix  $L \in End(\mathbf{Z}^L)$ ,

(1.15) 
$$\operatorname{Log} T = \mathsf{L} \cdot \operatorname{Log} X.$$

Below the columns  $\vec{v}_i$  (resp.  $\vec{w}_i$ ) of the matrix L (resp. L<sup>-1</sup>) shall always be ordered in accordance with (1.11), (1.12), (1.13) unless otherwise stated.

For the polynomial mapping (0.2), the choice of monomials to be modified by supplementary variables is a bit delicate. Namely, we have to observe the following rules to avoid the degeneracy of the matrix L of the relation (1.15).

**Lemma 1.1.** For (0.2) and (1.8), we get a non-degenerate matrix L if we observe the following rules:

**a.** For the fixed index  $q \in \{1, \ldots, k\}$ , it is necessary to choose at least one of monomials  $x^{\vec{\alpha}_{i,q}}, 1 \leq i \leq \tau_q$  that remains without modification.

**b.** For the fixed index  $j \in \{1, ..., N\}$  it is necessary to choose at least one of monomials  $x^{\vec{\alpha}_{r,i}}$  such that  $\alpha_{r,i,j} \neq 0, 1 \leq i \leq k, 1 \leq r \leq \tau_i$ , that remains without modification.

We recall here the notion of non-degenerate hypersurface,

**Definition 2.** The hypersurface defined by a polynomial  $g(x) = \sum_{\alpha \in \text{supp}(g)} g_{\alpha} x^{\alpha} \in \mathbf{C}[x_1, \ldots, x_n]$  is said to be non-degenerate if and only if for any  $\xi \in \mathbf{R}^n$  the following inclusion takes place,

$$\left\{x \in \mathbf{C}^n; x_1 \frac{\partial g^{\xi}}{\partial x_1} = \dots = x_n \frac{\partial g^{\xi}}{\partial x_n} = 0\right\} \subset \{x \in \mathbf{C}^n; x_1 \cdots x_n = 0\}$$

where  $g^{\xi}(x) = \sum_{\{\beta; \langle \beta, \xi \rangle \leq \langle \alpha, \xi \rangle, \text{ for all } \alpha \in \text{supp}(g)\}} g_{\alpha} x^{\alpha}$ . We call the CI  $X_0$  for (0.2) non-degenerate if the hypersurface  $Z_{F(x, \mathbf{1}, 0, y)+1}$  is non-degenerate.

The following is an easy consequence of the above Definition.

**Proposition 1.2.** If the matrix L is non-degenerate, the hypersurface  $Z_{F(x, 1, 0, y)+1}$  and the CI  $X_0$  are non-degenerate in the sense of the Definition 2.

#### $\S 2.$ Horn's hypergeometric functions

From this section, we change the name of variables  $x'' = (x'_1, \ldots, x'_m)$ into  $s' := (s'_1, \ldots, s'_m)$ . We use both of the notations  $X = (x, x^n) = (x, s')$ .

Let us consider the Leray's coboundary (see [14]) to define the fibre integral,  $\gamma \subset H_N(\mathbf{T}^N \setminus \bigcup_{i=1}^k \{x \in \mathbf{T}^N \colon f_i(X) + s_i = 0\})$  such that  $\Re(f_i(X) + s_i)|_{\gamma} < 0$ . Further on central object of our study is the following fibre integral,

(2.1)  

$$I_{x^{\mathbf{i}},\gamma}^{\zeta}(s,\,s') = \int_{\gamma} (f_1(x,\,s') + s_1)^{-\zeta_1 - 1} \cdots (f_k(x,\,s') + s_k)^{-\zeta_k - 1} x^{\mathbf{i} + 1} \frac{dx}{x^1},$$

and its Mellin transform,

(2.2) 
$$M_{\mathbf{i},\gamma}^{\zeta}(\mathbf{z},\,\mathbf{z}') := \int_{\Pi} s^{\mathbf{z}} s'^{\mathbf{z}'} I_{x^{\mathbf{i}},\gamma}^{\zeta}(s,\,s') \frac{ds}{s^{\mathbf{1}}} \wedge \frac{ds'}{s'^{\mathbf{1}}},$$

for certain cycle II homologous to  $\mathbf{R}^{m+k}$  which avoids the singular loci of  $I_{x^i,\gamma}^{\zeta}(s, s')$  (cf. [11]). After Definition 1 above, we understand that  $s' \in \mathbf{D}(\tilde{\Sigma})$  is a variable on the Néron-Severi torus. Thus the fibre integral  $I_{x^i,\gamma}^{\zeta}(s, s')$  is a ramified function on the torus  $\mathbf{D}(\tilde{\Sigma}) \times \mathbf{T}^k$ . It is useful to understand the calculus of the Mellin transform in connection with the notion of the generalized HGF in the sense of Mellin-Barnes-Pincherle [1], [10]. After this formulation, the classical HGF of Gauss can be expressed by means of the integral,

$${}_{2}F_{1}(\alpha, \beta, \gamma|s) = \frac{1}{2\pi i} \int_{z_{0}-i\infty}^{z_{0}+i\infty} (-s)^{z} \frac{\Gamma(z+\alpha)\Gamma(z+\beta)\Gamma(-z)}{\Gamma(z+\gamma)} dz,$$
$$-\Re\alpha, -\Re\beta < z_{0}.$$

Next we modify the Mellin transform

$$M_{\mathbf{i},\gamma}^{\zeta}(\mathbf{z},\mathbf{z}')$$

$$= c(\zeta) \int_{S_{+}^{k-1}(w'')\times\gamma^{\Pi}} \frac{x^{\mathbf{i}}\omega^{\zeta}s^{\mathbf{z}-\mathbf{1}}s'^{\mathbf{z}'-\mathbf{1}}dx \wedge \Omega_{0}(\omega) \wedge ds \wedge ds'}{(\omega_{1}(f_{1}(X)+s_{1})+\dots+\omega_{k}(f_{k}(X)+s_{k}))^{\zeta_{1}+\dots+\zeta_{k}+k}}$$

$$= c(\zeta) \int_{\mathbf{R}_{+}} \sigma^{\zeta_{1}+\dots+\zeta_{k}+k} \frac{d\sigma}{\sigma} \int_{S_{+}^{k-1}(w'')} \omega^{\zeta}\Omega_{0}(\omega)$$

$$\int_{\gamma} x^{\mathbf{i}}dx \int_{\Pi} s^{\mathbf{z}}s'^{\mathbf{z}'}e^{\sigma(\omega_{1}(f_{1}(X)+s_{1})+\dots+\omega_{k}(f_{k}(X)+s_{k}))} \frac{ds}{s^{\mathbf{1}}} \frac{ds'}{s'^{\mathbf{1}}},$$

with  $c(\zeta) = \Gamma(\zeta_1 + \dots + \zeta_k + k) / (\Gamma(\zeta_1 + 1) \cdots \Gamma(\zeta_k + 1))$ . Here we made use of notations

$$S_{+}^{k-1}(w'') = \left\{ (\omega_1, \dots, \omega_k) \colon \omega_1^{\mathbf{w}''/w_1''} + \dots + \omega_k^{\mathbf{w}''/w_k''} = 1, \, \omega_\ell > 0$$
  
for all  $\ell, \, \mathbf{w}'' = \prod_{1 \le i \le k} w_i'' \right\}$ 

and  $\Omega_0(\omega)$  the (k-1) volume form on  $S^{k-1}_+(w'')$ ,

$$\Omega_0(\omega) = \sum_{\ell=1}^k (-1)^\ell w_\ell'' \omega_\ell d\omega_1 \wedge \cdots \wedge d\omega_k.$$

In the above transformation we used a classical interpretation of Dirac's delta function as a residue:

$$\int_{\gamma} \int_{\mathbf{R}_+} e^{y_j (f_j(X) + s_j)} y_j^{\zeta_j} dy_j \wedge dx = \Gamma(\zeta_j + 1) \int_{\gamma} (f_j(X) + s_j)^{-\zeta_j - 1} dx.$$

We introduce the notation  $\gamma^{\Pi} := \bigcup_{(s, s') \in \Pi} ((s, s'), \gamma)$ . One shall not confuse it with the thimble of Lefschetz, because  $\gamma^{\Pi}$  is rather a tube without thimble. We will rewrite the last expression,

$$\int_{(\mathbf{R}_{+})^{k} \times \gamma^{\Pi}} e^{\Psi(T)} x^{\mathbf{i}+1} y^{\zeta+1} s^{\mathbf{z}} s'^{\mathbf{z}'} \frac{dx}{x^{1}} \wedge \frac{dy}{y^{1}} \wedge \frac{ds}{s^{1}} \wedge \frac{ds'}{s'^{1}}$$

where

(2.3) 
$$\Psi(T) = T_1(X, s, y) + \dots + T_L(X, s, y) = F(X, s, y),$$

in which each term  $T_i(X, s, y)$  stands for a monomial in variables (X, s, y) of the phase function (1.4). We transform the above integral into the following form,

$$(2.4) \qquad \int_{(\mathbf{R}_{+})^{k} \times \gamma^{\Pi}} e^{\Psi(T(X, s, y))} x^{\mathbf{i}+\mathbf{1}} s^{\mathbf{z}} s'^{\mathbf{z}'} y^{\zeta+\mathbf{1}} \frac{dx}{x^{\mathbf{1}}} \wedge \frac{dy}{y^{\mathbf{1}}} \wedge \frac{ds}{s^{\mathbf{1}}} \wedge \frac{ds'}{s'^{\mathbf{1}}} = (\det \mathsf{L})^{-1} \int_{\mathsf{L}_{*}(\mathbf{R}_{+}^{k} \times \gamma^{\Pi})} e^{\sum_{a \in I} T_{a}} \prod_{a \in I} T_{a}^{\mathcal{L}_{a}(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)} \bigwedge_{a \in I} \frac{dT_{a}}{T_{a}} = (-1)^{\zeta_{1}+\dots+\zeta_{k}+k} (\det \mathsf{L})^{-1} \cdot \int_{-\mathsf{L}_{*}(\mathbf{R}_{+}^{k} \times \gamma^{\Pi})} e^{-\sum_{a \in I} T_{a}} \prod_{a \in I} T_{a}^{\mathcal{L}_{a}(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)} \bigwedge_{a \in I} \frac{dT_{a}}{T_{a}}.$$

Here  $L_*(\mathbf{R}_+{}^k \times \gamma^{\Pi})$  means the image of the chain in  $\mathbf{C}_X^M \times \mathbf{C}_s^k \times \mathbf{C}_y^k$  into that in  $\mathbf{C}_T^L$  induced by the transformation (1.15). We define  $-L_*(\mathbf{R}_+{}^k \times \gamma^{\Pi}) = \{(-T_1, \ldots, -T_L) \in \mathbf{C}^L; (T_1, \ldots, T_L) \in \mathbf{L}_*(\mathbf{R}_+{}^k \times \gamma^{\Pi}), \Re T_a < 0, a \in [1, L]\}$ . The second equality of (2.4) follows from Proposition 2.1, 3) below that can be proven in a way independent of the argument to derive (2.4). We will denote the set of columns and rows of the matrix  $\mathsf{L}$  by I,

$$I := \{1, \cdots, L\}.$$

Here we remember the relation L = N + m + 2k = M + 2k.

The following notion helps us to formulate the result in a compact manner.

**Definition 3.** A meromorphic function  $g(\mathbf{z}, \mathbf{z}')$  is called  $\Delta$ -periodic for  $\Delta \in \mathbf{Z}_{>0}$ , if

$$g(\mathbf{z},\,\mathbf{z}') = h(e^{2\pi\sqrt{-1\frac{z_1}{\Delta}}},\,\ldots\,,\,e^{2\pi\sqrt{-1\frac{z_k}{\Delta}}},\,e^{2\pi\sqrt{-1\frac{z_1'}{\Delta}}},\,\ldots\,,\,e^{2\pi\sqrt{-1\frac{z_m'}{\Delta}}}),$$

for some rational function  $h(\zeta_1, \ldots, \zeta_{k+m})$ .

For the simplicial CI (0.2) (i.e. we can construct F(X, s, y) for which the matrix L is non-degenerate), we have the following statement.

**Proposition 2.1.** 1) For any cycle

$$\Pi \in H_{k+m}(\mathbf{T}^{k+m} \setminus S.S.I_{x^{\mathbf{i}},\gamma}^{\zeta}(s,s'))$$

the Mellin transform (2.1) can be represented as a product of  $\Gamma$ -function factors up to a  $\Delta$ -periodic function factor  $g(\mathbf{z})$ ,

$$M_{\mathbf{i},\gamma}^{\zeta}(\mathbf{z},\,\mathbf{z}') = g(\mathbf{z}) \prod_{a \in I} \Gamma(\mathcal{L}_a(\mathbf{i},\,\mathbf{z},\,\mathbf{z}',\,\zeta)),$$

with

(2.5) 
$$\mathcal{L}_{a}(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)$$
  
=  $\frac{\sum_{j=1}^{N} A_{j}^{a}(i_{j}+1) + \sum_{j=1}^{m} C_{j}^{a} z_{j}' + \sum_{\ell=1}^{k} (B_{\ell}^{a} z_{\ell} + D_{\ell}^{a}(\zeta_{\ell}+1))}{\Delta}, \ a \in I.$ 

Here the following matrix  $\Delta^{-1}T = (L)^{-1}$  has integer elements,

(2.6) 
$${}^{t}\mathsf{T} = (A_{1}^{a}, \dots, A_{N}^{a}, C_{1}^{a}, \dots, C_{m}^{a}, B_{1}^{a}, \dots, B_{k}^{a}, D_{1}^{a}, \dots, D_{k}^{a})_{1 \le a \le L},$$

with  $\operatorname{GCD}(A_1^a, \ldots, A_N^a, C_1^a, \ldots, C_m^a, B_1^a, \ldots, B_k^a, D_1^a, \cdots, D_k^a) = 1$ , for all  $a \in [1, L]$ . In this way  $\Delta > 0$  is uniquely determined. The coefficients of (2.5) satisfy the following properties for each index  $a \in I$ :

 $\begin{array}{cc} \boldsymbol{a} \quad Either \ \mathcal{L}_a(\mathbf{i}, \ \mathbf{z}, \ \mathbf{z}', \ \zeta) = \frac{\Delta}{\Delta} z_{\ell}, \ i.e. \ A_1^a = \cdots = A_N^a = 0, \ B_1^a = \stackrel{\checkmark}{\overset{\lor}{\cdots}} = \\ B_k^a = 0, \ B_\ell^a = 1. \\ \boldsymbol{b} \quad Or \end{array}$ 

$$\mathcal{L}_{a}(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta) = \frac{\sum_{j=1}^{N} A_{j}^{a}(i_{j}+1) + \sum_{j=1}^{m} C_{j}^{a} z_{j}' + \sum_{\ell=1}^{k} B_{\ell}^{a}(z_{\ell} - \zeta_{\ell} - 1)}{\Delta}$$

2) For each fixed index  $1 \leq \ell \leq N, 1 \leq q \leq k, 1 \leq j \leq m$  the following equalities take place:

(2.7) 
$$\sum_{a \in I} A^a_{\ell} = 0, \ \sum_{a \in I} B^a_q = 0, \ \sum_{a \in I} C^a_j = 0.$$

3) The following relation holds among the linear functions  $\mathcal{L}_a, a \in I$ :

$$\sum_{a \in I} \mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta) = \zeta_1 + \dots + \zeta_k + k.$$

*Proof.* 1) First of all we recall the definition of the  $\Gamma$ -function,

$$\int_{C_a} e^{-T_a} T_a^{\sigma_a} \frac{dT_a}{T_a} = (1 - e^{2\pi i \sigma_a}) \Gamma(\sigma_a),$$

for the unique non-trivial cycle  $C_a$  that turns around  $T_a = 0$  with the asymptotes  $\Re T_a \to +\infty$ . We consider a transformation of the integral (2.4) induced by the change of cycle  $\lambda \colon C_a \to \lambda(C_a)$  defined by the relation,

$$\int_{\lambda(C_a)} e^{-T_a} T_a^{\sigma_a} \frac{dT_a}{T_a} = \int_{C_a} e^{-T_a} (e^{2\pi\sqrt{-1}}T_a)^{\sigma_a} \frac{dT_a}{T_a}.$$

By the aid of this action the chain  $L_*(\mathbf{R}_+^{\ k} \times \gamma^{\Pi})$  turns out to be homologous to a chain,

$$\sum_{(j_1^{(\rho)},\ldots,j_L^{(\rho)})\in[1,\,\Delta]^L} m_{j_1^{(\rho)},\ldots,j_L^{(\rho)}} \prod_{a=1}^k \lambda^{j_a^{(\rho)}}(\mathbf{R}_+) \prod_{a'=k+1}^L \lambda^{j_{a'}^{(\rho)}}(C_{a'}),$$

with  $m_{j_1^{(\rho)}, \dots, j_L^{(\rho)}} \in \mathbf{Z}$ . This fact explains the appearance of the factor

$$g(\mathbf{z}, \mathbf{z}') = \sum_{\substack{(j_1^{(\rho)}, \dots, j_L^{(\rho)}) \in [1, \Delta]^L \\ \cdot \prod_{a=1}^k e^{2\pi \sqrt{-1} j_a^{(\rho)} \mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)}} \\ \cdot \prod_{a'=k+1}^k e^{2\pi \sqrt{-1} j_{a'}^{(\rho)} \mathcal{L}_{a'}(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)} (1 - e^{2\pi \sqrt{-1} \mathcal{L}_{a'}(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)})$$

apart from the factors of type  $\Gamma(\bullet)$ .

In the sequel we analyze the  $\Gamma$ - function factors that arise from the integral (2.4). To this end, we represent the matrix L (resp. L<sup>-1</sup>) as a set of L columns properly ordered:

(2.8) 
$$\mathsf{L} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_L), \ \mathsf{L}^{-1} = (\vec{w}_1, \vec{w}_2, \dots, \vec{w}_L), \vec{w}_a = {}^t(w_{a, 1}, \dots, w_{a, L}).$$

The interior product of vectors  $(\mathbf{i} + \mathbf{1}, \mathbf{z}, \mathbf{z}', \zeta + \mathbf{1})$  and  $\vec{w}_a$  defines the linear function in question:

(2.9) 
$$\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta) = (\mathbf{i} + \mathbf{1}, \mathbf{z}, \mathbf{z}', \zeta + \mathbf{1}) \cdot \vec{w}_a.$$

The vector columns of  $L^{-1}$  are divided into 3 groups:

1 the columns with all formally non-zero elements.

**2** with unique non-zero element (= 1) that produces  $z_i$ ,  $1 \le i \le k$  and  $z'_i$ ,  $1 \le j \le m$  in (2.9).

**3** with the non-zero elements that produce a function linear in  $\zeta + 1$ ,  $\mathbf{i} + \mathbf{1}$  after (2.5).

In the further argument, only the first two groups of columns are important.

The column that corresponds to  $\log s_i$  of L contains the unique nonzero element (= 1) at the position  $\tau_1 + \cdots + \tau_i + i$ . Meanwhile the column of L that corresponds to the variable  $\log x'_{\ell}$  consists also of an unique non-zero element (= 1) outside the positions  $\tau_1 + \cdots + \tau_i + i$ ,  $(1 \le i \le k)$ . Let us denote this correspondence by

$$\vec{v}_{\rho(i)} = {}^{t}(0, \ldots, 0, \stackrel{\sigma(i)}{\overset{\vee}{\dots}}, 1, 0, \ldots, 0),$$

that yields in  $L^{-1}$ ,

$$\vec{w}_{\sigma(i)} = {}^{t}(0, \ldots, 0, \overset{\rho(i)}{\overset{\vee}{\ldots}}, 1, 0, \ldots, 0).$$

Here the mappings  $\rho$ ,  $\sigma$ :  $\{N + 1, \ldots, M + k\} \to I$  are injections that send the number of columns corresponding to the variables s, x' to the total set of indices I. We divide the columns of  $\mathsf{L}^{-1}$  into k groups  $\Lambda_1, \ldots, \Lambda_k \subset I$  each of which corresponds to  $\Lambda_b = \{\tau_1 + \cdots + \tau_{b-1} + b, \cdots, \tau_1 + \cdots + \tau_b + b\} \subset I$ . For this group, one can claim following assertions. a) The column

$$\vec{v}_{M+k+b} = {}^{t}(0, \dots, 0, 0, \dots, 0, 1, 1, \dots, \dots, 1, \dots, 1, 0, \dots, 0),$$

with  $\tau_b + 1$ ,  $(1 \le b \le k)$  non-zero elements (= 1). b) For the vectors  $\vec{w}_a$  of the case **1** above,

(2.10) 
$$\sum_{a \in \Lambda_b} w_{a,j} = 0 \quad \text{if} \quad j \neq M + k + b, \ 1 \le b \le k,$$

and there exists another vector of the same group  $\Lambda_b$  that satisfies:

(2.11) 
$$w_{\sigma(i),j} = \delta_{\rho(i),j},$$

where  $\delta_{\cdot,*}$  is the Kronecker delta symbol. The vector (2.11) corresponds to the group **2**.

Thus the columns of the group 2 (resp. 1) give rise to the linear functions of the group  $\mathbf{b}$  (resp.  $\mathbf{a}$ ).

2) The 1-st, ..., M + k-th vector rows of the matrix  $\mathsf{L}^{-1}$  are orthogonal to the vectors  $\vec{v}_{M+k+1}, \cdots, \vec{v}_{M+2k}$  above. This means the relations (2.7).

3) The statement can be deduced from 2). Q.E.D.

In view of the Proposition 2.1, we introduce the subsets of indices  $a \in \{1, 2, \ldots, M\}$  as follows.

**Definition 4.** The subset  $I_q^+ \subset \{1, 2, \ldots, k\}$  (resp.  $I_q^-, I_q^0$ ) consists of the indices *a* such that the coefficient  $B_q^a$  of  $\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)$  (2.5) is positive (resp. negative, zero). Analogously we define the subset  $J_r^+ \subset \{1, 2, \ldots, m\}$  (resp.  $J_r^-, J_r^0$ ) that consists in such indices *a* that the coefficient  $C_r^a$  of  $\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)$  is positive (resp. negative, zero).

To assure the convergence of the Mellin inverse transform of  $M_{\mathbf{i},\gamma}^{\zeta}(\mathbf{z}, \mathbf{z}')$  from (2.1) in a properly chosen angular sector in the variables  $(s, s') \in \mathbf{C}^{k+m}$ , we shall verify that the Mellin transform  $M_{\mathbf{i},\gamma}^{\zeta}(\mathbf{z}, \mathbf{z}')$  admits the following estimation modulo multiplication by a  $\Delta$ -periodic function  $g(\mathbf{z}, \mathbf{z}')$ .

$$|M_{\mathbf{i},\gamma}^{\zeta}(\mathbf{z}, \mathbf{z}')| < C_{\mathbf{i}} \exp(-\epsilon |\operatorname{Im} z|) \text{ while}$$
  
 $\operatorname{Im} z \to \infty, \text{ in a sector of aperture} < 2\pi$ 

for certain  $\epsilon > 0$ ,

Here we remember an elementary lemma for the integral:

(2.12) 
$$\int_{z_0-i\infty}^{z_0+i\infty} s^z g(z) \prod_{j=1}^{\nu} \frac{\Gamma(z+\alpha_j)}{\Gamma(z+\rho_j)} dz.$$

**Lemma 2.2.** If one chooses one of the following functions  $g^+(z)$ (resp.  $g^-(z)$ ) in terms of  $g(\mathbf{z}, \mathbf{z}')$ , then the integrand of (2.12) is exponentially decaying as Im z tends to  $\infty$  within the sector  $0 \leq \arg z < 2\pi$ , (resp.  $-\pi \leq \arg z < \pi$ .)

$$g^{\pm}(z) = 1 + e^{\pm 2\pi i\beta_{\nu}} \prod_{j=1}^{\nu} \frac{\sin 2\pi (z+\alpha_j)}{\sin 2\pi (z+\rho_j)},$$

with  $\beta_{\nu} = -1 + \sum_{j=1}^{\nu} (\rho_j - \alpha_j)$ 

*Proof.* It is enough to recall

$$\prod_{j=1}^{\nu} \frac{\Gamma(x+iy+\alpha_j)}{\Gamma(x+iy+\rho_j)} \to \text{const.} |y|^{-(\beta_{\nu}+1)}$$

while  $y \to \pm \infty$ . Here we used the formula of Binet:

$$\log \Gamma(z+a) = \log \Gamma(z) + a \log z - \frac{a-a^2}{2z} + \mathcal{O}(|z|^{-2})$$

if |z| >> 1, The factor  $|s^{-(x+iy)}| = r^{-x}e^{\theta y}$ , for  $s = re^{i\theta}$  gives the exponentially decreasing contribution in each cases. Q.E.D.

Let us introduce a simplified notation,

$$\mathcal{L}_{j}(z) = A_{j1}z_{1} + A_{j2}z_{2} + \dots + A_{jk}z_{k} + A_{j0}, \ 1 \le j \le p,$$
  
$$\mathcal{M}_{j}(z) = B_{j1}z_{1} + B_{j2}z_{2} + \dots + B_{jk}z_{k} + B_{j0}, \ 1 \le j \le r.$$

Lemma 2.3. The sufficient conditions so that

(2.13) 
$$\int_{\Pi} s^{\mathbf{z}} g(z) \frac{\prod_{j=1}^{p} \Gamma(\mathcal{L}_{j}(z))}{\prod_{j=1}^{r} \Gamma(\mathcal{M}_{j}(z))} dz_{1} \wedge \dots \wedge dz_{k}$$

defines a polynomially increasing function with g(z) a properly chosen  $\Delta$ -periodic function (including the infinity  $\infty$ ) are the following.

i) For every i > 0

$$\sum_{j=1}^{p} A_{j,i} = \sum_{j=1}^{r} B_{j,i}$$

ii) The real number

$$\alpha = \min_{z \in S^{k-1}} \left( \sum_{j=1}^{p} |\mathcal{L}_j(z) - A_{j0}| - \sum_{j=1}^{r} |\mathcal{M}_j(z) - B_{j0}| \right)$$

is non negative.

To see the exponential decay property of the integrand, one shall make reference to Nörlund's trick [10]. Further we apply the Stirling's formula on the asymptotic behaviour of the  $\Gamma$ -function (Whittaker-Watson, Chapter XII, Example 44).

If we apply this lemma to our integral, we see that there exists a cycle  $\check{\Pi}$  such that

(2.14) 
$$I_{\mathbf{x}^{\mathbf{i}},\gamma}^{\zeta}(s,s') = \int_{\tilde{\Pi}} g(\mathbf{z},\mathbf{z}') \frac{\prod_{a \in I_q^+ \cup I_q^0} \Gamma(\mathcal{L}_a(\mathbf{i},\mathbf{z},\mathbf{z}',\zeta))}{\prod_{\bar{a} \in I_q^-} \Gamma(1-\mathcal{L}_{\bar{a}}(\mathbf{i},\mathbf{z},\mathbf{z}',\zeta))} s^{-\mathbf{z}} s'^{-\mathbf{z}'} d\mathbf{z} \wedge d\mathbf{z}',$$

with a  $\Delta$ -periodic function  $g(\mathbf{z}, \mathbf{z}')$  rational with respect to  $e^{2\pi\sqrt{-1}\mathcal{L}_a(\mathbf{i}, \mathbf{z}, \mathbf{z}', \zeta)}, a \in I$ . Here we remember the relation  $e^{\pi\sqrt{-1}z}\Gamma(z)$  $\Gamma(1-z) = \pi/(1-e^{-2\pi\sqrt{-1}z})$ . Thus we get the theorem on the Horn type system.

**Theorem 2.4.** The integral  $I_{x^{i},\gamma}^{\zeta}(s, s')$  satisfies the hypergeometric system of Horn type as follows:

$$(2.15)_{1} \qquad L_{q,\mathbf{i}}(\vartheta_{s},\,\vartheta_{s'}s,\,s',\,\zeta)I_{x^{\mathbf{i}},\,\gamma}^{\zeta}(s,\,s')$$
$$:= \left[P_{q,\mathbf{i}}(\vartheta_{s},\,\vartheta_{s'},\,\zeta) - s_{q}^{\Delta}Q_{q,\mathbf{i}}(\vartheta_{s},\,\vartheta_{s'},\,\zeta)\right]I_{x^{\mathbf{i}},\,\gamma}^{\zeta}(s,\,s') = 0, \ 1 \le q \le k$$

with

$$(2.15)_2 \qquad P_{q,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta) = \prod_{a\in I_q^+} \prod_{j=0}^{B_q^a-1} \left( \mathcal{L}_a(\mathbf{i},\,-\vartheta_s,\,-\vartheta_{s'},\,\zeta) + j \right),$$

$$(2.15)_3 \qquad Q_{q,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta) = \prod_{\bar{a}\in I_q^-} \prod_{j=0}^{-B_q^{\bar{a}}-1} \left(\mathcal{L}_{\bar{a}}(\mathbf{i},\,-\vartheta_s,\,-\vartheta_{s'},\,\zeta)+j\right),$$

where  $I_q^+$ ,  $I_q^-$ ,  $1 \le q \le k$  are the sets of indices defined in Definition 4.

$$(2.15)_{4} \qquad L'_{r,\mathbf{i}}(\vartheta_{s},\,\vartheta_{s'},\,s,\,s',\,\zeta)I^{\zeta}_{x^{\mathbf{i}},\,\gamma}(s,\,s')$$
$$:= \left[P'_{r,\mathbf{i}}(\vartheta_{s},\,\vartheta_{s'},\,\zeta) - s'_{r}{}^{\Delta}Q'_{r,\mathbf{i}}(\vartheta_{s},\,\vartheta_{s'},\,\zeta)\right]I^{\zeta}_{x^{\mathbf{i}},\,\gamma}(s,\,s') = 0, \ 1 \le q \le k$$

$$(2.15)_5 \qquad P_{r,\mathbf{i}}'(\vartheta_s,\,\vartheta_{s'},\,\zeta) = \prod_{a\in J_r^+} \prod_{j=0}^{C_r^a-1} \left(\mathcal{L}_a(\mathbf{i},\,-\vartheta_s,\,-\vartheta_{s'},\,\zeta)+j\right)$$

$$(2.15)_6 \qquad Q'_{r,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta) = \prod_{\bar{a}\in J_r^-} \prod_{j=0}^{-C_r^{\bar{a}}-1} \left(\mathcal{L}_{\bar{a}}(\mathbf{i},\,-\vartheta_s,\,-\vartheta_{s'},\,\zeta)+j\right).$$

where  $J_r^+$ ,  $J_r^-$ ,  $1 \leq r \leq m$  are the sets of indices defined in the Definition 4. The degree of two operators  $P_{q,\mathbf{i}}(\vartheta_s, \vartheta_{s'}, \zeta)$ ,  $Q_{q,\mathbf{i}}(\vartheta_s, \vartheta_{s'}, \zeta)$  are equal. Namely,

(2.16) 
$$\deg P_{q,\mathbf{i}}(\vartheta_{s},\,\vartheta_{s'},\,\zeta) = \sum_{a\in I_{q}^{+}} B_{q}^{a} = -\sum_{\bar{a}\in I_{q}^{-}} B_{q}^{\bar{a}} = \deg Q_{q,\mathbf{i}}(\vartheta_{s},\,\vartheta_{s'},\,\zeta).$$

Analogously,

$$\deg P'_{r,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta) = \sum_{a\in J_r^+} C_r^a = -\sum_{\bar{a}\in J_r^-} C_r^{\bar{a}} = \deg Q'_{r,\mathbf{i}}(\vartheta_s,\,\vartheta_{s'},\,\zeta).$$

The proof is mainly based on the Proposition 2.1. To deduce (2.15) from the Mellin transform  $M_{\mathbf{i},\gamma}^{\zeta}(\mathbf{z}, \mathbf{z}')$  we use the following well known recurrence relation:

$$\Gamma\left(\frac{\alpha(n+\Delta)}{\Delta}+\zeta\right)$$
  
=  $\Gamma\left(\frac{\alpha n}{\Delta}+\zeta\right)\left(\frac{\alpha n}{\Delta}+\zeta\right)\left(\frac{\alpha n}{\Delta}+1+\zeta\right)\cdots\left(\frac{\alpha n}{\Delta}+\alpha-1+\zeta\right),$ 

if  $\alpha > 0$  a positive integer.

$$\Gamma\left(\frac{\alpha(n+\Delta)}{\Delta}+\zeta\right)$$
  
=  $\Gamma\left(\frac{\alpha n}{\Delta}+\zeta\right)\left(\frac{\alpha n}{\Delta}+\zeta-1\right)^{-1}\left(\frac{\alpha n}{\Delta}+\zeta-2\right)^{-1}\cdots\left(\frac{\alpha n}{\Delta}+\zeta+\alpha\right)^{-1},$ 

if  $\alpha < 0$  a negative integer.

The evident compatibility (i.e. integrability) of the above system  $(2.15)_*$  in the sense of Ore-Sato ([12]) can be formulated like the following cocycle condition. To state the proposition we introduce the notation  $\mathbf{z} + \Delta e_r = (z_1, \ldots, z_{r-1}, z_r + \Delta, z_{r+1}, \ldots, z_k).$ 

**Proposition 2.5.** The rational expression

(2.17) 
$$R_q(\mathbf{z}, \mathbf{z}') = \frac{P_{q, \mathbf{i}}(\mathbf{z}, \mathbf{z}', \zeta)}{Q_{q, \mathbf{i}}(\mathbf{z} + \Delta e_q, \mathbf{z}', \zeta)},$$

defined for the operators  $(2.15)_2$ ,  $(2.15)_3$  satisfies the following relation:

(2.18) 
$$R_q(\mathbf{z} + \Delta e_r, \mathbf{z}')R_r(\mathbf{z}, \mathbf{z}') = R_r(\mathbf{z} + \Delta e_q, \mathbf{z}')R_q(\mathbf{z}, \mathbf{z}'), \quad q, r = 1, \dots, k.$$

Similarly for

(2.19) 
$$R'_{\kappa}(\mathbf{z}, \mathbf{z}') = \frac{P'_{\kappa, \mathbf{i}}(\mathbf{z}, \mathbf{z}', \zeta)}{Q'_{\kappa, \mathbf{i}}(\mathbf{z}, \mathbf{z}' + \Delta e'_{\kappa}, \zeta)},$$

satisfies the following relation:

(2.20) 
$$\begin{aligned} R'_{\kappa}(\mathbf{z},\,\mathbf{z}'+\Delta e'_{\rho})R'_{\rho}(\mathbf{z},\,\mathbf{z}') \\ &= R'_{\rho}(\mathbf{z},\,\mathbf{z}'+\Delta e'_{\kappa})R'_{\kappa}(\mathbf{z},\,\mathbf{z}'), \quad \kappa,\,\rho=1,\,\ldots,\,m. \end{aligned}$$

**Remark 1.** As  $m = \dim \mathbf{D}(\tilde{\Sigma})$  (see (1.3)), one can consider that the above system  $(2.15)_*$  is defined on  $\mathbf{T}^k \times \mathbf{D}(\tilde{\Sigma})$  for  $\mathbf{D}(\tilde{\Sigma})$ : the Néron-Severi torus associated to the fan  $\tilde{\Sigma}$ .

We introduce here the main object of our study: the discriminantal loci of the CI defined by the polynomials  $f_1(x, s')+s_1, \ldots, f_k(x, s')+s_k$ .

(2.21) 
$$D_{s,s'} := \{(s,s') \in \mathbf{T}^{k+m};$$
  
 $f_1(x,s') + s_1$   
 $= \cdots$   
 $= f_k(x,s') + s_k, \quad \operatorname{rank} \begin{pmatrix} \operatorname{grad}_x f_1(x,s') \\ \vdots \\ \operatorname{grad}_x f_k(x,s') \end{pmatrix} < k,$   
for certain  $x \in \mathbf{T}^N \}.$ 

As it is easy to see [5],  $D_{s,s'}$  coincides with the discriminantal loci of F(x, s', s, y).

Let us define the  $\Delta$ -th roots of rational functions associated with the linear functions (2.5) as follows.

$$(2.22) \quad \psi_{q}(\mathbf{z}, \, \mathbf{z}') = \left( \frac{\prod_{a \in I_{q}^{+}} (\sum_{\ell=1}^{k} B_{\ell}^{a} z_{\ell} + \sum_{j=1}^{m} C_{j}^{a} z_{j}')^{B_{q}^{a}}}{\prod_{\bar{a} \in I_{q}^{-}} (\sum_{\ell=1}^{k} B_{\ell}^{\bar{a}} z_{\ell} + \sum_{j=1}^{m} C_{j}^{\bar{a}} z_{j}')^{-B_{q}^{\bar{a}}}} \right)^{\frac{1}{\Delta}},$$

$$(2.23) \quad \phi_{r}(\mathbf{z}, \, \mathbf{z}') = \left( \frac{\prod_{a \in J_{r}^{+}} (\sum_{\ell=1}^{k} B_{\ell}^{a} z_{\ell} + \sum_{j=1}^{m} C_{j}^{a} z_{j}')^{C_{r}^{a}}}{\prod_{\bar{a} \in J_{r}^{-}} (\sum_{\ell=1}^{k} B_{\ell}^{\bar{a}} z_{\ell} + \sum_{j=1}^{m} C_{j}^{\bar{a}} z_{j}')^{-C_{r}^{\bar{a}}}} \right)^{\frac{1}{\Delta}},$$

$$(2.24) \quad h: \mathbf{C}^{k+m} \setminus \{0\} \to (\mathbf{C}^{\times})^{k+m},$$

(2.24) *n*. C ({0}) 
$$\rightarrow$$
 (C) ,  
( $\mathbf{z}, \mathbf{z}'$ )  $\rightarrow$  ( $\psi_1(\mathbf{z}, \mathbf{z}'), \ldots, \psi_k(\mathbf{z}, \mathbf{z}'), \phi_1(\mathbf{z}, \mathbf{z}'), \ldots, \phi_m(\mathbf{z}, \mathbf{z}')$ ).

By virtue of the property (2.7), the rational function  $\psi_q(\mathbf{z}, \mathbf{z}')^{\Delta}$  (resp.  $\phi_r(\mathbf{z}, \mathbf{z}')^{\Delta}$ ) is of weight zero with respect to the variables  $(\mathbf{z}, \mathbf{z}')$  and thus it is possible to consider the mapping h defined on  $\mathbf{C}P^{k+m-1}$  instead of  $\mathbf{C}^{k+m}$ .

Let  $\Delta_f(s, s')$  be a polynomial that defines the discriminantal loci  $D_{s,s'}$  without multiplicity.

**Theorem 2.6.** The image of  $h: \mathbb{CP}^{k+m-1} \to (\mathbb{C}^{\times})^{k+m}$  is identified with the discriminantal loci  $D_{s,s'}$  if we choose a proper  $\Delta$ -th branch in the equations (2.2), (2.3).

*Proof.* From the system of equations (2.15) we see that  $D_{s,s'}$  is contained in the set:

(2.25) 
$$\nabla_{s,s'} := \{(s,s') \in \mathbf{T}^{k+m}; \\ \sigma(L_{q,-1})(s\xi, s'\xi', s, s', -1) = 0, \ 1 \le q \le k, \\ \sigma(L'_{r,-1})(s\xi, s'\xi', s, s', -1) = 0, \ 1 \le r \le m \\ \text{for some}(\xi, \xi') \in \mathbf{T}^{k+m}\}.$$

here we use the notation

$$(s\xi, s'\xi') = (s_1\xi_1, \ldots, s_k\xi_k, s'_1\xi'_1, \ldots, s'_m\xi'_m).$$

The existence of  $(\xi, \xi') \in \mathbf{T}^{k+m}$  in (2.25) is equivalent to the existence of  $(\mathbf{z}, \mathbf{z}') = (s\xi, s'\xi') \in \mathbf{T}^{k+m}$ . Thus the set  $\nabla_{s,s'}$  admits a representation,

$$\left\{ (s, s') \in \mathbf{T}^{k+m}; \begin{array}{l} s_q^{\Delta} = \frac{P_{q,-1}(\mathbf{z}, \mathbf{z}', -\mathbf{1})}{Q_{q,-1}(\mathbf{z}, \mathbf{z}', -\mathbf{1})}, & 1 \le q \le k, \\ (s, s') \in \mathbf{T}^{k+m}; \\ (s'_r)^{\Delta} = \frac{P'_{r,-1}(\mathbf{z}, \mathbf{z}', -\mathbf{1})}{Q'_{q,-1}(\mathbf{z}, \mathbf{z}', -\mathbf{1})}, & 1 \le r \le m \end{array} \right\}.$$

While after Theorem 2.1, a) and Remark 2.4 of [7], this set  $\nabla_{s,s'}$  coincides with  $D_{s,s'}$  if  $\Delta = 1$ . As for the case  $\Delta > 1$ , it is natural to consider the  $\Delta$ -covering  $\tilde{h}$  of the mapping h,

$$\tilde{h}: \mathbf{CP}^{k+m-1} \to (\mathbf{C}^{\times})^{k+m},$$

while the branch of the image of h shall be specified in a proper way. To do that we remark that  $h(\mathbf{CP}^{k+m-1}) \subset \nabla_{s,s'}$  where the difference  $\nabla_{s,s'} \setminus h(\mathbf{CP}^{k+m-1})$  consists of the divisors that arise from the  $\Delta$ -branching effect  $\tilde{h}(\mathbf{CP}^{k+m-1})$ . In considering  $D_{s,s'}$  we shall discard the superfluous  $\Delta$ -branching effect  $\tilde{h}(\mathbf{CP}^{k+m-1}) \setminus h(\mathbf{CP}^{k+m-1})$ . Q.E.D. The mapping (2.24) is nothing but the inverse mapping of the logarithmic Gauss map;

$$D_{s,s'} \to \mathbf{C}P^{k+m-1},$$

$$(s,s') \to \left(s_1 \frac{\partial}{\partial s_1} \Delta_f(s,s') \colon \dots \colon s_k \frac{\partial}{\partial s_k} \Delta_f(s,s') \atop s'_1 \frac{\partial}{\partial s'_1} \Delta_f(s,s') \colon \dots \colon s'_m \frac{\partial}{\partial s'_m} \Delta_f(s,s')\right).$$

This is a direct consequence of the cocycle property (2.18), (2.20) of the operators  $L_{q,\mathbf{i}}(\vartheta_s, \vartheta_{s'}, s, s', \zeta)$  and  $L'_{r,\mathbf{i}}(\vartheta_s, \vartheta_{s'}, s, s', \zeta)$ , see [7], Theorem 2.1, b).

## §3. A-Hypergeometric function of Gel'fand-Kapranov-Zelevinski

Let us consider the set of polynomials with deformation parameter coefficients  $(a_{0,1}, \ldots, a_{\tau_k,k})$  associated to the polynomial system (0.2),

(3.1) 
$$\bar{f}_{\ell}(x, \mathbf{a}) = a_{1,\ell} x^{\vec{\alpha}_{1,\ell}} + \dots + a_{\tau_{\ell},\ell} x^{\vec{\alpha}_{\tau_{\ell},\ell}} + a_{0,\ell}. \ 1 \le \ell \le k.$$

For the sake of simplicity we will further make use of the notation  $\mathbf{a} := (a_{0,1}, \ldots, a_{\tau_k,k}) \in \mathbf{T}^L$ . We consider the Leray coboundary  $\partial \gamma_{\mathbf{a}}$  of a cycle  $\gamma_{\mathbf{a}} \in H_n(X_{\mathbf{a}}, \mathbf{Z})$  of the CI  $X_{\mathbf{a}} = \{x \in \mathbf{T}^N; \bar{f}_1(x, \mathbf{a}) = \cdots = \bar{f}_k(x, \mathbf{a}) = 0\}.$ 

Then we can define the A-hypergeometric function  $\Phi_{x^{\mathbf{i}},\gamma_{\mathbf{a}}}^{\zeta}(a_{0,1},\ldots,a_{\tau_k,k})$  introduced by Gel'fand-Zelevinski-Kapranov [4] associated to the polynomials,

$$f_{\ell}(x) = x^{\vec{\alpha}_{1,\ell}} + \dots + x^{\vec{\alpha}_{\tau_{\ell},\ell}}, \quad 1 \le \ell \le k,$$
  
$$x^{\mathbf{i}} = x_{1}^{i_{1}} \cdots x_{N}^{i_{N}}, \quad x^{\vec{\alpha}_{j,\ell}} = x_{1}^{\alpha_{j,\ell,1}} \cdots x_{N}^{\alpha_{j,\ell,N}},$$

Namely it is defined as a kind of multiple residue along  $X_{\mathbf{a}}$ ,

(3.2) 
$$\Phi_{x^{\mathbf{i}},\gamma_{\mathbf{a}}}^{\zeta}(a_{0,1},\ldots,a_{\tau_{k},k}) := \int_{\partial\gamma_{\mathbf{a}}} \prod_{\ell=1}^{k} \bar{f}_{\ell}(x,\mathbf{a})^{-\zeta_{\ell}-1} x^{\mathbf{i}+1} \frac{dx}{x^{\mathbf{i}}}.$$

We impose here the non-degeneracy condition of the Definition 2 for the complete intersection  $X_s$  after the procedure described in §1.

In the sequel we consider a lattice  $\Lambda \subset \mathbf{Z}^L$  of *L*-vectors defined by the system of following linear equations:

$$\sum_{q=1}^{\tau_q} b(j, q, \nu) = 0, \ 1 \le q \le k,$$
$$\sum_{q=1}^k \sum_{j=1}^{\tau_q} \alpha_{jq\ell} b(j, q, \nu) = 0, \ 1 \le \ell \le N$$

Here we denoted by  $(b(0, 1, \nu), \dots, b(\tau_1, 1, \nu), b(0, 2, \nu), \dots, b(\tau_2, 2, \nu), \dots, b(\tau_k, k, \nu)), 1 \le \nu \le m + k$ , a **Z** basis of  $\Lambda$ .

For the subset  $\mathbf{K} \subset \{(0, 1), \ldots, (k, \tau_k)\}$  such that the columns  $\vec{m}_{j,q}(A), (j, q) \in \mathbf{K}$  of the matrix  $\mathsf{M}(A)$  (1.7) span  $\mathbf{R}^{N+k}$  over  $\mathbf{R}$  and  $|\mathbf{K}| = N+k$  we define the set of indices (a generalisation of the Frobenius' method) after [4],

$$\Pi((\zeta + \mathbf{1}, \mathbf{i} + \mathbf{1}), \mathbf{K}) = \{((\lambda(0, 1, \nu), \dots, \lambda(\tau_1, 1, \nu), \dots, \lambda(\tau_k, k, \nu)))\}_{1 \le \nu \le |\det(\vec{m}_{j, q}(A))_{(j, q) \in \mathbf{K}}|},$$

which satisfy the following system of equations,

$$\sum_{j=0}^{\tau_{\nu}} \lambda(j, q, \nu) + \zeta_q + 1 = 0, \ 1 \le q \le k,$$
$$\sum_{q=1}^k \sum_{j=1}^{\tau_q} \alpha_{jq\ell} \lambda(j, q, \nu) - (i_\ell + 1) = 0, \ 1 \le \ell \le N.$$

Let T be a triangulation of the Newton polyhedron  $\Delta(F(x, \mathbf{1}, \mathbf{1}, y) + 1)$ for  $F(x, \mathbf{1}, \mathbf{1}, y)$  of (1.4) after the definition [4], 1.2. Here we impose that  $\lambda(j, q, \nu) \in \mathbf{Z}$  for  $(j, q) \notin \mathbf{K}$ . Let  $\mathbf{K}_1, \mathbf{K}_2 \in T$  be two different simplices of the triangulation T. We suppose that  $\vec{\lambda}(\nu_p) :=$  $(\lambda(0, 1, \nu_p), \ldots, \lambda(k, \tau_k, \nu_p)) \in \Pi((\zeta + \mathbf{1}, \mathbf{i} + \mathbf{1}), \mathbf{K}_p), \lambda(j, q, \nu_p) \in \mathbf{Z}$ for  $(j, q) \notin \mathbf{K}_p, (p = 1, 2)$  with  $1 \leq \nu_p \leq |\det(\vec{m}_{\rho}(A))_{\rho \in \mathbf{K}_p}|$ . We introduce the condition of T-non-resonance on  $(\zeta + \mathbf{1}, \mathbf{i} + \mathbf{1})$ 

(3.3) 
$$(\lambda(0, 1, \nu_1), \ldots, \lambda(k, \tau_k, \nu_1)) \not\equiv (\lambda(0, 1, \nu_2), \ldots, \lambda(k, \tau_k, \nu_2))$$
  
mod  $\Lambda$ ,

for any pair  $\vec{\lambda}(\nu_p) = (\lambda(0, 1, \nu_p), \dots, \lambda(k, \tau_k, \nu_p)) \in \Pi((\zeta + 1, \mathbf{i} + 1), \mathbf{K}_p), p = 1, 2$ . An adaptation of Theorem 3 [4] to our situation can be formulated as follows.

**Theorem 3.1.** 1) The A-HGF  $\Phi_{x^{\mathbf{i}}, \gamma_a}^{\zeta}(\mathbf{a})$  satisfies the following system of equations.

$$(3.4) \qquad \left(\sum_{j=0}^{\gamma_q} a_{ji} \frac{\partial}{\partial a_{ji}} + \zeta_q + 1\right) \Phi_{x^{\mathbf{i}}, \gamma_a}^{\zeta}(\mathbf{a}) = 0, \ 1 \le q \le k,$$

$$\left(\sum_{1 \le q \le k, \ 1 \le j \le \tau_q} \alpha_{jq1} a_{jq} \frac{\partial}{\partial a_{jq}} - (i_1 + 1)\right) \Phi_{x^{\mathbf{i}}, \gamma_a}^{\zeta}(\mathbf{a}) = \cdots$$

$$= \left(\sum_{1 \le q \le k, \ 1 \le j \le \tau_q} \alpha_{jqN} a_{jq} \frac{\partial}{\partial a_{jq}} - (i_N + 1)\right) \Phi_{x^{\mathbf{i}}, \gamma_a}^{\zeta}(\mathbf{a}) = 0,$$

$$\left(\prod_{\{(j,q); b(j,q,\nu) > 0\}} \left(\frac{\partial}{\partial a_{jq}}\right)^{b(j,q,\nu)} - \prod_{\{(j,q); b(j,q,\nu) < 0\}} \left(\frac{\partial}{\partial a_{jq}}\right)^{-b(j,q,\nu)}\right)$$

$$\cdot \Phi_{x^{\mathbf{i}}, \gamma_a}^{\zeta}(\mathbf{a}) = 0, \ 1 \le \nu \le L - (k+N).$$

2) The dimension of solutions of the system above at a generic point  $\mathbf{a} \in \mathbf{T}^L$  is equal to

$$(N+k)! \operatorname{vol}_{N+k} \Delta(F(x, \mathbf{1}, \mathbf{1}, y) + 1) = |\chi(Z_{F(x, \mathbf{1}, \mathbf{1}, y)})|$$

if the T-non-resonant condition (3.3) is satisfied.

In the sequel we shuffle the variables  $\mathbf{a} = (a_{0,1}, \ldots, a_{\tau_k,k})$  in accordance with the order of their appearance and we define anew the indexed parameters  $a_1 = a_{1,1}, \ldots, a_{\tau_1} = a_{\tau_1,1}, a_{\tau_1+1} = a_{0,1}, \ldots, a_{L-1} = a_{\tau_k,k}, a_L = a_{0,k}$ . Let us introduce notations analogous to (1.14),

$$(3.5) \quad \Xi(A) := {}^{t}(\log X_{1}, \ldots, \log X_{N}, \log a_{1}, \ldots, \log a_{L}, \log U_{1}, \ldots, \log U_{k}). \log T_{1} = \langle \vec{\alpha}_{1,1}, \log X \rangle + \log a_{1} + \log U_{1}, \\ \vdots \\ \log T_{\tau_{1}} = \langle \vec{\alpha}_{1,\tau_{1}}, \log X \rangle + \log a_{\tau_{1}} + \log U_{1}, \\ \vdots \\ \log T_{L} = \log a_{L} + \log U_{k}.$$

We consider the equation

$$\mathsf{L}(A) \cdot \operatorname{Log} \Xi(A) = \mathsf{L} \cdot \operatorname{Log} \Xi,$$

where the matrix L(A) is constructed as follows. The columns  $\vec{\ell}_i(A) = \vec{v}_i$ ,  $1 \leq i \leq N$  with vectors  $\vec{v}_i$  defined like the column of the matrix L in

(1.15). For the columns of number N + 1 to N + L

$$(\vec{\ell}_{N+1}(A), \ldots, \vec{\ell}_{N+L}(A)) = \mathrm{id}_L.$$

The columns

$$\vec{\ell}_{N+L+j}(A) = {}^t (\underbrace{0, \dots, 0, 0}_{\tau_1, \tau_1, \tau_2, \tau_1}, \underbrace{\tau_{j+1}}_{\tau_{j+1}, \tau_{j+1}}, \underbrace{\tau_{j+1}}_{\tau_{j+1}, \tau_{j+1}}, 0, \dots, 0), \quad 1 \le j \le k,$$

the matrix L(A) is obtained after implementation of the matrix  $id_L$  into the transposed matrix  ${}^tM(A)$  between the k-th and the (k+1)-th column up to necessary permutations necessary after the implementation.

**Proposition 3.2.** There exists a cycle  $\gamma_a$  such that the following equality holds for the integral defined in (3.2),

(3.6) 
$$\Phi_{x^{\mathbf{i}},\gamma_{\mathbf{a}}}^{\zeta}(\mathbf{a}) = B_{\mathbf{i}}^{\zeta}(\mathbf{a})I_{x^{\mathbf{i}},\gamma}^{\zeta}(s(\mathbf{a}), s'(\mathbf{a})),$$

here

$$s_{\ell}(\mathbf{a}) = \prod_{j=1}^{L} a_{j}^{w_{j, N+\ell}}, \quad 1 \le \ell \le k,$$
  
$$s_{\rho}'(\mathbf{a}) = \prod_{j=1}^{L} a_{j}^{w_{j, N+k+\rho}}, \quad 1 \le \rho \le m,$$
  
$$B_{\mathbf{i}}^{\zeta}(\mathbf{a}) = \prod_{\ell=1}^{N} \left(\prod_{j=1}^{L} a_{j}^{w_{j, \ell}}\right)^{i_{\ell}+1} \prod_{\nu=1}^{k} \left(\prod_{j=1}^{L} a_{j}^{w_{j, N+k+m+\nu}}\right)^{\zeta_{\nu}+1}.$$

The exponents  $w_{j,\ell}$  are determined by the following relation,

$$(3.7) \quad \mathsf{L}^{-1} \cdot \mathsf{L}(A) = \begin{bmatrix} 1 & \cdots & 0 & w_{1,1} & \cdots & w_{L,1} & 0 & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & w_{1,N} & \cdots & w_{L,N} & 0 & & \\ 0 & \cdots & 0 & w_{1,N+1} & \cdots & w_{L,N+1} & 0 & \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & w_{1,N+k+m} & \cdots & w_{L,N+k+m} & 0 & & \\ 0 & \cdots & 0 & w_{1,N+k+m+1} & \cdots & w_{L,N+k+m+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & w_{1,L} & \cdots & w_{L,L} & 0 & \cdots & 1 \end{bmatrix},$$

that has been essentially introduced in (2.8). The transition of the cycle  $\gamma(a)$  to  $\gamma$  is controlled by the transformations,

$$X_i = \left(\prod_{j=1}^L a_j^{w_{j,i}}\right)^{-1} \cdot x_i.$$

*Proof.* It is enough to remark the following property,

$$x^{\mathbf{i}+\mathbf{1}}y^{\zeta+\mathbf{1}}\frac{dx}{x^{\mathbf{1}}} \wedge \frac{dy}{y^{\mathbf{1}}} = B_{\mathbf{i}}^{\zeta}(\mathbf{a})x^{\mathbf{i}+\mathbf{1}}U^{\zeta+\mathbf{1}}\frac{dx}{x^{\mathbf{1}}} \wedge \frac{dU}{U^{\mathbf{1}}}.$$
Q.E.D

One can thus conclude (at least locally on the chart  $a_j \neq 0$  for  $j \in I$ , |I| = k + m) A-HGF of GZK (3.2) is expressed by means of a fibre integral annihilated by the Horn system (2.15). One can find a similar statement in [7] where Kapranov restricts himself to a power series expansion of the solution to (3.2).

**Corollary 3.3.** The dimension of the solution space of the system (3.3) at the generic point is equal to  $|\chi(Z_{F(x, 1, 1, y)})|$  if the *T*-non-resonance condition (3.3) is satisfied.

*Proof.* We shall consider the convex hull of vectors that correspond to the vertices of the Newton polyhedron of the polynomial  $y_1(f_1(x) + 1) + \cdots + y_k(f_k(x) + 1)$ . That is to say

$$(\vec{\alpha}_{1,1}, 1, 0, \dots, 0), \dots, (\vec{\alpha}_{\tau_1,1}, 1, 0, \dots, 0),$$
  
 $(\vec{\alpha}_{1,2}, 0, 1, 0, \dots, 0), \dots, (\vec{\alpha}_{\tau_k,k}, 0, \dots, 0, 1) \in \mathbf{Z}^{N+k}.$ 

They are located on the hyperplane  $\zeta_1 + \cdots + \zeta_k = 1$ . Thus it is possible to measure (N + k - 1) dimensional volume

$$(N+k-1)! \operatorname{vol}_{N+k-1}(\Delta(F(x, 1, 1, y)))$$

that is equal to  $(N + k)! \operatorname{vol}_{N+k}(\Delta(F(x, \mathbf{1}, \mathbf{1}, \mathbf{y}) + 1))$ . The Euler characteristic admits the following expression

$$\begin{aligned} |\chi(Z_{F(x,\,\mathbf{1},\,\mathbf{1},\,\mathbf{1},\,y)})| &= \sum_{p} |\det \mathsf{M}_{\mathbf{K}_{p}}| \\ &= (N+k-1)! \operatorname{vol}_{N+k-1} \big( \Delta(F(x,\,\mathbf{1},\,\mathbf{1},\,y)) \big), \end{aligned}$$

after Khovanski [8].

Q.E.D.

We define the A-discriminantal loci  $\nabla^0_{\mathbf{a}}$  in  $\mathbf{T}^L$  like following,

(3.8) 
$$\nabla_{\mathbf{a}}^{0} = \left\{ \begin{aligned} & \bar{f}_{1}(x, \, \mathbf{a}) \\ & = \cdots \\ & = \bar{f}_{k}(x, \, \mathbf{a}) \\ & = 0 \end{aligned} \right. \quad \operatorname{rank} \left( \begin{aligned} & \operatorname{grad}_{x} \, \bar{f}_{1}(x, \, \mathbf{a}) \\ & \vdots \\ & \operatorname{grad}_{x} \, \bar{f}_{k}(x, \, \mathbf{a}) \end{aligned} \right) < k \right\}.$$

As it is seen from (3.7) the uniformisation equations (2.22), (2.23) give rise to an uniformisation of A-discriminantal loci  $\nabla^0_{\mathbf{a}}$  without  $\Delta$ -branching effect.

**Corollary 3.4.** We have the following relations among  $\mathbf{a} \in \mathbf{T}^L$  located on the discriminantal loci  $\nabla_{\mathbf{a}}^0$ ,

(3.9)<sub>1</sub> 
$$\prod_{j=1}^{L} \left( \frac{a_j}{\mathcal{L}_j(-1, \mathbf{z}, \mathbf{z}', -1)} \right)^{B_j^q} = 1, \quad 1 \le q \le k,$$

(3.9)<sub>2</sub> 
$$\prod_{j=1}^{L} \left( \frac{a_j}{\mathcal{L}_j(-1, \mathbf{z}, \mathbf{z}', -1)} \right)^{C_j^r} = 1, \quad 1 \le r \le m.$$

This allows us to express  $\nabla^0_{\mathbf{a}}$  by means of the deformation parameters  $(\mathbf{z}, \mathbf{z}') \in \mathbf{CP}^{k+m-1}$  and  $\mathbf{a}' \in \mathbf{T}^{L-k}/\mathbf{D}(\Sigma) \cong \mathbf{T}^{L-(k+m)}$ .

#### $\S4.$ Examples

**4.1.** Deformation of  $D_4$ .

Let us consider the versal deformation of  $D_4$  singularity of the following form,

(4.1) 
$$f(x, s_0, s_1, s_2, s_3) = x_1^3 + x_1 x_2^2 + s_3 x_1^2 + s_2 x_1 + s_1 x_2 + s_0.$$

By means of the resultant calculus on computer, we get a defining equation of the discriminantal loci as follows,

$$(4.2) \quad \Delta_f(s) = 1024s_1^6 (432s_0^4 + 64s_1^6 + 576s_0^2s_1^2s_2 + 128s_1^4s_2^2 + 64s_0^2s_2^3 + 64s_1^2s_2^4 + 192s_0s_1^4s_3 - 288s_0^3s_2s_3 - 320s_0s_1^2s_2^2s_3 - 24s_0^2s_1^2s_3^2 - 144s_1^4s_2s_3^2 - 16s_0^2s_2^2s_3^2 - 16s_1^2s_2^3s_3^2 + 64s_0^3s_3^3 + 72s_0s_1^2s_2s_3^3 + 27s_1^4s_3^4).$$

This is a polynomial with quasihomogeneous weight 24 if we assign to the variables  $(x_1, x_2; s_0, s_1, s_2, s_3)$  the weights (1, 1; 3, 2, 2, 1). Here we remark that  $s_1 = 0$  branch of the discriminantal locus  $D_s = \{s \in \mathbf{C}^3; \Delta_f(s) = 0\}$  corresponds to the deformation of  $A_2$  singularity.

On the other hand, our Theorem 2.6 states that the uniformisation equation of the discriminantal loci for the deformation (i.e. torus action quotient of the deformation parameter space  $(s_0, s_1, 0, s_3)$  on the chart  $s_3 \neq 0$ ),

$$f(x, s_0, s_1, 0, 1) = x_1^3 + x_1 x_2^2 + x_1^2 + s_1 x_2 + s_0,$$

has the following form,

(4.3) 
$$s_0 = -\frac{z_2(3z_1 + 4z_2)^2}{4(2z_1 + 3z_2)^3},$$
$$s_1 = \left(-\frac{z_1(3z_1 + 4z_2)^3}{4(2z_1 + 3z_2)^4}\right)^{1/2}.$$

If we eliminate the variables  $(z_1, z_2)$  from the expressions (4.3), we get an equation

$$64s_0^3 + 432s_0^4 - 24s_0^2s_1^2 + 27s_1^4 + 192s_0s_1^4 + 64s_1^6 = 0.$$

We recall here that our method requires that the expression yf(x, s) contains so much terms as the variables in it. The reason why the value  $(s_2, s_3) = (0, 1)$  has been chosen is of purely technical character. In substituting the special value (0, 1) for  $(s_2, s_3)$  in (4.2) we get,

$$\frac{\Delta_f(s_0, s_1, 0, 1)}{1024s_1^6} = 64s_0^3 + 432s_0^4 - 24s_0^2s_1^2 + 27s_1^4 + 192s_0s_1^4 + 64s_1^6.$$

# **4.2.** Deformation of a non-quasihomogeneous complete intersection.

Let us consider the following pair of polynomials that define a nondegenerate complete intersection  $X_s$  in  $\mathbb{C}^2$ ,

(4.4) 
$$f_1 = x_1^3 + x_2^2 + s_1, \ f_2 = x_1^2 + x_2^3 + s_2.$$

The discriminant of this CI in  $\mathbf{C}^2$  can be calculated as follows,

$$(4.5) \quad (s_1^3 + s_2^2)^3 (s_2^3 + s_1^2)^3 (800000 + 387420489s_1^5 - 43740000s_1s_2 + + 438438825s_1^2s_2^2 + 387420489s_1^3s_2^3 + 387420489s_2^5).$$

Evidently the fibres corresponding to the parameter values on the divisor  $(s_1^3 + s_2^2)^3(s_2^3 + s_1^2)^3 = 0$  are contained in  $\{(x_1, x_2) \in \mathbf{C}^2; x_1x_2 = 0\}$ . Thus the discriminant of CI  $X_s \cap \mathbf{T}^2$  is given by the third factor of (4.5). After Theorem 2.6, we can find an uniformisation equation of the discriminantal loci  $D_s$ ,

(4.6)  
$$s_{1} = -\left(\frac{(4z_{1} + 6z_{2})^{4}(5z_{1})^{5}(6z_{1} + 4z_{2})^{6}}{(9z_{1} + 6z_{2})^{9}(6z_{1} + 9z_{2})^{6}}\right)^{1/5},$$
$$s_{2} = -\left(\frac{(4z_{1} + 6z_{2})^{6}(5z_{2})^{5}(6z_{1} + 4z_{2})^{4}}{(9z_{1} + 6z_{2})^{6}(6z_{1} + 9z_{2})^{9}}\right)^{1/5}.$$

If we eliminate the variables  $(z_1, z_2)$  from the expressions (4.6), we get an equation of  $\nabla_s$ ,

$$(800000 + 387420489s_1^5 - 43740000s_1s_2 + 438438825s_1^2s_2^2 + 387420489s_1^3s_2^3 + 387420489s_2^5)R(z_1, z_2),$$

where  $R(z_1, z_2)$  is a polynomial whose Newton polyhedron is contained in a four sided rectilinear figure with vertices (0, 0), (20, 0), (12, 12), (0, 20). This factor contains the image of  $\tilde{h}(\mathbf{CP}^1)$  outside of  $D_s$ .

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## Duality of Euler data for affine varieties

## Mihai Tibăr

#### Abstract.

We compare the Euler-Poincaré characteristic to the global Euler obstruction, in case of singular affine varieties, and point out a certain duality among their expressions in terms of strata of a Whitney stratification.

The local Euler obstruction was defined by MacPherson [MP], as a key ingredient for introducing Chern classes for singular spaces. Results on the local Euler obstruction have been obtained during the time by, among others, A. Dubson, M.-H. Schwartz, J.-P. Brasselet, G. Gonzalez-Sprinberg, B. Teissier, Lê D.T, J. Schürmann, J. Seade. Some of them are surveyed in [Br] and [Sch2]. For more recent results and generalizations one can look up [BLS, BMPS, Sch1, STV1, STV2].

For a connected singular algebraic closed affine space  $Y \subset \mathbb{C}^N$  we have defined in [STV1] a global Euler obstruction  $\operatorname{Eu}(Y)$ . The definition in the global setting can be traced back to Dubson's viewpoint [Du]. It immediately follows that, for a non-singular Y,  $\operatorname{Eu}(Y)$  equals the Euler characteristic  $\chi(Y)$ . The natural question that we address here is how these two "Euler data" compare to each other whenever Y is singular.

Both objects, Eu and  $\chi$ , can be viewed as constructible functions with respect to some Whitney (b)-regular algebraic stratification of Y. Let us fix such a stratification  $\mathcal{A} = {\mathcal{A}_i}_{i \in \Lambda}$  on Y. We first show how Eu(Y) and  $\chi(Y)$  can be expressed in terms of strata such that the formulas are, in a certain sense, dual:

(0.1) 
$$\operatorname{Eu}(Y) = \sum_{i \in \Lambda} \chi(\mathcal{A}_i) \operatorname{Eu}_Y(\mathcal{A}_i),$$

(0.2) 
$$\chi(Y) = \sum_{i \in \Lambda} \operatorname{Eu}(\mathcal{A}_i)\chi(\operatorname{NMD}(\mathcal{A}_i)).$$

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The duality consists in the observation that the formulas are obtained one from another by interchanging Eu with  $\chi$ . To the Euler characteristic  $\chi(\mathcal{A}_i)$  of some stratum  $\mathcal{A}_i$  in formula (0.1) corresponds the global Euler obstruction Eu( $\mathcal{A}_i$ ) of the same stratum in formula (0.2). The latter has the following meaning: as it will be explained in §1, the Euler obstruction Eu( $\bar{\mathcal{A}}_i$ ) of the algebraic closure  $\bar{\mathcal{A}}_i$  of  $\mathcal{A}_i$  in  $\mathbb{C}^N$  is well defined and depends only on the open part  $\mathcal{A}_i$ . We may therefore set Eu( $\mathcal{A}_i$ ) := Eu( $\bar{\mathcal{A}}_i$ ). In case of a point-stratum {y}, we set Eu({y}) = 1.

Let us explain how the "normal Euler data"  $\chi(\text{NMD}(\mathcal{A}_i))$  and Eu<sub>Y</sub>( $\mathcal{A}_i$ ) fit into this correspondence. Both data are attached to a general slice  $\mathcal{N}_i$  of complementary dimension of the stratum  $\mathcal{A}_i$  at some point  $p_i \in \mathcal{A}_i$ .

Firstly, NMD( $\mathcal{A}_i$ ) stands for the normal Morse data of the stratum  $\mathcal{A}_i$  (after Goresky-MacPherson's [GM]), i.e. the Morse data of ( $\mathcal{N}_i, p_i$ ), see §2.

Secondly,  $\operatorname{Eu}_Y(\mathcal{A}_i)$  denotes the normal Euler obstruction of the stratum  $\mathcal{A}_i$ , i.e. the local Euler obstruction of  $\mathcal{N}_i$  at  $p_i$ .

It is known that both data are independent on the choices of  $\mathcal{N}_i$  and of  $p_i$ . We refer to §2 for the definitions and more details.

We finally consider the case when Y is a locally complete intersection with arbitrary singularities. We show (Proposition 3.1) how the difference  $\chi(Y) - \operatorname{Eu}(Y)$  can be expressed in terms of Betti numbers of complex links and the polar invariants  $\alpha_Y$  defined in §1. If the singularities are isolated then the difference  $\chi(Y) - \operatorname{Eu}(Y)$  measures the total "quantity of slice-singularities" of Y, see (3.3).

For another comparison of the Euler characteristic, namely to the total curvature, in case of an affine hypersurface, we send the reader to [ST].

#### $\S1$ . Global Euler obstruction

Since  $Y \subset \mathbb{C}^N$  is affine, one has a well defined *link at infinity* of Y, denoted by  $K_{\infty}(Y) := Y \cap S_R$ . It follows from Milnor's finiteness argument [Mi, Cor. 2.8] and from standard isotopy arguments that  $K_{\infty}(Y)$  does not depend on the radius R, provided that R is large enough.

Let  $\tilde{Y} = \text{closure}\{(x, T_x Y_{\text{reg}}) \mid x \in Y_{\text{reg}}\} \subset Y \times G(d, N)$  be the Nash blow-up of Y, where G(d, N) is the Grassmannian of complex d-planes in  $\mathbb{C}^N$ . Let  $\nu \colon \tilde{Y} \to Y$  denote the natural projection and let  $\tilde{T}$  denote the restriction over  $\tilde{Y}$  of the bundle  $\mathbb{C}^N \times U(d, N) \to \mathbb{C}^N \times G(d, N)$ , where U(d, N) is the tautological bundle over G(d, N). This is the "Nash bundle" over  $\tilde{Y}$ . We next consider a continuous, stratified vector field  $\mathbf{v}$  on a subset  $V \subset Y$ . The restriction of  $\mathbf{v}$  to V has a well-defined
canonical lifting  $\tilde{\mathbf{v}}$  to  $\nu^{-1}(V)$  as a section of the Nash bundle  $\widetilde{T} \to \widetilde{Y}$  (see e.g. [BS], Prop. 9.1).

We refer to [STV1] for other details concerning the following definition (which can be traced back to Dubson's approach), and in particular for the discussion on the independence on the choices:

**Definition 1.1.** Let  $\tilde{\mathbf{v}}$  be the lifting to a section of the Nash bundle  $\tilde{T}$  of a stratified vector field  $\mathbf{v}$  over  $K_{\infty}(Y) = Y \cap S_R$ , which is radial with respect to the sphere  $S_R$ . The obstruction to extend  $\tilde{\mathbf{v}}$  as a nowhere zero section of  $\tilde{T}$  within  $\nu^{-1}(Y \cap B_R)$  is a relative cohomology class  $o(\tilde{\mathbf{v}}) \in H^{2d}(\nu^{-1}(Y \cap B_R), \nu^{-1}(Y \cap S_R)) \simeq H^{2d}(\tilde{Y})$ .

One calls global Euler obstruction of Y, and denotes it by  $\operatorname{Eu}(Y)$ , the evaluation of  $o(\tilde{\mathbf{v}})$  on the fundamental class of the pair  $(\nu^{-1}(Y \cap B_R), \nu^{-1}(Y \cap S_R))$ .

By obstruction theory,  $\operatorname{Eu}(Y)$  is an integer and does not depend on the radius of the sphere defining the link at infinity  $K_{\infty}(Y)$ . We have shown in [STV1, Theorem 3.4] that  $\operatorname{Eu}(Y)$  can be expressed in terms of polar multiplicities as follows, denoting  $d = \dim Y$ :

(1.1) 
$$\operatorname{Eu}(Y) = \sum_{j=1}^{d+1} (-1)^{d-j+1} \alpha_Y^{(j)},$$

where:

(1.2)  $\alpha_Y^{(1)} :=$  the number of Morse points of a global generic linear function on  $Y_{\text{reg}}$ .

After taking a general hyperplane slice  $H \cap Y$ , the second number is  $\alpha_Y^{(2)} := \alpha_{H \cap Y}^{(1)}$ . This continues by induction and yields a sequence of non-negative integers:

$$\alpha_Y^{(1)}, \, \alpha_Y^{(2)}, \, \dots, \, \alpha_Y^{(d)},$$

which we complete by  $\alpha_Y^{(d+1)}$  := the number of points of the intersection of  $Y_{\text{reg}}$  with a global generic codimension d plane in  $\mathbb{C}^N$ .

Of course  $\alpha_Y^{(k)}$  depends on the embedding of Y into  $\mathbb{C}^N$ . Nevertheless, these invariants (and therefore, by the equality (1.1), Eu(Y) too) depend only on some Zariski open part of Y. Now, for a stratum  $\mathcal{A}_i$ from the stratification  $\mathcal{A} = \{\mathcal{A}_i\}_{i \in \Lambda}$  of Y, the global Euler obstruction  $\operatorname{Eu}(\bar{\mathcal{A}}_i)$  of its Zariski closure  $\bar{\mathcal{A}}_i$  is well-defined. However, since we have seen that this depends only on the open part  $\mathcal{A}_i$ , we can use the notation  $\operatorname{Eu}(\mathcal{A}_i)$  for  $\operatorname{Eu}(\overline{\mathcal{A}}_i)$ . This convention explains the occurrence of  $\operatorname{Eu}(\mathcal{A}_i)$  instead of  $\operatorname{Eu}(\overline{\mathcal{A}}_i)$  in formula (0.2).

If the highest dimensional stratum is denoted by  $\mathcal{A}_0$ , then we have  $\overline{\mathcal{A}}_0 = Y$  and therefore  $\operatorname{Eu}(Y) = \operatorname{Eu}(\mathcal{A}_0)$ .

## $\S 2.$ The dual formula

The equality (0.1) was explained in [STV1]. It follows by Dubson's [Du, Theorem 1] applied to our setting. In case of germs of spaces a similar formula was proved in [BLS, Theorem 3.1] by using the Lefschetz slicing method. A different proof may be derived from [BS, Theorem 4.1]. For a more general proof, in terms of constructible functions, we send to [Sch2, (5.65)].

We now give a proof of the equality (0.2). This can be viewed as a global index theorem, similar to Kashiwara's local index theorem (see for this [Sch2, (5.38), (5.38)]). Our proof will only use the equality (1.1).

**Definition 2.1** (cf. [GM]). The *complex link* of a space germ (X, x) is the general fibre in the local Milnor-Lê fibration defined by a general (linear) function germ at x. Up to homotopy type, this does not depend on the stratification or the choices of the representatives of the space or of the general function.

Let  $\operatorname{CL}_Y(\mathcal{A}_i)$  denote the *complex link of the stratum*  $\mathcal{A}_i$  of Y. This is by definition the complex link of the germ  $(\mathcal{N}_i, p_i)$ , where  $\mathcal{N}_i$  is a generic slice of Y at some  $p_i \in \mathcal{A}_i$ , of codimension equal to the dimension of  $\mathcal{A}_i$ . Let us remark that the complex link of a point-stratum  $\{y\}$  is precisely the complex link of the germ (Y, y).

Let  $\operatorname{Cone}(\operatorname{CL}_Y(\mathcal{A}_i))$  denote the cone over this complex link. We denote by  $\operatorname{NMD}(\mathcal{A}_i)$  the normal Morse data at some point of  $\mathcal{A}_i$ , that is the pair of spaces ( $\operatorname{Cone}(\operatorname{CL}_Y(\mathcal{A}_i))$ ,  $\operatorname{CL}_Y(\mathcal{A}_i)$ ). After Goresky and MacPherson [GM], the local normal Morse data are local invariants up to homotopy and do not depend on the various choices in cause. The complex link of the highest dimensional stratum  $\mathcal{A}_0$  is empty, and we set by definition  $\chi(\operatorname{NMD}(\mathcal{A}_0)) = 1$ . In the same case, for the normal Euler obstruction we have  $\operatorname{Eu}_Y(\mathcal{A}_0) = 1$  by definition.

**Theorem 2.2.** Let  $Y \subset \mathbb{C}^N$  be an algebraic closed affine space and let  $\mathcal{A} = \{\mathcal{A}_i\}_{i \in \Lambda}$  be some Whitney stratification of Y. Then:

(2.1) 
$$\chi(Y) = \sum_{i \in \Lambda} \operatorname{Eu}(\mathcal{A}_i) \chi(\operatorname{NMD}(\mathcal{A}_i)).$$

*Proof.* Take an affine Lefschetz pencil of hyperplanes in  $\mathbb{C}^N$  defined by a linear function  $l_H : \mathbb{C}^N \to \mathbb{C}$ . By the genericity of the pencil, there are only finitely many stratified Morse singularities of the pencil, each one contained in a different slice. By the definition (1.2), the number of stratified Morse points on a stratum  $\mathcal{A}_i$  of dimension > 0 is precisely  $\alpha_{\bar{\mathcal{A}}_i}^{(\dim \mathcal{A}_i)}$ .

According to the Lefschetz slicing method applied to singular spaces (see e.g. [GM]), the space Y is obtained from a generic hyperplane slice  $Y \cap \mathcal{H}$  of the pencil, to which are attached cones over the complex links of each singularity of the pencil. Goresky and MacPherson have proved that the Milnor data of a stratified Morse function germ is the  $(\dim \mathcal{A}_i)$ times suspension of  $\text{NMD}(\mathcal{A}_i)$ . At the level of Euler characteristic, we then have:

(2.2) 
$$\chi(Y) = \chi(Y \cap \mathcal{H}) + \sum_{i \in \Lambda} (-1)^{\dim \mathcal{A}_i} \alpha_{\bar{\mathcal{A}}_i}^{(1)} \chi(\text{NMD}(\mathcal{A}_i)),$$

The sign  $(-1)^{\dim \mathcal{A}_i}$  is due to the repeated suspension of the normal Morse data. By convention, for 0 dimensional strata  $\mathcal{A}_i$  we put  $\alpha_{\bar{\mathcal{A}}_i}^{(1)} := 1$ , and therefore  $\operatorname{Eu}(\bar{\mathcal{A}}_i) = 1$ . We apply formula (2.2) to  $Y \cap \mathcal{H}$  and to the successive generic slicings in decreasing dimensions. In the resulting equality, we get the sum of all the coefficients of  $\chi(\operatorname{NMD}(\mathcal{A}_i))$ , for each  $i \in \Lambda$ . We may then identify this sum to  $\operatorname{Eu}(\bar{\mathcal{A}}_i)$  via the formula (1.1). This ends our proof. Q.E.D.

## $\S$ **3.** Case of locally complete intersections

We consider here the case of a locally complete intersection  $Y \subset \mathbb{C}^N$ of dimension d, with arbitrary singularities. Being a locally complete intersection implies however that the complex link of any stratum  $\mathcal{A}_i$ is homotopy equivalent to a bouquet of spheres of dimension equal to  $\operatorname{codim}_Y \mathcal{A}_i - 1$ , by Lê's result [Lê]. Let  $b_{d-\dim \mathcal{A}_i-1}(\operatorname{CL}_Y(\mathcal{A}_i))$  denote the Betti number of this complex link. One can then write the formula (2.2) in the following form:

(3.1) 
$$\chi(Y) = \chi(Y \cap \mathcal{H}) + (-1)^d (\alpha_Y^{(1)} + \beta_Y^{(1)})$$

where  $\beta_Y^{(1)}$  collects the contributions from all the lower dimensional strata in the sum (2.2), more precisely, under our assumption we have:

$$\beta_Y^{(1)} := \sum_{i \in \Lambda \setminus \{0\}} \alpha_{\bar{\mathcal{A}}_i}^{(1)} b_{d-\dim \mathcal{A}_i - 1}(\operatorname{CL}_Y(\mathcal{A}_i)).$$

According to their definitions,  $\alpha_Y^{(1)}$  and  $\beta_Y^{(1)}$  are both non-negative integers. Their sum represents the number of *d*-cells which have to be attached to  $Y \cap \mathcal{H}$  in order to obtain Y.

Let us define  $\beta_Y^{(k)}$  for  $k \ge 2$ , by:

$$\beta_Y^{(2)} := \beta_{Y \cap \mathcal{H}}^{(1)}$$

and so on by induction, for successive slices of Y, as in case of the  $\alpha_{Y}^{(k)}$ -series defined before.<sup>1</sup>

After repeatedly applying (3.1), and then using (1.1), we get the following expression of the difference among the two Euler data:

## Proposition 3.1.

(3.2) 
$$\chi(Y) - \operatorname{Eu}(Y) = \sum_{k=1}^{d} (-1)^{d-k+1} \beta_Y^{(k)}.$$

**Remark 3.2.** Let us see what becomes this difference in case Y is a hypersurface, or a locally complete intersection, with *isolated sin*gularities. For an isolated singular point  $q \in Y$ , let  $\mu_q^{\langle d-1 \rangle}(Y)$  denote the Milnor number of the local complete intersection  $(Y \cap \mathcal{H}, q)$  which is the result of slicing Y by a generic hyperplane  $\mathcal{H}$ . In case Y is a hypersurface, this is the second highest Milnor-Teissier number in the sequence  $\mu_q^*(Y)$ . We get:

(3.3) 
$$\chi(Y) - \operatorname{Eu}(Y) = (-1)^d \sum_{q \in \operatorname{Sing} Y} \mu_q^{\langle d-1 \rangle}(Y).$$

Since by convention  $\alpha_{\{q\}}^{(1)} = 1$ , and since  $b_{d-1}(\operatorname{CL}_Y(\{q\})) = \mu_q^{\langle d-1 \rangle}(Y)$ , formula (3.3) is indeed a particular case of formula (3.2). This can be also proved by using the local Euler obstruction formula [BLS, Theorem 3.1].

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<sup>&</sup>lt;sup>1</sup>We send to [ST, §6] for examples where the integers  $\beta_Y^{(k)}$  are computed (but beware that we use a different convention for the indices k).

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# Algèbre graduée associée à une valuation de K[x]

## Michel Vaquié

## Abstract.

We extend some results on augmented valuations and key-polynomials to limit augmented valuations and limit key-polynomials. As any valuation  $\mu$  of K[x] is obtained as a limit of an admissible family of valuations, we deduce a description of the graded algebra associated to  $\mu$ .

## § Introduction

Dans cet article nous donnons une description de l'algèbre graduée  $\operatorname{gr}_{\mu} K[x]$  associée à une valuation  $\mu$  de l'anneau des polynômes K[x] sur un corps K. La nature de cette algèbre graduée dépend essentiellement du fait que la valuation  $\mu$  est ou n'est pas bien spécifiée, c'est-à-dire du fait que l'extension  $(K(x), \mu)/(K, \nu)$  de corps valués vérifie ou ne vérifie pas l'égalité d'Abhyankar.

Nous caractérisons cette propriété de la valuation  $\mu$  en utilisant la notion de famille admissible associée à la valuation  $\mu$ , telle qu'elle a été introduite et étudiée dans [Va 1] et [Va 2].

Dans la première partie, pour étudier certaines propriétés des familles admissibles, nous étendons aux valuations augmentées limites et aux polynômes-clé limites quelques résultats déjà connus pour les valuations augmentées et les polynômes-clés. Nous donnons ensuite plusieurs propriétés caractéristiques des valuations bien spécifiées.

Dans la deuxième partie, nous étudions plus précisément l'algèbre graduée  $\operatorname{gr}_{\mu} K[x]$ .

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## §1. Valuation bien spécifiée

Nous considérons un corps K muni d'une valuation  $\nu$ , de corps résiduel  $\kappa_{\nu}$  et de groupe des valeurs  $\Gamma_{\nu}$ . Nous choisissons un plongement de  $\Gamma_{\nu}$  dans un groupe totalement ordonné  $\bar{\Gamma}$  suffisamment grand et toutes les valeurs finies  $\gamma$  que nous considèrerons seront dans  $\bar{\Gamma}$ .

Nous appelons  $\mathcal{E} = \mathcal{E}(K[x], \nu)$  l'ensemble des valuations ou pseudovaluations de l'anneau des polynômes K[x] dont la restriction à K est égale à  $\nu$ . Dans la suite toutes les valuations de K[x] appartiendront à  $\mathcal{E}$ .

Une famille *admissible* de valuations de K[x] est une famille de la forme  $\mathcal{A} = (\mu_i)_{i \in I}$ , où I est un ensemble totalement ordonné, obtenue comme réunion de familles *admissibles simples*  $\mathcal{S}^{(j)}$ , pour j parcourant J, avec  $J = \{1, \ldots, N\}$  ou  $J = \mathbb{N}^*$ , chaque famille simple  $\mathcal{S}^{(j)}$ étant constituée d'une partie *discrète*  $\mathcal{D}^{(j)}$  et d'une partie *continue*  $\mathcal{C}^{(j)}$ , la dernière famille continue  $\mathcal{C}^{(N)}$  pouvant être éventuellement vide.

Les valuations  $\mu_i$  de la famille  $\mathcal{A}$  apparaissant dans les parties discrètes  $\mathcal{D}^{(j)}$ , sauf la première de chaque partie, ainsi que celles apparaissant dans les parties continues  $\mathcal{C}^{(j)}$  sont définies comme valuations augmentées, et les valuations apparaissant comme première valuations d'une partie discrète  $\mathcal{D}^{(j)}$  sont définies comme valuations augmentées limites. Dans le premier cas nous avons

$$\mu_i = [\mu_{i-1}; \mu_i(\phi_i) = \gamma_i],$$

et  $\phi_i$  est un *polynôme-clé* définissant la valuation  $\mu_i$  à partir de la valuation  $\mu_{i-1}$ , et dans le deuxième cas nous avons

$$\mu_i = \left[ \left( \mu_\alpha \right)_{\alpha \in A}; \mu_i(\phi_i) = \gamma_i \right],$$

et  $\phi_i$  est un *polynôme-clé limite* définissant la valuation  $\mu_i$  à partir de la famille continue  $\mathcal{C}^{(j-1)} = (\mu_{\alpha})_{\alpha \in A}$ .

Nous disons que la famille  $\mathcal{A}$  est *close* si l'ensemble I possède un plus grand élément  $\bar{\iota}$ , dans ce cas la valuation  $\mu$  est la valuation  $\mu_{\bar{\iota}}$ . Sinon, nous disons que la famille  $\mathcal{A}$  est *ouverte* et dans ce cas la valuation  $\mu$  n'appartient pas à la famille  $\mathcal{A}$ . Dans [Va 2] nous appelions famille admissible *complète* une famille close.

A toute valuation ou pseudo-valuation  $\mu$  de  $\mathcal{E}$  nous pouvons associer une famille admissible  $\mathcal{A}$  que nous notons  $\mathcal{A}(\mu)$ , cette famille n'est pas unique mais définie à équivalence près, le fait que la famille  $\mathcal{A}(\mu)$  soit close ou ouverte ne dépend que de la valuation  $\mu$  donnée.

Nous renvoyons le lecteur aux articles [Va 1] et [Va 2] de l'auteur pour des définitions précises et pour les propriétés de ces valuations, de ces polynômes et de ces familles.

Définition. Si la famille  $\mathcal{A}$  associée à la valuation  $\mu$  est close, nous disons que  $\mu$  est bien spécifiée. Le polynôme-clé ou polynôme-clé limite  $\phi = \phi_{\bar{\iota}}$  définissant  $\mu$  comme valuation augmentée ou comme valuation augmentée limite est appelé le polynôme définissant  $\mu$ .

Remarque 1.1. Les valuations bien spécifiées sont les valuations  $\mu$  de K(x) dont le corps résiduel  $\kappa_{\mu}$  est une extension transcendante de  $\kappa_{\nu}$  ou dont le groupe des valeurs  $\Gamma_{\mu}$  est tel que le groupe quotient  $\Gamma_{\mu}/\Gamma_{\nu}$  contient des éléments sans torsion. Nous en déduisons que les valuations bien spécifiées  $\mu$  sont les valuations telles que l'extension  $(K(x), \mu)/(K, \nu)$  de corps valués vérifie l'égalité d'Abhyankar:

dim.alg.<sub>K</sub>
$$K(x) = \text{dim.alg.}_{\kappa_{\mu}}\kappa_{\mu} + \text{rang.rat.}\Gamma_{\mu}/\Gamma_{\nu} = 1.$$

Soit  $\mu$  une valuation bien spécifiée et soit  $\phi$  le polynôme qui la définit, nous pouvons alors écrire soit  $\mu = [\mu_0; \mu(\phi) = \gamma]$  si  $\mu$  est une valuation augmentée pour la valuation  $\mu_0$ , soit  $\mu = [(\mu_\alpha)_{\alpha \in A}; \mu(\phi) = \gamma]$  si  $\mu$ est une valuation augmentée limite pour la famille continue  $(\mu_\alpha)_{\alpha \in A}$ . Dans le dernier cas, pour tout polynôme f tel que la famille de valeurs  $(\mu_\alpha(f))_{\alpha \in A}$  devient stationnaire, nous notons  $\mu_0(f) = \mu_A(f)$  la valeur limite de cette famille. Alors pour tout polynôme f de K[x], si nous notons  $f = f_m \phi^m + \ldots + f_0$  le développement de f selon les puissances de  $\phi$ , nous avons par définition:

$$\mu(f) = \operatorname{Inf}(\mu_0(f_j) + j\gamma, \ 0 \le j \le m),$$

et de plus nous avons pour tout j,  $\mu(f_j) = \mu_0(f_j)$ .

Nous avons alors la proposition suivante qui généralise les résultats de MacLane ([McL 1] Lemma 9.2 et [McL 2] Lemma 4.3).

**Proposition 1.1.** Si  $\mu$  est une valuation bien spécifiée, le polynôme  $\phi$  définissant  $\mu$  est un polynôme-clé pour la valuation  $\mu$ .

*Preuve.* Soit  $f = f_m \phi^m + \ldots + f_0$  le développement de f selon les puissances de  $\phi$ , alors nous avons:

$$\mu(f_0) \ge \mu(f) \quad \text{et} \quad \mu(f_0) > \mu(f) \iff \phi \mid f.$$

Nous en déduisons immédiatement que tout polynôme  $f \mu$ -divisible par  $\phi$  est de degré supérieur ou égal au degré de  $\phi$ , c'est-à-dire que  $\phi$  est  $\mu$ -minimal.

Soient f et g deux polynômes qui ne sont pas  $\mu$ -divisibles par  $\phi$ , alors nous avons d'après ce qui précède  $\mu(f_0) = \mu(f)$  et  $\mu(g_0) = \mu(g)$ , où nous notons  $f_0$  et  $g_0$  les restes de la division euclidienne respectivement de f et g par  $\phi$ . Nous avons alors  $f_0g_0 = h'\phi + h_0$ , avec h' et  $h_0$  de degré strictement inférieur au degré de  $\phi$  et  $h_0$  est le reste de la division de h = fg par  $\phi$ . Nous avons donc l'inégalité  $\mu(h_0) \ge \mu(f_0g_0)$  et si nous montrons que nous avons  $\mu(f_0g_0) = \mu(h_0)$ , alors nous pourrons en déduire l'égalité  $\mu(h) = \mu(h_0)$ , donc que h = fg n'est pas  $\mu$ -divisible par  $\phi$ , cela nous donnera la  $\mu$ -irréductibilité de  $\phi$ .

Dans le cas où  $\mu$  est une valuation augmentée, comme  $\phi$  est un polynôme-clé pour la valuation  $\mu_0$ , le produit  $f_0g_0$  n'est pas  $\mu_0$ -divisible par  $\phi$  par conséquent nous avons bien  $\mu(f_0g_0) = \mu_0(f_0g_0) = \mu_0(h_0) = \mu(h_0)$ .

Dans le cas où  $\mu$  est une valuation augmentée limite, il existe  $\alpha$  dans A tel que pour tout  $\beta$  dans A vérifiant  $\beta \geq \alpha$  nous avons les égalités  $\mu_{\beta} = \mu_0 = \mu$  pour  $f_0, g_0$  et  $h_0$ . Si nous avions l'inégalité stricte  $\mu(h_0) > \mu(f_0g_0)$ , alors pour tout  $\beta \geq \alpha$  nous aurions  $\mu_{\beta}(f_0g_0) = \mu_{\beta}(h'\phi)$ , ce qui est impossible car la famille  $(\mu_{\beta}(\phi))$  est strictement croissante.

La proposition 1.1 énonce une propriété commune aux valuations augmentées et aux valuations augmentées limites. Nous allons étendre ce résultat et montrer que les valuations augmentées limites et les polynômes-clés limites ont d'autres propriétés équivalentes à celles des valuations augmentées et des polynômes-clés, telles qu'elles ont été données par MacLane ([McL 1] et [McL 2]).

Soit S une famille admissible simple de valuations de K[x], constituée de la partie discrète finie  $\mathcal{D} = (\mu_i)_{i \in I}$ , et de la partie continue  $\mathcal{C} = (\mu_\alpha)_{\alpha \in A}$ . Les valuations  $\mu_\alpha$  sont des valuations augmentées de la forme  $\mu_\alpha = [\mu_0; \mu_\alpha(\phi_\alpha) = \gamma_\alpha]$ , où les polynômes-clés  $\phi_\alpha$  sont tous de même degré, deg  $\phi_\alpha = d_A$ , et où la famille de valeurs  $(\gamma_\alpha)_{\alpha \in A}$  est strictement croissante, de plus ces valuations ont même groupe des ordres  $\Gamma_{\mathbf{A}}$ .

Nous supposons que l'ensemble  $\tilde{\Phi}(A)$  des polynômes f vérifiant  $\mu_{\alpha}(f) < \mu_{\beta}(f)$  pour tout  $\alpha < \beta$  dans A est non vide. Rappelons que si un polynôme f n'appartient pas à  $\tilde{\Phi}(A)$ , c'est-à-dire s'il existe un couple  $\alpha < \beta$  dans A tel que  $\mu_{\alpha}(f) = \mu_{\beta}(f)$ , alors pour tout  $\alpha' \geq \alpha$  nous avons l'égalité  $\mu_{\alpha'}(f) = \mu_{\alpha}(f)$ , et nous notons  $\mu_A(f)$  cette valeur. Nous appelons  $m_A$  le degré minimal d'un polynôme de  $\tilde{\Phi}(A)$  et nous définissons l'ensemble  $\Phi(A)$  par:

$$\Phi(A) = \left\{ \phi \in K[x] \mid \begin{array}{c} \mu_{\alpha}(\phi) < \mu_{\beta}(\phi) \ \forall \alpha < \beta \in A, \\ \phi \text{ unitaire, } \deg \phi = m_A \end{array} \right\}.$$

Alors tout polynôme  $\phi$  de  $\Phi(A)$  est un polynôme-clé limite pour la famille  $\mathcal{C} = (\mu_{\alpha})_{\alpha \in A}$  (cf. [Va 1] Proposition 1.21), et pour tout  $\gamma$ vérifiant  $\gamma > \mu_{\alpha}(\phi)$  pour tout  $\alpha$  dans A, nous pouvons définir la valuation, ou la pseudo-valuation dans le cas  $\gamma = +\infty$ , augmentée limite  $\mu$  associée à  $\phi$  et à  $\gamma$ , que nous notons:

$$\mu = \left[ \left( \mu_{\alpha} \right)_{\alpha \in A}; \mu(\phi) = \gamma \right].$$

Dans la suite nous supposons que nous avons choisi un polynômeclé limite  $\phi$  pour la famille C et que nous avons défini une valuation augmentée limite  $\mu$  associée à  $\phi$  et à une valeur  $\gamma$ .

**Lemme 1.2.** Pour tout polynôme f n'appartenant pas à  $\Phi(A)$ , par exemple pour f vérifiant deg  $f < \deg \phi$ , il existe un polynôme g, avec deg  $g < \deg \phi$ , tel que fg soit  $\mu$ -équivalent à 1.

*Preuve.* Si le polynôme f n'appartient pas à  $\tilde{\Phi}(A)$ , f est premier à  $\phi$  et nous pouvons trouver des polynômes h et g, avec deg  $g < \deg \phi$ , tels que  $fg + h\phi = 1$ , et nous supposons  $f \notin K$ , d'où  $h \neq 0$ . Pour  $\alpha$ suffisamment grand nous avons alors

 $\mu(h\phi) > \mu_{\alpha}(h\phi) \ge \operatorname{Inf}\left(\mu_{\alpha}(fg), \, \mu_{\alpha}(1)\right) = \operatorname{Inf}\left(\mu(fg), \, \mu(1)\right)$ 

d'où  $fg \mu$ -équivalent à 1.

**Proposition 1.3.** Soit  $\phi'$  un polynôme unitaire de K[x] vérifiant  $\deg \phi' \geq \deg \phi$  et non  $\mu$ -équivalent à  $\phi$ , et soit  $\phi' = f_m \phi^m + \cdots + f_0$  son développement selon les puissances de  $\phi$ . Alors  $\phi'$  est un polynôme-clé pour la valuation  $\mu$  si et seulement si

-  $\phi'$  est  $\mu$ -irréductible,

-  $f_m = 1$ , c'est-à-dire  $\phi' = \phi^m + \dots + f_0$ , et  $\mu(\phi') = m\gamma = \mu_A(f_0)$ .

*Preuve.* La démonstration est identique à celle dans le cas où  $\mu$  est une valuation augmentée associée à un polynôme-clé  $\phi$  (cf. [McL 1] Theorem 9.4, [Va 1] Théorème 1.11).

Remarque 1.2. Soit  $\mu$  une valuation bien spécifiée définie par le polynôme  $\phi$ , et nous reprenons la notation  $\mu_0$  définie plus haut, alors pour tout f dans K[x] nous avons les implications (i)  $\Longrightarrow$  (ii)  $\Longrightarrow$  (iii) avec:

(i)  $\mu(f) = \mu_0(f)$ ,

(ii) il existe  $f_0$  avec deg  $f_0 < \deg \phi \mu$ -équivalent à f,

(iii) f est  $\mu$ -unitaire, c'est-à-dire il existe f' dans K[x] tel que ff' soit  $\mu$ -équivalent à 1, et nous pouvons choisir f' avec deg  $f' < \deg \phi$ .

Nous avons sur l'ensemble  $\mathcal{E}$  des valuations ou pseudo-valuations de K[x] prolongeant  $\nu$  la relation d'ordre partiel  $\leq$  définie de la manière suivante:

 $\mu \leq \mu'$  si et seulement si  $\mu(f) \leq \mu'(f)$  pour tout f dans K[x].

La Proposition 1.1 montre que toute valuation bien spécifiée admet un polynôme-clé, nous avons en fait le résultat plus précis suivant qui répond en particulier à la question de savoir à quelle condition une valuation  $\mu$  de K[x] possède un polynôme-clé.

**Proposition 1.4.** Les propositions suivantes sont équivalentes:

- 1) La valuation  $\mu$  est bien spécifiée.
- 2) La valuation  $\mu$  n'est pas maximale pour la relation d'ordre  $\leq$ .
- 3) La valuation  $\mu$  admet un polynôme-clé.
- 4) La valuation  $\mu$  peut être obtenue comme valuation augmentée

$$\mu = [\mu_0; \mu(\phi) = \gamma],$$

ou comme valuation augmentée limite

$$\mu = \left[ \left( \mu_{\alpha} \right)_{\alpha \in A}; \mu(\phi) = \gamma \right].$$

Nous allons d'abord rappeler le lemme suivant (cf. [Va 2] Lemme 2.8).

**Lemme 1.5.** Soient  $\mu_0$ ,  $\mu$  et  $\mu'$  trois valuations de K[x] vérifiant  $\mu_0 < \mu \leq \mu'$ . Nous appelons  $\tilde{\Phi}$ , respectivement  $\tilde{\Phi}'$ , l'ensemble des polynômes f de K[x] vérifiant  $\mu_0(f) < \mu(f)$ , respectivement  $\mu_0(f) < \mu'(f)$ . Alors les ensembles  $\tilde{\Phi}$  et  $\tilde{\Phi}'$  sont égaux.

*Preuve de la proposition.* Par définition nous avons 1) implique 4), et nous avons montré à la Proposition 1.1 que 4) implique 3).

Les propriétés 2) et 3) sont équivalentes, en effet si la valuation  $\mu$  admet un polynôme-clé  $\phi$  alors pour toute valeur  $\gamma'$  strictement plus grande que  $\mu(\phi)$  nous pouvons définir la valuation augmenté  $\mu' = [\mu; \mu'(\phi) = \gamma']$ qui vérifie  $\mu \leq \mu'$  et  $\mu \neq \mu'$ . Réciproquement si  $\mu'$  est une valuation avec  $\mu \leq \mu'$  et  $\mu \neq \mu'$ , nous pouvons en déduire l'existence d'un polynômeclé  $\phi$  pour  $\mu$ , il suffit de choisir un polynôme unitaire de degré minimal vérifiant  $\mu'(\phi) > \mu(\phi)$ .

Montrons que 2) implique 1). Soit  $\mu$  une valuation qui n'est pas bien spécifiée et nous supposons qu'il existe une valuation  $\mu'$  vérifiant  $\mu \leq \mu'$ . La famille admise  $\mathcal{A}(\mu)$  associée à  $\mu$  est alors une famille de la forme  $\mathcal{A} = (\mu_i)_{i \in I}$  où l'ensemble I n'a pas de plus grand élément, et pour tout i dans I nous avons  $\mu_i \leq \mu \leq \mu'$ . Pour tout polynôme f de K[x] il existe  $i_0$  dans I tel que pour tout  $i \geq i_0$  nous ayons  $\mu_{i_0}(f) = \mu_i(f) = \mu(f)$ , par conséquent si nous appelons comme précédemment  $\tilde{\Phi}(\mu_i)$ , respectivement  $\tilde{\Phi}'(\mu_i)$ , l'ensemble des polynômes f vérifiant  $\mu_i(f) < \mu(f)$ , respectivement  $\mu_i(f) < \mu'(f)$ , l'ensemble  $\bigcap_{i \in I} \tilde{\Phi}(\mu_i)$  est vide. Nous déduisons alors du lemme que l'ensemble  $\bigcap_{i \in I} \tilde{\Phi}'(\mu_i)$  est vide lui aussi, par conséquent la valuation  $\mu'$  est forcément égale à la valuation  $\mu$ . Définition. Soient  $\mu$  une valuation bien spécifiée de K[x] et  $\phi$  le polynôme qui la définit. Un polynôme-clé  $\phi'$  pour la valuation  $\mu$  est dit admissible s'il vérifie deg  $\phi' \ge \deg \phi$  et s'il n'est pas  $\mu$ -équivalent à  $\phi$ .

Soit  $\mathcal{A}$  une famille admissible de valuations de K[x], nous considérons deux valuations  $\mu$  et  $\mu'$  appartenant à la même sous-famille admissible simple  $\mathcal{S}$  de  $\mathcal{A}$  telle que  $\mu'$  est obtenue comme valuation augmentée  $\mu' = [\mu; \mu'(\phi') = \gamma']$ . Cela correspond au cas  $\mu = \mu_i$  et  $\mu' = \mu_{i+1}$ deux valuations successives appartenant à la partie discrète  $\mathcal{D}$ , au cas  $\mu = \mu_n$  dernière valuation de la partie discrète  $\mathcal{D}$  et  $\mu' = \mu_{\alpha}$  une valuation quelconque de la partie continue  $\mathcal{C}$ , ou au cas de deux valuations  $\mu = \mu_{\alpha}$  et  $\mu' = \mu_{\beta}$  de la partie continue  $\mathcal{C}$  avec  $\alpha < \beta$ . En particulier le polynôme-clé  $\phi'$  est un polynôme-clé admissible pour la valuation  $\mu$ .

Définition. Nous appelons un couple de valuations  $(\mu, \mu')$  vérifiant la propriété précédente un couple de valuations successives de la famille admissible  $\mathcal{A}$ .

Soit  $\mu$  une valuation bien spécifiée définie par le polynôme  $\phi$  et la valeur  $\gamma$ , nous reprenons la notation  $\mu_0$  précédente et nous appelons  $\Gamma_0$  soit le groupe des valeurs  $\Gamma_{\mu_0}$  de la valuation  $\mu_0$  si  $\mu$  est une valuation augmentée, soit le groupe des valeurs  $\Gamma_{\mathbf{A}}$  commun aux valuations de la famille continue  $(\mu_{\alpha})_{\alpha \in A}$  si  $\mu$  est une valuation augmentée limite.

**Proposition 1.6.** Pour tout  $\delta$  dans  $\Gamma_0$  il existe p appartenant à K[x] avec deg  $p < \text{deg } \phi$  tel que  $\mu(p) = \mu_0(p) = \delta$ .

*Preuve.* Appelons  $(p_0)$  la propriété

 $\forall \delta \in \Gamma_0 \exists p \in K[x] \text{ avec } \deg p < \deg \phi \text{ tel que } \mu(p) = \delta.$ 

La propriété  $(p_0)$  est évidemment vérifiée par toute valuation  $\mu$ définie par  $\mu(f) = \text{Inf}(\nu(a_j) + j\gamma; 0 \le j \le d)$  pour  $f = a_d x^d + \cdots + a_0$ , c'est-à-dire pour la première valuation de toute famille admissible. Comme toute valuation est obtenue à partir d'une famille admissible, il suffit alors de montrer les deux résultats suivants.

- Si une valuation bien spécifiée  $\mu$  vérifie  $(p_0)$  alors toute valuation augmentée  $\mu' = [\mu; \mu'(\phi') = \gamma']$  avec  $\phi'$  polynôme-clé admissible pour  $\mu$ , vérifie encore la propriété  $(p_0)$ , c'est-à-dire:

$$\forall \delta' \in \Gamma_{\mu} = \Gamma_0 + \gamma \mathbb{Z} \exists p' \in K[x] \text{ avec } \deg p' < \deg \phi' \text{ tel que } \mu(p') = \delta'.$$

En effet si la valuation  $\mu$  possède un polynôme-clé admissible  $\phi'$ , alors d'après le théorème de MacLane ([McL 1] Theorem 9.4) et la Proposition 1.3 la valeur  $\gamma$  vérifie  $m\gamma \in \Gamma_0$  avec  $m = \deg \phi' / \deg \phi$ . Par conséquent pour tout  $\delta'$  dans  $\Gamma_{\mu}$  il existe  $\delta \in \Gamma_0$  et t avec  $0 \leq t < m$  tel que

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 $\delta' = \delta + t\gamma$ , et le résultat est une conséquence de la propriété  $(p_0)$  pour la valuation  $\mu$ .

- Si  $(\mu_{\alpha})_{\alpha \in A}$  est une famille continue de valuations qui vérifient  $(p_0)$ alors toute valuation augmentée limite  $\mu' = [(\mu_{\alpha}); \mu'(\phi') = \gamma']$  avec  $\phi'$ polynôme-clé limite pour  $\mu$ , vérifie encore la propriété  $(p_0)$ , c'est-à-dire:

 $\forall \delta' \in \Gamma_{\mathbf{A}} \exists p' \in K[x] \text{ avec } \deg p' < \deg \phi' \text{ tel que } \mu(p') = \delta'.$ 

Il suffit de considérer  $\alpha < \beta$  dans A, alors la valuation  $\mu_{\beta}$  est une valuation augmentée  $\mu_{\beta} = [\mu_{\alpha}; \mu_{\beta}(\phi_{\beta}) = \gamma_{\beta}]$  qui vérifie la propriété  $(p_0)$ . Le résultat est alors une conséquence de l'égalité  $\Gamma_{\alpha} = \Gamma_{\mathbf{A}}$  et de deg  $\phi_{\beta} < \deg \phi'$ .

**Corollaire.** Soit  $\mathcal{A}$  une famille admissible de valuations de K[x], alors pour tout couple  $(\mu, \mu')$  de valuations successives de  $\mathcal{A}$ , avec  $\mu' = [\mu; \mu'(\phi') = \gamma']$ , il existe q et q' dans K[x] vérifiant qq'  $\mu$ -équivalent à 1 et  $\mu(q) = -\mu(q') = \mu(\phi')$ .

De plus si  $\mu'$  n'est pas la dernière valuation de la famille  $\mathcal{A}$ ,  $\gamma'$ appartient à  $\Gamma_{\mu} \otimes_{\mathbb{Z}} \mathbb{Q}$  et si nous appelons  $\tau$  le plus petit entier t > 0 tel que  $t\gamma' \in \Gamma_{\mu}$ , alors il existe p et  $p' = p'(\tau\gamma')$  dans K[x] vérifiant pp' $\mu'$ -équivalent à 1 et  $\mu'(p) = \mu(p) = -\mu'(p') = -\mu(p') = \tau\gamma'$ .

*Preuve.* Si  $\phi'$  est un polynôme-clé admissible pour la valuation  $\mu$  la valeur  $\mu(\phi')$  appartient au groupe  $\Gamma_0$ , par conséquent le résultat est une conséquence de la proposition précédente et de la Remarque 1.2.

De même, si  $\mu'$  n'est pas la dernière valuation de la famille, elle admet un polynôme-clé admissible  $\phi''$  et nous pouvons appliquer le théorème de MacLane ([McL 1] Theorem 9.4) ou la Proposition 1.3, et comme la valeur  $\tau \gamma'$  est dans  $\Gamma_{\mu}$  nous avons encore l'existence des polynômes p et p'.

## §2. Algèbre graduée

Pour toute valuation  $\mu$  de K[x] et pour tout  $\gamma$  dans  $\Gamma$ , nous définissons les groupes  $\mathcal{P}_{\gamma} = \{f \in K[x] \mid \mu(f) \geq \gamma\}$  et  $\mathcal{P}_{\gamma}^+ = \{f \in K[x] \mid \mu(f) > \gamma\}$ . Par définition l'algèbre graduée  $\operatorname{gr}_{\mu} K[x]$  associée à la valuation  $\mu$  est égale à:

$$\operatorname{gr}_{\mu}(K[x]) = \bigoplus_{\gamma \in \overline{\Gamma}} \mathcal{P}_{\gamma} / \mathcal{P}_{\gamma}^+.$$

Nous notons  $H_{\mu}$  l'application de K[x] dans  $\operatorname{gr}_{\mu}K[x]$  qui à tout polynôme f avec  $\mu(f) = \gamma$  associe l'image de f dans  $\mathcal{P}_{\gamma}/\mathcal{P}_{\gamma}^+$ , et nous notons  $\Delta_{\mu}$  la composante  $(\operatorname{gr}_{\mu}K[x])_0$  de degré 0.

Rappelons que si  $\mu'$  est une valuation augmentée  $\mu' = [\mu; \mu'(\phi) = \gamma]$ ou une valuation augmentée limite  $\mu' = [(\mu_{\alpha})_{\alpha \in A}; \mu'(\phi) = \gamma]$ , nous pouvons déterminer l'algèbre graduée  $\operatorname{gr}_{\mu'} K[x]$  associée à la valuation  $\mu'$  à partir de celle associée à la valuation  $\mu$ , ou à celles associées aux valuations  $\mu_{\alpha}$  ([Va 1] Théorème 1.7 et Théorème 1.26).

Plus précisément si  $\mu'$  est une valuation augmentée pour  $\mu$  définie par le polynôme-clé  $\phi$ ,  $\mu' = [\mu; \mu'(\phi) = \gamma]$ , l'application naturelle g de  $\operatorname{gr}_{\mu} K[x]$  dans  $\operatorname{gr}_{\mu'} K[x]$  induit un isomorphisme d'algèbres graduées

$$G: \left( \operatorname{gr}_{\mu} K[x] / (H_{\mu}(\phi)) \right) [T] \longrightarrow \operatorname{gr}_{\mu'} K[x],$$

qui envoie T sur  $G(T) = H_{\mu'}(\phi)$ , où  $\operatorname{gr}_{\mu} K[x]/(H_{\mu}(\phi))$  est muni de la structure d'algèbre graduée induite par celle de  $\operatorname{gr}_{\mu} K[x]$  et où T est muni du poids  $\gamma$ .

D'après le corollaire à la Proposition 1.6 il existe un polynôme  $\mu$ unitaire q' tel que  $\mu(q'\phi) = 0$  et le noyau de la composante de degré 0  $g_0: \Delta_{\mu} \to \Delta_{\mu'}$  est l'idéal engendré par  $\varphi = H_{\mu}(q'\phi)$ . Nous avons alors: si  $\alpha$  n'appartient pas à  $\Gamma_{\mu} \otimes_{\mathbb{Z}} \mathbb{O}$ 

si  $\gamma$  n'appartient pas à  $\Gamma_{\mu} \otimes_{\mathbb{Z}} \mathbb{Q}$ 

$$\Delta_{\mu'} \simeq (\Delta_{\mu}/(\varphi)),$$

si  $\gamma$  appartient à  $\Gamma_{\mu} \otimes_{\mathbb{Z}} \mathbb{Q}$ , en utilisant la deuxième partie du corollaire à la Proposition 1.6,

$$\Delta_{\mu'} \simeq \left( \Delta_{\mu} / (\varphi) \right) [S],$$

avec  $S = H_{\mu'}(p'(\tau\gamma)\phi^{\tau})$  (cf. [Va 1] Remarque 1.5).

Nous avons un résultat similaire pour une valuation augmentée limite  $\mu' = [(\mu_{\alpha})_{\alpha \in A}; \mu'(\phi) = \gamma]$ . Soit  $\mathcal{C} = (\mu_{\alpha})_{\alpha \in A}$  une famille continue de valuations, et nous pouvons toujours supposer que A admet un plus petit élément  $\theta$ , ce qui permet d'écrire toute valuation  $\mu_{\alpha}$  de  $\mathcal{C}$  comme valuation augmentée  $\mu_{\alpha} = [\mu_{\theta}; \mu_{\alpha}(\phi_{\alpha}) = \gamma_{\alpha}]$ , où tous les polynômes-clés  $\phi_{\alpha}$  sont de même degré d.

Nous remarquons que pour tout  $\alpha$  dans A, l'image  $H_{\mu_{\alpha}}(\phi_{\beta})$  du polynôme-clé  $\phi_{\beta}$  dans l'algèbre graduée  $\operatorname{gr}_{\mu_{\alpha}} K[x]$  ne dépend pas de  $\beta > \alpha$ . De plus grâce au corollaire à la Proposition 1.6, nous pouvons trouver  $p'(\gamma_{\alpha})$  dans K[x] dont l'image dans  $\operatorname{gr}_{\mu_{\alpha}} K[x]$  est inversible et de poids  $-\gamma_{\alpha} = -\mu_{\alpha}(\phi_{\beta})$ . Nous notons  $\varphi_{\alpha^{+}}$  l'image de  $p'(\gamma_{\alpha})\phi_{\beta}$  dans  $\operatorname{gr}_{\mu_{\alpha}} K[x]$ , c'est un élément de degré 0 qui engendre le même idéal que  $H_{\mu_{\alpha}}(\phi_{\beta})$ .

Si nous posons:

$$\operatorname{\mathbf{gr}}_{\mathbf{A}} = \operatorname{gr}_{\mu_{\theta}} K[x]/(\varphi_{\theta^+}),$$

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alors pour tout  $\alpha > \theta$ , l'algèbre  $\operatorname{gr}_{\mu_{\alpha}} K[x]$  est isomorphe à l'anneau de polynômes  $\operatorname{gr}_{\mathbf{A}}[T_{\alpha}]$  avec  $T_{\alpha} = H_{\mu_{\alpha}}(\phi_{\alpha})$ .

Alors pour tout  $\alpha > \theta$  l'algèbre quotient  $\operatorname{gr}_{\mu_{\alpha}} K[x]/(\varphi_{\alpha^{+}})$  est aussi isomorphe à  $\operatorname{\mathbf{gr}}_{\mathbf{A}}$  et pour tout  $\beta > \alpha$ , le morphisme d'algèbres graduées  $\operatorname{gr}_{\mu_{\alpha}} K[x] \longrightarrow \operatorname{gr}_{\mu_{\beta}} K[x]$  se factorise par:

$$\operatorname{gr}_{\mu_{\alpha}} K[x] \xrightarrow{U_{\alpha}} \operatorname{gr}_{\mu_{\alpha}} K[x] / (\varphi_{\alpha^{+}}) \xrightarrow{V_{\beta}} \operatorname{gr}_{\mu_{\beta}} K[x]$$

$$\uparrow^{\simeq} \qquad \uparrow^{\simeq} \qquad \uparrow^{\simeq}$$

$$\operatorname{gr}_{\mathbf{A}}[T_{\alpha}] \xrightarrow{u_{\alpha}} \operatorname{gr}_{\mathbf{A}} \xrightarrow{v_{\beta}} \operatorname{gr}_{\mathbf{A}}[T_{\beta}]$$

où  $v_{\beta}$  est le morphisme naturel mais où  $u_{\alpha}$  n'est pas un morphisme de  $\mathbf{gr}_{\mathbf{A}}$ -algèbres, en particulier son noyau n'est pas  $(T_{\alpha})$ . Mais nous pouvons remarquer que le morphisme composé  $U_{\beta} \circ V_{\beta}$  de  $\mathbf{gr}_{\mu_{\alpha}} K[x]/(\varphi_{\alpha^{+}}) \simeq \mathbf{gr}_{\mathbf{A}}$  dans  $\mathbf{gr}_{\mu_{\beta}} K[x]/(\varphi_{\beta^{+}}) \simeq \mathbf{gr}_{\mathbf{A}}$  est un isomorphisme.

Nous appelons  $\Delta_{\mathbf{A}}$  la partie homogène de degré 0 de  $\mathbf{gr}_{\mathbf{A}}$ , alors comme  $\varphi_{\theta^+}$  est de degré nul, nous avons:

$$\Delta_{\mathbf{A}} \simeq \Delta_{\theta} / (\varphi_{\theta^+}),$$

où nous notons comme précédemment  $\Delta_{\theta}$  la partie homogène de degré 0 de gr<sub> $\mu_{\theta}$ </sub> K[x].

En prenant les parties homogènes de degré 0 des algèbres du diagramme précédent nous trouvons le nouveau diagramme:

où  $S_{\alpha} = H_{\mu_{\alpha}}(p'(\gamma_{\alpha})\phi_{\alpha})$ , et nous avons encore le morphisme composé  $(U_{\beta})_0 \circ (V_{\beta})_0$  qui induit un isomorphisme de  $\Delta_{\mathbf{A}}$  dans lui-même.

Si l'ensemble  $\Phi(A)$  est vide, c'est-à-dire si pour tout f dans K[x] il existe  $\alpha$  dans A tel que  $\mu_{\alpha}(f) = \mu_{\beta}(f)$  pour tout  $\beta \geq \alpha$ , nous définissons une valuation limite  $\mu_A$  de K[x] par  $\mu_A(f) = \operatorname{Sup}_{\alpha \in A}(\mu_{\alpha}(f))$ . Dans ce cas la famille C n'admet pas de valuation augmentée limite et la valuation  $\mu_A$  n'est pas une valuation bien spécifiée.

**Proposition 2.1.** Si  $\tilde{\Phi}(A)$  est vide et si  $\mu_A$  est la valuation limite de la famille continue  $\mathcal{C} = (\mu_{\alpha})_{\alpha \in A}$ , alors pour tout  $\alpha$  dans A le morphisme naturel de  $\operatorname{gr}_{\mu_{\alpha}} K[x]$  dans  $\operatorname{gr}_{\mu_{A}} K[x]$  induit un isomorphisme

d'algèbres graduées:

$$Q \colon \mathbf{gr}_{\mathbf{A}} \xrightarrow{\sim} \mathrm{gr}_{\mu_A} K[x].$$

Cet isomorphisme induit un isomorphisme entre les parties homogènes de degré 0:

$$Q_0: \Delta_{\mathbf{A}} \xrightarrow{\sim} \Delta_{\mu_A}.$$

*Preuve.* Cf. [Va 1] Corollaire à la Proposition 1.25.

Si l'ensemble  $\Phi(A)$  n'est pas vide, pour  $\phi$  appartenant à  $\Phi(A)$  et pour  $\gamma$  vérifiant  $\gamma > \mu_{\alpha}(\phi)$  pour tout  $\alpha$  dans A, nous définissons une valuation augmentée limite  $\mu' = [(\mu_{\alpha})_{\alpha \in A}; \mu'(\phi) = \gamma].$ 

**Proposition 2.2.** Soit  $\mu' = [(\mu_{\alpha})_{\alpha \in A}; \mu'(\phi) = \gamma]$  une valuation augmentée limite pour la famille continue  $\mathcal{C} = (\mu_{\alpha})_{\alpha \in A}$ , alors pour tout  $\alpha$  dans A le morphisme naturel de  $\operatorname{gr}_{\mu_{\alpha}} K[x]$  dans  $\operatorname{gr}_{\mu'} K[x]$  induit un isomorphisme d'algèbres graduées:

$$Q\colon \mathbf{gr}_{\mathbf{A}}[T] \xrightarrow{\sim} \mathrm{gr}_{\mu'} K[x],$$

qui envoie T sur  $Q(T) = H_{\mu'}(\phi)$ .

De plus nous avons:

- si  $\gamma$  n'appartient pas à  $\Gamma_{\mathbf{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$ , ce morphisme induit un isomorphisme en degré 0:

$$Q_0\colon \Delta_{\mathbf{A}} \xrightarrow{\sim} \Delta_{\mu'}.$$

- si  $\gamma$  appartient à  $\Gamma_{\mathbf{A}} \otimes_{\mathbb{Z}} \mathbb{Q}$ , ce morphisme induit un isomorphisme en degré 0:

$$Q_0 \colon \Delta_{\mathbf{A}}[S] \xrightarrow{\sim} \Delta_{\mu'}.$$

qui envoie S sur  $H_{\mu'}(p'\phi^{\tau})$ , où nous appelons  $\tau$  le plus petit entier positif t tel que  $t\gamma$  appartienne à  $\Gamma_{\mathbf{A}}$  et où p' est un polynôme  $\mu_{\alpha}$ -unitaire pour  $\alpha$  suffisamment grand tel que  $\mu_{\alpha}(p') = -\tau\gamma$ .

*Preuve.* L'existence de l'isomorphisme Q d'algèbres graduées entre  $\mathbf{gr}_{\mathbf{A}}[T]$  et  $\mathbf{gr}_{\mu_{A}}K[x]$  est démontrée dans [Va 1] Théorème 1.26.

Grâce au corollaire à la Proposition 1.6, la démonstration des résultats qui s'en déduisent pour les parties homogènes de degré 0 est identique au cas d'une valuation augmentée (cf. [Va 1] Corollaire au Théorème 1.7).

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**Proposition 2.3.** Soit  $\mu$  une valuation de l'anneau des polynômes K[x], alors l'algèbre graduée associée  $gr_{\mu}K[x]$  est de la forme suivante: i) si la valuation  $\mu$  n'est pas bien spécifiée

$$\operatorname{gr}_{\mu}K[x] = \overline{G_0}$$

où  $\overline{G_0}$  est une algèbre graduée simple, c'est-à-dire telle que tout élément homogène non nul admette un inverse;

ii) si la valuation  $\mu$  est bien spécifiée

$$\operatorname{gr}_{\mu}K[x] = \overline{G_0}[T],$$

où  $\overline{G_0}$  est une algèbre graduée simple et T est l'image  $H_{\mu}(\phi)$  du polynôme  $\phi$  définissant la valuation  $\mu$ .

De plus un élément homogène  $\psi$  de  $\operatorname{gr}_{\mu}K[x]$  est irréductible si et seulement si il existe f polynôme-clé pour la valuation  $\mu$  dans K[x] et  $\varepsilon$ élément homogène inversible de  $\operatorname{gr}_{\mu}K[x]$  tels que  $\varepsilon\psi$  soit égal à l'image  $H_{\mu}(f)$  de f dans  $\operatorname{gr}_{\mu}K[x]$ .

Preuve. Considérons d'abord le cas d'une valuation augmentée  $\mu = [\mu_0; \mu(\phi) = \gamma]$ , alors l'algèbre graduée  $\operatorname{gr}_{\mu} K[x]$  est isomorphe à  $\overline{G_0}[T]$  avec  $\overline{G_0} = \operatorname{gr}_{\mu_0} K[x]/(H_{\mu_0}(\phi))$ . Nous pouvons identifier  $\overline{G_0}$  à la sous-algèbre graduée engendrée par les éléments homogènes  $\psi$  de la forme  $H_{\mu}(f)$  avec f tels qu'il existe g dans K[x]  $\mu$ -équivalent à f vérifiant  $\mu_0(g) = \mu(g)$ . Il existe alors g' dans K[x] vérifiant  $\mu_0(g') = \mu(g')$  tel que gg' soit  $\mu$ -équivalent à 1 ([Va 1] Lemme 1.4), par conséquent  $\psi' = H_{\mu}(g')$  est un inverse de  $\psi$  dans  $\overline{G_0}$ .

Supposons maintenant que nous avons une famille continue de valuations  $(\mu_{\alpha})_{\alpha \in A}$ , et nous notons comme précédemment  $\mathbf{gr}_{\mathbf{A}}$  l'algèbre graduée  $\operatorname{gr}_{\mu_{\theta}} K[x]/(\varphi_{\theta^+})$ . Nous déduisons de ce qui précède que  $\mathbf{gr}_{\mathbf{A}}$  est aussi une algèbre graduée simple, et la première partie de la proposition est une conséquence des Propositions 2.1 et 2.2.

Si f est un polynôme-clé pour une valuation  $\mu$  alors par définition il est  $\mu$ -irréductible, c'est-à-dire que son image  $H_{\mu}(f)$  est un élément irréductible de l'algèbre graduée  $\operatorname{gr}_{\mu} K[x]$ .

Réciproquement soit  $\psi$  un élément homogène irréductible de l'algèbre graduée  $\operatorname{gr}_{\mu} K[x]$ . Nous déduisons de la première partie de la proposition que la valuation  $\mu$  est bien spécifiée, c'est-à-dire  $\mu$  est soit une valuation augmentée  $\mu = [\mu_0; \mu(\phi) = \gamma]$ , soit une valuation augmentée limite  $\mu = [(\mu_{\alpha})_{\alpha \in A}; \mu(\phi) = \gamma]$ . Nous choisissons f dans K[x] tel que  $H_{\mu}(f) = \psi$ , et nous écrivons le développement de f selon les puissances de  $\phi$ ,  $f = f_m \phi^m + \ldots + f_0$ . Quitte à remplacer f par un polynôme  $\mu$ équivalent nous pouvons supposer que nous avons  $\mu(f) = \mu_0(f_m) + m\gamma$ , et quitte à multiplier f par un polynôme h avec deg  $h < \deg \phi$  nous pouvons supposer que f est  $\mu$ -équivalent à un polynôme de la forme  $\phi^m + \ldots + f_0$  avec  $\mu(f) = m\gamma$ . Comme  $\psi$  est  $\mu$ -irréductible nous avons aussi  $\mu(f_0) = m\gamma$ , par conséquent nous déduisons de [McL 1] Theorem 9.4 ou de [Va 1] Théorème 1.11 dans le cas d'une valuation augmentée, et de la Proposition 1.3 dans le cas d'une valuation augmentée limite, que f est un polynôme-clé pour  $\mu$ .

Nous disons qu'un polynôme e de K[x] est  $\mu$ -unitaire s'il existe un polynôme e' dans K[x] tel que ee' soit  $\mu$ -équivalent à 1, c'est-à-dire si son image  $H_{\mu}(e)$  dans  $\operatorname{gr}_{\mu}K[x]$  est inversible. Nous déduisons alors de la proposition précédente la généralisation suivante du résultat de MacLane ([McL 2] Theorem 4.2).

**Corollaire.** Soit  $\mu$  une valuation de K[x], alors pour tout polynôme f il existe un polynôme  $\mu$ -unitaire e et des polynômes-clés pour la valuation  $\mu \phi_1, \ldots, \phi_t, t \ge 0$ , tels que nous ayons:

$$f \mathop{\sim}_{\mu} e \phi_1 \cdots \phi_t.$$

De plus cette décomposition est unique à  $\mu$ -équivalence près.

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# Plane curve singularities whose Milnor and Tjurina numbers differ by three

## Masahiro Watari

## Abstract.

Bayer and Hefez described irreducible plane curve singularities whose Milnor and Tjurina numbers differ by one or two, modulo analytic equivalence. After their work, we classify the case in which their difference is three.

## § Introduction

We first define a *plane curve singularity*, which is the main subject in the present paper. Let f be an irreducible element of  $\mathbb{C}[[X, Y]]$  such that its partial derivatives  $f_X$  and  $f_Y$  belong to the maximal ideal (X, Y). Set

$$C := \{ u \cdot f \mid u \text{ is a unit of } \mathbb{C}[[X, Y]] \}.$$

If f is a convergent power series, f = 0 defines a singular germ of a plane curve at the origin. So it is natural that we call C an irreducible plane curve singularity. The Milnor and Tjurina numbers of C at the origin are defined by,

$$\mu := \dim_{\mathbb{C}} \mathbb{C}[[X, Y]]/(f_X, f_Y) \text{ and } \tau := \dim_{\mathbb{C}} \mathbb{C}[[X, Y]]/(f, f_X, f_Y).$$

It follows from these definitions that  $\mu \geq \tau$ . We set  $r := \mu - \tau$ . Let *n* be the multiplicity of *C* at the origin. Then there exists a positive integer *m* with m > n and  $n \nmid m$  such that *C* has the following parametrization at the origin:

(1) 
$$x = t^n, \quad y = t^m + a_{m+1}t^{m+1} + \cdots,$$

where  $x \equiv X \mod (f)$  and  $y \equiv Y \mod (f)$ . The local ring of C is defined by  $\mathcal{O}_C := \mathbb{C}[[X, Y]]/(f)$ . Using the parametrization (1), we have the

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following isomorphism:  $\mathcal{O}_C \cong \mathbb{C}[[x, y]] = \mathbb{C}[[t^n, t^m + a_{m+1}t^{m+1} + \cdots]]$ . Let D be an irreducible plane curve singularity. Then C and D are said to be *analytically equivalent*, if there exists a  $\mathbb{C}$ -algebra isomorphism  $\mathcal{O}_C \cong \mathcal{O}_D$ .

Zariski ([7]) showed that r = 0 if and only if C is analytically equivalent to the singularity  $Y^n - X^m = 0$  with gcd(n, m) = 1. When  $r \neq 0$ , he introduced an important invariant  $\lambda$ . Recently, Bayer and Hefez ([3]) classified irreducible plane curve singularities with r = 1 and 2. Their work was reviewed by Azevedo in [2]. The aim of this paper is to classify irreducible plane curve singularities with r = 3.

**Theorem.** Let C be an irreducible plane curve singularity whose parametrization is of the form (1). Then we have r = 3 if and only if gcd(n, m) = 1 and the parametrization takes one of the following three types. We write m = pn + q with 0 < q < n.

 $\begin{array}{ll} \text{Type (i):} & \lambda = (n-1)m - 4n. \\ (A) \ x = t^n, \ y = t^m + t^{\lambda}, & where \ n \geq 3, \ p \geq 2. \\ (B) \ x = t^n, \ y = t^m + t^{\lambda} + at^{(n-2)m-2n}, \\ where \ n \geq 5, \ p = 1 \ and \ a \in \mathbb{C}. \end{array} \\ \\ \text{Type (ii):} & \lambda = (n-2)m - 2n. \\ (C) \ x = t^n, \ y = t^m + t^{\lambda} + at^{(n-1)m-4n} + bt^{(n-1)m-3n}, \\ where \ n \geq 5, \ p \geq 2 \ and \ a \ (\neq 0), \ b \in \mathbb{C}. \\ (D) \ x = t^4, \ y = t^m + t^{\lambda} + at^{3m-16} + bt^{3m-12}, \\ where \ p \geq 2 \ and \ a \ (\neq (3m-8)/2m), \ b \in \mathbb{C}. \end{array} \\ \\ \\ \text{Type (iii)} \quad \lambda = (n-3)m - 2n \\ (E) \ x = t^n, \ y = t^m + t^{\lambda} + \sum_{i=1}^p (a_i t^{m_i} + b_i t^{n_i}) + \sum_{i=p+1}^{2p} b_i t^{n_i}, \\ where \ n > 2q, \ n \geq 5, \ m > 2n/(n-4), \ a_i, \ b_i \in \mathbb{C}, \ m_i = (n-2)m - (p+3-i)n \ and \ n_i = (n-1)m - (2p+3-i)n. \\ \\ \\ (F) \ x = t^n, \ y = t^m + t^{\lambda} + \sum_{i=1}^p (a_i t^{m_i} + b_i t^{n_i}) + \sum_{i=p+1}^{2p+1} a_i t^{m_i}, \\ where \ n < 2q, \ n \geq 5, \ m > 2n/(n-4), \ a_i, \ b_i \in \mathbb{C}, \ m_i = (n-1)m - (2p+4-i)n \ and \ n_i = (n-2)m - (p+4-i)n. \end{array}$ 

Furthermore, the coefficients in the parametrizations (E) and (F) must satisfy the relations given in Tables 1 and 2 in Section 4, respectively.

The present paper is organized as follows: In Section 1, we recall some results on the parametrization of plane curve singularities. The notion "genus" g of an irreducible plane curve singularity plays an important role. We infer from a result of Bayer and Hefez that if r = 3, then g = 1 or g = 2. In Section 2, we study the properties of plane curve singularities of genus one. In particular, we consider the certain types of  $\lambda$  which are needed in the proof of Theorem. In Section 3, we prove the following fact. **Proposition 1.** If r = 3, then we have g = 1.

In Section 4, we develop the method in [3] and prove Theorem by using it.

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## $\S1$ . Semigroups and differentials

Let  $\widetilde{C}$  be the nonsingular model of C and we denote by  $\mathcal{O}_{\widetilde{C}}$  its local ring. Since  $\mathcal{O}_{\widetilde{C}} \cong \mathbb{C}[[t]]$ , the order function  $\nu$  on  $\mathbb{C}((t))$  gives a discrete normalized valuation of  $\mathcal{O}_{\widetilde{C}}$ . We define the *semigroup* of C to be  $S := \{\nu(A) \mid A \in \mathcal{O}_C\}$ . The *conductor* c of S is characterized by the following properties:

(2) 
$$c-1 \notin S \text{ and } c+n \in S \text{ for any } n \in \mathbb{N}.$$

It is well known that  $\mu = c$  (See [6], Theorem 1). An element of  $G := \mathbb{N} \cup \{0\} \setminus S$  is called a *gap* of *S*. The properties (2) implies that c - 1 is the biggest gap of *S*. We define two sequences  $(e_i)$  and  $(\beta_i)$  associated to the parametrization (1) as follows:

$$e_0 = \beta_0 = n, \quad \beta_j = \min\{i \mid i \neq 0 \mod e_{j-1} \text{ and } a_i \neq 0\},$$
$$e_j = \gcd(e_{j-1}, \beta_j).$$

It follows that  $\beta_1 = m$ . Since the relevant exponents in a parametrization of C are coprime, there exists an integer g such that  $e_{g-1} \neq 1$  and  $e_g = 1$ . We call this integer g and the set  $\{\beta_0, \ldots, \beta_g\}$  the genus of C and the *characteristic* of C respectively. The characteristic of C is denoted by Ch(C). Define the integers  $n_i$  by

$$n_0 = 1$$
 and  $e_{i-1} = n_i e_i$ ,  $(i = 1, \dots, g)$ .

It follows that  $n = n_1 \cdots n_g$ . The semigroup S of C is minimally generated by the set of integers  $\{v_0, v_1, \ldots, v_g\}$ , defined by

$$v_0 = n$$
 and  $v_i = n_{i-1}v_{i-1} + \beta_i - \beta_{i-1}, \quad (i = 1, \dots, g).$ 

(See [8], Theorem 3.9) We easily see that  $v_1 = m$  and  $v_0 < v_1 < \cdots < v_g$ .

We denote by  $\Omega_C^1$  and  $\Omega_{\widetilde{C}}^1$  the module of differentials of  $\mathcal{O}_C$  and that of  $\mathcal{O}_{\widetilde{C}}$ , respectively. Note that  $\Omega_C^1$  is the  $\mathcal{O}_C$ -module generated by

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dx and dy, modulo  $f_x dx + f_y dy = 0$ . Similarly,  $\Omega^1_{\widetilde{C}}$  is the  $\mathcal{O}_{\widetilde{C}}$ -module generated by dt. Consider the map  $\pi^*$  from  $\Omega^1_C$  to  $\Omega^1_{\widetilde{C}}$  defined by

$$\pi^* \big( A(x, y) dx + B(x, y) dy \big) = A \left( t^n, \varphi(t) \right) dt^n + B(t^n, \varphi(t)) d(\varphi(t)),$$

where  $(t^n, \varphi(t))$  is the parametrization of C. We naturally extend the valuation  $\nu$  of  $\mathcal{O}_{\widetilde{C}}$  to  $\Omega_{\widetilde{C}}^1$  through  $\pi^*$ . Namely, for  $\zeta = H(t)dt \in \Omega_{\widetilde{C}}^1$ , we define  $\nu(\zeta)$  to be  $\nu(H(t))$ . Let  $\xi$  be an element of  $\Omega_C^1$ . Since  $\Omega_C^1$ can be regarded as a submodule of  $\Omega_{\widetilde{C}}^1$  through its image of  $\pi^*$ , we define  $\nu(\xi)$  to be  $\nu(\pi^*(\xi))$ . A differential  $\xi$  is said to be *exact*, if there exists an element  $A \in \mathcal{O}_C$  such that  $\xi = dA$ . We denote by  $d\mathcal{O}_C$ the set of all exact differentials. Set  $V := \nu(\Omega_C^1) \setminus \nu(d\mathcal{O}_C)$ . Since  $\nu(d\mathcal{O}_C) = \{l-1 \in \mathbb{N} \mid l \in S\}$ , we have  $V = \{l-1 \in \nu(\Omega_C^1) \mid l \in G\}$ . Zariski ([7]) showed that

(3) 
$$r = \dim_{\mathbb{C}} \left( \Omega_C^1 / d\mathcal{O}_C \right) = \sharp(V).$$

For the case where r > 0, he also showed that  $\lambda = \min\{V\} - n + 1$  is an analytic invariant with the following property:

(4) 
$$\lambda, \lambda + n \notin S \text{ and } m < \lambda \leq \beta_2 = v_2 - (n_1 - 1)v_1.$$

We call  $\lambda$  the Zariski invariant of C. The differential  $\omega := mydx - nxdy$  gives the minimal order  $\lambda + n - 1$  in V (See [7]). Furthermore, C is analytically equivalent to the plane curve singularity given by

(5) 
$$x = t^n, \quad y = t^m + t^\lambda + \cdots.$$

By the way, any integer t can be written in a unique way as

(6) 
$$t = t_g v_g + \dots + t_1 v_1 - t_0 v_0,$$

where  $t_0, \ldots, t_g$  are integers such that  $0 \le t_i \le n_i - 1$  for  $i = 1, \ldots, g$ (See [1], Lemma I.2.4). It follows from (6) that t belongs to S if and only if  $t_0 \le 0$ . The biggest gap of S, c - 1, is expressed as  $(n_g - 1)v_g + \cdots + (n_1 - 1)v_1 - v_0$ . The Zariski invariant is also written as  $\lambda = \lambda_g v_g + \cdots + \lambda_1 v_1 - \lambda_0 v_0$  where  $0 \le \lambda_i \le n_i - 1$  for  $i = 1, \ldots, g$ . By the properties (4), we easily see that  $\lambda_0 \ge 2$ .

The  $\mathbb{C}$ -vector space  $\Omega_C^1$  is expressed as the following form ([3], Proposition 2):

(7) 
$$\Omega_C^1 = \mathcal{O}_C \omega + d\mathcal{O}_C.$$

So any element of  $\Omega^1_C$  can be written as  $A\omega + dB$  for some  $A, B \in \mathcal{O}_C$ .

**Definition 2.** We define the subsets  $V_0$ ,  $V_1$  of V by

$$V_0 := \nu(\mathcal{O}_C \omega) \setminus \nu(d\mathcal{O}_C), \quad V_1 := V \setminus V_0.$$

Furthermore, define the sets  $V^+$ ,  $V_0^+$  and  $V_1^+$  by

$$V^{+} := \{ \alpha + 1 \mid \alpha \in V \}, \quad V_{0}^{+} := \{ \alpha + 1 \mid \alpha \in V_{0} \},$$
$$V_{1}^{+} := \{ \alpha + 1 \mid \alpha \in V_{1} \}.$$

Note that we have  $V^+ = V_0^+ \cup V_1^+ \subset G$  where  $V_0^+ \cap V_1^+ = \emptyset$ . A positive integer  $\alpha$  is contained in V if and only if  $\alpha + 1$  is contained in  $V^+$ . So we have  $\sharp(V) = \sharp(V^+)$ . It is also clear that the relations  $\sharp(V^+) = \sharp(V) = \sharp(V_0) + \sharp(V_1), \ \sharp(V_0^+) = \sharp(V_0)$  and  $\sharp(V_1^+) = \sharp(V_1)$  hold. The formula (3) can be rewritten as

(8) 
$$r = \sharp(V_0^+) + \sharp(V_1^+).$$

**Lemma 3.** Let the genus of C be 1. Then we have

$$\sharp(V_0^+) = (\lambda_0 - 1)(n - \lambda_1).$$

*Proof.* Recall that  $v_0 = n$ ,  $v_1 = m$  and  $\nu(\omega) = \lambda_1 m - (\lambda_0 - 1)n - 1$ . If  $\gamma \in V_0^+$ , then we have  $\gamma = \nu(A\omega) + 1$  for some  $A \in \mathcal{O}_C$ . The gap  $\gamma$  is expressed as  $(l_1 + \lambda_1)m - (\lambda_0 - l_0 - 1)n$  where  $\nu(A) = l_1m + l_0n$ . So we have

$$V_0^+ = \left\{ (l_1 + \lambda_1)m - (\lambda_0 - l_0 - 1)n \in V^+ \ \middle| \ \begin{array}{l} 0 \le l_0 \le \lambda_0 - 2, \\ l_1 \ge 0 \end{array} \right\}.$$

Define a subset  $U_0^+$  of  $V_0^+$  by

$$U_0^+ = \left\{ (l_1 + \lambda_1)m - (\lambda_0 - l_0 - 1)n \mid \begin{array}{l} 0 \le l_0 \le \lambda_0 - 2, \\ 0 \le l_1 + \lambda_1 \le n - 1 \end{array} \right\}.$$

We prove that  $V_0^+ = U_0^+$ , which gives the desired result. It is enough to show that  $V_0^+ \subset U_0^+$ . Take an element  $\gamma = (l_1 + \lambda_1)m - (\lambda_0 - l_0 - 1)n$ from  $V_0^+$ . If  $l_1 + \lambda_1 \leq n - 1$ , then there is nothing to prove. So we assume that  $l_1 + \lambda_1 > n - 1$ . Then there exists a positive integer k such that  $0 \leq l_1 + \lambda_1 - kn \leq n - 1$ . By using this k, we rewrite  $\gamma$  as the form of (6). That is,

$$\gamma = (l_1 + \lambda_1 - kn)m - (\lambda_0 - km - l_0 - 1)n.$$

Since  $\gamma \in V^+$ , the inequality  $\lambda_0 - km - l_0 - 1 > 0$  holds. Then  $\gamma$  is given by  $\nu(x^{km+l_0}\omega) + 1$ , so  $\gamma \in U_0^+$ . Q.E.D.

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**Remark 4.** We infer from Lemma 3 that  $\sharp(V_0^+)$  is determined by  $\lambda$  for the case where g = 1.

For any genus, the following relations hold (See [3], Proposition 1, Corollary 5):

(9) 
$$r \ge (\lambda_0 - 1)(n_1 - \lambda_1) \cdots (n_g - \lambda_g),$$

 $(10) r \ge 2^{g-1}.$ 

Let  $\zeta = (at^{\nu(\zeta)} + \text{terms of higher degree})dt$  be an element of  $\Omega^1_{\widetilde{C}}$ . We denote by  $LT(\zeta)$  the leading term  $at^{\nu(\zeta)}$ . Let  $LC(\zeta)$  denote the leading coefficient a. For  $\xi \in \Omega^1_C$ , we simply write  $LT(\xi) = LT(\pi^*(\xi))$  and  $LC(\xi) = LC(\pi^*(\xi))$ .

**Lemma 5.** Let C be an irreducible plane curve singularity of genus g. If  $\xi = A\omega + dB$  is an element of  $\Omega_C^1$  with  $\nu(\xi) + 1 \in V_1^+$ , then  $\xi$  satisfies the following conditions:

(11) 
$$LT(A\omega) + LT(dB) = 0,$$

(12) 
$$\nu(\xi) + 1 < \sum_{i=1}^{g} (n_i - 1)v_i - v_0.$$

*Proof.* If  $\operatorname{LT}(A\omega) + \operatorname{LT}(dB) \neq 0$ , then  $\nu(\xi)$  belongs to  $V_0$  or to  $\nu(d\mathcal{O}_C)$ . Hence the condition (11) must occur. Let  $\lambda = \sum_{i=1}^g \lambda_i v_i - \lambda_0 v_0$  be the Zariski invariant of C. Then  $\nu(\omega) + 1$  is expressed as  $\sum_{i=1}^g \lambda_i v_i - (\lambda_0 - 1)v_0$ . We know that  $\max\{V^+\} \leq \sum_{i=1}^g (n_i - 1)v_i - v_0$ . Let  $z_i$  be an element of  $\mathcal{O}_C$  with  $\nu(z_i) = v_i$  for  $i = 0, \ldots, g$ . Then the differential  $z_0^{\lambda_0 - 2} \prod_{i=1}^g z_i^{(n_i - \lambda_i - 1)} \omega$  gives the order  $\sum_{i=1}^g (n_i - 1)v_i - v_0 - 1$ . Hence  $\sum_{i=1}^g (n_i - 1)v_i - v_0 \in V_0^+$ . We have the desired consequence. Q.E.D.

There are some criteria for simplifying the parametrization of C modulo analytic equivalence (See [4] and [8], Ch.III, Proposition 1.2; Ch.IV, Lemma 2.6 and Proposition 3.1).

**Lemma 6.** Let  $a_s t^s$  be a term of y in the parametrization (5) where  $s > \lambda$  and  $a_s \neq 0$ . If either

- (EC 1): s belongs to S, or
- **(EC 2):** s + n = lm for some  $l \in \mathbb{N}$ , or
- **(EC 3):**  $s \lambda$  belongs to the subset of S generated by n and m,

then C is analytically equivalent to an irreducible plane curve singularity given by a parametrization of the same form, but with  $a_s = 0$  and  $a_i$ unchanged for i < s. Applying Lemma 6 to the parametrization (5), we have the following parametrization:

$$x = t^n$$
,  $y = t^m + t^\lambda + \sum_{i \in G} a_i t^i$ .

We always consider such parametrizations of C in this paper.

#### $\S 2$ . Singularities of genus one

In this section, we consider irreducible plane curve singularities of genus 1. Note that g = 1 if and only if gcd(n, m) = 1. In this case, we have  $v_0 = n = n_1$  and  $v_1 = m$ . We write m = pn + q where 0 < q < n. We first prove the following proposition:

**Proposition 7.** If C is given by

(13) 
$$x = t^n, \quad y = t^m + t^{(n-1)m - (R+1)n},$$

where  $1 \leq R \leq p+1$ , then we have r = R.

*Proof.* Note that  $\lambda = (n-1)m - (R+1)n$  in (13). If  $1 \le R \le p$ , then we have  $(n-2)m - n < (n-1)m - (p+1)n \le (n-1)m - (R+1)n$ . So the gaps which are greater than  $\lambda$  are

(14) 
$$(n-1)m - Rn, \ldots, (n-1)m - n.$$

If R = p + 1, then we have (n - 1)m - (R + 1)n < (n - 2)m - n < (n - 1)m - Rn. The gaps which are greater than  $\lambda$  are

$$(n-2)m-n, (n-1)m-Rn, \dots, (n-1)m-n.$$

For both cases, clearly, we have  $V_0 = \{\nu(\omega), \nu(x\omega), \ldots, \nu(x^{R-1}\omega)\}$ . Note that  $\nu(x^i\omega) + 1 = (n-1)m - (R-i)n$  for  $i = 0, \ldots, R-1$ . Since  $\min\{V^+\} = \nu(\omega) + 1$ , we conclude that  $V_1^+ = \emptyset$ . Hence we conclude that r = R by (8). Q.E.D.

**Remark 8.** Since  $\lambda = (n-1)m - (R+1)n$ , we infer from Proposition 7 that r is determined by  $\lambda$  for the plane curve singularity given by the parametrization (13).

**Remark 9.** The cases where R = 1 and R = 2 in Proposition 7 correspond to Theorems 7 and 17 in [3], respectively.

**Corollary 10.** Fix a positive integer  $n (\geq 3)$ . For any positive integer R, there exists an irreducible plane curve singularity of g = 1 with multiplicity n and r = R.

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*Proof.* Put m = (R+1)n+1 and  $\lambda = (n-1)m - (R+1)n$ . Then we have  $\lambda > m$  and (n-1)m - (R+1)n > (n-2)m - n. Hence the gaps which are greater than  $\lambda$  are same as (14). Therefore the parametrization

$$x = t^n, \quad y = t^m + t^{(n-1)m - (R+1)n}$$

Q.E.D.

gives the desired singularity.

In what follows, we consider three types of the values of  $\lambda$ : (i)  $\lambda = (n-1)m - 4n$ , (ii)  $\lambda = (n-2)m - 2n$ , (iii)  $\lambda = (n-3)m - 2n$ , which will be used in the proof of Theorem.

## 2.1. Type (i): $\lambda = (n-1)m - 4n$

Since  $\lambda > m$ , we must have (n-2)m > 4n, hence  $n \ge 3$ . We first consider the case in which  $p \ge 2$ . Furthermore, if  $p \ge 3$ , then the gaps which are greater than  $\lambda$  are

$$(n-1)m-3n$$
,  $(n-1)m-2n$ ,  $(n-1)m-n$ .

On the other hand, if p = 2, then we have the following gaps:

$$(n-2)m-n, (n-1)m-3n, (n-1)m-2n, (n-1)m-n.$$

For both cases, by Lemma 6, the parametrization of C can be taken as

(15) 
$$x = t^n, \quad y = t^m + t^{\lambda}.$$

Next we consider the case in which p = 1. This case occurs only when  $n \ge 5$ . The gaps which are greater than  $\lambda$  are

$$(n-2)m-2n, (n-1)m-3n, (n-2)m-n,$$
  
 $(n-1)m-2n, (n-1)m-n.$ 

By Lemma 6, the parametrization of C can be taken as

(16) 
$$x = t^n, \quad y = t^m + t^\lambda + at^{(n-2)m-2n}, \quad (a \in \mathbb{C}).$$

**Remark 11.** According to the conditions: (1) gcd(n, m) = 1, (2) m > 4n/(n-2), we have some restrictions on p, q. First of all, we must have  $q \ge 1$ . We also infer that gcd(n, q) = 1.

## 2.2. Type (ii): $\lambda = (n-2)m - 2n$

It follows from  $\lambda > m$  that  $n \ge 4$  and m > 2n/(n-3). Since  $\nu(\omega) + 1 = (n-2)m - n$ , we find that  $V_0 = \{\nu(\omega), \nu(y\omega)\}$ . The gaps which are greater than  $\lambda$  are

$$(n-1)m - (p+2)n, (n-2)m - n, (n-1)m - (p+1)n,$$
  
 $(n-1)m - pn, \dots, (n-1)m - n.$ 

By Lemma 6, the parametrization of C can be taken as

(17) 
$$x = t^n, \quad y = t^m + t^\lambda + \sum_{i=1}^p a_i t^{m_i},$$

where  $m_i = (n-1)m - (p+3-i)n$  and  $a_i \in \mathbb{C}$ .

**Definition 12.** Define the differentials  $\eta_k$  for  $k \ge 1$  by

$$\eta_k := x^k \omega + d \left( u_k x^{k-1} y^{n-2} \right)$$
 where  $u_k = \frac{-n(m-\lambda)}{(k-1)n + (n-2)m}$ 

Then we have

$$\eta_k = x^k \omega + u_k \left\{ (k-1) x^{k-2} y^{n-2} dx + (n-2) x^{k-1} y^{n-3} dy \right\}.$$

Furthermore, we see that

(

$$\pi^{*}(\eta_{k}) = n \left[ (m - \lambda)t^{(n-2)m + (k-1)n - 1} + \sum_{i=1}^{p} a_{i}(m - m_{i})t^{(n-1)m - (p+2-i-k)n - 1} \right] dt$$

$$+ u_{k}(k - 1)n \left[ t^{(n-2)m + (k-1) - 1} + (n - 2)t^{(2n-5)m - (3-k)n - 1} + \cdots \right] dt$$

$$+ u_{k}(n - 2) \left[ mt^{(n-2)m + (k-1) - 1} + \{m(n - 3) + \lambda\}t^{(2n-5)m - (3-k)n - 1} + \cdots \right] dt$$

So we have  $LT(x^k\omega) + LT(d(u_kx^{k-1}y^{n-2})) = 0$  for any k. Comparing (n-1)m - (3-k)n - 1 with (2n-5)m - (3-k)n - 1 in (18), we have the following relations according to n.

$$(n-1)m - (3-k)n - 1 < (2n-5)m - (3-k)n - 1$$
 for  $n \ge 5$ ,  
 $(n-1)m - (3-k)n - 1 = (2n-5)m - (3-k)n - 1$  for  $n = 4$ .

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So we consider the cases (C)  $n \ge 5$  and (D) n = 4 separately.

**Lemma 13.** Let  $\xi$  be an element of  $\Omega_C^1$  with  $\nu(\xi) + 1 \in V_1^+$ . Then we have  $\nu(\xi) \ge \nu(\eta_k)$  for some k.

*Proof.* Put  $\xi = A\omega + dB$  where  $A, B \in \mathcal{O}_C$ . There exists only one term  $c_1 x^{k_1} y^{l_1}$  in A such that  $\nu(A) = \nu(c_1 x^{k_1} y^{l_1})$ . Then we must have  $l_1 = 0$ . Indeed, if not, we have

$$\nu(A\omega) + 1 = (l_1 + n - 2)m + (k_1 - 1)n \ge (n - 1)m - n$$

By (12), we see that  $\nu(\xi) + 1 \notin V^+$ , which is a contradiction.

Since the cancellation (11) occurs, we have  $\nu(dB) = (n-2)m + (k_1 - 1)n - 1$ . So the function B contains only one term  $h_1 x^{k_1 - 1} y^{n-2}$  such that  $\nu(dB) = \nu(h_1 x^{k_1 - 1} y^{n-2})$ . Since LT  $(c_1 x^{k_1} \omega) + \text{LT} \left( d \left( h_1 x^{k_1 - 1} y^{n-2} \right) \right) = 0$ , we easily see that  $h_1 = c_1 u_{k_1}$ . Hence  $\xi$  can be written as  $c_1 \eta_{k_1} + \xi_1$  where  $\xi_1 = \left( A - c_1 x^{k_1} \right) \omega + d \left( B - h_1 x^{k_1 - 1} y^{n-2} \right)$ . If  $\nu(\xi_1) + 1 \in V_1^+$ , then we can apply the same argument to  $\xi_1$ . Namely, there exists  $\eta_{k_2}$  such that  $\xi = c_1 \eta_{k_1} + c_2 \eta_{k_2} + \xi_2$  where  $\xi_2 = \left( A - c_1 x^{k_1} - c_2 x^{k_2} \right) \omega + d \left( B - h_1 x^{k_1 - 1} y^{n-2} - h_2 x^{k_2 - 1} y^{n-2} \right)$ . Note that  $\nu(A - c_1 x^{k_1} - c_2 x^{k_2}) > \nu(A - c_1 x^{k_1})$ . We can continue this procedure successively. After the j-th step, we have

$$\xi = \sum_{i=1}^{j} c_i \eta_{k_i} + \xi_j,$$
  
where  $\xi_j = \left(A - \sum_{i=1}^{j} c_i x^{k_i}\right) \omega + d\left(B - \sum_{i=1}^{j} h_i x^{k_i - 1} y^{n-2}\right).$ 

Since we have  $\nu \left(A - \sum_{i=1}^{j+1} c_i x^{k_i}\right) > \nu \left(A - \sum_{i=1}^{j} c_i x^{k_i}\right)$ , there exists a positive integer j such that  $\nu(\xi_j) \ge \nu \left(\left(A - \sum_{i=1}^{j} c_i x^{k_i}\right)\omega\right) > (n - 1)m - n - 1$ . It follows from (12) that  $\nu(\xi_j) + 1 \notin V_1^+$ . So we have  $\xi = \sum c_i \eta_{k_i} + \xi_j$  where  $\nu(\sum c_i \eta_{k_i}) < \nu(\xi_j)$ . Thus we obtain  $\nu(\xi) \ge \min\{\nu(\eta_{k_i})\}$ . Q.E.D.

**Lemma 14.** Let C be an irreducible plane curve singularity given by (17). If  $n \ge 5$  and p = 1, then we have  $V_1^+ = \emptyset$ .

*Proof.* Assume that  $V_1 \neq \emptyset$ . Let  $\xi$  be a differential with  $\nu(\xi) \in V_1$ . Then we have  $\nu(\xi) \ge \nu(\eta_k)$  for some k by Lemma 13. However we have

$$\nu(\eta_k) \ge \begin{cases} (n-1)m - n - 1 & \text{for } k = 1, \\ (n-1)m + (k-2)n - 1 & \text{for } k \ge 2. \end{cases}$$

Since  $\nu(\eta_k) + 1 \ge (n-1)m - n$  for any k, we have  $\nu(\xi) + 1 \notin V_1^+$  by (12). This is a contradiction. Q.E.D.

**Lemma 15.** If  $V_1^+ \neq \emptyset$ , then we have  $\nu(\eta_1) + 1 = \min\{V_1^+\}$ .

*Proof.* We here prove this lemma for Case (C). We can similarly deal with Case (D). We have

(19) 
$$\pi^*(\eta_1) = \left[ n \sum_{i=1}^{p-1} a_i (m-m_i) t^{(n-1)m-(p+1-i)n-1} + \cdots \right] dt,$$

where we abbreviate the terms whose degree is greater than (n-1)m - 2n - 1. Assume that  $V_1^+ \neq \emptyset$ . We must have  $p \geq 2$  by Lemma 14. We first show that  $\nu(\eta_1) \in V_1$ . If  $\nu(\eta_1) \notin V_1$ , then  $\nu(\eta_1) \in V_0$  or  $\nu(\eta_1) \in \nu(d\mathcal{O}_C)$ . Now we have  $V_0 = \{\nu(\omega), \nu(y\omega)\}$ . If  $\nu(\eta_1) \in V_0$ , then we have  $\nu(\eta_1) = \nu(y\omega)$  by the definition of  $\eta_1$ . At least the coefficients in (19) must satisfy

(20) 
$$a_i = 0 \text{ for } i = 1, \dots, p-1.$$

Let  $\xi$  be a differential with  $\nu(\xi) \in V_1$ . By Lemma 13, we have  $\nu(\xi) \ge \nu(\eta_k)$  for some k. Under the conditions (20), if  $k \ge 2$ , then we have

$$\pi^*(\eta_k) = \left[a_p n(m - m_p) t^{(n-1)m + (k-2)n - 1} + \cdots\right] dt.$$

Since  $\nu(\eta_k) + 1 \ge (n-1)m - n$  for all k, we have  $\nu(\xi) + 1 \notin V_1^+$  by (12), which is a contradiction. On the other hand, if  $\nu(\eta_1) \in \nu(d\mathcal{O}_C)$ , then (20) must hold again. Since same contradiction occurs, we have  $\nu(\eta_1) \in V_1$ .

Next we show that  $\min\{V_1^+\} = \nu(\eta_1) + 1$ . It suffices to consider the case where  $\sharp(V_1^+) \ge 2$ . Let  $\xi$  be an element of  $\Omega_C^1$  with  $\nu(\xi) \in V_1$  and  $\nu(\xi) \neq \nu(\eta_1)$ . By Lemma 13, we have  $\nu(\xi) \ge \nu(\eta_k)$  for some  $k (\ge 2)$ . We have

(21) 
$$\pi^*(\eta_k) = \left[ n \sum_{i=1}^p a_i (m - m_i) t^{(n-1)m - (p+2-i-k)n-1} + \cdots \right] dt.$$

Set  $N := \min\{i \mid a_i \neq 0\}$ . Then we have  $\nu(\eta_k) = (n-1)m - (p+2-N-k)n - 1$ . It follows from (12) that (n-1)m - (p+2-N-k)n - 1 < (n-1)m - n - 1. It yields the inequality

$$(22) N$$

On the other hand, it follows from (19) that  $\nu(\eta_1) = (n-1)m - (p+1-N)n - 1$ . We see that  $\nu(\eta_1) < \nu(\eta_k)$  by (22), which gives the desired consequence. Q.E.D.

## 2.3. Type (iii): $\lambda = (n-3)m - 2n$

It follows from  $\lambda > m$  that  $n \ge 5$  and m > 2n/(n-4). Since  $\nu(\omega) = (n-3)m - n - 1$ , we find that  $V_0 = \{\nu(\omega), \nu(y\omega), \nu(y^2\omega)\}$ . We divide Type (iii) into two cases: (E) n > 2q, (F) n < 2q. (E): n > 2q. The following gaps are greater than  $\lambda$ :

$$(n-2)m - (p+2)n, (n-1)m - (2p+2)n, (n-3)m - n,$$
  

$$(n-2)m - (p+1)n, (n-1)m - (2p+1)n, (n-2)m - pn,$$
  

$$(n-1)m - 2pn, (n-2)m - (p-1)n, \dots, (n-2)m - n,$$
  

$$(n-1)m - (p+1)n, \dots, (n-1)m - n.$$

By Lemma 6 with the above gaps, we see that C has the parametrization

(23) 
$$x = t^n, \quad y = t^m + t^\lambda + \sum_{i=1}^p (a_i t^{m_i} + b_i t^{n_i}) + \sum_{i=p+1}^{2p} b_i t^{n_i},$$

where

$$m_i = (n-2)m - (p+3-i)n, n_i = (n-1)m - (2p+3-i)n$$

and 
$$a_i, b_i \in \mathbb{C}$$
.  
(F):  $n < 2q$ . The gaps which are greater than  $\lambda$  are  
 $(n-1)m - (2p+3)n, (n-2)m - (p+2)n, (n-3)m - n,$   
 $(n-1)m - (2p+2)n, (n-2)m - (p+1)n,$   
 $(n-1)m - (2p+1)n, (n-2)m - pn, (n-1)m - 2pn, \dots$   
 $(n-2)m - n, (n-1)m - (p+1)n, \dots, (n-1)m - n.$ 

Then C has the following parametrization:

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(24) 
$$x = t^n, \quad y = t^m + t^\lambda + \sum_{i=1}^p (a_i t^{m_i} + b_i t^{n_i}) + \sum_{i=p+1}^{2p+1} a_i t^{m_i},$$

where

$$m_i = (n-1)m - (2p+4-i)n, \ n_i = (n-2)m - (p+3-i)n$$
  
and  $a_i, b_i \in \mathbb{C}$ .

**Definition 16.** Define the differentials  $\zeta_{kl}$   $(k \ge 1, l \ge 0)$  by

$$\zeta_{kl} := x^k y^l \omega + d \left( s_{kl} x^{k-1} y^{n+l-3} \right),$$
  
here  $s_{kl} = \frac{-n(m-\lambda)}{(k-1)n + (n+l-3)m}.$ 

We rewrite the differentials  $\zeta_{kl}$  as follows:

$$\zeta_{1l} = y^l \phi_{1l} \text{ and } \zeta_{kl} = x^{k-2} y^l \phi_{kl},$$
  
where 
$$\begin{cases} \phi_{1l} = x\omega + s_{kl} (n+l-3) y^{n-4} dy, \\ \phi_{kl} = x^2 \omega + s_{kl} \{ (k-1) y^{n-3} dx \\ + (n+l-3) x y^{n-4} dy \} (k \ge 2). \end{cases}$$

We can easily check that  $\operatorname{LT}(x^k y^l \omega) + \operatorname{LT}(d(s_{kl}x^{k-1}y^{n+l-3})) = 0$ . Note that  $\phi_{10} = \zeta_{10}$  and  $\phi_{20} = \zeta_{20}$ . The following lemma is an analogue of Lemma 13.

**Lemma 17.** If  $\xi$  is an element of  $\Omega_C^1$  with  $\nu(\xi) + 1 \in V_1^+$ , then  $\xi$  has the form  $a\zeta_{kl} + \xi'$  for some  $\zeta_{kl}$  where  $\nu(\zeta_{kl}) \leq \nu(\xi')$  and  $a \in \mathbb{C}$ .

*Proof.* This proof is similar to that of Lemma 13. So we omit it. Q.E.D.

## $\S$ **3.** Singularities of genus two

We consider irreducible plane curve singularities of genus 2 in this section. The aim of this section is to prove Proposition 1. We first prove some technical auxiliary results needed in the proof of Proposition 1. Recall that if g = 2, then we have  $S = \langle v_0, v_1, v_2 \rangle$  where  $v_0 < v_1 < v_2$ ,  $v_0 = n = n_1 n_2$  with  $n_i \ge 2$  (i = 1, 2),  $v_1 = m = e_1 m_1$  for some positive integer  $m_1$  and  $e_1 = n_2$ . Set  $\lambda = \lambda_2 v_2 + \lambda_1 v_1 - \lambda_0 v_0$ . In case r = 3, by (9), we have

(25) 
$$3 \ge (\lambda_0 - 1)(n_1 - \lambda_1)(n_2 - \lambda_2) > 0.$$

**Lemma 18** (Nishiyama). If C is an irreducible plane curve singularity of genus 2 with r = 3, then we have  $\lambda = (n_1 - 1)m - 2n$  and  $Ch(C) = \{3n_1, 3m_1, \beta_2\}$  where  $n_1$  and  $m_1$  are coprime,  $n_1 < m_1$  and  $\beta_2$  is not divisible by 3.

*Proof.* We first show that  $n_2 \neq 2$  (cf. Lemma 10 in [3]). If  $n_2 = 2$ , then we have  $S = \langle 2p, 2q, v_2 \rangle$  where p < q, gcd(p, q) = 1,  $p = n_1$  and  $v_2 > n_1v_1 = 2pq$ . Furthermore, we can rewrite S with some positive and odd integer d as

$$S = \langle 2p, \, 2q, \, 2pq + d \rangle.$$

Luengo and Pfister ([5]) showed that the irreducible plane curve singularity C with such semigroup has  $\tau = \mu - (p-1)(q-1)$ . That is, r = (p-1)(q-1). So if we set r = (p-1)(q-1) = 3, then we have p = 2 and q = 4. This implies that m = 2n, which is a contradiction.

Let  $\lambda = \lambda_2 v_2 + \lambda_1 v_1 - \lambda_0 v_0$  be the Zariski invariant of C. We first consider the case where  $\lambda_2 \neq 0$ . Recall that  $\lambda \leq \beta_2$  (See (4)). Assume that  $\lambda < \beta_2$ . Since  $v_2 = n_1 v_1 + \beta_2 - \beta_1$  and  $\beta_2$  can not be divisible by  $e_1$ ,  $\lambda$  is also not divisible by  $e_1$ . This contradicts the definition of  $\beta_2$ . Hence we have  $\lambda = \beta_2 = v_2 + v_1 - m_1 v_0$ . It follows that  $\lambda_2 = 1$ ,  $\lambda_1 = 1$ ,  $\lambda_0 = m_1$ . Since  $m_1 > n_1 \geq 2$ , we easily see that  $n_2 = 2$  by (25). So the case in which  $\lambda_2 \neq 0$  does not occur by the above argument.

On the other hand, if  $\lambda_2 = 0$ , then we must have  $n_2 = 3$  by the above argument and (25). It follows that  $S = \langle 3n_1, 3m_1, v_2 \rangle$ . We also obtain  $\lambda_1 = n_1 - 1$  and  $\lambda_0 = 2$  by (25). The corresponding characteristic is  $Ch(C) = \{3n_1, 3m_1, \beta_2\}$  where  $\beta_2 = v_2 - (n_1 - 1)m$ . We have completed the proof of Lemma 18. Q.E.D.

By Lemma 18, we have only to consider the case where

$$\lambda = (n_1 - 1)m - 2n$$
 and  $Ch(C) = \{3n_1, 3m_1, \beta_2\}.$ 

In this case, We have  $S = \langle v_0, v_1, v_2 \rangle$  where  $v_0 = n = 3n_1$ ,  $v_1 = m = 3m_1$  and  $v_2 = 2m + \beta_2$ . We also have  $n_1 \ge 3$  by  $\lambda > m$  and the following conditions are satisfied:

$$m_1 \ge \begin{cases} n_1 + 1 & \text{for } n_1 \ge 4. \\ 7 & \text{for } n_1 = 3. \end{cases}$$

**Lemma 19.** Let C be an irreducible plane curve singularity with  $Ch(C) = \{3n_1, 3m_1, \beta_2\}$  and  $\lambda = (n_1-1)m-2n$ . Then the parametrization of C can be taken as

(26) 
$$x = t^n, \quad y = t^m + t^\lambda + at^{\beta_2} + \cdots, \quad (a \neq 0).$$

*Proof.* Let  $h_1$  be the biggest positive integer satisfying  $m + h_1 e_1 < \beta_2$ . Note that  $e_1 = 3$  and  $\lambda = m + 3\{(n_1 - 2)m_1 - 2n_1\}$ . So we can take the parametrization of C as

(27) 
$$x = t^n$$
,  $y = t^m + t^{\lambda} + \sum_{(n_1 - 2)m_1 - n_1 \le i \le h_1} a_i t^{m+3i} + a t^{\beta_2} + \cdots$ ,

where  $m + 3i \in G$  for any *i*. Since each m + 3i is a gap, it is written in a unique way as

(28) 
$$m + 3i = t_2 v_2 + t_1 v_1 - t_0 v_0,$$

where  $0 \le t_2 \le 2$ ,  $0 \le t_1 \le n_1 - 1$  and  $t_0 > 0$  (See (6)). Since the left hand side of (28) is divisible by 3 and  $v_2$  is not divisible by 3, we must have  $t_2 = 0$ . Since  $\lambda = (n_1 - 1)m - 2n$ , no integer satisfies this condition other than  $(n_1 - 1)m - n$ . If  $\beta_2 < (n_1 - 1)m - n$ , then we obtain (26). On the other hand, if  $(n_1 - 1)m - n < \beta_2$ , then (27) becomes

$$x = t^n$$
,  $y = t^m + t^\lambda + a_{(n_1-2)m_1-n_1}t^{(n_1-1)m-n} + at^{\beta_2} + \cdots$ 

By using (EC 2) in Lemma 6, we can rewrite this as (26). Q.E.D.

**Lemma 20.** If a positive integer k = am + bn  $(a, b \in \mathbb{Z})$  is greater than  $(n_1 - 1)m - n$ , then we have  $k \in \langle n, m \rangle \subset S$ .

*Proof.* By (6), we can rewrite k as  $l_2v_2+l_1v_1-l_0v_0$  where  $0 \le l_2 \le 2$  and  $0 \le l_1 \le n_1 - 1$ . Now we have  $l_2 = 0$ . Indeed, if not, then we have

$$l_2 v_2 = 3\{(a - l_1)m_1 + (b + l_0)m_1\}.$$

Since  $l_2$  is equal to 1 or 2, the integer  $v_2$  must be divisible by 3, which is a contradiction. Thus we have  $k = l_1m - l_0n$ . Since the biggest gap of such form is  $(n_1 - 1)m - n$ , the positive integer k is contained in S. Q.E.D.

Proof of Proposition 1. It follows from (10) that if r = 3, then g = 1 or 2. We shall show that if g = 2, then  $r \neq 3$ . It is enough to consider the plane curve singularity C with  $\lambda = (n_1 - 1)m - 2n$  and  $Ch(C) = \{3n_1, 3m_1, \beta_2\}$  by Lemma 18. By Lemma 19, we may assume that C is given by (26). Since  $(\lambda_0 - 1)(n_1 - \lambda_1)(n_2 - \lambda_2) = 3$ , there exist three distinct elements of  $V_0$ . They are given by  $\nu(\omega)$ ,  $\nu(z\omega)$  and  $\nu(z^2\omega)$  where  $z \in \mathcal{O}_C$  with  $\nu(z) = v_2$ . We shall inductively construct a differential  $\xi$  such that

(29) 
$$\pi^*(\xi) = \left\{ am(m-\beta_2)t^{\beta_2+2n-1} + \cdots \right\} dt.$$

Since  $\beta_2 + 2n - 1 = v_2 + m - (m_1 - 2)n - 1$  is different from  $\nu(\omega)$ ,  $\nu(z\omega)$ and  $\nu(z^2\omega)$ , we would have  $r \ge 4$  by (8). We first set  $\xi_0 = (m/n)x\omega$ . Then we have

$$\pi^* (\xi_0) = \left\{ m(m-\lambda)t^{(n_1-1)m-1} + am(m-\beta_2)t^{\beta_2+2n-1} + \cdots \right\} dt$$

Next we set  $\xi_1 = \xi_0 - (m - \lambda)y^{n_1 - 2}dy$  as the first step. We have

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(30)  
$$y^{n_{1}-2} = t^{(n_{1}-2)m} \left[ 1 + \binom{n_{1}-2}{1} \left\{ t^{\lambda-m} + at^{\beta_{2}-m} + \cdots \right\} + \binom{n_{1}-2}{2} \left\{ t^{2(\lambda-m)} + 2at^{\beta_{2}+\lambda-2m} + \cdots \right\} + \binom{n_{1}-2}{3} \left\{ t^{3(\lambda-m)} + \cdots \right\} + \cdots + \binom{n_{1}-2}{n_{1}-2} \left\{ t^{(n_{1}-2)(\lambda-m)} + \cdots \right\} \right].$$

We consider the cases where  $\beta_2 - m < 2(\lambda - m)$  and where  $2(\lambda - m) < \beta_2 - m$  separately.

If  $\beta_2 - m < 2(\lambda - m)$ , then we have

$$\pi^* \left( y^{n_1 - 2} dy \right) = \left[ m t^{(n_1 - 1)m - 1} + \left\{ m(n_1 - 2) + \lambda \right\} t^{(2n_1 - 3)m - 2n - 1} \right. \\ \left. + a \left\{ m(n_1 - 2) + \beta_2 \right\} t^{\beta_2 + (n_1 - 2)m - 1} + \cdots \right] dt.$$

Since  $(2n_1 - 3)m - 2n > \beta_2 + 2n$ , we have

$$\pi^*(\xi_1) = \left[am(m - \beta_2)t^{\beta_2 + 2n - 1} + \cdots\right] dt,$$

which is the desired differential. In particular, if  $n_1 = 3$ , then this case always occurs.

Next we consider the case where  $2(\lambda - m) < \beta_2 - m$ . This case occurs only when  $n_1 \ge 4$ . Set  $N_1 := \max\{i \mid i(\lambda - m) < \beta_2 - m \text{ and } 2 \le i \le n_1 - 2\}$ . Then (30) becomes

$$y^{n_1-2} = t^{(n_1-2)m} + \sum_{i=1}^{N_1} \binom{n_1-2}{i} t^{(i+1)(n_1-2)m-2in} + a(n_1-2)t^{\beta_2+(n_1-3)m} + \cdots$$

So we have

$$\pi^* \left( y^{n_1 - 2} dy \right) = \left[ m t^{(n_1 - 1)m - 1} + \sum_{i=1}^{N_1} \left\{ m \binom{n_1 - 2}{i} + \lambda \binom{n_1 - 2}{i - 1} \right\} t^{n_i - 1} + \cdots \right] dt,$$

where  $n_i = \{(i+1)n_1 - 2i - 1\}m - 2in$ . In a similar manner as in the previous case, we set  $\xi_1 = \xi_0 - (m-\lambda)y^{n_1-2}dy$ . Since  $n_{N_1-1} < \beta_2 + 2n < 1$
$n_{N_1}$  holds, we have

$$\pi^*(\xi_1) = \left[ -(m-\lambda) \sum_{i=1}^{N_1-1} \left\{ m \binom{n_1-2}{i} + \lambda \binom{n_1-2}{i-1} \right\} t^{n_i-1} + am(m-\beta_2) t^{\beta_2+2n-1} + \cdots \right] dt.$$

Note that  $n_i \in \langle n, m \rangle$  by Lemma 20. Starting with  $\xi_1$ , we inductively define a differential  $\xi_k$ . Assume that  $\xi_k$   $(k \ge 1)$  satisfies the following condition:

(31) 
$$\pi^*(\xi_k) = \left\{ \sum_{\text{finite sum}} c_{k,\,\alpha} t^{m_\alpha - 1} + am(m - \beta_2) t^{\beta_2 + 2n - 1} + \cdots \right\} dt,$$

where  $m_{\alpha} \in \langle n, m \rangle$ . Putting  $\nu(\xi_k) = a_k m + b_k n - 1$ , we set

$$\xi_{k+1} := \begin{cases} \xi_k - (\mathrm{LT}(\xi_k)/n) x^{b_k} dx, & \text{if } a_k = 0 \text{ and } b_k \neq 0. \\ \xi_k - (\mathrm{LT}(\xi_k)/n) x^{b_k - 1} y^{a_k} dx, & \text{if } a_k \neq 0 \text{ and } b_k \neq 0. \\ \xi_k - (\mathrm{LT}(\xi_k)/m) y^{a_k - 1} dy, & \text{if } a_k \neq 0 \text{ and } b_k = 0. \end{cases}$$

It follows from this definition that  $\nu(\xi_{k+1}) > \nu(\xi_k)$ . We prove that  $\xi_{k+1}$  above satisfies the condition (31).

**Case 1)**  $a_k = 0$  and  $b_k \neq 0$ . We have

$$\pi^*(\xi_{k+1}) = \left\{ \sum c_{k+1,\alpha} t^{m_\alpha - 1} + am(m - \beta_2) t^{\beta_2 + 2n - 1} + \cdots \right\} dt,$$

where  $m_{\alpha} \in \langle n, m \rangle$ . Note that the number of  $m_{\alpha}$  is finite. The differential  $\xi_{k+1}$  satisfies the condition (31).

**Case 2)**  $a_k \neq 0$  and  $b_k \neq 0$ . Consider the differential  $x^{b_k-1}y^{a_k}dx$ . Writing

$$y^{a_k} = t^{a_k m} \left[ 1 + \binom{a_k}{1} \left\{ t^{\lambda - m} + at^{\beta_2 - m} + \cdots \right\} \right.$$
$$\left. + \binom{a_k}{2} \left\{ t^{2(\lambda - m)} + 2at^{\beta_2 + \lambda - 2m} + \cdots \right\} \right.$$
$$\left. + \binom{a_k}{3} \left\{ t^{3(\lambda - m)} + \cdots \right\} + \cdots$$
$$\left. + \binom{a_k}{a_k} \left\{ t^{a_k(\lambda - m)} + \cdots \right\} \right],$$

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we set  $N_{k+1} := \max\{i \mid i(\lambda - m) < \beta_2 - m \text{ and } 2 \le i \le a_k\}$ . Then we have

$$\pi^* \left( x^{b_k - 1} y^{a_k} dx \right) = n \left[ t^{a_k m + b_k n - 1} + \sum_{i=1}^{N_{k+1}} c_i t^{\{(n_1 - 2)i + a_k\}m + (b_k - 2i)n - 1} + aa_k t^{\beta_2 + (a_k - 1)m + b_k n - 1} + \cdots \right] dt.$$

By Lemma 20, the integers  $\{(n_1 - 2)i + a_k\}m + (b_k - 2i)n$  belong to  $\langle n, m \rangle$ . It is easy to see that  $\beta_2 + (a_k - 1)m + b_kn - 1 > \beta_2 + 2n - 1$ . So  $\xi_{k+1}$  satisfies the condition (31).

**Case 3)**  $a_k \neq 0$  and  $b_k = 0$ . We consider the differential  $y^{a_k-1}dy$ . By the same argument as Case 2, we define  $N_{k+1} := \max\{i \mid i(\lambda - m) < \beta_2 - m \text{ and } 2 \leq i \leq a_k - 1\}$  for  $y^{a_k-1}$ . We have

$$\pi^* \left( y^{a_k - 1} dy \right) = \left[ m t^{a_k m - 1} + \sum_{\text{finite sum}} c_\alpha t^{m_\alpha - 1} \right]$$
$$+ a(m + \beta_2) t^{\beta_2 + (a_k - 1)m - 1} + \cdots dt,$$

where  $m_{\alpha} \in \langle n, m \rangle$ . So we see that  $\xi_{k+1}$  satisfies the condition (31).

We can therefore inductively construct  $\xi_{k+1}$  from  $\xi_k$ . Since there exist finitely many elements of  $\langle n, m \rangle$  which are smaller than  $\beta_2 + 2n - 1$  and  $\nu(\xi_0) < \nu(\xi_1) < \cdots < \nu(\xi_k) < \cdots$  holds, we obtain  $\xi$  with  $\pi^*(\xi) = [am(m-\beta_2)t^{\beta_2+2n-1}+\cdots] dt$  after finitely many steps. Q.E.D.

### $\S4.$ Proof of Theorem

By Proposition 1, it is enough to consider the case where g = 1. Substituting r = 3 and g = 1 to (9), we obtain  $3 = r \ge (\lambda_0 - 1)(n - \lambda_1)$ . This inequality yields the following possible five types of  $\lambda$ :

(i) 
$$\lambda = (n-1)m - 4n$$
, (ii)  $\lambda = (n-2)m - 2n$ , (iii)  $\lambda = (n-3)m - 2n$ ,  
(iv)  $\lambda = (n-1)m - 2n$ , (v)  $\lambda = (n-1)m - 3n$ .

**Lemma 21.** If  $\lambda$  is either of type (iv) or of type (v), then  $r \neq 3$ .

*Proof.* For type (iv) (resp. type (v)), letting R = 1 (resp. R = 2) in Proposition 7, we conclude that r = 1 (resp. r = 2). Q.E.D.

We consider the remaining three types separately. We freely use the notations and the results in Section 2.

**Type (i)**:  $\lambda = (n-1)m - 4n$ . We show that r = 3. We first consider the case in which  $p \ge 2$ . We may assume that C is given by (15). By Proposition 7, we have r = 3.

Next we consider the case in which p = 1. The parametrization of C has the form (16). We have

$$\pi^*(\omega) = n \left\{ (m-\lambda)t^{(n-1)m-3n-1} + a(m-m_1)t^{(n-2)m-n-1} \right\} dt.$$

It follows from  $\nu(\omega) = (n-1)m - 3n - 1$  that  $V_0 = \{\nu(\omega), \nu(x\omega), \nu(x^2\omega)\}$ . By (8), we have r = 3 if and only if  $V_1^+ = \emptyset$ . Assume that  $V_1^+ \neq \emptyset$ . Let  $\xi = A\omega + dB$  be an element of  $\Omega_C^1$  with  $\nu(\xi) \in V_1$ . If we set  $(A\omega) = (ux^k y^l + \cdots) \omega$  where  $u \in \mathbb{C}$ , then we have

(32) 
$$\nu(A\omega) = (n+l-1)m + (k-3)n - 1.$$

Since  $\nu(A\omega) \in d\mathcal{O}_C$  by (11), we have  $k \geq 3$  or  $l \geq 1$ . Suppose  $k \geq 3$ . Then we see from (32) that  $\nu(A\omega) > (n-1)m - n - 1$ . So the order  $\nu(\xi)$  can not belong to  $V_1$  by (12). Thus we must have  $l \geq 1$ . If  $l \geq 2$ , then we have  $\nu(A\omega) > (n-1)m - n - 1$  again. Hence l = 1. Then (12) yields (k-1)n + q < 0. We infer from this that k = 0. Thus we have  $A\omega = (uy + \text{terms of higher degree})\omega$ . We have

$$\pi^*(uy\omega) = un \left[ (m-\lambda)t^{(m-3)n-1} - abt^{(n-1)m-n-1} \right] dt.$$

where b := (n-3)m - 2n. Since (11) holds, the differential dB has the form  $d(-u(m-\lambda)x^{m-3}/(m-3) + \cdots)$ . So  $\xi$  can be rewritten as  $uy\omega + d(-u(m-\lambda)x^{m-3}/(m-3)) + \xi'$ . If we write  $\xi' = (u'x^ky^l + \cdots)\omega + dB'$ , then  $(k, l) \neq (0, 1)$  and hence  $\nu(\xi') \notin V_1$ . This fact implies that  $\nu(\xi) = \nu(y\omega) + d(-u(m-\lambda)x^{m-3}/(m-3))$  If n = 5, 6, we find that  $\nu(A\omega) > (n-2)m - n - 1$ . Since (n-1)m - n is the only gap greater than (n-2)m - n, there exists no element of  $V_1$ . That is,  $V_1 = \emptyset$ . For  $7 \geq n$ , we have

$$\pi^*\left(uy\omega + d\left(\frac{-u(m-\lambda)}{(m-3)}x^{m-3}\right)\right) = -abunt^{(n-1)m-n-1}dt.$$

By (12),  $\nu(\xi)$  can not be in  $V_1$ . We conclude that  $V_1 = \emptyset$  for  $7 \ge n$ . **Type (ii)**:  $\lambda = (n-2)m - 2n$ . By Lemma 3,  $\sharp(V_0^+) = 2$  holds. So we have r = 3 if and only if  $\sharp(V_1^+) = 1$  by (8). Furthermore, by Lemma 15, we obtain  $\sharp(V_1^+) = 1$  if and only if  $V_1^+ = \{\nu(\eta_1) + 1\}$ . (C):  $n \ge 5$ . It follows from the inequality  $\lambda > m$  that  $p \ge 1$  for  $n \ge 5$ . However we must have  $p \ge 2$  by Lemma 14. If  $\sharp(V_1^+) = 1$ , then the coefficients in (19) must satisfy

(33) 
$$a_i = 0 \text{ for } i = 1, \dots, p-2 \text{ and } a_{p-1} \neq 0.$$

Conversely, assume that the coefficients in the parametrization (17) satisfy (33). Then we have  $\nu(\eta_1) = (n-1)m - 2n - 1$ . Since (n-1)m - n is the only one gap of S which is greater than (n-1)m - 2n, by Lemma 15, we have  $V_1^+ = \{\nu(\eta_1) + 1\}$ .

(D): n = 4. It follows from the inequality  $\lambda > m$  that  $p \ge 2$ . We have

$$\pi^*(\eta_1) = \left[4\sum_{i=1}^{p-2} a_i(m-m_i)t^{3m-4(p+1-i)-1} + 4\left\{a_{p-1}(m-m_{p-1}) - \frac{m^2 - \lambda^2}{m}\right\}t^{3m-8-1} + \cdots\right]dt.$$

If  $\sharp(V_1^+) = 1$ , then we must have the following condition:

(34) 
$$a_i = 0 \text{ for } i = 1, \dots, p-2 \text{ and } a_{p-1} \neq \frac{3m-8}{2m}$$

Conversely, if C is given by the parametrization (17) with (34), then we find that  $\sharp(V_1^+) = 1$  by the same argument as in (C).

**Type (iii)**:  $\lambda = (n-3)m - 2n$ . Since  $\sharp(V_0^+) = 3$ , we have r = 3 if and only if  $V_1^+ = \emptyset$  (See (8)). We here prove Case (E). We can similarly deal with Case (F). Now we have

 $\pi$ 

Comparing the exponent (2n-7)m-2n-1 with (n-1)m-n-1, the following three subcases occur:

- (E1): (2n-7)m 2n 1 > (n-1)m n 1 for  $n \ge 7$ .
- (E2): (2n-7)m 2n 1 = (n-1)m 2n 1 for n = 6.
- (E3): (2n-7)m 2n 1 < (n-1)m 2n 1 for n = 5.

It follows from  $\lambda > m$  and n > 2q that the conditions (i)  $p \ge 1$  for  $n \ge 7$ , (ii)  $p \ge 1$  and q = 1 for n = 6, (iii)  $p \ge 2$  and q = 1, 2 for n = 5. (E1):  $n \ge 7$ . The differential  $\pi^*(\zeta_{10})$  becomes

(35)  

$$\pi^{*}(\zeta_{10}) = \left[ n \sum_{i=1}^{p} a_{i}(m-m_{i})t^{(n-2)m-(p+1-i)n-1} + n \sum_{i=1}^{p} b_{i}(m-n_{i})t^{(n-1)m-(2p+1-i)n-1} + n \sum_{i=p+1}^{2p} b_{i}(m-m_{i})t^{(n-1)m-(2p+1-i)n-1} + \cdots \right] dt.$$

If  $V_1^+ = \emptyset$ , then the order  $\nu(\zeta_{10})$  must belong to  $\nu(d\mathcal{O}_C)$  or  $V_0$ . Furthermore, if  $\nu(\zeta_{10}) \in V_0$ , then  $\nu(\zeta_{10})$  equals  $\nu(y\omega)$  or  $\nu(y^2\omega)$ . **E1.1**:  $\nu(\zeta_{10}) \neq \nu(y\omega)$ . If  $V_1^+ = \emptyset$ , then the coefficients in (35) must satisfy the following conditions.

(36) 
$$a_i = 0 \text{ for } i = 1, \dots, p$$
$$b_i = 0 \text{ for } i = 1, \dots, 2p - 1 \text{ and } \forall b_{2p}.$$

Conversely, assume that (23) has (36). If  $V_1^+ \neq \emptyset$ , then there exists a differential  $\xi$  with  $\nu(\xi) + 1 \in V_1^+$ . By Lemma 17,  $\xi$  has the form  $\xi = a\zeta_{kl} + \xi'$ . Recall that there exists the following relation between  $\nu(\zeta_{kl})$  and  $\nu(\phi_{kl})$ :

(37) 
$$\nu(\zeta_{kl}) = \begin{cases} \nu(\phi_{1l}) + lm, & \text{if } k = 1.\\ \nu(\phi_{kl}) + lm + (k-2)m, & \text{if } k \ge 2. \end{cases}$$

(See Subsection 2.3). If k = 1, then we have

$$\pi^*(\phi_{1l}) = \left[b_{2p}n(m-n_{2p})t^{(n-1)m-n-1} + \text{higher degree terms}\right]dt.$$

So we have  $\nu(\xi) \ge \nu(\phi_{1l}) \ge (n-1)m - n - 1$ . By Lemma 12,  $\nu(\xi) + 1$  can not be in  $V_1^+$ .

On the other hand, if  $k \ge 2$ , then we have

$$\pi^*(\phi_{kl}) = \left[b_{2p}(m-n_p)t^{(n-1)m-1} + \text{higher degree terms}\right]dt.$$

Since  $\nu(\xi) \ge \nu(\phi_{kl}) > (n-1)m - n - 1$ , we have  $\nu(\xi) + 1 \notin V_1^+$  again. Thus, we conclude that  $V_1^+ = \emptyset$ .

**E1.2**:  $\nu(\zeta_{10}) = \nu(y\omega)$ . If  $V_1^+ = \emptyset$ , then the parametrization (35) must have the following coefficients:

(38) 
$$a_i = b_i = 0 \text{ for } i = 1, \dots, p-1 \text{ and } a_p \neq 0.$$

For the parametrization (35) with the condition (38), we consider the differential  $\zeta_{10} - (LT(\zeta_{10})/LT(y\omega))y\omega$ .

$$\pi^* \left( \zeta_{10} - \frac{\operatorname{LT}(\zeta_{10})}{\operatorname{LT}(y\omega)} y\omega \right) = \left[ \sum_{i=p}^{2p-2} b_i n(m-n_i) t^{(n-1)m-(2p+1-i)n-1} + \left\{ b_{2p-1}(m-n_{2p-1}) - \frac{a_p^2(m-m_p)^2}{(m-\lambda)} \right\} n t^{(n-1)m-2n-1} - \frac{a_p b_p m n(m-m_p)(m-n_p)}{(m-\lambda)} t^{(m-p-2)n-1} + \cdots \right] dt.$$

In (39), we must put

(40)  
$$b_{i} = 0 \text{ for } i = p, \dots, 2p - 2$$
$$b_{2p-1} = \frac{a_{p}^{2}(m - m_{p})^{2}}{(m - n_{2p-1})(m - \lambda)}.$$

Conversely, assume that the parametrization (23) has (38) and (40). If  $V_1^+ \neq \emptyset$ , then we take a differential  $\xi$  with  $\nu(\xi) + 1 \in V_1^+$ . By Lemma 17,  $\xi$  has the form  $c_1\zeta_{k_1l_1} + \xi_1$  where  $\nu(\zeta_{k_1l_1}) \leq \nu(\xi_1)$ . Note that  $\xi_1$  does not contain  $\zeta_{k_1l_1}$ . We first consider the case where  $k_1 = 1$ . Then we have

$$\pi^*(\phi_{1l_1}) = \left[a_p n(m-m_p) t^{(n-2)m-n-1} + \cdots \right] dt.$$

If  $l_1 \geq 1$ , then we have  $\nu(\zeta_{11}) \geq (n-1)m-n-1$  by (37). Since  $\nu(\xi) + 1$ can not be an element of  $V_1^+$  by (12), it contradicts assumption. So we must have  $l_1 = 0$ . Then we have  $\nu(\zeta_{10}) = \nu(y\omega) = (n-2)m-n-1$ . So the relation  $\operatorname{LT}(\zeta_{10}) + \operatorname{LT}(\xi_1) = 0$  must need for  $\nu(\xi) + 1 \in V_1^+$ . We show that  $\xi_1$  has the form  $\{-c_1 \operatorname{LT}(\zeta_{10})/\operatorname{LT}(y\omega)\}y\omega + \xi'_1$ . Set  $\xi_1 = A_1\omega + dB_1$ . If  $\nu(A_1\omega) > \nu(dB_1)$ , then  $\nu(\xi_1) \in d\mathcal{O}_C$ . This case does not occur. If  $\nu(A_1\omega) = \nu(dB_1)$ , then  $\operatorname{LT}(A_1\omega) + \operatorname{LT}(dB_1) = 0$  holds. By the same argument as in the proof of Lemma 13, we have the expression  $\xi_1 = a\zeta_{kl} + \xi'_1$  for some  $\zeta_{kl}$ . It is clear that  $k \neq 1$ . For  $k \geq 2$ , we have

(41) 
$$\pi^*(\phi_{kl}) = \left[a_p(m-m_p)t^{(n-2)m-1} + \cdots\right]dt.$$

It follows from (37) and (41) that  $\nu(\zeta_{kl}) > \nu(y\omega)$ . Thus, only the case where  $\nu(A_1\omega) < \nu(dB_1)$  occurs. By the same argument as in the proof of Lemma 13, there exists only one term  $ax^ky^l$  in  $A_1$  such that  $\nu(ax^ky^l) =$  $\nu(A_1\omega)$ . It follows from  $\nu(\xi_1) = \nu(A_1\omega) = \nu(y\omega)$  that k = 0 and l = 1. Since  $\operatorname{LT}(\zeta_{10}) + \operatorname{LT}(\xi_1) = 0$ , we must set  $a = -c_1 \operatorname{LT}(\zeta_{10}) / \operatorname{LT}(y\omega)$ . Putting  $\xi'_1 = \xi_1 + \{c_1 \operatorname{LT}(\zeta_{10}) / \operatorname{LT}(y\omega)\}y\omega$ , we obtain the desired expression. Now we have  $\xi = c_1\zeta_{10} - \{c_1 \operatorname{LT}(\zeta_{10}) / \operatorname{LT}(y\omega)\}y\omega + \xi'_1$ . Since

$$\pi^* \left( c_1 \zeta_{10} - \frac{c_1 \operatorname{LT}(\zeta_{10})}{\operatorname{LT}(y\omega)} y\omega \right) = c_1 \left[ b_{2p} n(m - n_{2p}) t^{(n-1)m-n-1} + \cdots \right] dt$$

holds, the order  $\nu(\xi'_1)$  must equal  $\nu(\xi)$ .

(\*) By Lemma 17, there exists  $\zeta_{k_2l_2}$  such that  $\xi'_1 = c_2\zeta_{k_2l_2} + \xi_2$  where  $\xi_2$  does not contain  $\zeta_{k_2l_2}$ . Now  $\zeta_{k_2l_2}$  is different from  $\zeta_{10}$  and  $\zeta_{11}$ . So we must have  $k_2 \geq 2$ . If  $l_2 \geq 1$ , then  $\nu(\zeta_{kl}) > (n-1)m - n - 1$  by (37). We must have  $l_2 = 0$ . Note that we have

(42) 
$$\pi^*(\zeta_{k0}) = \left[a_p n(m-m_p) t^{(n-2)m+(k-2)n-1} + b_{2p-1} n(m-n_{2p-1}) t^{(n-1)m+(k-3)n-1} + \cdots\right] dt,$$

for  $k \geq 2$ . Since  $\nu(\zeta_{k_20}) \in \nu(d\mathcal{O}_C)$ , the equality  $\operatorname{LT}(c_2\zeta_{k_20}) + \operatorname{LT}(\xi_2) = 0$ must hold for  $\nu(\xi) \in V_1$ . Write  $\xi_2 = A_2\omega + dB_2$ . By the same argument as in the proof of Lemma 13, there exists only one term  $e_2x^ky^l$  in  $A_2$ such that  $\nu(e_2x^ky^l) = \nu(A_2\omega)$ . Similarly,  $B_2$  contains only one term  $h_2x^ky^l$  such that  $\nu(h_2x^ky^l) = \nu(dB_2)$ . It is easily checked that  $\xi_2$  has the form

$$\xi_2 = \left(e_2 x^{k_2 - 1} y + \cdots\right) \omega + d\left(h_2 x^{k_2 - 2} y^{n - 2} + \cdots\right),$$

where  $LT(c_2\zeta_{k_20}) + LT(e_2x^{k_2-1}y\omega + d(h_2x^{k_2-2}y^{n-2})) = 0$ . Furthermore, if we set  $\xi'_2 = \xi_2 - \{e_2x^{k_2-1}y\omega + d(h_2x^{k_2-2}y^{n-2})\}$ , then we have  $\nu(\xi'_2) > \nu(e_2x^{k_2-1}y\omega + d(h_2x^{k_2-2}y^{n-2})) = \nu(\zeta_{k_20})$  holds. The differential  $\xi'_1$  is expressed as

$$\xi_1' = c_2 \zeta_{k_2 l_2} + e_2 x^{k_2 - 1} y \omega + d \left( h_2 x^{k_2 - 2} y^{n-2} \right) + \xi_2'.$$

Since  $\xi_2$  dose not contain  $\zeta_{k_20}$ , so does not  $\xi'_2$ . We easily see that  $\nu(\xi'_1 - \xi'_2) \ge (n-1)m - n - 1$ . Hence we must have  $\nu(\xi) = \nu(\xi'_2)$ . The argument started from (\*) is applicable to  $\xi'_2$ . So we obtain

$$\xi_2' = c_3\zeta_{k_30} + e_3x^{k_3-1}y\omega + d\left(h_3x^{k_3-2}y^{n-2}\right) + \xi_3',$$

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where  $\nu(\xi'_3) > \nu(c_3\zeta_{k_30} + e_3x^{k_3-1}y\omega + d(h_3x^{k_3-2}y^{n-2})) \in \nu(d\mathcal{O}_C)$ . Note that  $\nu(\zeta_{k_0}) < \nu(\zeta_{k'0})$  if and only if k < k' by (42). Since  $\nu(\zeta_{k_20}) < \nu(\xi'_3)$ , we obtain  $k_2 < k_3$ . We continue this process successively and after *j*-th step we have

$$\xi_{j-1}' = c_j \zeta_{k_j 0} + e_j x^{k_j - 1} y \omega + d \left( h_j x^{k_j - 2} y^{n-2} \right) + \xi_j'.$$

where  $\nu(\xi'_j) > \nu\left(c_j\zeta_{k_j0} + e_jx^{k_j-1}y\omega + d\left(h_jx^{k_j-2}y^{n-2}\right)\right) \in \nu(d\mathcal{O}_C).$ Then  $\xi$  is rewritten as

$$\xi = c_1 \zeta_{10} - \frac{c_1 \operatorname{LT}(\zeta_{10})}{\operatorname{LT}(y\omega)} y\omega + \sum_{i=2}^j \left\{ c_i \zeta_{k_i 0} + e_i x^{k_i + 1} y\omega + d\left(h_i x^{k_i} y^{n-2}\right) \right\} + \xi'_j.$$

where  $k_2 < k_3 < \ldots < k_j$  and  $\nu(\xi'_j) > \nu(\zeta_{k_j0})$ . Since  $\nu(\xi - \xi'_j) \notin V_1$ , we must have  $\nu(\xi) = \nu(\xi'_j)$ . However, the inequalities  $\nu(\xi'_j) > \nu(\zeta_{k_j0}) >$ (n-1)m - n - 1 occur after finitely many steps. It contradicts the assumption  $\nu(\xi) + 1 \in V_1^+$ . Hence we have  $V_1^+ = \emptyset$ .

For the case where  $k_1 \geq 2$ , we can apply the argument started from (\*) to  $\xi$  by replacing  $\xi'_1$  by  $\xi$ . Then we find  $V_1^+ = \emptyset$ , so r = 3.

The proofs of (E2), (E3) and Case (F) are essentially same. So we omit them. Q.E.D.

We summarized the consequences for Case (E) and Case (F). If r = 3, then the parametrizations (23) and (24) have the coefficients in Table 1 and in Table 2 respectively.

#### Table 1

No.	Conditions	Coefficients
E1.1	$n \ge 7$	$a_i = 0 \ (i = 1, \ldots, p),$
		$b_i = 0 \ (i = 1, \dots, 2p - 1), \ \forall b_{2p}.$
E1.2	$n \ge 7$	$a_i = 0 \ (i = 1, \dots, p-1), \ a_p \neq 0,$
		$b_i = 0 \ (i = p, \dots, 2p - 2),$
		$b_{2p-1} = a_p^2 (m - m_p)^2 / (m - n_{2p-1})(m - \lambda), \ \forall b_{2p}.$
E2.1	n = 6	$a_i = 0$ $(i = 1,, p), b_i = 0$ $(i = 1,, 2p - 2),$
	m = 6p + 1	$b_{2p-1} = (5m - 12)/2m, \ \forall b_{2p}.$
E2.2	n = 6	$a_i = 0 \ (i = 1, \dots, p-1), \ a_p \neq 0,$
	m = 6p + 1	$b_i = 0 \ (i = 1, \dots, 2p - 2),$
		$b_{2p-1} = \left(9a_p^2m + 20m - 48\right)/8m, \ \forall b_{2p}.$

E3.1	n = 5	$a_i = 0 \ (i = 1, \dots, p - 2, p),$
		$a_{p-1} = (3m - 10)/2m,$
	$p \ge 3$	$b_i = 0 \ (i = 1, \dots, 2p - 4, 2p - 2, 2p - 1),$
		$b_{2p-3} = 4(m-5)(2m-5)/3m^2, \ \forall b_{2p}.$
E3.2	n = 5	$a_i = 0 \ (i = 1, \dots, p-2), \ a_{p-1} = (3m-10)/2m,$
	$p \ge 3$	$a_p \neq 0, \ b_i = 0 \ (i = 1, \dots, 2p - 4),$
		$b_{2p-3} = 4(m-5)(2m-5)/3m^2,$
		$b_{2p-2} = 3a_p(4m^2 - 45m + 100)/m(3m - 25),$
		$b_{2p-1} = a_p^2 (2m - 15)^2 / (3m - 20)(m - 10), \ \forall b_{2p}.$
E3.3	n = 5	$a_1 = 23/22, \ \forall a_2, \ b_1 = 136/121,$
	m = 11	$b_2 = 440215/56689952 + 267a_2/88, \ \forall b_4,$
		$b_3 = -103195941517/43159874536064$
		$-440813a_2/66997216 + 49a_2^2/13.$
E3.4	n = 5	$a_1 = 13/12, \ a_2 = 0,$
	m = 12	$b_1 = 133/108, \ b_2 = 0, \ b_3 = 5225/559872, \ \forall b_4.$
E3.5	n = 5	$a_1 = 13/12, a_2 \neq 0, b_1 = 133/108, b_2 = 34a_2/11,$
	m = 12	$b_3 = 81a_2^2/32 + 5225/559872, \forall b_4.$

Table 2

No.	Conditions	Coefficients
F1.1	$n \ge 7$	$a_i = 0 \ (i = 1, \dots, 2p), \ \forall a_{2p+1},$
		$b_i = 0 \ (i = 1, \dots, p).$
F1.2	$n \ge 7$	$a_i = 0 \ (i = 1, \dots, 2p - 1),$
		$b_i = 0 \ (i = 1, \dots, p-1), \ b_p \neq 0,$
		$a_{2p} = b_p^2 (m - n_p)^2 / (m - m_{2p})(m - \lambda), \ \forall a_{2p+1}.$
F2.1	n = 6	$a_i = 0 \ (i = 1, \dots, 2p - 1), \ a_{2p} = (5m - 12)/2m,$
	m = 6p + 5	$\forall a_{2p+1}, b_i = 0 \ (i = 1, \dots, p).$
F2.2	n = 6	$a_i = 0 \ (i = 1, \dots, 2p - 1)$
	m = 6p + 5	$a_{2p} = \left(9b_p^2m + 20m - 48\right)/8m, \ \forall a_{2p+1},$
	m = 6p + 5	$b_i = 0 \ (i = 1, \dots, p-1), \ b_p \neq 0.$
F3.1	n = 5	$a_i = 0 \ (i = 1, \dots, 2p - 3, 2p - 1, 2p),$
	$p \ge 2$	$a_{2p-2} = 4(m-5)(2m-5)/3m^2,$
		$\forall a_{2p+1}, b_i = 0 \ (i = 1, \dots, p-2, p),$
		$b_{p-1} = (3m - 10)/2m.$
F3.2	n = 5	$a_i = 0 \ (i = 1, \dots, 2p - 3),$
	$p \ge 3$	$a_{2p-2} = 4(m-5)(2m-5)/3m^2,$
		$a_{2p-1} = 3b_p(4m^2 - 45m + 100)/m(3m - 25),$
		$a_{2p} = b_p^2 (2m - 15)^2 / (3m - 20)(m - 10),$
		$\forall a_{2p+1}, b_i = 0 \ (i = 1, \cdots, p-2),$
		$b_{p-1} = (3m - 10)/2m, \ b_p \neq 0.$

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# Characteristic classes of (pro)algebraic varieties

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Dedicated to Jean-Paul Brasselet on the occasion of his sixtieth birthday

### §1. Introduction

Various characteristic classes of singular varieties have been introduced and studied. One of them is the so-called Chern–Schwartz– MacPherson class. Its unique existence was conjectured by P. Deligne and A. Grothendieck and it was affirmatively solved by R. MacPherson. This characteristic class is a fundamental and important characteristic class from the viewpoint of investigation of other characteristic classes.

In this paper, in the first half we make a quick survey on three interesting characteristic classes of singular varieties with a naïve motivation of constructing a "singular version" of the so-called generalized Hirzebruch–Riemann–Roch theorem behind, and state a "unification" theorem concerning these three characteristic classes and its bivarianttheoretic version. And in the latter half we make a quick survey on characrteristic classes of proalgebraic varieties, which are very much related to motivic measure and motivic integration.

## §2. Hirzebruch–Riemann–Roch and Grothendieck–Riemann– Roch

A characteristic class of a vector bundle over a topological space X is defined to be a map from the set of isomorphism classes of vector bundles over X to the cohomology group (ring)  $H^*(X; \Lambda)$  with a coefficient ring

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 $\Lambda$ , which is supposed to be compatible with the pullback of vector bundle and cohomology group for a continuous map. Namely, it is an assignment  $c\ell\colon \operatorname{Vect}(X) \to H^*(X;\Lambda)$  which satisfies that for a continuous map  $f\colon X \to Y$  the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Vect}(Y) & \stackrel{cl}{\longrightarrow} & H^*(Y;\Lambda) \\ & & & & \downarrow f^* \\ & & & \downarrow f^* \\ \operatorname{Vect}(X) & \stackrel{\phantom{}}{\longrightarrow} & H^*(X;\Lambda). \end{array}$$

Here Vect(W) is the set of isomorphism classes of vector bundles over W. In this paper we only deal with complex vector bundles.

If cl is multiplicative, i.e., cl satisfies the Whitney sum condition

$$cl(E \oplus F) = cl(E)cl(F),$$

then the contravariant functor Vect can be replaced by the Grothendieck K-theory:



For complex vector bundles, the Chern class is essential in the sense that any characteristic class is expressed as a polynomial of Chern classes. And furthermore any multiplicative characteristic class can be described via Hirzebruch's multiplicative sequence of Chern classes [Hir1].

For a complex manifold M its complex tangent bundle  $T_M$  is available and thus we can define a characteristic class  $cl(T_M)$ , which is called a *characteristic class* cl(M) of the manifold M.

Let X be a non-singular complex projective variety and E a holomorphic vector bundle over X. Let

$$\chi(X, E) = \sum_{i \ge 0} (-1)^i \dim_{\mathbb{C}} H^i(X; \Omega(E))$$

be the Euler-Poincaré characteristic, where  $\Omega(E)$  is the coherent sheaf of germs of sections of E. J.-P. Serre conjectured (in his letter to Kodaira and Spencer, dated September 29, 1953): There exists a polynomial

P(X, E) of Chern classes of the base variety X and the vector bundle E such that

$$\chi(X, E) = \int_X P(X, E) \cap [X].$$

Within three months (December 9, 1953) F. Hirzebruch solved this conjecture: the above looked-for polynomial P(X, E) can be expressed as

$$P(X, E) = ch(E) \cup td(X)$$

where ch(E) is the total Chern character of E and  $td(T_X)$  is the total Todd class of the tangent bundle  $T_X$  of X. For the sake of later use, we recall that for a complex vector bundle V the total cohomology classes ch(V) and td(V) are defined as follows:

$$ch(V) = \sum_{i=1}^{\operatorname{rank} V} e^{\alpha_i}$$

and

$$td(V) = \prod_{i=1}^{\operatorname{rank} V} \frac{\alpha_i}{1 - e^{-\alpha_i}}$$

where  $\alpha_i$ 's are the Chern roots of V. Namely, we have the following celebrated theorem of Hirzebruch:

Theorem (2.1) (Hirzebruch–Riemann–Roch)(HRR).

$$\chi(X, E) = T(X, E) := \int_X (ch(E) \cup td(X)) \cap [X].$$

T(X, E) is called the *T*-characteristic ([Hir1]). For a more detailed historical aspect of **HRR**, see [Hir2].

A. Grothendieck (cf. [BoSe]) generalized **HRR** for non-singular quasi-projective algebraic varieties over any field and proper morphisms with Chow cohomology ring theory instead of ordinary cohomology theory. For the complex case we can still take the ordinary cohomology theory (or the homology theory by the Poincaré duality). Here we stick ourselves to complex projective algebraic varieties for the sake of simplicity. For a variety X, let  $\mathbf{G}_0(X)$  denote the Grothendieck group of algebraic S. Yokura

coherent sheaves on X and for a morphism  $f: X \to Y$  the pushforward  $f_!: \mathbf{G}_0(X) \to \mathbf{G}_0(Y)$  is defined by

$$f_!(\mathcal{F}) := \sum_{i \ge 0} (-1)^i \mathbf{R}^i f_* \mathcal{F},$$

where  $\mathbf{R}^i f_* \mathcal{F}$  is (the class of ) the higher direct image sheaf of  $\mathcal{F}$ . Then  $\mathbf{G}_0$  is a covariant functor with the above pushforward (see [Grot1] and [Man]). Then Grothendieck showed the existence of a natural transformation from the covariant functor  $\mathbf{G}_0$  to the Q-homology covariant functor  $H_*(; \mathbb{Q})$  (see [BoSe]):

**Theorem (2.2)** (Grothendieck–Riemann–Roch)(**GRR**). Let the transformation  $\tau$ :  $\mathbf{G}_0() \to H_*(; \mathbb{Q})$  be defined by  $\tau(\mathcal{F}) = td(X)ch(\mathcal{F}) \cap [X]$  for any smooth variety X. Then  $\tau$  is actually natural, i.e., for any morphism  $f: X \to Y$  the following diagram commutes:

$$\mathbf{G}_0(X) \xrightarrow{\tau} H_*(X;\mathbb{Q}) \\
 \begin{array}{ccc}
 f_! & & \downarrow f_* \\
 \mathbf{G}_0(Y) \xrightarrow{\tau} H_*(Y;\mathbb{Q})
 \end{array}$$

*i.e.*,

$$td(T_Y)ch(f_!\mathcal{F})\cap [Y] = f_*(td(T_X)ch(\mathcal{F})\cap [X]).$$

Clearly **HRR** is induced from **GRR** by considering a map from X to a point.

Note that the target of the transformation of the original **GRR** is the cohomology  $H^*(;\mathbb{Q})$  with the Gysin homomorphism instead of the homology  $H_*(;\mathbb{Q})$ , but, by the definition of the Gysin homomorphism the original **GRR** can be put in as above.

### §3. The Generalized Hirzebruch–Riemann–Roch

In Hirzebruch's book [Hir1, §12.1 and §15.5] he has generalized the characteristics  $\chi(X, E)$  and T(X, E) to the so-called  $\chi_y$ -characteristic  $\chi_y(X, E)$  and  $T_y$ -characteristic  $T_y(X, E)$  as follows, using a parameter y (see also [HBJ, Chapter 5]).

Definition (3.1).

$$\chi_y(X, E) := \sum_{p \ge 0} \left( \sum_{q \ge 0} (-1)^q \dim_{\mathbf{C}} H^q(X, \Omega(E) \otimes \Lambda^P T_X^{\vee}) \right) y^p$$
$$= \sum_{p \ge 0} \chi(X, E \otimes \Lambda^P T_X^{\vee}) y^p$$

where  $T_X^{\vee}$  is the dual of the tangent bundle  $T_X$ , i.e., the cotangent bundle of X.

$$T_{y}(X, E) := \int_{X} \widetilde{td}_{(y)}(T_{X}) ch_{(1+y)}(E) \cap [X],$$
  
$$\widetilde{td}_{(y)}(T_{X}) := \prod_{i=1}^{\dim X} \left( \frac{\alpha_{i}(1+y)}{1-e^{-\alpha_{i}(1+y)}} - \alpha_{i}y \right),$$
  
$$ch_{(1+y)}(E) := \sum_{j=1}^{\operatorname{rank} E} e^{\beta_{j}(1+y)},$$

where  $\alpha_i's$  are the Chern roots of  $T_X$  and  $\beta_j's$  are the Chern roots of E.

F. Hirzebruch [Hir1, §21.3] showed the following theorem:

**Theorem (3.2)** (The generalized Hirzebruch–Riemann–Roch)(g-HRR).

$$\chi_y(X, E) = T_y(X, E).$$

The above modified Todd class  $\widetilde{td}_{(y)}(T_X)$  defined above unifies the following three important characteristic cohomology classes:

(y = -1) the total Chern class

$$\widetilde{td}_{(-1)}(T_X) = c(T_X),$$

(y=0) the total Todd class

$$\widetilde{td}_{(0)}(T_X) = td(T_X),$$

(y = 1) the total Thom–Hirzebruch L-class

$$\widetilde{td}_{(1)}(T_X) = L(T_X).$$

In particular, for E=the trivial line bundle, for these special values y = -1, 0, 1 the **g-HRR** reads as follows:

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(y = -1) Gauss-Bonnet-Chern Theorem:

$$e(X) = \int_X c(T_X) \cap [X],$$

(y = 0) Riemann-Roch:

$$\chi(X) = \int_X t d(T_X) \cap [X],$$

(y = 1) Hirzebruch's Signature Theorem:

$$\sigma(X) = \int_X L(T_X) \cap [X].$$

#### $\S4$ . Characteristic classes of singular varieties

In the following we consider only compact spaces.

For a singular complex algebraic or analytic variety X its tangent bundle is not available any longer because of the existence of singularities, thus one cannot define its characteristic class cl(X) as in the above case of manifolds, although a "tangent-like" bundle such as Zariski tangents is available. A main theme for defining reasonable characteristic classes for singular varieties is that reasonable ones should be interesting enough; for example, they must be geometrically or topologically interesting, and they should be quite well related to other well-known interesting invariants of varieties (see [Mac3]).

The theory of characteristic classes of vector bundles is nothing but saying that the assignment  $c\ell \colon \operatorname{Vect}(X) \to H^*(X; \Lambda)$  is a natural transformation from the contravariant functor Vect to the contravariant cohomology functor  $H^*(\ ; \Lambda)$ . This naturality is a key for various theories of characteristic classes for singular varieties.

The first example of a characteristic class formulated as a natural transformation was the Stiefel–Whitney class transformation due to Dennis Sullivan [Sull] (also see [Fu-Mc]). And the complex version of the Stiefel–Whitney class, i.e., the first characteristic class of singular complex varieties formulated as a natural transformation is MacPherson's Chern class transformation [Mac2].

Let F(X) be the abelian group of constructible functions on a variety X. Then the assignment  $F: \mathcal{V} \to \mathcal{A}$  is a contravariant functor (from the category of varieties to the category of abelian groups) by the usual functional pullback: for a morphism  $f: X \to Y$ 

$$f^* \colon F(Y) \to F(X)$$
 defined by  $f^*(\alpha) := \alpha \circ f$ .

For a constructible set  $Z \subset X$ , we define

$$\chi(Z;\alpha) := \sum_{n \in \mathbb{Z}} n \chi(Z \cap \alpha^{-1}(n)).$$

Then it turns out that the assignment  $F: \mathcal{V} \to \mathcal{A}$  also becomes a covariant functor by the following pushforward:

$$f_*: F(X) \to F(Y)$$
 defined by  $f_*(\alpha)(y) := \chi(f^{-1}(y); \alpha).$ 

To show this requires a stratification theory (see [Mac2]).

P. Deligne and A. Grothendieck conjectured (around 1969) and R. MacPherson [Mac2] solved the following:

**Theorem (4.1).** There exists a unique natural transformation

$$c_* \colon F \to H_*$$

from the constructible function covariant functor F to the homology covariant functor  $H_*$  satisfying the "normalization" that the value of the characteristic function  $\mathbb{1}_X$  of a smooth complex algebraic variety X is the Poincaré dual of the total Chern cohomology class:

$$c_*(\mathbb{1}_X) = c(T_X) \cap [X].$$

The main ingredients are Chern–Mather classes, local Euler obstructions (also see [Br3], [Gon] and [Sa]) and "graph construction" (also see [Mac1]). The uniqueness follows from the resolution of singularities. For recent investigations on local Euler obstruction, e.g. see [BLS], [BMPS] and [STV1, STV2], etc.

J.-P. Brasselet and M.-H. Schwartz [BrSc] showed that the distinguished value  $c_*(\mathbb{1}_X)$  of the characteristic function of a variety embedded into a complex manifold is isomorphic under this transformation to the Schwartz class [Sc1, Sc2] via the Alexander duality. Thus, for a complex algebraic variety X, singular or nonsingular,  $c_*(\mathbb{1}_X)$  is called the total *Chern–Schwartz–MacPherson class* of X and denoted simply by  $c_*(X)$ . By considering mapping X to a point, one can get

$$e(X) = \int_X c_*(X)$$

which is a singular version of the Gauss–Bonnet–Chern theorem.

Motivated by the formulation of MacPherson's Chern class transformation, P. Baum, W. Fulton and R. MacPherson [BFM] have extended **GRR** to singular varieties, by introducing the so-called *localized Chern* 

character  $ch_X^M(\mathcal{F})$  of a coherent sheaf  $\mathcal{F}$  with X embedded into a nonsingular quasi-projective variety M, as a substitute of  $ch(F) \cap [X]$  in the above **GRR**. Note that if X is smooth  $ch_X^X(\mathcal{F}) = ch(F) \cap [X]$ . In [BFM] they showed the following theorem:

**Theorem (4.2)** (Baum–Fulton–MacPherson's Riemann–Roch) (**BFM-RR**). (i)  $td_*(\mathcal{F}) := td(i_M^*T_M) \cap ch_X^M(\mathcal{F})$  is independent of the embedding  $i_M : X \to M$ .

(ii) Let the transformation  $td_*: \mathbf{G}_0(\ ) \to H_*(\ ;\mathbb{Q})$  be defined by

$$td_*(\mathcal{F}) = td(i_M^*T_M) \cap ch_X^M(\mathcal{F})$$

for any variety X. Then  $td_*$  is actually natural, i.e., for any morphism  $f: X \to Y$  the following diagram commutes:

$$\mathbf{G}_0(X) \xrightarrow{td_*} H_*(X;\mathbb{Q})$$

$$\begin{array}{c}
 f_! \\
 & \downarrow f_* \\
 \mathbf{G}_0(Y) \xrightarrow{td_*} H_*(Y;\mathbb{Q})
 \end{array}$$

*i.e.*, for any embeddings  $i_M : X \to M$  and  $i_N : Y \to N$ 

$$td(i_N^*T_N) \cap ch_Y^N(f_!\mathcal{F}) = f_*(td(i_M^*T_M) \cap ch_X^M(\mathcal{F})).$$

For a complex algebraic variety X, singular or nonsingular,  $td_*(X) := c_*(\mathcal{O}_X)$  is called the Baum–Fulton–MacPherson's Todd homology class of X. And we get

$$\chi(X) = \int_X t d_*(X)$$

which is a singular version of the Riemann–Roch.

Using the notion of "perversity", M. Goresky and R. MacPherson [GM1, GM2] have introduced *Intersection Homology Theory*, in which almost all properties, such as the Poincaré duality, of the (co)homology of smooth manifolds are satisfied. Note that the intersection homology group is not a homotopy invariant unlike the (co)homology group. For the intersection homology theory, e.g., see also [Bor], [Br2] and [Kir].

In [GM1], they introduced a homology *L*-class  $L_*^{\text{GM}}(X)$  such that if X is nonsingular it becomes the Poincaré dual of the original Thom– Hirzebruch *L*-class:

$$L^{\mathrm{GM}}_*(X) = L(TX) \cap [X].$$

Later, S. Cappell and J. Shaneson [CS1] (see also [CS2] and [Sh]), using some topological aspects of perverse sheaves [BBD], introduced a homology *L*-class transformation  $L_*$ , which turns out to be a natural transformation from the abelian group  $\Omega$  of cobordism classes of selfdual constructible complexes to the rational homology group [BSY2] (cf. [Y1]):

**Theorem (4.3)** (Cappell–Shaneson's homology L-class). There exists a natural transformation

$$L_*: \Omega \to H_*(\ ; \mathbb{Q})$$

such that for X smooth

$$L_*(\mathbb{Q}_X[2\dim X]) = L(TX) \cap [X].$$

Here  $\mathbb{Q}_X$  is the constant sheaf (considered as a complex concentrated at degree 0) of X.

For a complex algebraic variety X, singular or nonsingular, the value  $L_*(IC_X)$  of the middle intersection cohomology complex  $IC_X$  is the total Goresky-MacPherson's homology L-class  $L_*^{\text{GM}}(X)$  of X and simply denoted by  $L_*(X)$ . And we get

$$\sigma(X) = \int_X L_*(X)$$

which is a singular version of Hirzebruch's signature theorem. Here  $\sigma(X)$  is defined by the pairing of the intersection homology group with middle perversity.

For a survey concerning characteristic classes of singular varieties other than MacPherson's survey article [Mac3], there are now various articles available, e.g., [Alu1], [Br4], [Pa] (also see [PP]), [Su3] (also see [Su1, Su2]), [Sch2] (also see [Sch4]), [SY] etc., and also consult various papers therein.

### $\S 5.$ A "unification" theorem

So far we have seen that the generalized Hirzebruch–Riemann–Roch **g-HRR** unifies the three important and distinguished characteristics (or genera):

(y = -1) the topological Euler–Poincaré characteristic e(X), (y = 0) the arithmetic genus  $\chi(X)$ , (y = 1) the signature  $\sigma(X)$ , and that corresponding to these three invariants there are three distinguished natural transformations of characteristic homology classes of possibly singular varieties, which are respectively,

(y = -1) MacPherson's Chern class transformation  $c_* \colon F() \to H_*(;\mathbb{Z}),$ 

(y = 0) Baum–Fulton–MacPherson's Riemann–Roch  $td_* : \mathbf{G}_0() \to H_*( ; \mathbb{Q}),$ 

(y = 1) Cappell–Shaneson's homology L-class  $L_* \colon \Omega() \to H_*(; \mathbb{Q}).$ 

It seems to be natural to pose the following naïve problem (cf. [Mac2] and [Y2]):

**Problem (5.1).** Is there a theory of characteristic homology classes unifying the above three characteristic homology classes of possibly singular varieties? A naïve question is whether or not there is a reasonable "singular version"  $\bigcirc_{y}$  of the generalized Hirzebruch-Riemann-Roch **g-HRR** such that

(y = 0) ? gives rise to the Baum-Fulton-MacPherson's Riemann-Roch  $td_*$ , and

(y=1) ? gives rise to the Cappell-Shaneson's homology L-class  $L_*$ .

An obvious problem for this unification problem is that the source covariant functors of these three natural transformations are all different!

A "reasonable" answer for the above problem has been obtained [BSY2] (cf. [BSY3] and [SY]) via the so-called *relative Grothendieck ring* of complex algebraic varieties over X, denoted by  $K_0(\mathcal{V}/X)$ . This ring was introduced by E. Looijenga in [Lo] and further studied by F. Bittner in [Bit].

The relative Grothendieck group  $K_0(\mathcal{V}/X)$  (of morphisms over a variety X) is the quotient of the free abelian group of isomorphism classes of morphisms to X (denoted by  $[Y \to X]$  or  $[Y \xrightarrow{h} X]$ ), modulo the following relation:

$$[Y \xrightarrow{h} X] = [Z \hookrightarrow Y \xrightarrow{h} X] + [Y \setminus Z \hookrightarrow Y \xrightarrow{h} X]$$

for  $Z \subset Y$  a closed subvariety of Y. The ring structure is given by the fiber square: for  $[Y \xrightarrow{f} X], [W \xrightarrow{g} X] \in K_0(\mathcal{V}/X)$ 

$$[Y \xrightarrow{f} X] \cdot [W \xrightarrow{g} X] := [Y \times_X W \xrightarrow{f \times_X g} X].$$

Here  $Y \times_X W \xrightarrow{f \times_X g} X$  is  $g \circ f' = f \circ g'$  where f' and g' are as in the following diagram

$$\begin{array}{cccc} Y \times_X W & \xrightarrow{f'} & W' \\ g' \downarrow & & \downarrow^g \\ Y & \xrightarrow{f} & X. \end{array}$$

The relative Grothendieck ring  $K_0(\mathcal{V}/X)$  has the unit  $1_X := [X \xrightarrow{\operatorname{id}_X} X].$ 

Note that when X = pt is a point, the relative Grothendieck ring  $K_0(\mathcal{V}/pt)$  is nothing but the usual Grothendieck ring  $K_0(\mathcal{V})$  of  $\mathcal{V}$ , which is the free abelian group generated by the isomorphism classes of varieties modulo the subgroup generated by elements of the form  $[V] - [V'] - [V \setminus V']$  for a subvariety  $V' \subset V$ , and the ring structure is given by the Cartesian product of varieties.

For a morphism  $f: X' \to X$ , the pushforward

$$f_* \colon K_0(\mathcal{V}/X') \to K_0(\mathcal{V}/X)$$

is defined by

$$f_*[Y \xrightarrow{h} X'] := [Y \xrightarrow{f \circ h} X].$$

With this pushforward, the assignment  $X \mapsto K_0(\mathcal{V}/X)$  is a covariant functor. The pullback

$$f^* \colon K_0(\mathcal{V}/X) \to K_0(\mathcal{V}/X')$$

is defined as follows: for a fiber square

$$\begin{array}{cccc} Y' & \stackrel{g'}{\longrightarrow} & X' \\ f' \downarrow & & \downarrow f \\ Y & \stackrel{g}{\longrightarrow} & X \end{array}$$

the pullback  $f^*[Y \xrightarrow{g} X] := [Y' \xrightarrow{g'} X']$ . With this pullback, the assignment  $X \longmapsto K_0(\mathcal{V}/X)$  is a contravariant functor.

**Theorem (5.2).** Let  $K_0(\mathcal{V}/X)$  be the Grothendieck group of morphisms over X. Then there exists a unique natural transformation

$$T_y \colon K_0(\mathcal{V}/ ) \to H_*( ) \otimes \mathbb{Q}[y]$$

such that for X nonsingular

$$T_y([X \xrightarrow{\mathrm{id}} X]) = \widetilde{td_{(y)}}(X) \cap [X].$$

And we have the following theorem:

**Theorem (5.3).** (y = -1) There exists a unique natural transformation  $\epsilon$ :  $K_0(\mathcal{V}/ ) \to F( )$  such that for X nonsingular  $\epsilon([X \xrightarrow{\text{id}} X]) = \mathbb{1}_X$ . And the following diagram commutes



(y = 0) There exists a unique natural transformation  $\gamma \colon K_0(\mathcal{V}/) \to \mathbf{G}_0()$  such that for X nonsingular  $\gamma([X \xrightarrow{\mathrm{id}} X]) = [\mathcal{O}_X]$ . And the following diagram commutes



(y = 1) There exists a unique natural transformation  $\omega \colon K_0(\mathcal{V}/) \to \Omega()$  such that for X nonsingular  $\omega([X \xrightarrow{\mathrm{id}} X]) = [\mathbb{Q}_X[2 \dim X]]$ . And the following diagram commutes



An original proof of Theorem (5.2) uses Saito's theory of mixed Hodge modules [Sai] and it turns out that it can be also proved without it and, instead, via a Bittner-Looijenga's theorem about the relative Grothendieck group [Bit].

### $\S 6.$ Bivariant Theories

In [FM] W. Fulton and R. MacPherson introduced the notion of *Bivariant Theory*, which is a simultaneous generalization of a pair of covariant and contravariant functors. Most pairs of covariant and contravariant theories, e.g., such as homology theory, K-theory, etc, extend

to bivariant theories. They also introduced the *operational bivariant the*ory (also see [Fu]), which can be always constructed from any covariant functor.

A bivariant theory  $\mathbb{B}$  on a category  $\mathcal{C}$  with values in the category of abelian groups is an assignment to each morphism  $X \xrightarrow{f} Y$  in the category  $\mathcal{C}$  a graded abelian group  $\mathbb{B}(X \xrightarrow{f} Y)$ , which is equipped with the following three basic operations:

(Product operations): For morphisms  $f: X \to Y$  and  $g: Y \to Z$ , the product operation

•: 
$$\mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \to \mathbb{B}(X \xrightarrow{gf} Z)$$

is defined.

(Pushforward operations): For morphisms  $f\colon X\to Y$  and  $g\colon Y\to Z$  with f proper, the pushforward operation

$$f_{\bigstar} \colon \mathbb{B}(X \xrightarrow{gf} Z) \to \mathbb{B}(Y \xrightarrow{g} Z)$$

is defined.

(Pullback operations): For a fiber square

$$\begin{array}{cccc} X' & \stackrel{g'}{\longrightarrow} & X \\ f' \downarrow & & \downarrow f \\ Y' & \stackrel{g}{\longrightarrow} & Y, \end{array}$$

the pullback operation

$$g^{\bigstar} \colon \mathbb{B}(X \xrightarrow{f} Y) \to \mathbb{B}(X' \xrightarrow{f'} Y')$$

is defined. And these three operations are required to satisfy the seven compatibility axioms (see [FM, Part I, §2.2] for details).

Let  $\mathbb{B}$ ,  $\mathbb{B}'$  be two bivariant theories on a category  $\mathcal{C}$ . Then a *Grothen*dieck transformation from  $\mathbb{B}$  to  $\mathbb{B}'$ 

$$\gamma \colon \mathbb{B} \to \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \to Y) \to \mathbb{B}'(X \to Y)$$

for a morphism  $X \to Y$  in the category  $\mathcal{C}$ , which preserves the above three basic operations:

(i) 
$$\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta),$$
  
(ii)  $\gamma(f_{\bigstar} \alpha) = f_{\bigstar} \gamma(\alpha),$  and  
(iii)  $\gamma(g^{\bigstar} \alpha) = g^{\bigstar} \gamma(\alpha).$ 

 $B_*(X) := \mathbb{B}(X \to pt)$  and  $B^*(X) := \mathbb{B}(X \xrightarrow{\mathrm{id}} X)$  become a covariant functor and a contravariant functor, respectively. And a Grothendieck transformation  $\gamma : \mathbb{B} \to \mathbb{B}'$  induces natural transformations  $\gamma_* : B_*$  $\to B'_*$  and  $\gamma^* : B^* \to B'^*$ . If we have a Grothendieck transformation  $\gamma : \mathbb{B} \to \mathbb{B}'$ , then via a bivariant class  $b \in \mathbb{B}(X \xrightarrow{f} Y)$  we get the commutative diagram



This is called the Verdier-type Riemann-Roch formula associated to the bivariant class b.

Fulton-MacPherson's bivariant group  $\mathbb{F}(X \xrightarrow{f} Y)$  of constructible functions consists of all the constructible functions on X which satisfy the local Euler condition with respect to f. Here a constructible function  $\alpha \in F(X)$  is said to satisfy the *local Euler condition with respect to* fif for any point  $x \in X$  and for any local embedding  $(X, x) \to (\mathbb{C}^N, 0)$ the equality  $\alpha(x) = \chi (B_{\epsilon} \cap f^{-1}(z); \alpha)$  holds, where  $B_{\epsilon}$  is a sufficiently small open ball of the origin 0 with radius  $\epsilon$  and z is any point close to f(x) (cf. [Br1], [Sa]). In particular, if  $\mathbb{1}_f := \mathbb{1}_X$  belongs to the bivariant group  $\mathbb{F}(X \xrightarrow{f} Y)$ , then the morphism  $f: X \to Y$  is called *an Euler morphism*. For example, a holomorphic submersion between complex spaces is an Euler morphism.

The three operations on  $\mathbb{F}$  are defined as follows:

(i) the product operation  $\bullet \colon \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \to \mathbb{F}(X \xrightarrow{gf} Z)$  is defined by

$$\alpha \bullet \beta := \alpha \cdot f^* \beta,$$

(ii) the pushforward operation  $f_{\bigstar} \colon \mathbb{F}(X \xrightarrow{gf} Z) \to \mathbb{F}(Y \xrightarrow{g} Z)$  is the usual pushforward  $f_*$ , i.e.,

$$f_{\bigstar}(\alpha)(y) := \int c_*(\alpha|_{f^{-1}}),$$

(iii) for a fiber square

$$\begin{array}{cccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation  $g^{\bigstar} \colon \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{F}(X' \xrightarrow{f'} Y')$  is the functional pullback  ${g'}^*$ , i.e.,

$$g^{\bigstar}(\alpha)(x') := \alpha(g'(x')).$$

Note that for any bivariant constructible function  $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ , the Euler–Poincaré characteristic  $\chi(f^{-1}(y); \alpha) = \int c_*(\alpha|_{f^{-1}(y)})$  of  $\alpha$ restricted to each fiber  $f^{-1}(y)$  is locally constant, i.e., constant along connected components of the base variety Y; in particular, if  $f: X \to Y$ is an Euler morphism, then the Euler–Poincaré characteristic of the fibers are locally constant.

The correspondence  $\mathbb{F}^s(X \to Y) := F(X)$  assigning to a morphism  $f: X \to Y$  the abelian group F(X) of the source variety X, whatever the morphism f is, becomes a bivariant theory with the same operations above. This bivariant theory is called the *simple* bivariant theory of constructible functions (see [Y3] and [Sch3]). In passing, what we need to do to show that the Fulton–MacPherson's group of constructible functions satisfying the local Euler condition with respect to a morphism is a bivariant theory is to show that the local Euler condition with respect to a morphism is preserved by each of the above three operations.

Let  $\mathbb{H}$  be Fulton-MacPherson's bivariant homology theory, constructed from the cohomology theory [FM, §3.1]. W. Fulton and R. MacPherson conjectured or posed as a question the existence of a socalled bivariant Chern class and J.-P. Brasselet [Br1] solved it:

**Theorem (6.1)** (J.-P. Brasselet). For the category of embeddable complex analytic varieties with cellular morphisms, there exists a Grothendieck transformation

 $\gamma\colon \mathbb{F}\to\mathbb{H}$ 

such that for a morphism  $f: X \to pt$  from a nonsingular variety X to a point pt and the bivariant constructible function  $\mathbb{1}_f := \mathbb{1}_X$  the following normalization condition holds:

$$\gamma(\mathbb{1}_f) = c(TX) \cap [X].$$

In [Z1, Z2] J. Zhou showed that the bivariant Chern classes constructed by J.-P. Brasselet [Br1] and by C. Sabbah [Sa] are identical in the case when the target variety is a nonsingular curve. And the present author showed the following uniqueness theorem of bivariant Chern classes for morphisms whose target varieties are nonsingular and of any dimension:

**Theorem (6.2)** ([Y4, Theorem (3.7)]). If there exists a bivariant Chern class  $\gamma \colon \mathbb{F} \to \mathbb{H}$ , then it is unique when restricted to morphisms whose target varieties are nonsingular; explicitly, for a morphism  $f \colon X$  $\to Y$  with Y nonsingular and for any bivariant constructible function  $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$  the bivariant Chern class  $\gamma(\alpha)$  is expressed by

$$\gamma(\alpha) = f^* s(TY) \cap c_*(\alpha)$$

where  $s(TY) := c(TY)^{-1}$  is the Segre class of the tangent bundle.

See [Sch3] and [Y5, Y6, Y7, Y8, Y9] for other related results.

And in [BSY1] (see also [BSY4]) the above theorem is furthermore generalized to the case when the target variety can be singular but is "like a manifold":

**Theorem (6.3).** Let Y be a complex analytic variety which is an oriented A-homology manifold. If there exists a bivariant Chern class  $\gamma \colon \mathbb{F} \to \mathbb{H}$ , then for any morphism  $f \colon X \to Y$  the bivariant Chern class  $\gamma_f \colon \mathbb{F}(X \xrightarrow{f} Y) \otimes A \to \mathbb{H}(X \xrightarrow{f} Y) \otimes A$  is uniquely determined and it is described by

$$\gamma_f(\alpha) = f^* c^* (Y)^{-1} \cap c_*(\alpha).$$

Here  $c^*(Y)$  is the unique cohomology class such that  $c_*(\mathbb{1}_Y) = c^*(Y) \cap [Y]$ . (Note that  $c^*(Y)$  is invertible.)

Notice that  $c^*(Y) = c(TY)$  for Y smooth and thus Theorem (6.3) indeed generalizes Theorem (6.2).

**Remark (6.4).** As to the uniqueness of operational bivariant Chern class [EY1, EY2] and operational bivariant Riemann–Roch [FM], one can also use a result due to S.-I. Kimura [Kim1] (also see [Kim2]).

**Remark (6.5).** In [BSY1] we have also shown that a natural transformation of covariant theories extends uniquely to a Grothendieck transformation of suitable bivariant subtheories associated to them, provided that the given transformation commutes with exterior products. This gives in a sense a positive solution to [FM, §10.9 Uniqueness questions]. For more details of this result and other results, see [BSY1].

Hence it follows from this general result that our natural transformation  $T_y: K_0(\mathcal{V}/ ) \to H_*( ) \otimes \mathbb{Q}[y]$  can be extended to a suitable bivariant version. Here, to get the suitable bivariant subtheories, the bivariant theories associated to the covariant functors which we consider are respectively the *simple* bivariant theory  $\mathbb{K}_0^s(\mathcal{V}/X \xrightarrow{f} Y) := K_0(\mathcal{V}/X)$ , just like the simple bivariant theory  $\mathbb{F}^s$  of constructible functions as above, and the Fulton-MacPherson's bivariant homology theory  $\mathbb{H}$  described above.

## §7. Proconstructible functions and Euler–Poincaré characteristics of proalgebraic varieties

Let I be a directed set and let C be a given category. Then a projective system is, by definition, a system

$$\{X_i, \pi_{ii'} : X_{i'} \to X_i (i < i'), I\}$$

consisting of objects  $X_i \in \text{Obj}(\mathcal{C})$ , morphisms  $\pi_{ii'} \colon X_{i'} \to X_i \in \text{Mor}(\mathcal{C})$ for each i < i' and the index set I. The object  $X_i$  is called a *term* and the morphism  $\pi_{ii'} \colon X_{i'} \to X_i$  a *bonding morphism* or *structure morphism* ([MS]). The projective system

$$\{X_i, \pi_{ii'} : X_{i'} \to X_i (i < i'), I\}$$

is sometimes simply denoted by  $\{X_i\}_{i \in I}$ .

Given a category  $\mathcal{C}$ , Pro- $\mathcal{C}$  is the category whose objects are projective systems  $X = \{X_i\}_{i \in I}$  in  $\mathcal{C}$  and whose set of morphisms from  $X = \{X_i\}_{i \in I}$  to  $Y = \{Y_j\}_{j \in J}$  is

$$\operatorname{Pro-} \mathcal{C}(X, Y) := \varprojlim_{J} (\varinjlim_{I} \mathcal{C}(X_{i}, Y_{j})).$$

Note that given a projective system  $X = \{X_i\}_{i \in I} \in \text{Pro-}\mathcal{C}$ , the projective limit  $X_{\infty} := \varprojlim X_i$  may not exist or may not belong to the source category  $\mathcal{C}$ ; for a certain sufficient condition for the existence of the projective limit in the category  $\mathcal{C}$ , see [MS] for example.

An object in Pro-C is called a *pro-object*. A projective system of algebraic varieties is called a *pro-algebraic variety* or simply *pro-variety* and its projective limit is called a *proalgebraic variety* or simply *prova-riety*, which may not be an algebraic variety but simply a topological space.

**Remark (7.1).** In Etale Homotopy Theory [AM] and Shape Theory (e.g., see [Boru], [Ed], [MS]) they stay in the pro-category and do not consider limits and colimits, because doing so throw away some geometric informations (also see [Grot2]).

A pro-morphism between two pro-objects is quite complicated. However, it follows from [MS] that the pro-morphism can be described more naturally as a so-called *level preserving pro-morphism*. Suppose that we have two pro-algebraic varieties  $X = \{X_{\gamma}\}_{\gamma \in \Gamma}$  and  $Y = \{Y_{\lambda}\}_{\lambda \in \Lambda}$ . Then a pro-algebraic morphism  $\Phi = \{f_{\lambda}\}_{\lambda \in \Lambda} : X \to Y$  is described as follows: there is an order-preserving map  $\xi \colon \Lambda \to \Gamma$ , i.e.,  $\xi(\lambda) < \xi(\mu)$  for  $\lambda < \mu$ , and for each  $\lambda \in \Lambda$  there is a morphism  $f_{\lambda} \colon X_{\xi(\lambda)} \to Y_{\lambda}$  such that for  $\lambda < \mu$  the following diagram commutes:

$$\begin{array}{cccc} X_{\xi(\mu)} & \xrightarrow{f_{\mu}} & Y_{\mu} \\ & & & & \downarrow^{\pi_{\lambda\mu}} \\ & & & & \downarrow^{\pi_{\lambda\mu}} \\ & & & X_{\xi(\lambda)} & \xrightarrow{f_{\lambda}} & Y_{\lambda}, \end{array}$$

Then, the projective limit of the system  $\{f_{\lambda}\}$  is a morphism from the provariety  $X_{\infty} = \lim_{\lambda \in \Lambda} X_{\lambda}$  to the provariety  $Y_{\infty} = \lim_{\lambda \in \Gamma} Y_{\lambda}$ . It is called a promorphism and denoted by  $f_{\infty} \colon X_{\infty} \to Y_{\infty}$ .

From here on, for the sake of simplicity, we only deal with the case when the directed set  $\Lambda$  is the natural numbers  $\mathbb{N}$  and a pro-morphism  $\{f_n\}$  of two pro-varieties  $\{X_n\}$  and  $\{Y_n\}$  is such that for each n the following diagram commutes:

$$\begin{array}{cccc} X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \\ & & & & \downarrow^{\pi_{n(n+1)}} \\ & & & & \downarrow^{\pi_{n(n+1)}} \\ & & & X_n & \xrightarrow{f_n} & Y_n. \end{array}$$

The projective system  $\{X_n\}$  induces the projective system of abelian groups of constructible functions:

$$\{F(X_n), \pi_{nm} * \colon F(X_m) \to F(X_n)(n < m)\}.$$

And a system of morphims  $f_n \colon X_n \to Y_n$  induces the system of homomorphisms

$$f_{n_*}\colon F(X_n)\to F(Y_n).$$

Thus the system of commutative diagrams

induces the homomorphism

$$f_{*\infty} \colon \varprojlim_n F(X_n) \to \varprojlim_n F(Y_n)$$

Similarly we get the homomorphism of the projective limits of homology groups

$$f_{*\infty}: \lim_{n} H_*(X_n) \to \lim_{n} H_*(Y_n).$$

Furthermore the commutative diagram of Chern–Schwartz–MacPherson class homomorphisms

induces the projective limit of MacPherson's Chern class transformations:

$$c_{*\infty} \colon \varprojlim_n F(X_n) \to \varprojlim_n H_*(X_n)$$

So, we define, for the proalgebraic variety  $X_{\infty} = \lim_{\lambda \in \Lambda} X_{\lambda}$ ,

pro 
$$F(X_{\infty}) := \varprojlim_{n} F(X_{n})$$
 and pro  $H_{*}(X_{\infty}) := \varprojlim_{n} H_{*}(X_{n}).$ 

If we define  $\operatorname{pro} c_*$ :  $\operatorname{pro} F \to \operatorname{pro} H_*$  to be the above  $c_{*\infty}$  and define  $f_{\infty*}$  to be the above  $f_{*\infty}$ , then we have a naïve proalgebraic version of MacPherson's Chern class transformation

pro 
$$c_*$$
: pro  $F \to \text{pro } H_*$ ,

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i.e., for a proalgebraic morphism  $f_{\infty} \colon X_{\infty} \to Y_{\infty}$  we have the commutative diagram

Although the above construction by taking the projective limits is quite easy, the structure of the progroup  $\operatorname{pro} F(X^{\infty})$  is not so obvious and also it is not obvious how to capture an element of  $\varprojlim_n F(X_n)$  as a function on the proalgebraic variety  $X_{\infty} = \varprojlim_n X_n$ .

**Remark (7.2).** In [Alu3] P. Aluffi considered the above projective limit for a certain special projective system of morphisms called *modification system*, which is more precisely a projective system of birational morphisms.

So, we consider the inductive limits:

**Definition (7.3).** For a proalgebraic variety  $X_{\infty} = \varprojlim_n X_n$ , the inductive limit of the inductive system  $\{F(X_n), \rho_{nm}^* \colon F(X_n) \to F(X_m) \ (n < m)\}$  is denoted by  $F^{\text{pro}}(X_{\infty})$ ;

$$F^{\mathrm{pro}}(X_{\infty}) := \varinjlim_{n} F(X_{n}) = \bigcup_{n} \rho^{n} \left( F(X_{n}) \right)$$

where  $\rho^n \colon F(X_n) \to \varinjlim_n F(X_n)$  is the homomorphism sending  $\alpha_n$  to its equivalence class  $[\alpha_n]$  of  $\alpha_n$ . An element of the group  $F^{\text{pro}}(X_\infty)$  is called a *proconstructible* function on the proalgebraic variety  $X_\infty$ . As a function on  $X_\infty$ , the value of  $[\alpha_n]$  at a point  $(x_m) \in X_\infty$  is defined by

$$[\alpha_n]((x_m)) := \alpha_n(x_n).$$

The terminology *proconstructible* is used in [Grom1] (cf. [Grom2]), but its definition is not given there.

**Lemma (7.4).** For each positive integer n, let  $G_n = \mathbb{Z}$  be the integers and  $\pi_{n,n+1} \colon G_n \to G_{n+1}$  be the homomorphism defined by multiplication by a non-zero integer  $p_n$ , i.e.,  $\pi_{n,n+1}(m) = mp_n$ . Then there exists a unique (injective) homomorphism

$$\Psi\colon \varinjlim_n G_n \to \mathbb{Q}$$

such that the following diagram commutes



Here we set  $p_0 := 1$ .

Using this lemma we can show the following theorem:

**Theorem (7.5).** Let  $X_{\infty} = \varprojlim_{n \in \mathbb{N}} X_n$  be a provariety such that for each *n* the structure morphism  $\pi_{n(n+1)} \colon X_{n+1} \to X_n$  satisfies the condition that the Euler-Poincaré characteristics of the fibers of  $\pi_{n,n+1}$ are non-zero (which implies the surjectivity of the morphism  $\pi_{n(n+1)}$ ) and the same; for example,  $\pi_{n(n+1)} \colon X_{n+1} \to X_n$  is a locally trivial fiber bundle with fiber variety being  $F_n$  and  $\chi(F_n) \neq 0$ . Let us denote the constant Euler-Poincaré characteristic of the fibers of the morphism  $\pi_{n(n+1)} \colon X_{n+1} \to X_n$  by  $\chi_n$  and we set  $\chi_0 := 1$ .

(i) The canonical Euler-Poincaré (pro)characteristic homomorphism, i.e., a "canonical realization" of the inductive limit of the Euler-Poincaré characteristic homomorphisms  $\{\chi: F(X_n) \to \mathbb{Z}\}_{n \in \mathbb{N}}$ , is described as the homomorphism

$$\chi^{\mathrm{pro}} \colon F^{\mathrm{pro}}(X_{\infty}) \to \mathbb{Q}$$

defined by

$$\chi^{\text{pro}}\left(\left[\alpha_{n}\right]\right) = \frac{\chi(\alpha_{n})}{\chi_{0} \cdot \chi_{1} \cdot \chi_{2} \cdots \chi_{n-1}}$$

(Here "canonical realization" means "through the injective homomorphism in the above lemma".)

(ii) In particular, if the Euler-Poincaré characteristics  $\chi_n$  are all the same, say  $\chi_n = \chi$  for any n, then the canonical Euler-Poincaré (pro)characteristic homomorphism  $\chi^{\text{pro}}$ :  $F^{\text{pro}}(X_{\infty}) \to \mathbb{Q}$  is described by

$$\chi^{\mathrm{pro}}\left(\left[\alpha_{n}\right]\right) = \frac{\chi(\alpha_{n})}{\chi^{n-1}}$$

In this special case, the target ring  $\mathbb{Q}$  can be replaced by the ring  $\mathbb{Z}[1/\chi]$ .

In a more special case, the target ring  $\mathbb{Q}$  in the above theorem can be replaced by the Grothendieck ring of varieties.

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Let  $K_0(\mathcal{V}_{\mathbb{C}})$  be the Grothendieck ring of algebraic varieties, i.e., the free abelian group generated by the isomorphism classes of varieties modulo the subgroup generated by elements of the form  $[V] - [V'] - [V \setminus V']$ for a closed subset  $V' \subset V$  with the ring structure  $[V] \cdot [W] := [V \times W]$ . There are distinguished elements in  $K_0(\mathcal{V}_{\mathbb{C}})$ :  $\mathbb{1}$  is the class [p] of a point p and  $\mathbb{L}$  is the Tate class  $[\mathbb{C}]$  of the affine line  $\mathbb{C}$ . From this definition, we can see that any constructible set of a variety determines an element in the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$ . Provisionally the element [V] in the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$  is called the *Grothendieck "motivic" class of* V and let us denote it by  $\Gamma(V)$ . Hence we get the following homomorphism, called the *Grothendieck "motivic" class homomorphism*: for any variety X

$$\Gamma \colon F(X) \to K_0(\mathcal{V}_{\mathbb{C}}),$$

which is defined by

$$\Gamma(\alpha) = \sum_{n \in \mathbb{Z}} n \left[ \alpha^{-1}(n) \right].$$

Or  $\Gamma(\sum a_V 1_V) := \sum a_V[V]$  where V is a constructible set in X and  $a_V \in \mathbb{Z}$ . From now on, we sometimes write  $[\alpha]$  for  $\Gamma(\alpha)$  for a constructible function  $\alpha$ .

This Grothendieck "motivic" class homomorphism is *tautological* and its more "geometric" one is the Euler-Poincaré characteristic homomorphsim  $\chi: F(X) \to \mathbb{Z}$ . The above theorem is about extending the Euler-Poincaré characteristic homomorphsim  $\chi: F(X) \to \mathbb{Z}$ to the category of proalgebraic varieties. Thus a very natural problem is to generalize the Grothendieck "motivic" class homomorphism  $\Gamma: F(X) \to K_0(\mathcal{V}_{\mathbb{C}})$  to the category of proalgebraic varieties. Here one should be a bit careful; the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$  is not a domain unlike the ring  $\mathbb{Z}$  of integers as shown recently by B. Poonen [Po, Theorem 1].

**Theorem (7.6).** Let  $X_{\infty} = \lim_{n \in \mathbb{N}} X_n$  be a proalgebraic variety such that each structure morphism  $\pi_{n(n+1)} \colon X_{n+1} \to X_n$  satisfies the condition that for each n there exists a  $\gamma_n \in K_0(\mathcal{V}_{\mathbb{C}})$  such that  $[\pi_{n(n+1)}^{-1}(S_n)] = \gamma_n \cdot [S_n]$  for any constructible set  $S_n \subset X_n$ , for example,  $\pi_{n(n+1)} \colon X_{n+1} \to X_n$  is a Zariski locally trivial fiber bundle with fiber variety being  $F_n$  (in which case  $\gamma_n = [F_n]$ ).

(i) The canonical Grothendieck "motivic" proclass homomorphism,

$$\Gamma^{\mathrm{pro}} \colon F^{\mathrm{pro}}(X_{\infty}) \to K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{G}}$$

is described by

$$\Gamma^{\mathrm{pro}}\left(\left[\alpha_{n}\right]\right) = \frac{\left[\alpha_{n}\right]}{\gamma_{0} \cdot \gamma_{1} \cdot \gamma_{2} \cdots \gamma_{n-1}}.$$

Here  $\gamma_0 := 1$  and  $K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{G}}$  is the localization of  $K_0(\mathcal{V}_{\mathbb{C}})$  with respect to the multiplicative set consisting of all the finite products of  $\gamma_i^{m_j}$ , i.e,

$$\mathcal{G} := \left\{ \gamma_{j_1}^{m_1} \gamma_{j_2}^{m_2} \cdots \gamma_{j_s}^{m_s} | j_i \in \mathbb{N}, m_i \in \mathbb{N} \right\}.$$

(ii) In particular, if all the fibers are the same, say  $\gamma_n = \gamma$  for any n, then the canonical Grothendieck "motivic" (ind)class homomorphism

$$\Gamma^{\mathrm{pro}} \colon F^{\mathrm{pro}}(X_{\infty}) \to K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{G}}$$

is described by

$$\Gamma^{\mathrm{pro}}\left(\left[\alpha_{n}\right]\right) = \frac{\left[\alpha_{n}\right]}{\gamma^{n-1}}.$$

In this special case the quotient ring  $K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{G}}$  shall be simply denoted by  $K_0(\mathcal{V}_{\mathbb{C}})_{\gamma}$ .

Thus one can see that the so-called motivic measure (e.g., see [Bit], [Cr], [DL1, DL2], [Kon], [Loo], [Ve], etc., and also see [Na]) is a natural and reasonable object from the viewpoint of proconstructible functions. For a more general case when  $\pi_{n(n+1)}: X_{n+1} \to X_n$  is not necessarily a Zariski locally trivial fiber bundle, see [Y10]. In this sense, our definition of proconstructible function is quite reasonable.

### $\S$ 8. Characteristic classes of proalgebraic varieties

In this section we make a quick review of the author's recent work on characteristic classes of proalgebraic varities (for more details see [Y10, Y11]).

Theorem (7.5) can be extended to a class version  $c_*^{\text{pro}}$  via the Bivariant Theory, in particular a bivariant Chern class [Br1]. Note that for a morphism  $f: X \to pt$  from a variety X to a point  $pt, \gamma: \mathbb{F}(X \to pt) \to \mathbb{H}(X \to pt)$  is nothing but the original MacPherson's Chern class transformation  $c_*: F(X) \to H_*(X)$ .

**Theorem (8.1)**(Verdier-type Riemann–Roch formula for Chern classes) For a bivariant constructible function  $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$  we have the following commutative diagram:

In particular, for an Euler morphism we have the following diagram:

(The homomorphism  $\gamma(\mathbf{1}_f) \bullet_{\mathbb{H}}$  shall be denoted by  $f^{**}$ .)

For example, for a holomorphic submersion  $f: X \to Y$  of complex varieties one gets  $\gamma(1_f) \bullet_{\mathbb{H}} = c(T_f) \cap f^*$ , where  $f^*$  is the smooth pullback in homology and  $T_f$  is the relative tangent bundle of the morphism f.

Using this Verdier–Riemann–Roch for Chern class (also see [FM] and [Sch1]), we can get the following theorem:

**Theorem (8.2).** Let  $X_{\infty} = \varprojlim_n X_n$  be a proalgebraic variety such that for each n < m the structure morphism  $\pi_{nm} \colon X_m \to X_n$  is an Euler proper morphism (hence surjective) of topologically connected algebraic varieties. Let  $H_{**}^{\text{pro}}(X_{\infty})$  be the inductive limit of the inductive system  $\{\pi_{nm}^{**} \colon H_*(X_n) \to H_*(X_m)\}$ . Then there exists a proalgebraic MacPherson's Chern class homomorphism

$$c_*^{\operatorname{pro}} \colon F^{\operatorname{pro}}(X_\infty) \to H^{\operatorname{pro}}_{**}(X_\infty) \quad defined \ by \quad c_*^{\operatorname{pro}}\left([\alpha_n]\right) = \rho^n\left(c_*(\alpha_n)\right)$$

What we have done so far is the proalgebraic Chern–Schwartz– MacPherson class homomorphism, and our eventual problem is whether one can capture this homomorphism as *a natural transformation* as in the original MacPherson's Chern class transformation.

If the commutative diagram

is a fiber square, then we call the pro-morphism  $\{f_n: Y_n \to X_n\}$  a fibersquare pro-morphism, abusing words.

**Theorem (8.3).** Let  $\{f_n: Y_n \to X_n\}$  be a fiber-square pro-morphism between two pro-algebraic varieties with structure morphisms being Euler morphisms. Then we have the following commutative diagram:

This can be furthermore generalized. First we introduce the following notion. For a morphism  $f: X \to Y$  and a bivariant class  $b \in \mathbb{B}(X \xrightarrow{f} Y)$ , the pair (f; b) is called a *bivariant-class-equipped morphism* and we just express  $(f; b): X \to Y$ . If a system  $\{b_{nm}\}$  of bivariant classes satisfies that

$$b_{nm} \bullet b_{ln} = b_{lm} \quad (l < n < m),$$

then we call the system a projective system of bivariant classes, abusing words. If  $\{\pi_{nm} \colon X_m \to X_n\}$  and  $\{b_{nm}\}$  are projective systems, then the system  $\{(\pi_{nm}; b_{nm}) \colon X_m \to X_n\}$  shall be called a projective system of bivariant-class-equipped morphisms.

For a bivariant theory  $\mathbb{B}$  on the category  $\mathcal{C}$  and for a projective system  $\{(\pi_{\lambda\mu}; b_{\lambda\mu}): X_{\mu} \to X_{\lambda}\}$  of bivariant-class-equipped morphisms, the inductive limit

$$\varinjlim_{n} \{ B_*(X_n), b_{nm} \bullet \colon B_*(X_n) \to B_*(X_m) \}$$

shall be denoted by

$$B^{\mathrm{pro}}_*(X_\infty; \{b_{nm}\})$$

emphasizing the projective system  $\{b_{nm}\}$  of bivariant classes, because the above inductive limit surely depends on the choice of it. For example, in Theorem (7.4) we have that

$$F^{\mathrm{pro}}(X_{\infty}) = F^{\mathrm{pro}}_{*}(X_{\infty}; \{\mathbb{1}_{\pi_{nm}}\}).$$

Our more general theorem is the following

**Theorem (8.4).** (i) Let  $\gamma \colon \mathbb{B} \to \mathbb{B}'$  be a Grothendieck transformation between two bivariant theories  $\mathbb{B}, \mathbb{B}' \colon \mathcal{C} \to \mathcal{C}'$  and let

$$\left\{ (\pi_{nm}; b_{nm}) \colon X_m \to X_n) \right\}$$

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be a projective system of bivariant-class-equipped morphisms. Then we get the following pro-version of the natural transformation  $\gamma_* \colon B_* \to B'_*$ :

$$\gamma^{\mathrm{pro}}_* \colon B^{\mathrm{pro}}_* (X_\infty; \{b_{nm}\}) \to B'^{\mathrm{pro}}_* (X_\infty; \{\gamma(b_{nm})\}).$$

(ii) Let  $\{f_n: Y_n \to X_n\}$  be a fiber-square pro-morphism between two projective systems of bivariant-class-equipped morphisms such that  $b_{nm} = f_n^{\bigstar} b_{nm}$ . Then we have the following commutative diagram:

As remarked in Remark (6.5), the "motivic" characteristic class  $T_y: K_0(\mathcal{V}/ ) \to H_*( ) \otimes \mathbb{Q}[y]$  can be extended to a Grothendieck transformation of suitable bivariant theories. Therefore it follows from the above general Theorem (8.4) that the "motivic" characteristic class  $T_y: K_0(\mathcal{V}/ ) \to H_*( ) \otimes \mathbb{Q}[y]$  can be extended in a suitable way to a category of provarieties. More details and some other related work will be done in a different paper.

We hope to do further investigations on (motivic) characteristic classes of proalgebraic varieties and some applications of them. (Also, see recent articles [Alu2, Alu3], [dFLNU], [O1, O2], [PM], [To], [Ve] etc.)

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